Cost of Carry and Regime shifts in the commodity futures market

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ABSTRACT

This paper investigates the pricing theories in the storable commodity futures market using a model that can implement both the cost-of-carry and convenience yield theory. Given the findings of stochastic trend in the cost-of-carry elements, a long-run equilibrium relationship is found among the spot, futures price, carrying costs and stock level. A Markov Regime Switching model is estimated to account for the regime switching in the cost-of-carry relationship and supportive evidence is found. Though the spot and future prices are cointegrated in the long run, structural changes are detected over the sample period. The contribution of this paper is twofold. Firstly, it examines the long-run cointegration relationship between futures and spot price with other cost-of-carry elements presence of stochastic trend. Secondly, structural changes in the cost-of-carry relationship are investigated using a Regime Switching model.

KEYWORDS: commodity futures, cost-of-carry, Error Correction model, Markov Regime Switching

JEL: G13, G14

I. INTRODUCTION

The relationship between the commodity futures market and the cash market is linked through a *Cost-of-Carry* argument. When the futures price is much higher than the cash price, an arbitrage opportunity may exist and can be exploited through purchasing the physical asset and simultaneously sell futures at the higher price, and then deliver the physical commodity to settle the futures contract at maturity. The transaction profit then would be the difference between the futures and cash prices minus any costs associated with the transaction and carrying the commodity from the time purchasing till futures contract maturity. On the other hand, when the cash price is much higher than the futures price, one can easily buy a futures contract at a lower price and short selling the underlying physical commodity at a higher price. After taking physical delivery from the futures contract, he could cover the short-selling[1] in the cash market by returning the physical commodity that he gets delivered. This arbitrage opportunity is referred as *cash-and-carry* arbitrage[2]. This arbitrage helps to correct the price discrepancy between the cash and futures prices. Buying the futures pushes up futures prices, while selling the spot pushes down the cash price. Constant trading to pursue the cash-and-carry arbitrage assures that the futures and cash price have a well-defined relationship to one another, namely the Cost-of-Carry relationship.

The term, cost-of-carry, refers to the costs associated with purchasing and carrying (or holding) a physical asset for a pre-determined period of time. In theory, at any given time, the futures price should equal the cash price of a physical asset plus an allowance for carrying the asset from that time to the futures contract maturity. The carrying costs would include the financing costs for purchasing the physical asset, the storage costs (warehouse costs), shipping costs (delivery costs), insurance costs and any other associated costs. In other words, the theory implies that the difference between the futures and cash price should be the cost-of-carry, i.e. $coc = F_{t,t+n} - S_t$[3]. The futures price calculated from this formula is called full-carry futures price. The

[1] It is assumed that there is no bound on short selling.

[2] Arbitrage opportunity is mutually compatible set of net trades which are utility nondecreasing and, at most, costless to make. Allouch and Van (2002)

[3] *coc* represents cost-of-carry, $F_{t,t+n}$ is the futures price at time *t* matures at *t*+*n*, and S_t is the spot price.

actual futures price might differ from the full-carry futures price significantly under certain circumstances.

If futures prices are precisely described by the full-carry relationship, the basis[4], which is the difference between the futures price and cash price, should be positive. This condition is referred as contango market, meaning that the futures and cash market is determined by the cost-of-carry. However, it has been shown very often that the basis can be negative, i.e. the futures price is below the cash price, in many (commodity) futures markets, for instance, the Copper futures[5] contract traded on the New York Mercantile Exchange. This condition is referred as backwardation. This market condition can only occur when the futures prices are determined by reasons other than the cost-of-carry, otherwise the futures price cannot be lower than the cash price. A shortage in the physical market is the most likely reason for backwardation market condition to exist. In cases when the physical inventory is low, a reverse cash-and-carry arbitrage[6] is not easily to pursue even though the futures price is lower than the cash price. Because under such circumstance, the market participants who have the possession of the physical assets would be unwilling to lend the assets due to the advantage by holding them.

Over the last two decades or so, a large body of research has been focusing on the dynamic relationship between cash and futures prices, in particular, the relationship between the futures price and the realised spot price in the efficient market aspect.[7] Given that many financial time series are found to have a stochastic trend, i.e. it's nonstationary, the cointegration technique developed by Engle and Granger (1987) has been widely used to investigate the long-run cointegration between futures and spot price. For instance, Chowdhurry (1991) and Franses and Kofman (1991) apply VECM model to test the cointegration relationship between futures and spot price. Brenner and Kroner (1995) argue that the cointegration between futures and cash price critically depends on the time-series property of the difference between them,

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[4] $Basis_{t,t+n} = F_{t,t+n} - S_t$.
[5] As suggested by Edwards and Ma (1992), the US copper futures market is a classic backwardation market.
[6] A reverse cash-and-carry arbitrage allows arbitrage opportunity to exist by taking long position in the futures and short (borrow) the spot.
[7] See previous chapter for a detailed literature review on the general market efficiency related research

which they refer as the cost-of-carry. They argue that if the cost-of-carry element is stationary, spot and futures price is cointegrated. Heaney (1998) examines the relationship among the major cost-of-carry elements, namely the spot and futures price, interest rate and stock level. In his paper, it is assumed that the storage cost is a fix proportion of the spot price. He finds that in the long run the cost-of-carry relationship holds based on the cointegration between the futures and spot. Moreover, he models the stock level effect on the cost-of-carry relationship and finds supportive evidence that the stock level dose have an effect on the convenience yield.

Recent literature has developed that the dynamic relationship between cash and futures prices may be characterised by a nonlinear equilibrium-correction model (see, for instance, Sarno and Valente 2000). Examining foreign exchange markets, Sarno and Valente (2000) suggest that the nonlinearity may be due to factors such as non-zero transaction costs, infrequent trading, or simply the existence of structural changes in the dynamic adjustment of cash and futures price changes towards the long-run equilibrium.

This paper follows Brenner and Kroner (1995) argument and tests the cost-of-carry relationship in a five systematic equation framework. Moreover, we instigate the possibility of structural changes in the cost-of-carry estimation system and apply a Markov Regime Switching (MRS) model to account for the regime switches in the cost-of-carry relationship estimation. It contributes to the literature by testing the cointegration relationship between spot and futures price with the presence of other cost-of-carry elements, which in particular present stochastic trend. Also a Markov Regime Switching model is applied to examine the structural changes in the cost-of-carry relationship in the storable commodity futures market. The remaining of the paper is organised as follows. Section two explains the methodology, i.e. the Granger causality, cointegration test and the cost-of-carry model, and the Markov-Regime Switching model. Section three outlines the data which is followed by section four, the presentation of the results. It ends with the conclusion section.

II. METHODOLOGY

In this section we explain the econometric procedures employed to investigate the spot and futures relationship, i.e. the Granger Causality test and the cointegration test in exploring the dynamic relationship between integrated variables.

