Two-Dimensional Risk-Neutral Valuation Relationships for the Pricing of Options.

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Abstract

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The Black-Scholes model is the prime example of a risk-neutral valuation relationship, where the function relating the price of the option to the price of the underlying asset is consistent with risk neutrality. We generalize the concept of a risk-neutral valuation relationship in order to price options in cases where the restrictive conditions required for a traditional one-dimensional risk-neutral valuation relationship do not apply. We derive conditions under which a two-dimensional risk-neutral valuation relationship exists, relating the price of an option on an asset to the prices of the underlying asset and one other option on the asset. This allows us to price contingent claims in economies where the pricing kernel exhibits non-constant elasticity.

1 Introduction

A risk-neutral valuation relationship (RNVR) exists for the pricing of contingent claims on an underlying asset, if the relationship between the price of the claim and the price of the underlying asset is the same as it would be under risk neutrality. The most celebrated RNVR is the Black-Scholes model which prices options on an asset with a lognormal distribution. In this paper we extend the idea of the RNVR to two dimensions.

Recent work has emphasised the role of the pricing kernel in the pricing of options. Given the price of the underlying asset, the price of a call (or a put) option with strike price K is constrained by the possible shapes of the pricing kernel function. Bernardo and Ledoit (2000) and Cochrane and Saa-Requejo (2000) both use the pricing kernel approach to derive bounds for option prices. Cochrane and Saa-Requejo (2000) show that the option price can be bounded by limiting the variance of the pricing kernel. In similar vein, Bernardo and Ledoit (2000) show that the option price can be bounded by limiting the convexity of the pricing kernel. We also take the pricing kernel approach. However, rather than deriving option pricing bounds, we look for precise option prices that exist if the pricing kernel has specific properties. As the analysis of Heston (1993) makes clear and earlier work by Brennan (1979) and Rubinstein (1976) discovered, constant elasticity of the pricing kernel is a sufficient condition for the Black-Scholes RNVR to obtain for options on a lognormally distributed asset price. Hence a generalization of the Black-Scholes model involves a less rigorous condition on the pricing kernel. Here we show that if the pricing kernel has non-constant elasticity but is determined by two parameters, it may be possible to generalize the Black-Scholes and similar models, by employing a two-dimensional RNVR.

In other recent work which is closely related to our own, Jackwerth (2000) and Rosenberg and Engle (2002) develop methods for estimating the pricing kernel (and hence the utility function of the representative investor) using observed option prices together with the empirical distribution of asset prices. Jackerth's analysis is, in a sense, the inverse of what we do in this paper. We assume general characteristics of the pricing kernel and then we are able to price options in relation to the underlying asset price and to the price of options on the same asset. In our model, a functional relationship exists between the price of an option with strike price K and the price of the underlying asset and one other option.

As noted above, any generalization of the one-dimensional RNVRs of Black and Scholes (1973) and others must be related to the work on option price bounds. In a recent paper, Ryan (2000) considers the bounds on an option price given the price of the underlying asset and the price of one other option on the asset. Ryan finds much tighter bounds for the option price given these two other prices than the bounds that exist given only the underlying asset price. Our two-dimensional RNVR option price in this paper can be thought of as the unique price within the Ryan bounds.

Before proceeding with our analysis, we should relate to the large volume of literature on the pricing of options in a dynamically complete market. Following the original paper by Cox, Ross and Rubinstein (1979), these models assume that the asset price follows a given process, which allows the computation of 'pseudo' probabilities or state prices. These are often referred to as 'risk-neutral' probabilities. They are then used to price the option as if investors used these as probabilities and were risk neutral. This general methodology differs from that used in this paper, where we assume, as in Rubinstein (1976), Brennan (1979), Camara (2003) and Schroeder (2004) a discrete time model in which the market is closed between the valuation date and the option maturity date. When we refer here to the existence of a RNVR this should not be confused with the existence of risk-neutral probabilities in the dynamically complete market.

The outline of the article is as follows. In section 2, we define the traditional (one-dimensional) risk-neutral valuation relationship and Heston's extension to the case of missing parameters. We also derive *necessary* and sufficient conditions for the existence of these relationships. In section 3, we extend the idea of a RNVR to two dimensions. Section 4 concentrates on the case where the underlying asset has a generalised lognormal distribution. Here we derive an option pricing formula which is a logical generalisation of the Black-Scholes formula for the value of a European-style option. In Section 5, we illustrate option prices using a numerical example. The example is calibrated using recent estimates of implied volatility for options traded on the S&P 100. In section 6, we summarise our conclusions and relate our results to other recent work on option bounds and on the pricing kernel.

2 Risk-Neutral Valuation Relationships: A Review and Generalization

We consider the valuation of European-style contingent claims, with maturity T, paying $c_T(x_T)$, which depend on a non-dividend paying underlying asset, whose payoff is x_T. For convenience we write $x_T \equiv x$. We assume a no-arbitrage economy in which there exists a stochastic discount factor, ψ_T, that prices all assets. The price of the contingent claim, at time $t = 0$, is

$$c = E[c_T(x)\psi_T]. \tag{1}$$

For convenience, we now define a variable $\phi(x)$ by the relation

$$c = e^{-rT}E[c_T(x)\phi(x)],$$

where

$$\phi(x) = e^{rT}E[\psi_T|x],$$

where e^{-rT} is the price of a zero coupon bond. We refer to this asset specific pricing function, $\phi(x)$, simply as the pricing kernel. This is unambiguous, since throughout our analysis we are pricing claims on a particular underlying asset with payoff x. The contingent claim could, for example, be a call option with strike price K. In this case, we denote the price as $c(K)$. The price of the underlying asset is the price of a call with strike price $K = 0$, hence this price is $S = c(0)$. In this case,

$$S = e^{-rT}E[x\phi(x)]. \tag{2}$$

First, we define the concepts of a risk-neutral valuation relationship for contingent claims and Heston's generalization to 'missing parameters' valuation. Following Brennan(1979), we say that a *risk-neutral valuation relationship* (RNVR) exists for the valuation of contingent claims on an asset if the relationship between the price of the claim in (1) and the price of the asset in (2) is the same as it would be under risk neutrality. If a RNVR exists, it follows that the formula for the price of the contingent claim can be written as a function of S, and investor preference parameters do not enter the relationship.

