THE CREATION OF WEALTH

by Klaus Hellwig

Faculty of Mathematics and Economics, University of Ulm, 89069 Ulm, Germany. Phone: ++49731-5023540, Fax: ++49731-5023584, Email: hellwig@mathematik.uni-ulm.de.

Key words: Growth, Value

JEL classification: D90

Abstract

This paper is concerned with the existence of a consumption sequence that implies wealth to grow at a given rate. It is shown that under reasonable assumptions such a sequence exists and can be determined by solving a fixed-point problem.

THE CREATION OF WEALTH

1. INTRODUCTION

Wealth is created by generating future consumption through the use of capital. Thus the creation of wealth can be understood as a problem of planning consumption.

Assuming uncertainty the standard approach to solve this problem is to determine a consumption stream that maximizes expected utility. However, the approach has a number of deficiencies that limit its applicability considerably. First, an intertemporal utility function respresenting the time and risk preferences of the decision maker is assumed to exist and to be derivable. Second, a probability distribution of the future consequences of every feasible decision is assumed to exist and to be derivable. Third, as emphasized by Sen (1997), the utility function in particular is assumed not to depend upon the menu over which choice is being made. As a consequence, Sen argues that "the regularities of choice behavior assumed in standard models of rational choice will need significant modification" (Sen, 1997, p. 745). Finally, expected utility maximization can lead to an inefficient consumption sequence. That is, there may exist a feasible consumption sequence that guarantees an equally high consumption in all instances and for at least one instance a higher consumption (see, for example, Copeland et al, 2005, pp 66).

The deficiencies of the expected utility maximizing approach necessitate the development of alternative approaches with less disadvantages.

Such an approach has been analysed by Hellwig (1996, 1998, 2004), Hellwig, Speckbacher and Wentges (2000), Korn (1997, 1998, 2000), Korn and Schäl (1999) and Speckbacher (1998). Instead of maximizing utility they propose a two step procedure. In the first step all solutions are elimininated that are (intertemporal) inefficient. Every remaining solution can be associated with a certain growth pattern of the associated initial present value. In the second step all solutions among these that are incompatible with the growth preference of the decision maker are eliminated.

The procedure can be applied by defining the present value either cum or ex present consumption. In the articles cited above the present value is defined ex present consumption. This requires to fix present consumption a priori. Furthermore, even value preservation can not guarantee consumption to be non-negative.

Recently, Hellwig (2002a) analysed the approach more generally and in particular considered the case where the present value is defined cum present consumption. However, the underlying model is linear and therefore of limited applicability. Moreover the paper focuses on computational aspects and does not provide economic meaningful conditions for the existence of a solution satisfying given growth requirements.

The aim of this paper is to fill this gap. In section two the choice problem is formulated. In section three some applications are discussed. In section four it is shown that a desired growth pattern of wealth can be realized under reasonable assumptions. Finally, in section five two modifications of the approach are analysed.

2. THE CHOICE PROBLEM

We assume a finite-state, discrete-time standard approach (e.g. Magill/Quinzii, 1996) where uncertainty is modelled by an event-tree with a finite set of events (nodes). The following notation will be used:

$S = \{0, \ldots, n\}$:	Set of nodes
S_t :	Set of nodes at time t, $t = 0, \ldots, T$ where $S_0 = \{0\}$
$N(s)$:	Set of nodes succeeding s
$F(s)$:	Set of nodes, immediately following s
s^- :	Immediate predecessor of s. It will be assumed that s^- is uniquely given.
$c = (c_0, \ldots, c_n)$:	Consumption sequence
C :	Set of feasible consumption sequences. C will be assumed to be non void, convex and compact.

Assume $c \in C$ and let $p = (p_0, \ldots, p_n) > 0$ be a vector of node prices (price vector). Then wealth in node s will be defined as

$$V_s = V_s(c, p) = c_s + \sum_{k \in N(s)} \frac{p_k}{p_s} c_k = c_s + \sum_{k \in F(s)} \frac{p_k}{p_s} V_k. \tag{1}$$

But how should c and p be chosen?