2.1 GRANGER CAUSALITY

Firstly Granger causality test is conducted on the futures price and spot price. The Granger causality test has been widely used in applied economics and finance in order to investigate the lead-lag relationship between two variables. As originally specified, the general formalization of Granger (1969) causality for the case of two scalar-valued, stationary, and ergodic time series $\{x_t\}$ and $\{y_t\}$ is defined as follows. Let $F(x_t | I_{t-1})$ be the conditional probability distribution of X_t given the bivariate information set I_{t-1} consisting of an *Lx*-length lagged vector of x_t, say $x_{t-Lx}^{Lx} \equiv (x_{t-Lx}, x_{t-Lx+1}, \ldots, x_{t-1})$, and an *Ly*-length lagged vector of y_t, say $y_{t-Lx}^{Lx} \equiv (y_{t-Lx}, y_{t-Lx+1}, \ldots, y_{t-1})$. Given lags *Lx* and *Ly*, the time series $\{y_t\}$ does not strictly Granger cause $\{X_t\}$ if:

(1)

$$F(x_t | I_{t-1}) = F\left(x_t \middle| \left(I_{t-1} - y_{t-Ly}^{Ly}\right)\right), \qquad t = 1,2, \ldots,$$

If the equality in equation (1) does not hold, then knowledge of past *y* values helps to predict current and future *x* values, and *y* is said to strictly Granger cause *x*. As shown in equation (1), strict Granger causality relates to the past of one time series influencing the present and future of another time series.

$\{y_t\}$ is said to be Granger-caused by $\{x_t\}$, if $\{x_t\}$ helps in the prediction of $\{y_t\}$, or, in other words, the coefficient of the lagged x_t's are statistically significant. This Granger-causality test is conducted in a Vector Autoregressive (VAR) model as follows:

$$
\begin{aligned}
x_t &= \alpha_0 + A(L)x_{t-L} + B(L)y_{t-L} + \varepsilon_t \\
y_t &= \alpha_0 + A'(L)y_{t-L} + B'(L)x_{t-L} + \varepsilon_t
\end{aligned} \tag{2}
$$

where, *A(L), B(L), A'(L), B'(L)* are the lag operators. The test of whether *y* strictly Granger causes *x* is simply a test of the joint restriction that all the coefficients contained in the lag polynomial $B(L)$ are zero. Similarly, a test of whether *x* strictly Granger causes *y* is a test of the restriction that all the coefficients contained in the lag polynomial $B'(L)$ are jointly zero.

2.2 TESTING FOR COINTEGRATION: A GENERAL FRAMEWORK

Since the cointegration relationship exists in two or more unit root variables, it is useful to start the discussion by considering the univariate time-series model:

$$y_t - \mu = \rho(y_{t-1} - \mu) + e_t \tag{3}$$

where, y_t is a univariate time-series, μ is the mean of it and e_t is a pure white noise with mean zero and constant variance. The coefficient ρ measures the degree of persistence of y_t deviations of from μ. When ρ=1, these deviations are permanent. In this case, y_t is said to follow a random walk – it can wander arbitrarily far from any given constant if enough time passes. In fact, when $|\rho| > 1$ the variance of y_t is infinite as time *t* increases and μ the mean of y_t is not defined. Alternatively, when $|\rho| < 1$, the series is said to be mean reverting and the variance of y_t is finite. Put it in another way, a stochastic process $\{y_t\}$ is said to be strictly stationary if the joint distribution of $y_{t_1}, \cdots, y_{t_n}$ is the same as the joint distribution of $y_{t_1+\tau}, \cdots, y_{t_n+\tau}$ for all $t_1, \cdots, t_n$ and τ. The distribution of the stationary process remains unchanged when shifted in time by an arbitrary value τ. Thus the parameters (i.e. the mean, and variance[8]) which characterise the distribution of the process do not depend on *t*, but on the lag τ. Many economic variables are found to be integrated I(1) process, i.e. the first and the second

[8] Mean: $\mu_t = E(y_t)$, variance: $\sigma_t^2 = \text{var}(y_t)$.

moments of the time series are time dependent and the first difference of the time series are stationary.

Given the conventional regression analysis are based on the assumption that the variables under examination are all stationary, a method to investigate the relationship between nonstationary variables are desired. Cointegration technique, which is pioneered by Engle and Granger (1987), is just to explore the long run relationship between two or more integrated variables. It states that give any two or more variables that have been found to be a unit root process, there is one or more linear combination of these variables can be found to be stationary, then it is said to have one (or more) cointegration relationship(s) between the variables.

Consider the Johansen (1991, 1995) testing methodology, which applies maximum likelihood to the Vector Autoregressive (VAR) model and assumes the residuals are Gaussian to test the cointegration in a system.

(4)
$$\mathrm{Y}_t = A_1 \mathrm{Y}_{t-1} + A_2 \mathrm{Y}_{t-2} + \cdots + A_K \mathrm{Y}_{t-K} + \mathrm{E}_t$$

where, Y_t is an n-vector of I(0) variables. By taking the first difference, equation (4) becomes:

(5)
$$\Delta \mathrm{Y}_t = B_1 \mathrm{Y}_{t-1} + B_2 \Delta \mathrm{Y}_{t-1} + \cdots + B_k \Delta \mathrm{Y}_{t-k+1} + \mathrm{E}_t$$

where, $B_1 = \sum_{i=1}^{k} A_i - \mathrm{I}$ and $B_j = -\sum_{i=j}^{k} A_i$ for j = 2, ….. , k. Since the first differences $\Delta \mathrm{Y}_t$, …, $\Delta \mathrm{Y}_{t-k+1}$ are all I(0) but Y_{t-1} is I(1), in order that this equation be consistent, B_1 should not be of full rank. Let its rank be r. $\mathrm{B}_1 = \alpha\beta`$ where $\alpha$ is an $n$ x $r$ matrix and $\beta_1$ is a $r$ x $n$ matrix. Then $\beta`Y_{t-1}$ are the r cointegrated variables, $\beta`$ is the matrix of coefficients of the cointegrating vectors and α has the interpretation of the matrix of error correction terms.