In the literature, well known RNVRs are the Black-Scholes formula for options on assets with log-normal distributions and the Brennan (1979) formula for the price of an option on a normally distributed asset. Other RNVRs applying to assets following a displaced diffusion process and to options on multiple assets have been established by Rubinstein (1983), Stapleton and Subrahmanyam (1984), and Camara (2003).

However, the set of known RNVRs is limited both by the type of probability distribution function assumed and by the form of the pricing kernel required to derive a RNVR, given the probability distribution. This has led Heston (1993) to propose a generalization (of the set of RNVRs). Heston assumes that the pricing kernel depends on a preference parameter, γ; $\phi(x) = \phi(x;\gamma)$ and the probability distribution of x depends on a parameter q; $f(x) = f(x;q)$.

Heston proposes a set of contingent claims formulae, where the price of the claim is independent of one preference parameter γ and of one parameter of the probability distribution, q. To be precise, he assumes that the probability distribution function of x 'belongs to a family of densities that depend on an additional parameter' and establishes a 'missing parameters' relationship. Formally, a missing parameters relationship exists for contingent claims on an asset, if the relationship between the price of the claim and the price of the asset is independent of a preference parameter, γ and a probability distribution parameter q.

Heston (1993) establishes necessary and sufficient conditions for the existence of such a missing parameters relationship. He shows: given the pricing kernel $\phi(x;q)$ and a probability density function $f(x,q)$, the pricing of contingent claims is independent of q and γ, if and only if the pricing kernel and the probability density function have the form

$$\phi(x;\gamma) = b(\gamma)h(x)e^{\gamma k(x)} \tag{3}$$

and

$$f(x;q) = a(q)g(x)e^{qk(x)}, \tag{4}$$

where $h(x)$ is not dependent on γ and $g(x)$ is not dependent on q.

Note that the set of these missing parameters relationships includes the set of RNVRs. This must be the case, since the parameter representing risk aversion drops out in the case of a RNVR. We now investigate the set of possible RNVRs. Since these must be within the wider set of Heston's missing parameters relationships, we restrict ourselves to assets with probability density functions of the form in equation (4).

Option pricing models typically assume that the type of distribution of x is known (for example lognormal) and that the type of function $\phi(x)$ is also known (for example, a declining power function)[1]. When strong enough assumptions are made it is possible to use the

[1] The typical assumption made in continuous time models is that the asset price follows a geometric Brownian motion. This implies that the pricing kernel is a declining power function, see Franke, Stapleton and Subrahmanyam (1999).

price of the underlying asset to back out the risk-neutral density and establish a RNVR. We assume a general class of distributions of the form:

$$f(x;q) = a(q)g(x)e^{qk(x)},$$

where q is a parameter of the distribution, $a(q)$ is a deterministic function of q, $g(x)$ is a positive function, and where $k(x)$ is monotonic. As noted by Heston, this set leaves considerable flexibility in the specification of the probability density. The following example illustrates this.

Suppose that x is 'generalized lognormal', that is $f(x)$ is of the form

$$f(x;q) = a(q)\hat{g}(x)e^{qk(x)}. \tag{5}$$

with

$$\begin{aligned} a(q) &= \frac{1}{\int \hat{g}(x)e^{qk(x)}dx} \\ \hat{g}(x) &= \frac{e^{-(\ln x)^2/2\sigma^2}}{x}G(x) \\ k(x) &= \ln x, \end{aligned}$$

where $G(x)$ is any positive function of x. This example clearly conforms to the general form of $f(x)$ in equation (4). In this case, $f(x)$ reduces to the lognormal distribution when $G(x) = 1$.

We now consider the necessary and sufficient conditions for the existence of RNVRs for assets with distributions in the set defined by equation (4). First, we formalise this idea of a RNVR. Above, we defined a RNVR in terms of the prices of the asset and of the contingent claim. More formally, in the case of the set of distributions introduced above, we can characterize the RNVR as follows. Since, under risk neutrality the time-0 forward price of the asset would be given by the expected value:

$$Se^{rT} = \int xf(x;q)dx, \tag{6}$$

and since $f(x)$ is of the form

$$f(x;q) = a(q)g(x)e^{qk(x)}, \tag{7}$$

we can solve (6) for the parameter q. If we call this 'risk neutral' value q_0, then a RNVR exists for the value of contingent claims if and only if

$$ce^{rT} = \int c_T(x) f(x; q_0) dx, \tag{8}$$

for all claims. That is, the forward price of the contingent claim has to be be given by the expected value of the claim using the risk-neutral distribution $f(x; q_0)$, where q_0 is found from solving the forward price of the asset equation (6). This characterization provides us with an operational definition of a RNVR. We can now derive conditions for the existence of a RNVR. The following proposition provides the complete set of possible RNVRs.

Proposition 1 *[Necessary and Sufficient Conditions for a Risk-Neutral Valuation Relationship]*

Assume that the distribution of x, the payoff of the underlying asset, is of the form

$$f(x; q) = a(q)g(x)e^{qk(x)},$$

where q is a parameter and $k(x)$ is a monotonic function. Then there exists a RNVR for contingent claims on x, if and only if the pricing kernel $\phi(x)$ is of the form

$$\phi(x; \gamma) = b(\gamma)e^{\gamma k(x)}.$$

This proposition is proved in the appendix. Using the example above, if $f(x)$ is 'generalized lognormal' as in (5), then a RNVR exists if and only if

$$\phi(x; \gamma) = b(\gamma)x^{\gamma}.$$

Note that the same condition: constant elasticity of the pricing kernel, guarantees a RNVR for this generalized log-normal distribution, as it does in the case of the lognormal distribution.

Proposition 1 characterizes the set of possible RNVRs that can exist for the pricing of European-style contingent claims. We now investigate how large this set is. The function $k(x)$ is common to both the probability distribution of the asset and to the pricing kernel. It is this commonality that restricts the range of possible RNVRs. It is easy to establish the following implications of Proposition 1.

