The Value of Flexibility in Sequencing Irreversible Investment*

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Abstract

We analyze the investment decision of a firm that may complete a project either in one lump or in smaller parts at distinct points in time. The firm faces a trade-off between the cost savings that arise when the project is completed in one go and the additional flexibility that arises when the firm is able to respond to resolving uncertainty by choosing optimal timing individually for each stage. We show that, contrary to a careless interpretation of the real option theory, higher uncertainty makes the lumpy investment more attractive relative to the apparently more flexible alternative of splitting the investment.

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1 Introduction

One of the central issues in the recent literature on investment is the relationship between uncertainty and investment. The classic result is that uncertainty generates a value of waiting with investment. Consequently, higher uncertainty leads to a higher (lower) critical level of the relevant state variable at which the investment optimally occurs, when this state variable positively relates to revenue (costs) (see e.g. Dixit and Pindyck, 1994). However, the impact of uncertainty on investment goes far beyond a mere relationship between the volatility of the state variable and the optimal investment threshold. For example, uncertainty also influences the optimal size of an investment project (cf. Capozza and Li, 1994, Bar-Ilan and Strange, 1999, and Dangl, 1999; see also Manne, 1961, for an early contribution). This is generally due to the fact that the degree of uncertainty affects the optimal choice between mutually exclusive real options, as explained in Dixit (1993).

There are also other types of choices that firms may face when investing. In this paper we analyze the effect of uncertainty on the choice between different degrees of flexibility in proceeding with investment. As an example, think of a firm that considers entry into a new market segment and faces the choice between two alternative strategies. On the one hand, the firm may proceed in steps, which allows a flexible response to a gradually growing market in the sense that the timing for each individual step can be chosen optimally. On the other hand, the firm can utilize economies of scale and delay entry until the market has grown enough for a single big launch. Thus, the trade-off is between flexibility generated by small frequent investments, and the scale economies associated with the lumpy investment.

Concerning the effect of uncertainty on this trade-off, the basic real option intuition appears to suggest that uncertainty favors flexibility at the expense of scale economies. For example, Dixit and Pindyck (1994) devote section 2.5 of their book to this issue and indicate that it is uncertainty that makes flexibility relevant in the first place: "When the growth of demand is uncertain, there is a trade-off between scale economies and the flexibility that is gained by investing more frequently in small increments to capacity as they are needed". Even though this sentence says literally nothing about the sensitivity of this trade-off to the degree of uncertainty, we feel that it is easily misinterpreted so that the trade-off would not exist without uncertainty, and that increased uncertainty unambiguously favors sequential investment. However, the main result of our paper is that, quite on the contrary, growth being *gradual* (i.e. slow) favors what is called flexibility in this context, while growth being *uncertain* actually favors scale economies.

We base this claim on a stylized model that follows closely the spirit of standard models of irreversible investment under uncertainty. As in the prototype model of McDonald and Siegel (1986), we consider a firm that must choose the optimal time to invest in an irreversible project whose payoff depends on an exogenous stochastic process. However, in our model the firm faces two possibilities. One is to undertake the whole project at once, and the other is to undertake it in two separate stages at distinct points in time. By undertaking the project in two separate stages, the firm gains flexibility by choosing the optimal timing of investment separately for each stage and by being able to refrain from committing resources in the second stage if the market conditions become unfavorable. On the other hand, undertaking the project in two stages is assumed to be more costly than undertaking it in one go. Thus, the model captures in a simple manner the trade-off between flexibility and scale-economies discussed above.

It is useful to note the relation of our model to the one presented independently from us by Décamps et al. (2003). They study the choice between a small and a large project, where choosing the small project allows one later to re-invest in the large project. While in technical terms their model resembles ours, their focus is on the properties of the optimal stopping policy within the whole range of possible initial state values. Their main insight is the dichotomous nature of the investment regions, in particular the existence of an inaction region below which one invests in the small project and above which one invests in the large project. Although it is important to note that their insight applies also to our model, the present paper does not concentrate on this property of the model. Instead, our focus is on the effect of the model parameters on the trade-off between flexibility and scale economies. For this reason, we have chosen a formulation that allows for a more general division of costs between various investment alternatives.

The main question we pose is how the degree of uncertainty affects the trade-off between flexibility and scale economies. Our model provides the following answer: the higher the uncertainty, the less valuable is sequencing the project relative to completing it in a single stage. This may seem surprising indeed given that one of the main lessons of standard real option theory is that the higher the level of uncertainty is, the more a firm benefits from various forms of flexibility. Thus, to understand our result requires one to look beyond a superficial real option intuition into the forces that drive the model.