Suppose that a suitable price vector $\bar{p} > 0$ is given. Under this assumption the desired consumption sequence $\bar{c} \in C$ should clearly maximize present wealth:

$$\text{(C1)} \quad \bar{c} \text{ is an optimal solution of } V_0(\bar{c}, \bar{p}) = \max\{V_0(c, \bar{p}) \mid c \in C\}. \tag{2}$$

Now suppose on the contrary that a suitable consumption sequence $\bar{c} \in C$ is given. Under this assumption the valuation of $\bar{c}$ should be consistent with the desired increase of wealth. Thus the price vector $\bar{p} > 0$ should satisfy

$$\text{(C2)} \quad V_s(\bar{c}, \bar{p}) = (1 + g_s)V_{s^-}(\bar{c}, \bar{p}), s = 1, \ldots, n. \tag{3}$$

where g_s is the required growth rate of wealth between nodes s^- and s.

<u>Definition</u>: $\bar{c}$ is called growth-oriented (with respect to $g_1, \ldots, g_n$), if a price vector $\bar{p} > 0$ exists such that (C1) and (C2) are satisfied.

Contrary to the expected utility maximizing approach the concept of a growth-oriented consumption sequence neither requires a utility function nor a probability distribution. Furthermore, efficiency is guaranteed by (C1). Finally, the concept is not independent of the menu over which the choice is made. A growth-oriented consumption sequence in principle may be found by expected utility maximization. However - contrary to the standard approach - if C is changed, then expected utility maximization with the same utility function may not lead to a growth-oriented consumption sequence with respect to the same growth rates (Hellwig, 2002b).

3. APPLICATIONS

The growth model developed in the last section can be applied to a variety of intertemporal choice problems. These include:

Multiperiod portfolio selection. Assume that an investor can choose among m investment and financing activities. Let $b = (b_0, \ldots, b_n)'$ be the vector of initial endowments, $A \in \mathbb{R}^{(n+1)\times m}$ the payoff matrix, $c = (c_0, \ldots, c_n)'$ the consumption vector, $x = (x_1, \ldots, x_m)'$ the vector characterizing the activity levels chosen by the investor and X the set of restrictions (such as short selling restrictions or upper bounds) that have to be taken into account. Then the set of feasible consumption sequences is given by $C = \{c \mid c = Ax + b, x \in X\}$.

Economic growth. In models of optimal economic growth (e.g. Arrow, 1968, Koopmans, 1967) it is generally assumed that an optimal growth path for an economy can be determined by maximizing the discounted utility of consumption.

However, discounting utility has been questioned. Sen (1961) and Rawls (1971, p. 294) argue that there is no ethical justification to discount the utility of our descendants while Koopmans (1960) and Koopmans, Diamond and Williamson (1964) show that a utility function of all consumption sequences, which exhibits time neutrality and satisfies other reasonable postulates on utility functions, does not exist.

The concept presented in the last section provides a possible solution to such problems.

Sustainable development. Intergenerational equity is also of interest in models of sustainable development. Following the well known Brundtland report of the WCED, sustainable development can be defined as "development that meets the needs of the present without comprising the ability of future generations to meet their own needs".

Clearly this means that present generation should restrict their decisions such that initial wealth does not decline.

Although the concept of growth oriented consumption sequences and utility maximization are incompatible in the sense outlined above, utility considerations may make sense. As an example assume that a decision is sought where the growth rates are required to be bounded from below. In this case generally more than one consumption sequence satisfying these growth requirements exists and one may be chosen that maximizes utility.

4. EXISTENCE

Assume $c \in C$ and $p > 0$. Combining (1) with the growth requirements yields

$$c_s = \{1 - \sum_{k \in F(s)} \frac{p_k}{p_s} (1 + g_k)\} \Pi_{\tau \in T(0,s)} (1 + g_\tau) V_0(c, p) \tag{4}$$

where $T(0, s)$ denotes the set of nodes between 0 and s (excluding 0 and including s). Let $\hat{p} > 0$ be an arbitrary price vector. Then the optimal solution $c^{su}(\hat{p})$ of (2) can be understood as the consumption sequence that is supplied by $\hat{p}$ and $c^d(\hat{p})$ given by

$$c^d_s(\hat{p}) = \{1 - \sum_{k \in F(s)} \frac{\hat{p}_k}{\hat{p}_s} (1+g_k)\} \Pi_{\tau \in T(0,s)} (1+g_\tau) V_0 (c^{su}, \hat{p})$$

as the consumption sequence that is demanded by $\hat{p}$. Clearly, if excess demand $z(\hat{p}) := c^d(\hat{p}) - c^{su}(\hat{p})$ is zero, $c^d(\hat{p}) = c^{su}(\hat{p})$ is a growth-oriented consumption sequence.

