Portfolio Selection Subject to Experts' Judgments

K. Smimou[1,a], C. R. Bector[b], G. Jacoby[c]

[a]*Faculty of Management, University of Lethbridge, Lethbridge,*

Alberta, T1K-3M4, Canada.

[b]*Department of Business Administration, University of Manitoba, Winnipeg,*

Manitoba, R3T- 5V4, Canada

[c]*Department of Accounting and Finance, University of Manitoba, Winnipeg,*

Manitoba, R3T-5V4, Canada

[1]Corresponding author. Tel.: 1-403-329-2439; *Email address*: kamal.smimou@uleth.ca

Abstract

This paper is written with two purposes in mind. First, it brings together some recent results in the area of mean variance theory model validation for fuzzy systems in the existence of subjective measures suggested by experts. The central idea of the methods presented here is to map random uncertainty given a portfolio selection model into fuzzy random uncertainty description which is useful from an application and analysis point of view. Secondly, this paper also presents a brief self-contained glimpse of empirical representations to practitioners unfamiliar with the field of fuzzy modeling. It is hoped that the expositions such as this one will open new collaborations between other branches of fuzzy mathematics (in particular, operations research which deals with large scale static uncertainty modeling) and asset pricing theories.

Keywords: Finance, Portfolio Selection, Fuzzy Theory, Mean Variance Theory, Subjective Measures, Experts' judgments

JEL Classification: G11, G12, G15 , C61

Portfolio Selection Subject to Experts' Judgments

1 Introduction

The pioneer work in the mean variance theory has been presented by (Markowitz (1952), Markowitz (2003)) and Tobin (1985). Later, Sharpe (1964) and Lintner (1965) presented the Capital Asset Pricing Model (CAPM) which was built on the foundation of the mean-variance theory. The logic behind this correlation is that the identification of the efficient frontier of risky assets with the risk-free asset is provided by the mean-variance theory. That efficient frontier is singled out by the risk-free asset and the tangency frontier portfolio. In equilibrium, after asserting the assumption that *all* investors have identical probability beliefs (share the same information) the amalgamation of the risk-free asset and the portfolio would hold. Therefore, if the portfolio of all risky assets represents the market, then the CAPM is developed and it is empirically measured. Obviously, without ignoring the Roll's critique (1977) that CAPM's view of the market portfolio as it contains every asset is not always available. For example, data of real state or real asset investments are not available, yet are crucial elements in the market. Thus, the applicability of CAPM in its existing form is questionable, because the use of different proxies for market return will reshape the empirical implications.

In this paper, we question one important assumption made by Markowitz (Markowitz (1952), Markowitz (2003)), which remains a fundamental "*hidden*" assumption in mean-variance theory literature today: that assets are normally distributed or that random uncertainty is the sole way of modeling uncertainty.

Markowitz (Markowitz (2003), p.193) discussed the reasons behind the use of variance as a measure of dispersion in asset pricing instead of other dispersion measures.

> "Many considerations influence the choice of V or S as the measure of variability in a portfolio analysis. These considerations include cost, convenience, familiarity, and the desirability of the portfolios produced by the analysis."

Following Markowitz's articulation of the importance of using variance as a measure of dispersion, in this paper, the variance analysis is considered. Knowing that the analyses based on S (semi-variance) tend to produce better portfolios than those based on V, the analyses based on semi-variance can be considered in future research endeavors after experience is gained with simpler measures in our context.

Although Markowitz (1952) ignores the experts' judgments in the derivation of the efficient frontier, he emphasizes the merit of such a combination of statistical techniques and the judgment of experts in the portfolio selection process. Yet, Markowitz does not propose a method to tackle that issue, and he does not study the efficient set of portfolios for the investor in the presence of fuzziness or any subjective information. Recently, It has been noted that the number of financial papers dealing with fuzzy theory and human judgement is growing, for example Zhou and Dong (2004).

White (1969) has presented a viable conceptual framework for the uncertainty theories which will be used in this paper. White (1969) divides the uncertainty into so-called "subjective" and "objective". Subjective measures are derivable from observation of choice, whereas objective measures are derived, once the basic data are given, by specific procedures, independent of the problem faced. White (1969) has suggested that measures of uncertainty are either formally derived from specified data, or are imputed by observing choice in a given class of problems. Also, he said:

> "It is perhaps not an unreasonable prerequisite that objective and subjective measures should be correlated to some extent."

The objective of this paper is to re-examine mean-variance theory in the presence of fuzziness articulated by fuzzy returns (LR type). We rederive the Markowitz efficient set and present the Fuzzy Capital Market Line (FCML). By illustrating these ideas with an empirical example, a comparative study is obtainable.

The suggested method will serve the interest of investors who select their portfolios using a Markowitz-based model with the induction of fuzziness or any other subjective techniques like the judgment of experts.

The remainder of the paper is organized as follows. Section 2 describes background and the mathematical preliminaries. Section 3 presents the problem setting and derives analytically the efficient frontier when all securities are risky and when one of the assets is riskless. Section 4 empirically investigates the impact of experts judgments on the efficient frontiers. Section 5 concludes with a summary.

2 Mathematical Background/Preliminaries

Inferences and decisions in statistics are based on information supplied by a random experiment associated with a population and on additional information about the experiment. To achieve a statistical inference in terms of certainty and precision is almost impossible. Since the development of fuzzy set theory, many studies have tackled the combination of both fuzzy set and probability theory.

The aim of this paper is to examine methods for handling statistical problems involving fuzziness in the elements of the random experiment, and serves as a point from which to derive the Markowitz frontier in the presence of fuzzy uncertainty and random uncertainty. Gebhardt, Gil, and Kruse (1991) presented two illustrative figures showing the elements and stages in a random experiment and involving the observation of random variables and fuzziness in the observed report.

In statistics, we traditionally assume that the experimental performance and the parameter value, or state specification in a Bayesian setting, are accomplished under randomness, whereas the remaining stages in the experiment are handled under certain and well-defined conditions. However, fuzziness can arise in some of these remaining stages, that is, in the assessment of the experimental and/or prior distribution.

The notion of a fuzzy random variable (see for example, Kwakernaak (1978), Puri and Ralescu (1986), Kruse and Meyer (1987)) provides a valuable model that is manageable in a probabilistic framework. Also, the concept of fuzzy information presented by Zadeh (1978) can formalize either the experimental data or the events involving fuzziness. The concept of a fuzzy random variable Puri and Ralescu (1986) was defined as a tool for establishing relationships between the outcomes of a random experiment and inexact data. By inexactness, we mean non-statistical inexactness that is due to subjectivity and to imprecision of human knowledge rather than to the occurrence of random events. Korner (1997) pointed out that the variability is given by two kinds of uncertainties: randomness (stochastic variability) and imprecision (vagueness). Randomness models the stochastic variability of all possible outcomes of an experiment. Fuzziness describes the vagueness of the given or realized outcome. Randomness answers the question: What will happen in the future? Whereas fuzziness answers the question: What has happened? or What is meant by the data?

Kwakernaak (1978) presented another explanation for the difference between randomness and fuzziness. He pointed out that when we consider an opinion poll in which a number of people are questioned, randomness occurs because it is not known which response may be expected from any given individual. Once the response is available, there still is uncertainty about the precise meaning of the response. The latter uncertainty will be characterized by fuzziness.

2.1 Fuzzy Random Variables and Properties

In this case, we deal with two types of uncertainty, namely, randomness and possibility (fuzzy). Randomness refers to the description of a random experiment by a probability space (Ω, A, P), where Ω is the set of all possible outcomes of this experiment, A is $\sigma-$field of subsets of Ω (the set of all possible events), and the set-function P, defined on A, is a probability measure. We assume that all the information that is relevant for further analysis of any outcome of the random experiment can be expressed with the aid of a real number, so that we can specify a mapping $U : \Omega \to \mathbb{R}$, which assigns to each outcome in Ω its random value in $\mathbb{R}$. U is called a random variable and is expected to be measurable with respect to the $\sigma-$field A and the Borel $\sigma-$ field B of the real line.

The possibility of a second kind of uncertainty in our discussion of a random experiment has to be involved whenever we are not in the position to fix the random values $U(w)$ as crisp numbers in $\mathbb{R}$, but only to imperfectly specify these values by a possibility distribution on $\mathbb{R}$. In this case the random variable $U : \Omega \to \mathbb{R}$ changes to fuzzy random variable $X : \Omega \to \mathcal{F}(\mathbb{R})$ with $\mathcal{F}(\mathbb{R}) = \{\widetilde{x}/\mu_x : \mathbb{R} \to [0,1]\}$ denoting the class of all fuzzy subsets. Fuzzy random variable (f. r.v.) is interpreted as a fuzzy perception of an inaccessible usual random variable, $U : \Omega \to \mathbb{R}$, which is the original of X. The idea is that the corresponding description of a random experiment $U_0(w)$ is imperfect in the sense that its most specific specification is the possibility distribution $X_w = X(w)$. In this case, for any $r \in \mathbb{R}$ the value $X_w(r)$ quantifies the degree of possibility with which the proposition $U_0(w) = r$ is regarded as being true. $X_w(r) = 0$ implies that there is no supporting evidence for the possibility of the truth of $U_0(w) = r$, whereas $X_w(r) = 1$ implies that there is no evidence against the possibility of the truth of $U_0(w) = r$, so that this proposition is fully possible. $X_w(r) \in [0, 1)$ reflects that there is evidence that supports the truth of the proposition as well as evidence that contradicts it. A way proposed by Gebhardt et al. Gebhardt, Gil, and Kruse (1991) of interpreting a possibility distribution X_w:$\mathbb{R} \to [0, 1]$ is viewing X_w in terms of the context approach.

The concept of a fuzzy random variable is a reasonable extension of the concept of a usual random variable in the many practical applications of random experiments, where the implicit assumption of data precision seems to be an inappropriate simplification rather than an adequate modeling of the real physical conditions. Considering possibility distribution allows us to involve uncertainty (due to the probability of occurrence of competing specification contexts) and imprecision (due to the context-dependent set-valued specifications of $U_0(w)$).

Definition 1 *Let (Ω, A, P) be a probability space. A function X: $\Omega \to \mathcal{F}(\mathbb{R})$ is called a fuzzy random variable if and only if:*

$$\begin{aligned}\underline{X}_\alpha &: \Omega \to \mathbb{R}, w \to \inf(X(w)_\alpha) \text{ and} \\ \overline{X}_\alpha &: \Omega \to \mathbb{R}, w \to \sup\ (X(w)_\alpha)\end{aligned}$$

are A-B- measurable for all $\alpha \in [0, 1]$, with B being the Borel $\sigma - field$ of $\mathbb{R}$.

The notion of a probabilistic set and fuzzy random variable was introduced by several authors in different ways. Kwakernaak's theory (1978) is similar to that presented here. Puri and Ralescu (1986) considered fuzzy random variables whose values are fuzzy subsets of R^n, or more generally of Banach space.

Kwakernaak (1978) defines the concept of fuzzy random variable as follows:

Let $I_i : R \to [0,1]$ be the characteristic function of the set w_i. Also, let S be the space of all piecewise continuous functions $R \to [0,1]$. We then define the perception of the random variable U, as described above, as the mapping $X : \Omega \to S$ given by

$$w \xrightarrow{X} X_w$$

with $X_w = I_i$ if and only if $U(w) \in W_i$. This means that we associate with each $w \in \Omega$, not a real number $U(w)$, as in the case of an ordinary random variable, but a characteristic function X_w , which is an element of S.

The map $X : \Omega \to S$ described above characterizes a special type of fuzzy random variable. The random variable U, of which this fuzzy random variable is a perception, is called an *original* of the fuzzy random variable. Many originals may exist. Kwakernaak (1978) introduced the notion of a fuzzy random variable as a function F

$$F : \Omega \to F(\mathbb{R})$$

subject to certain measurability conditions, where (Ω, A, P) is a probability space and $F(\mathbb{R})$ denotes all piecewise continuous functions:

$$u : \mathbb{R} \to [0,1]$$

Puri and Ralescu (1986) defined fuzzy random variable slightly differently from Kwakernaak (1978). In Puri and Ralescu (1986), fuzzy random variable is defined as a function $X : \Omega \to F_0(\mathbb{R}^n)$, where (Ω, A, P) is probability space, and $F_0(\mathbb{R}^n)$ denotes all functions (fuzzy subsets of $\mathbb{R}^n$)

$u : \mathbb{R}^n \to [0,1]$ such that $\{x \in \mathbb{R}^n : u(x) \geq \alpha\}$ is non-empty and compact for each $0 < \alpha \leq 1$

2.2 Fuzzy Variables and Their Expectations

Let (Ω, A, P) be a probability space where P is a probability measure. Let $F_0(\mathbb{R}^n)$ denote the set of fuzzy subsets $\mu : \mathbb{R}^n \to [0,1]$ with the following properties:

(a) $\{x \in \mathbb{R}^n ; \mu(x) \geq \alpha\}$ is compact for each $\alpha > 0$

(b) $\{x \in \mathbb{R}^n ; \mu(x) = 1\} \neq \phi$

Definition 2 Korner (1997). *A fuzzy random variable (fuzzy variable) is a function*

$X : \Omega \to F_0(\mathbb{R}^n)$

such that: $\{(w,x) : x \in X_\alpha(w)\} \in A \times B$ *for every* $\alpha \in [0,1]$

Where $X_\alpha : \Omega \to P(\mathbb{R}^n)$ *is defined by*

$X_\alpha(w) = \{x \in \mathbb{R}^n : X(w)(x) \geq \alpha\}$

Definition 3 Nather (1997). *A fuzzy variable X is called integrably bounded if* X_α *is integrably bounded for all* $\alpha \in [0,1]$, *i.e. for any* $\alpha \in [0,1]$ *there exists* $h_\alpha \in L^1(\Omega)$ *such that* $\|x\| \leq h_\alpha(w)$ *for each* x, w *with* $x \in X_\alpha(w)$. $L^1(\Omega)$ *denotes all functions* $h : \Omega \to \mathbb{R}$ *which are integrable with respect to the probability measure P. Then, expected value* $E[X]$ *of a fuzzy variable X is defined as:*

$X : \Omega \to F_0(\mathbb{R}^n)$; $\{x \in \mathbb{R}^n : \ (E[X])(x) \geq \alpha\} = \int X_\alpha$ *for each* $\alpha \in [0,1]$

Theorem 4 (Puri and Ralescu (1986), Korner (1997)). *If* $X : \Omega \to F_0(\mathbb{R}^n)$ *is an integrably bounded fuzzy variable, there exists a unique fuzzy set* $v \in F_0(\mathbb{R}^n)$ *such that* $\{x \in \mathbb{R}^n : v(x) \geq \alpha\} = \int X_\alpha$ *for every* $\alpha \in [0,1]$. *This theorem was used to define expected value of a fuzzy random variable* $X : \Omega \to F_0(\mathbb{R}^n)$ *which is integrably bounded.*

Definition 5 *The expected value of X, denoted by E[X], is the fuzzy set* $v \in F_0(\mathbb{R}^n)$[1] *such that* $\{x \in \mathbb{R} : v(x) \geq \alpha\} = \int X_\alpha$ *for every* $\alpha \in [0,1]$. *Existence and uniqueness of v are established*

[1] Sets of fuzzy subsets

in the following theorem $(E[X])(x) = Sup\left\{\alpha \in [0,1] : x \in \int X_\alpha\right\}$ *and its level sets are given by :* $\{x : (E[X])(x) \geq \alpha\} = \int X_\alpha,\ \alpha \in [0,1]$

2.3 Variance of Fuzzy Random Variables

Fuzzy random variable introduced by Puri and Ralescu (1986) as a generalization of compact random sets, combines both randomness and imprecision. Stochastic variability is described by use of probability theory and the vagueness by use of fuzzy sets introduced by Zadeh (1965).