Proposition 1 allows us to specify the function $k(x)$. Assuming that

$$f(x; q) = a(q)g(x)e^{qk(x)},$$

let $R(x) = \frac{-\phi'(x)}{\phi(x)}$ be defined as the absolute risk aversion of the pricing kernel. Then Proposition 1 implies that a RNVR exists for the pricing of contingent claims if and only if

$$k(x) = -\int \frac{R(x)}{\gamma} dx.$$

For example, if $R(x)$ is a constant (CARA), then $k(x)$ is linear in x, i.e. $f(x)$ is the normal distribution as in the Brennan (1979) model. If $R(x) = a/x$ (constant elasticity), $k(x)$ is linear in $\ln x$ and the Black-Scholes RNVR applies. However, if $R(x) = \frac{1}{x+a}$, $k(x)$ is linear in $\ln(x+a)$ so that a RNVR may exist if x is a shifted lognormal with a threshold parameter a. This is the case considered by Camara (1999).

This implication of Proposition 1 also shows the limited set of possible RNVRs. Suppose, for example, that we wish to price an option on a lognormal asset, where $k(x) = \ln x$. However, suppose that we suspect that the pricing kernel has declining elasticity with respect to $\ln x$. Then it follows that no RNVR exists. In order to price options in such economies, we need to generalize the concept of the RNVR. This is the task of the following section.

Proposition 1 is a direct implication of Heston's more general result on missing parameters. If $h(x) = 1$ in the pricing kernel

$$\phi(x;\gamma) = b(\gamma)h(x)e^{\gamma k(x)},$$

then

$$\phi(x;\gamma) = b(\gamma)e^{\gamma k(x)}$$

and according to Heston's proposition a RNVR exists. This emphasises the point that if a missing parameters relationship exists and if there is only one unknown preference parameter, then a RNVR must exist.

3 Two-Dimensional Risk-Neutral Valuation Relationship: General Results

The set of risk-neutral valuation relationships is severely restricted, as shown by the implications of Proposition 1 above. In order to price contingent claims in less restricted economies, where the pricing kernel is more complex, we need to relax the conditions imposed on the pricing kernel. For example, how can we price an option on a log-normally distributed asset, if the pricing kernel has non-constant elasticity? One possible answer lies in a generalization of Heston's concept of a missing parameters valuation relationship.

First, we consider an extension of Heston's result to the case where *two* preference parameters are invisible in the option pricing formula. We define:

Definition 1 *[Two-Dimensional Missing Parameters Relationship]*

A two-dimensional missing parameters valuation relationship exists for contingent claims on an asset if the relationship between the price of the claim, and the price of the asset and one other contingent claim on the asset, is independent of two preference parameters, γ_1 and γ_2 and two probability distribution parameters q_1 and q_2.

There are two differences between this two-dimensional missing parameters valuation relationship and Heston's one-dimensional case. First, the relationship here is between the claim price and two other prices: the price of the underlying asset and the price of one other contingent claim. Secondly, two preference parameters and two distribution parameters rather than one are missing in the pricing relationship.

A generalization of Heston's now yields:

Proposition 2 *[Necessary and Sufficient Conditions for a Two-Dimensional Missing-Parameters Valuation Relationship]*

Given the pricing kernel $\phi(x)$ and a probability density function $f(x)$, the relationship between the price of a contingent claim on the asset and the price of the asset and of one other claim on the asset is independent of two preference parameters, γ_1 and γ_2 and two probability distribution parameters, q_1 and q_2, if and only if the pricing kernel and the probability density function have the forms

$$\phi(x) = b(\gamma_1, \gamma_2)h_2(x)e^{\gamma_1 k_1(x)+\gamma_2 k_2(x)} \tag{9}$$

and

$$f(x) = a(q_1, q_2)g_2(x)e^{q_1 k_1(x)+q_2 k_2(x)},$$

where $h_2(x)$ is not dependent on γ_1 and γ_2, and $g(x)$ is not dependent on q_1 and q_2.

<u>*Proof*</u>:

To show sufficiency, assume we know the price of the underlying asset, S, and of one other claim, c_a. If the pricing kernel and the probability density function have the above form, then

$$\begin{aligned}\phi(x)f(x) &= a(q_1, q_2)b(\gamma_1, \gamma_2)h_2(x)g_2(x)e^{(\gamma_1+q_1)k_1(x)+(\gamma_2+q_2)k_2(x)} \\ &= \hat{a}(q_1+\gamma_1, q_2+\gamma_2)h_2(x)g_2(x)e^{(\gamma_1+q_1)k_1(x)+(\gamma_2+q_2)k_2(x)}.\end{aligned}$$

Given the forward prices:

$$Se^{rT} = \int_0^{+\infty} xf(x)\phi(x)dx,$$

$$c_a e^{rT} = \int_0^{+\infty} c_a(x)\phi(x)f(x)dx$$

we can solve for $(\gamma_1 + q_1) = m(S, c_a)$ and $(\gamma_2 + q_2) = n(S, c_a)$[2]. Any other claim then has a price

$$c_b e^{rT} = \int_0^{+\infty} c_b(x)\phi(x)f(x)dx,$$

which is independent of γ_1 and γ_2. Necessity is shown in the appendix. □

Also, we could further define and establish conditions for three-dimensional and in general n-dimensional relationships. The n-dimensional generalization is shown in the appendix. However, our main purpose in introducing this generalization of Heston's missing parameters valuation relationship is that it leads to more general risk-neutral valuation relationships. The two-dimensional missing parameters valuation relationship implies a two-dimensional RNVR when $h_2(x) = 1$, i.e. when only two parameters determine the pricing kernel. This leads us to define:

Definition 2 *[A Two-Dimensional-Risk-Neutral Valuation Relationship]*

A two-dimensional risk-neutral valuation relationship exists for the pricing of contingent claims on an asset, if the relationship between the price of a claim and the prices of the underlying asset and one other claim on the asset, is the same as it would be under risk neutrality.

As in the traditional one-dimensional case, we interpret prices in the definition as forward prices. We can now establish conditions for the existence of a two-dimensional RNVR:

Proposition 3 *[Necessary and Sufficient Conditions for a Two-Dimensional-Risk-Neutral Valuation Relationship]*

Assume that Proposition 2 holds. Then, a two-dimensional risk-neutral valuation relationship exists for the valuation of contingent claims on x if and only if $h_2(x) = 1$ in equation (9).

