# Exotic Options Pricing under Stochastic Volatility

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# Exotic Options Pricing under Stochastic Volatility


## Abstract

This paper proposes an analytical approximation to price exotic options within a stochastic volatility framework. Assuming a general mean reverting process for the underlying asset and a square-root process for the volatility, we derive an approximation for option prices using a Taylor expansion around two average defined volatilities. The moments of the average volatilities are computed analytically at any order using a Frobenius series solution to some ordinary differential equation. Pricing some exotics such as barrier and digital barrier options, the approximation is found to be very efficient and convergent even at low Taylor expansion order.



***Keywords***: *Option pricing; Exotic options; Digital barrier options; Mean reversion; Stochastic volatility; Frobenius series.*



## Résumé

Cet article propose une approximation analytique pour évaluer les options exotiques dans un cadre de volatilité stochastique. En considérant un processus avec retour à la moyenne pour l'actif sous-jacent et un processus racine-carrée pour la volatilité, on dérive une approximation pour les options en utilisant un développement de Taylor autour de deux volatilités « moyennes » qui seront définies. Les moments des volatilités moyennes sont calculés analytiquement en utilisant une solution en séries de Frobenius d'une certaine équation différentielle ordinaire. En évaluant certaines options exotiques comme les options barrières et les options barrières digitales, on montre que l'approximation converge rapidement et qu'elle est très précise.



***Mots-clés***: *Évaluation d'options; Options exotiques; Options barrières digitales; Retour à la moyenne; Volatilité stochastique; Séries de Frobenius.*

## Introduction

Several papers propose pricing formulas for plain vanilla options on stocks within different stochastic volatility frameworks. Heston (1993) is the first one who proposes a closed-form solution for a standard European call when using a square-root volatility process by inverting the characteristic function seen as a Fourier transform. Bakshi, Cao and Chen (1997) propose an empirical performance study of some alternative option pricing models including stochastic volatility and jumps processes by deriving closed-form solutions in the same way as Heston (1993). Schöbel and Zhu (1999) and Zhu (2000) derive, in a very elegant way, a modular pricing method which includes the square-root and the Ornstein-Uhlenbeck volatility processes mixed eventually with jumps. For some volatility models however, no closed-form solutions can be derived and some numerical techniques are used instead. Hull and White (1987) and Sabanis (2002, 2003) obtain an approximate solution for a standard European option using a Taylor series expansion based on the underlying asset's distribution conditional on the average value of the stochastic variance.

Slightly little work was done for pricing exotic derivatives such as path-dependent options in non-constant volatility models. Davydov and Linetsky (2001) derive closed-form solutions, in terms of Bessel and Whittaker functions, for barrier and lookback options under a constant elasticity of variance (CEV) diffusion model. Henderson and Hobson (2001) price passport options[1] in Hull and White (1987) and Stein and Stein (1991) stochastic volatility frameworks using the series expansion technique. In both cases, very simple closed-form solutions for the central moments of the average stochastic variance are proposed.[2] This is possible only for some volatility processes such as the geometric Brownian motion and the mean reverting diffusion.[3] Unfortunately, for other stochastic volatility processes such as the square-root diffusion, no simple closed-form formulas for the moments of the average variance can be found. In that case, other techniques such as numerical approximations or Monte Carlo simulation may be used to

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[1] A passport option is a call option on the profits of a trading account, it can be priced as a lookback option.

[2] Sabanis (2002, 2003) derives an iterative procedure to compute these moments.

[3] Sabanis (2003) calls "mean reverting" the following volatility diffusion $dV_t = (\kappa\theta - \lambda V_t)dt + \sigma V_t dW_t$.

price derivatives. Apel, Winkler and Wystup (2001) propose a finite elements method to price plain vanilla and barrier options under a square-root stochastic volatility model.