Expectation as defined by Puri and Ralescu (1986) is the unique fuzzy set $E\widetilde{X}$ with

$$\left(E\widetilde{X}\right)_\alpha = E\left[\widetilde{X}_\alpha\right] \qquad\qquad 0 \leq \alpha \leq 1$$

Further, we can define:

$$\int_A \widetilde{X} dP = E\left(\widetilde{X}_{\chi_A}\right) \qquad\qquad \forall\ \mathrm{A} \in A,$$

where χ_A denotes the indicator of A$\in A$

Following Korner (1997), the variance of frv $\widetilde{X}$ is defined as $Var\widetilde{X} = Ed_2^2(\widetilde{X}, E\widetilde{X})$.

Using $\left(E\widetilde{X}\right)_\alpha = E\widetilde{X}_\alpha$ and $s_{E\widetilde{X}_\alpha} = E_{s_{\widetilde{X}_\alpha}}$, this can be written as

$$Var\widetilde{X} = n\int_0^1 \int_{S^{n-1}} Var\ s_{\widetilde{X}_\alpha}(t)\mu(dt)d\alpha$$

Analogously, the covariance between two frv's $\widetilde{X}$ and $\widetilde{Y}$ is defined as[2]:

$$Cov(X,Y) = n\int_0^1 \int_{S^{n-1}} Cov(s_{\widetilde{X}_\alpha}(t), s_{\widetilde{Y}\alpha}(t))\mu(dt)d\alpha$$

[2]For details see Nather (2000)

2.4 LR-Fuzzy Numbers

If $l > 0$ and $r > 0$, then the membership function of an LR-fuzzy number $\langle \mu, l, r \rangle_{LR}$, A is

$$m_A(x) = \begin{cases} L(\frac{\mu - x}{l}) & if \quad x < \mu \\ 1 & if \quad x = \mu \\ R(\frac{x-\mu}{r}) & if \quad x > \mu \end{cases}$$

Here $L,\ R:\ \mathbb{R}^+ \rightarrow [0,1]$ are fixed left-continuous and non-increasing functions with $L(0) = R(0) = 1$. The functions L and R are called left and right shape functions, μ the modal point and $l\ , r \geq 0$ are respectively the left and right spreads of the LR-fuzzy number. The most commonly used LR-fuzzy numbers are **triangular fuzzy numbers** $\langle \mu, l, r \rangle_{\Delta}$ with linear shape functions $L(x) = R(x) = Max\,\{0, 1-x\}$ and, especially, the **symmetric triangular fuzzy numbers** $\langle \mu, l \rangle_{\Delta}$ with $l = r$.

2.5 Random LR Fuzzy Numbers

Denote $\widetilde{Y} = \langle \mu_Y, l_Y, r_Y \rangle_{LR}$ a random LR-fuzzy number with left/right shape function L/R, with the random central value μ_Y and the positive random left and right spreads l_Y and r_Y. The result for $E\widetilde{Y}$ is known:

$$E\widetilde{Y} = \langle E\mu_Y, El_Y, Er_Y \rangle_{LR}$$

Following (Nather (1997), Korner (1997)) for random LR-fuzzy numbers $Var\widetilde{X}$ and $Cov(\widetilde{X}, \widetilde{Y})$ is given by:

$$Var\widetilde{X} = Var(\mu_X) + a_{l_2}Var(l_X) + a_{r_2}Var(r_X) - 2a_{l_1}Cov(\mu_X, l_X) + 2a_{r_1}Cov(\mu_X, l_X) \qquad (1)$$

and

$$Cov(\widetilde{X},\widetilde{Y}) = Cov(\mu_X,\mu_Y) + a_{l_2}Cov(l_X,l_Y) + a_{r_2}Cov(r_X,r_Y) \tag{2}$$
$$-2a_{l_1}\left[Cov(\mu_X,l_Y) + Cov(\mu_Y,l_Y)\right] + 2a_{r_1}\left[Cov(\mu_X,r_Y) + Cov(\mu_X,r_X)\right],$$

where

$$a_{l_1} = \frac{1}{2}\int L^{-1}(\alpha)d\alpha, \qquad a_{l_2} = \frac{1}{2}\int \left(L^{-1}(\alpha)\right)^2 d\alpha$$
$$a_{r_1} = \frac{1}{2}\int R^{-1}(\alpha)d\alpha, \qquad a_{r_2} = \frac{1}{2}\int \left(R^{-1}(\alpha)\right)^2 d\alpha$$

2.6 Fast computation of the parameters $\mathbf{a}_{l_1}$, $\mathbf{a}_{r_1}$, $\mathbf{a}_{l_2}$, $\mathbf{a}_{r_2}$

$\alpha - Cuts$ of $A = (\mu, l, r)_{LR}$ are given by the intervals

$$A_\alpha = \left[\mu - L^{-1}(\alpha)l, \mu + R^{-1}(\alpha)r\right]; \qquad \alpha \in [0,1],$$

An LR-fuzzy number $A = (\mu, l, r)_{LR}$ with $L = R$ and $l = r \overset{def}{=} \Delta$ is called symmetric and abbreviated by:

$$A \overset{def}{=} (\mu, \Delta)_L.$$

For a random symmetric fuzzy number (Nather (1997), Korner (1997)):

$$Y = (\mu, \Delta)_L$$

$$E\left(\mu, \Delta\right)_L = \left(E\mu, E\Delta\right)_L,$$

and

$$Var\left(\mu, \Delta\right)_L = Var\left(\mu\right) + 2a_{l_2}Var(\Delta)$$

In particular, the variance of a random triangular fuzzy number is simply given by

$$Var(X) = Var(\mu) + \frac{1}{6}Var(l) + \frac{1}{6}Var(r),$$

and the variance of random bell-kind fuzzy numbers is:

$$Var(X) = Var(\mu) + Var(l) + Var(r).$$

The covariance of two random LR-fuzzy numbers X, Y is given by equation (2). This form is more convenient under additional assumptions:

- If $L = R$ (shape symmetric LR-fuzzy number) then

$$\begin{aligned} Cov(X,Y) &= Cov(\mu_{X,}\mu_Y) + a_{l_2}(Cov(l_X, l_Y) + Cov(r_X, r_Y)) \\ &\quad +2a_{l_1}(Cov(\mu_X, r_Y - l_Y) + Cov(\mu_Y, r_X - l_X)) \end{aligned}$$

- If $L = R$, $l_X = r_X$, $l_Y = r_Y$ (symmetric LR-fuzzy number $\mathrm{A}_S := A_L - A_L$). Then,

$$Cov(X,Y) = Cov(\mu_X, \mu_Y) + a_{s_2}Cov(l_x, l_Y)$$

2.7 Expected Utility Maximum

First, mathematicians formulated principles of behavior in chance situations by assuming that the proper objective of the individual was to maximize expected monetary return. However, later on, some researchers found that the expected return maximum is not the proper methodology Savage (1954) Herstein and Milnor (1953). Therefore, the expected utility rule was proposed as a substitute for the expected return rule (Alchian (1953), Dorfman, Samuelson, and Solow (1958)). Instead of maximizing expected return, the rational investor would maximize the expected value of the utility of return Allais (1953).

Markowitz (Markowitz (2003), p. 209) says:

> "Some recent commentators, on the other hand, have agreed that the expected utility maxim is not the essence of rational behavior. They show instances in which human

action differs from that dictated by the maxim... At least two well-known economists who first wrote as opponents later became adherents of the expected utility maxim. The writer knows of no equally famous conversion in the other direction..."

Thus, following Markowitz (2003) we use the expected utility maximum approach to rederive the efficient frontier in the presence of fuzzy random returns.

Following (Huang and Litzenberger (1988), p 60-61), an individual's utility function may be expanded as a Taylor series around his expected end of period wealth.

$$U(\widetilde{w}) = U(E\left[\widetilde{w}\right]) + U^{'}(E\left[\widetilde{w}\right])(\widetilde{w} - E\left[\widetilde{w}\right]) + \frac{1}{2}U^{"}\left(E\left[w\right]\right)\left(\widetilde{w} - E\left[\widetilde{w}\right]\right)^{2} + R_{3},$$

where the remainder is:

$$R_{3} = \sum_{n=3}^{\infty} \frac{1}{n!} U^{(n)}\left(E\left[\widetilde{w}\right]\right)\left(\widetilde{w} - E\left[\widetilde{w}\right]\right)^{n},$$

and where $U^{(n)}$ denotes the n^{th} derivative of U. Assuming that the Taylor series converges and that the expectation and summation operations are interchangeable, the individual's expected utility may be expressed as

$$E\left[U\left(\widetilde{w}\right)\right] = U(E[\widetilde{w}]) + \frac{1}{2!}U^{"}(\widetilde{w})\sigma^{2}(\widetilde{w}) + E[R_{3}],$$

where

$$E[R_{3}] = \sum_{n=3}^{\infty} \frac{1}{n!} U^{(n)}\left(E\left[\widetilde{w}\right]\right) m^{n}(\widetilde{w})$$

$m^{n}(\widetilde{w})$ denotes the n^{th} central moment of $\widetilde{w}$. Assuming quadratic utility (or jointly normal returns), the third and higher order derivatives are zero and, therefore, $E[R_{3}] = 0$. Hence, an individual's expected utility is defined over the first two central moments of his end of period wealth, $\widetilde{w}$,

$$E[U(\widetilde{w})] = E[\widetilde{w}] - \frac{b}{2}E[\widetilde{w}^{2}] = E[w] - \frac{b}{2}\left((E[\widetilde{w}])^{2} + \sigma^{2}(\widetilde{w})\right).$$

3 Analytical Derivation of the Efficient Frontier with Fuzzy Random Returns

In this section, we analytically derive the efficient frontier in the presence of subjective information indicated by LR-fuzzy random returns. Firstly, the efficient frontier has been developed assuming an economy consisting of no riskless assets. Then the derivation of the Fuzzy Capital Market Line assuming an economy with both risky and riskless assets is achieved.

Throughout most of this paper we will use the following set of maintained assumptions:

(A1) Perfect markets: The markets for all assets are perfect with no taxes or transaction costs. Unlimited borrowing and short sales are not permitted. Each asset is infinitely divisible.

(A2) Competition: All investors act as price takers in all markets.

(A3) Homogenous expectations: All investors have identical probability beliefs.

(A4) State-independent utility: Investors are risk averse and maximize the expectation of a Von Neuman-Morgenstern utility function, which depends solely on wealth.

(A5) Complete markets: Each competitive investor can obtain any pattern of returns through the purchase of marketed assets (subject only to his/her own budget constraint) if the number of marketed assets with linearly independent returns is equal to the number of states. Under assumptions A1 through A4 it is known that the CAPM will obtain if investor's utility function is quadratic over the relevant range of outcomes or if all asset returns are drawn from one of the class of "separating distributions" defined by Ross (1978).

Following Markowitz (1952) in assuming a one-period economy, we assume that the investor applies a buy-and-hold strategy during the entire period. Of course, it is noticeable that the usual variations which we observe in a continuous framework are ignored here. As they are under a multiperiod setting, the investors are willing to rebalance their portfolios over time and single period investment models are not appropriate to help investors to make the optimal allocation of their wealth. Still, it is plausible that the analysis under the one-period model assists in understanding the mean-variance theory in the presence of subjective measure, articulated by the use of fuzzy random returns.

3.1 Investor optimization problem

Let us assume that we have N risky assets, indexed by j, where $j = 1, 2, \ldots, N$. Let the symbol " ~" and "$*$ " designate a random fuzzy variable. Let $\tilde{R}_j$ represent the one-period gross return on asset j, where the "gross" return is equivalent to one plus the rate of return. Let $\tilde{a}_j$ and $\tilde{b}_j$ represent the lower limit and maximum limit return of security j.

For example, when the investor faces a situation in which returns are not sharply defined but rather vague, she/he will establish, based on the experts' judgments, an aspiration interval in which the returns are located . In that context, the membership function which measures his/her degree of precision has a symmetric LR linear form. Thus, when $\widetilde{R}_j^*$ is assumed to be vague, we construct the fuzzy random return in the following fashion

$$\widetilde{R}_j^* = \widetilde{R}j \pm \text{width } (l_j),\ \text{thus, } \widetilde{a}_j = \widetilde{R}_j - \widetilde{l}_j \text{ and } \widetilde{b}j = \widetilde{R}_j + \widetilde{l}_j.$$

The experts' judgments provide the investor with the level of tolerance (width) she/he needs to develop the efficient frontier and $\tilde{a}_j$ and $\tilde{b}_j$ represent left-hand width and right-hand returns respectively. The fuzzy random return can be abbreviated by $\widetilde{R}_j^* = \left\langle \widetilde{R}_j, \widetilde{l}_j \right\rangle$

Let R_f represent the gross risk-free rate of return. Let W represent initial wealth, $\tilde{Y}$ represent terminal wealth, B represent the investment in a riskless asset, and V_j represent the investment in a risky asset j.

