Executive Stock Options: Value to the Executive and Cost to the Firm*

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ABSTRACT

We develop a continuous time utility-based model for valuing executive stock options (ESOs). We solve for the optimal exercise policy and the value of ESOs from an executive's perspective. Assuming ESOs to be perpetual and the executive to have Constant Absolute Risk Aversion, we derive explicit formulas for the optimal exercise price and the executive's value of a vested ESO. We also prove the verification theorem for the optimal stopping problem related to this valuation. Using the optimal exercise policy that emerges, we derive a simple formula for the cost of ESOs to the firm at the grant date. From an accounting perspective, this cost formula has the advantage of being transparent and straightforward to implement using data on previous ESO exercises. Using numerical analysis, we find that our model may provide cost estimates significantly lower than cost estimates found using the Black-Scholes-Merton model, especially for firms with low expected returns and high volatilities.

1. Introduction

"It is important to point out that current valuation methods available for expensing stock options are not ideal." General Motors CFO John Devine in the Wall Street Journal, Pg. A 20, Sept 3, 2002.

The bankruptcy of Enron along with accounting irregularities in several large US corporations has lead to calls for greater transparency of accounting statements. An important part of this debate revolves around the expensing of Executive Stock Options (henceforth ESOs).[1] On one hand, press and practitioner literature seem to indicate that excessive ESO grants as well as the lack of expensing of ESO's were, in part, responsible for the internet bubble of the late 90's as well as the accounting fraud in several large corporations subsequent to the bubble.[2] On the other hand, there is a large theoretical literature starting with Jensen and Meckling (1976) that suggests that stock options have the beneficial effect of aligning managerial incentives with shareholder interests. Several empirical papers find support for the incentive alignment role of employee stock options.[3] Thus, while there are clear benefits of granting executive stock options, there also appears to be a critical need

[1] Several policy makers, public figures, and economists support expensing executive stock options. For example, the Federal Reserve Chairman, Alan Greenspan, is a strong proponent of expensing ESOs. (See article titled "Greenspan urges new rules on stock options" in the New York Times, pg C4, May 4, 2002).

[2] See for example, articles titled: "These stock options just did'nt add up" in New York Times, January 30, 2005; "Ending options is just a start" in the Financial Times, July 11, 2003; and "Origins of the Crash: The great bubble and its undoing" by Roger Lowenstein, Penguin Publishers, 2004.

[3] Recent papers that provide such evidence include Rajagopal and Shevlin (2002), Ittner, Lambert, and Larcker (2003), Hanlon, Rajagopal, and Shevlin (2003) among others

from the perspective of the investing public for correctly reporting stock option expense.

In response to this perceived need, the Financial Accounting Standards Board (FASB) as well as the International Accounting Standards Board (IASB) have recently issued mandates requiring expensing of ESOs. However, both the FASB as well as the IASB allow companies substantial latitude in deriving the fair value of these ESO's. This is because ESO's are difficult to value using conventional methods of option pricing, which are applicable principally to traded stock options. For example, a traded American call option on a non-dividend paying stock can be valued using the pricing formulas developed for traded European call options in the seminal papers of Black and Scholes (1973) and Merton (1973).[4] However, certain critical assumptions needed to derive these option pricing formula's are not met by ESOs.[5] In particular, the Black-Scholes-Merton models assume that the holder can buy and sell short the company's stock, and that the options themselves can be freely traded. However, ESOs are not transferable and cannot be traded. Further, SEC rules and company policies prevent executives from selling their own company's stock short. Therefore, valuing an ESO and computing the optimal exercise policy using the Black-Scholes-Merton framework is likely to lead to substantial errors.[6]

[4]Merton (1973) shows that early exercise of a traded American call option on a non-dividend paying stock is always sub-optimal. This implies that a traded American call option on a non-dividend paying stock can be valued as a traded European call option.

[5]See Rubinstein (1995) for a detailed discussion on how ESOs differ from traded stock options.

[6]For instance, Huddart and Lang (1996) find that over 90 percent of ESOs are exercised before maturity. This demonstrates that the early exercise feature of ESOs is valuable and that ESOs cannot be valued as European call options. See also footnote 4.

Lambert, Larcker, and Verrechia (1991) were among the first to depart from the Black-Scholes-Merton methodology by proposing a model that explicitly incorporates the trading restrictions faced by executives. They show that a utility-based approach is more appropriate in the context of ESO valuation. Subsequent to this, other authors developed utility-based models that provide several excellent insights into the determinants of option value from the executive's perspective.[7] However, most of these insights are based on numerical analysis. In contrast, this article provides analytical closed form solutions and analytical derivation of comparative statics, while also offering a proof of the verification theorem for the candidate solution proposed. Another key contribution of our paper is that our formula for the firm's cost at grant date can be easily implemented using data on past exercises. It is a transparent conservative formula with advantages from an accounting perspective. This simplicity and transparency is not observed in any previous ESO model.

The model that comes closest to ours in terms of providing a closed-form solution for ESO valuation and cost is Hemmer, Matsunaga, and Shevlin (1998). They derive a closed form solution for valuing reload options (a special type of ESO where the employee gets additional options at the time of exercise if the exercise price is paid for by previously owned shares and the employee exercise prior to maturity). While this problem may appear to be more complicated than the

[7] See, for example, Huddart (1994), Kulatilaka and Marcus (1994), Carpenter (1998), DeTemple and Sundaresan (1999), Hall and Murphy (2000), Hall and Murphy (2002) and Jain and Subramanian (2004).

ESO valuation problem, Hemmer, Matsunaga, and Shevlin (1998) demonstrate that there is a dominant exercise strategy for reload options which consists of exercising whenever the stock price exceeds the strike price. We derive a similar result with respect to exercise strategy in our paper for ESO's. In particular, the executive exercises whenever the stock price exceeds a stationary optimal exercise price.[8] Our method for deriving the firm's cost of an ESO is similar to theirs.[9]

One drawback of our model is that it cannot be applied to value reload options. However, the above cited work by Hemmer, Matsunaga, and Shevlin (1998) can be used for valuing such options. Another drawback of our model is the assumption that all options are exercised at one point in time rather than in separate chunks across time. This has been carefully addressed by Jain and Subramanian (2004) who value ESOs allowing for the possibility that a risk-averse employee strategically exercises her options over time rather than at a single date. Lastly, a third drawback is the assumption that the ESO's maturity is perpetual, which makes our work similar to Kadam, Lakner, and Srinivasan (2005) and Henderson (2005). Both of these focus on valuation of American type contingent claims in an incomplete market, and are motivated in the context of real options where infinite maturity is more easily justified. However, given the abundance of early exercise

[8]This optimal exercise price however is not equal to the strike price in our model as is the case in Hemmer, Matsunaga, and Shevlin (1998).

[9]The dominant exercise strategy that is derived in their model critically depends on the "reload" feature of the option. Thus, it cannot be applied directly to value ESO's or to derive the cost of an ESO to the firm.

evidence, it is an open empirical question as to what extent the (long) maturities of ESOs really matter.[10]

We now enumerate some important aspects of our model. The executive's optimization problem is to maximize the expected present value of utility derived from wealth generated by the option exercise in future; the control for maximization being the timing of option exercise. We assume the stock price process to be a geometric Brownian motion with constant drift and volatility parameters. We choose the negative exponential utility to model the executive's preferences.[11] The executive discounts her utility from wealth in future using an impatience parameter that captures her preference for future vs. immediate consumption. This impatience parameter and the constant absolute risk aversion parameter of negative exponential utility completely characterize the executive's preferences.

Thus, we model the value of the ESO to the executive as the value function of an optimal stopping problem. In general, identifying this value would amount to solving a (difficult) free boundary problem. In fact with the finite maturity constraint on exercise times, there is no known closed-form solution. To simplify the problem, we assume that the ESO has infinite maturity. While this assumption limits the applicability of our model, an ESO value thus computed yields an upper

[10]Huddart and Lang (1996) find a very weak relation between time remaining to maturity and exercises. This mitigates to some extent our assumption that the executive's exercise behavior is not affected by time to maturity. Hemmer, Matsunaga, and Shevlin (1996) and Carpenter (1998) also provide evidence of early exercise of ESO's.

[11]Haubrich (1994) and Garen (1994) use this utility function to evaluate models of executive compensation.

bound on the 'true' value to the executive of an otherwise identical ESO with finite maturity.

For a vested ESO, the executive's optimal exercise policy is extremely simple. The executive exercises whenever the stock price attains or exceeds a stationary critical value. We denote this optimal exercise barrier by b^*. It turns out that b^* depends only on the executive's risk aversion, the option exercise price and a quantity p^*. We show that p^* can be interpreted as the elasticity of the ESO value with respect to the underlying price process.[12] This elasticity coefficient p^*, in turn, depends solely on the drift and volatility of the price process and on the executive's time preference. Further, these parameters (the drift, the volatility and the time preference) influence the optimal exercise barrier and ESO value exclusively via the elasticity.

The cost of an ESO grant from the perspective of a diversified shareholder can be computed based on the knowledge of the executive's exercise decisions. This cost is exactly equal to the cost of taking a short position in a market of traded stock options, conditional on the knowledge of the executive's optimal exercise policy. Since the diversified shareholder has no trading restrictions, the cost of the short position can be computed using a no-arbitrage approach. This enables us to express the cost of the ESO as $\left(\frac{x}{b^*}\right)[b^* - K]$ where x is the current stock price, K is the exercise price and b^* is the optimal exercise price.

[12] This elasticity can also be interpreted as one measure of the incentive effect of the ESO, i.e., as the percentage change in the executive's value of the ESO for a unit percentage change in the underlying stock price.

The advantage of the cost expression above is that it is easily measured. Historical values of b^* are easily available to the company, and the values of x and K are known at the time of the grant. Thus, our model provides a transparent and easily implementable method to expense stock options at the time of the grant.[13]

Lastly, we use numerical analysis to compute the executive's values and the cost implied by our model. Confirming results in several previous papers, we find a significant divergence in the value of the ESO to the executive and the cost to the firm. We also compare the cost value implied by our model to the cost value implied by the Black-Scholes-Merton model. We find that the two models yield values that are comparable for cost of the firm, but also that our model may provide cost estimates significantly lower than cost estimates found using the Black-Scholes-Merton model, especially for firms with low expected returns and high volatilities.

The remainder of this paper is organized as follows. In Section 2, we review related literature. In Section 3, we develop the model to equate the valuation and optimal exercise of a vested ESO to the solution of an optimal stopping problem. We also provide therein the solution to the optimal stopping theorem and the corresponding verification theorem. Based on this solution, in Section 4 we provide analytical results and comparative statics for the ESO value to the executive. In Section 5, we use the results for the optimal exercise policy for the executive to

[13] Bartov, Mohanram, and Nissim (2004) find that managers tend to opportunistically manipulate volatility estimate to reduce option value. Our model has relatively fewer inputs, thereby reducing the possibility of manipulation.

derive the firm's cost. In Section 6, we document numerical results. In Section 7, we summarize our findings and conclude.

2. Literature Review

Early work on ESOs such as Foster, Koogler, and Vickery (1991) recognized that the value of an ESO from an executive's perspective could significantly differ from the cost of an ESO grant as incurred by the firm. This divergence in valuation perspectives has led to two streams in the ESO valuation literature: first, those that value ESOs solely from the company's perspective (a modified Black-Scholes-Merton approach) and second, those that value ESOs from the executive's perspective (a utility-based approach).

