Sensitivity Analysis of Portfolio Volatility: an Application to Financial Risk Management[¤]

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Abstract

This work discusses the application of the Di®erential Importance Measure to the Sensitivity Analysis (SA) of portfolio volatility. Some recent work in this ¯eld has shown that volatility SA based on partial derivatives (PD) can provide guidance in: a) the evaluation of the impact of changes in portfolio composition on the portfolio volatility (σ_p); and b) in asset allocation in order to match the minimum portfolio volatility. In this work we focus on point a, discussing the issues related to PD based SA. In particular we show that: 1) Utilizing PDs one makes the implicit assumption of uniform portfolio changes 2) the impact of the change in one or more weights in the portfolio composition cannot be directly assessed. It is evident that the above points 1) and 2) can pose some limitations in the evaluation of the impact of trading strategies on portfolio volatility, since usually a trading strategy is composed by arbitrary portfolio weight changes and by the change in several rather than individual weights. The above limitations can be overcome by exploiting the properties of DIM as an SA

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tool. We therefore propose a portfolio volatility SA method based on DIM. We first derive the expression of the relevance of individual weights and weight groups on the portfolio volatility. We then apply the result to the evaluation of the influence of weights in trading strategies. We first consider the two simplest trading schemes: uniform and proportional variations of portfolio weights. We then devote out attention to optimal trading strategies, i.e. trading strategies that aim at minimizing portfolio volatility. The proposed method is applied to a portfolio of 30 stocks composing the Dow Jones Index.

Keywords: Portfolio Management, Trading Strategies, Sensitivity Analysis, Asset Allocation

1 Introduction

This work discusses the Sensitivity Analysis (SA) of portfolio models and proposes a SA scheme that is capable of assessing the joint impact of changes in portfolio composition on portfolio volatility (σ_p).

The recent years have seen the fast development of models for the estimation of σ_p. Especially after the seminal works of Bollerslev and Engle, ([2], [?]). Generalized Autoregressive Conditional Heteroschedasticity Models (GARCH) models have nowadays become fundamental tools in investment management. Thanks to the use of computers these models have become increasingly complex and the need for the appropriate SA techniques to fully utilized the information produced by the models is felt.

In a recent paper, Saltelli ([28]) has demonstrated how SA can be thought of as an essential ingredient in portfolio management. Partial Derivatives (PD) based SA has been used in the case of the sensitivity of the Value at Risk (VaR) models by McNeil and Frey ([22]) and by Gourieroux et al. ([15]). These authors derive analytically the expressions for the ¯rst and second derivatives of the VaR, and explain how they can be used to simplify statistical inference and to perform a local analysis of the VaR. An application of this technique can be found in Drudi et al. ([13]), where the sensitivity of risk assessment is tested with respect to (w.r.t) the number of factors employed, the measures of volatility (conditional versus unconditional) and correlations (stable versus unstable), and the linearization of non-linear payo®s. Manganelli et al. [20] propose a tool based on the calculation of the partial derivatives of σ_p estimated via the GARCH model. The tool is aimed at helping \risk managers to ¯nd out what the major sources of risk are, or allow them to evaluate the impact on the portfolio variance of a certain transaction." ([20]). In a next work by Manganelli the approach is extended to asset allocation [21].

Recent works in the SA ¯eld have shown that the PD approach su®ers of some limitations when used for the purposes mentioned in the italicized sentence above ([12], [3], [4], [5], [6], [9]). In general, performing PD based SA to evaluate the impact of parameter changes:

1 is equivalent to make the assumption of uniform parameter changes

2 does not allow the appreciation of the model sensitivity to changes in groups of parameters

Since a trading strategy involves simultaneous changes in more than one weight and such changes are generally not uniform, then the shortfalls in using PD based SA for the evaluation of the impact of trading strategies appear evident. On the one hand the impossibility to evaluate the impact of weight changes other than the uniform ones, and on the other hand, the impossibility of testing the sensitivity on weight groups.

In this work we suggest that the use of the Di®erential Importance Measure (DIM) leads to a solution to the two above mentioned limitations. DIM was, in fact, introduced in the SA realm to cope with the SA of model output in response to arbitrary changes in several parameters ([12], [4], [3], [5], [?], [10], [6], [9]). DIM generalizes traditional PD based SA techniques as, for example, Elasticity ([6], [9]) and shares two important properties | (i) additivity and (ii) relative changes consideration. We show that these two properties are capable of overcoming the theoretical limitations inherent to the PD approach illustrated in points 1) and 2) above. In particular, additivity enables the computation of the sensitivity of σ_p given a change in a group of portfolio weights, and the de¯nition of DIM itself to accommodate generic portfolio composition changes.

We ¯rst investigate the application of DIM to GARCH models from a mathematical point of view. We provide the de¯nition of portfolio weight di®erential importance w.r.t. σ_p. We compare the volatility response to uniform and to proportional portfolio weights. As a result a method for valuing the impact of simultaneous changes in subsets of portfolio weights is proposed.