In order to get the coefficient matrices α and β that we are interested in, we need to eliminate B_2, … , B_k first. This is done by regressing $\Delta \mathrm{Y}_t$ on $\Delta \mathrm{Y}_{t-1}$, … , $\Delta \mathrm{Y}_{t-k+1}$ to get the residuals, which are called R_{0t}. Then regress Y_{t-1} on these same variables and get the residuals, which are called R_{1t}. Hence the regression equation is reduced to:

(6)

$$R_{0t} = \alpha\beta' R_{1t} + e_t$$

Let us define the matrices of the sums of squares and sums of products of R_{0t} and R_{1t} as $\begin{bmatrix} S_{00} & S_{01} \\ S_{10} & S_{11} \end{bmatrix}$ (each of the matrices is n x n). Taking partial difference against α and β and maximising the likelihood function and solve the likelihood function, we get the Maximum Likelihood (ML) $L_{\max}^{-\frac{2}{T}} = |S_{00}| \cdot \prod_{i-1}^{n}(1-\lambda_i)$, λ_i are the roots of the determinant equation of $\left|S_{00}S_{00}^{-1}S_{01} - \lambda S_{11}\right| = 0$. The Likelihood Ratio (LR) test statistic for the hypothesis of at most r cointegrating vectors is:

$$\lambda_{trace} = -T\sum_{i=r+1}^{n}\ln(1-\hat{\lambda}_i)$$

Where $\hat{\lambda}_{n+1}, \cdots, \hat{\lambda}_n$ are the (n –r) smallest eigenvalues of the determinant equation $\left|S_{00}S_{00}^{-1}S_{01} - \lambda S_{11}\right| = 0$.

To test the null hypothesis of r cointegrating vectors versus the alternative of (r + 1) cointegrating vectors the LR test statistic is:

(7)

$$\lambda_{\max} = -T\ln(1-\hat{\lambda}_{r+1})$$

2.3 Cost-of-Carry and Convenience Yield model

The cost-of-carry pricing relationship suggests the following formula links between futures and spot prices:

(8)

$$F_{t,t+n} = S_t \cdot \exp(r_{t,t+n} + c_{t,t+n} + cy_{t,t+n} + \theta_{t,t+n})$$

where, $F_{t,t+n}$ is the futures price at time t maturity at $t+n$, S_t is the underlying spot price at time t, $r_{t,t+n}$ is the compounding risk-free interest over the period, $c_{t,t+n}$ is the storage costs from time t to $t+n$, $cy_{t,t+n}$ is the convenience yield over the period, and θ_t is the marking-to-market term[9].

Formula (8) represents that the futures price is implied by the cost-of-carry relationship to be the spot price plus any financing, storage costs and marking-to-market profit and the convenience yield by holding the physical asset. Set the marking-to-market term to be zero (following Watkins and McAleer 2003) and take logarithm, an estimable version of equation (8) is written as:

(9)

$$f_{t,t+n} = \alpha_0 + \alpha_1 s_t + \alpha_2 r_{t,t+n} + \alpha_3 c_{t,t+n} + \alpha_4 cy_{t,t+n} + \varepsilon_t$$

where, $f_{t,t+n}$ is the logarithm of futures price, s_t is the underlying spot price at time t, εt is the white noise error term .

However, the convenience yield $cy_{t,t+n}$ is not observable. Since the convenience yield is dependent on the stock level, i.e. when the stock level is high the convenience yield is low and *vice versa*. Heaney (1998) proposes to include the stock level effect in the model specification. Accordingly, equation (9) is estimated on the (logarithm) stock level l_t instead of the convenience yield.

(10)

$$f_{t,t+n} = \alpha_0 + \alpha_1 s_t + \alpha_2 r_{t,t+n} + \alpha_3 c_{t,t+n} + \alpha_4 ls_t + \varepsilon_t$$

where, ls_t is the logarithm of the inventory level at time t.

Due to the time series property in equation (10), it cannot be estimated using conventional simple regression analysis. An Error Correction (EC) model developed by Engle and Granger (1989) and the Johansen (1991,1995) cointegration testing methodology discussed above is applied in the cost-of-carry test. Equation (10), accordingly, is transformed into an Error Correct model.

[9] Marking-to-market means the daily profit and loss is transferred between traders at the end of a trading day. In empirical studies, this term is generally regarded as having little impact on the cost-of-carry relationship test, hence is often omitted.

$$\Delta f_{t,t+n} = \tau_0 + \begin{pmatrix}\tau_1 & \tau_2 & \tau_3 & \tau_4\end{pmatrix} \begin{pmatrix}\Delta s_t \\ \Delta r_{t,t+n} \\ \Delta c_{t,t+n} \\ \Delta ls_t\end{pmatrix} + \alpha \cdot \beta' \cdot \begin{pmatrix} s_t \\ r_{t,t+n} \\ c_{t,t+n} \\ ls_t \end{pmatrix} + \varepsilon_t \tag{11}$$

where, τ_i ($i = 0,\ldots,4$) are the coefficients of the lagged terms, α is the weighting matrix of the error correction term, β is the cointegration vector, and Δ is the lag operator.

The existence of a cointegration vector and statistically significant β should suggest the cost-of-carry or convenience yield theory hold.

2.4 Cost-of-carry cointegration in a Markov Regime Switching framework

The possible existence of a cointegration vector in the cost-of-carry relationship suggests that there is a long-run relationship between the five cost-of-carry elements. However, the short-run coefficients or matrices are subject to possible structure changes. This thinking is motivated by the fact that different market behaviours should be observed when the commodity futures market is in backwardation or in contango. For instance, the short-term adjustment of the cost-of-carry elements to the futures price may vary from the contango market to the backwardation where the price volatility seems relatively lower in the former market condition. In other words, the shock influences on the price supposedly to be different in different market conditions. Consequently, the short-term coefficients in equation (11) are proposed to be state dependent.

$$\Delta f_{t,t+n} = \tau_{0,s_t} + \begin{pmatrix}\tau_{1,s_t} & \tau_{2,s_t} & \tau_{3,s_t} & \tau_{4,s_t}\end{pmatrix} \begin{pmatrix}\Delta s_t \\ \Delta r_{t,t+n} \\ \Delta c_{t,t+n} \\ \Delta ls_t\end{pmatrix} + \alpha_{s_t} \cdot \beta' \cdot \begin{pmatrix} s_t \\ r_{t,t+n} \\ c_{t,t+n} \\ ls_t \end{pmatrix} + \varepsilon_t \tag{12}$$

where, s_t represents different states . In this model, we let the long-run cointegration vector remain the same across states, which is the same assumption in Francis and Owyang (2004).

III. DATA

The data set comprises daily three-month futures price of the seven metal futures contracts traded on the London Metal Exchange (LME) and the corresponding cash prices of the underlying physical assets. The specification of the London Metal Exchange provides the authors the opportunity to observe both spot and futures price in the same day. The data is obtained from the exchange. The sample period is from April 1, 1994 to July 31, 2004. Figure 1 shows the spot and three-month futures prices of the seven metals markets. In general, futures and spot prices move along very closely to each other. The descriptive statistics of the weekly spot and futures prices are shown in Table 1. The first moments of most of the futures prices are larger than that of the spot prices except copper and nickel market which reveals the opposite. It's also shown that spot prices are generally more volatile than the futures prices.