[2]Note that, in order to solve for $(\gamma_1 + q_1)$ and $(\gamma_2 + q_2)$, the claims must differ in their payoffs expressed in terms of the $k_1(x)$ and $k_2(x)$ functions

Proof:

Since a two-dimensional RNVR must also be a two-dimensional missing parameters valuation relationship, then Proposition 2 must hold. The necessity of $h_2(x) = 1$ follows, since otherwise the pricing kernel depends on more than two parameters. In this case pricing would depend on those parameters, which contradicts the existence of a two-dimensional RNVR. Sufficiency follows from an argument along the same lines as the proof of Proposition 1. $\square$

Some care is required in interpreting the two-dimensional RNVR. Although the relationship is the same as it would be under risk neutrality, it is not a RNVR in the usual sense. It is a valuation relationship that requires both the price of the underlying asset and the *price of an additional contingent claim on the asset* in order to price any other claim. The second claim required could be an at-the-money call option, for example. If a two-dimensional RNVR exists, any option can be priced if we know the price of the underlying asset and the price of the at-the-money call option.

4 A Two-Dimensional RNVR: The Generalized Lognormal Case

The most important special case of the two-dimensional RNVR is the example where the underlying asset has a generalized lognormal distribution and the pricing kernel has two unknown parameters. In this case, the two-dimensional risk-neutral valuation relationship leads to a straightforward generalization of the Black-Scholes formula for the price of a European-style call option.

Assume that the underlying asset has a generalized lognormal distribution of the form:

$$f(x) = a(q_1, q_2)g_2(x)e^{q_1 k_1(x) + q_2 k_2(x)}, \tag{10}$$

with

$$\begin{aligned} g_2(x) &= \frac{e^{-(\ln x)^2/2\sigma^2}}{x} \\ k_1(x) &= \ln(x). \end{aligned}$$

The distribution of x is generalized lognormal in the sense of the example in section 2 above. It departs from lognormal in having a 'fatter' left tail. Note that, if $q_2 = 0$, $f(x)$ is lognormal with

$$a(q_1) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\mu^2/2\sigma^2}.$$

Also, assume that the pricing kernel is of the form:

$$\phi(x) = b(\gamma_1, \gamma_2) e^{\gamma_1 k_1(x) + \gamma_2 k_2(x)},$$

again with

$$k_1(x) = \ln(x).$$

Note that for $\gamma_2 \neq 0$, we know from Proposition 1 that a one-dimensional RNVR does not exist.

From Proposition 3, for a two-dimensional RNVR to exist, the functions $k_1(x)$ and $k_2(x)$ have to be common to the probability function and the pricing kernel and they are so in this example. Also, the value of the underlying asset and any option on the asset depend on

$$q_{1,0} = q_1 + \gamma_1,$$

and

$$q_{2,0} = q_2 + \gamma_2.$$

Although the forward prices of the underlying asset and contingent claims on the asset depend upon the two preference parameters, γ_1 and γ_2, it is only the sums $q_{1,0}$ and $q_{2,0}$ that are relevant, since a two-dimensional RNVR exists. Also, we can use, for example, the price of the underlying asset, S, and the price of the at-the-money call option, c_a, to solve for $q_{1,0}$ and $q_{2,0}$, and then price any other contingent claim on the same asset.

In this example, the risk-neutral probability distribution is given by

$$\hat{f}(x) = a(q_{1,0}, q_{2,0}) g_2(x) e^{q_{1,0} k_1(x) + q_{2,0} k_2(x)},$$

The price of the underlying asset, the at-the-money call option, c_a, and the call option with strike price K, are given by the equations

$$S = e^{rT} \int_0^{+\infty} x \hat{f}(x) dx, \tag{11}$$

$$c_a = e^{rT} \int_0^{+\infty} c_a(x) \hat{f}(x) dx, \tag{12}$$

$$c_K = e^{rT} \int_0^{+\infty} c_K(x) \hat{f}(x) dx, \tag{13}$$

where the risk-neutral distribution $\hat{f}(x)$ depends only on the parameters $q_{1,0}$ and $q_{2,0}$. Hence we can solve (11) and (12) for these parameters and then use (13) to price the remaining options. Since $q_{1,0}$ determines $\hat{\mu}$, in the following we replace $q_{1,0}$ with $\hat{\mu}$.

In the appendix, we show that the value of a call option with strike price K is given by the 'generalized Black-Scholes' equation:

$$c_K = S\left[1 - \frac{G(K, \hat{\mu} + \sigma^2, \sigma, q_{2,0})}{G(\hat{\mu} + \sigma^2, \sigma, q_{2,0})}\right] - Ke^{-rT}\left[1 - \frac{G(K, \hat{\mu}, \sigma, q_{2,0})}{G(\hat{\mu}, \sigma, q_{2,0})}\right], \tag{14}$$

where

$$\begin{aligned} G(\hat{\mu}, \sigma, q_{2,0}) &\equiv \int_{-\infty}^{\infty} e^{q_{2,0}k_2(x)} n(\ln x, \hat{\mu}, \sigma) d\ln x, \\ G(K, \hat{\mu}, \sigma, q_{2,0}) &\equiv \int_{-\infty}^{\ln k} e^{q_{2,0}k_2(x)} n(\ln x, \hat{\mu}, \sigma) d\ln x, \end{aligned}$$

and $\hat{\mu}$ is the mean of the risk-neutral distribution. $G(\hat{\mu}+\sigma^2, \sigma, q_{2,0})$ and $G(K, \hat{\mu}+\sigma^2, \sigma, q_{2,0})$ are defined in a similar manner. When $q_{2,0} = 0$, we have $G(\hat{\mu}, \sigma) = 1$ and

$$G(K, \hat{\mu}, \sigma) = N\left(\frac{\ln K - \hat{\mu}}{\sigma}\right).$$

Also, $G(\hat{\mu} + \sigma^2, \sigma) = 1$ and

$$1 - G(K, \hat{\mu} + \sigma^2, \sigma) = N\left(\frac{\hat{\mu} - \ln K + \sigma^2}{\sigma}\right).$$

In this case the general formula (14) reduces to the Black formula for the forward price of the call option.

The essence of the two-dimensional RNVR is that given $q_1 + \gamma_1$ and $q_2 + \gamma_2$, the value of any contingent claim is independent of the values of the preference parameters, γ_1 and γ_2. It follows that we can take the values that these parameters would have under risk neutrality and value any claim.