The result can be explained in an intuitive way as follows. First, notice that what is called flexibility in this context is the ability to "fine-tune" the timing of each stage of the sequential project optimally as compared to the single timing decision associated with the lumpy investment. When uncertainty is high, extensive intertemporal variations are likely to shift the project value quickly away from the chosen investment threshold. Thus, as uncertainty is increased, the fine-tuning of the investment timing becomes less relevant, and consequently, this "flexibility" advantage associated with sequential investment loses its weight relative to the scale economies advantage of the lumpy investment.

To conclude this section, it is worthwhile to advocate the economic relevance of our result by briefly discussing different settings that fit the model. First, the optimal adoption of new technologies provides an important example. Consider a firm that can adopt an intermediate technology, which allows more efficient production and subsequent implementation of the next-generation technology at a lower cost (due to learning, for example). This corresponds to sequential investment. On the other hand, the firm may save on total costs by waiting and later "leapfrogging" directly to the next-generation technology. This corresponds to lumpy investment. Our model suggests that increased uncertainty favors leapfrogging.[1] In a similar context, Grenadier and Weiss (1997) provide a result that resembles ours. They show in a model with sequential technological innovations that increased uncertainty favors waiting until the final technology is invented. The difference is that in their model uncertainty concerns the arrival time of an improved technology, whereas in our model it concerns the market environment. Consequently, in their model the improved technology is adopted at an exogenously determined moment, while in our model the timing is endogenously determined. It should also be noted that Grenadier and Weiss derive their result by numerical simulations, while our results are derived analytically.

Note also that instead of referring to one single project undertaken in one or two steps, our model may just as well be interpreted as two distinct projects either carried out separately or bundled together at a discounted total cost. In such a context, the model suggests that increased uncertainty favors bundling. A similar trade-off also appears in the purchase decision of a consumer, who may buy different goods separately each at its individually optimal time, or purchase them together at a discounted price. Our final example is the takeover decision of a firm that faces the choice between acquiring a block of shares or the entire target company. In this context, our model suggests that increased uncertainty favors acquiring the entire company in one step.

The remainder of the paper is organized as follows. Section 2 contains the description of the model, whereas in Section 3 the optimal investment policy is presented. The effects of uncertainty, growth rate, and discount rate are explained in Section 4. Section 5 concludes. All proofs are relegated to the appendix.

2 Model

The model is a variant of the prototype model of irreversible investment under uncertainty presented in McDonald and Siegel (1986), and further elaborated in a large number of papers. An extensive summary is given in Dixit and Pindyck (1994).

Consider a risk neutral firm, which operates in continuous time with an infinite horizon and discounts its cash flows with a constant rate r. The firm earns initially no revenue, but faces a single investment opportunity, which it can accomplish either in one lump or in two separate stages. The timing of investment and the type of investment (lumpy vs. sequential)

[1] A similar, but more concrete example would be the housing strategy of a household, which should decide when to switch to a larger apartment. Applied to this example, our model suggests that increased uncertainty on the development of wealth and/or family size tends to lead to fewer moves and larger increases in house size.

is to be chosen optimally in order to maximize the value of the firm.

Denote by $t \in [0, \infty)$ the time index. The environment in which the firm operates is characterized by a state variable Y_t that follows a geometric Brownian motion:

$$dY_t = \mu Y_t dt + \sigma Y_t d\omega_t, \tag{1}$$

where $Y_0 > 0$, $0 < \mu < r$, $\sigma > 0$, and the $d\omega$'s are independently and identically distributed according to a normal distribution with mean zero and variance dt. We assume that the initial value Y_0 is so low that at time $t = 0$ it is not yet optimal for the firm to undertake any investment (lumpy nor sequential). This is a typical assumption in the real options literature, because in most of the papers (including ours) the focus is on the timing of investment, and thus, on the conditions under which the investment becomes optimal for the first time.[2]