Papers in the first strand of literature that value ESOs solely from the company's perspective, such as Foster, Koogler, and Vickery (1991), Jennergren and Naslund (1993), Cuny and Jorion (1995) and Meulbroek (2001) essentially modify the Black-Scholes-Merton pricing framework to allow for a forced early exercise or termination of the option. The advantage of this approach is that the Black-Scholes-Merton formula is very well understood. However, the determinants of early exercise in this branch of literature are exogenously specified, unlike in this paper where the exercise policy is endogenously determined. Equally important is the distinction that a Black-Scholes-Merton like model assumes exactly one fixed time instant when the option may be exercised whereas most ESOs are American. The modified Black-Scholes-Merton models, in spite of adjustments, do not completely capture the executive's flexibility in the timing of ESO exercise.[14] In contrast, our model explicitly views the ESO as an American option and not as a European option.

The second strand of literature that focuses on executive's valuation has used a utility-based certainty-equivalence framework developed by Lambert, Larcker, and Verrechia (1991).[15] Most of the papers in this category, such as Huddart (1994), Kulatilaka and Marcus (1994), Aboody (1996), Carpenter (1998), Hall and Murphy (2000), Tian (2001) and Hall and Murphy (2002) use the binomial method developed by Cox, Ross, and Rubinstein (1979) to value ESOs. The papers in this category provide several excellent insights (based on numerical analysis) into the determinants of option value from the executive's perspective.[16]

The only paper that solves the ESO problem in terms of providing an explicit analytical solution for the exercise barrier is Hemmer, Matsunaga, and Shevlin (1998) who find that the optimal exercise barrier for a reload option is to exercise whenever the stock price hits the strike price. However, their methodology cannot

[14] Note that the executive's exercise policy has a direct impact on the cost to the firm.

[15] Most utility based models assume that the executive cannot hedge the option payoff. This assumption was called into question by Ofek and Yermack (2000) who find that executives are partially able to hedge their ESOs by selling stock that they currently own. While such hedging no doubt goes mitigates the effect of executive risk aversion and the utility function of executives, Huddart and Lang (1996) find that the overwhelming majority (over 90 percent) of executives, exercise their options well before maturity. Others such as Carpenter (1998), Heath, Huddart, and Lang (1999), Core and Guay (2001) find similar results with respect to early exercise. If executives are truly able to hedge their entire option portfolio, such early exercises should be a rare occurrence.

[16] Some of the utility-based models also derive the Black-Scholes-Merton cost from the firm's perspective using the prevailing stock prices at the time of exercise.

be applied to solve for the exercise barrier for standard ESO's (i.e. ESO's without a reload feature).

We differ from all the above mentioned papers in our emphasis on easily implementable explicit solutions, analytical derivation of comparative statics and a rigorous proof of the verification theorem for the candidate solution proposed. In that sense this work is very similar to Kadam, Lakner, and Srinivasan (2005). While we focus exclusively on the exponential utility, they solve an identical problem for a more general class of utility functions. As a consequence, in implementing their model they require parametric restrictions whereas we do not. These parametric restrictions make the solution of that model more applicable to real option valuation than to ESO valuation. Further we link the executive's valuation to firm cost, they do not. From an accounting perspective, this cost formula has the tremendous advantage of being transparent and straightforward to implement using data on previous ESO exercises.

In valuing the ESO, the utility based models surveyed so far do not allow the executive to optimally choose the exercise and investment policy simultaneously. DeTemple and Sundaresan (1999) introduce a simple binomial framework in which such simultaneous optimization is possible. Our model, as the rest of the literature, ignores the investment decision and focuses on optimal exercise policy in isolation.

A significant departure from the ESO literature was made by Jain and Subramanian (2004). Their valuation method permits a risk-averse employee to strategically exercise her set of options in separate chunks, with option exercise happening at different points in time. In contrast we effectively assume that all options are exercised at one point in time. The latter is a more realistic scenario, but the tradeoff is a more realistic model accompanied by an increase in the complexity of the option valuation procedure.

We discovered recently that the approach we are taking in this paper is fairly similar to that being taken (independently) by Henderson (2005), in the context of real options. She values American type contingent claims in an incomplete market by allowing a risk averse claim holder (with exponential utility) to trade in a correlated asset. This is particularly close to our work because of the perpetuity assumption she makes.[17] The valuation approach that results is more general and is likely to be more useful to the executive. For accounting purposes however, the cost to the firm is quite important as well. The cost to the firm is different from the market price of a traded option, since most ESOs have American features and the cost to the firm is affected by early exercise behavior of executives. If the approach in Henderson (2005) is extended to compute cost to the firm, the resulting cost expression that emerges is more complicated than ours, and further assumptions are necessary to ensure an explicit solution.

Our simple cost formula is derived from a utility-based model that is consistent with results in literature. For example, it has been observed that the value of the ESO and the optimal exercise boundary increase with the drift of the underlying

[17] See Henderson (2002) and Henderson (2004) for similar work without the perpetuity assumption.

stock price.[18] We also derive this result. Similarly, it has been found that ESO value may increase or decrease with volatility.[19] We identify simple conditions to characterize the effect of volatility. Our sensitivity analysis results are also consistent with the incentive effect identified in Carpenter (2000) who finds that executives may prefer to increase the variance of stock prices when granted ESOs.

To summarize the positioning of this paper in literature, we propose a cost formula that may be better than prior models in literature in its accounting related advantages (easier to implement, more transparent and conservative). Further, it is derived from a utility based model that is consistent with stylized facts mentioned in ESO valuation literature.

3. Model for executive's valuation

3.1. Price process

Consistent with the Black-Scholes-Merton setup, we assume that the price of the underlying stock follows a geometric Brownian motion with constant drift and volatility parameters μ and σ respectively. Treating the option exercise time as a stopping time, we now define the price process until exercise.

[18]See for instance Lambert, Larcker, and Verrechia (1991) and DeTemple and Sundaresan (1999), Hall and Murphy (2002).

[19]See for instance Lambert, Larcker, and Verrechia (1991) and DeTemple and Sundaresan (1999). In a Black-Scholes-Merton framework the option value strictly increases with volatility.

Let X_t denote the price process of an asset. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space. Let $\mathcal{S} = \{\tau : \Omega \mapsto [0, \infty]\}$ be the class of admissible stopping times.

For an arbitrary exercise time τ in this class we define the dynamics of the price process until option exercise as follows [20]

$$dX_t = \mu X_t \; dt + \sigma X_t \; dB_t \qquad \forall t \in [0, \tau]$$

where B_t is a standard Brownian motion on $\mathbb{P}$ and where $\sigma^2 > 0, \; \mu \geq 0$ are known constants. Note that $\mu < \frac{1}{2}\sigma^2$ is allowed so $\mathbb{P}(\tau = \infty)$ may be positive.

We assume that the drift and volatility of the stock price process are not affected by grant or exercise of ESOs. Thus we do not model the effect of stock options on the executive's effort and incentives and assume that this effect is negligible.[21] We also assume that exercise of ESOs by executives does not convey any information about the company nor does it change the executive's incentives.[22]

3.2. Terminal wealth

We assume that the executive sells the stock immediately on exercise, and hence, the wealth upon exercise is $X_\tau - K$ where K is the exercise price of the option.

[20] Implicitly, we assume that exercise times are not restricted by option maturity. We justify this in section 3.4.

[21] See Jin (2000) for a model of the incentive effects for ESOs. His paper does not focus on the valuation of ESOs. Agrawal and Mandelker (1987) among others find that managers tend to increase asset variance after being granted ESOs. Modelling such effects is outside the scope of this paper.

[22] Seyhun (1986) finds that insiders are able to identify mispricing of their company stock and trade on this information using stock they own. On the other hand, Carpenter and Remmers (2001) find little effect of option exercises by insiders on the stock prices of their companies. The latter finding is of more relevance in the context of our model.

This is consistent with actual exercise decisions of employees. Employees often use the mechanism of cashless exercise where the broker exercises the ESO and sells the stock simultaneously.[23] For analytical tractability we exclude the possibility of resetting and reloading of options on the exercise date.[24]

In our model, the effect of incorporating income tax would be to lower the net cash proceeds at the time of exercise, i.e., to lower the terminal wealth. If the tax were to be imposed on the the date of exercise, then for most reasonable tax policies, this merely results in an affine transformation of the terminal wealth. The nature of our solution and the solution procedure remain unchanged with an affine transformation of terminal wealth, so we do not model the implications of taxes explicitly (although such an extension would be trivial). However, we realize that the taxes are typically due later than the exercise date. The interest earned on taxes will affect the optimal exercise policy. In an attempt to retain the simplicity of the model, we ignore this effect.

3.3. Executive's utility preferences

We assume that the executive has a negative exponential utility. As is well known, this utility function exhibits Constant Absolute Risk Aversion. Use of exponential utility for evaluating executive compensation has been done in Haubrich

[23]Huddart and Lang (1996) provide evidence that suggests the prevalence of such cashless exercises. This is probably because of the tax treatment of capital gains on account of ESO exercise.

[24]See Acharya, John, and Sundaram (2000) for implications of the reset features and Hemmer, Matsunaga, and Shevlin (1998) and Dybvig and Loewenstein (2003) for implications of the reload feature in option based employee compensation.

(1994) and Garen (1994). If α is the Constant Absolute Risk Aversion parameter (assumed to be greater than zero) the typical form of this utility function is $U(w) = -e^{-\alpha w}$ but we introduce an offset of 1. Thus an option exercise gives rise to $1 - e^{-\alpha(X_\tau - K)}$ as the utility of wealth upon exercise. The offset of 1 serves the purpose of making the utility of non-consumption to become zero and simplifies the problem setting.[25]

3.4. The value to the executive

We model the value of the ESO to the executive as the expected present value of utility from wealth obtained by exercising the option optimally. To this end, we first define the reward for stopping at time t as the discounted utility of terminal wealth

$$g(t, X_t) = e^{-\rho t}\left(1 - e^{-\alpha(X_t - K)}\right) \tag{1}$$

where ρ is a positive discount rate. The parameter ρ captures the executive's 'opportunity cost' of a delay in exercise. It may also be interpreted as an impatience parameter that captures the executive's preference for future vs. immediate consumption. In terms of equation (1), the value of an ESO can now be defined as the supremum of expected reward $\sup_{\tau\in\ \mathcal{S}} E^x\ [g(\tau, X_\tau)1_{\{\tau<\infty\}}]$ where the supremum is computed over the domain of admissible ESO exercise times. Thus, we model

[25]The value of the ESO will in general depend on the offset if it is different from 1. Making the utility of non-consumption as an additional free parameter (constrained to exceed 1) only complicates the problem further and leaves room for ambiguity as to what is the appropriate range for the values it can take. Such an extension is straightforward as it does not impact the solution procedure in any way, merely the numerical value of the solution obtained.

the ESO value as the value function of an optimal stopping problem and the ESO exercise time as an admissible stopping time.

Solving for the value function of such a problem is quite hard in general. With the finite option maturity restriction on exercise times, the value function has no closed form solution. To simplify the problem, we assume that the option has infinite maturity. This ensures that the value function does not depend on the time elapsed.

The assumption of infinite option maturity that leaves τ unrestricted yields tremendous analytical tractability (as well as closed form solutions). In Appendix G, we show that the ESO value and the optimal exercise barrier obtained from this model (by assuming infinite maturity) are upper bounds for their analogues when the option maturity is constrained to be finite.