It is interesting to compare the two-dimensional RNVR in equation (14) with the Black-Scholes one-dimensional RNVR. In the latter case, the price of the asset is used to solve for the risk-neutral mean. In the two-dimensional RNVR, the asset price and the price of one other option allows us to solve for $\hat{\mu}$ and for $q_{2,0}$. We now illustrate this two-dimensional RNVR with a calibrated numerical example.

5 A Numerical Example of the Two-Dimensional RNVR

In this section we illustrate the two-dimensional RNVR in the case of the generalized lognormal distribution, using a specific numerical example. First, we assume that $k_1(x) = \ln x$ as in section 4, above. Second, we assume that

$$k_2(x) = \frac{1}{(x+\varepsilon)^t},$$

where $\varepsilon > 0$.

For simplicity, and to contrast our model with the Black-Scholes case, we first take the case where the *true* distribution is lognormal. Also, for illustrative purposes we assume the asset has a mean equal to 1. We assume that the options have one year to maturity and that the forward price of the underlying asset is $x_0 = 0.9400$, representing a 6% risk premium. Note however, that the option prices in the model do not depend on this assumption regarding the true mean, which is in fact irrelevant, just as it is in the Black-Scholes model.

We calibrate the example to the case of an option on a stock price index. Hence, we assume that the volatility of the underlying asset is 25%, which approximates to current estimates of the historical volatility of the typical index. However, there is some evidence that the implied volatility for at-the-money options on the index exceeds the historical volatility. In the examples below, we assume that this 'excess volatility' is in the range of 2% to 4%, which is in line with recent estimates. For example, Corrado and Miller (2003) provide estimates of realised volatility and implied volatility for the S&P 100 index for the periods 1988-94 and 1995-2002. In the earlier period the mean realised volatility was 14.75% and the mean implied volatility was 17.84%, an excess volatility of 3.09%. In the later period the mean realised volatility was 21.11% and the mean implied volatility was 24.01%, an excess volatility of 2.90%.

We proceed as follows:

1. First, we use the Black-Scholes model to price the at-the-money option, given a forward price of $Se^{rT} = 0.94$, with a given excess volatility.

2. Next, we solve the pair of equations , (11), (12) for values of $q_{1,0}$ and $q_{2,0}$, given the true volatility.

3. Then, using $q_{1,0}$ and $q_{2,0}$, we solve for option prices in (13), for a range of strike prices, K, given the true volatility.

4. Finally, we invert the Black-Scholes formula, using these prices, in order to report the prices in terms of the conventional implied volatility measure.

In Tables 1 and 2, we show the option prices that result from these calculations, for two cases, where $t = 4$ and $t = 8$. In each table we assume that the excess volatility of the at-the-money option is 2% and 4%. These two sub cases are then compared with the Black-Scholes prices.

First with $t = 4$, solving for the values of $q_{1,0}$ and $q_{2,0}$ in the 29% case yields $q_{1,0} = 0.0460$ and $q_{2,0} = 0.1058$, when there are 21 nodes. Figure 1 shows the risk-neutral PDF and contrasts it with a lognormal PDF for the case where $\sigma = 25\%$. In the case of 22 nodes, $q_{1,0} = 0.0424$ and $q_{2,0} = 0.1033$. Using these parameter values and valuing the options with strike prices ranging from 0.5 to 1.6, and averaging over the cases of 21 and 22 nodes yields the call prices shown in column 3 of the table. The implied volatilities are shown in brackets. Similarly, option prices are then shown for the case where the at-the-money option sells at a 27% implied volatility and then the Black-Scholes prices are shown in the final column. The results show a 'smile' in the implied volatility function with a shape that is similar to that documented in studies of index option prices. This is illustrated in Figure 2. It shows volatilities declining relatively steeply and then flattening out. The function is steeper in the 29% case than in the 27% case.

In Table 2 and Figure 3, we show results for the case where $t = 8$. The model here produces a fatter left tail in the risk-neutral distribution. The result is a steeper fall in the implied volatility function, both in the case of the 27% at-the-money implied volatility case and in the 29% case.

5.1 The Non-Lognormal Case

Two problems are associated with the lognormal example above. First, there is some evidence that the empirical distribution of the S&P index has fat left tail, not just the risk-neutral distribution. We may wish to assume therefore that the true distribution has a form such as that in equation (10). Second, unless the empirical distribution has this form, with a known $k_2(x)$, then there is a difficulty in assuming knowledge of $k_2(x)$ for the pricing kernel function. Hence, we assume now that the parameter q_2 in (10) is non zero.

We first illustrate the distribution, $f(x)$, for the case where $\gamma_2 = 0$ and contrast it with the case where $q_2 = 0$. In this example we assume that $t = 4$. The resulting distributions are illustrated for binomial distributions with $n = 20$ bifurcations in Figure 1. When $q_2 = 0$, the distribution is a binomial approximation to the lognormal distribution. In the case where

$\gamma_2 = 0$, the distribution has significantly higher probabilities in the left tail. However, the pricing of options in Table 1 is unaffected by the change in the PDF assumption, as long as $q_{1,0} = q_1 + \gamma_1$ and $q_{2,0} = q_2 + \gamma_2$ are held constant.

Returning to the case where $t = 8$, we now illustrate the two-dimensional RNVR by pricing a given option at different asset forward prices. The option has a strike price of $K = 1.1$. In Table 3, we show prices computed by recalibrating the model so that at each asset forward price, the at-the-money option has an implied volatility of 25%, 27%, and 29% respectively. We then use the model in equation (14) to price the option with strike price $K = 1.1$. The two-dimensional nature of the pricing relationship is illustrated in Figure 4. In this case the prices for a given forward price are in a range whose maximum is the price in the case of the 29% at-the-money case and whose minimum is the Black-Scholes value when the at-the-money option sells at an implied volatility of 25%.

6 Conclusions

The idea of a risk-neutral valuation relationship is at the core of option pricing theory. However, as shown by our Proposition 1, the set of one-dimensional RNVRs is quite limited. For example, if the underlying asset has a log-normal distribution, a necessary condition for a one-dimensional RNVR is that the pricing kernel has constant elasticity. When looking for new tractible option pricing models, a more promising approach is to develop Heston's idea of a missing-parameters valuation relationship. However, missing-parameters relationships are of limited use for practical application. There are many cases where one preference parameter drops out of the valuation relationship, but option valuation is still not possible. To obtain preference-free valuation we need to consider a more restricted set of relationships.