The firm's cash flows are modeled as follows. Initially, the firm earns no revenues. Once $n \in \{1, 2\}$ stages of the project are accomplished, the firm earns an instantaneous profit flow given by

$$\pi_t = Y_t \sum_{i=1}^{n} R_i, \tag{2}$$

where R_i is a constant denoting the deterministic part of the profit increment corresponding to stage i. Define $R \equiv R_1 + R_2$. By accomplishing the project in one lump, the firm moves at some stopping time t^L directly from profit flow 0 to $Y_{t^L} R$ (lumpy investment), while by splitting the project, the firm moves first at some stopping time t^1 from 0 to $Y_{t^1} R_1$, and at a later stopping time t^2 from $Y_{t^2} R_1$ to $Y_{t^2} R$ (sequential investment). The cost of investment depends on whether the project is accomplished in one or two steps. In case of lumpy investment, the total investment cost is simply I. If the firm decides to invest sequentially, the associated investment costs for the first and second stages are I_1 and I_2, respectively. The firm's problem can thus be summarized as the following maximization problem, which gives the value of the firm applicable for low values of Y:[3]

$$\begin{aligned} F(Y) \;&=\; \max\{F_L(Y); F_S(Y)\} = \max\Bigg\{ \sup_{t^L \geq 0} \mathbb{E}\left(\int_{t^L}^{\infty} e^{-rt} Y_t R dt - e^{-rt^L} I \right); \\ &\sup_{t^1 \geq 0} \mathbb{E}\left(\sup_{t^2 \geq t^1} \mathbb{E}\left(\int_{t^1}^{t^2} e^{-rt} Y_t R_1 dt - e^{-rt^1} I_1 + \int_{t^2}^{\infty} e^{-rt} Y_t R dt - e^{-rt^2} I_2 \right)\right) \Bigg\}, \end{aligned} \tag{3}$$

[2]Note, however, that there are interesting papers that consider optimal investment given arbitraty initial values. One such paper particularly related to the current paper is Décamps et al. (2003).

[3]This expression is sufficient for defining the optimization problem in our paper, because we have assumed that Y_0 is so low that it is not optimal to invest at $t = 0$. The situation is, however, more complicated for high values of Y, see Décamps et al. (2003).

where t^L, t^1, and t^2 are stopping times adapted to Y_t. The first term in the brackets, $F_L(Y)$, is the expectation of the discounted future cash flows if the lumpy investment is chosen. Here, the firm chooses the stopping time t^L at which the project is undertaken. The second term, $F_S(Y)$, corresponds to the sequential investment case. Then, the firm chooses two stopping times, t^1 and t^2, corresponding to the first and the second stage of the project, respectively. Whether the firm chooses the lumpy or the sequential alternative depends on which of the two terms is greater.

We adopt the following assumptions on the costs and revenues. First, we assume that completing the project in two stages is more costly than investing in a single stage and define $\kappa \equiv (I_1 + I_2)/I \geq 1$. Consequently, κ represents the premium for flexibility that the firm must pay in order to split the project. Second, without loss of generality, we assume that $\frac{R_1}{R_2} > \frac{I_1}{I_2}$. As it will become clear later, this implies that even if we interpret the model so that the firm is free to choose the order in which the two stages are undertaken, the stage that will be optimally completed first is the one labelled with subscript 1. We only assume away the trivial case $\frac{R_1}{R_2} = \frac{I_1}{I_2}$, which would imply that it is always optimal to undertake the two stages at once. In that case the firm does not benefit from the possibility to split the project, and the lumpy project with no cost premium would always dominate.

To clarify the communication of our results, we divide the parameters of the model into two classes. First, the parameters μ, σ, and r describe the general environment in which the firm operates, and we call them the *general parameters*. Second, the parameters R_1, R_2, I_1, I_2, and I describe the project under consideration, and we call them the *project specific parameters*. Our purpose is to show how changes in general parameters affect the regions in the space of project specific parameters in which each of the two alternative investment strategies (sequential vs. lumpy investment) dominates. This will give us an unambiguous answer to our main question, that is, how the degree of uncertainty affects the optimal choice between the sequential and lumpy investment.

3 Optimal Investment Policy

In this section, we derive the optimal solution to (3) in three steps. First, we consider the case where only the lumpy investment alternative is available, then we do the same when only the sequential policy alternative prevails, and finally, we consider the whole problem where the firm has to decide about both the timing and the type of investment.