Suppose that $z(\hat{p}) \neq 0$. Then a new price vector may be chosen, for example, as an optimal solution $p(z)$ of $\max\{\sum_{s=0}^n z_s(\hat{p})p_s \mid p \in P\}$ where P is a suitable set of price vectors. This means that prices should be increased if demand exceeds supply and decreased if supply exceeds demand. Performing $z(p) : P \to Z$ where Z denotes the image of z und thereafter $p(z) : Z \to P$ leads to a multivalued mapping $\varphi = p(z(p)) : P \to P$. As shown in the appendix (Lemma 1), P can be chosen such that φ has a fixed point $\bar{p}$. Furthermore, it is shown (Lemmas 2 and 3) that $z(\bar{p}) = 0$ if the following assumptions hold:

(A1) (Deferred consumption). By reducing consumption in any node $s \notin S_T$ by Δc_s consumption in every node $k \in F(s)$ can be increased by $(1 + r_{1k})\Delta c_s$ where $r_{1k} > -1$.

(A2) (Anticipated consumption). Let s and $k \in F(s)$ be two arbitrary succeeding nodes. By reducing consumption in node k by $(1+r_{2k})\Delta c_s$ (where $r_{2k} > r_{1k}$) consumption in node s can be increased by Δc_s.

(A3) There exists a consumption vector $c^* \in C$ with $c^* > 0$.

(A4) $-1 < g_k < r_{1k} \quad (k = 1, \dots, n)$.

(Because C is assumed to be compact, the opportunities in (A1) and (A2) have to be upper bounded. These bounds are chosen such that they never become active. See also the proof of Lemma 3 in the appendix).

This establishes the main result of the paper which is proved in the appendix.

Theorem 1: Given (A1) - (A4) a growth-oriented consumption sequence $\bar{c} \geq 0$ exists.

5. MODIFICATIONS

Additional insights can be obtained by modifying assumptions (A1) - (A4).

As a first modification (A1) and (A2) are substituted by the following assumption:

(A5) (Perfectly transferable consumption). Let s and $k \in F(s)$ be two arbitrary succeeding nodes. By reducing (increasing) consumption in node k by $(1 + r_k)\Delta c_s$ consumption in node s can be increased (decreased) by Δc_s where Δc_s can be chosen arbitrary. (This assumption, for example, underlies the well known Cox-Ross-Rubinstein formula for the valuation of a European call option.)

Given (A3), (A4) and (A5) $-p_{s^-} + (1 + r_s)\, p_s = 0 \quad (s = 1, \ldots, n)$ for every price vector $p > 0$ is a necessary condition for (1) to have a finite optimum. Recursive application yields

$$\frac{p_s}{p_0} = \Pi_{k \in T(0,s)} \ (1 + r_k)^{-1} =: d_s.$$

Thus a growth oriented consumption vector $\bar{c}$ can be determined in two steps. First, V_0 is determined by $V_0 = \max\{\sum_{s=0}^{n} \ d_s c_s \mid c \in C\}$.

Second, $\bar{c}$ is determined by

$$\bar{c}_s = \{1 - \sum_{k \in F(s)} \frac{1 + g_k}{1 + r_k}\} \ \Pi_{\tau \in T(0,s)} \ (1 + g_\tau) \ V_0.$$

As a second modification in addition to (A1) - (A4) the following assumption is made:

(A6) For every node s a probability $\pi_s > 0$ (where $\sum_{s \in S_t} \ \pi_s = 1, \ t = 0, \ldots, T$) is given.

Given (A6) every price vector $\frac{p}{p_0}$ can be decomposed into π_s and a discount factor q_s : $\frac{p_s}{p_0} = \pi_s \ q_s$ where $q_s = \Pi_{\tau \in T(0,s)} (1 + i_\tau)^{-1}$.

Let $\pi(s/s^-)$ be the conditional probability of s given s^-. Then (4) can be written as

$$\bar{c}_s = (\sum_{k \in F(s)} \ \pi(k/s) \ \frac{i_k - g_k}{1 + i_k}) \ \Pi_{\tau \in T(0,s)} \ (1 + g_\tau) \ V_0 \qquad (5)$$

Assume that C is given by

(A7) $C = \{c \mid c_s = e_s(x) + b_s, \ s = 0, \ldots, n, \ x \in X\}$

where $x \in I\!R^m$ is a decision vector and $b_s \in I\!R$. Then the following theorem holds:

Theorem 2: Let (A1)-(A4), (A6) and (A7) hold. $\bar{c}$ is growth-oriented with respect to the growth rates $g_s = i_s \quad (s = 1, \ldots, n)$ if and only if $\bar{c}$ is an optimal solution of

$(P) \quad \max\{\sum_{s \in S_T} \pi_s \ln c_s \mid c \in C\}$.