Given the infinite maturity assumption, time *per se* is no more a state variable in the optimization that determines the value of a vested ESO. At each time instant, the executive faces the same optimization decision, the only change is the current price of the underlying. Hence the value $V(\cdot)$ of the ESO is a function only of the current price x and not of time t elapsed since the grant date. In the reward expression (1) the random variable X_t captures the state of the process at time t. The value function in state x is given by

$$V(x) = \sup_{\tau \in \mathcal{S}} E^x \left[g(\tau, X_\tau) 1_{\{\tau < \infty\}} \right] = E^x \left[g(\tau^*, X_{\tau^*}) 1_{\{\tau^* < \infty\}} \right] \tag{2}$$

where τ^* is an optimal stopping time. We solve this optimal stopping problem in Section 3.5 to arrive at both the optimal exercise policy and the value of an ESO.

Note that we assume that the executive solves this maximization problem independent of any other assets that she holds. Thus, we do not consider the executive's optimization problem taking into account all other assets in her portfolio. Hall and Liebman (1998) and Murphy (1999) find that ESOs are the largest single component of compensation for US executives. Therefore, assuming the stock options are the only assets that the executive holds may be a reasonable first approximation.

3.5. The Optimal Stopping Problem

3.5.1. Problem

In the absence of vesting restrictions, the value $V(\cdot)$ of the ESO, given the current price x of the underlying, is the value function of the optimal stopping problem implicit in equation (2). The problem at hand is to solve for this value function and also to solve for an optimal stopping time that gives this value function. Characterizing the behavior of the optimal stopping time so obtained yields an optimal exercise policy.

3.5.2. Solution

Our solution methodology has the following three steps:

1. We first make whatever assumptions necessary to arrive at some candidate solution <optimal stopping time, value function> that *might* solve the original optimal stopping problem implicit in equation (2).
2. We then prove that if certain variational inequalities are satisfied by an arbitrary candidate solution then it is indeed a solution to the original optimal stopping problem implicit in equation (2).
3. Finally we verify that the candidate in Step 1 satisfies the variational inequalities in Step 2.

The details of Step 1 are contained in Appendix A where we derive the continuation region for the optimal stopping problem and show that the optimal exercise barrier is a constant b. We also show that value function $V(\cdot)$ of the solution must satisfy the following ODE inside this stationary continuation region

$$\mu x \frac{\partial V}{\partial x} + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 V}{\partial x^2} - \rho V = 0 \quad \forall \quad x \in \ [0, b) \tag{3}$$

At the boundary b the value from continuing equals the reward from stopping.[26] Using this boundary condition, we solve the above ODE in Appendix B and obtain a candidate solution. We state and prove the theorem on variational inequalities as required by Step 2 in Appendix C. Variational inequalities provide sufficient conditions for the value function to satisfy so as to solve the optimal stopping problem. For instance, below the optimal exercise barrier, the value from continuing should exceed the reward from stopping immediately, whereas at or above the

[26]We note that wealth upon optimal exercise makes sense only if it is positive. This implies $b > K$.

barrier, the value is the same as the immediate reward. In Appendix D we verify that the candidate satisfies the variational inequalities as required by Step 3. This concludes the solution of the optimal stopping problem implicit in equation (2).

4. Analytical Results for executive's valuation

4.1. Vested ESO value and the optimal exercise barrier

The solution described in Section 3.5 to the optimal stopping problem implicit in equation (2) implies that the unique value $V(\cdot)$ of a vested ESO, as a function of the current stock price x, is given by

$$V(x) = V(b^*)\Big(\frac{x}{b^*}\Big)^{p^*} = \Big(1 - e^{-\alpha(b^*-K)}\Big)\left(\frac{x}{b^*}\right)^{p^*} \quad \forall x \in [0, b^*) \tag{4}$$

Here the stationary optimal exercise barrier b^* is the unique root of equation[27]

$$b^* = K + \frac{1}{\alpha}\log\left(1 + \frac{\alpha}{p^*}b^*\right) \tag{5}$$

and p^* is the only positive root of quadratic equation[28]

$$\frac{1}{2}\sigma^2 p^2 + (\mu - \frac{1}{2}\sigma^2)p - \rho = 0 \tag{6}$$

[27]In Appendix E we show that equation (5) has a unique root.

[28]Parametric restrictions $\rho, \sigma^2 > 0$ together ensure that there exists exactly one positive root of quadratic equation (6).

An optimal exercise policy that yields the value $V(\cdot)$ is to exercise when the price process first hits the optimal exercise barrier. From equation (4) it follows that the value function $V(\cdot)$ satisfies $\frac{\partial V/V}{\partial x/x} = p^*$ so that p^* can be interpreted as the price elasticity of the value function.

4.2. Certainty equivalence and vesting

The value of the ESO to the executive is stated in utility terms. As proposed by Lambert, Larcker, and Verrechia (1991), we can find a price (certainty-equivalent in dollar terms) that corresponds to the level of executive's utility. We refer to the certainty-equivalent price inferred from the ESO value as the ESO price. We denote the certainty equivalent prices of vested and unvested ESOs by ψ and ψ_u respectively. For a vested ESO

$$1 - e^{-\alpha\psi} = V \;\therefore\; \psi = -\frac{1}{\alpha}\log(1 - V)$$

The value V of a vested ESO in the above expression ignores the vesting restriction. If it is known that the ESO will remain unvested for a time interval T_u in future (in particular T_u is zero for a vested ESO) one can obtain a 'fair' price ψ_u for the unvested ESO by certainty equivalence

$$1 - e^{-\alpha\psi_u} = e^{-\rho T_u} E[V(X_{T_u})] \tag{7}$$

Table 1: Summary of first order sensitivity analysis

	as ρ increases	as μ increases	as σ increases	as K increases
p^*	increases	decreases	decreases if $\mu < \rho$ increases if $\mu > \rho$	-
b^*	decreases	increases	increases if $\mu < \rho$ decreases if $\mu > \rho$	increases
$V(\cdot)$, ψ	decreases	increases	increases if $\mu < \rho$ decreases if $\mu > \rho$	decreases

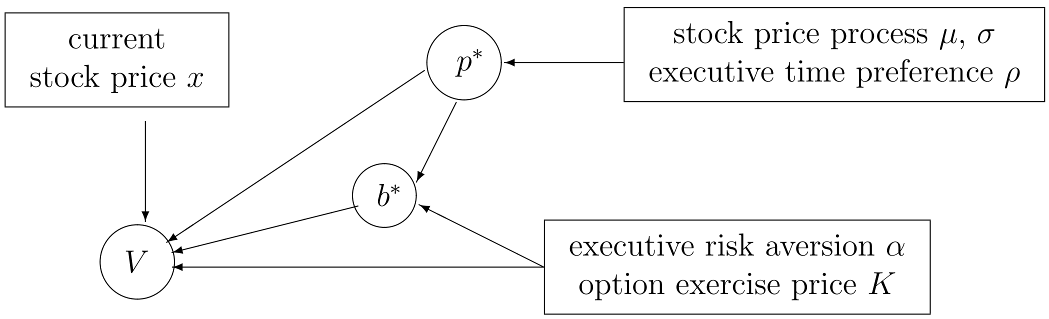
This table summarizes the results of the analytical sensitivity analysis. $V(\cdot)$ is the value of a vested ESO to the executive. ψ is the certainty equivalent in dollars of the vested ESO to the executive. b^* is the optimal exercise barrier, i.e., the price at which the executive optimally exercises the ESO. p^* is the elasticity of the ESO value with respect to the price of the underlying stock. μ is the drift and σ is the variance of the underlying stock price process. ρ is the executive's impatience parameter,i.e., the time discount rate of the utility function of the executive. K is the exercise price of the ESO.

The expectation on the right hand side of equation (7) becomes mathematically tractable because of the properties of geometric Brownian motion and because of the simple form of the value function. In Appendix H, we elaborate on how to compute this expectation. Discounting this expected future value at the discount rate ρ over a time period T_u gives the present value of an unvested ESO. The result is the right hand side of equation (7); it can be easily inverted for ψ_u, the 'fair' price of an unvested ESO.

4.3. Comparative Statics for value to the executive

The first order sensitivity analysis results for the outputs of the model are summarized in Table 1. These results are derived in Appendix L. To avoid (unnecessary) complications in presenting analytical results, we have focussed the exposition on

Figure 1: Simple pattern of dependence for the ESO value



This figure demonstrates the nature of the solution for ESO value when the executive exercises the ESO optimally. Variables in ovals are endogenous and those in rectangles are exogenous. p^* is the elasticity of ESO value with respect to the underlying stock price. b^* is the stationary optimal exercise price at which the executive exercises the ESO. V is the value of the ESO to the executive in utility terms. The stock price process is assumed to be a geometric Brownian motion with constant drift μ and constant volatility σ. The executive is assumed to have a constant absolute risk aversion of α and an impatience parameter ρ. The exercise price of the ESO is K. The current price of the underlying stock is x.

the ESO value rather than on the certainty equivalent price, but such an extension is straightforward.

4.4. Discussion on the value to the executive

4.4.1. Simple dependence patterns

Equations (4) and (5) imply that the stock price process parameters μ and σ as well as the executive impatience parameter ρ exert their impact only and entirely through the elasticity coefficient p^*. As shown in Figure 1, the elasticity coefficient p^* completely captures the dependence of the executive's optimal exercise barrier b^* as well as of the ESO value $V(\cdot)$ on these problem parameters.

4.4.2. Sensitivity to volatility

The variation in the ESO value, as the volatility parameter changes, depends on the relative magnitude of μ and ρ. When $\mu > \rho$, the executive perceives the rate of price appreciation to be fast enough so that there is a high probability of hitting the exercise barrier within a reasonable (limited) holding period. Hence, she is averse to disturbances due to volatility in the process. When $\mu < \rho$, the executive perceives the rate of price appreciation to be too slow and hence favors an increase in volatility hoping to increase the probability of hitting the exercise barrier within a reasonable (limited) holding period.[29]

Our results confirm the intuition in Lambert, Larcker, and Verrechia (1991) who construct an example where marginal utility with respect to volatility is positive when the ESO has a low probability of expiring in the money. Above a certain cut-off probability, the marginal utility with respect to volatility becomes negative. In other words, when the option has a high enough probability of expiring in the money, volatility reduces ESO value. Otherwise, volatility increases ESO value. DeTemple and Sundaresan (1999) analytically derive a similar result with respect to volatility. Similarly, Carpenter (2000) finds that an executive compensated with a European call option on assets that he controls, optimally chooses to increase the

[29] Although there is no limit on the time to maturity in our model, discounting naturally results in valuing earlier consumption more. Therefore, an increase in the chance of hitting a high price within a reasonable holding period will matter more at low values of drift.

underlying asset variance when the asset value is low, and decrease the underlying asset variance when the asset value is high.[30]

5. Model for the firm's cost

5.1. Cost as expected discounted option payout

An extremely useful implication of the analysis from the executive's perspective is the simple and explicit nature of the optimal exercise policy adopted by the executive. For the purpose of computing the cost to the firm, the magnitude of the exercise barrier is exogenous, as this barrier is determined by the executive. In the context of our model, the firm could think of b^* as a statistic that summarizes the time preference and risk aversion parameters governing the executive's exercise behavior. From the firm's perspective there is cash outflow of $b^* - K$ at an uncertain time τ^* in the future, where τ^* is the first hitting time of the barrier b^* by the stock price process.