The possibilities of that importance measure in the context of asset management are empirically presented by considering a portfolio composed of 30 stocks composing the Dow Jones index, as of March 2002.

In Section 2 the de¯nition of DIM and some SA background related to the recent development on this ¯eld are discussed. In Section 3 some preliminary considerations on the SA of portfolio models highlighting the e®ect of relative portfolio changes are presented. Section 4 presents volatility Estimation Models. Section 5 discusses the application of DIM to GARCH models for the estimation of the importance of portfolio weights. Section 6 presents numerical results focusing on the ¯nancial management aspects of the analysis. Section 7 o®ers some conclusions.

2 Sensitivity Analysis Background

In the recent past, the activity in the scientific field of SA of Model Output has been steadily growing. Due to the increasing complexity of numerical models, SA has acquired a key role in testing the correctness and corroborating the robustness of models in several disciplines. This has led to the development and application of several new SA techniques ([4], [17], [26], [27], [?], [29], [31],[?]). Since the SA of portfolio volatility based on PDs has a character that can be defined as Local from a SA point of view, it is on Local SA techniques that we focus in this introduction. In the next paragraphs we present the Differential Importance Measure (DIM) and discuss in detail its relation to other local SA techniques.

Let us consider the generic model output:

$$Y = f(\mathbf{x}) \tag{1}$$

where $\mathbf{x} = \{x_i,\ i = 1, 2, ...n\}$ is the set of the input parameters. Let also

$$d\mathbf{x} = [dx_1, dx_2, \ldots, dx_n]^T$$

denote the vector of (infinitesimal) changes.

Then the differential importance of x_s at $\mathbf{x}^0$ is defined as ([4]):

$$D_s(\mathbf{x}^0, d\mathbf{x}) = \frac{f_s(\mathbf{x}^0)dx_s}{\sum_{j=1}^n f_j(\mathbf{x}^0)dx_j} = \frac{df_s(\mathbf{x}^0)}{df(\mathbf{x}^0)} \tag{2}$$

D_s is the ratio of the (infinitesimal) change in Y caused by a change in x_s and the total change in Y caused by a change in all the parameters. Thus, D_s is the normalized change in Y provoked by a change in parameter x_s.

It can be shown that ([4], [5], [6], [9]):

- D_s shares the additivity property with respect to the various inputs, i.e. the impact of the change in some set of parameters coincides with the sum of the individual parameter impacts. More formally, let $S \subseteq \{1, 2, ., n\}$ identify some subset of interest of the input set. We have:

$$D_S(\mathbf{x}^0, d\mathbf{x}) = \frac{\sum_{s \in S} f_s(\mathbf{x}^0)dx_s}{\sum_{j=1}^n f_j(\mathbf{x}^0)dx_j} = \sum_{s \in S} D_s(\mathbf{x}^0, d\mathbf{x}) \tag{3}$$

As a consequence,

$$\sum_{s=1} D_s(\mathbf{x}^0, d\mathbf{x}) = 1 \qquad (4)$$

i.e. the sum of all the individual parameter DIMs ($i = 1...n$) is always equal to unity [4].

² Eq. (2) shows that DIM accounts for the way parameters are varied through the dependence on $d\mathbf{x}$. Therefore, it is eliminated the arbitrariness of the selection of the parameter changes, that was one of the drawbacks in using other techniques, such as Tornado Diagrams, one way SA, or PDs. In the hypothesis of uniform parameter changes (H1), one ¯nds ([4], [6], [9]):

$$D1_s(\mathbf{x}^0) = \frac{f_s(\mathbf{x}^0)}{\sum_{j=1}^{n} f_j(\mathbf{x}^0)} \qquad (5)$$

In the hypothesis of proportional changes (H2), one ¯nds:

$$D2_s(\mathbf{x}^0) = \frac{f_s(\mathbf{x}^0) \cdot x_s^0}{\sum_{j=1}^{n} f_j(\mathbf{x}^0) \cdot x_j^0} \qquad (6)$$

DIM generalizes other local SA techniques as the Fussell-Vesely importance measure and Local Importance Measures based on normalized partial derivatives, also known as measures of Criticality Importance or Elasticity ([4],[6], [5], [9],[12],[17]). The discussion of the relationship between DIM and the Fussel-Vesely importance can be found in [4], the discussion on the relationship between DIM and Elasticity can be found in ([4], [6], [9]). In particular such works show that Elasticity produces the importance of parameters when proportional changes in their values are considered.

3 E®ect of relative weight changes

In this Section, we show that the relative weight variations have to be considered when evaluating the impact of a trading strategy on a portfolio.

To appreciate the e®ects of assumptions on weight changes, let us consider a simple example.