The LME has over 400 approved warehouse around the world. The storage level data is obtained from the exchange over the period December 1992 to July 2004. The major warehouses report warehouse rent on a yearly basis as US cents per ton per day in every April. The storage costs are obtained from the LME-approved warehouses via the LME and the average rent is taken across the warehouses to get the yearly storage costs. Three-month London Inter Bank Offered Rate (LIBOR)[10] is chosen as the short-term risk free rate. The BBA Libor rates between January 1989 and July 2004 are obtained from Datastream and British Bankers Association.

[10] LIBOR is the rate of interest at which banks borrow funds from other banks, in marketable size, in the London interbank market.

Table 1. Descriptive statistics of the daily spot and futures prices

		Average	Std Dev.	Skewness	Kurtosis	Obs.
	Interests	17.1	8.1	-0.4	1.8	2606
Al alloy	Cash	1347.4	190.8	0.9	3.8	2603
	3-Month	1369.3	190.9	1.0	4.0	2603
	Stocks	63942.5	25487.3	0.2	2.1	2603
	Storage costs	19.7	9.0	-0.3	1.3	2603
Al.	Cash	1504.5	171.0	0.7	3.4	2606
	3-Month	1523.1	171.6	0.7	3.5	2606
	Stocks	950551.6	497357.2	1.7	6.1	2606
	Storage costs	18.8	2.7	-0.3	1.7	2606
Copper	Cash	2025.3	506.0	0.7	2.1	2606
	3-Month	2015.3	469.1	0.7	2.2	2606
	Stocks	472317.8	251181.5	0.4	1.8	2606
	Storage costs	15.4	3.7	0.0	1.4	2606
Lead	Cash	569.6	125.3	1.0	3.3	2606
	3-Month	574.3	116.6	0.9	2.7	2606
		157789.7	75324.7	1.5	4.6	2606
		15.3	2.8	0.1	1.5	2606
Nickel	Cash	7483.4	2312.3	1.4	5.9	2606
	3-Month	7501.8	2276.9	1.4	5.8	2606
	Stocks	48223.2	36531.6	1.4	4.3	2606
	Storage costs	19.9	3.3	-0.3	1.4	2606
Tin*	Cash	5495.7	1014.1	1.3	6.9	2606
	3-Month	5504.1	966.8	1.0	6.0	2606
	Stocks	16031.7	8878.3	1.0	2.8	2606
	Storage costs	16.4	3.4	0.0	1.4	2606
Zinc	Cash	1012.6	167.3	0.9	5.6	2606
	3-Month	1027.9	155.3	0.4	3.7	2606
	Stocks	557728.9	272497.4	0.9	3.3	2606
	Storage costs	15.7	3.0	0.2	1.5	2606

** The cash and 3-m futures prices are presented as USD/ton, stock level is presented in ton, storage costs is US cents/ton over the 3 months period.

IV. EMPIRICAL RESULTS

4.1 unit root and Granger Causality test

The time series of interest are the first difference of the underlying series. The reason for choosing the first difference is that, as it is often found in financial data, the prices are nonstationary but first-difference stationary when stationarity is tested using Augmented Dickey-Fuller (ADF) test and the Kwiatkowski, Phillips, Schmidt, and Shin (KPSS) Test. The stationarity test results are shown in Table 2. As expected, the futures and spot prices are found to be integrated at level one. So are the stock levels and the storage costs data[11]. It's safe to conclude that all the variables under examination are integrated $I(1)$ process.

The Granger causality test results are mixed. In the aluminium alloy and copper market, the lagged futures prices help to predict the spot prices and *vice versa*. In the lead and tin markets, there is Granger causality from the spot to the futures price but not the other way around. The zinc lagged futures prices help to predict the spot price, i.e. there is Granger causality from futures to spot only. In aluminium market, there is no clear evidence of Granger causality in both directions according the tests on the examining period. There is Granger causality from the futures to spot market at 10% confidence level in the nickel market.

4.2 Cointegration test

Two (or more) non-stationary time series are said to be cointegrated if a linear combination of the terms results in a stationary time series. For example, if X_t and Y_t are non-stationary but X_t - C Y_t is stationary (for some constant C), then the two series are cointegrated (and there's an underlying, common trend). This would be the case if the "error", $e_n = X_n - C\, Y_n$ is stationary and therefore has time-independent statistical parameters: Mean, Variance and Autocovariance.

[11] The stock level of lead is found to be stationary at levels according to ADF test, but fails to pass the KPSS unit root test, i.e. KPSS test suggests that it is integrated at levels.

We apply Johansen (1991, 1995a) cointegration test methodology to examine the long-run relationship between the five variables in equation (10), namely the spot and futures prices, interests, storage costs and inventory level. The cointegration test is conducted in a Vector Error Correction model (VECM):

(13)

$$\Delta Y_t = \Pi Y_{t-1} + \sum_{i=1}^{p-1} \Gamma_i \Delta Y_{t-i} + e_t$$

where $\mathbf{Y}_t$ is the 5 x 1 vector of variables under examination, $Y_t = \left(f_{t,t+n} \quad s_t \quad r_{t,t+n} \quad c_{t,t+n} \quad ls_t \right)'$. Π is the coefficient matrix. The cointegration test is to test whether there exists a vector Π with reduced rank $r < k$**,** which returns $k \times r$ matrices α and β, such that $\Pi = \alpha\beta'$ and $\beta' Yt$ is stationary. If so, r is the number of cointegration relations (the rank) and each column of β is the cointegrating vector. If there are r cointegration relationships, then we say that there are r long-run equilibrium relationships among the variables under examination. The null hypothesis is that there are r cointegration relationships and likelihood ratio is calculated. The test is based on a sequential testing, from $r = 0$ to $r = k - 1$, and terminates when it fails to reject. Once the number of cointegration relationship r is identified, matrix α represents the speed of adjustment or loading attached to the variables and thus generates an estimate of the rate at which the variables react to departures from long-run relationship. If the variables are exogenous, the speed of adjustment is zero. We present cointegration test results on the number of cointegration relationship and the cointegrating vector coefficients of the five major cost-of-carry elements in Table 4.

The evidence of using Johansen cointegration test in the cost-of-carry relationship has been little. Theoretically, Brenner and Kroner (1995) argues that if the cost-of-carry elements (e.g. interests) are not stationary, the spot and futures price tend not to be cointegrated which is (they state) mostly the case in commodity markets. Moreover, they argue that the differential, i.e. the rest cost-of-carry elements apart from spot and futures price, must be included in the testing system to find cointegration. However, we argue that there should exist a long-run relationship in the cost-of-carry elements even though interests to maturity and storage costs are nonstationary. Heaney (1998) uses four elements in the cost-of-carry model, namely spot, futures price, interest rate and stock level in the lead contract traded on the LME and finds one cointegration relationship among the variables. Even though the interest rate and stock level are found to be nonstationary.