We now compute the cost a diversified shareholder incurs by granting an ESO, conditional on the knowledge of the executive's optimal exercise policy. This is exactly equal to the cost of taking a short position in a market of traded stock

[30] Thus, our results suggest that similar incentive effects exist even when the executive is not constrained to exercise at a fixed point in time, as is the case for a European call option.

options, conditional on the knowledge of the executive's optimal exercise policy. Let $C(x)$ denote this cost, where x is the current stock price. Then[31]

$$C(x) = E^Q[e^{-r\tau^*}(b^* - K)]$$

where E^Q is the expectation computed under the risk neutral measure and r is the risk free rate.[32] The time τ^* in the above cost equation depends on the current price x and the target price b^*. It also depends on the drift and volatility parameters r and σ respectively that govern the dynamics of the price process (under the risk neutral measure) which will carry the price from x to b^*. In fact, this expectation can be explicitly computed[33] and yields $C(x) = (\frac{x}{b^*})^{q^*}[b^* - K]$ where q^* is the only positive root of the quadratic equation $\frac{1}{2}\sigma^2 q^2 + (r - \frac{1}{2}\sigma^2)q - r = 0$. The fact that the the stock price drift is also the same as the discount rate r makes q^* identically equal to 1. The ESO cost simply becomes

$$C(x) = (\frac{x}{b^*})[b^* - K] \tag{8}$$

The characterization of the cost to the firm as given in equation (8) lends itself to a nice interpretation. The term b^*-K is the cash outflow to the firm at the time the executive exercises the option. The term $(\frac{x}{b^*})$ is always less than one and therefore can be interpreted as a discount factor. A lower value of x, ceteris

[31]The R.H.S. of this cost expression depends on x via τ^*.

[32]Under the risk neutral measure Q, the discounted stock price $e^{-rt}S_t$ is a martingale and therefore the stock price process has a drift of r.

[33]See Karlin and Taylor (1975) for a similar result.

paribus, implies a smaller value for the discount factor, i.e. the present value of the cash outflow will be lower. A lower value for the current stock price x implies that it will take a longer time for the stock price to hit the exercise barrier b^*, and therefore the option will, on average, be exercised later. The cash outflow at the time of exercise is fixed. When the period over which a fixed cash outflow is discounted increases, the present value obviously decreases.

In order to facilitate comparison with the value of a traded perpetual American call option, we rewrite equation (8) as $C(x) = x\left(1 - \frac{K}{b^*}\right)$. From Merton (1973), we know that the value of a traded perpetual American call option in the absence of dividends is simply the stock price x, and the optimal exercise policy for such an option is never to exercise, i.e., it is a dominant strategy to hold the option rather than to exercise. Another way of stating the same result is that the optimal exercise price barrier for a traded perpetual American call option is infinity. In the ESO cost formula above, as b^* tends to infinity, i.e. as the executive's behavior tends to that of a diversified risk-neutral investor with no trading restrictions, the term in parentheses tends to 1 and the cost to the company tends to the cost of the corresponding traded stock option. If b^* is close to K, the executive exercises at a relatively low price, and consequently, the cost to the company is much lower than that of the corresponding traded perpetual American call option.

5.2. Comparative statics for the ESO cost

From the firm's perspective the exercise barrier $b^* > K$ is exogenous so the cost expression in equation (8) is increasing in x and b^*, decreasing in K. Therefore, ceteris paribus, higher prevailing stock prices, lower strike prices and higher exercise barriers determined by executives lead to higher ESO costs to the firm.

5.3. Accounting transparency

The advantages of our approach for accounting purposes are evident. The cost formula we propose is transparent and computationally straightforward. Rubinstein (1995) illustrates in great detail the ambiguities associated with estimating inputs required for valuing ESOs using either the Black-Scholes-Merton formula or the standard binomial approach. He demonstrates that even after using reasonable estimates of the inputs, a firm seeking to overvalue its options can report an option expense almost twice as much as that reported by a similar firm seeking to undervalue options.

In contrast, the inputs to our model as well as the computations are transparent. Closed form solution obviates the need for a binomial tree implementation. b^* is observable from past data, the strike and stock prices are known at the time of the option grant. Therefore, the company cost of the ESO can be easily estimated. The company does not have to estimate future expected return or future volatility nor does it have to estimate the executive's preferences.

Core and Guay (2002) recently developed a method to estimate the average strike price of outstanding stock options that are vested. Using this method, one can estimate the ratio of b^* to K. Thus, this method will allow companies to have a contemporaneous estimate of the value of an ESO based on current executive preferences. Alternately, companies can use historical values of market prices at the time of exercise to calculate expected stock option costs.

6. Some Numerical Estimates

In this section, we empirically examine the exercise, pricing and costing of Executive Stock Options at grant date using our option pricing model and compare estimates of cost using our model to that implied by the Black-Scholes-Merton model.[34]

6.1. Simulation Study

As a first attempt at numerical analysis we performed a simulation study, details of which are presented in Appendix M. Simulation is unavoidable for executive's valuation since a real dataset on executive exercises cannot contain executive preferences as observables. In this simulation study, by simulating price paths and executive preferences we obtained a distribution of market-to-strike ratios that was compatible with values tabulated in literature. An ESO's certainty equivalent price was roughly between one fourth and one third of its current stock price,

[34] The scope of our analysis is meant to be exploratory rather than a full-fledged test of our model versus the Black-Scholes-Merton model.

and even lower when the stock price drift was small. In comparison the Black-Scholes-Merton price for a ten year maturity and reasonable parametrization is around half the stock price. Focusing on the median ESO cost to the firm, the simulation results indicate that the median cost from our model is about 20% lower than that computed using the Black-Scholes-Merton model. However, if the Black-Scholes-Merton model with adjusted (shorter) maturity is used then the median of simulated costs to the firm is about the same in both cases.

6.2. Dow Jones 30 Sample Study

Armed with the preliminary insights from the simulation study, we proceeded to perform a more realistic examination of ESO cost formula from our model using a real data set. Our approach to compare the prices of the two models is similar in spirit to that suggested by Marquardt (2002). The details of this sample study can be found in Appendix N. We find that based on ex-post computations from realized values, the cost from our ESO cost model is lower than the Black-Scholes-Merton estimate for about a third of the stock option grants in our sample. For the sub-sample where our estimate is lower, we also find that our model estimates are about 20% lower on average than the Black-Scholes-Merton estimate. We find that companies where our estimate is lower tend to have lower expected return and higher volatility. We also find that options that yield a lower price by our model relative to Black-Scholes-Merton tend to be exercised earlier and have a lower market to strike ratio. Given the preliminary nature of our empirical analysis, it is premature to conclude as to the types of companies where our estimate would differ from the Black-Scholes-Merton estimate one way or the other. However, our preliminary analysis does suggest that companies with high volatility and lower expected return may find costs lower using our model formula than by using the Black-Scholes-Merton model.

7. Conclusion

We developed a continuous time model for computing the optimal exercise policy of ESOs, valuing ESOs from an executive's perspective, and computing the cost of the ESO from the firm's perspective. Assuming infinite maturity, we obtained closed-form solutions for all of these, and we were also able to offer a rigorous proof of the verification theorem for the candidate solution proposed. Our solution from the executive's perspective enabled us to analytically derive some of the (numerical) findings in literature regarding the impact of drift and volatility. For instance, we found that vested ESO value increases with the drift of the stock price process. We also found that the ESO value decreases with volatility when drift is high and that it increases with volatility when drift is low.

In solving the problem from the firm's perspective we were able to derive an extremely simple formula for the cost of the ESO to the firm at the grant date. This cost depends only on the executive's optimal exercise price b^*, the current stock price and the exercise price. We hope that the transparent ESO cost formula we propose will enable companies to calculate the expense of ESOs more easily.

The advantage of our model is accounting transparency. It requires the estimation of only one quantity viz. the market-to-strike ratio, this acts as a summary statistic for several Black-Scholes-Merton inputs as well as executive preferences that affect exercise behavior. The perpetuity assumption keeps the model tractable enough to incorporate vesting as shown in this paper, and potentially other features like reloads and resets as discussed elsewhere in literature.

The disadvantage of our model on the theoretical side is that it does not incorporate the option maturity restriction. Surely this factor will affect the executive's exercise behavior and hence affect the cost to the firm. With respect to the option maturity, our model gives an overestimate of the true cost, but this is consistent with conservative accounting. The disadvantage on the empirical side is that the point estimation of the market-to-strike ratio, while transparent, may give an estimate with high variance. This in turn may translate to widely varying costs. The wide variation in costs in our numerical analysis is almost entirely due to our simulation of a widely varying market-to-strike ratio. This may or may not be true empirically; the evidence from literature, as summarized in Table 2, is unclear. Besides, being forced to make a point estimate for one widely varying market-to-strike multiple is probably a lesser evil than having widely varying estimates for several Black-Scholes-Merton inputs (such as a volatility estimate for the next 10 years) or allowing for widely different binomial lattice structures from one firm to another. In the former case at least we retain accounting transparency.

Our simulation results indicate that the median cost from our model is about 20% lower than that computed using the Black-Scholes-Merton model. However, if the Black-Scholes-Merton model with adjusted (shorter) maturity is used then the median of simulated costs to the firm is about the same in both cases. A preliminary numerical analysis using our data sample of Dow Jones 30 company stock exercises indicates that using our model results in ex-post cost estimates that may be significantly lower, especially in the case of firms with low expected returns and high volatilities. In most such cases, ESOs get exercised earlier and have lower market-to-strike ratios. A more thorough empirical analysis is required to compare the cost estimates of our model with that using the Black-Scholes-Merton model.

References

Aboody, David, 1996, Market valuation of employee stock options, *Journal of Accounting and Economics* 22, 357–391.

Acharya, Viral V., Kose John, and Rangarajan K. Sundaram, 2000, On the optimality of resetting executive stock options, *Journal of Financial Economics* 57, 65–101.

Agrawal, A., and G. Mandelker, 1987, Managerial Incentives and Corporate Investment and Financing Decisions, *Journal of Finance* 42, 823–837.

Bartov, Eli, Partha Mohanram, and Doron Nissim, 2004, Stock option expense, forward-looking informatio, and implied volatilities of traded options, *Journal of Accounting Research, forthcoming.*

Black, F., and M. Scholes, 1973, The Pricing of Options and Corporate Liabilities, *Journal of Political Economy* 81, 637–654.

Carpenter, Jennifer N., 1998, The exercise and valuation of executive stock options, *Journal of Financial Economics* 48, 127–158.

Carpenter, Jennifer N., 2000, Does option compensation increase managerial risk appetite, *Journal of Finance* 55, 2311–2331.

Carpenter, Jennifer N., and Barbara Remmers, 2001, Executive Stock Option Exercises and Inside Information, *Journal of Business* 74, 513–534.

Core, John E., and Wayne R. Guay, 2001, Stock option plans for non-executive employees, *Journal of Financial Economics* 61, 253–287.

Core, John E., and Wayne R. Guay, 2002, Estimating the Value of Employee Stock Option Portfolios and their Sensitivities to Price and Volatility, *Journal of Accounting Research* 40, 613–630.

Cox, John C., Stephen A. Ross, and Mark Rubinstein, 1979, Option Pricing: A simplified approach, *Journal of Financial Economics* 7, 229–263.

Cuny, Charles J., and Philippe Jorion, 1995, Valuing executive stock options with endogenous departure, *Journal of Accounting and Economics* 20, 193–205.