Many assets including interest rates, credit spreads [see Longstaff and Schwartz (1995), Tahani (2000), Prigeant et al. (2001) and Jacobs and Li (2003)] and some commodities (Schwartz, 1997) are shown to exhibit a mean reversion feature. But, there is little literature on pricing derivatives on this type of underlying assets. Under stochastic volatility models, most of the work is done on plain vanilla derivatives. Clewlow and Strickland (1997) price standard interest rate derivatives under a square-root volatility model using Monte Carlo simulations. Assuming the latter volatility process, Tahani (2004) prices credit spread options, caps, floors and swaps using Gaussian quadrature. Under a constant volatility assumption, Leblanc and Scaillet (1998) propose some path-dependent interest rate options formulas for the affine term structure model.

This paper proposes to price some exotic options on a mean reverting underlying asset in a square-root volatility model using a series expansion around two average defined volatilities. The choice of this power series method is encouraged by the findings of Ball and Roma (1994) about its accuracy and its easy implementation in comparison to other approaches. The key thing of this method is that the price of a contingent claim may be computed as the expectation of the corresponding constant-volatility model's price where the volatility and the spot price are random variables accounting for stochastic variance [see Hull and White (1987) and Romano and Touzi (1997)]. It remains though to derive the central moments of the average variances and use them in the series expansion. But since the closed-form formulas for these moments can only be derived in terms of Whittaker functions,[4] which are heavy-computational, it is preferable to compute them using a Frobenius series solution[5] which is very accurate, very fast and very easy to implement.

The next section presents the proposed model and introduces the series expansion method. Section 2 derives a Frobenius series solution to the moments of the average

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[4] In fact, we can derive the moments of the average variances in terms of derivatives of Whittaker functions w.r.t. the first and the third arguments.

[5] Selby and Strickland (1995) use the same technique to avoid the use of confluent hypergeometric functions in the Fong and Vasicek (1992) bond price formula.

variances. Section 3 presents valuation formulas for some exotic options. Section 4 presents some numerical results on convergence and efficiency. Section 5 will conclude.

## 1 The proposed model

Following Tahani (2004), we consider two stochastic differential equations (SDEs) for the state variable and its volatility under a risk-neutral measure *Q*:

$$dX_u = (\mu - \alpha X_u - \gamma V_u)dt + \sqrt{V_u}\left(\rho dB_u + \sqrt{1-\rho^2}\, dW_u\right) \tag{1}$$

$$dV_u = (\kappa\theta - \lambda V_u)dt + \sigma\sqrt{V_u}\, dB_u \tag{2}$$

where $\{\exp(X_u), t \le u \le T\}$ is the price process of a primitive asset such as a stock or a credit spread, $\{V_u, t \le u \le T\}$ is the volatility process and $\rho$ is the correlation between the state variable and its volatility. *W* and *B* are two independent Brownian motions on a probability space $(\Omega, \mathcal{F}, Q)$ and $\{\mathcal{F}_u, t \le u \le T\}$ is the *Q*-augmentation of the filtration generated by $(W, B)$. The parameters $\mu$, $\alpha$, $\gamma$, $\kappa$, $\theta$, $\lambda$ and $\sigma$ are constant.

Pricing theory allows us to write the price $p(X_t, V_t, t)$ of any European contingent claim on *X* as the expectation, under a risk-neutral measure, of the discounted payoff of the contract in order to get:

$$p(X_t, V_t, t) = E^Q\left(e^{-r(T-t)} H(\{X_s, t \le s \le T\}, T) \,\middle|\, \mathcal{F}_t\right) \tag{3}$$

where *r* is the constant risk-free rate, *T* the contract maturity, *H* is the payoff that could depends on the whole path of the state variable $(X_s)_{t \le s \le T}$. In order to develop a series expansion approximation to the contract price, we shall adapt the methodology in Romano and Touzi (1997) to our mean reverting process *X*. Referring to the details in Appendix A, we can write the solution to the SDE (1) at any time *T* as:

$$\begin{aligned} X_T = {} & e^{-\alpha(T-t)} X_t + Y_{t,T} + \mu \int_t^T e^{-\alpha(T-s)} ds - \gamma \int_t^T e^{-\alpha(T-s)} V_s ds \\ & + \frac{1}{2}\rho^2 \int_t^T e^{-2\alpha(T-s)} V_s ds + \sqrt{1-\rho^2} \int_t^T e^{-\alpha(T-s)} \sqrt{V_s}\, dW_s \end{aligned} \tag{4}$$

where

$$Y_{t,T} = \rho\int_t^T e^{-\alpha(T-s)}\sqrt{V_s}dB_s - \frac{1}{2}\rho^2\int_t^T e^{-2\alpha(T-s)}V_s ds \quad (5)$$