In Brenner and Kroner (1995) paper, they apply no-arbitrage asset pricing model to derive the relationship between contemporaneous spot and futures price in foreign exchange, commodity and equity markets. Specifically, in storable commodity market, the differential between futures and spot is the cost-of-carry, which consists of interests to maturity and storage costs (including convenience yield) to maturity. They state that if the differential is found to be nonstationary, the spot and futures price should not be cointegrated. However, the empirical results in this paper confront the theoretical anticipation from Brenner and Kroner. It is found that there is one cointegration relationship in six out of seven metals markets under examination. One exception is the aluminium market in which two cointegration relationships are found. In all the market there is no cointegration relationship is found between futures price and interests to maturity, storage costs. In the aluminium market the stock level is found to be cointegrated with both the futures price and spot price. Specifically, there are two cointegration relationships between the futures price and the stock level at both 5% and 1% confidence level. Granger and Lee (1989) suggest that in a bivariate I(1) system more than one cointegrating vector may exist such that the number of cointegrating vectors and the number of stochastic trends do not add up to the dimensions of the system as is the case with cointegrated I(1) models. The presence of such a relationship indicates that the bivariate system is bound together by two equilibrating forces rather than the more traditional single equilibrium relationship that characterises conventionally cointegrated systems. The results support the cost-of-carry relationship in all the seven markets under examination.

4.3 The Structural Changes in the short-run relationship between Spot and Futures price

It has been shown that in the long run, the cost-of-carry holds in all the metal markets given the fact that there is at least one cointegration relationship in the cost-of-carry model. In this session, possible structural changes in the short-run adjustment coefficients are investigated. The Error Correction model (ECM) of equation (11) is applied here. The estimation results are shown in Table 5. As suggested by the results, there are structural changes in the model according to the statistical significance of the estimated coefficients in two states. The coefficients of the error correction term, α_e varies in the two states in all the markets (except in tin and zinc they are insignificant), suggesting that the short-run adjustment of the cost-of-carry elements are subject to structural changes.

To support the assumption that the regime where the volatility is higher is backwardation market and the regime where the volatility is lower is contango market, Table 6 represents the coefficient of variance of basis when its positive (contango) and negative (backwardation). It shows that the coefficients of variance are consistently higher when the market is in backwardation than in contango across all seven metals markets.

Table 2. Stationarity tests on the logarithm of stock level, futures and spot prices, and on the actual value of interests and storage costs

	Intercept (I) /trend (T) §	ADF test (†) on levels	ADF test on 1^{st} difference	Intercept (I) /trend (T) §	KPSS test (‡) on levels	KPSS test on 1^{st} difference
Libor interests	I+T	-3.44	-54.79*		0.464	0.409*
Al alloy						
Cash	N/A	0.3442	-56.73*	I	1.636	0.111*
3-Month	N/A	0.4034	-55.86*	I	1.7229	0.118*
Stocks	I	-2.4138	-5.6439*	I	1.068	0.1756*
StorCosts	N/A	-1.0724	-20.819*	I	5.830	0.0989*
Al						
Cash	I	-2.6496	-52.658*	I + T	0.388	0.078*
3-Month	I	-2.471	-52.729*	I + T	0.376	0.093*
Stocks	I	-2.5599	-7.955*	I + T	0.896	0.489
StorCosts	I + T	-3.0336	-51.02*	I + T	0.2597	0.034*
Copper						
Cash	N/A	0.644	-41.013*	I + T	0.844	0.268*
3-Month	N/A	0.604	-54.659*	I + T	0.836	0.265*
Stocks	N/A	-0.837	-9.970*	I + T	0.447	0.345*
StorCosts	I + T	-3.019	-15.996*	I + T	0.503	0.066*
Nickel						
Cash	N/A	0.884	-52.448	I + T	0.692	0.154*
3-Month	N/A	0.878	-51.687*	I + T	0.711	0.152*
Stocks	N/A	-1.589	-15.299*	I + T	0.301	0.089*
StorCosts	I + T	-2.299	-15.725*	I + T	0.568	0.098*
Lead						
Cash	N/A	1.224	-34.036*	I + T	0.681	0.333*
3-Month	N/A	1.155	-34.338*	I + T	0.680	0.311*
Stocks	N/A	-2.12*		I + T	0.600	0.375*
StorCosts	I + T	-3.128	-16.047*	I + T	0.559	0.091*
Tin						
Cash	N/A	-0.901	-39.58*	I + T	0.431	0.37*
3-Month	N/A	-0.917	-39.23*	I + T	0.43	0.211*
Stocks	N/A	-1.046	-22.73*	I	0.814	0.208*
StorCosts	I + T	-2.579	-15.73*	I + T	0.632	0.134*
Zinc						
Cash	I	-1.907	-55.49*	I + T	0.50	0.078*
3-Month	I	-1.813	-55.89*	I + T	0.517	0.094*
Stocks	I + T	-0.421	-24.67*	I + T	1.879	1.693
StorCosts	I + T	-2.988	-16.20*	I + T	0.691	0.111*

The Critical value at 5% level	None	Intercept	Intercept + trend
ADF	-1.6166	-2.8625	-3.4115
KPSS		0.463	0.146

†‡ The null hypothesis for the ADF test is that there is a unit root in the testing series. The Null hypothesis for the KPSS test is that the time series is stationary.
‡ The KPSS unit root test always includes an intercept.
§ Whether to include an Intercept or trend is determined in the tests on levels.
* Asteroids beside the number represents that the series are stationary at 5%.

Table 3. Granger Causality tests of futures and spot price

	$f_t = \alpha_0 + A(L)f_{t-L} + B(L)s_{t-L} + \varepsilon_t$ H_0: s_t does not Granger causes f_t.		$s_t = \alpha_0 + A'(L)s_{t-L} + B'(L)f_{t-L} + \varepsilon_t$ H_0: f_t does not Granger causes s_t.	
	L * (d.f.)	χ^2 - stat	L * (d.f.)	χ^2 - stat
Al alloy	4	**15.10** **[0.005]**	4	**16.51** **[0.002]**
Aluminium	4	2.64 [0.619]	4	4.001 [0.406]
Copper	5	**21.854** **[0.001]**	5	**27.524** **[0.00]**
Lead	4	**10.898** **[0.028]**	4	3.645 [0.46]
Nickel	5	7.486 [0.187]	5	9.388 [0.09]
Tin	3	**8.139** **[0.04]**	3	3.593 [0.31]
Zinc	3	3.128 [0.372]	3	**14.995** **[0.00]**

- L represents the number of lags included in the VAR estimation, or the degree of freedom of the Wald test. AIC is the selection criterion for determining L.
- Figures in parenthesis [ ] are probability of the Wald test
- Figures in bold are statistically significant at 5% level

Table 4. Cointegration tests on the Cost-of-Carry elements (between $f_{t,t+n}$ and the variables)