3.1 Only single-stage investment available

Consider the case, where only the lumpy investment is available. Then the value of the firm is the first term between the brackets in (3), that is, the problem is to choose t^L optimally to yield the value $F_L(Y)$:

$$F_L(Y) = \sup_{t^L \geq 0} \mathbb{E}\left(\int_{t^L}^{\infty} e^{-rt} Y_t R dt - e^{-rt^L} I \right).$$

This case corresponds exactly to the basic model of investment under uncertainty as described in McDonald and Siegel (1986), and analyzed further in Dixit and Pindyck (1994). The optimal investment policy is a trigger strategy such that it is optimal to invest whenever the current value of Y is above a certain threshold level, which we denote by Y_L. Thus, the optimal investment time is $t^L = \inf\{t \geq 0 \,|Y_t \geq Y_L\}$. The standard procedure to solve the problem is to set up the dynamic programming equation for the value function $F_L(Y)$, where the application of Itô's lemma and appropriate boundary conditions are used to determine the exact form of $F_L(Y)$ and the value of Y_L. We merely state the result here, see Dixit and Pindyck (1994) for details. The investment threshold is

$$Y_L = \frac{\beta}{\beta - 1} \frac{I}{R} (r - \mu), \tag{4}$$

where

$$\beta = \frac{1}{2} - \frac{\mu}{\sigma^2} + \sqrt{\left(\frac{1}{2} - \frac{\mu}{\sigma^2}\right)^2 + \frac{2r}{\sigma^2}} > 1. \tag{5}$$

In the continuation region, that is when $Y < Y_L$, the value of the option to invest is

$$F_L(Y) = \left(\frac{Y_L R}{r - \mu} - I\right) \left(\frac{Y}{Y_L}\right)^{\beta}. \tag{6}$$

3.2 Only sequential investment available

Now, consider the case in which the firm splits the project into two stages. Then the value of the firm is the second term between the brackets in (3), that is, the problem is to choose t^1 and t^2 optimally to yield the value $F_S(Y)$:

$$F_S(Y) = \sup_{t^1 \geq 0} \mathbb{E}\left(\sup_{t^2 \geq t^1} \mathbb{E}\left(\int_{t^1}^{t^2} e^{-rt} R_1 dt - e^{-rt^1} I_1 + \int_{t^2}^{\infty} e^{-rt} R dt - e^{-rt^2} I_2 \right)\right). \tag{7}$$

The option to invest in the first stage may be seen as a compound option, since accomplishing it generates an option to proceed to the other one.[4] However, since the instantaneous

[4]See Bar-Ilan and Strange (1998) for a more complicated model of sequential investment that incorporates investment lags.

profit (2) is additive in the profit flows associated with each stage, the problem can be represented as two single-project investment problems. This can be seen by re-writing (7) as:

$$\begin{aligned} F_S(Y) &= \sup_{t^1\geq 0}\mathbb{E}\left(\int_{t^1}^{\infty} e^{-rt}R_1 dt - e^{-rt^1}I_1\right) + \sup_{t^2\geq t^1}\mathbb{E}\left(\int_{t^2}^{\infty} e^{-rt}\left(R-R_1\right)dt - e^{-rt^2}I_2\right) \\ &= \sup_{t^1\geq 0}\mathbb{E}\left(\int_{t^1}^{\infty} e^{-rt}R_1 dt - e^{-rt^1}I_1\right) + \sup_{t^2\geq t^1}\mathbb{E}\left(\int_{t^2}^{\infty} e^{-rt}R_2 dt - e^{-rt^2}I_2\right). \end{aligned} \tag{8}$$

Expression (8) implies that the problem is decomposed into two stopping problems, which are only linked through the constraint $t^2 \geq t^1$. For the moment, ignore this constraint, and note that the two resulting problems are identical to the one considered in section 3.1. Therefore, without constraint $t^2 \geq t^1$, the solution must consist of two investment thresholds, Y_1 and Y_2, given by:

$$Y_1 = \frac{\beta}{\beta-1}\frac{I_1}{R_1}\left(r-\mu\right), \tag{9}$$

$$Y_2 = \frac{\beta}{\beta-1}\frac{I_2}{R_2}\left(r-\mu\right). \tag{10}$$

Comparing these expressions, one can see immediately that $Y_1 < Y_2$ under our assumption $\frac{R_1}{R_2} > \frac{I_1}{I_2}$. Therefore, concerning the corresponding stopping times $t^1 = \inf\{t \geq 0\,|Y_t \geq Y_1\}$ and $t^2 = \inf\{t \geq 0\,|Y_t \geq Y_2\}$, it must hold that $t^2 > t^1$, which means that the constraint $t^2 \geq t^1$ linking the two problems is automatically satisfied. We conclude that the first stage is accomplished strictly earlier than the second stage, and the existence of the second stage has no effect on the optimal exercise time of the first stage[5], meaning that the two stages can be considered separately. We denote the values of the options to invest separately for the two stages as $F_1(Y)$ and $F_2(Y)$. Analogously to (6), these can be written as:

$$F_1(Y) = \left(\frac{Y_1 R_1}{r-\mu} - I_1\right)\left(\frac{Y}{Y_1}\right)^{\beta}, \tag{11}$$

$$F_2(Y) = \left(\frac{Y_2 R_2}{r-\mu} - I_2\right)\left(\frac{Y}{Y_2}\right)^{\beta}, \tag{12}$$

and they are applicable for $Y < Y_1$ and $Y < Y_2$, respectively. The value of the (compound) option to invest sequentially in stages 1 and 2 can be written as:

[5] This result is due to the special structure of optimal stopping problems that also underlies the main conclusions of Leahy (1993) and Baldursson and Karatzas (1997), according to which an investor, who must take into account subsequent investments of the competitors, employs the same investment policy as a monopolist who is not threatened by such future events.

$$F_S(Y) = F_1(Y) + F_2(Y) = \left(\frac{Y_1 R_1}{r-\mu} - I_1\right)\left(\frac{Y}{Y_1}\right)^{\beta} + \left(\frac{Y_2 R_2}{r-\mu} - I_2\right)\left(\frac{Y}{Y_2}\right)^{\beta}, \tag{13}$$

which is applicable in the continuation region, that is, when $Y < Y_1$.

3.3 General case

So far, we have determined the option values and the optimal investment thresholds for the lumpy and the sequential investment separately. Now we consider the general problem (3). Since we have assumed that the initial value Y_0 is so low that it is not optimal to undertake any investment at $t = 0$, the value of the firm being valid for low values of Y is simply $F(Y) = \max\{F_L(Y); F_S(Y)\}$, where the expressions for $F_L(Y)$ and $F_S(Y)$ applicable for low values of Y were given in (6) and (13), respectively. Our aim is to establish conditions that determine which of these expressions is greater. Since we are interested in the trade-off between cost efficiency and flexibility, we want to state the relation of the option values in terms of the parameter κ that represents the cost premium that must paid by the firm for the flexibility of splitting the investment.

The following proposition states that there is a single threshold value such that if κ is below that level, the option value of the sequential investment dominates that of the lumpy investment, while the converse is true for κ above that level.[6]

Proposition 1 *Consider values of Y in the interval $(0, Y_1)$. There exists a critical level of the investment cost premium $\widehat{\kappa}$ such that when $\kappa = \widehat{\kappa}$, we have $F_S(Y) = F_L(Y)$. The critical premium $\widehat{\kappa}$ can be expressed in terms of the other model parameters as follows:*

$$\widehat{\kappa} = (I_1 + I_2)\left[\left(\frac{R_1^{\beta}}{I_1^{\beta-1}} + \frac{R_2^{\beta}}{I_2^{\beta-1}}\right) R^{-\beta}\right]^{\frac{1}{\beta-1}}. \tag{14}$$

For $\kappa < \widehat{\kappa}$, we have $F_S(Y) > F_L(Y)$, whereas for $\kappa > \widehat{\kappa}$, we have $F_L(Y) > F_S(Y)$.

Proof. See the Appendix. ■

Note that $\widehat{\kappa}$ depends on the general parameters μ, σ, and r only through their effect on parameter β. Hence, β aggregates the effect of the environment in which the firm operates on the choice between the lumpy and sequential investment alternatives.

Proposition 1 gives us an unambiguous dominance relation between the lumpy and sequential investment alternatives. To see why, consider first the scenario where $\kappa > \widehat{\kappa}$. In

[6] More generally, we could present the threshold where the two options are equally valuable as the surface in the space of all model parameters, where function $f(r, \kappa, \mu, \sigma, I_1, I_2, R_1, R_2; Y) \equiv F_S(Y) - F_L(Y)$ gets the value zero for low values of Y. Thus, the threshold level $\widehat{\kappa}$ is implicitly defined by the condition $f(r, \widehat{\kappa}, \mu, \sigma, I_1, I_2, R_1, R_2; Y) = 0$, and is thus of course a function of all other parameters of the model.

that case, the lumpy project dominates the sequential project for low values of Y, but when Y is increased, the lumpy project becomes all the more attractive relative to the sequential project. Hence, the lumpy investment is clearly superior to the sequential investment at any state of the world. On the other hand, when $\kappa < \widehat{\kappa}$, the situation is slightly more complex, because even if the sequential project dominates the lumpy project for low values of Y, any investment alternative (lumpy or sequential) may actually be optimally chosen depending on the initial value of Y (see Décamps et al., 2003). However, since we have assumed that the initial value Y_0 is so low that at time $t = 0$ it is not optimal to invest in any project, the eventual choice of investment is always to choose the sequential investment. Thus, we say that the sequential investment dominates the lumpy investment in the case $\kappa < \widehat{\kappa}$.