Example 1 Let

$$v = a_1 x_1 + a_2 x_2 \tag{7}$$

be the value of a portfolio at a certain point in time. Let also $a_1 = 100$, $a_2 = 9900$, $x_1 = 10EUR$ and $x_2 = 5EUR$. The total value of the portfolio is then $v = 50500EUR$. Let us now assume that the first trading strategy is to buy one additional stock of 1 and 2. In this case we have a unitary change in a_1 and a_2, i.e. $da_1 = da_2 = 1$. Applying eq. (5), one gets: $D1_1 = 0.667$ and $D1_2 = 0.333$. This result means that asset 1 is the most influential if a trading strategy involving uniform weight changes is considered. Let us consider the case in which the trader opts for a proportional change in the two assets, i.e. he buys (or sells) ϖ% in each of them. Applying eq. (6), one gets: $D2_1 = 0.02$ and $D2_2 = 0.98$. In this case asset 2 would be the most influential on the portfolio value.

The above example clearly shows that considering the impact of changes in portfolio composition involves not only the consideration of the rate of change of the portfolio w.r.t. the weight (v_{a_i}), but also the relative way in which the weights are changed. We now show that evaluating the impact of portfolio changes by means of PDs only is in fact equivalent to assume uniform weight changes.

Proposition 1 Ranking weights based on partial derivatives is equivalent to consider trading strategies involving uniform weight changes.

Proof. Let us consider a set of n parameters, and let us assume that one ranks them utilizing PDs as importance measures. We use the symbol $x_i \succ x_j$ to state that parameter x_i is more important than parameter x_j. If one utilizes partial derivatives to rank parameters then one says that parameter x_i is more important than parameter x_j when the magnitude of the change in $f(\mathbf{x}^0)$ provoked by a change in x_i is greater than the magnitude of the change in $f(\mathbf{x}^0)$ provoked by a change in x_j:

$$x_i \succ x_j \Leftrightarrow \left|f_i(\mathbf{x}^0)\right| \geq \left|f_j(\mathbf{x}^0)\right| \tag{8}$$

Nothing changes in $|f_i(\mathbf{x}^0)| > |f_j(\mathbf{x}^0)|$ if one multiplies and divides both sides for $|\sum_{k=1}^{n} f_k(\mathbf{x}^0)|$, one gets:

$$x_i \succ x_j \Leftrightarrow \left|f_i(\mathbf{x}^0)\right| \geq \left|f_j(\mathbf{x}^0)\right| \Leftrightarrow \frac{|f_i(\mathbf{x}^0)|}{|\sum_{k=1}^{n} f_k(\mathbf{x}^0)|} \geq \frac{|f_j(\mathbf{x}^0)|}{|\sum_{k=1}^{n} f_k(\mathbf{x}^0)|} \tag{9}$$

The above is then equivalent to stating:

$$x_i \succeq x_j \iff \left| D1_i(\mathrm{x}^0) \right| \geq \left| D1_j(\mathrm{x}^0) \right| \qquad (10)$$

proving that ranking based on partial derivatives is equivalent to ranking based on $D1_s(\mathrm{x}^0)$, i.e. to stating an assumption of uniform parameter changes.
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4 Importance of portfolio weights in GARCH Volatility Estimation Models

Models of time-varying volatility have been popular since the early '90s in empirical research in finance, following an influential paper by Bollerslev ([2]). Models of this type are well known as generalized autoregressive conditional heteroscedasticity (GARCH) in the time series econometrics literature. Time-varying volatility was initially concerned with an economic phenomenon - time-varying and autoregressive variance of inflation.

An autoregressive conditional heteroscedastic (ARCH) process is a stochastic process if its time-varying conditional variance is heteroscedastic with autoregression, *i.e.*:

$$\begin{aligned} y_t &= \varepsilon_t, \qquad \varepsilon_t \sim N(0, \sigma^2) && (11) \\ \sigma_t^2 &= \alpha_0 + \alpha_1 \varepsilon_{t-1}^2 + \ldots + \alpha_q \varepsilon_{t-q}^2 && (12) \end{aligned}$$

The first equation is the mean equation where regressors can generally be added to the right-hand side along side ε_t. The second is the variance equation, which is an ARCH(q) process where autoregression in its squared residuals has an order of q, or has q lags.

A stochastic process is called GARCH if its time-varying conditional variance is heteroscedastic with both autoregression and moving average:

$$\begin{aligned} y_t &= \varepsilon_t, \qquad \varepsilon_t \sim N(0, \sigma^2) && (13) \\ \sigma_t^2 &= \alpha_0 + \sum_{i=1} \alpha_i \varepsilon_{t-i}^2 + \sum_{j=1} \beta_j \sigma_{t-j}^2 && (14) \end{aligned}$$

The above equations express a GARCH(p, q) process where autoregression in its squared residuals has an order of q, and the moving average component has an order of p.

The estimation of parameters of a GARCH model is straightforward as, conditional on an initial variance estimate and on model parameters, the data are normally distributed, that is a likelihood function can be easily constructed. The conditional log-likelihood of y_{t+1} is ([11]):

$$\ell_t(y_{t+1};\theta) = \log\left[N\left(\frac{y_{t+1}}{\sigma_t}\right)\right] - \frac{\log(\sigma_t^2)}{2}$$

where θ is a vector of the parameters of the model and $N(\cdot)$ is a standard normal density function. The log-likelihood of the whole dataset is:

$$L_T(y_1, ..., y_T) = \sum_{t=1}^{T} \ell_t(y_{t+1};\theta)$$

Thus, parameters can be estimated by numerically maximizing the previous expression.