Proof: Assume that $\bar{c}$ is growth-oriented with respect to the growth rates $g_s = i_s$ $(s = 1, \ldots, n)$. Let $\bar{p}, \bar{c}, \bar{x}$ satisfy (C1) and (C2) where $\bar{x}$ is optimal for (2). By (5) $\bar{c}_s = 0 \quad (s \notin S_T)$ and by assumption $\bar{c}_s = \frac{V_0}{q_s}$ $(s \in S_T)$.

Define $\hat{p}$ by $\hat{p}_0 = \frac{1}{V_0\,(\bar{p},\bar{c})}$, $\hat{p}_s = \hat{p}_0\ \pi_s\ q_s = \hat{p}_0\ \frac{\bar{p}_s}{\bar{p}_0} \quad (s = 1, \ldots, n)$.

Then $\frac{\hat{p}_s}{\hat{p}_0} = \frac{\bar{p}_s}{\bar{p}_0} \quad (s = 1, \ldots, n)$ and $\bar{c}_s = \frac{V_0}{q_s} = \frac{\hat{p}_0\ \pi_s}{\hat{p}_s}\ V_0 = \frac{\pi_s}{\hat{p}_s} \quad (s \in S_T)$.

It follows that $\bar{c}$ and $\bar{x}$ maximize the Lagrangean

$L = \sum_{s \in S_T}\ \pi_s \ln c_s - \sum_{s=0}^{m}\ \hat{p}_s(c_s - e_s(x) - b_s)$

and therefore (Everett, 1963, p. 401) are optimal for (P).

Now assume that $\bar{c}$ is optimal for (P). Let $\bar{p} = (\bar{p}_0, \ldots, \bar{p}_n)$ denote the vector of shadow prices of the first $m+1$ restrictions of (P). $\bar{p} > 0$ by (A1). Furthermore $\pi_s = \bar{p}_s\ \bar{c}_s \quad (s \in S_T)$. Thus $V_0 = \sum_{s \in S_T}\ \bar{c}_s\ \frac{\bar{p}_s}{\bar{p}_0} = \frac{1}{p_0}$. Let $N^*(s) := N(s) \cap S_T$. Then

$V_s = \sum_{k \in N^*(s)}\ \bar{c}_k\ \frac{\bar{p}_k}{\bar{p}_s} = \sum_{k \in N^*(s)}\ \frac{\pi_k}{\bar{p}_s} = \sum_{k \in N^*(s)}\ \frac{\pi_k}{\bar{p}_0 \pi_s q_s} = \frac{1}{\bar{p}_0 q_s} = \frac{V_0}{q_s}$.

Thus $V_s = (1 + r_s)V_{s^-} \quad (s = 1, \ldots, n)$. □

APPENDIX

Lemma 1: Let $P := \{p \mid p_0 = 1, p_s^l \leq \frac{p_s}{p_{s^-}}(s \notin S_0), 1 - \frac{1}{p_s}\sum_{k \in F(s)} p_k(1 + g_k) \geq 0\ (s \notin S_T)\}$ where $g_k > -1$ for all k and $p_s^l > 0$ are chosen such that $P \neq \emptyset$. Then φ has a fixed-point.

Proof: P is compact, non-void and convex where $p > 0$ for every $p \in P$. $c^{su}(\hat{p})$ is upper-semicontinuous and $V_0 = V_0(c(\hat{p}), \hat{p})$ (and consequently V_s for $s = 1, \ldots, n$) continuous (e.g., Luenberger, 1995, pp. 467). $c^d(\hat{p})$ is continuous. Therefore $z(\hat{p})$ is upper-semicontinuous. Because P is compact and non-void, $p(z)$ is upper-semicontinuous (Luenberger, 1995, p. 468). This implies that φ (as a combination of two upper-semicontinuous mappings) is upper-semicontinuous. φ is convex, because the set of optimal solutions of a convex optimization problem is convex. Therefore (applying Kakutani's fixed point theorem, Kakutani, 1948), a fixed point exists.

$\square$

Lemma 2: $z(\hat{p}) = 0$ for every fixed point $\hat{p}$ of φ if the following conditions hold:

(B1) $p \in P,\ 1 - \frac{1}{p_s} \sum_{k \in F(s)} p_k(1 + g_k) = 0 \Rightarrow z_s(p) \geq 0 \quad (s \notin S_T)$.