DeTemple, Jerome, and Suresh Sundaresan, 1999, Nontraded Asset Valuation with Portfolio Constraints: A Binomial Approach, *Review of Financial Studies* 12, 835–872.

Dybvig, Philip, and Mark Loewenstein, 2003, Employee Reload Options: Pricing, Hedging, and Optimal Exercise, *Review of Financial Studies* 16, 145–171.

Foster, Taylor W. III, Paul R. Koogler, and Don Vickery, 1991, Valuation of Executive Stock Options and the FASB Proposal, *The Accounting Review* 66, 595–610.

Garen, J., 1994, Executive Compensation and Principal-Agent Theory, *Journal of Political Economy* 106, 1175–1199.

Gately, Dermot, 1980, Individual discount rates and the purchase and utilization of Energy-Using Durables: Comment, *Bell Journal of Economics* 11, 373–374.

Hall, Brian J., and Jeffrey B. Liebman, 1998, Are CEO's really paid like Bureaucrats, *Quarterly Journal of Economics* 113, 653–691.

Hall, Brian J., and Kevin J. Murphy, 2000, Optimal Exercise Prices for Executive Stock Options, *American Economic Review* 90, 209–214.

Hall, Brian J., and Kevin J. Murphy, 2002, Stock options for undiversified executives, *Journal of Accounting and Economics* 33, 3–42.

Hanlon, Michelle, Shivaram Rajagopal, and Terry Shevlin, 2003, Are Executive Stock Options Associated with Future Earnings, *Journal of Accounting and Economics* 36.

Haubrich, Joseph G., 1994, Risk Aversion, Performance Pay, and the Principal Agent Problem, *Journal of Political Economy* 102, 258–276.

Hausman, Jerry, 1979, Individual Discount Rates and the purchase and utilization of Energy-Using Durables, *Bell Journal of Economics* 10, 33–54.

Heath, Chip, Steven Huddart, and Mark Lang, 1999, Psychological factors and stock option exercise, *Quarterly Journal of Economics* 114, 601–627.

Hemmer, Thomas, Steven Matsunaga, and Terry Shevlin, 1996, The Influence of Risk Diversification on the Early Exercise of Employee Stock Options by Executive Officers, *Journal of Accounting and Economics* 21, 45–68.

Hemmer, Thomas, Steven Matsunaga, and Terry Shevlin, 1998, Optimal exercise and the cost of granting employee stock options with a reload provision, *Journal of Accounting Research* 36, 231–255.

Henderson, V., 2002, Valuation of claims on non-traded assets using utility maximization, *Mathematical Finance* 12, 351–373.

Henderson, V., 2004, The Impact of the Market Portfolio on the Valuation, Incentives and Optimality of Executive Stock Options, *Quantitative Finance, forthcoming.*

Henderson, V., 2005, Valuing the option to invest in an incomplete market, Working Paper, ORFE, Princeton University.

Huddart, Steven, 1994, Employee Stock Options, *Journal of Accounting and Economics* 18, 207–231.

Huddart, Steven, and Mark Lang, 1996, Employee stock option exercises An empirical analysis, *Journal of Accounting and Economics* 21, 5–43.

Ittner, Christopher D., Richard A Lambert, and David F. Larcker, 2003, The structure and performance consequences of equity grants to employees of new economy firms, *Journal of Accounting and Economics* 34.

Jain, Ashish, and Ajay Subramanian, 2004, The Inter-Temporal Exercise and Valuation of Employee Options, *The Accounting Review* 79, 705–743.

Jennergren, L., and B. Naslund, 1993, A comment on Valuation of executive stock options and the FASB proposal, *The Accounting Review* 68, 179–183.

Jensen, Michael C., and William H. Meckling, 1976, Theory of the firm: Managerial Behavior, Agency Costs and Ownership Structure, *Journal of Financial Economics* 3, 305–360.

Jin, Li, 2000, CEO Compensation, Diversification and Incentives, Working Paper MIT Sloan School of Management.

Kadam, Ashay, Peter Lakner, and Anand Srinivasan, 2005, Perpetual Call Options with Non-Tradability, *Optimal Control Applications and Methods, forthcoming.*

Karatzas, I., and S. Shreve, 1991, *Brownian Motion and Stochastic Calculus.* (Springer) second edn.

Karlin, S., and H. Taylor, 1975, *A first course in stochastic processes.* (Academic Press) second edn.

Kulatilaka, Nalin, and Alan J. Marcus, 1994, Valuing Employee Stock Options, *Financial Analysts Journal* 50, 46–56.

Lambert, Richard A., David F. Larcker, and Robert E. Verrechia, 1991, Portfolio Considerations in Valuing Executive Compensation, *Journal of Accounting Research* 29, 129–149.

Marquardt, Carol, 2002, The cost of employee stock option grants: an empirical analysis, *Journal of Accounting Research* 40, 1191–1217.

Merton, Robert, 1973, Theory of rational option pricing, *Bell Journal of Economics and and Management Science* 4, 141–183.

Meulbroek, Lisa K., 2001, The efficiency of Equity-Linked Compensation: Understanding the Full Cost of Awarding Executive Stock Options, *Financial Management* 31, 5–44.

Murphy, Kevin J., 1999, Executive Compensation, *Handbook of Labor Economics* III, 2485–2563 in Orley Ashenfelter and David Card, eds, North-Holland.

Ofek, Eli, and David Yermack, 2000, Taking Stock: Equity Based Compensation and the Evolution of Managerial Ownership, *Journal of Finance* 55, 1367–1384.

Oksendal, B., 1998, *Stochastic Differential Equations : An Introduction with Applications.* (Springer) fifth edn.

Rajagopal, Shivaram, and Terry J. Shevlin, 2002, Empirical Evidence on the Relation Between Stock Option Compensation and Risk Taking, *Journal of Accounting and Economics* 33, 145–171.

Rubinstein, Mark, 1995, On the accounting valuation of employee stock options, *Jounral of Derivatives* 3, 8–24.

Seyhun, H. Nejat, 1986, Insiders' profits, costs of trading and market efficiency, *Journal of Financial Economics* 16, 189–212.

Shane, Frederick, George Lowenstein, and Ted O'Donoghue, 2002, Time Discounting and Time Preference: A critical review, *Journal of Economic Literature* 40, 351–401.

Tian, Yisong S., 2001, Optimal Contracting, Incentive Effects and the Valuation of Executive Stock Options, Working Paper, York University.

Warner, John T., and Saul Pleeter, 2001, The Personal Discount Rate: Evidence from Military downsizing programs, *The American Economic Review* 91, 33–53.

Appendix

A. Derivation of the value function ODE

A.1. Problem reformulation

To obtain the ODE for a candidate value function of the optimal stopping problem implicit in equation (2), we first reformulate the problem so as to have a time-homogenous reward and value functions. This makes it possible to invoke some results from optimal stopping theory that facilitate the solution procedure. We reformulate the problem in terms of another multi-dimensional diffusion $Y_t = \langle s+t, X_t \rangle$ where s is the initial time of the first component of Y_t. Thus Y_t incorporates time and wealth as its two different dimensions. Rephrasing it thus has the advantage of removing the explicit time-dependence of the reward function. Redefine the value function as

$$\forall y = \langle s, x \rangle \ \in \ [0,\infty) \times [0,\infty) \text{ let}$$

$$\widehat{V}(y) = \sup_{\tau \in \ \mathcal{S}} E^y \ [g(Y_\tau)] \quad \equiv \quad \widehat{V}(s,x) = \sup_{\tau \in \ \mathcal{S}} E^{s,x} \ [g(s+\tau, X_\tau)]$$

From the definitions of the reward function $g(\cdot)$ and the value function $V(\cdot)$ it is obvious that the effects of time and wealth can be isolated. In fact

$$g(s+\tau, X_\tau) = e^{-\rho s} g(\tau, X_\tau)$$

$$\therefore \quad \widehat{V}(y) = \widehat{V}(s,x) = e^{-\rho s} \sup_{\tau \in \mathcal{S}} E^x \left[g(\tau, X_\tau) \right] = e^{-\rho s} V(x)$$

The redefinition of the reward and value functions facilitates application of the rich theory already developed for time homogeneous reward functions. One such existing result, theorem 10.1.12 from Oksendal (1998) states that if an optimal stopping time τ^* for this problem exists, then the exit time $\tau_{\mathcal{D}}$ from the continuation region

$$\mathcal{D} = \{ y : g(y) < \widehat{V}(y) \}$$

is an optimal stopping time for this problem. Assuming the existence of τ^* the problem is reduced to identifying the exact form of $\mathcal{D}$.[35] The first step towards characterizing $\mathcal{D}$ is to show that it is invariant with respect to time. This is easily done as follows:

$$\begin{aligned} \mathcal{D} &= \{ \langle s, x \rangle \colon g(s+t, X_t) < \widehat{V}(s,x) \} \\ &= \{ \langle s, x \rangle \colon e^{-\rho s} g(t, X_t) < e^{-\rho s} V(x) \} \\ &= \{ x \colon g(t, X_t) < V(x) \} \end{aligned}$$

A.2. The form of $\mathcal{D}$

Given the 1-dimensional time invariant nature of the continuation region it may, in general, be an uncountable union of distinct points or disjoint intervals or both. However from the continuity and monotonicity of the utility function and of the discount function it follows that the reward function is continuous, strictly in-

[35]This derivation is part of step 1 of the solution procedure outlined in section 3.5. In this step any assumptions necessary can be made without any rigorous proof. The limited goal here is to arrive at a candidate solution which will be verified later.

creasing in wealth and strictly decreasing in time. Hence exiting the continuation region from below will be suboptimal. Thus the region has to be either a single point or a single interval with $\inf\{X_t\} = 0$ as its lower boundary. Thus it must be the origin or $[0, b)$ or $[0, b]$ for some non-negative real number b. We ignore[36] the possibility $b = 0$ as it forces $\tau_{\mathcal{D}}$ to become identically zero.

We further assume [37] that b is finite and that the value function $\widehat{V}(\cdot)$ is twice continuously differentiable. The continuation region then inverts a $<$ relation between two continuous functions and hence it cannot be closed on both sides. Thus we arrive at the form of the continuation region as $[0, b)$ for some $0 < b < \infty$.

A.3. The ODE for $\widehat{V}(\cdot)$ and hence for $V(\cdot)$

From the positivity and finiteness of b assumption above and from the definition of $g(\cdot)$ it follows that the reward function is a bounded measurable function on the boundary of the continuation region. Thus the conditions for lemma 9.2.4 in Oksendal (1998) are satisfied. By this lemma and by the assumption of twice continuous differentiability of $\widehat{V}(\cdot)$ it follows that $\widehat{V}(\cdot)$ must satisfy the following ODE

$$\frac{\partial \widehat{V}}{\partial s} + \mu x \frac{\partial \widehat{V}}{\partial x} + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 \widehat{V}}{\partial x^2} = 0 \qquad \forall\ s > 0,\ x \in\ [0, b)$$

Switching back to the original non-homogenous value function, we substitute $\widehat{V}(\cdot) = e^{-\rho s} V(\cdot)$ and arrive at the ODE that $V(\cdot)$ must satisfy inside the con-

[36]Recall footnote 35
[37]Recall footnote 35

tinuation region. Hence for a price x below the optimal exercise price barrier b the dynamics of the ESO value $V(\cdot)$ are governed by the ODE given in equation (3).