Throughout the discussion we consider the following GARCH(p, q) process [21]:

$$y_t = \sqrt{\sigma_t}\varepsilon_t \qquad \varepsilon_t \sim N(0, 1) \tag{15}$$

$$\sigma_t = z_t\theta \tag{16}$$

where y_t is the return of a portfolio composed by $n+1$ assets calculated as $y_t = \sum_{i=1}^{n+1} a_i y_{t,i}$, where a_i and $y_{t,i}$ are the weight and the return respectively of asset i; $z_t = \left[1, y^2_{t-1}, ..., y^2_{t-q}, h_{t-1}, ..., h_{t-q}\right]$ and $\theta = \left[\alpha_0, \alpha_1, ..., \alpha_q, \beta_1, ..., \beta_p\right]'$.

The partial derivative of σ_t w.r.t. the portfolio weights is [21]:

$$\frac{\partial \sigma_t}{\partial a_i} = \frac{\partial z_t}{\partial a_i} \cdot \theta + z_t \cdot \frac{\partial \theta}{\partial a_i} \tag{17}$$

$\frac{\partial \theta}{\partial a_i}$ are found by differentiating the log-likelihood function as follows ([20],[21]). The solution to the set of conditions:

$$\left.\frac{\partial L_T}{\partial \theta_i}\right| = 0, \qquad i = 1, 2, ..., m \tag{18}$$

namely, $\hat{\theta} = \left\{\hat{\theta}_1, \hat{\theta}_2, ..., \hat{\theta}_m\right\}$, can be regarded as an implicit function of the weights:

$$\hat{\theta} = g(a^0) \tag{19}$$

where g is an m-dimensional vector of n-dimensional functions at a^0. The requirements for eq. (19) to hold are that the conditions for the implicit function theorem are fulfilled. Applying the implicit function theorem, if the Hessian $H_{\theta\theta} = \left[\frac{\partial^2 L}{\partial\theta_i\theta_j}\right]$ $(i, j = 1...m)$ is non-singular, one finds that:

$$\begin{bmatrix} \frac{\partial\theta_1}{\partial a_1} & \frac{\partial\theta_2}{\partial a_1} & \cdots & \frac{\partial\theta_m}{\partial a_1} \\ \frac{\partial\theta_1}{\partial a_2} & \frac{\partial\theta_2}{\partial a_2} & \cdots & \frac{\partial\theta_m}{\partial a_2} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial\theta_1}{\partial a_n} & \frac{\partial\theta_2}{\partial a_n} & \cdots & \frac{\partial\theta_m}{\partial a_n} \end{bmatrix} = - \begin{bmatrix} \frac{\partial L_T}{\partial a_1\partial\theta_1}\Big| & \frac{\partial L_T}{\partial a_2\partial\theta_1}\Big| & \cdots & \frac{\partial L_T}{\partial a_n\partial\theta_1}\Big| \\ \frac{\partial L_T}{\partial a_1\partial\theta_2}\Big| & \frac{\partial L_T}{\partial a_2\partial\theta_2}\Big| & \cdots & \frac{\partial L_T}{\partial a_n\partial\theta_2}\Big| \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial L_T}{\partial a_1\partial\theta_m}\Big| & \frac{\partial L_T}{\partial a_2\partial\theta_m}\Big| & \cdots & \frac{\partial L_T}{\partial a_n\partial\theta_m}\Big| \end{bmatrix}^T \cdot H_{\theta\theta}^{-1} \tag{20}$$

More synthetically, let us denote one of the rows in the above matrix by $\frac{\partial}{\partial a_i}$.

We have now all the elements to come to the definition of the differential importance of weights on portfolio volatility. Combining eq. (20) with eqs. (17) and (2), one finds:

$$D_i(a^0, da) = \frac{\left(\frac{\partial z_t}{\partial a_i} \cdot \theta + z_t \cdot \frac{\partial\theta}{\partial a_i}\right) da_i}{\sum\limits_{j=1} \left(\frac{\partial z_t}{\partial a_j} \cdot \theta + z_t \cdot \frac{\partial\theta}{\partial a_j}\right) da_j + \left(\frac{\partial z_t}{\partial a_j} \cdot \theta + z_t \cdot \frac{\partial\theta}{\partial a_j}\right) da_{n+1}} \Bigg|_{a^0} \tag{21}$$