	Number of cointegration	Lags	Cointegrating coefficient			
			s_t	$r_{t,t+n}$	$c_{t,t+n}$	ls_t
Al alloy	One	6	**-1.0452** (0.018)	**-0.00088** (0.0003)	-0.0003 (0.0003)	**-0.029** (0.005)
Aluminium	Two	6	**-0.9568** (0.014)	**0.0008** (0.00033)	**0.0034** (0.00078)	**0.0133** (0.004)
Copper	One	6	**-0.9215** **(0.015)**	**-0.0011** **(0.00036)**	-0.002 (0.001)	**-0.01** **(0.0056)**
Lead	One	6	**-0.9408** (0.0092)	**-0.0024** (0.001)	-0.00045 (0.0012)	**-0.022** (0.0043)
Nickel	One	6	**-0.993** **(0.005)**	**0.00024** **(4.4e-5)**	**0.0028** **(0.00064)**	**-0.0084** **(0.0026)**
Tin	One	7	**-0.9788** **(0.0096)**	**-0.0023** **(7.5e-5)**	-0.0004 (0.00061)	**-0.0127** **(0.0032)**
Zinc	One -	6	**-0.9276** **(0.0257)**	0.0021 (0.0012)	0.0029 (0.0016)	**0.156** **(0.0067)**

- Standard errors are in brackets ( )
- Figures in bold are statistically significant at 5% level
- The CN in zinc full sample test is one at 1% level, two at 5% level.

Table 5. The short-run adjustment coefficient with and without structural breaks

$\Delta f_{t,t+n} = \alpha_0 + \alpha_1 \Delta s_t + \alpha_2 \Delta r_{t,t+n} + \alpha_3 \Delta c_{t,t+n} + \alpha_3 \Delta ls_t + a_e \cdot ecm_t + \varepsilon_t$							$\Delta f_t = \alpha_{s_t,0} + \alpha_{s_t,1} \Delta s_t + \alpha_{s_t,2} \Delta r_{t,t+n} + \alpha_{s_t,3} \Delta c_{t,t+n} + \alpha_{s_t,3} \Delta ls_t + a_{s_t,e} \cdot ecm_t + \varepsilon_t$													
	α_0	α_1	α_2	α_3	α_4	α_e	$\alpha_{1,0}$	$\alpha_{1,1}$	$\alpha_{1,2}$	$\alpha_{1,3}$	$\alpha_{1,4}$	$\alpha_{1,e}$	σ	$\alpha_{2,0}$	$\alpha_{2,1}$	$\alpha_{2,2}$	$\alpha_{2,3}$	$\alpha_{2,4}$	$\alpha_{1,e}$	σ
Al alloy	**0.034** **[0.00]**	**0.792** **[0.00]**	**0.002** **[0.00]**	-0.0003 [0.29]	**0.025** **[0.00]**	**0.053** **[0.00]**	**0.0317** **[0.00]**	**0.8868** **[0.00]**	**0.0012** **[0.005]**	-0.0003 [0.402]	0.0003 [0.97]	**0.0496** **[0.00]**	**0.0026** **[0.00]**	**0.0328** **[0.00]**	**0.6151** **[0.00]**	**0.0041** **[0.01]**	0.0002 [0.85]	**0.0804** **[0.01]**	**0.051** **[0.00]**	**0.0053** **[0.00]**
Al	**-0.018** **[0.00]**	**0.892** **[0.00]**	0.00 [0.39]	0.00 [0.87]	**0.023** **[0.00]**	**0.0315** **[0.00]**	**-0.042** **[0.00]**	**0.686** **[0.00]**	-0.00 [0.71]	0.001 [0.76]	0.053 [0.10]	**0.073** **[0.00]**	**0.0047** **[0.00]**	**-0.014** **[0.00]**	**0.9399** **[0.00]**	-0.00 [0.41]	-0.00 [0.85]	0.0076 [0.10]	**0.0237** **[0.00]**	**0.0014** **[0.00]**
Cu	**-0.011** **[0.00]**	**0.908** **[0.00]**	**-0.0031** **[0.00]**	-0.001 [0.20]	**0.0143** **[0.01]**	**0.0252** **[0.00]**	**-0.0138** **[0.00]**	**0.8548** **[0.00]**	**-0.0047** **[0.00]**	-0.0008 [0.72]	**0.0265** **[0.005]**	**0.0323** **[0.00]**	**0.0059** **[0.00]**	**-0.0068** **[0.00]**	**0.9507** **[0.00]**	-0.0002 [0.14]	**-0.0007** **[0.00]**	0.0055 [0.057]	**0.0153** **[0.00]**	**0.0014** **[0.00]**
Lead	**-0.0065** **[0.00]**	**0.8169** **[0.00]**	-0.0006 [0.63]	-0.0009 [0.19]	**0.0163** **[0.045]**	**0.0599** **[0.00]**	**-0.0064** **[0.00]**	**0.8243** **[0.00]**	**-0.0244** **[0.00]**	-0.0007 [0.72]	0.0415 [0.06]	**0.067** **[0.00]**	**0.007** **[0.00]**	**-0.0057** **[0.00]**	**0.8758** **[0.00]**	0.0018 [0.063]	-0.0011 [0.058]	-0.009 [0.19]	**0.0512** **[0.00]**	**0.0029** **[0.00]**
Ni	**-0.0037** **[0.00]**	**0.9592** **[0.00]**	**-0.0002** **[0.01]**	**-0.0010** **[0.01]**	-0.0008 [0.85]	**0.0709** **[0.00]**	**-0.0026** **[0.00]**	**0.9636** **[0.00]**	**0.0001** **[0.034]**	**-0.0007** **[0.00]**	-0.0007 [0.84]	**0.0492** **[0.00]**	**0.002** **[0.00]**	**-0.0044** **[0.00]**	**0.9388** **[0.00]**	**-0.0005** **[0.00]**	-0.0013 [0.34]	0.00 [0.99]	**0.0884** **[0.00]**	**0.0056** **[0.00]**
Tin	0.0001 [0.18]	**0.9003** **[0.00]**	0.00 [0.95]	-0.00 [0.20]	-0.002 [0.49]	0.0017 [0.10]	0.00 [0.28]	**0.964** **[0.00]**	**-0.0002** **[0.00]**	**-0.0006** **[0.045]**	0.0016 [0.49]	0.0006 [0.34]	**0.0018** **[0.00]**	0.0002 [0.74]	**0.788** **[0.00]**	-0.001 [0.28]	-0.001 [0.63]	**-0.013** **[0.047]**	0.004 [0.55]	**0.0061** **[0.00]**
Zinc	-0.0035 [0.056]	**0.9069** **[0.00]**	**-0.0024** **[0.00]**	**-0.0012** **[0.008]**	-0.0105 [0.18]	0.0014 [0.056]	-0.0112 [0.37]	**0.7285** **[0.00]**	**-0.005** **[0.00]**	-0.002 [0.51]	-0.023 [0.57]	0.0045 [0.36]	**0.0057** **[0.00]**	-0.0012 [0.33]	**0.9235** **[0.00]**	**0.0007** **[0.005]**	**-0.0014** **[0.00]**	**-0.0212** **[0.00]**	0.0004 [0.34]	**0.0016** **[0.00]**

❖ Figures in parenthesis [ ] are *p*-value.