Given the above assumptions, the investor selects an optimal portfolio that maximizes the expected utility of the investor's end period wealth. It follows that the investor solves the following optimization problem.

$$Max\ E\left[U\left(\tilde{Y}\right)\right]$$

Subject to

$$1 = \tfrac{B}{W} + \textstyle\sum_{j=1}^{N} \tfrac{V_j}{W}$$

$$\tilde{Y} = R_f B + \textstyle\sum_{j=1}^{N} V_j \tilde{R}_j$$

The first constraint is the investor's budget constraint, both sides of which are divided by the investor's initial wealth w. The second constraint is the wealth accumulation constraint, which incorporates fuzziness. The investor can hold an asset long or short. A short position implies $X_j < 0$. We denote the investment weights as $X_j = \frac{V_j}{W}$ for asset j and $X_f = \frac{B}{W}$ for the riskless asset. Restating the optimization problem:

$$Max\ E\left[U\left(\tilde{Y}\right)\right]$$

Subject to

$$1 = X_f + \textstyle\sum_{j=1}^{N} X_j$$

$$\tilde{Y} = R_f W X_f + \textstyle\sum_{j=1}^{N} W X_j \tilde{R}_j,$$

Using Taylor series expansion, we expand the investor's utility function around the expected end of period wealth.

$$\begin{aligned} U\left(\tilde{Y}\right) &= U\left(E\left[\tilde{Y}\right]\right) + U'\left(E\left[\tilde{Y}\right]\right)\left(\tilde{Y} - E\left[\tilde{Y}\right]\right) \\ &\quad + \frac{1}{2}U"\left(E\left[\tilde{Y}\right]\right)\left(\tilde{Y} - E\left[\tilde{Y}\right]\right)^2 + T_3 \end{aligned}$$

where

$$T_3 = \sum_{n=3}^{\infty} \frac{1}{n!} U^{(n)} \left(E\left[\tilde{Y}\right] \right) \left(\tilde{Y} - E\left[\tilde{Y}\right] \right)^n$$

Assuming that the Taylor series converges, and because the expectation and summation operations are interchangeable, the individual's expected utility can be expressed as

$$E\left[U\left(\tilde{Y}\right)\right] = U\left(E\left[\tilde{Y}\right]\right) + \frac{1}{2}U\text{"}\left(E\left[\tilde{Y}\right]\right)\sigma^2\left(\tilde{Y}\right) + E\left[T_3\right]$$

where

$$E[T_3] = \sum_{n=3}^{\infty} \frac{1}{n!} U^{(n)} \left(E\left[\tilde{Y}\right] \right) m^n \left(\tilde{Y} \right)$$

and $m^n(\tilde{Y})$ denotes the nth central moment of $\widetilde{Y}$.

To maximize expected utility of wealth, the investor will maximize a function of the moments of the portfolio return, taking into account the assumption A4 that all investors are risk averse.

In addition, we know from the previous section that the covariance of random LR-fuzzy random variable is:

$$\begin{aligned} Cov[X,Y] &= Cov[m_x, m_y] + a_{l_2}[Cov(l_X, l_Y) + Cov(r_X, r_Y)] \\ &\quad -2a_{l_1}[Cov(m_X, r_Y - l_Y) + Cov(m_Y, r_X - l_X), \end{aligned}$$

under the symmetric assumption of the fuzzy LR-fuzzy variable, we get:

$$Cov(X,Y) = Cov[m_x, m_y] + a_{l_2}[Cov(l_X, l_Y) + Cov(r_X, r_Y)] - 2a_{l_1}[Cov(m_X, l_Y) + Cov(m_Y, l_X),$$

assuming further that m, r and l are independent,

$$Var(X) = Var(\mu) + \frac{1}{6}Var(l) + \frac{1}{6}Var(r), \tag{3}$$

and

$$Cov(X,Y) = Cov(\mu_X, \mu_Y) + \frac{1}{3}Cov(l_X, l_Y). \tag{4}$$

Applying the equation (4) in the context of the fuzzy random returns, we get:

$$Var\left[\tilde{R}_p^*\right] = \sum_{i=1}^{N}\sum_{j=1}^{N} X_j X_i \left[Cov\left[\tilde{R}_j, \tilde{R}_i\right] + \frac{1}{3} Cov\left[\tilde{l}_j, \tilde{l}_i\right]\right],$$

where $\widetilde{R}_p^*$ is portfolio fuzzy random return, and $\widetilde{R}_j$, $\widetilde{R}_i$ are the individual returns of assets j and i respectively. $\widetilde{l}_j, \widetilde{l}_i$ represent their spreads.

Following Markowitz (1952), portfolio p is a mean-variance efficient portfolio if there is no portfolio q such that $E\left[\tilde{R}_q^*\right] \geq E\left[\tilde{R}_p^*\right]$ and $Var\left[\tilde{R}_q^*\right] < Var\left[\tilde{R}_p^*\right]$. Thus, the efficient frontier can be presented as the set of portfolios that satisfy the quadratic minimization problem:

$$\begin{cases} & Min\ Var\left[\tilde{R}_p^*\right] \\ \text{Subject to} & \mu_p^* = X_f R_f + \sum_{j=1}^{N} X_j E\left[\tilde{R}_j^*\right] \\ & X_f + \sum_{j=1}^{N} X_j = 1 \end{cases} \quad (5)$$

where, $\mu_p^* = E\left[\tilde{R}_p^*\right]$, is the expected portfolio fuzzy random return. Because of the linearity of the expectation in fuzzy random environment, the $E[\widetilde{R}_j^*]$ implies that the expectation of a random LR-fuzzy number $\widetilde{R}_j^*$ is again an LR-fuzzy number:

$$E[\widetilde{R}_j^*] = \left\langle E[\widetilde{R}_j], E[\widetilde{l}_j]\right\rangle_{LR}$$

Thus, the model (5) is equivalent to:

$$\begin{cases} & Min\ Var\left[\tilde{R}_p^*\right] \\ \text{Subject to} & \mu_p^* = X_f R_f + \sum_{j=1}^{N} X_j \left\langle E[\widetilde{R}_j], E[\widetilde{l}_j]\right\rangle \\ & X_f + \sum_{j=1}^{N} X_j = 1 \end{cases},$$

using the following notation:

$$\mu_p^* = \left\langle \mu_p, l_p \right\rangle;\ Cov(\widetilde{R}_j, \widetilde{R}_i) = \sigma_{ij};\ Cov(\widetilde{l}_j, \widetilde{l}_i) = L_{ij},$$

the investment problem with only risky assets under fuzzy random environment is as follows:

$$\begin{cases} & Min\ \sum_{i=1}^{N}\sum_{j=1}^{N} X_j X_i \left[\sigma_{ij} + \frac{1}{3}L_{ij}\right] \\ & \mu_p = \sum_{j=1}^{N} X_j E[\widetilde{R}_j] \\ \text{Subject to} & \\ & l_p = \sum_{j=1}^{N} X_j E[\widetilde{l}_j] \\ & \sum_{j=1}^{N} X_j = 1 \end{cases}.$$

We know from (Dubois and Prade (1980), Arnold and Madan (1985), **?**)) that the following multiplication has two different outcomes when k is negative versus a positive value.

$$k \odot (m, \alpha, \beta)_{LR} = \begin{cases} (km, k\alpha, k\beta)_{LR} & if\ k > 0 \\ (km, -k\alpha, -k\beta)_{LR} & if\ k < 0 \end{cases}$$

In response to this consideration, we will limit our investigation to the case when the proportions have positive values, which means we will be dealing with an investment problem without short sales. Specifically, many investors do not hold short sales due to either choice or regulation (see e.g., Jarrow (1980), Aitken, Frino, McCorry, and Swan (1998)).

We know from the existing literature that empirical derivation of the mean variance efficient set, when short sales are allowed, shows that most, if not all efficient frontiers contain some negative investment proportions. Levy (1983) empirically finds that without short sales, many securities do not enter the efficient frontier, and the larger N, the smaller the percentage of the securities that will appear in the efficient set. Thus, the efficient frontier grows slowly with an increased sample size. This finding has been duplicated here under fuzzy information.

Ross (1977) suggested that in the absence of short sales, except on a single riskless asset, using a geometric approach CAPM holds, as long as the market portfolio is efficient. That assumption is maintained here; so it is intended that we will be able to generate the CAPM.

Also, a portfolio model under a fuzzy random environment without consideration for non-negativity constraint is difficult to model. In response to these considerations, in this paper, we tackle the analytical derivation of the efficient frontier with fuzzy random returns, under the assumption that there are no short sales of risky assets. So, the model is a quadratic programming one in which some stocks are held long (positive proportions) while other stocks are omitted (held in zero proportions). Efficient frontier is a combination of assets if there are no other combinations with the same (higher) expected return with lower risk, and if there is no other portfolio with the same (or lower) risk and with higher expected return.

3.2 Efficient frontier in an economy with risky assets

In this section we want to solve the following utility minimization problem to find the efficient frontier:

$$Min \sum_{i=1}^{N}\sum_{j=1}^{N} X_j X_i \sigma_{ji} + X_j X_i L_{ji} \tag{6}$$

$s.t.$

$$\mu_p = \sum_{j=1}^{N} X_j E[\widetilde{R}_j] \tag{7}$$

$$l_p = \sum_{j=1}^{N} X_j E[\widetilde{l}_j] \tag{8}$$

$$\sum_{j=1}^{N} X_j = 1 \tag{9}$$

$$X_j \geq 0 \tag{10}$$

To find the optimal solution of this quadratic programming, we first write the Lagrangian form as

$$F(X,\lambda_1,\lambda_2,\lambda_3) = \sum_{i=1}^{N}\sum_{j=1}^{N} X_j X_i \sigma_{ij} + \sum_{i=1}^{N}\sum_{j=1}^{N} X_j X_i L_{ij}$$
$$+\lambda_1 \left(\mu_p - \sum_{j=1}^{N} X_j E\left[\tilde{R}_j\right]\right) + \lambda_2 \left(l_p - \sum_{j=1}^{N} X_j E\left[\widetilde{l}_j\right]\right) + \lambda_3 \left(1 - \sum_{j=1}^{N} X_j\right). \tag{11}$$

In what follow X is in $\mathbb{R}^n$ and is $X = (X_1, X_2, ..., X_N)$.

Organizing the previous equation (11) we obtain:

$$\begin{aligned} F(X,\lambda_1,\lambda_2,\lambda_3) &= \sum_{i=1}^{N}\sum_{j=1}^{N} X_j X_i \left(\sigma_{ij} + L_{ij}\right) \qquad (12)\\ &+\lambda_1 \left(\mu_p - \sum_{j=1}^{N} X_j E\left[\tilde{R}_j\right]\right) + \lambda_2 \left(l_p - \sum_{j=1}^{N} X_j E[\widetilde{l}_j]\right) + \lambda_3 \left(1 - \sum_{j=1}^{N} X_j\right). \end{aligned}$$

The Kuhn-Tucker conditions of equation (12) are

$$0 \le \sum_{j=1}^{N} X_i \sigma^*_{ij} - \lambda_1 E\left[\tilde{R}_j\right] - \lambda_2 E[\widetilde{l}_j] - \lambda_3, j = 1,..,N \tag{13}$$

$$0 = \mu_p - \sum_{j=1}^{N} X_j E\left[\tilde{R}_j\right], \tag{14}$$

$$0 = l_p - \sum_{j=1}^{N} X_j E\left[\widetilde{l}_j\right], \tag{15}$$

$$0 = 1 - \sum_{j=1}^{N} X_j. \tag{16}$$

$$0 = \frac{\partial L}{\partial X_j} X_j,\ j = 1,..,N \tag{17}$$

$$X_j \ge 0 \tag{18}$$

where

$$\sigma^*_{ij} = Cov\left[\tilde{R}_j, \tilde{R}_i\right] + \frac{1}{3} Cov\left[\widetilde{l}_j, \widetilde{l}_i\right] = \sigma_{ij} + \frac{1}{3} L_{ij}$$

If every variable is positive then inequalities (13) are equalities because of the complementarity conditions (16). The X_j's that satisfy the first order conditions minimize the variance for every given level of expected return and are unique. Equation (13) implies

$$\sum_{j=1}^{N} X_i \sigma^*_{ij} - \lambda_1 E\left[\tilde{R}_j\right] - \lambda_2 E[\widetilde{l_j}] - \lambda_3 = 0,$$

that implies:

$$X_k = \lambda_1 \sum_{i=1}^{N} M_{ki} E\left[\tilde{R}_i\right] + \lambda_2 \sum_{i=1}^{N} E[\widetilde{l_i}] M_{ki} + \lambda_3 \sum_{i=1}^{N} M_{ki}, \; k = 1, .., N. \tag{19}$$

Define Ω^* : Variance-Covariance of fuzzy returns, Ω^{*-1} : the inverse of the matrix Ω^* where M_{ki} denote the elements of the inverse of the variance-covariance matrix of fuzzy random returns, i.e., $\Omega^{*-1} \equiv [M_{ki}]$. Ω represents the sum of the two variance-covariance matrices, $\Omega = [\sigma_{ij}] + \frac{1}{3}[L_{ij}]$. Multiplying both sides of equation (19) by $E\left[\tilde{R}_k\right]$, and summing over $k = 1, .., N$, it follows

$$\begin{aligned} \sum_{k=1}^{N} X_k E\left[\tilde{R}_k\right] &= \lambda_1 \sum_{k=1}^{N}\sum_{i=1}^{N} M_{ki} E\left[\tilde{R}_i\right] E\left[\tilde{R}_k\right] \qquad (20) \\ &\quad + \lambda_2 \sum_{k=1}^{N}\sum_{k=1}^{N} M_{ki} E[\widetilde{R_k}] E[\widetilde{l_i}] + \lambda_3 \sum_{k=1}^{N}\sum_{i=1}^{N} M_{ki} E\left[\tilde{R}_k\right]. \end{aligned}$$