We have first established a two-dimensional missing-parameters relationship, where two preference parameters drop out of the valuation relationship. This then led us to a two-dimensional RNVR, which can be used to value options if the underlying asset price, and in addition the price of one other contingent claim on the asset, are known. The numerical examples developed illustrate that these results could be used to price options on assets with 'fat-tailed' distributions.

In the case of the Black-Scholes model, a valid procedure is to form a probability distribution for the underlying asset price which is constructed 'as if' the world was risk neutral. This distribution is then used to value options using the assumption of risk neutrality. In the case of our two-dimensional extension the procedure is the same. First we construct a distribution for the underlying asset which is consistent, in a risk-neutral world, with the observed prices of the asset and of one option on the asset (in our examples, the at-the-money option). We then proceed to value all options on the asset under the assumption of risk neutrality. These are then the correct option prices.

Although we have concentrated on examples extending the Black-Scholes RNVR to account for non-constant elasticity of the pricing kernel, a similar methodology could be applied to generalize the Brennan (1979) normal distribution option pricing model. Here we would assume non-constant *absolute* risk aversion of the pricing kernel. Similarly, extensions are possible for the Rubinstein (1983) displaced diffusion model.

The two-dimensional RNVR relates a contingent claim price to any two other contingent claim prices on the same underlying asset. This analysis admits to a further extension. We could price claims using an n-dimensional RNVR. However, as the number of claims used in the pricing increases, we are in effect merely describing the shape of the pricing kernel in more detail, as in the empirical studies of Jackwerth (2000). The key to option pricing is to price contingent claims with as few pieces of information as possible. However, the analysis here suggests a technique for extending no-arbitrage pricing of claims from the rather unrealistic search for one-dimensional RNVRs of the Black-Scholes type, to the less demanding and hopefully more available set of two-dimensional RNVRs.

7 Appendix

7.1 Proof of Proposition 1

Proof

Sufficiency

Since

$$\phi(x) = b(\gamma)e^{\gamma k(x)}$$

and $E[\phi(x)] = 1$, we have

$$\int_0^{+\infty} b(\gamma)a(q)g(x)e^{(q+\gamma)k(x)}dx = 1,$$

and it follows that

$$b(\gamma)a(q) = a(q+\gamma).$$

Hence we can write

$$f(x;q)\phi(x) = a(q+\gamma)g(x)e^{(q+\gamma)k(x)} = f(x;q+\gamma).$$

Since the forward price of the underlying asset is given by

$$\int_0^{+\infty} xf(x;q)\phi(x)dx = x_0,$$

we obtain

$$\int_0^{+\infty} xf(x;q+\gamma)dx = x_0.$$

From this we can infer $q+\gamma$. The forward price of a contingent claim paying $c_T(x)$ is

$$c_{0,T} = \int_0^{+\infty} c_T(x)\phi(x)f(x;q)dx = \int_0^{+\infty} c_T(x)f(x;q+\gamma)dx. \tag{15}$$

The contingent claim price in (15) is given by (8), with $q_0 = q+\gamma$ and a RNVR exists.

Necessity

Since a RNVR exists for every contingent claim, it follows that

$$\phi(x) = f(x;q_0)/f(x;q).$$

Since any RNVR is also a missing-parameters relationship, it follows from Heston (1993, Proposition 1) that we can write

$$f(x;q) = a(q)g(x)e^{qk(x)}.$$

Hence,

$$\begin{aligned} \phi(x) &= \frac{a(q_0)}{a(q)}e^{(q_0-q)k(x)} \\ &= b(\gamma)e^{\gamma k(x)} \end{aligned}$$

□

7.2 Proof of Proposition 2, Necessity

Since the time 0 price of any contingent claim paying $c(x)$, given by

$$c_0 = \int f(x)\phi(x)c(x)dx, \tag{16}$$

is independent of γ_1 and γ_2, differentiating with respect to $\gamma_i,\ i = 1, 2$, we have

$$\int c(x)\frac{\partial}{\partial \gamma_i}[f(x)\phi(x)]dx = 0,\ i = 1, 2.$$

Now, since this must hold for any contingent claim, it follows that

$$\frac{\partial}{\partial \gamma_i}[f(x)\phi(x)] = 0,\ i = 1, 2.$$

Re-writing in logarithms, it then follows that

$$\frac{\partial \ln f(x)}{\partial q_1}\frac{\partial q_1}{\partial \gamma_i} + \frac{\partial \ln f(x)}{\partial q_2}\frac{\partial q_2}{\partial \gamma_i} + \frac{\partial \ln \phi(x)}{\partial \gamma_i} = 0,\ i = 1, 2. \tag{17}$$

Now consider given values of the parameters $q_i = q_i^*,\ i = 1, 2$. Writing $k_i(x) \equiv \frac{\partial \ln f(x)}{\partial q_i}|q_i = q_i^*$,

we obtain

$$\frac{\partial \ln \phi(x)}{\partial \gamma_1} = -[k_1(x)a_1 + k_2(x)a_2] \tag{18}$$

$$\frac{\partial \ln \phi(x)}{\partial \gamma_2} = -[k_1(x)b_1 + k_2(x)b_2] \tag{19}$$

where

$$a_i = \frac{\partial q_i}{\partial \gamma_1}|q_i = q_i^*, \ i = 1, 2.$$

$$b_i = \frac{\partial q_i}{\partial \gamma_2}|q_i = q_i^*. \ i = 1, 2.$$

From (18), it follows that

$$\ln \phi(x) = a_0(x) - k_1(x) \int a_1 d\gamma_1 - k_2(x) \int a_2 d\gamma_1, \tag{20}$$

where a_0 is independent of γ_1. Differentiating (20) w.r.t. γ_2, we have

$$\frac{\partial \ln \phi(x)}{\partial \gamma_2} = \frac{\partial a_0(x)}{\partial \gamma_2} - k_1(x) \frac{\partial}{\partial \gamma_2} \int a_1 d\gamma_1 - k_2(x) \frac{\partial}{\partial \gamma_2} \int a_2 d\gamma_1.$$

Comparing this with (19), we obtain

$$a_0(x) = a_{00}(x) + a_{01} k_1(x) + a_{02} k_2(x),$$

where $a_{00}(x)$ is independent of γ_1 and γ_2.