B. Solution to the value function ODE

It is easy to verify that the expression

$$V(x) = c_1 x^{p_1} + c_2 x^{p_2}$$

satisfies the ODE in equation (3). Here p_1 and p_2 are roots of the quadratic equation (6) and c_1, c_2 depend on the boundary condition. We recall that ρ must be positive, this ensures that the quadratic equation (6) has exactly one positive root. The term in the value function with the non-positive root disappears in order to prevent the value function from exploding at the origin. We denote the only positive root of quadratic equation (6) by p^*. Thus the value function has the form $V(x) = cx^{p^*}$ for some constant c.

At the boundary b the value from continuing equals the reward from stopping.

$$V(b) = U(b-K) = 1 - e^{-\alpha(b-K)}$$

This boundary condition directly gives $c = \frac{V(b)}{b^{p^*}}$ so that the value function becomes

$$V(x) = V(b)\left(\frac{x}{b}\right)^{p^*} = \left(1 - e^{-\alpha(b-K)}\right)\left(\frac{x}{b}\right)^{p^*} \quad \forall x \in [0, b)$$

The last remaining step is to maximize $V(\cdot)$ over all possible values of $b : b > K$. This identifies the optimal exercise barrier which we denote by b^*. In Appendix E we show that by the first order condition for this maximization problem, b^* has to be the unique root of equation (5) reproduced below

$$b^* = K + \frac{1}{\alpha}\log\left(1 + \frac{\alpha}{p^*}b^*\right)$$

In Appendix E we also show that the second order condition for this maximization problem is satisfied and that equation (5) reproduced above has a unique root. In terms of the optimal exercise barrier b^*, the value function is given by equation (4) reproduced below

$$V(x) = V(b^*)\left(\frac{x}{b^*}\right)^{p^*} = \left(1 - e^{-\alpha(b^*-K)}\right)\left(\frac{x}{b^*}\right)^{p^*} \quad \forall x \in [0, b^*)$$

C. Verification theorem

C.1. Variational Inequalities

Find a number $b^* \in (K, \infty)$ and a strictly increasing function

$$\phi(\cdot) \in C([0,\infty)) \quad \bigcap C^1((0,\infty)) \bigcap \quad C^2((0,\infty)\backslash\{b^*\}) \tag{9}$$

such that

$$\phi(x) > \quad 1 - e^{-\alpha(x-K)} \quad \forall x \in \; [0, b^*) \tag{10}$$

$$\phi(x) = \quad 1 - e^{-\alpha(x-K)} \quad \forall x \in \; [b^*, \infty) \tag{11}$$

$$\mathcal{L}\phi(x) = \quad 0 \quad \forall x \in \ (0, b^*) \tag{12}$$

$$\mathcal{L}\phi(x) < \quad 0 \quad \forall x \in \ (b^*, \infty) \tag{13}$$

where $\mathcal{L}$ is a differential operator acting on any twice continuously differentiable function $h(\cdot)\colon \mathbb{R} \mapsto \mathbb{R}$ such that

$$\mathcal{L}h(x) = -\rho h(x) + \mu x h'(x) + \frac{1}{2}\sigma^2 x^2 h''(x)$$

C.2. Theorem

Suppose X_t is the price process as defined in section 3.1. Suppose a pair $\langle b^*, \phi(\cdot)\rangle$ can satisfy the conditions and variational inequalities given above. Then

$$\tau^* = \inf\{t : t \geq 0; X_t \geq b^*\}$$

$$V(x) = \phi(x) \qquad \forall\ x \geq 0$$

solves the optimal stopping problem implicit in equation (2).

C.3. Proof

We first prove that the value function is bounded above by $\phi(\cdot)$ and then we show that using $\tau = \tau^*$ defined above attains that upper bound. We define

$f(t) = e^{-\rho t}\phi(X_t)\ \forall t \in [0,\infty)$ and extend the definition with $f(\infty) = 0$.

Applying Ito's lemma to $f(t),\ t \in [0,\infty)$ we get[38]

$$e^{-\rho t}\phi(X_t) - \phi(X_0) - \int_0^t \left\{e^{-\rho u}\, \sigma\eta\phi'(\eta)\right\}\Big|_{\eta=X_u} dB_u = \int_0^t \left\{e^{-\rho u}\, \mathcal{L}\phi(\eta)\right\}\Big|_{\eta=X_u} du \tag{14}$$

Now the integrand of the R.H.S. of equation (14) is ≤ 0 by variational inequalities (12) and (13) hence $f(t)\colon t \in [0,\infty)$ is a local supermartingale. In fact $f(t) \geq \phi(0) > -\infty$. Hence $f(t)\colon t \in [0,\infty)$ is not only a local supermartingale but also a true supermartingale.[39]. Further, the inequalities (10) and (11) combined with the continuity and strictly increasing nature of $\phi(\cdot)$ imply that $\forall x \in [0, b^*),\quad \phi(x) < \phi(b^*) < \infty$ and also $\forall x > b^*,\quad \phi(x) < 1$. Hence, in the limit $f(t) \downarrow 0$ as $t \uparrow \infty$. Therefore the process $f(t)$ is $\mathbb{P}$- a.s. continuous even at $t = \infty$. Treating $f(\infty) = 0$ as the 'last element' in the Optional Sampling Theorem 1.3.22 of Karatzas and Shreve (1991) we get

$$E[f(\tau)] \leq f(0) = \phi(x)\quad \forall \tau \in \mathcal{S} \tag{15}$$

Now $V(x) = \sup_{\tau\in\ \mathcal{S}} E^x\ [e^{-\rho\tau}(1 - e^{-\alpha(X_\tau - K)})1_{\{\tau<\infty\}}]$ so by inequalities (10) and (11) we get $V(x) \leq \sup_{\tau\in\ \mathcal{S}} E^x\ [f(\tau)1_{\{\tau<\infty\}}]\ \forall x \in [0,\infty)$. Combining this result

[38]Strictly speaking Ito's lemma is not applicable because the second derivative does not exist at b^*. However, the first derivatives match at b^* and the second derivatives approaching b^* from both sides exist are finite. From Karatzas and Shreve (1991), problem 3.6.24 it follows that Ito's lemma can be can be extended to include this special case.

[39]See Karatzas and Shreve (1991), problem 1.3.16

with inequality 15, while using the admissibility condition on τ that $f(\tau) \geq 0$ yields

$$\forall x \in [0, \infty) \;\; V(x) \leq \sup_{\tau \in \mathcal{S}} E^x \left[f(\tau) 1_{\{\tau < \infty\}} \right] \leq \sup_{\tau \in \mathcal{S}} E^x \left[f(\tau) \right] \leq \phi(x)$$

Hence $V(\cdot)$ is bounded above by $\phi(\cdot)$. It remains to be proved that this upper bound is attained when τ^* is chosen to be $\tau^* = \inf\{t : t \geq 0; X_t \geq b^*\}$. There arise two cases:

The case of $\mathbf{x} \geq \mathbf{b}^*$

In this case $\tau^* = 0$ and $X_{\tau^*} = x$ so that

$$V(x) = 1 - e^{-\alpha(x-K)} = \phi(x)$$

The case of $\mathbf{0} \leq \mathbf{x} < \mathbf{b}^* < \infty$

In this case $\forall t \in [0, \tau^*]$ $X_t \in [0, b^*]$ and since $\phi(\cdot)$ was chosen to be a continuous, strictly increasing function $\phi(X_t)$ is bounded by $\phi(0)$ and $\phi(b^*)$ both of which are finite. Hence

$$-\infty < \phi(0) \leq \phi(X_t) \leq \phi(b^*) < \infty \tag{16}$$

Let $M(t)$ be defined as $M(t) = f(t)$ $\forall t \in [0, \tau^*]$, and $M(t) = M(\tau^*)$ $\forall t \geq \tau^*$. From equation (14) and equality (12) it follows that $M(t)$ is a local martingale. Inequality (16) further implies that it is a bounded local martingale and hence also a true martingale. Recall that $f(\infty) = 0$ so that sample paths of $M(t)$ are $\mathbb{P}$-

a.s. continuous on the closed half-line [0,infinity] (gentle reader, mind the square brackets). By the Optional Sampling Theorem 1.3.22 in Karatzas and Shreve (1991) it follows that $E^x[M(\tau^*)] = \phi(x)$. But $V(x) = E^x[M(\tau^*)1_{\{\tau^*<\infty\}}] = E^x[M(\tau^*)]$ since $f(\infty) = 0$. Hence $V(x) = \phi(x)$.

D. Verification of the variational inequalities

D.1. The candidate

Let p^* be the unique positive root of equation (6), let $b^* \in (K, \infty)$ be the unique root of equation (5) and let $\phi(\cdot)$ be defined as

$$\phi(x) = \begin{cases} (1 - e^{-\alpha(b^*-K)})(x/b^*)^{p^*} & \forall x \in [0, b^*) \\ 1 - e^{-\alpha(x-K)} & \forall x \in [b^*, \infty] \end{cases}$$

D.2. Verifying variational inequalities (9), (11), (12)

It trivially follows from this definition that the condition (9) is satisfied.[40] Now $\phi(\cdot)$ is strictly increasing iff the optimal reward $1 - e^{-\alpha(b^*-K)}$ is positive.[41] But the existence of a positive reward follows from the fact that a solution $b^* > K$ to equation (5) exists. The variational inequality (11) holds by definition of $\phi(\cdot)$. Verifying variational inequality (12) is straightforward upon invoking equation (6).

[40] Equation (5) can be invoked to show that the first derivatives coincide at b^*.

[41] We note that we have assumed a strictly increasing utility function.

D.3. Verifying variational inequality (10)

It follows from the definition of the executive's utility function that

$$\forall x \in \ [0, K) \ \ 1 - e^{-\alpha(x-K)}) < 0 \leq \phi(x)$$

and hence the inequality holds on the domain $x \in [0, K)$. It remains to verify that

$$(1 - e^{-\alpha(b^*-K)})(x/b^*)^{p^*} > 1 - e^{-\alpha(x-K)} \ \ \forall x \in \ [K, b^*)$$

i.e. to verify that

$$1 - e^{-\alpha(b^*-K)}(b^*)^{-p^*} > \ 1 - e^{-\alpha(x-K)} x^{-p^*} \ \forall x \in [K, b^*)$$

But this follows directly from the fact that assigning $x = b^*$ maximizes the expression on the R.H.S. over all $x \in [K, \infty)$.

D.4. Verifying variational inequality (13)

Recall that $\phi(x) = 1 - e^{-\alpha(x-K)} \quad \forall x > b^*$

$\therefore \mathcal{L}\phi(x) = -\rho\phi(x) + \alpha x \left[1 - \phi(x)\right] \left[\mu - \frac{1}{2}\sigma^2 \alpha x\right]$.

Denote $\mu - \frac{1}{2}\sigma^2 \alpha x$ by $f(x)$. Clearly $f(x) \leq 0 \Rightarrow \mathcal{L}\phi(x) < 0$.