Note that the assumption of a low initial value for Y is a natural one in growing economies, since what we really want to model is the conditions under which the investment becomes optimal for the first time. If the initial value were higher, there should be some explanation for why the investment has not yet taken place before the "initial" time. Note that a similar interpretation on the domination relation of mutually exclusive options is implicitly adopted, for example, in Dixit (1993).

4 Role of Model Parameters

Our main objective is to show how the choice between the lumpy and the sequential investment depends on the parameters related to the environment in which the firm operates. According to (14), the effects of these parameters on the dominance relation of these two alternatives are aggregated in parameter β. Hence, at first instance it is sufficient to examine the effect of changes in β on the threshold level $\widehat{\kappa}$.

An increase in $\widehat{\kappa}$ is equivalent to a reduction (expansion) of the set of project specific parameter values under which the lumpy investment dominates (is dominated by) the sequential investment. This leads to the interpretation that $\widehat{\kappa}$ represents the cost advantage for the lumpy investment required to compensate for the loss of flexibility associated with splitting the investment. Thus, an increase (a decrease) in $\widehat{\kappa}$ is equivalent to a higher (lower) value of flexibility in sequencing the investment, because it results in the cost premium making the alternatives equally attractive being larger (smaller). The next proposition states our main result:

Proposition 2 *Consider the critical cost premium $\widehat{\kappa}$ as a function of β. Then, the following relationship holds:*

$$\frac{\partial \widehat{\kappa}}{\partial \beta} > 0.$$

This implies that the relative value of flexibility in sequencing the investment is negatively *related to volatility and drift rate of the process (1), but positively related to the discount rate:*

$$\frac{\partial \widehat{\kappa}}{\partial \sigma} < 0, \tag{15}$$

$$\frac{\partial \widehat{\kappa}}{\partial \mu} < 0, \tag{16}$$

$$\frac{\partial \widehat{\kappa}}{\partial r} > 0. \tag{17}$$

Proof. See the Appendix. ■

Equation (15) embodies the most interesting result of this paper. It means that increasing uncertainty *reduces* the relative value of flexibility in sequencing the project. This contradicts the basic real options intuition according to which the value of flexibility increases with uncertainty. The intuition for the result is as follows. As uncertainty increases, large intertemporal variations in the value of the project are likely to shift it quickly away from the value corresponding to the optimal investment threshold. Thus, the fine-tuning of the timing of the project by sequencing the investment becomes less relevant. From the mathematical point of view the result follows from the fact that the real investment options are convex functions of the project values. To see this, notice that for the lumpy project substituting (4) into (6) gives

$$F_L(Y) = \frac{(\beta-1)^{\beta-1}}{\beta^{\beta} I^{\beta-1}} \left(\frac{RY}{r-\mu} \right)^{\beta}. \tag{18}$$

(For stage 1 and 2 of the sequential project analogous formulae hold.) In the Appendix we show that adding up the option values of each stage of the sequential project gives a greater value than the option on the sum of each stage of the sequential project. These two values can be made equal by making the sequential project more expensive, thus having as investment expenditure $\hat{\kappa} I$ with $\hat{\kappa} > 1$. Now, an increase in uncertainty (σ) leads to a reduction in β and, as a result, in the convexity of the option values as functions of the project payoffs. As a consequence, the critical level of premium, $\hat{\kappa}$, becomes smaller.

Equation (16) says that the lower the drift rate of process (1) is, the more valuable in relative terms is the possibility to sequence the project. The intuition is that if the value of the project grows only slowly, the cost of delaying investment until it is optimal to undertake both stages together is high. This is a rather obvious result, but it completes our main argument: it is rather the fact that the growth is *gradual* that makes sequencing the investment valuable in this context, not the fact that growth is *uncertain*. Of course, the effect of the discount rate, as expressed in (17), can be explained in a similar way: increased discounting makes it more costly to delay investment until both stages are optimally undertaken together, thus the relative value of sequential investment is increased.

5 Conclusions

We have analyzed the choice between completing a project in one step and completing it sequentially. We have determined the optimal investment rule as a function of the premium the firm has to pay for the possibility to split the project (and not having to commit the cost of the entire project up-front).