Also, multiplying both sides of equation (19) by $E\left[\widetilde{l_k}\right]$, and summing over $k = 1, .., N$, it follows

$$\begin{aligned} \sum_{k=1}^{N} X_k E\left[\widetilde{l_k}\right] &= \lambda_1 \sum_{k=1}^{N}\sum_{i=1}^{N} M_{ki} E\left[\tilde{R}_i\right] E\left[\widetilde{l_k}\right] \qquad (21) \\ &\quad + \lambda_2 \sum_{k=1}^{N}\sum_{k=1}^{N} M_{ki} E[\widetilde{l_k}] E[\widetilde{l_i}] + \lambda_3 \sum_{k=1}^{N}\sum_{i=1}^{N} M_{ki} E\left[\widetilde{l_k}\right]. \end{aligned}$$

Then, summing equation (19) over $k = 1, .., N$, it follows

$$\sum_{k=1}^{N} X_k = \lambda_1 \sum_{k=1}^{N}\sum_{i=1}^{N} M_{ki} E\left[\tilde{R}_i\right] + \lambda_2 \sum_{k=1}^{N}\sum_{i=1}^{N} E[\widetilde{l_i}] M_{ki} + \lambda_3 \sum_{k=1}^{N}\sum_{i=1}^{N} M_{ki}. \tag{22}$$

Next, we define

$$\begin{aligned}
A &= \sum_{k=1}^{N}\sum_{i=1}^{N} M_{ki} E\left[\tilde{R}_i\right], \\
B &= \sum_{k=1}^{N}\sum_{i=1}^{N} M_{ki} E\left[\tilde{R}_i\right] E\left[\tilde{R}_k\right], \\
C &= \sum_{k=1}^{N}\sum_{i=1}^{N} M_{ki}. \\
A_1 &= \sum_{k=1}^{N}\sum_{i=1}^{N} M_{ki} E\left[\tilde{R}_i\right] E[\widetilde{l}_k], \\
B_1 &= \sum_{k=1}^{N}\sum_{i=1}^{N} M_{ki} E[\widetilde{l}_i] E[\widetilde{l}_k] \\
C_1 &= \sum_{k=1}^{N}\sum_{i=1}^{N} M_{ki} E[\widetilde{l}_k]
\end{aligned} \tag{23}$$

From equations (14), (15), (16), (20), (21) and (22), it follows:

$$\mu_p = \lambda_1 B + \lambda_2 A_1 + \lambda_3 A \tag{24}$$

$$l_p = \lambda_1 A_1 + \lambda_2 B_1 + \lambda_3 C_1 \tag{25}$$

$$1 = \lambda_1 A + \lambda_2 C_1 + \lambda_3 C. \tag{26}$$

Noting here,

$$\begin{aligned}
\sum_{k=1}^{N}\sum_{i=1}^{N} M_{ki} E\left[\tilde{R}_i\right] &= \sum_{k=1}^{N}\sum_{i=1}^{N} M_{ki} E\left[\tilde{R}_k\right], \\
\sum_{k=1}^{N}\sum_{i=1}^{N} M_{ki} E[\widetilde{l}_i] E[\widetilde{R}_k] &= \sum_{k=1}^{N}\sum_{i=1}^{N} M_{ki} E[\widetilde{l}_k] E[\widetilde{R}_i], \\
\sum_{k=1}^{N}\sum_{i=1}^{N} M_{ki} E\left[\tilde{R}_i\right] E[\widetilde{l}_k] &= \sum_{k=1}^{N}\sum_{i=1}^{N} M_{ki} E[\widetilde{l}_i] E[\widetilde{R}_k].
\end{aligned}$$

Solving system of equations (24), (25) and (26) for λ_1, λ_2 and λ_3, and defining $\Delta \equiv B(B_1C - C_1^2) - A_1(A_1C - AC_1) + A(A_1C_1 - AB_1)$, and as $\sum_k \sum_i M_{ki}$ is positive because of the positive definiteness of matrix $M \equiv [M_{ki}]$, we obtain

$$\begin{aligned}
\lambda_1 &= \frac{\mu_p(B_1C - C_1^2) - l_p(A_1C - C_1A) + (A_1C_1 - B_1A)}{\Delta} \\
\lambda_2 &= \frac{-\mu_p(A_1C - AC_1) + l_p(BC - A^2) - (BC_1 - A_1A)}{\Delta} \\
\lambda_3 &= \frac{\mu_p(A_1C_1 - AB_1) - l_p(BC_1 - AA_1) + (BB_1 - A_1^2)}{\Delta}
\end{aligned} \tag{27}$$

Next, we substitute for λ_1, λ_2 and λ_3, from equation (27) into equation (19) to solve for X_k. X_k is the proportion of each risky asset k held in a portfolio on the minimum-variance for a given expected return, which is as follows:

$$\begin{aligned}
X_k &= \frac{\begin{array}{c}\mu_p \sum_{i=1}^{N} M_{ki} \left[\left(B_1C - C_1^2\right) E(\widetilde{R}_i) - (A_1C - AC_1)E(\widetilde{l}_i) + (A_1C_1 - AB_1)\right] \\ -l_p \sum_i M_{ki} \left[(A_1C - C_1A)E(\widetilde{R}_i) - (BC - A^2)E(\widetilde{l}_i) + (BC_1 - AA_1)\right] \\ + \sum_{i=1}^{N} M_{ki} \left[(A_1C_1 - B_1A)\, E(\widetilde{R}_i) - (BC_1 - A_1A)E(\widetilde{l}_i) + (BB_1 - A_1^2)\right]\end{array}}{\Delta}, \\
k &= 1, .., N.
\end{aligned} \tag{28}$$

Using the following notations: $\left(B_1C - C_1^2\right) = \alpha$; $(A_1C - AC_1) = \beta$; $(A_1C_1 - AB_1) = \gamma$; $(BC - A^2) = \delta$; $(BC_1 - AA_1) = \varphi$; $(BB_1 - A_1^2) = \psi$, the equation (28) is equivalent to:

$$\begin{aligned}
X_k &= \frac{\begin{array}{c}\mu_p \sum_{i=1}^{N} M_{ki} \left[\alpha E(\widetilde{R}_i) - \beta E(\widetilde{l}_i) + \gamma\right] - l_p \sum_i M_{ki} \left[\beta E(\widetilde{R}_i) - \delta E(\widetilde{l}_i) + \varphi\right] \\ + \sum_{i=1}^{N} M_{ki} \left[\gamma E(\widetilde{R}_i) - \varphi E(\widetilde{l}_i) + \psi\right]\end{array}}{\Delta}, \\
k &= 1, .., N.
\end{aligned} \tag{29}$$

Because $M \equiv [M_{ki}]$ is positive definite and Δ is zero if and only if $\mu^* = \lambda 1$ such that $\mu^* = [\mu_1^*, ..., \mu_n^*]'$; $\mu_j^* = \left\langle E\left[\widetilde{R}_j\right], E[\widetilde{l}_j]\right\rangle$, otherwise $\Delta > 0$.

Theorem 6 (Voros (1986)). *Let* $\mu^* = [\mu_1^*, ...\mu_j^*]' \neq 1$ *for all* λ. *In the model there exists an open* *interval* $(\mu_{p_0}^*, \mu_{p_1}^*)$ *of* μ_p^* *in which every variable is positive if and only if :*

$$\left(\sum_i M_{pi}\right)\left(\sum_i M_{qi}\mu_i^*\right) < \left(\sum_i M_{pi}\mu_i^*\right)\left(\sum_i M_{qi}\right)$$

for all $p \in I^-$ *and* $q \in I^+$ *and*

$$\left(\sum_k \sum_i M_{ki}\mu_k^*\right)\left(\sum_k M_{ki}\mu_k^*\right) - \left(\sum_k \sum_i M_{ki}\mu_k^*\mu_i^*\right)\left(\sum_i M_{ki}\right) < 0$$

for all $i \in I^0$

such that:

$$\begin{aligned}
I^+ &= \left\{ k / \left(\sum_k \sum_i M_{ki}\right)\left(\sum_i M_{ki}\mu_i^*\right) - \left(\sum_i \sum_k M_{ki}\mu_i^*\right)\left(\sum_i M_{ki}\right) > 0 \right\} \\
I^- &= \left\{ k / \left(\sum_k \sum_i M_{ki}\right)\left(\sum_i M_{ki}\mu_i^*\right) - \left(\sum_i \sum_k M_{ki}\mu_i^*\right)\left(\sum_i M_{ki}\right) < 0 \right\} \\
I^0 &= \left\{ k / \left(\sum_k \sum_i M_{ki}\right)\left(\sum_i M_{ki}\mu_i^*\right) - \left(\sum_i \sum_k M_{ki}\mu_i^*\right)\left(\sum_i M_{ki}\right) = 0 \right\}
\end{aligned}$$

Proof. Similar to what Vörös (1986) presented in his paper. ■

Because of the positivity of the variables and of Δ it follows that:

$$\left[\begin{array}{c} \mu_p \sum_{i=1}^N M_{ki}\left[\alpha E(\widetilde{R}_i) - \beta E(\widetilde{l}_i) + \gamma\right] - l_p \sum_i M_{ki}\left[\beta E(\widetilde{R}_i) - \delta E(\widetilde{l}_i) + \varphi\right] \\ + \sum_{i=1}^N M_{ki}\left[\gamma E(\widetilde{R}_i) - \varphi E(\widetilde{l}_i) + \psi\right] \end{array} \right] > 0 \qquad (30)$$

If we define

$$h_k = \sum_i M_{ki} E[\widetilde{R}_i]; \ f_k = \sum_i M_{ki} E(\widetilde{l}_i) \text{ and } g_k = \sum_i M_{ki},$$

then, the equation (30) is equivalent to:

$$\mu_p(\alpha h_k - \beta f_k + \gamma g_k) - l_p(\beta h_k - \delta f_k + \varphi g_k) + (\gamma h_k - \varphi f_k + \psi g_k) > 0 \quad k = 1, ..., n$$

If $i \in I^0$ then $l_p(\beta h_k - \delta f_k + \varphi g_k) + (\gamma h_k - \varphi f_k + \psi g_k) < 0$, indices $q \in I^+ \rightarrow (\alpha h_k - \beta f_k + \gamma g_k) > 0$, and for $p \in I^-$ and $p \in I^- \rightarrow (\alpha h_k - \beta f_k + \gamma g_k) < 0$, then the following inequality holds:

$$\frac{l_p(\beta h_k - \delta f_k + \varphi g_k) + (\gamma h_k - \varphi f_k + \psi g_k)}{(\alpha h_k - \beta f_k + \gamma g_k)} < \mu_p < \frac{l_p(\beta h_k - \delta f_k + \varphi g_k) + (\gamma h_k - \varphi f_k + \psi g_k)}{(\alpha h_k - \beta f_k + \gamma g_k)} \quad (31)$$

In line with Vörös (1986), from the inequality (31), the interval in which every variable is positive is given by:

$$\mu_p^1 = \min_{p \in I^-} \left\{ \frac{l_p(\beta h_p - \delta f_p + \varphi g_p) + (\gamma h_p - \varphi f_p + \psi g_p)}{(\alpha h_p - B f_p + \gamma g_p)} \right\}$$
$$\mu_p^0 = \max_{q \in I^+} \left\{ \frac{l_p(\beta h_q - \delta f_q + \varphi g_q) + (\gamma h_q - \varphi f_q + \psi g_q)}{(\alpha h_q - B f_q + \gamma g_q)} \right\},$$

We next multiply equation (13) by X_j and sum over j for $j = 1, \ldots, N$, to derive the following:

$$\sum_{j=1}^{N} \sum_{i=1}^{N} X_j X_i \sigma_{ij}^* = \lambda_1 \sum_{j=1}^{N} E\left[\tilde{R}_j\right] X_j + \lambda_2 \sum_j X_j E(\widetilde{l}_j) + \lambda_3 \sum_{j=1}^{N} X_j. \quad (32)$$

From the definition of $\sigma^2\left(\tilde{R}_p^*\right)$, equations (14), and (15), equation (32) implies

$$Var\left[\tilde{R}_p^*\right] = \lambda_1 \mu_p^* + \lambda_2 l_p + \lambda_3. \quad (33)$$

Substituting for λ_1, λ_2 and λ_3 from (27) into (33), to obtain the equation for the minimum-variance frontier. So, for the interval (μ_{p1}^*, μ_{p0}^*), we obtain the functional form of return-variance:

$$\sigma^2\left(\tilde{R}_p^*\right) = \frac{\left(\mu_p^2 \alpha + l_p^2 \delta - 2\mu_p l_p \beta - 2 l_p \varphi + \mu_p \gamma + \psi\right)}{\Delta}, \quad (34)$$

Once all fuzzy components ($l_p = 0$, l_i and $l_k = 0$) have been discarded in the equation (34), we will get the standard functional form of return-variance. Thus, the model is a special case of the Markowitz frontier. Next, for the sake of completeness of the analysis, the minimum-variance portfolio in the presence of fuzzy random uncertainty is presented below. Since, the equation (34) is a function of two variables of degree 2, partial derivatives and all other properties of multiple variables are applicable. The differentiability is achieved as follows

$$
\begin{aligned}
\frac{\partial \sigma^2\left(\tilde{R}_p^*\right)}{\partial \mu_p} &= \frac{2\alpha\mu_p - 2l_p\beta + \gamma}{\Delta} = 0 \Longrightarrow \mu_{p\min} = \frac{2l_p\beta - \gamma}{2\alpha}, \text{ and} \\
\frac{\partial^2 \sigma^2\left(\tilde{R}_p^*\right)}{\partial^2 \mu_p} &= \frac{2\alpha}{\Delta} > 0.
\end{aligned}
\tag{35}
$$