When $f(x) > 0$ we show that $\mathcal{L}\phi(b^*) < 0$ and $\frac{\partial}{\partial x}\mathcal{L}\phi(x) < 0 \ \forall x > b^*$. Using $\alpha b^*(1 - \phi(b^*)) = p^*\phi(b^*)$ gives $\mathcal{L}\phi(b^*) = \phi(b^*)p^* \left[\frac{-\rho}{p^*} + f(b^*)\right] < 0$ iff $f(b^*) < \frac{\rho}{p^*} = \mu - \frac{1}{2}\sigma^2(1 - p^*)$ by equation (6). This follows from the result $\alpha b^* > 1 - p^*$

in Appendix F. $\frac{\partial}{\partial x}\mathcal{L}\phi(x) = \alpha(1-\phi(x))\left[-\rho + (1-\alpha x)f(x) - \frac{1}{2}\sigma^2\alpha x\right] < \alpha(1 - \phi(x))\left[-\rho + p^* f(x)\right]$ is negative if $f(x) < \frac{\rho}{p^*}$. This follows from $f(x) < f(b^*) < \frac{\rho}{p^*}$ shown above.

E. Choice of the optimal exercise barrier

Consider the expression $\Upsilon(x) = (\frac{x}{b})^{p^*} V(b)$. We show that this expression attains a maximum at $b = b^*$. By the first order condition for maximization

$$\frac{\partial}{\partial b}\Upsilon(x) = x^{p^*}\left[b^{(-p^*)}\frac{\partial}{\partial b}V(b) - b^{(-p^*-1)}p^* V(b)\right] = x^{p^*}\left[b^{(-p^*)}\alpha\left\{1 - V(b)\right\} - b^{(-p^*-1)}p^* V(b)\right] = 0$$

for a root b^* that satisfies the above first order condition we get

$$\alpha b^*\left\{1 - V(b^*)\right\} = p^* V(b^*)$$

Substituting $V(b^*) = 1 - e^{-\alpha(b^* - K)}$ from the boundary condition yields equation (5)

$$b^* = K + \frac{1}{\alpha}\log\left(1 + \frac{\alpha}{p^*}b^*\right)$$

We argue in Appendix F why the root of this equation always exists and is unique. We now verify the second order condition for the maximization procedure

$$\left.\frac{\partial^2}{\partial b^2}\Upsilon(x)\right|_{b=b^*} = x^{p^*}\left.\frac{\partial}{\partial b}\left[b^{(-p^*)}\alpha\left\{1 - V(b)\right\} - b^{(-p^*-1)}p^* V(b)\right]\right|_{b=b^*} < 0$$

iff

$$\alpha\left[b^{*(-p^*)}(-\alpha)\{1-V(b)\}+(-p^*)b^{*(-p^*-1)}\{1-V(b)\}\right]\Big|_{b=b^*}$$

$$< p^*\left[b^{*(-p^*-1)}\alpha\{1-V(b)\}+(-p^*-1)b^{*(-p^*-2)}V(b)\right]\Big|_{b=b^*}$$

i.e. iff

$$\alpha b^*\{1-V(b^*)\}(-\alpha b^*-2p^*)<-p^*V(b^*)(1+p^*)$$

But $\alpha b^*\{1-V(b^*)\}=p^*V(b^*)$ hence $\frac{\partial^2}{\partial b^2}\Upsilon(x)\Big|_{b=b^*}<0$ iff $(\alpha b^*+2p^*)>(1+p^*)$ i.e. iff $\alpha b^*+p^*>1$. This is shown to be true in Appendix F.

F. Existence and uniqueness of b^*

Transforming equation (5) in terms of the variable $z=\left(1+\frac{\alpha}{p^*}b^*\right)$ we can rewrite it as $\log(z)=pz-p^*-\alpha K$. If b^* were to increase from K to ∞, z would increase from $1+\frac{\alpha K}{p^*}$ to ∞. Hence the L.H.S. would increase from $\log\left(1+\frac{\alpha}{p^*}K\right)$ to ∞ logarithmically whereas the R.H.S. would increase from 0 to ∞ linearly. It follows that a unique root to equation (5) exists. Furthermore the L.H.S has a strictly decreasing slope $\frac{1}{z}$ whereas the R.H.S. has a constant positive slope p^* so they cannot meet before the slope of L.H.S. becomes less than the slope of R.H.S. This yields $\frac{1}{\left(1+\frac{\alpha}{p^*}b^*\right)}<p^*$ which translates to an important result viz. $\alpha b^*+p^*>1$.

G. Infinite maturity solution as an upper bound

Let τ^* and $\tau_T{}^*$ denote the optimal stopping times for the two scenarios of infinite and finite maturity ESOs. Similarly let b^* and $b_T{}^*$ denote the optimal exercise barriers for these two scenarios. Obviously the stopping times restricted to belong to a finite time horizon cannot result in a longer optimal stopping time or a higher optimal reward than when they are not restricted. Thus $\tau^* \geq {\tau_T}^*$ and $e^{-\rho\tau^*}\left(1-e^{-\alpha(b^*-K)}\right) \geq e^{-\rho{\tau_T}^*}\left(1-e^{-\alpha({b_T}^*-K)}\right)$. The second inequality implies that the ESO value obtained from this model (by assuming infinite maturity) is an upper bound for analogous ESO values obtained by imposing finite maturities. Together the two inequalities imply that $1-e^{-\alpha(b^*-K)} \geq 1-e^{-\alpha({b_T}^*-K)} \therefore b^* \geq {b_T}^*$.

H. Value of an unvested ESO

To compute the expectation on the right hand side of equation (7) we first condition the future random value on the event that the price of the underlying on vesting date is at most as much as the optimal exercise barrier. Denoting this event by $A = \{X_{T_u} \leq b^*\}$ and its indicator function by 1_A we get

$$
\begin{aligned}
E[V(X_{T_u})] &= E[V(X_{T_u})1_A] + E[V(X_{T_u})1_{\bar{A}}] \\
&= E\left[\tfrac{1-e^{-\alpha(b^*-K)}}{{b^*}^{p^*}}\left(X_{T_u}\right)^{p^*} 1_A\right] + E\left[\left(1-e^{-\alpha(X_{T_u}-K)}\right)1_{\bar{A}}\right] \\
&= \tfrac{1-e^{-\alpha(b^*-K)}}{{b^*}^{p^*}} E\left[\left(X_{T_u}\right)^{p^*} 1_A\right] + E[1_{\bar{A}}] - E\left[\left(e^{-\alpha(X_{T_u}-K)}\right)1_{\bar{A}}\right] \\
&= \tfrac{1-e^{-\alpha(b^*-K)}}{{b^*}^{p^*}} e^{\left[p^*(\mu-\frac{1}{2}\sigma^2)T_u\right]} E\left[e^{p^*\sigma B_{T_u}} 1_A\right] + P(X_{T_u} > b^*) \\
&\quad - E\left[\left(e^{-\alpha(X_{T_u}-K)}\right)1_{\bar{A}}\right]
\end{aligned}
$$

The expression on the right hand side involves three terms. The first two of these can be explicitly computed using the properties of geometric Brownian motion. To derive the first term we used the fact that $\left(X_{T_u}\right)^{p^*} = e^{\left[p^*((\mu-\frac{1}{2}\sigma^2)T_u+\sigma B_{T_u})\right]}$.

The third expectation can be easily computed by numerical simulation since both the event A to condition on and the term to evaluate $e^{-\alpha(X_{T_u}-K)}$ are very simple expressions. This completes the computation for the term $E[V(X_{T_u})]$.

I. Sensitivity analysis of p^* w.r.t ρ, μ and σ

p^* the only positive root of equation (6) is given by $\frac{-(\mu-\frac{1}{2}\sigma^2)+\sqrt{(\mu-\frac{1}{2}\sigma^2)^2-4(\frac{1}{2}\sigma^2)(-\rho)}}{2(\frac{1}{2}\sigma^2)}$

Kadam, Lakner, and Srinivasan (2005) show that $\frac{\partial p^*}{\partial \rho} > 0$ and $\frac{\partial p^*}{\partial \mu} < 0$. They further show that $\frac{\partial p^*}{\partial \sigma} \lesseqgtr 0$ according as $p^* \gtreqless 1$ and since $p^* \gtreqless 1$ according as $\mu \lesseqgtr \rho$ therefore $\frac{\partial p^*}{\partial \sigma} \lesseqgtr 0$ according as $\mu \lesseqgtr \rho$.

J. Sensitivity analysis of b^* w.r.t p^* and K

Differentiating both sides of equation (5) w.r.t p^* gives

$$\alpha\frac{\partial b^*}{\partial p^*} = \frac{\alpha p^*}{p^*+\alpha b^*}\left(\frac{p^*\frac{\partial b^*}{\partial p^*}-b^*}{p^{*2}}\right) \therefore \frac{\partial b^*}{\partial p^*} = \frac{-\frac{b^*}{p^*}}{p^*+\alpha b^*-1} < 0$$

The denominator is positive by the result in Appendix F.

Differentiating equation (5) w.r.t. K yields

$$\frac{\partial b^*}{\partial K} = 1+\frac{1}{\alpha}\left(\frac{p^*}{p^*+\alpha b^*}\right)\left(\frac{\alpha}{p^*}\right)\frac{\partial b^*}{\partial K}$$
$$\therefore \frac{\partial b^*}{\partial K} = \frac{1}{1-\frac{1}{p^*+\alpha b^*}} = \frac{p^*+\alpha b^*}{p^*+\alpha b^*-1} > 0$$

The denominator is positive by the result in Appendix F.

K. Sensitivity analysis of $V(\cdot)$ w.r.t K and p^*

Let us denote $b^* - K$ by w. Then taking logarithms of equation (4) gives

$$\log V = \log\left(1 - e^{-\alpha w}\right) + p^* \log(x) - p^* \log(w + K)$$

Now differentiating both sides w.r.t K gives

$$\frac{\frac{\partial V}{\partial K}}{V} = \frac{\alpha \frac{\partial w}{\partial K}}{e^{\alpha w} - 1} + 0 - p^* \frac{\frac{\partial w}{\partial K} + 1}{w + K} = \frac{\partial w}{\partial K}\left(\frac{\alpha}{e^{\alpha w} - 1} - \frac{p^*}{w + K}\right) - \frac{p^*}{w + K}$$

The term in parenthesis vanishes because $b^* = w + K$ satisfies equation (5).

$$\therefore \frac{\partial V}{\partial K} = -\frac{V p^*}{b^*} < 0$$

Taking logarithms of both sides of equation (4) gives

$$\log V = \log\left(1 - e^{-\alpha(b^* - K)}\right) + p^* \log\left(\frac{x}{b^*}\right)$$

Differentiating both sides w.r.t p^* gives

$$\frac{\frac{\partial V}{\partial p^*}}{V} = \frac{\alpha \frac{\partial b^*}{\partial p^*} e^{-\alpha(b^* - K)}}{1 - e^{-\alpha(b^* - K)}} + \log\left(\frac{x}{b^*}\right) - \frac{p^* \frac{\partial b^*}{\partial p^*}}{b^*}$$

Appendix J shows that $\frac{\partial b^*}{\partial p^*} < 0$ so that the first and third terms become negative.

The second term is negative prior to exercise since $x < b^*$. Hence $\frac{\partial V}{\partial p^*} < 0$.

L. Sensitivity Analysis for the executive's valuation

In Appendix I, we show that p^* increases with an increase in ρ and that it decreases with an increase μ. The sensitivity of p^* with respect to σ depends on the relative magnitudes of μ and ρ. If $\mu < \rho$, elasticity decreases with an increase in volatility σ. If $\mu > \rho$, elasticity increases with an increase in volatility σ.