Our main result is that increasing uncertainty favors the lumpy investment relative to sequential investment. This is in contrast with a careless interpretation of the real option theory. Depending on the interpretation of our model, this means that increasing uncertainty favors *a*) building one big plant rather than two small ones, *b*) entering a growing market through a single launch rather than taking smaller steps *c*) leapfrogging rather than implementing a progressive technology adoption, *d*) bundling two projects together rather than undertaking them separately, or *e*) taking over an entire firm rather than purchasing a partial stake as a first step possibly followed by a complete takeover.

A Appendix

Proof of Proposition 1. We begin by comparing the two option values $F_L(Y)$ and $F_S(Y)$. It holds (cf. (6) and (13)) that

$$\begin{aligned}\frac{F_L(Y)}{F_S(Y)} &= \frac{\left(\frac{Y_L R}{r-\mu} - I\right)\left(\frac{Y}{Y_L}\right)^{\beta}}{\left(\frac{Y_1 R_1}{r-\mu} - I_1\right)\left(\frac{Y}{Y_1}\right)^{\beta} + \left(\frac{Y_2 R_2}{r-\mu} - I_2\right)\left(\frac{Y}{Y_2}\right)^{\beta}} \\ &= R^{\beta}\left(\frac{\kappa}{I_1+I_2}\right)^{\beta-1}\left(\frac{R_1^{\beta}}{I_1^{\beta-1}} + \frac{R_2^{\beta}}{I_2^{\beta-1}}\right)^{-1}, \qquad \text{(A.1)}\end{aligned}$$

since $I_1 + I_2 \equiv \kappa I$, and invstment thresholds Y_L, Y_1, and Y_2, are given by (4), (9), and (10), respectively. Equation (14) follows directly from (A.1). Since β is always greater than 1, and all terms in (A.1) are positive, it holds that $\frac{\partial}{\partial\kappa}\left(\frac{F_L(Y)}{F_S(Y)}\right) > 0$. This implies that $\frac{F_L(Y)}{F_S(Y)} = 1$ if and only of $\kappa = \widehat{\kappa}$ and that the inequalities stated in the proposition hold.

Now, in order to prove that $\widehat{\kappa} > 1$, we show that the $F_L(Y) < F_S(Y)$ for $\kappa = 1$. Define

$$D(\beta) \equiv \frac{R^{\beta}}{(I_1+I_2)^{\beta-1}}\left(\frac{R_1^{\beta}}{I_1^{\beta-1}} + \frac{R_2^{\beta}}{I_2^{\beta-1}}\right)^{-1}, \qquad \text{(A.2)}$$

that is, the ratio of $F_L(Y)$ and $F_S(Y)$ for $\kappa = 1$. It can easily be seen that $\lim_{\beta\to 1} D(\beta) = 1$. Now, define α and γ such that

$$\begin{aligned} I_1 &\equiv \alpha I \kappa, \qquad &\text{(A.3)} \\ R_1 &\equiv \gamma R. \qquad &\text{(A.4)}\end{aligned}$$

(Note that (A.3) and (A.4) imply that $I_2 = (1-\alpha) I\kappa$ and $R_2 = (1-\gamma) R$.) Then, $D(\beta)$ can be expressed as

$$D(\beta) = \left(\frac{\gamma^{\beta}}{\alpha^{\beta-1}} + \frac{(1-\gamma)^{\beta}}{(1-\alpha)^{\beta-1}} \right)^{-1}. \tag{A.5}$$

To show that $D(\beta) < 1$ for $\beta > 1$, let us calculate the following derivative

$$\begin{aligned} \frac{\partial D(\beta)}{\partial \beta} &= \frac{\partial}{\partial \beta}\left[\left(\frac{\gamma^{\beta}}{\alpha^{\beta-1}} + \frac{(1-\gamma)^{\beta}}{(1-\alpha)^{\beta-1}} \right)^{-1} \right] \qquad \text{(A.6)} \\ &= \left(\frac{\gamma^{\beta}}{\alpha^{\beta-1}} + \frac{(1-\gamma)^{\beta}}{(1-\alpha)^{\beta-1}} \right)^{-2} \left[\gamma \left(\frac{\gamma}{\alpha}\right)^{\beta-1} \ln \frac{\alpha}{\gamma} + (1-\gamma)\left(\frac{1-\gamma}{1-\alpha}\right)^{\beta-1} \ln \frac{1-\alpha}{1-\gamma} \right]. \end{aligned}$$