3.3 Efficient frontier in an economy where one asset is risk-free

For all investors to achieve the efficient frontier by lending or borrowing against the risky portfolio, and for the separation theorem to hold, following Ross'analysis (1977), by permitting the investor to short sale the riskless asset, the analytical derivation of the efficient frontier is presented. The risk-free asset offers a riskless return of R_f. With short sales restrictions, all assets will appear in positive amounts in the market portfolio. The investor's utility minimization problem is formulated as follows.

$$
\begin{cases}
\underset{X_j}{Min} \sum_i \sum_j X_i X_j \sigma_{ij}^* \\
\text{Subject to} \quad \mu_p^* = X_f R_f + \sum_{j=1} X_j E[\widetilde{R}_j^*] \\
X_f + \sum_j X_j = 1 \\
X_j \geq 0 \ , \quad j = 1, ..., N
\end{cases}
\tag{36}
$$

The above model is equivalent to:

$$
\begin{cases}
\underset{X_j}{Min} \sum_i \sum_j X_i X_j \sigma_{ij}^* \\
\text{Subject to} \quad \left\langle \mu_p, l_p \right\rangle = X_f \left\langle R_f, l_f \right\rangle + \sum_{j=1} X_j \left\langle E[\widetilde{R}_j], E[\widetilde{l}_j] \right\rangle \\
X_f + \sum_j X_j = 1 \\
X_j \geq 0 \ , \quad j = 1, ..., N
\end{cases}
\tag{37}
$$

which is equivalent to:

$$\begin{cases} \underset{X_j}{Min} \sum_i \sum_j X_i X_j \sigma^*_{ij} \\ \text{Subject to} \quad \begin{array}{c} \mu_p = X_f R_f + \sum_{j=1} X_j E[\widetilde{R}_j] \\ l_p = X_f l_f + \sum_{j=1} X_j E[\widetilde{l}_j] \\ X_f + \sum_j X_j = 1 \\ X_j \geq 0 \ , \quad j = 1, ..., N \end{array} \end{cases}, \tag{38}$$

For simplicity, we assume that R_f is sharply defined, which means that $l_f = 0$. In order to find the optimal solution of this quadratic programming, we write the Lagrangian form:

$$\begin{aligned} \Psi(X_j, \lambda_1, \lambda_2) &= \sum_i \sum_j X_i X_j \sigma^*_{ij} \\ &\quad + \lambda_1 \left(\mu_p - R_f - \sum_{j=1} X_j \left(E[\widetilde{R}_j] - R_f \right) \right) \\ &\quad + \lambda_1 \left(l_p - \sum_{j=1} X_j \left(E[\widetilde{l}_j] - R_f \right) \right) \end{aligned} \tag{39}$$

The Kuhn-Tucker conditions of (39) are:

$$\frac{\partial \Psi}{\partial X_j} = \sum_{i=1}^{N} X_i \sigma^*_{ij} - \lambda_1 \left(E\left[\tilde{R}^*_j\right] - R_f \right) - \lambda_2 E\left[\widetilde{l}_j\right] \geq 0 \quad j = 1, ..., N \tag{40}$$

$$\frac{\partial \Psi}{\partial \lambda_1} = \mu_p - R_f - \sum_{j=1}^{N} X_j \left(E\left[\tilde{R}_j\right] - R_f \right) = 0 \tag{41}$$

$$\frac{\partial \Psi}{\partial \lambda_2} = l_p - \sum_{j=1}^{N} X_j E\left[\widetilde{l}_j\right] = 0 \tag{42}$$

$$\frac{\partial \Psi}{\partial X_j} X_j = 0 \tag{43}$$

$$X_j \geq 0 \qquad j = 1, ..., N \tag{44}$$

If every variable X_j is positive (inequalities (44) hold) then inequalities (40) are equalities because of the complementarity conditions (43). So, the equations (40) imply that:

$$X_k = \lambda_1 \sum_{i=1} M_{ki}(E[\widetilde{R}_i] - R_f) + \ \lambda_2 \sum_i M_{ki} E(\widetilde{l}_i) \qquad k = 1, ..., N \tag{45}$$

Multiplying both sides of the equation (45) by $\left[E[\widetilde{R}_k] - R_f\right]$ and summing over $k = 1, ..., N$, it follows:

$$\begin{aligned} \sum X_k \left(E\left[\widetilde{R}^*_k\right] - R_f\right) \quad &= \quad \lambda_1 \sum_k \sum_i M_{ki} \left(E\left[\widetilde{R}^*_i\right] - R_f\right)\left(E\left[\widetilde{R}^*_k\right] - R_f\right) \\ &\quad + \lambda_2 \sum_k \sum_i M_{ki} \left(E\left[\widetilde{l}_i\right]\right)\left(E[\widetilde{R}_k] - R_f\right) \end{aligned} \tag{46}$$

Multiplying both sides of the equation (45) by $E[\widetilde{l}_k]$ and summing over $k = 1, ..., N$, it follows:

$$\begin{aligned} \sum X_k E\left[\widetilde{l}_k\right] \quad &= \quad \lambda_1 \sum_k \sum_i M_{ki} \left(E\left[\widetilde{R}^*_i\right] - R_f\right) E\left[\widetilde{l}_k\right] \\ &\quad + \lambda_2 \sum_k \sum_i M_{ki} E(\widetilde{l}_i) E\left[\widetilde{l}_k\right] \end{aligned} \tag{47}$$

From equation (41), we deduce that the equation (46) using the implied parameters A, B, C, A_1, B_1 and C_1, it follows that:

$$\mu_p - R_f = \lambda_1 \left[B - 2R_f A + R_f^2 C\right] + \lambda_2 \left[A_1 - R_f C_1\right] \tag{48}$$

Also, from equation (42), the equation (47) implies that:

$$l_p = \lambda_1 \left[A_1 - R_f C_1\right] + \lambda_2 B \tag{49}$$

So, both equations (48) and (49) imply:

$$\begin{aligned} \lambda_1 \quad &= \quad \frac{(\mu_p - R_f)B - l_p(A_1 - R_f C_1)}{(B - 2R_f A + R_f^2 C)B - (A_1 - R_f C_1^2)} \\ \lambda_2 \quad &= \quad \frac{(B - 2R_f A + R_f^2 C)l_p - (A_1 - R_f C_1)(\mu_p - R_f)}{(B - 2R_f A + R_f^2 C)B - (A_1 - R_f C_1^2)} \end{aligned} \tag{50}$$

Defining $D = (B - 2R_f A + R_f^2 C)B - (A_1 - R_f C_1^2)$ and substituting for $\lambda_1$and λ_2 from previous equation (50) into equation (45) to solve for X_k:

$$\begin{aligned} X_k &= \frac{1}{D}\left[\begin{array}{c} \left[(\mu_p - R_f)B - l_p(A_1 - R_f C_1)\right] \sum_{i=1} M_{ki}(E[\widetilde{R}_i] - R_f) + \\ \left[\ (B - 2R_f A + R_f^2 C)l_p - (A_1 - R_f C_1)(\mu_p - R_f)\right] \sum_i M_{ki} E(\widetilde{l}_i) \end{array}\right] \\ k &= 1, ..., N \end{aligned} \tag{51}$$

Using the notation indicated in the previous section,

$$g_k = \sum_i M_{ki}\ ;\ \ f_k = \sum_i M_{ki} E[\widetilde{l}_i] \text{ and } \ h_k = \sum_i M_{ki} E[\widetilde{R}_i^*],$$

equation (51) is equivalent to:

$$\begin{aligned} X_k &= \frac{1}{D}\left[\begin{array}{c} \left[(\mu_p - R_f)B - l_p(A_1 - R_f C_1)\right](h_k - R_f g_k)\ + \\ \left[\ (B - 2R_f A + R_f^2 C)l_p - (A_1 - R_f C_1)(\mu_p - R_f)\right] f_k \end{array}\right] \\ k &= 1, ..., N \end{aligned} \tag{52}$$

Under the positive condition of the previous equation (52), and in a fashion similar to the previous section, we derive the equation for the frontier using Vörös's method Voros (1986). Because of the positivity of the variables and of the dominator, multiplying equation (40) by X_j, summing from $j = 1, ..., N$ and rearranging, we find that:

$$\sum_j \sum_i X_j X_i \sigma_{ij}^* = \lambda_1 \sum_j \left(E\left[\widetilde{R}_i\right] - R_f\right) X_j + \lambda_2 \sum_j X_j E(\widetilde{l}_j) \tag{53}$$

From the definition of $\sigma^2(\widetilde{R}_p^*)$ and equations (41), (42) , equation (53) implies that:

$$Var(\tilde{R}_p^*) = \lambda_1 \left(\mu_p - R_f\right) + \lambda_2 l_p. \tag{54}$$

Substituting for $\lambda_1$and λ_2 from (50) into (54), we obtain the equation for the minimum-variance frontier.

$$\sigma^2\left(\tilde{R}_p^*\right)=\frac{1}{D}\left[\begin{array}{c}\left(\mu_p-R_f\right)\left[(\mu_p-R_f)B-l_p(A_1-R_fC_1)\right]\\ +l_p\left[(B-2R_fA+R_f^2C)l_p-(A_1-R_fC_1)(\mu_p-R_f)\right]\end{array}\right] \tag{55}$$

Arranging the above equation we get:

$$\sigma^2\left(\tilde{R}_p^*\right)=\frac{\left[\left(\mu_p-R_f\right)^2B+l_p^2\left(B-2AR_f+R_f^2C\right)-2l_p(A_1-R_fC_1)(\mu_p-R_f)\right]}{(B-2R_fA+R_f^2C)B-(A_1-R_fC_1^2)}. \tag{56}$$

In the mean-standard deviation space, we get the following equation:

$$\sigma\left(\tilde{R}_p^*\right)=\sqrt{\frac{\left[\left(\mu_p-R_f\right)^2B+l_p^2\left(B-2AR_f+R_f^2C\right)-2l_p(A_1-R_fC_1)(\mu_p-R_f)\right]}{(B-2R_fA+R_f^2C)B-(A_1-R_fC_1^2)}} \tag{57}$$

Thus, the minimum-variance frontier in mean-standard space is *non*linear, and equation (57) is the Fuzzy Capital Market Line (FCML). We believe with the absence of fuzziness (in every single return $l_j=0$ and in the portfolio mean $l_p=0$) in the model, the equation (57) will offer the classical capital market line. An empirical implication of this conclusion is shown in the next section.

4 Empirical Implications of the Model

In this section we analyze the relationship between risk and return in the presence of fuzzy information, revealed by the use of fuzzy returns, in NASDAQ stocks in the 1990-2000 period.

4.1 The impact of the subjective measure on the location of capital market line

In this subsection we use NASDAQ stock data to show the impact of the introduction of fuzziness on the location of Capital Market Line. In real life, the investor will be faced with more than just 15 assets as presented here. However, we limit our investigation in this section to 15 stocks to compare the location of fuzzy capital market line with respect to the location of the original CML. The model can be solved using any optimization software to construct the market line of 15 risky assets. The randomly selected 15 stocks are traded on the NASDAQ. The data, which covered the monthly rate of returns of these stocks for the 10-year period 1990-2000, were taken from the Center for Research and Security Prices (CRSP) and used to estimate the mean and standard deviation of returns. The following tables (1 and 2) show the returns and the widths (spreads) of fuzzy returns for 15 stocks over the 10-year period.

Table 1: 15 NASDAQ returns randomly selected

Permno	M2	M3	M4	M5	M6	M7	M8	M9	M10
10025	6.38E-07	-0.010696	-0.02174	-0.11641	0.036368	0.068993	-0.022473	0.07654	-0.043018
10078	0.121184	-0.182324	-0.044449	0.018019	-0.071705	0.028368	0.00465	0.146501	0.120673
10200	0.033226	-0.210156	-0.365934	0.088947	0.056863	0.014963	-0.184304	0	-0.241162
10271	0.152579	0	0.050586	-0.009009	-0.018265	-0.018605	-0.053024	0.02927	-0.034233
10290	0.030214	-0.05506	-0.045024	0.006557	-0.013158	0.181217	-0.068598	0.111848	0.147325
10588	0.223143	-0.123059	-0.04879	0.150572	-0.242139	0.140356	-0.011976	0.102948	0.042559
10772	0.179152	0.038916	-0.14781	-0.017857	-0.018183	-0.066375	-0.159065	0.188052	0.150061
11293	0	0.098801	0.064103	-0.006558	0.066797	0.085472	-0.014225	-0.002869	0
11701	-0.174717	0	-0.105361	-0.020409	-0.020834	0.010471	-0.053489	0.094311	-0.094311
11917	0.179693	0.013072	-0.102479	-0.082444	-0.677643	0.129678	-0.176931	0.190354	-0.127833
12063	-0.158057	-0.147325	-0.057158	0.028988	-0.264816	0.051825	-0.051825	-0.161268	0.012423
12068	-0.037272	0.033198	-0.00409	-0.012371	0	0.024591	0.031875	-0.015811	-0.003992
12189	0.133531	0.020618	0	0.185719	-0.107245	0.072759	-0.072759	0	0.055059
19546	-0.11441	-0.06252	0.279585	0.115069	-0.079137	0.13206	0.030459	0.04879	0.258695
23318	-0.106314	-0.0977	-0.165619	0.312223	-0.009132	0.04485	-0.146142	-0.015346	-0.020834

Due to the space limitation, the above table does not contain all the observations over the 10-year period; it is a subset from the complete data set. Permno is a number identifying the issuing company

Because there are an infinite number of ways to characterize fuzziness, there are an infinite number of ways to graphically depict the membership and to generate the data. Normally, experts should be able to offer decision makers or investors information regarding the measure of fuzziness. In this context, fuzziness has been used under the following conditions, that it reflects the experts' judgments and that the returns should be around those values. For example, the company with permn 10078 has a 0.211 return. After getting a subjective recommendation from experts, the return that should be used is 0.211 ± 0.000645. In a fuzzy setting with TR type fuzzy membership, that means that the membership function equals 1 for a return 0.211, and it is linearly decreasing on the right and left. Ross (1995) pointed out that there are more ways to assign membership function values to fuzzy variables than for random variables. The literature on this topic is rich with references, for example Dubois and Prade (1980). The assignment can be intuitive or based on algorithms or logical operations. We established the table (2) based on a combination of the intuition and inference methods presented by Ross (1995).