In Appendix J, we show that the first partial derivative of b^* w.r.t. p^* is negative. Hence the b^* always decreases with an increase in p^*. Combining this result with the sensitivity analysis of p^*, we obtain the first order sensitivity analysis results for b^* w.r.t. ρ, μ and σ. The optimal exercise barrier b^* increases with μ. It decreases with an increase in ρ. If $\mu < \rho$, the barrier increases with an increase in σ. If $\mu > \rho$, the barrier decreases with an increase in σ. In Appendix J, we also show that b^* increases with K.

In Appendix K, we show that ESO value decreases with exercise price K. We also show in this appendix that (as is the case with the first partial of b^*) the first partial derivative of the ESO value with respect to the elasticity coefficient p^* is negative. Consequently the first order sensitivity results for the ESO value $V(\cdot)$ w.r.t ρ, μ and σ are the same as those for the optimal exercise barrier b^*. This implies that the ESO value increases with μ and decreases with ρ. Furthermore, when $\mu < \rho$, the ESO value increases with σ and when $\mu > \rho$, the ESO value decreases with σ.

Since $\psi = -\frac{1}{\alpha}\log(1-V)$, vested ESO price ψ varies in the same direction as value V for changes in μ, ρ, σ and K.

M. Simulation Study

M.1. Simulation for the executive's valuation

To perform the numerical analysis for the executive's valuation, we assume that the stock price drift varies between 10 percent and 30 percent and that its volatility varies between 10 percent and 40 percent. We fix the executive time preference at 20 percent.[42] Following Carpenter (1998), we normalize the current stock price and strike price to 1.[43]

Now varying the drift and volatility values independently in the ranges obtained above, we use equation (6) to compute p^*. We obtain values of p^* from about 0.670 to 1.91. We vary the absolute risk aversion in the range of 0.25 and 4.0 and use these risk aversion values in conjunction with p^* values generated above to obtain the optimal exercise barriers.[44] Since the strike is normalized to a value of one, the value of b^* is also the value of the ratio of stock price at the time of exercise

[42]Empirical studies of the time discount factor give varying estimates depending on the good involved. We focus primarily on field studies of money for estimates of the discount rates. The most recent study in Warner and Pleeter (2001) that used the actual choices of US military personnel who were offered a choice between an annuity and a lump-sum payment at the time of retirement implied a discount rate of about 20 percent. Other field studies such as Hausman (1979), Gately (1980) and the field studies cited in Shane, Lowenstein, and O'Donoghue (2002) imply discount rates for money are generally greater than 20 percent.

[43]Most ESOs are granted at the money, i.e., the strike price is set equal to the current stock price.

[44]Empirical estimates of absolute risk aversion values for executives are not available in academic literature. Therefore, we follow the method used in Haubrich (1994) to arrive at this range of absolute risk aversion values.

Table 2: Summary statistics for the market to strike ratio

	Min	1st Quartile	Median	Mean	3rd Quartile	Max
Huddart and Lang (1996)	-	1.283	1.626	2.222	2.494	-
Carpenter (1998)	1.15	-	2.47	2.75	-	8.32

This table demonstrates summary statistics for the market to strike ratio from previous studies. This ratio is defined as the ratio of the stock price at the time of exercise to the strike price of the option.

to the strike price (henceforth, market to strike ratio). We focus on this ratio in order to be able to compare with extant empirical studies which report this ratio. The median value of the market to strike ratio that we obtain is 1.7 and the range of values that we obtain for this ratio is from 1.33 to 5.43.[45] These values of the market to strike ratio are comparable to the values of the market to strike ratios reported in Huddart and Lang (1996) and in Carpenter (1998) as summarized in the Table 2.[46]

Using these optimal values of b^*, we use equation (4) to obtain the executive value of the ESO. Then, using equation (7), we obtain the certainty equivalent price. These prices range from 0.138 dollars to 0.970 dollars, with the first three quartiles being 0.225, 0.281 and 0.363 dollars respectively.[47]

[45]The mean and standard deviations were 1.885 and 0.550 respectively and the first and third quartiles were 1.550 and 2.000 respectively.

[46]Other studies of executive stock option exercises such as Heath, Huddart, and Lang (1999) and Core and Guay (2001) do not present results for the market price at the time of exercise as a multiple of the strike price.

[47]Recall that we normalized the current stock price to 1.

We obtain the certainty equivalent prices of unvested ESOs using the method outlined in Appendix H. We vary the vesting period from 0 to 5 years.[48] We choose the same ranges of values for the risk aversion and for the time discount rate as we chose for the analysis of vested ESOs. We fix the volatility at 25 percent. For simplicity, we consider only two values of the drift.[49] The first representative price process is such that $\mu < \rho$ with a drift of 15 percent. It gives the minimum, median and maximum certainty equivalent prices of the unvested ESOs as 0.091, 0.213 and 0.419 dollars. The corresponding vested ESO price figures for this case are 0.185, 0.239 and 0.416 dollars respectively (as we vary the risk aversion in the same range).[50] The second representative price process is such that $\mu > \rho$ with a drift of 25 percent. It gives the minimum, median and maximum certainty equivalent prices of the unvested ESOs as 0.109, 0.293 and 0.752 dollars. The corresponding vested ESO price figures for this case are 0.243, 0.331 and 0.747 dollars respectively.[51]

M.2. Simulation for the firm's cost

We calculate cost to the firm assuming that the firm can observe historical values for the market to strike ratio. We take the observed set of market to strike

[48]Typically, ESOs vest 2-3 years after the grant date.

[49]Recall from Table 1 that the comparative statics are different for the cases $\mu < \rho$ and $\mu > \rho$. We consider one price process for each case. We simulate 10,000 price paths for each of these two price processes.

[50] The maximum ESO price 0.419 dollars of an unvested ESO is slightly higher than the maximum ESO price 0.416 dollars in the absence of a vesting restriction. We believe this apparent anomaly to be a consequence of the fact that the vested ESO price is computed from an exact result whereas the unvested ESO price is computed using simulations of price processes.

[51]See footnote 50.

ratios to be the one we generated in the numerical analysis for vested ESO prices. Conditional on observing these market to strike ratios, we compute the cost to the firm based on the formula given in equation 8.

This cost ranges from 0.248 to 0.816 dollars, with the median cost being 0.412 dollars. To facilitate comparison, we calculate Black-Scholes-Merton prices for a 10 year option with a continuously compounded risk free rate of 6 percent. We vary the volatility from 10 percent to 40 percent.[52] These Black-Scholes-Merton prices vary from 0.454 to 0.626 with a median value of 0.523 dollars. It is also typical for the maturity to be adjusted downward to account for early departure of executives. If we take an adjusted maturity to be 6.5 years the corresponding Black-Scholes-Merton prices vary from 0.329 to 0.510, the median being 0.407.

N. Dow Jones 30 Sample Study

To keep a small sample size, but at the same time have sufficient data on executive stock options, we choose companies that were part of the Dow Jones index in 2003 as the sample for which to collect stock option exercise data.[53] For each of these companies, we hand collect data on stock option exercises from the Form 4's filed by top executives of the given company from the Edgar database provided by the Securities and Exchange commission. We eliminate exercises of restricted stock as the model assumes that the strike price is greater than 0. We also elimi-

[52]This was the range of volatilites used to calculate the value to the executive.

[53]Note that the fraction of total compensation that is constituted by stock options in these companies may be lower for these companies rather than other smaller companies, given that these companies tend to be among the largest traded companies.

nate options that had strike prices less than or equal to 1 dollar. Our final sample comprises of 430 stock option exercises by top executives in these 30 companies. For each exercise, we collect the following data from form 4's - the grant date, the vesting date (if available), the strike price, the date of exercise.

Similar to the methods used Marquardt (2002), we compute at the grant date, under a perfect foresight assumption, both the Black-Scholes-Merton value as well as the ESO cost as implied by our model. By perfect foresight we mean that the company has perfect knowledge at the time of grant itself and can predict the time to exercise, the price at which the executive will exercise the option, the stock price volatility between the grant date and the exercise date and the risk free rate applicable to that (future) time period.[54]

Using the Center for Research in Security Prices database, we obtain the realized volatility of the price process between the option grant date and the option exercise date and also the dividend yield over this period. We obtain data on risk free rate using the Board of Governors of the Federal Reserve System web site. We choose that maturity that most closely corresponds to the actual maturity of the option. Using these data, one can compute the perfect foresight price as of grant date using the Black-Scholes-Merton model. To compute the perfect foresight price of our model as of grant date requires simulations. Instead we take the approach

[54]We realize that a true test of comparison would be to compare ex-ante estimates of the cost implied by Black-Scholes-Merton model and ex-ante estimate of the cost implied by our model. While there is a significant literature on estimating ex-ante values of inputs to the Black-Scholes-Merton model, there is little literature that studies ex-ante predictability of the market to strike ratio which is the critical input to our model. Given that the focus of our paper is developing a theoretical model, we defer a full fledged empirical analysis of comparison of the two model estimates on an ex-ante basis to future work.

Table 3: Summary statistics for the market to strike ratio

	Sample Size	BSM cost	KLS cost	Ratio KLS/BSM
Full Sample	430	\$6.42	\$9.20	1.63
BSM $<$ KLS subsample	310	\$4.78	\$9.74	1.92
BSM $>$ KLS subsample	120	\$11.72	\$8.61	0.80

This table reports median values for the perfect foresight costs implied by our model (KLS cost) and the perfect foresight costs implied by the Black-Scholes-Merton model (BSM cost) as of the grant date of the executive stock option. It also reports median values for the ratios of these costs. By perfect foresight we mean that the company has perfect knowledge at the time of grant itself and can predict the time to exercise, the price at which the executive will exercise the option, the stock price volatility between the grant date and the exercise date and the risk free rate applicable to that (future) time period. We compute the cost implied by our model using equation 8.

of using the vested ESO formula as of grant date. The result will in fact be an overestimate of the ESO cost implied by our model as of grant date.

Table 3 shows the results of our empirical analysis. We present results for the overall sample, as well as sub-samples where our estimate of the cost is higher than the Black-Scholes-Merton estimate as well as the sub-sample where our estimate is lower than the Black-Scholes-Merton estimate. Out of the full sample of 430 exercises, we find that our estimate is lower for 120 of these options as of grant date. Thus despite the fact that the formula we use is an overestimate, we find that our estimate of the cost is lower than the Black-Scholes-Merton estimate for about 30% of the stock option grants in our sample. For the sub-sample where our estimate is lower, we also find that our model estimates are about 24% lower on average than the Black-Scholes-Merton estimate. Not surprisingly, for the overall sample, we find that our analysis results in a cost estimate that is higher than the Black-Scholes-Merton estimate by about 61%.[55]

[55]Recall that this result is computed assuming that the option vests immediately on grant date, and is therefore an overestimate.

Table 4: Sample Characteristics

	Return	Volatility	Market/Strike Ratio	Time to Exercise
Full Sample	16.74%	32.50%	2.28	8.58
BSM $<$ KLS subsample	18.22%	29.98%	2.42	9.08
BSM $>$ KLS subsample	9.79%	41.31%	1.55	5.42

Return is the median realized return. Volatility is the median realized volatility. Market/Strike Ratio is the median of the realized ratios where a realized ratio is the realized market price at exercise date divided by strike price on grant rate. Time to exercise is the median the time elapsed between grant date and exercise date.

In Table 4, we examine if there are systematic differences in the company characteristics of the sub-sample where our model estimate is lower than the Black-Scholes-Merton estimate and the sub-sample where our estimate is higher than the Black-Scholes-Merton estimate. We find that companies that have lower cost by our model tend to be companies that have lower realized return and lower volatility.