Since the first factor is always positive, we are interested in the sign of the second factor in (A.6). For $\gamma \downarrow \alpha$, it is equal to zero. Therefore, in order to prove that (A.6) is negative, it is sufficient to show that

$$\frac{\partial}{\partial \gamma}\left[\gamma \left(\frac{\gamma}{\alpha}\right)^{\beta-1} \ln \frac{\alpha}{\gamma} + (1-\gamma)\left(\frac{1-\gamma}{1-\alpha}\right)^{\beta-1} \ln \frac{1-\alpha}{1-\gamma} \right] \tag{A.7}$$

is negative. Differentiating (A.7) and rearranging yields

$$-\left(\frac{\gamma}{\alpha}\right)^{\beta-1} \beta \ln \frac{\gamma}{\alpha} + \left(\frac{1-\gamma}{1-\alpha}\right)^{\beta-1} \beta \ln \frac{1-\gamma}{1-\alpha} - \left(\frac{\gamma}{\alpha}\right)^{\beta-1} + \left(\frac{1-\gamma}{1-\alpha}\right)^{\beta-1} < 0.$$

The last inequality results from the fact that the first three components are negative and that $\frac{\gamma}{\alpha} > 1 > \frac{1-\gamma}{1-\alpha}$. Consequently, for $\kappa = 1$ and $\beta > 1$, the value of the sequential investment opportunity is higher than the value of the lumpy project. Since (A.1) increases with κ, $\widehat{\kappa}$ is greater than 1. ■

Proof of Proposition 2. After rearranging (14), $\widehat{\kappa}$ can be expressed as

$$\widehat{\kappa} = \left(\gamma \left(\frac{\gamma}{\alpha}\right)^{\beta-1} + (1-\gamma)\left(\frac{1-\gamma}{1-\alpha}\right)^{\beta-1} \right)^{\frac{1}{\beta-1}}. \tag{A.8}$$

Let us choose two arbitrary values of β, say β' and β'', such that $\beta' > \beta''$, and define

$$\delta \equiv \frac{\beta' - 1}{\beta'' - 1} > 1.$$

It holds that

$$\begin{aligned} \gamma \left(\frac{\gamma}{\alpha}\right)^{\beta''-1} + (1-\gamma)\left(\frac{1-\gamma}{1-\alpha}\right)^{\beta''-1} &= \gamma \left(\frac{\gamma}{\alpha}\right)^{\frac{\beta'-1}{\delta}} + (1-\gamma)\left(\frac{1-\gamma}{1-\alpha}\right)^{\frac{\beta'-1}{\delta}} \\ &< \left(\gamma \left(\frac{\gamma}{\alpha}\right)^{\beta'-1} + (1-\gamma)\left(\frac{1-\gamma}{1-\alpha}\right)^{\beta'-1} \right)^{\frac{1}{\delta}}, \end{aligned}$$

where the last inequality results from the fact that $y^{\frac{1}{\delta}}$ is a concave function. This implies that the following inequality holds:

$$\left(\gamma\left(\frac{\gamma}{\alpha}\right)^{\beta''-1}+(1-\gamma)\left(\frac{1-\gamma}{1-\alpha}\right)^{\beta''-1}\right)^{\delta}<\gamma\left(\frac{\gamma}{\alpha}\right)^{\beta'-1}+(1-\gamma)\left(\frac{1-\gamma}{1-\alpha}\right)^{\beta'-1}.$$

It follows immediately that

$$\left(\gamma\left(\frac{\gamma}{\alpha}\right)^{\beta''-1}+(1-\gamma)\left(\frac{1-\gamma}{1-\alpha}\right)^{\beta''-1}\right)^{\frac{1}{\beta''-1}}<\left(\gamma\left(\frac{\gamma}{\alpha}\right)^{\beta'-1}+(1-\gamma)\left(\frac{1-\gamma}{1-\alpha}\right)^{\beta'-1}\right)^{\frac{1}{\beta'-1}}.$$

Defining $\beta' \equiv \beta'' + \Delta\beta_1$ and letting $\Delta\beta$ tend to zero leads to the conclusion that $\partial\widehat{\kappa}/\partial\beta > 0$. Results (15)-(17) follow from the fact that $\partial\beta/\partial\sigma < 0$, $\partial\beta/\partial\mu < 0$, and $\partial\beta/\partial r > 0$, respectively. ■

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