Figure 1. The probability in each state and the basis in aluminium market

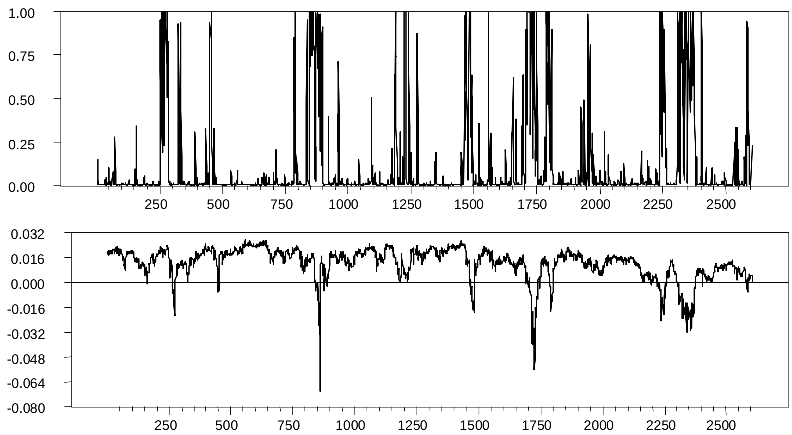
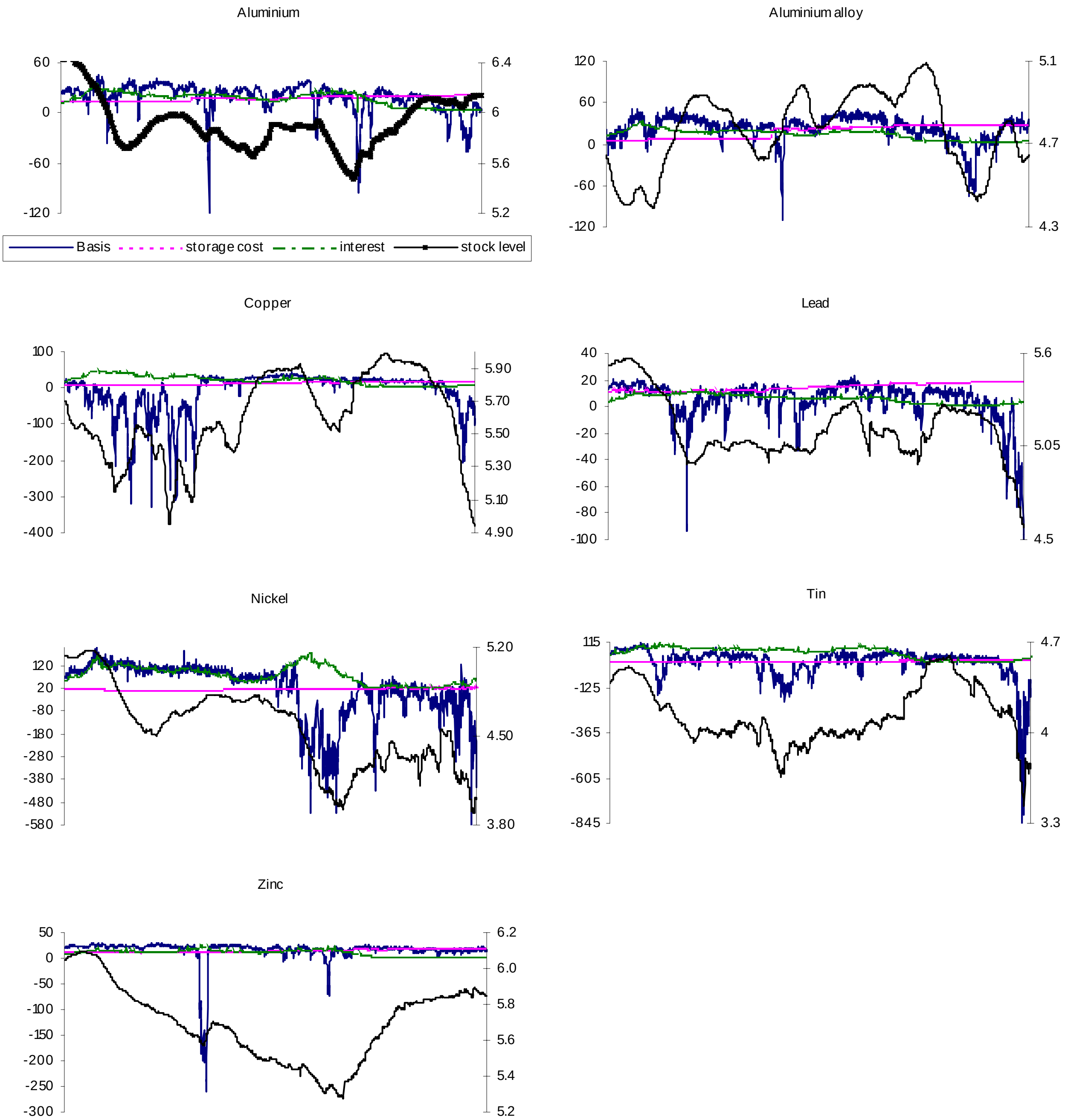


Table 6. The coefficient of variance of the basis in backwardation and contango markets

Coefficient of variance	Backwardation	Contango
Aluminium alloy	1.261	0.399
Aluminium	0.964	0.413
Copper	0.937	0.373
Lead	1.331	0.433
Nickel	0.919	0.449
Tin	1.339	0.500
Zinc	0.880	0.270

Figure 2. Cost-of-carry elements, basis, interests to maturity, storage cost and stock level of seven metal futures markets



V. CONCLUSION REMARKS

The futures pricing theory for storable commodity futures has usually been explained by the cost-of-carry relationship, which states that the futures price should consist of the underlying spot price plus the carrying costs. Some indirect tests have been applied to test of the cost-of-carry relationship in the literature. In particular, the Engle and Granger (1989) cointegration technique has been applied to test whether there exists a long run equilibrium between the futures price and its underlying spot price. Given the findings of a cointegration relationship is found, many researchers would assume that the cost-of-carry hold. However, Brenner and Kroner (1995) argue that give the stochastic trend in the cost-of-carry elements, such as the interests to maturity and storage costs, to test the cointegration between spot and futures price, one must include the cost-of-carry elements.