From the above equation and (20), we conclude that $\phi(x)$ can be written as

$$\phi(x) = h(x) \exp\{B_1 k_1(x) + B_2 k_2(x)\},$$

where $h(x)$ is independent of γ_1 and γ_2. We can view B_1 and B_2 as two preference parameters. Thus $\phi(x)$ has the following form

$$\phi(x) = b(\gamma_1, \gamma_2) h(x) \exp\{\gamma_1 k_1(x) + \gamma_2 k_2(x)\}.$$

Substituting this into (17), we obtain

$$\frac{\partial \ln f(x)}{\partial q_1} \frac{\partial q_1}{\partial \gamma_i} + \frac{\partial \ln f(x)}{\partial q_2} \frac{\partial q_2}{\partial \gamma_i} + k_i(x) + \frac{\partial b}{\partial \gamma_i} = 0, \ i = 1, 2.$$

Since γ_1 and γ_2 are two independent parameters, $k_1(x)$ and $k_2(x)$ must be linearly independent. Hence we must have

$$\frac{\partial q_1}{\partial \gamma_1} \frac{\partial q_2}{\partial \gamma_2} - \frac{\partial q_1}{\partial \gamma_2} \frac{\partial q_2}{\partial \gamma_1} \neq 0. \tag{21}$$

This implies that we can write (at least for a neighborhood)

$$\gamma_i = \gamma_i(x_0, c_{a0}; q_1, q_2), \ i = 1, 2.$$

The last three equations imply that

$$\frac{\partial \ln f(x)}{\partial q_i} = u_i k_1(x) + v_i k_2(x) + w_i,\ i = 1, 2.$$

It follows that

$$f(x) = \beta_0 g(x) \exp\{\beta_1 k_1(x) + \beta_2 k_2(x)\},$$

where $g(x)$ is independent of q_1 and q_2. We can view β_1 and β_2 as two distribution parameters; thus $f(x)$ has the following form

$$f(x) = a(q_1, q_2) g(x) \exp\{q_1 k_1(x) + q_2 k_2(x)\}.$$

Q.E.D.

7.3 An n-Dimensional Missing-Parameters Valuation Relationship

Corollary 1 *Given the pricing kernel $\phi(x) = \phi(x : \gamma_1, \gamma_2, ..., \gamma_n)$ and a probability density function $f(x) = f(x : q_1, q_2, ..., q_n)$, the relationship between the price of a contingent claim on an asset and the prices of any n other contingent claims on the asset is independent of $(\gamma_1,\ \gamma_2,...,\gamma_n)$ if and only if the pricing kernel and the probability density function have the form*

$$\phi(x) = b(\gamma) h_n(x) e^{\gamma_1 k_1(x) + \gamma_2 k_2(x) + ... + \gamma_n k_n(x)}$$

and

$$f(x) = a(q) g_n(x) e^{q_1 k_1(x) + q_2 k_2(x) + ... + q_n k_n(x)}. \qquad (22)$$

Proof The proof follows the same lines as the proof of Proposition 2.

7.4 Derivation of Two-Dimensional RNVR Call Option Price for a Generalised Lognormal Distribution.

We assume that the risk-neutral distribution has the form:

$$\hat{f}(\ln x) = a(\hat{\mu}, q_{2,0})e^{q_{2,0}k_2(x)}n(\ln x, \hat{\mu}, \sigma).$$

First we define:

$$\begin{aligned} G(\hat{\mu}, \sigma, q_{2,0}) &= \int_{-\infty}^{\infty} e^{q_{2,0}k_2(x)}n(\ln x, \hat{\mu}, \sigma)d\ln x \\ G(K, \hat{\mu}, \sigma, q_{2,0}) &= \int_{-\infty}^{\ln K} e^{q_{2,0}k_2(x)}n(\ln x, \hat{\mu}, \sigma)d\ln x. \end{aligned}$$

It follows that

$$a(\hat{\mu}, q_{2,0}) = \frac{1}{G(\hat{\mu}, \sigma, q_{2,0})}.$$

The forward price of the call option with strike price K is then given by

$$\begin{aligned} c(K)e^{rT} &= \int_{\ln K}^{\infty} xa(\hat{\mu}, q_{2,0})e^{q_{2,0}k_2(x)}n(\ln x, \hat{\mu}, \sigma) \\ &- K\int_{\ln K}^{\infty} a(\hat{\mu}, q_{2,0})e^{q_{2,0}k_2(x)}n(\ln x, \hat{\mu}, \sigma). \end{aligned}$$

Note that, from Rubinstein (1976),

$$xn(\ln x, \hat{\mu}, \sigma) = e^{\hat{\mu}+\frac{1}{2}\sigma^2}n(\ln x, \hat{\mu}+\sigma^2, \sigma).$$

Hence,

$$\begin{aligned} a(\hat{\mu}, q_{2,0})\int_{\ln K}^{\infty} xe^{q_{2,0}k_2(x)}n(\ln x, \hat{\mu}, \sigma) &= a(\hat{\mu}, q_{2,0})\int_{\ln K}^{\infty} e^{\hat{\mu}+\frac{1}{2}\sigma^2+q_{2,0}k_2(x)}n(\ln x, \hat{\mu}+\sigma^2, \sigma) \\ &= a(\hat{\mu}, q_{2,0})e^{\hat{\mu}+\frac{1}{2}\sigma^2}\int_{\ln K}^{\infty} e^{q_{2,0}k_2(x)}n(\ln x, \hat{\mu}+\sigma^2, \sigma). \end{aligned}$$

Now, using the definitions above, the forward price of the call option is

$$c(K)e^{rT} = e^{\hat{\mu}+\frac{1}{2}\sigma^2}\frac{a(\hat{\mu}, q_{2,0})}{a(\hat{\mu}+\sigma^2, q_{2,0})}\left[1 - \frac{G(K, \hat{\mu}+\sigma^2, \sigma, q_{2,0})}{G(\hat{\mu}+\sigma^2, \sigma, q_{2,0})}\right] - K\left[1 - \frac{G(K, \hat{\mu}, \sigma, q_{2,0})}{G(\hat{\mu}, \sigma, q_{2,0})}\right]$$