Assuming a non-zero correlation, Romano and Touzi (1997)[6] derive the price of a standard European call option as the expectation of the Black and Scholes (1973) price where the underlying asset price is replaced by $\exp(X_t + Y_{t,T})$ and the volatility parameter is replaced by $\sqrt{\frac{1}{T-t}\int_t^T (1-\rho^2)V_s ds}$. But their formula is not explicit since one still has to compute the expectation, which is almost impossible in the non-zero correlation case. Lewis (2000) notices that even in the case of a zero correlation, the pricing formula in Equation (3) does not always lead to an analytical solution because the integrated volatility density is difficult to derive in closed-form.[7] Assuming a zero correlation,[8] Hull and White (1987) and Sabanis (2002, 2003) derive an approximation to the European call price, while Henderson and Hobson (2001) derive an approximation to a passport option. Taking into account this findings and since our aim is to get some explicit pricing formulas for exotic options, we will assume for the remainder of the article that the correlation between the state variable and its volatility is zero.

In the zero correlation case, the process $Y_{t,T}$ given in Equation (5) is always 0 and Equation (4) becomes:

$$X_T = e^{-\alpha(T-t)}X_t + \mu\int_t^T e^{-\alpha(T-s)}ds - \gamma\int_t^T e^{-\alpha(T-s)}V_s ds + \int_t^T e^{-\alpha(T-s)}\sqrt{V_s}dW_s \quad (6)$$

Let $\mathcal{G}_t = \sigma\{W_t, B_u : t \le u \le T\}$ be a $\sigma$-algebra which assumes that the movements of the volatility over the entire life of the contract are known at time *t*, and define two average variances by:

[6] Romano and Touzi (1997) model can be recovered by putting $\alpha = 0$, $\mu = r$ and $\gamma = 0.5$.

[7] In Lewis (2000), page 116.

[8] In Forex and interest rates markets, it was found that the volatility smile is symmetric and thus the correlation between the volatility and the asset can be taken as zero.

$$\overline{V_1} = \frac{\int_t^T e^{-\alpha(T-s)} V_s ds}{\int_t^T e^{-\alpha(T-s)} ds} \quad ; \quad \overline{V_2} = \frac{\int_t^T e^{-2\alpha(T-s)} V_s ds}{\int_t^T e^{-2\alpha(T-s)} ds} \tag{7}$$

Conditional on $\mathcal{G}_t$, i.e. on the path of the volatility, the process $X_T$ in Equation (6) is Gaussian with mean:

$$e^{-\alpha(T-t)} X_t + \mu \int_t^T e^{-\alpha(T-s)} ds - \gamma \overline{V_1} \int_t^T e^{-\alpha(T-s)} ds \tag{8}$$

and variance:

$$\overline{V_2} \int_t^T e^{-2\alpha(T-s)} ds \tag{9}$$

The price of a contingent claim as given in Equation (3) can thus be rewritten as:

$$\begin{aligned} p(X_t, V_t, t) &= E^Q\left(E^Q\left(e^{-r(T-t)} H(\{X_s, t \le s \le T\}, T) \mid \mathcal{G}_t\right) \mid \mathcal{F}_t\right) \\ &\equiv E^Q\left(\overline{p}\left(\overline{V_1}, \overline{V_2}\right) \mid \mathcal{F}_t\right) \end{aligned} \tag{10}$$