Following an inference approach, we use the bid-ask spread to get the width of the fuzzy returns. the logic behind that technique is that a bid-ask spread creates vagueness and imprecision in the investor's choice. It is the irregularities, which may arise from the lack of imprecision in the data, that are a concern here.

Moreover, market-created uncertainty results from the interaction (directly or indirectly) among participants who form their expectations in an ill-defined market. Consequently, each participant will form his/her expectations based on their subjective prediction of other participants' expectations.

We use the bid-ask spread because it affects the stock returns (see, Koski (1998)). There are considerable theoretical justifications to the use of a bid-ask spread and to its effects on returns. Heinkel and Kraus (1988) pointed out that a component of the bid-ask spread, which is based on information asymmetries, could be considered part of true returns. Hence, the effect of bid-ask spread is that the observed returns differ from the true returns.

Moreover, as pointed out by Amihud and Mendelson (1989) rational investors select their assets to maximize their expected return net. These authors showed a strong effect of the spread on returns.

Because the bid-ask spread is related to the availability of information about the asset, the greater the amount of information about an asset, the narrower the spread, which means the closer the true return is to the observable return (see, Demsetz (1968)). In contrast, the more information about an asset is vague, the greater the distance between the true return and the observed return. In this sense, the width between the observed return and the net return, taking into consideration the bid-ask spread has been identified (see, Merton (1987)). Also, Merton (1987) pointed out that incomplete information about a stock, which is a major factor, is reflected in its bid-ask spread. This conclusion has been supported by the effect of Amihud and Mendelson's spread (1986).

In a statement Merton (1987) says:

> "I also believe that financial models based on frictionless markets and complete information are often inadequate to capture the complexity of rationality in action."

That lead to the development of the so-called width (tolerance level), which means that the investor uses the net and observable returns to form his/her fundamental returns, assuming that the fuzzy random return sways between them. We employ a method comparable to Amihud and Mendelson (1986) in developing the net return, to allow the investors the ability to compress information into fuzzy notions that they can analyse using fuzzy theory. Under these considerations, the following formulas have been derived to generate the widths data in Table (2).

$$P_{mt} = \frac{Ask\ price_t - Bid\ price_t}{2},$$

and

$$R_t = \ln\left(\frac{Pmt}{P_{mt-1}}\right), R_{net} = \left(\frac{1 - Spread_t}{1 + Spread_{t-1}}\right) R_t,$$

such that

$$Spread_t = \frac{Ask\ price_t - Bid\ price_t}{Ask\ price_t + Bid\ price_t},$$

then

$$width = l_t = |R_{net} - R_t|$$

Table 2: 15 widths of the 15 NASDAQ stocks 1990-2000

Permno	M2	M3	M4	M5	M6	M7	M8	M9	M10
10025	6.38E-08	0.000771	0.001388	0.008332	0.002098	0.003002	0.000999	0.00392	0.002747
10078	0.000645	0.001587	0.000402	0.00016	0.000508	0.000132	2.15E-05	0.000588	0.000428
10200	0.000647	0.005252	0.016981	0.003711	0.001969	0.000365	0.006733	0	0.014312
10271	0.004359	0	0.001566	0.000241	0.000498	0.000517	0.001818	0.000832	0.000846
10290	0.001234	0.00305	0.00199	0.000211	0.000513	0.004135	0.001642	0.003502	0.003996
10588	0.016129	0.010004	0.004648	0.011011	0.019282	0.011222	0.00124	0.009616	0.003406
10772	0.004632	0.001458	0.004932	0.000641	0.000974	0.003143	0.009105	0.010504	0.00611
11293	0	0.005047	0.003068	0.000336	0.002629	0.002612	0.000521	0.000105	0
11701	0.007638	0	0.005277	0.000812	0.000434	0.000323	0.001711	0.002763	0.003
11917	0.005927	0.000335	0.003676	0.003125	0.057284	0.008544	0.011111	0.012208	0.009176
12063	0.005244	0.00462	0.001405	0.000585	0.00999	0.002554	0.002659	0.007898	0.00045
12068	0.00108	0.000803	8.28E-05	0.000255	0	0.00059	0.000741	0.000374	7.89E-05
12189	0.015748	0.002018	0	0.018062	0.011509	0.007281	0.007801	0	0.004692
19546	0.006743	0.00391	0.013475	0.004907	0.004586	0.005464	0.001495	0.003145	0.009637
23318	0.001278	0.001265	0.002054	0.002871	8.34E-05	0.000392	0.001481	0.000157	0.000218

Due to the space limitation, the above table does not contain all the observations over the 10 year period; it is a subset from the complete data set.

The following graph (1), which has been plotted in two dimensions, shows the location of the capital market line with fuzziness (blue line) and without accounting for fuzziness (red line); the value of 0.031 has been used for the portfolio width $l_p = 0.031$ and 0.07 as the risk-free rate $R_f = 0.07$ to be able to show the graph in two dimensions[3]. The y axis represents σ and the x axis represents μ_p. By increasing from $l_p = 0.031$ (blue line) to 0.045 (navy line) and 0.061(brown

[3]Although it appears very high, risk-free rate of 0.07 has been used only for illustration purpose. The average of T-Bill rate over the period 1990-2000 could be more appropriate.

line), the FCML is moving upward, which means that an increase of fuzziness manifested by the portfolio width l_p will cause the market line to be more dominated by the original market line. On the contrary, a small degree of fuzziness in the model, measured by the portfolio width, shows that fuzzy capital market lines are dominated by the standard linear capital market line. Also, as presented in the previous section, it is obvious in the following figure that the FCML is nonlinear. The introduction of fuzzy information, then, shifts the intercept of the line relating μ_p and the slope from R_f to another positive value (value $< R_f$).

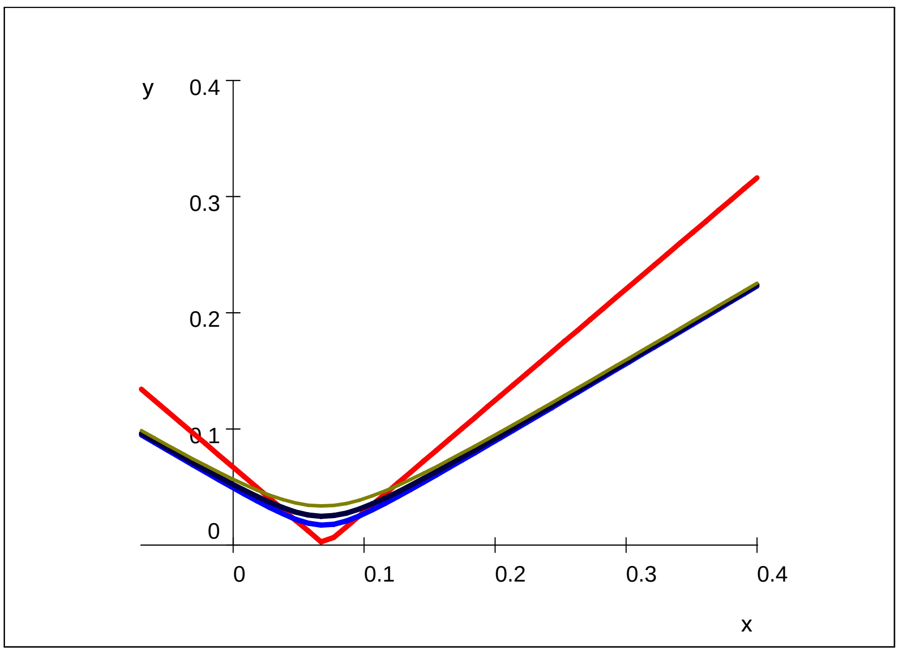

Figure 1: Capital Market Line (CML) without accounting for fuzziness (red line) and fuzzy CML (others)

The next graph (2) plots the capital market line (blue plane) with the risk free rate ($R_f = 0.07$) and the efficient frontier without the risk free-rate (green plane) in three dimensions; the y-axis represents the portfolio width l_p, x -axis represents the portfolio mean μ_p and z-axis represents the σ_p.

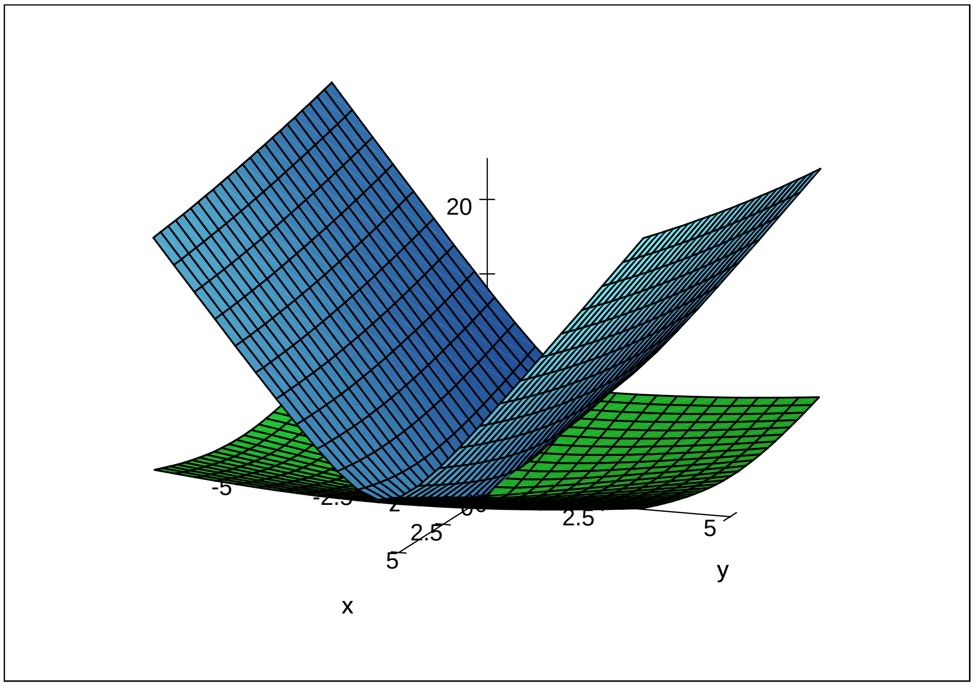

Figure 2: 3 D graphical representation of the fuzzy frontier with and without risk-free rate

4.2 The impact of fuzziness on the location of efficient frontier

Using another set of data we randomly selected 15, 30 and 50 stocks traded on the NASDAQ. Following the same method discussed in the previous subsection, we generate the widths (spreads) for all the complete data in the form of 15, 30 and 50 widths. Similarly to the case of 15 assets presented previously, the efficient frontiers have been presented for 15, 30 and 50 assets.

The mathematical problem without short sales presented previously (6-10) has two new constraints (8) and (10), so it requires programming techniques to handle the problem. With computer capability (Visual Basic (VBA) program) we are able to achieve the efficient investment strategy for each portfolio return level μ_p with width l_p for all different groups of assets. (15, 30 and 50 assets).

One aim of the computer program is to not only do all the necessary computations from stock prices and solve the optimization problem but also to generate a graph of the efficient frontier without short sales. The following figures [(3), (4), (5)] show the efficient frontier without short sales for all three sample sizes (15, 30 and 50 assets). One major element worth elaborating on is that all efficient frontiers are concave arcs, which is consistent with the finding of Szegö. However, the boundary of each sample size turns out not to be a parabola. It is also clearly observed that the arc, which is between minimum and maximum points does not coincide with the original boundary. The minimum (maximum) point represents, as discussed previously and supported by Szegö's finding (1986) can be achieved by investing the capital in the investment option with lowest (highest) return.

For comparison, the following graphs represent the case when fuzziness is not included. In accordance with Levy (1983), the figure (6) plots the efficient frontiers constructed with and without short sales; the efficient frontier without short sales lies inside the efficient frontier with short sales. An investor with short sales will attain a lower utility than an investor with both short and long positions. Also, in all cases (15, 30 and 50 assets), it is clear that the frontier is not a parabola, but an arc of a parabola as suggested by Szegö (1980), see figures [(7), (8)].