Eq. (21) determines the analytical expression of the importance of portfolio weights w.r.t. σ_p estimated via a GARCH model for the generic trading strategy. From eq. (21), it is then straightforward to estimate the importance of weights for strategies that foresee a uniform or a proportional change in weights. The importance of individual weights for a trading strategy that assumes of uniform weight changes is:

$$D1_i(a^0) = \frac{\frac{\partial z_t}{\partial a_i} \cdot \theta + z_t \cdot \frac{\partial\theta}{\partial a_i}}{\sum\limits_{j=1} \left(\frac{\partial z_t}{\partial a_j} \cdot \theta + z_t \cdot \frac{\partial\theta}{\partial a_j}\right)} \Bigg|_{a^0} \tag{22}$$

The importance of individual weights for a trading strategy that assumes proportional weight changes is:

$$D2_i(\mathrm{a}^0) = \frac{\left(\frac{\partial z_t}{\partial a_i}\cdot\theta + z_t\cdot\frac{\partial\theta}{\partial a_i}\right)a_i^0}{\sum\limits_{j=1}\left(\frac{\partial z_t}{\partial a_j}\cdot\theta + z_t\cdot\frac{\partial\theta}{\partial a_j}\right)a_j^0}\Bigg|_{\mathrm{a}^0} \tag{23}$$

Suppose now that the analyst wants to evaluate the impact of changing group A vs. group B of portfolio weights. Let $S_A = \{a_{1_A}, a_{2_A}, ..., a_{k_A}\}$ and $S_B = \{a_{1_B}, a_{2_B}, ..., a_{m_B}\}$, with kA and mB lower than n. Then, combining eq. (21) with eq. (3), the influence of a change in set A weights is determined by:

$$D_{S_A}(\mathrm{a}^0, \mathrm{da}) = \frac{\sum\limits_{i=1}^{kA}\left(\frac{\partial z_t}{\partial a_{iA}}\cdot\theta + z_t\cdot\frac{\partial\theta}{\partial a_{iA}}\right)da_{iA}}{\sum\limits_{j=1}\left(\frac{\partial z_t}{\partial a_j}\cdot\theta + z_t\cdot\frac{\partial\theta}{\partial a_j}\right)da_j + \left(\frac{\partial z_t}{\partial a_j}\cdot\theta + z_t\cdot\frac{\partial\theta}{\partial a_j}\right)da_{n+1}}\Bigg|_{\mathrm{a}^0} \tag{24}$$

i.e. it is the sum of the importance of the weights in set A. Similarly, the influence of a change in set B weights is determined by:

$$D_{S_B}(\mathrm{a}^0, \mathrm{da}) = \frac{\sum\limits_{i=1}^{mB}\left(\frac{\partial z_t}{\partial a_{iA}}\cdot\theta + z_t\cdot\frac{\partial\theta}{\partial a_{iA}}\right)da_{iB}}{\sum\limits_{j=1}\left(\frac{\partial z_t}{\partial a_j}\cdot\theta + z_t\cdot\frac{\partial\theta}{\partial a_j}\right)da_j + \left(\frac{\partial z_t}{\partial a_j}\cdot\theta + z_t\cdot\frac{\partial\theta}{\partial a_j}\right)da_{n+1}}\Bigg|_{\mathrm{a}^0} \tag{25}$$

Thus, if $D_{S_B}(\mathrm{a}^0, \mathrm{da}) \geq D_{S_A}(\mathrm{a}^0, \mathrm{da})$ then set B is more influential or as influential as set A on σ_p. The above result is a consequence of the additivity property of the differential importance measure and cannot be obtained utilizing the sole PDs as a means for computing the sensitivity of σ_p on the portfolio weights.

5 Sensitivity Analysis and portfolio management

As recently pointed out by Saltelli ([28]), Sensitivity Analysis can be thought as an important tool for asset allocation as it provides extremely useful information on the source of uncertainty affecting portfolio returns. In particular, in that paper, the author pays special attention to the volatility of stock returns and thus provides an importance measure ranking aiming at estimating the relevance of the stocks for the aggregate return. In the present paper, we consider the dual problem, i.e. the sensitivity of portfolio volatility w.r.t. weights.

Let us consider the classical portfolio choice program:

$$\mathbf{a}^* = \arg\max_{\mathbf{a}} [E(\mathbf{a}'\mathbf{y}) - kVar(\mathbf{a}'\mathbf{y})] \tag{26}$$

where $E(\mathbf{a}'\mathbf{y})$ is the expected value of the portfolio return; $\mathbf{y}$ is the vector of asset returns, k is a scale variable. For given values of k, problem (26) can be splitted into:

$$\begin{aligned} \mathbf{a}^* &= \arg\max_{\mathbf{a}} [E(\mathbf{a}'\mathbf{y})] \\ & s.t. \\ Var(\mathbf{a}'\mathbf{y}) &= \sigma^2 \end{aligned} \tag{27}$$

and the dual problem:

$$\begin{aligned} \mathbf{a}^* &= \arg\min_{\mathbf{a}} [Var(\mathbf{a}'\mathbf{y})] \\ & s.t. \\ E(\mathbf{a}'\mathbf{y}) &= \mu \end{aligned} \tag{28}$$