(B2) $p \in P,\ \frac{p_s}{p_{s^-}} = p^l_s \Rightarrow z_s(p) \geq 0 \quad (s \notin S_0)$.

Proof: Since $\hat{p}$ is a fixed point of φ, $\max\{\sum_s z_s(\hat{p})p_s \mid p \in P\} = \sum_s z_s(\hat{p})\hat{p}_s$. Furthermore $\sum_{s=0}^n c^d_s(\hat{p}_s)\hat{p}_s = \sum_{s=0}^n V_s\hat{p}_s - \sum_{s \notin S_T} \sum_{k \in F(s)} \hat{p}_k(1 + g_k)V_s = \sum_{s=0}^n V_s\hat{p}_s - \sum_{s \notin S_T} \sum_{k \in F(s)} V_k\hat{p}_k = V_0$. Because $\sum_{s=0}^n c^{su}_s(\hat{p}_s)\hat{p}_s = V_0$ this implies $\sum_{s=0}^n z_s(\hat{p})\hat{p}_s = 0$.

The dual of $\max\{\sum_{s=0}^n z_s(\hat{p})p_s \mid p \in P\}$ is

$$\min y_0$$

$$\text{s.t.} \qquad \sum_{k \in F(0)} p^l_k v_k - w_0 + y_0 \geq z_0(\hat{p}) \tag{6}$$

$$-v_s + \sum_{k \in F(s)} p^l_k v_k - w_s + (1 + g_s)w_{s^-} \geq z_s(\hat{p}) \quad (s \notin S_0, S_T) \tag{7}$$

$$-v_s + (1 + g_s)w_{s^-} \geq z_s(\hat{p}) \quad (s \in S_T) \tag{8}$$

$$v_s \geq 0 \quad (s \notin S_0), w_s \geq 0 \quad (s \notin S_T), y \in I\!R. \tag{9}$$

Since $\hat{p} > 0$, (6), (7) and (8) hold as equalities for every optimal solution $\bar{v}_s$ ($s \notin S_0$), $\bar{w}_s$ ($s \notin S_T$), $\bar{y}_0$ where $\bar{y}_0 = 0$. Therefore $z_s(\hat{p}) < 0$ implies $\bar{v}_s > 0$ and/or $\bar{w}_s > 0$. By complementary slackness $\frac{\hat{p}_s}{\hat{p}_{s^-}} = p_s^l$ and/or $\sum_{k \in F(s)} \hat{p}_k(1 + g_k) = \hat{p}_s$ which contradicts (B1) and (B2). Thus $z_s(\hat{p}) \geq 0$. Noting $\sum_s z_s(\hat{p}_s)\hat{p}_s = 0$ and $\hat{p} > 0$ completes the proof.

□

Lemma 3: (B1) and (B2) hold if (A1) and (A2) are met.

Proof: Assume, that (A1) holds. Choose $\hat{p} \in P$ such that $\sum_{k \in F(0)} \hat{p}_k(1 + g_k) = 1$. Then the net present value of deferring one unit of consumption in $t = 0$ is $-1 + \sum_{k \in F(0)} \hat{p}_k(1 + r_{1k}) > -1 + \sum_{k \in F(0)} \; p_k(1 + g_k) = 0$. Value maximization therefore requires to defer consumption as much as possible. As a result $c_0(\hat{p})$ strictly decreases with the amount of consumption that is deferred. On the other hand $c^d(\hat{p}) = (1 - \sum_{k \in F(0)} \hat{p}_k(1 + g_k))V_0 = 0$. Thus $z_0(\hat{p}) \geq 0$ if the amount of consumption that is deferred is increased sufficiently. A similar argumentation applies to all nodes $s \in S_1$ and subsequently to all nodes $s \in S_t$, $t = 2, \ldots, T-1$. This proves (B1).