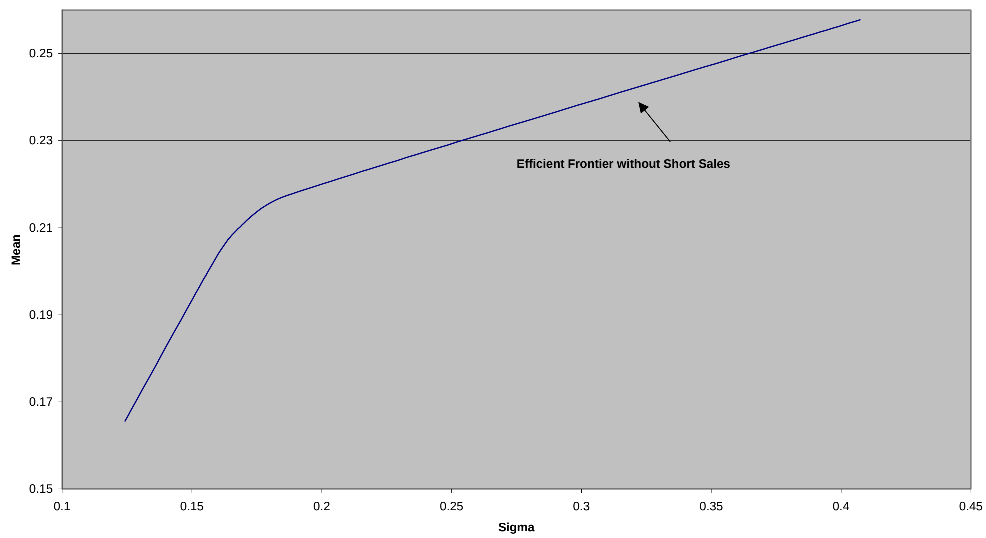

Figure 3: Efficient frontier (EF) without short sales for 15 asset prices

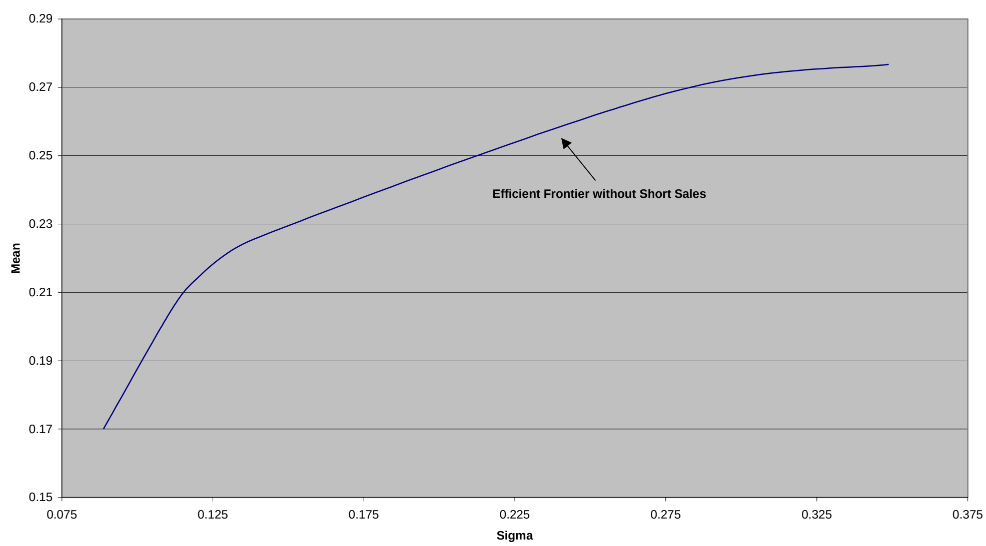

Figure 4: Efficient frontier (EF) without short sales for 30 asset prices

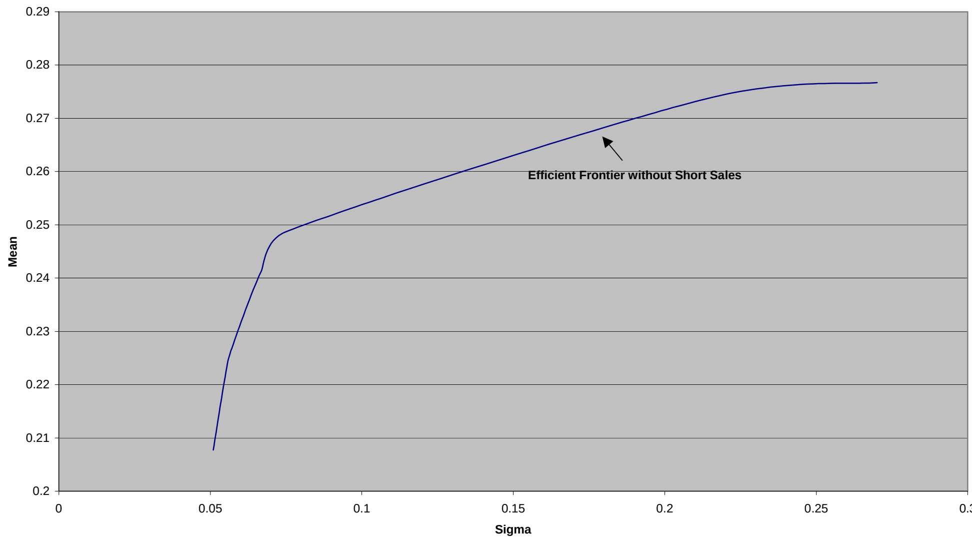

Figure 5: Efficient frontier (EF) without short sales for 50 asset prices

Gathering the information together in one graph will generate the following figures [(6), (7) and (8)]. We note here that the efficient frontier without short sales does not coincide with the one with short sales. Yet we may have attached a part of the efficient boundary to the original, so we need to identify the remaining parts of the new efficient frontier. Also, the next three figures reveal that, for all sample sizes, the efficient frontier with short sales dominates the one without short sales. This statement appeared in much of the literature. Because short sales restrictions add a new constraint, it is obvious that the efficient frontier will be dominated. Moreover, various sample sizes show that the efficient frontiers with short sales are parabolas.

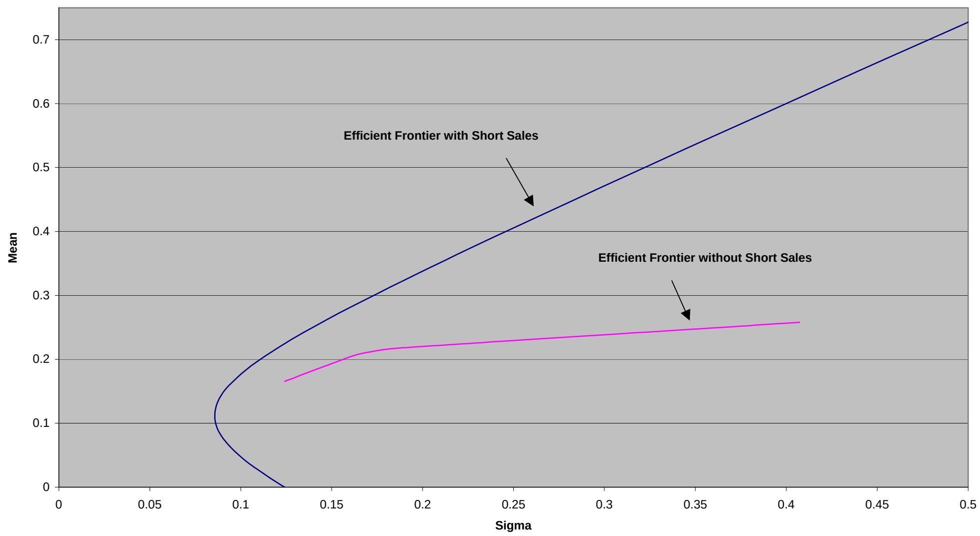

Figure 6: Efficient frontier with and without short sales for 15 asset prices

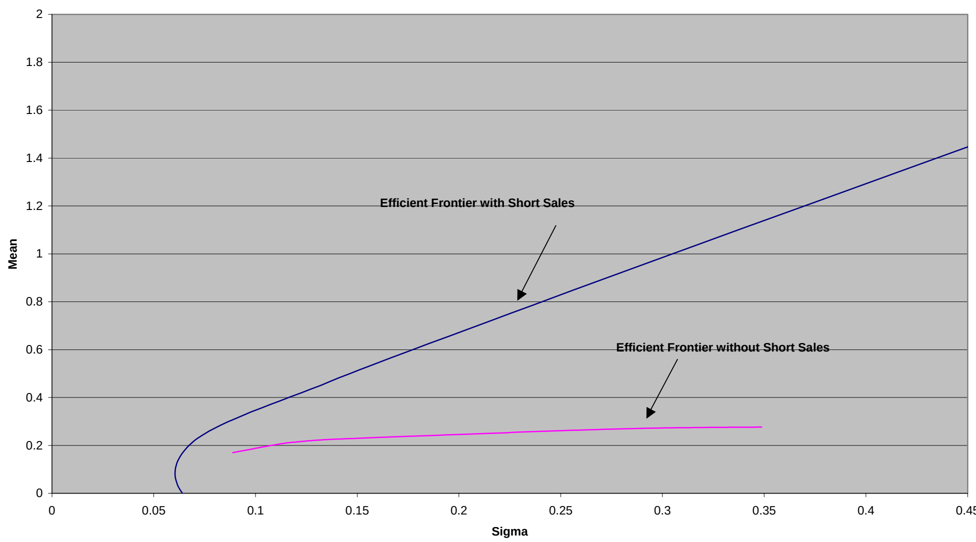

Figure 7: Efficient frontier with and without short sales for 30 assets prices

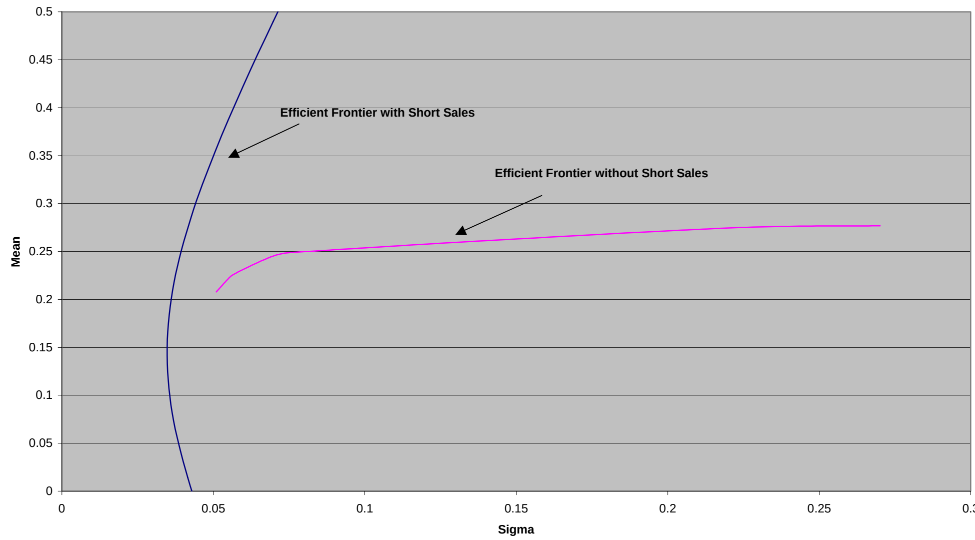

Figure 8: Efficient frontier with and without short sales for 50 assets prices

Taking into account various sample sizes, the data suggest a conclusion consistent with Levy's findings (1983) that as the sample increases, the efficient frontier with and without shift from the left. It is clear that the distance between is proportional to the data. Levy (1983) used a small sample size up to 15 assets; here we expand that finding with a larger sample size. He empirically finds that without short sales, many securities do not enter the efficient portfolios, and the larger N, the smaller the percentage of the securities that appear in the efficient portfolios out of the total number of available securities, N. This finding is supported by the collected data.

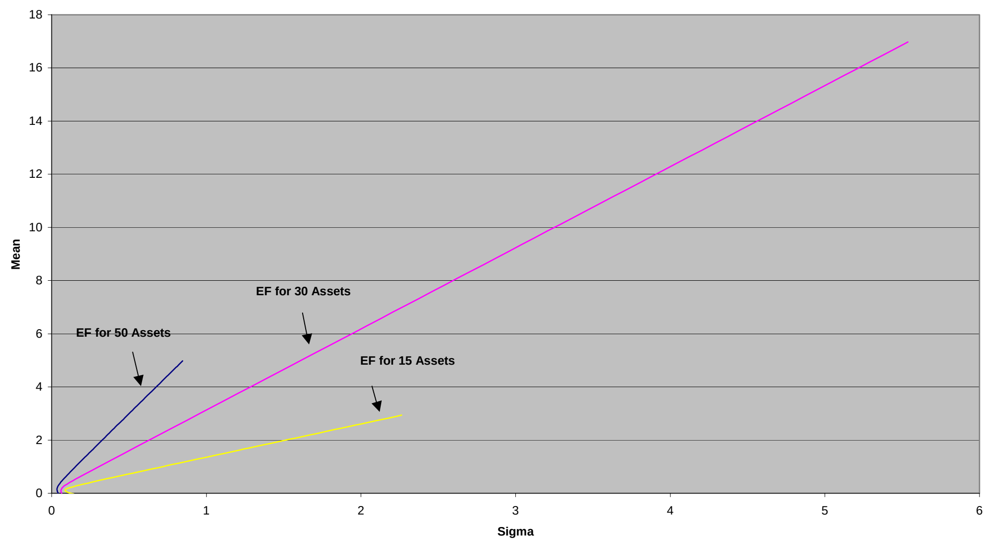

Figure 9: Portfolios' efficient frontier(s) (EF)

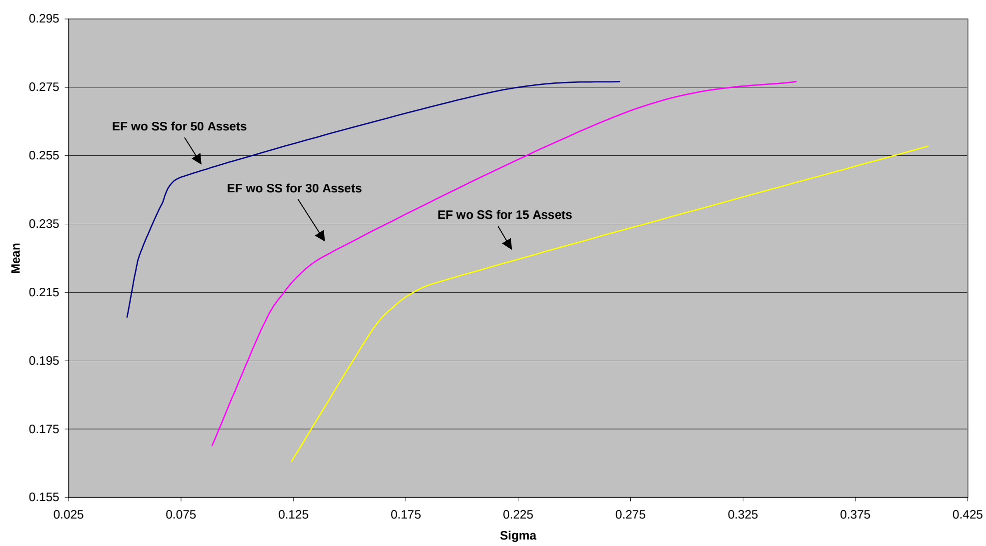

Figure 10: Portfolios' efficient frontier(s) without short sales (EF wo SS)

Taking into account fuzzy information, the fuzzy efficient frontiers are represented in XYZ plane as follows for various sample sizes, see figures [(11), (12) and (13)]:

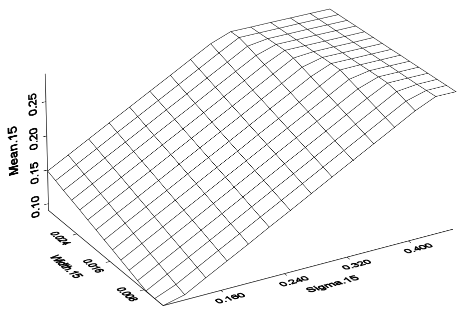

Figure 11: Efficient frontier with subjective fuzzy information without short sales (15 assets)