Note that the stock is the call option with a strike price of 0. Hence from the above equation, the price of the stock, S, is given by

$$S = e^{-rT} e^{\hat{\mu}+\frac{1}{2}\sigma^2} \frac{a(\hat{\mu}, q_{2,0})}{a(\hat{\mu}+\sigma^2, q_{2,0})}.$$

Hence the price of the option is

$$c(K) = S\left[1 - \frac{G(K, \hat{\mu}+\sigma^2, \sigma, q_{2,0})}{G(\hat{\mu}+\sigma^2, \sigma, q_{2,0})}\right] - Ke^{-rT}\left[1 - \frac{G(K, \hat{\mu}, \sigma, q_{2,0})}{G(\hat{\mu}, \sigma, q_{2,0})}\right].$$

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Table 1 : Two-Dimensional RNVR: Call Option Prices
$t = 4$ case

Strike K	K/x_0	$c_a(29\%)$	σ_I	$c_a(27\%)$	σ_I	$c_a(25\%)$	σ_I
0.50	0.53	0.4438	(36)	0.4411	(29)	0.4401	(25)
0.60	0.64	0.3501	(33)	0.3451	(28)	0.3418	(25)
0.70	0.74	0.2635	(31)	0.2565	(28)	0.2504	(25)
0.80	0.85	0.1887	(30)	0.1807	(28)	0.1737	(25)
0.94	1.00	0.1084	(29)	0.1009	(27)	0.0940	(25)
1.00	1.06	0.0830	(29)	0.0763	(27)	0.0699	(25)
1.10	1.17	0.0513	(28)	0.0460	(27)	0.0414	(25)
1.20	1.28	0.0305	(28)	0.0267	(26)	0.0239	(25)
1.40	1.49	0.0097	(27)	0.0081	(26)	0.0067	(25)
1.60	1.70	0.0027	(27)	0.0022	(26)	0.0014	(25)

1. Column 1 and two respectively show the strike price and the strike price ratio (given a forward price of the asset of $x_0 = 0.9400$. Columns 3, 5 and 7 show the forward prices of call options, with implied volatilities shown in brackets for the cases where the implied volatilities of the at-the-money option are 29%, 27%, and 25% respectively. In the last case, prices are those from a binomial approximation to the Black-Scholes model.

2. Prices are computed using a binomial approximation with 21 and 22 states and then taking the average price across these two approximations. The risk-neutral density approximated is

$$\hat{f}(x) = a_0 e^{\frac{1}{(x+\varepsilon)^t}} n(\ln x; \mu_0, \sigma)$$

   with $t = 4$ and $\sigma = 0.25$.

3. The model is calibrated so that the at-the-money option, with strike price $K = 0.9400$ sells at a Black-Scholes implied volatility of 29%, 27%, or 25%. This determines the values of $q_{1,0}$ and $q_{2,0}$. For example, given an implied volatility of 29%, in the case of 21 states $q_{1,0} = 0.0460$ and $q_{2,0} = 0.1058$ and in the case of 22 states $q_{1,0} = 0.0424$ and $q_{2,0} = 0.1033$.

Table 2 : Two-Dimensional RNVR: Call Option Prices
$t = 8$ case

Strike K	K/x_0	$c_a(29\%)$	σ_I	$c_a(27\%)$	σ_I	$c_a(25\%)$	σ_I
0.50	0.53	0.4492	(43)	0.4443	(37)	0.4401	(25)
0.60	0.64	0.3561	(38)	0.3488	(32)	0.3418	(25)
0.70	0.74	0.2686	(34)	0.2595	(29)	0.2504	(25)
0.80	0.85	0.1914	(31)	0.1823	(28)	0.1737	(25)
0.94	1.00	0.1084	(29)	0.1009	(27)	0.0940	(25)
1.00	1.06	0.0826	(29)	0.0758	(27)	0.0699	(25)
1.10	1.17	0.0503	(28)	0.0453	(26)	0.0414	(25)
1.20	1.28	0.0294	(27)	0.0260	(26)	0.0239	(25)
1.40	1.49	0.0090	(27)	0.0077	(26)	0.0067	(25)
1.60	1.70	0.0024	(26)	0.0020	(25)	0.0014	(25)

1. Column 1 and two respectively show the strike price and the strike price ratio (given a forward price of the asset of $x_0 = 0.9400$. Columns 3, 5 and 7 show the forward prices of call options, with implied volatilities shown in brackets for the cases where the implied volatilities of the at-the-money option are 29%, 27%, and 25% respectively. In the last case, prices are those from a binomial approximation to the Black-Scholes model.

2. Prices are computed using a binomial approximation with 21 and 22 states and then taking the average price across these two approximations. The risk-neutral density approximated is

$$\hat{f}(x) = a_0 e^{\frac{1}{(x+\varepsilon)^t}} n(\ln x; \mu_0, \sigma)$$

   with $t = 8$ and $\sigma = 0.25$.

3. The model is calibrated so that the at-the-money option, with strike price $K = 0.9400$ sells at a Black-Scholes implied volatility of 29%, 27%, or 25%. This determines the values of $q_{1,0}$ and $q_{2,0}$.

Table 3 : Two-Dimensional RNVR:
Call Option Price and Asset Forward Price
Strike Price $K = 1.1$

Asset price: x_0	$\sigma_{I,a} = 25\%$	$\sigma_{I,a} = 27\%$	$\sigma_{I,a} = 29\%$
0.8	0.0112	0.0140	0.0161
0.9	0.0285	0.0332	0.0370
1.0	0.0641	0.0711	0.0732
1.1	0.1100	0.1181	0.1261
1.2	0.1724	0.1791	0.1887
1.3	0.2451	0.2548	0.2651

1. Column 1 show the underlying asset's forward price. Columns 2, 3 and 4 show the forward prices of the call option, with strike price $K = 1.1$, given that the implied volatilities of at the money options are 25%, 27%, and 29% respectively. In the first case, prices are those from a binomial approximation to the Black-Scholes model.
2. Prices are computed using a binomial approximation with 11 states. The risk-neutral density approximated is
$$\hat{f}(x) = a_0 e^{\frac{1}{(x+\varepsilon)^t}} n(\ln x; \mu_0, \sigma)$$
with $t = 8$ and $\sigma = 0.25$.
3. For each $\sigma_{I,a}$, the model is calibrated for each x_0, so that the at-the-money options, with $K = x_0$ sell at the given Black-Scholes implied volatility.