where $\overline{p}\left(\overline{V_1}, \overline{V_2}\right)$, called pseudo-price hereafter, is the price of the same contingent claim under an Ornstein-Uhlenbeck process with long-run mean $\frac{\mu - \gamma \overline{V_1}}{\alpha}$, mean reversion parameter $\alpha$ and volatility $\sqrt{\overline{V_2}}$.[9] For path-dependent options, the pseudo-price should depend on the integrated variance processes $\left(\int_t^u e^{-\alpha(u-s)} V_s ds \, , \int_t^u e^{-2\alpha(u-s)} V_s ds\right)_{t \le u \le T}$, rather than only on $\left(\overline{V_1}, \overline{V_2}\right)$. Equation (10) is therefore only an approximation for path-dependent options. To obtain the price in Equation (10), the pseudo-price is expanded in a Taylor series around the expected values of $\overline{V_1}$ and $\overline{V_2}$. Thus, the expectation on the right-hand side of Equation (10) takes the following form:

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[9] In the case $\alpha = 0$, $\mu = r$ and $\gamma = 0.5$, the pseudo-price is the price of the contingent claim under a geometric Brownian process with drift $r$ and a volatility $\sqrt{\overline{V}} = \sqrt{\frac{1}{T-t} \int_t^T V_s ds}$

$$p(X_t, V_t, t) = \sum_{n=0}^{+\infty}\sum_{m=0}^{+\infty}\left\{\begin{array}{l}\frac{1}{n!}\frac{1}{m!}E\left[\left(\overline{V_1} - E(\overline{V_1})\right)^n\left(\overline{V_2} - E(\overline{V_2})\right)^m\right] \\ \times\partial_{n,m}\overline{p}\left(E(\overline{V_1}), E(\overline{V_2})\right)\end{array}\right\} \tag{11}$$

where all the expectations are taken under the risk-neutral measure $Q$ conditional on $\mathcal{F}_t$. We must compute the $(n,m)$-derivatives of the pseudo-price w.r.t. the average variances and the cross-moments $E\left(\overline{V_1}^n\,\overline{V_2}^m\right)$ for all orders $(n,m)$.[10] Since the differentiation of the pseudo-price depends on the contingent claim specifications, it will be done later and we start by deriving the cross-moments given the square-root volatility diffusion in Equation (2).

## 2 The Frobenius series solution

Define the cross-moment generating function to be given by:

$$g(a,b;t,T,V) \equiv E\left(\exp\left[-a\int_t^T e^{-\alpha(T-s)}V_s\,ds - b\int_t^T e^{-2\alpha(T-s)}V_s\,ds\right]\right) \tag{12}$$

where $E(\cdot)$ denotes $E^Q(\cdot\,|\,\mathcal{F}_t)$. Given the expressions of $\overline{V_1}$ and $\overline{V_2}$ in Equation (7), we can easily see that the $(n,m)^{\text{th}}$ cross-moment can be written as:

$$E\left(\overline{V_1}^n\,\overline{V_2}^m\right) = (-1)^{n+m}\times\left(\int_t^T e^{-\alpha(T-s)}ds\right)^{-n}\times\left(\int_t^T e^{-2\alpha(T-s)}ds\right)^{-m}\times\partial_{n,m}g(a,b)\Big|_{(0,0)} \tag{13}$$

Considering $g$ as a function of $(\tau, V)$, Feynman-Kac theorem allows us to write $g(\tau, V)$ as the solution to a partial differential equation (PDE) that takes the following form:

$$\begin{cases}\dfrac{\partial g}{\partial\tau} = \dfrac{1}{2}\sigma^2 V\dfrac{\partial^2 g}{\partial V^2} + (\kappa\theta - \lambda V)\dfrac{\partial g}{\partial V} - \eta(\tau)Vg \\ g(0,V) = 1\end{cases} \tag{14}$$

where $\tau = T - t$ and $\eta(\tau) = a\exp(-\alpha\tau) + b\exp(-2\alpha\tau)$. Following Tahani (2004), we assume that $g$ is log-linear in $(\tau, V)$ and can be written as:

$$g(\tau, V) = \exp\left(VD(\tau) + C(\tau)\right) \tag{15}$$

[10] Hopefully, we won't need to get to very large values of $n$ and $m$. Sabanis (2002, 2003) and Apel et al (2001) only need the second order and find the third of negligible impact.