This paper tests the cost-of-carry relationship in a systematic cointegration framework, in which four cost-of-carry elements namely the spot, futures price, financing costs and storage costs are estimated together with the stock level effect accounting for convenience yield. The authors use available three-month futures contract and the underlying cash price from the London Metal Exchange, and the LIBOR three-month rate, average LME-approved warehouse storage costs, and the inventory level for the underlying physical metals to direct test the cost-of-carry and convenience yield theory in this market. Even though the cost-of-carry elements are found to be nonstationary, cointegration relationship is found in all the seven metal futures markets that traded on the London Metal Exchange, implying a long-run equilibrium relationship existence in the cost-of-carry elements. In other words, in the long run, the cost-of-carry holds in all the seven metal futures markets. In particular, the stock level effect is found to be significant in all the markets, suggesting that the supply and demand of the physical asset have an impact on the futures pricing. This finding is consistent with the literature (see, for instance, Heaney 1998).

After detecting the long-run cointegration vector in the cost-of-carry relationship, the short run adjustment of the elements is investigated in terms of the possibility of structural changes. A Markov Regime Switching Error Correction (MRSEC) model is implemented. Obvious structural changes are observed. We observe that the market response to shocks differently under different market conditions. For instance, when the market is in backwardation, i.e. the futures price is lower than the spot price, the adjustments from the cost-of-carry elements to the futures price are quicker than in the contango market condition. This is supported by the relatively larger value of the statistically significant coefficients in the higher volatility regime. This higher volatility regime happens to be when the basis is negative, i.e. backwardation market.

The implications of this paper are important to the literature. Contradict to the theory suggested by Brenner and Kroner (1995), the findings of this paper support the cost-of-carry model, even the underlying elements are found to be nonstationary. Moreover, the short-run adjustments of the cost-of-carry elements to the shocks are found to be different in different market conditions. In the backwardation market condition, the adjustments are found to be quicker than in the contango market. This implies that the prices are more sensitive to shocks when in backwardation than in contango.

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APPENDIX

THE ESTIMATION OF THE MARKOV REGIME SWITCHING MODEL

Consider a *K*-dimensional *M*-regime *p*-th order Markov Regime Switching Vector Autoregression (MRS-VAR) model:

(14)
$$\mathbf{y}_{s_t} = \mathbf{B}_{s_t} + \sum_{i=1}^{p} \Gamma_i \mathbf{y}_{s_t - i} + \boldsymbol{\varepsilon}_{s_t}$$

where $\boldsymbol{y}_{st}$ is the K-dimensional observed time series vector $\mathbf{y}_{s_t} = \left[\mathbf{y}_{1s_t}, \mathbf{y}_{2s_t}, \cdots \mathbf{y}_{Ks_t}\right]'$; $\mathbf{B}_{\mathrm{st}}$ is the intercept vector $\mathbf{B}_{s_t} = \left[\mathbf{B}_{1s_t}, \mathbf{B}_{2s_t}, \cdots, \mathbf{B}_{Ks_t}\right]'$; Γ_{i} is the K×K variance-covariance matrices; $\boldsymbol{\varepsilon}_{s_t}$ is the White-Noise vector with covariance matrix Σ, $\boldsymbol{\varepsilon}_{s_t} \sim NID(0, \Sigma)$; $s_t \in \{1, 2, \cdots, M\}$ is the finite number of states which are governed by the transition probabilities $p_{ij} = \Pr(s_{t+1} = j \mid s_t = i)$, $\sum_{j=1}^{M} p_{ij} = 1 \;\; \forall i, j \in \{1, 2, \cdots, M\}$. This model allows for structural changes in both the intercepts and the slopes.

In our case, i.e. examining the cost-of-carry relationship with five variables, $\mathbf{y}_{s_t} = \left[f_{s_t}, s_{s_t}, \mathrm{int}_{s_t}, ls_{s_t}, storc_{s_t}\right]'$. Let us assume that there are two states during the observing period, $s_t \in \{1, 2\}$, consequently the transition probabilities are:

$$\begin{aligned} P_{11} &= \Pr[s_t = 1 \mid s_{t-1} = 1], \quad P_{12} = \Pr[s_t = 2 \mid s_{t-1} = 1] \\ P_{21} &= \Pr[s_t = 1 \mid s_{t-1} = 2], \quad P_{22} = \Pr[s_t = 2 \mid s_{t-1} = 2] \end{aligned}$$

where the transition probability p_{12} gives the probability that state 1 will be followed by state 2, and the transition probability p_{21} gives the probability that state 2 will be followed by state 1. Transition probabilities p_{11} and *p22* give the probabilities that there will be no change in the state of the market in the following period. These transition probabilities are assumed to remain constant between successive periods and can be estimated along with the other parameters of the model.

Based on these time-varying transition probabilities, the conditional regime probabilities that the process will be in a given state at a point in time can be written as

(15) $$\Pr(s_t = 1) = P_{1,t} = \frac{1 - P_{22,t}}{2 - P_{11,t} - P_{22,t}} \quad , \quad \Pr(s_t = 2) = P_{2,t} = \frac{1 - P_{11,t}}{2 - P_{11,t} - P_{22,t}}$$

Thus, assuming normality, the density function for each regime (state of the market) can be written as follows:

(16) $$f(f_{t,t+n} \mid s_t ; \boldsymbol{\theta}) = \frac{1}{\sqrt{2\pi\sigma_{s_t}^2}} \exp\left\{ \frac{-(f_{t,t+n} - \alpha_{0,s_t} - \alpha_{1,s_t} s_t - \alpha_{2,s_t} \operatorname{int}_t - \alpha_{3,s_t} storc_t - \alpha_{4,s_t} ls_t)^2}{2\sigma_{s_t}^2} \right\}$$

where, θ , s_t=1, 2 , is the vector of parameters to be estimated.

Once the density functions for each state of the market and probabilities of being in respective states are defined, the likelihood function for the entire sample is formed by a mixture of the probability distribution of the state variable and the density function for each regime as follows:

(17) $$f(f_{tt,+n};\boldsymbol{\theta}) = \frac{P_{1,t}}{\sqrt{2\pi\sigma_{1,t}^2}} \exp\left\{ \frac{-(f_{tt,+n} - \mathrm{A}_1 \cdot coc_t)^2}{2\sigma_{1,t}^2} \right\} + \frac{P_{2,t}}{\sqrt{2\pi\sigma_{2,t}^2}} \exp\left\{ \frac{-(f_{t,+nt} - \mathrm{A}_2 \cdot coc_t)^2}{2\sigma_{2.t}^2} \right\}$$

where $P_{1,t}$, $P_{2,t}$ are the probabilities of the regime being in state 1 or 2, respectively. Coc_t represent the RHS equation. The log-likelihood of the above density function can then be defined as:

$$L(\boldsymbol{\theta}) = \sum_{t=1}^{T} \log f(f_{tt,+n};\boldsymbol{\theta})$$

(18)

which can be maximized using numerical optimization methods, subject to the constraint that $P_{1,t} + P_{2,t} = 1$ and $0 \leq P_{1,t}$, $P_{2,t} \leq 1$.