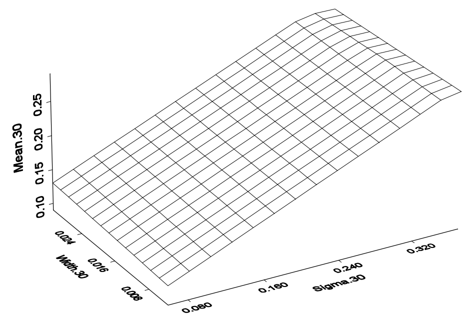

Figure 12: Efficient frontier with subjective fuzzy information without short sales (30 assets)

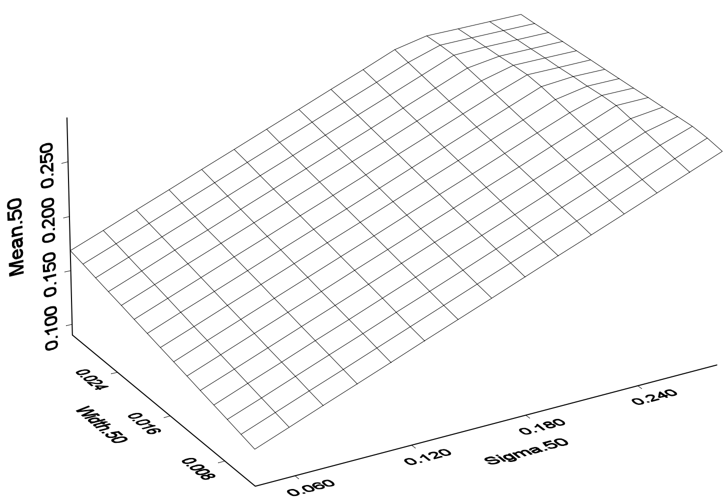

Figure 13: Efficient frontier with subjective fuzzy information without short sales (50 assets)

Under a fuzzy information environment, the efficient frontier without short sales has been derived and plotted for various sample sizes. The portfolio width has been included as a third parameter, and the frontier has been plotted in a three-dimensional graph. In this section, the relationship between risk, return and width, which is used as proxy for the subjective comment of the experts, has been represented by a surface. The efficient frontier portfolios are plotted on a graph with the σ_p in the x-axis, width in the y-axis and the mean in the z-axis. Projecting the graphical representation into a two standard deviation-mean plane figure (14) shows an arc, not a parabola, which is consistent with the result reported earlier when the subjective fuzzy measure was discarded from the model. Also, for 15, 30 and 50 asset sample sizes, similar to the case of short sales, we still observe that in the larger sample size, the efficient frontier is shifted to the left; the dominance of the large size sample still holds. In general, the efficient frontier is a combination of assets, if there is no other combination with the same (higher) expected return with lower risk, and if there is no other portfolio with the same (or lower) risk and with higher expected return. In this context, a higher (lower) risk is associated with a higher (lower) return.

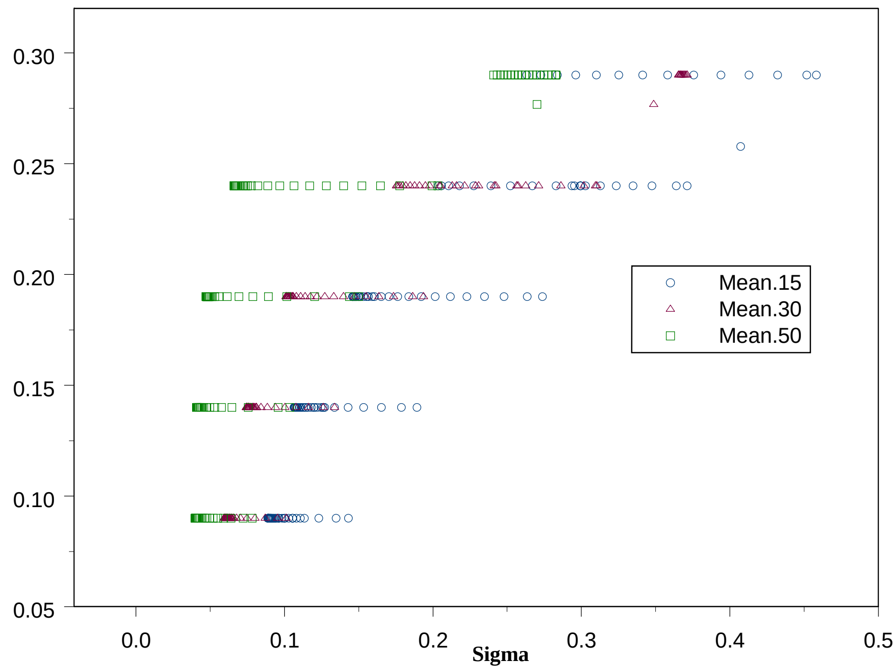

Figure 14: Efficient frontiers in a mean-standard deviation plane with subjective fuzzy measure

Also, the following graph (15) shows that as the degree of fuzziness increases (flexibility with respect to the portfolio mean improves) there is a slight decrease in the level of risk. Note here that the graph *does not suggest* a strong negative relationship for various sample sizes[4]. Because the widths in our samples are correlated with the returns, we could not see a strong visible (either positive or negative) relationship. Thus, we suggest that as soon as the investor starts getting new subjective information from experts, which is to some extent not primarily correlated with the historical data, we will be able to spot a strong visible relationship between the width size and the

[4]Also, due to the limited plotted number of observations, we could not see a very strong relationship.

risk level. Thus, an investor who is flexible and is acquiring additional subjective information to support the historical data will be flexible to accept a higher risk. Thus, we anticipate a negative relationship. The following graph (15) shows a slight negative trend, mainly for a larger size sample. In contrast, an investor with small portfolio width (not flexible with respect to the portfolio mean) tends to accept less risk. For instance, it has been shown in the figures that sometimes there is not a conclusive relationship between portfolio width and risk.

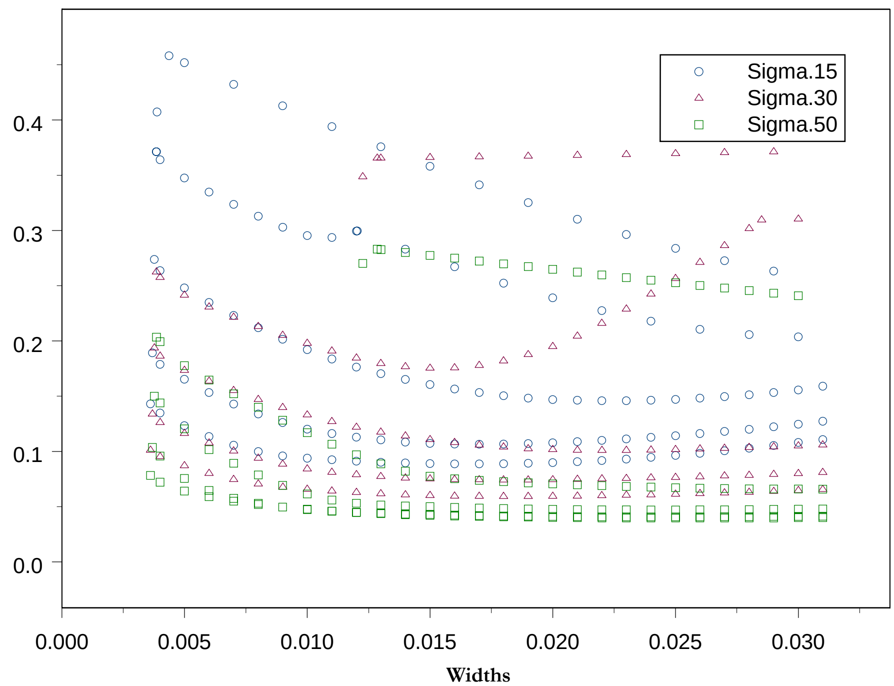

Figure 15: Relationship between the widths and sigma for different sample sizes

5 Conclusion

This study addressed the implications of relaxing one of the fundamental assumptions associated with mean-variance theory as set down in Markowitz (Markowitz (1952), Markowitz (2003)) and Tobin Tobin (1985) that asset returns are sharply defined. Theoretical arguments in fuzzy mathematics assume that there are cases in which random uncertainty alone may not serve the purpose and indicate that fuzziness may impact the first two moments of asset return. This suggests that the lack of information associated with market-traded securities challenges the usefulness of standard mean-variance theory for other research and practical portfolio management.

To make the link between existing theory and the subjectivity measure of expert's judgments, we rederived the Markowitz efficient set and dealt with the implications of the rederivation on the Capital Market Line (CML). The contribution of this paper is the presentation of a methodology for the derivation of the attainable efficient frontier in the presence of fuzzy information in the data or when the fuzzy information is imposed in the modeling environment to reflect a subjective measure.

References

Aitken, M. J., A. Frino, M. S. McCorry, and P. L. Swan (1998, December). Short sales are almost instantaneously bad news: Evidence from the australian stock exchange. *The Journal of Finance 53*(6), 2205–2223.

Alchian, A. A. (1953, March). The meaning of utility measurement. *American Economic Review XLIII*(1), 26–50.

Allais, M. (1953, October). Le comportement de l' homme rational devant le risque: Critique des postulats et axiomes de l'ecole americaine. *Econometrica XXI*(4), 503–546.

Amihud, Y. and H. Mendelson (1986). Asset pricing and the bid-ask spread. *Journal of Financial Economics 17*, 223–249.

Amihud, Y. and H. Mendelson (1989, June). The effects of beta, bid-ask spread, residual risk, and size on stock returns. *The Journal of Finance 44*(2), 479–486.

Arnold, K. and G. M. Madan (1985). *Introduction to Fuzzy Arithmetic: Theory and Applications.* New York, N. Y.: Van Nostrand Reinhold Co.

Demsetz, H. (1968, February). The cost of transaction. *Quarterly Journal of Economics 82*, 33–53.

Dorfman, R., P. A. Samuelson, and R. M. Solow (1958). *Linear Programming and Economic Analysis.* New York: McGraw-Hill.

Dubois, D. and H. Prade (1980). *Fuzzy Sets and Systems: Theory and Applications.* New York: Academic Press.

Gebhardt, J., M. A. Gil, and R. Kruse (1991). Fuzzy set–theoretic methods in statistics. In R. Slowinski (Ed.), *Fuzzy Sets in Decision Analysis, Operations Research and Statistics.* Boston: Kluwer Academic Publishers.

Heinkel, R. and A. Kraus (1988, March). Measuring event impacts in thinly traded stocks. *The Journal of Financial and Quantitative Analysis 23*(1), 71–88.

Herstein, I. N. and J. Milnor (1953, April). An axiomatix approach to measurable utility. *Econometrica XXI*(2), 291–297.

Huang, C. and R. H. Litzenberger (1988). *Foundations for Financial Economics*. New York: North Holland.

Jarrow, R. (1980, December). Heterogeneous expectations, restrictions on short sales, and equilibrium asset prices. *The Journal of Finance 35*(5), 1105–1113.

Korner, R. (1997). On the variance of fuzzy random variables. *Fuzzy Sets and Systems 92*, 83–93.

Koski, J. L. (1998, Spring). Measurement effects and the variance of returns after stock splits and stock dividends. *The Review of Financial Studies 11*(1), 143–162.

Kruse, R. and K. Meyer (1987). *Statistics with Vague Data*. Holland: D. Reidel Publishing Company.

Kwakernaak, H. (1978). Fuzzy random variables: Definition and theorems. *Inform. Sci. 15*, 1–29.

Levy, H. (1983, March). The capital asset pricing model: Theory and empiricism. *The Economic Journal 93*, 145–165.

Lintner, J. (1965). The valuation of risk assets and the selection of risky investments in stock portfolio and capital budgets. *Review of Economics and Statistics 47*, 13–37.

Markowitz, H. M. (1952). Portfolio selection. *The Journal of Finance 7*, 77–91.

Markowitz, H. M. (2003). *Portfolio Selection: Efficient Diversification of Investments*. Cambridge, MA, Oxford, UK: Blackwell.

Merton, R. C. (1987, July). A simple model of capital market equilibrium with incomplete information. *Journal of Finance 42*, 483–510.

Nather, W. (1997). Linear statistical inference for random fuzzy data. *Statistics 29*, 221–240.

Nather, W. (2000). On random fuzzy variables of second order and their application to linear statistical inference with fuzzy data. *Metrika 51*, 201–221.

Puri, M. and D. Ralescu (1986). Fuzzy random variables. *Journal of Mathematical Analysis and Applications 114*, 409–422.

Ross, S. A. (1977, March). The capital asset pricing model (CAPM), short-sale restrictions and related issues. *Journal of Finance 32*(1), 177–183.

Ross, S. A. (1978, April). Mutual fund separation in financial theory–the separating distributions. *Journal of Economic Theory 17*, 254–86.

Ross, T. (1995). *Fuzzy Logic with Engineering Applications.* New York: McGraw-Hill.

Savage, L. J. (1954). *The Foundations of Statistics.* New York and Chapman and Hall and London: John Wiley and Sons.

Sharpe, W. F. (1964). Capital asset prices: A theory of market equilibrium under conditions of risk. *Journal of Finance 19*, 425–442.

Tobin, J. (1985). Liquidity preference as behavior towards risk. *Review of Economic Studies 25*, 68–85.

Voros, J. (1986). Portfolio analysis–an analytic derivation of the efficient portfolio frontier. *European Journal of Operational Research 23*, 294–300.

White, D. J. (1969). *Decision Theory.* Boston: George Allen Unwin Ltd.

Zadeh, L. A. (1965). Fuzzy sets. *Inform. and Control 8*, 338–353.

Zadeh, L. A. (1978). Fuzzy sets as a basis for a theory of possibility. *Fuzzy Sets and Systems 1*, 3–28.

Zhou, X. S. and M. Dong (2004, August). Can fuzzy logic make technical analysis 20/20? *Financial Analysts Journal 60*(4), 54–75.