If we rewrite $E(\mathbf{a}'\mathbf{y})$ as $\mathbf{a}'\mathbf{1} = \mu$, then the solution to (28) can be drawn from the maximization of:

$$H = \frac{1}{2}\mathbf{a}'\Sigma\mathbf{a} - \lambda(\mathbf{a}'\mathbf{1} - \mu)$$

thus:

$$\frac{\partial H}{\partial \mathbf{a}} = \Sigma \mathbf{a} - \lambda \mu \tag{29}$$

that, set equal to zero, gives the optimal portfolio:

$$\mathbf{a}^* = \lambda \Sigma^{-1} \mu \tag{30}$$

As stated in Example 1, DIMs are extremely useful in calibrating trading strategies w.r.t. a specific stock or a group of assets, that is the relevance of an asset in determining the volatility of portfolio return changes according to the size of the weight but also to the considered change (da_i). As a particular case, Proposition 1 has demonstrated that trading strategies set up according to uniform changes over the whole set of weights have the same output as if they were produced by considering only PDs.

Manganelli ([21]) provides numerical analysis arguments for the computation of first derivatives of problem (28) when the volatility of the portfolio can be approximated by a GARCH(1,1).

5.1 An empirical application

Let us consider a portfolio of 30 stocks composing the Dow Jones Industrial Average index (table 1), as of March 2002. Daily returns cover the period ranging from January 2, 1992 through March 11, 2002.

[Table 1]

As a starting point we need to generate the sub-optimal weights of the portfolio because if (30) holds, than DIMs are undefined. Thus, we estimate the weights of the optimal portfolio as defined by the first order conditions of an exponentially weighted moving average (EWMA). The choice of that particular stochastic process as a generator of the weights has been made as Manganelli ([21]) demonstrates that the volatility of the 30 stocks estimated by an EWMA is only 7.34% lower than the one estimated by a GARCH(1,1) model in $\mathbf{a}^*$. This result is particularly interesting for our purposes as the EWMA optimal portfolio can be thought as a local deviation (i.e. a small change) in the GARCH(1,1) computed in the optimum. Notice that the first derivative and thus $\mathbf{a}^*$ is computed as of March 11, 2002.

[¯gures 1 and 2]

Figures 1 and 2 depicts daily returns of the EWMA optimal portfolio and the time series of the estimated variance respectively.

Table 2 shows the DIMs for the considered stocks under the hypothesis of uniform and proportional changes. As expected, values vary greatly according to the di®erent strategies, and what is more evident is the fact that the ranking of stocks presents signi¯cant di®erences (table 3).

[Tables 2 and 3]

Those di®erences re°ect the fact that there are several multi-step paths to reach the minimum variance optimum (but only one-step path) to reach the minimum variance optimum, a¤, and they are in°uenced by two main factors:

² initial conditions (in our case the EWMA optimal portfolio);

² the choice of the feasible adjustment strategy.

On the second point, even if it is not the proper object of the present paper, it should be stated that by solving the problem (28), no information are provided on the "optimal strategy to reach the optimum point". Computing DIMs provides information on how the optimum point can be reached, by calculating the sign of the impact of a given stock and by ranking them according to that impact.

Given a suboptimal portfolio, an optimal strategy might consist in the fastest or cheapest way to reach the minimum variance (as of March 2002). The ranking of strategies can be made on the bases of the total di®erential that provides, given the estimated GARCH(1,1) model, the magnitude of the impact of weights changes. In the case shown in table 2, the trading strategy imposing uniform changes will result in a decrease of -7.04 in the volatility, whilst the proportional changes strategy will increase volatility by 0.172.

In all the previous cases, the DIM attributes the relative importance of a given single asset in inducing changes in volatility. For example and having equations (24) and (25) in mind, in the uniform changes case, the former ten stocks account for 46% of the result (recall: -7.04), whilst the latter ten only for 18.38%. This implies that the former assets are the most important for the considered strategy.

It is worthnoting that the fastest strategy is the result of (da)¤= a¤ ¡ a, where a is the initial point. Third columns in tables 2 and 3 report the DIMs for the stocks involved in that strategy. As expectded, the values greatly vary, but what is interesting is the fact that the total di®erential of the "optimal portfolio strategy"[1] lies between the ones of the uniform changes and proportional changes strategy. This is not surprising as we are interested in solving the maximin problem (26), thus other strategies di®erent from (da)¤ are suboptimal because they lead to too low and too high risk respectively.

6 Conclusions

In this work we have illustrated the sensitivity analysis of portfolio volatility estimation models. We have suggested that performing the SA based on the sole PDs, one encounters the following limitations:

1. Impossibility of testing the sensitivity on simultaneous changes in several parameters

2. Impossibility of testing the sensitivity of σ_p for changes in weights other than uniform ones.

We have illustrated that the previous two limitations do prevent one from evaluating the impact of changes in portfolio composition, since a trading strategy usually involves a change in one or more of the portfolio weights and the changes are usually not uniform. We have seen that these limitations can be overcome if, instead of using a PDs approach, one utilizes the Di®erential Importance Measure (DIM) as a SA method. We have shown that, due to the way it is de¯ned, DIM shares the following properties:

1. Additivity

2. Relative parameter changes considerations

[1]We use the word "optimal" to indicate the fastest strategy.