where $D(.)$ and $C(.)$ are solutions to the two following ordinary differential equations (ODEs):

$$\begin{cases} D'(\tau) - \frac{1}{2}\sigma^2 D^2(\tau) + \lambda D(\tau) + \eta(\tau) = 0 \\ D(0) = 0 \end{cases} \tag{16}$$

and

$$\begin{cases} C'(\tau) = \kappa\theta\, D(\tau) \\ C(0) = 0 \end{cases} \tag{17}$$

The exact solutions to these ODEs are given by:

$$\begin{cases} D(\tau) = -\frac{2}{\sigma^2}\frac{U'(\tau)}{U(\tau)} \\ C(\tau) = -\frac{2\kappa\theta}{\sigma^2}\ln\big(U(\tau)\big) \end{cases} \tag{18}$$

where $U$ solves the following linear homogeneous second-order ODE:

$$\begin{cases} U''(\tau) + \lambda U'(\tau) - \frac{1}{2}\sigma^2 U(\tau)\eta(\tau) = 0 \\ U'(0) = 0 \quad , \quad U(0) = 1 \end{cases} \tag{19}$$

Tahani (2004) provides the exact solution to this ODE that involves Whittaker functions:

$$\begin{aligned} U(a,b;\tau) = \Phi \exp\left[-\frac{1}{2}(\lambda-\alpha)\tau\right] M\left(-\frac{\sqrt{2}}{4}\frac{a\sigma}{\alpha\sqrt{b}}, \frac{1}{2}\frac{\lambda}{\alpha}, \frac{\sigma\sqrt{2b}}{\alpha}e^{-\alpha\tau}\right) \\ + \Psi \exp\left[-\frac{1}{2}(\lambda-\alpha)\tau\right] W\left(-\frac{\sqrt{2}}{4}\frac{a\sigma}{\alpha\sqrt{b}}, \frac{1}{2}\frac{\lambda}{\alpha}, \frac{\sigma\sqrt{2b}}{\alpha}e^{-\alpha\tau}\right) \end{aligned} \tag{20}$$

where $M(.)$ and $W(.)$ are Whittaker functions and constants[11] $\Phi$ and $\Psi$ can be determined using the initial conditions in ODE (19).

To compute the cross-moments, we must compute the derivatives of functions $D$ and $C$ w.r.t variables $a$ and $b$ that appear in the first and the third arguments in Whittaker functions as well as in $\Phi$ and $\Psi$, which is very heavy. Instead, we will develop a

[11] $\Phi$ and $\Psi$ will depend on the variables $a$ and $b$.

Frobenius[12] series solution to the ODE (19) (see Appendix B for details). Making the change of variable $z = \exp(-\alpha\tau)$ and defining the function $S(z)$ by:

$$\begin{cases} S(z) \equiv z^{-\beta} U(\tau) \\ \beta = \dfrac{\lambda - \alpha}{2\alpha} \end{cases} \tag{21}$$

leads to the following series solution:

$$S(z) = \Phi \sum_{n=0}^{+\infty} k_n(\beta+1) z^{n+\beta+1} + \Psi \sum_{n=0}^{+\infty} k_n(-\beta) z^{n-\beta} \tag{22}$$

where constants $\Phi$ and $\Psi$ are determined using the fact that $S(1) = 1$ and $S'(1) = -\beta$, and functions $(k_n(.))_{0 \le n}$ are computed recursively by the following formulas:

$$\begin{cases} k_1(\varepsilon) = \dfrac{a\sigma^2}{4\alpha^2 \varepsilon} k_0 \\ k_n(\varepsilon) = \dfrac{\sigma^2}{2\alpha^2} \dfrac{\left(a k_{n-1}(\varepsilon) + b k_{n-2}(\varepsilon)\right)}{(n^2 + n(2\varepsilon - 1))} \qquad ; \quad n \ge 2 \end{cases} \tag{23}$$

where $k_0$ is an arbitrary constant. Once functions *D* and *C* are computed according to Equation (18), the computation of the moments can be achieved by differentiating the function *g* in Equation (12). We only need to differentiate the series solution in Equation (22) by truncating it at a finite order instead of dealing with Whittaker functions in Equation (20). The computation of the cross-(centered)-moments in Equation (11) is thus straightforward.