Now assume that (A2) holds. For $k \in F(0)$ choose p_k^l such that $p_k^l < (1+r_{2k})^{-1}$ and $\hat{p} \in P$ such that $\frac{\hat{p}_k}{\hat{p}_0} = p_k^l$. Then the time zero value of anticipating one unit of consumption between nodes $s = 0$ and k is $1 - \hat{p}_k(1+r_{2k}) = 1 - p_k^l(1+r_{2k}) > 0$. Therefore consumption should be anticipated as much as possible and $c_k(\hat{p})$ strictly decreases with the amount of anticipated consumption. On the other hand $c_k^d(\hat{p}) = (1 - \frac{1}{\hat{p}_k} \sum_{\tau \in F(k)} \hat{p}_\tau(1+g_\tau))V_0 \geq 0$ since $\hat{p} \in P$ and $V_0 \geq 0$ by (A3). Thus $z_0(\hat{p}) \geq 0$ if the upper bound for anticipating consumption is chosen sufficiently high. A similar argumentation subsequently applies to the succeeding nodes s and $k \in F(s)$ where $s \in S_t$, $t = 1, \ldots, T-1$. This proves

(B2).

□

Lemma 4: Given (A1) - (A4). Then $\bar{c} \geq 0$ for every growth-oriented consumption sequence $\bar{c}$.

Proof: (A3) and (A4) imply $V_s \geq 0$ for $s = 1, \ldots, n$. The assertion then follows from $1 - \frac{1}{p_s} \sum_{k \in F(s)} p_k (1 + g_k) \geq 0$ for every $p \in P$ and $s \notin S_T$.

□

REFERENCES

Arrow, K.J. (1968). *Applications of Control Theory to Economic Growth*, in Mathematics of the Decision Sciences, Pt. 2nd ed. by C.B. Dantzig and A.F. Veinolt (American Mathematical Society).

Copeland, T.E; Weston, J.F. and Shastri, K. (2005). *Financial Theory and Corporate Policy (4th ed.).*, Addison-Wesley Series in Finance.

Everett, H. (1963). *Generalized Lagrange Multiplier Method for Solving Problems of Optimum Allocation of Resources*, Operations Research, 11, pp. 399-417.

Hellwig, K. (1996). *Portfolio Selection Under the Condition of Value Preservation*, Review of Quantitative Finance and Accounting, 7 (1), pp. 299-305.

Hellwig, K. (1998). *Creating Value*, International Review of Economics and Finance, 7, pp. 141-147.

Hellwig, K.; Speckbacher, G. and Wentges, P. (2000). *Utility Maximization Under Capital Growth Constraints*, Journal of Mathematical Economics, 33, pp. 1-22.

Hellwig, K. (2002a). *Value Management*, Quantitative Finance, 2, pp. 133-139.

Hellwig, K. (2002b). *Growth and utility maximization*, Economics Letters, 77, pp. 377-380.

Hellwig, K. (2004). *Portfolio Selection Subject to Growth Objectives*, Journal of Economic Dynamics and Control 28, pp. 2119-2128.

Kakutani, S. (1948). *A Generalization of Brower's Fixed Point Theorem*, Duke Mathematical Journal 8, pp. 457-459.

Koopmans, T.C. (1960). *Stationary Ordinal Utility and Impatience*, Econometrica, 28, pp. 287-309.

Koopmans, T.C., Diamond P.A. and Williamson, R.E. (1964). *Stationary Utility and Time Perspective*, Econometrica, 32, pp. 82-100.

Koopmans, T.C. (1967). *Objectives, Constraints and Outcomes in Optimal Growth Models*, Econometrica, 35, pp. 1-15.

Korn, R. (1997). *Value preserving portfolio stratetgies in continuous-time models*, Mathematical Methods of Operations Research, 45, pp. 1-42.

Korn, R. (1998). *Value preserving portfolio strategies and the minimal martingale measure*, Mathematical Methods of Operations Research, 47, pp. 169-179.

Korn, R. and Schäl, M. (1999). *On value preserving and growth optimal portfolios*, Mathematical Methods of Operations Research, 50, pp. 189-218.

Korn, R. (2000). *Value preserving strategies and a general framework for local approaches to optimal portfolios*, Mathematical Finance, 10, pp. 227-241.

Luenberger, D.C. (1995). *Microeconomic Theory*, (McGraw-Hill).

Magill, M. and Quinzii, M. (1996). *Theory of incomplete markets*, Vol. 1 (MIT Press).

Rawls, J. (1971). *A Theory of Justice*, Massachusetts: Harvard University Press.

Sen, A. (1961). *On Optimising the Rate of Saving*, Economic Journal, 71, pp. 479-495.

Sen, A, (1997). *Maximization and the act of choice*, Econometrica, 65, pp. 745-780.

Speckbacher, G. (1998). *Maintaining capital intact and WARP*, Mathematical Social Sciences, 36, pp. 145-155.