We have shown that the above two properties enable one to solve the limitations encountered in the use of the sole PDs. In fact, thanks to the additivity property the impact of a change in several weights can be computed as sum of the individual weight DIMs, and the e®ect of arbitrary relative changes in portfolio weights is automatically taken into account in the de¯nition of DIM.

We have then discussed the application to the SA of σ_p estimated through GARCH models. We have utilized the result of theorem 1 in [21] and combined it with the de¯nition of DIM to ¯nd the analytical expression of the importance of portfolio weights w.r.t. σ_p. We have provided the speci¯c analytical expression of the weight importance for the cases of the two simplest possible strategies: the uniform change case and the proportional change case. We have discussed the calculation of the joint importance of weights with respect to σ_p by exploiting DIM additivity property.

We have then illustrated the above results by means of the numerical computation of the SA of a portfolio composed by 30 stocks | the same portfolio as in [21] |. It has been shown that the ranking of the importance of assets changes according to the considered strategy.

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Table 1: Composition of the Dow Jones Industrial Average Index

This table presents the list of 30 stocks composing the portfolio considered in the empirical application. The composition of the Dow Jones Index should be meant as of March 11, 2002.

Stock
Alcoa In.
American Express Co.
AT&T Corp.
Boeing Co.
Caterpillar Inc.
Citigroup Inc.
Coca-Cola Co.
Walt Disney Co.
E.I. DuPont de Nemours & Co.
Eastman Kodak Co.
Exxon Mobil Corp.
General Electric Co.
General Motors Corp.
Hewlett-Packard Co.
Hope Depot Inc.
Honeywell International Inc.
Intel Corp.
International Business Machines Corp.
International Paper Co.
Johnson&Johnson
J.P. Morgan Chase & Co.
McDonald's Corp.
Merck & Co.
Microsoft Corp.
3M Co.
Altria Group Inc.
Procter & Gamble
SBS Communications Inc.
United Technologies Corp.
Wal-Mart Stores Inc.

Table 2: DIMs for different strategies

The table presents the computation of DIMs for the stocks in the portfolio for the uniform changes and the proportional changes strategy. GARCH(1,1) model is estimated by considering the optimal EWMA portfolio. First derivatives are valued as of March 11, 2002. The stock Wal-Mart Stores Inc. does not have the derivative as it has a weight depending on all other stock weights.

Stock	**Uniform changes**	**Proportional changes**	**Optimal strategy**
Alcoa In.	0.051073	0.15398633	0.436356
American Express Co.	0.069683	0.12372132	0.010644
AT&T Corp.	0.044879	0.1134629	0.009414
Boeing Co.	0.056514	0.05225033	0.00903
Caterpillar Inc.	0.061969	0.03523841	-0.01418
Citigroup Inc.	0.049297	-0.0453762	0.088237
Coca-Cola Co.	0.014888	-0.1265676	-0.01352
Walt Disney Co.	0.043174	-0.0264934	-0.00836
E.I. DuPont de Nemours & Co.	0.040858	-0.1400711	-0.06707
Eastman Kodak Co.	0.029535	-0.0074914	0.00501
Exxon Mobil Corp.	0.037164	0.13972337	0.300262
General Electric Co.	0.0334	0.0073784	0.01815
General Motors Corp.	0.046271	0.0660631	0.008305
Hewlett-Packard Co.	-0.03921	0.23467628	0.020369
Hope Depot Inc.	0.038471	0.0291163	0.000542
Honeywell International Inc.	0.091107	0.45881233	0.044955
Intel Corp.	0.035303	-0.0385615	0.001764
International Business Machines Corp.	0.054383	0.20823972	0.017341
International Paper Co.	0.035303	-0.0466493	-0.01083
Johnson&Johnson	0.005725	-0.0539873	-0.00714
J.P. Morgan Chase & Co.	0.039949	-0.077956	-0.01063
McDonald's Corp.	-0.00473	0.00894133	0.161835
Merck & Co.	0.026424	-0.0438889	-0.01404
Microsoft Corp.	0.023015	-0.0013181	-0.00261
3M Co.	0.035772	-0.1002449	0.001536
Altria Group Inc.	-0.01725	0.26155141	0.032909
Procter & Gamble	0.01172	0.01668587	0.000512
SBS Communications Inc.	0.027916	-0.1251667	-0.0148
United Technologies Corp.	0.057395	-0.076075	-0.004
Total Differential	***-7.039***	***0.1720617***	***-0.01537***

Table 3: Rankings of stocks according to the DIM

The table presents the ranking of the stocks according to the importance measures presented in table 2.