**The no mean reversion case**

In the case of no mean reversion (i.e. $\alpha = 0$), the function $\eta$ in PDE (14) becomes constant and the moment generating function[13] is defined by:

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[12] Under some regularity conditions, an ODE may have a series solution taking the form $z^{\varepsilon} \sum_{n=0}^{+\infty} k_n z^n$.

[13] It can be seen as a zero-coupon price by considering $\left(\eta V_s\right)_{t \le s \le T}$ as the instantaneous interest rate in Cox, Ingersoll and Ross (1985) model.

$$g(\eta;t,T,V) \equiv E\left(\exp\left[-\eta\int_t^T V_s\,ds\right]\right) \tag{24}$$

We do not need a Frobenius series solution since the moment generating function $g$ is given by a simple closed-form expression [see Cox, Ingersoll and Ross (1985)]:

$$g(\tau,V) = \exp\big(VD(\tau) + C(\tau)\big) \tag{25}$$

where

$$\begin{cases} D(\tau) = \dfrac{-2\eta}{\lambda+\omega}\dfrac{1-e^{-\omega\tau}}{1+\delta\, e^{-\omega\tau}} \\ C(\tau) = \dfrac{-2\kappa\theta}{\sigma^2}\log\left(\dfrac{1+\delta\, e^{-\omega\tau}}{1+\delta}\right) - \dfrac{\kappa\theta}{\sigma^2}(\omega-\lambda)\tau \\ \omega = \sqrt{\lambda^2 + 2\eta\sigma^2} \quad ; \quad \delta = \dfrac{\omega-\lambda}{\omega+\lambda} \end{cases} \tag{26}$$

Notice that the solution $U$ to the ODE (20) is also very easy to compute by:

$$U(\eta;\tau) = \frac{1}{2}\frac{\lambda+\omega}{\omega}\exp\left(\frac{1}{2}(\lambda-\omega)\tau\right) + \frac{1}{2}\frac{\omega-\lambda}{\omega}\exp\left(-\frac{1}{2}(\lambda+\omega)\tau\right) \tag{27}$$

These formulas will be used later for pricing some exotic derivatives on stocks.

At this stage, we are able to use Equation (11) to compute the approximate price for any contingent claim using a Taylor expansion as long as we can compute the derivatives of the pseudo-price w.r.t. the average variances either analytically or numerically.

The next section will present some standard and some exotic options on both mean reverting and non-mean reverting underlying assets. The computation of standard option prices in a stochastic volatility model will allow us to check for the accuracy of the series expansion in simple cases where (semi)-closed-form solutions exist, among which Heston (1993) and Tahani (2004) models.[14]

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[14] The constant volatility counterparts of Heston (1993) and Tahani (2004) models are respectively Black and Scholes (1973) and Longstaff and Schwartz (1995) models, which will be used to compute the pseudo-prices.

# 3 Valuation formulas for exotic options

In this section, we will recall some well-known closed-form pricing formulas for path-dependent stock options under a constant volatility model, which will be used in the series expansion for pricing the same path-dependent options under the square-root stochastic volatility model. In the case of a mean reverting asset and a constant volatility, Leblanc and Scaillet (1998) propose closed-form solutions (up to an inversion of Fourier transform) for some path-dependent options on affine yields among which the arithmetic average option. This approach will be used to derive a pricing formula for credit spread average options. We also will use the distribution of the first passage time for an Ornstein-Uhlenbeck process to a boundary derived in Leblanc et al. (2000) to price digital barrier credit spread options. Once these pseudo-prices of exotic options are computed, we will use them in Equation (11) in order to price the same exotics under a stochastic volatility model.