Rank	Uniform changes	Proportional changes	Optimal strategy
1	Honeywell International Inc.	Honeywell International Inc.	Alcoa In.
2	American Express Co.	Altria Group Inc.	Exxon Mobil Corp.
3	Caterpillar Inc.	Hewlett-Packard Co.	McDonald's Corp.
4	United Technologies Corp.	International Business Machines Corp.	Citigroup Inc.
5	Boeing Co.	Alcoa In.	Honeywell International Inc.
6	International Business Machines Corp.	Exxon Mobil Corp.	Altria Group Inc.
7	Alcoa In.	American Express Co.	Hewlett-Packard Co.
8	Citigroup Inc.	AT&T Corp.	General Electric Co.
9	General Motors Corp.	General Motors Corp.	International Business Machines Corp.
10	AT&T Corp.	Boeing Co.	American Express Co.
11	Walt Disney Co.	Caterpillar Inc.	AT&T Corp.
12	E.I. DuPont de Nemours & Co.	Hope Depot Inc.	Boeing Co.
13	J.P. Morgan Chase & Co.	Procter & Gamble	General Motors Corp.
14	Hope Depot Inc.	McDonald's Corp.	Eastman Kodak Co.
15	Exxon Mobil Corp.	General Electric Co.	Intel Corp.
16	3M Co.	Microsoft Corp.	3M Co.
17	Intel Corp.	Eastman Kodak Co.	Hope Depot Inc.
18	International Paper Co.	Walt Disney Co.	Procter & Gamble
19	General Electric Co.	Intel Corp.	Microsoft Corp.
20	Eastman Kodak Co.	Merck & Co.	United Technologies Corp.
21	SBS Communications Inc.	Citigroup Inc.	Johnson&Johnson
22	Merck & Co.	International Paper Co.	Walt Disney Co.
23	Microsoft Corp.	Johnson&Johnson	J.P. Morgan Chase & Co.
24	Coca-Cola Co.	United Technologies Corp.	International Paper Co.
25	Procter & Gamble	J.P. Morgan Chase & Co.	Coca-Cola Co.
26	Johnson&Johnson	3M Co.	Merck & Co.
27	McDonald's Corp.	SBS Communications Inc.	Caterpillar Inc.
28	Altria Group Inc.	Coca-Cola Co.	SBS Communications Inc.
29	Hewlett-Packard Co.	E.I. DuPont de Nemours & Co.	E.I. DuPont de Nemours & Co.

Figure 1 : Portfolio returns

This figure shows the time series of the EWMA optimal portfolio returns. First order conditions for EWMA optimization have been valued as of March 11, 2002.

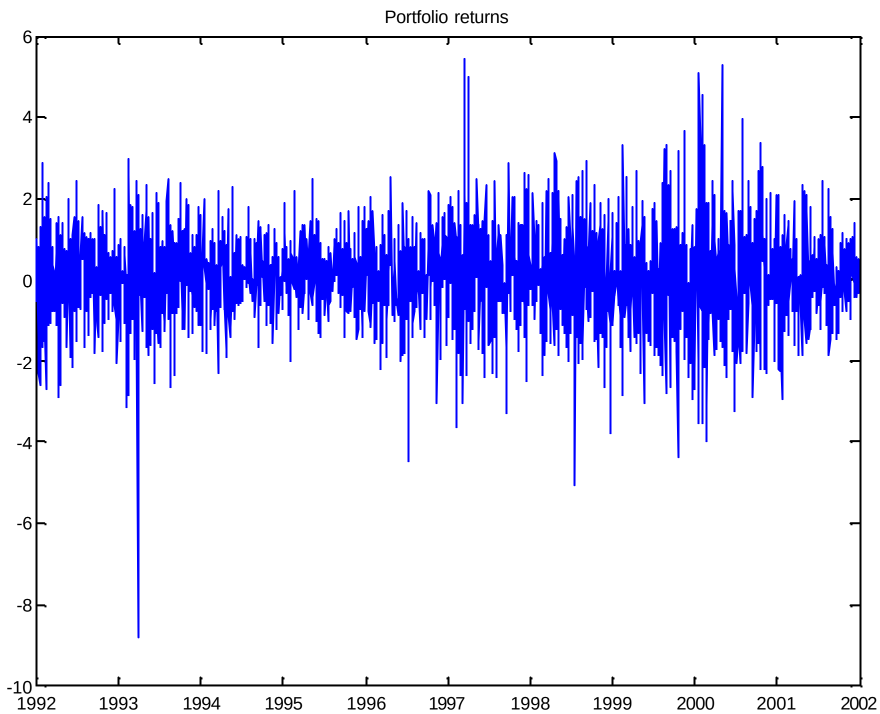

Figure 2 : Volatility of the portfolio

The figure shows the time series of the predicted variance of the GARCH(1,1) model estimated in the EWMA optimal portfolio. The low value of the function in t = 11/3/2002 indicates that the difference between the EWMA and the GARCH(1,1) optimal portfolios is relatively small. This difference can considered as a local change, thus SA based on differential measures provides appropriate indicators of relative importance..