## 3.1 Barrier and digital asset-or-nothing stock options

These formulas are derived in details in Reiner and Rubinstein (1991) and Haug (1997). The stock price is given by $e^X$ and the pseudo-price of a down-out call with strike price $K$ and barrier $L$ when $K \leq L$ is given by:

$$\begin{aligned} C_{DO}\left(X_t, \overline{V}, K, L, r, t, T\right) = {} & e^{X_t}\mathrm{N}(d) - Ke^{-r(T-t)}\mathrm{N}\left(d - \sqrt{\overline{V}(T-t)}\right) \\ & - e^{X_t}\left(\frac{L}{e^{X_t}}\right)^{2\psi}\mathrm{N}\left(2\psi\sqrt{\overline{V}(T-t)} - d\right) \\ & + Ke^{-r(T-t)}\left(\frac{L}{e^{X_t}}\right)^{2\psi-2}\mathrm{N}\left((2\psi-1)\sqrt{\overline{V}(T-t)} - d\right) \end{aligned} \tag{28}$$

where

$$\overline{V} = \frac{1}{T-t}\int_t^T V_s\,ds \quad ; \quad \psi = \frac{r}{\overline{V}} + \frac{1}{2} \quad ; \quad d = \frac{\ln\left(e^{X_t}/L\right)}{\sqrt{\overline{V}(T-t)}} + \psi\sqrt{\overline{V}(T-t)} \tag{29}$$

and $\mathrm{N}(\cdot)$ is the standard normal cumulative function.

The pseudo-price of a digital down-out asset-or-nothing option can be obtained by taking a strike price equal to $K = 0$ in Equation (28):

$$Dig_{DO}\left(X_t,\overline{V},L,r,t,T\right)=e^{X_t}\mathrm{N}(d)-e^{X_t}\left(\frac{L}{e^{X_t}}\right)^{2\psi}\mathrm{N}\left(2\psi\sqrt{\overline{V}(T-t)}-d\right) \tag{30}$$

The pseudo-price of an up-out put with $K \le L$ is given by:

$$\begin{aligned} P_{UO}\left(X_t,\overline{V},K,L,r,t,T\right)= & -e^{X_t}\mathrm{N}(-d)+Ke^{-r(T-t)}\mathrm{N}\left(-d+\sqrt{\overline{V}(T-t)}\right) \\ & +e^{X_t}\left(\frac{L}{e^{X_t}}\right)^{2\psi}\mathrm{N}\left(d-2\psi\sqrt{\overline{V}(T-t)}\right) \\ & -Ke^{-r(T-t)}\left(\frac{L}{e^{X_t}}\right)^{2\psi-2}\mathrm{N}\left(d-(2\psi-1)\sqrt{\overline{V}(T-t)}\right) \end{aligned} \tag{31}$$

### 3.2 Digital cash-or-nothing credit spread option

In the case of mean reversion, i.e. $\alpha \neq 0$, the diffusion of the state variable $X$ under the filtration $(\mathcal{G}_t)$ is an Ornstein-Uhlenbeck process and the average variances $\left(\overline{V_1},\overline{V_2}\right)$ are given in Equation (7). Defining the first hitting time $T_L=\inf\{s : X_s \ge L\}$, the density of $T_L$ is derived in Leblanc et al. (2000):

$$\chi\left(\overline{V_1},\overline{V_2};L,s\right)\equiv Q\left(T_L\in ds\right) \tag{32}$$

$$=\frac{(L-X)}{\sqrt{2\pi}}\exp\left(\frac{\alpha}{2\overline{\overline{V_2}}}\left[\begin{array}{l}\left(X-\dfrac{\mu-\gamma\overline{V_1}}{\alpha}\right)^2-\left(L-\dfrac{\mu-\gamma\overline{V_1}}{\alpha}\right)^2\\ +s-(L-X)^2\coth\left(\dfrac{\alpha}{\overline{\overline{V_2}}}s\right)\end{array}\right]\right)\times\left(\frac{\alpha}{\overline{\overline{V_2}}}\frac{1}{\sinh\left(\dfrac{\alpha}{\overline{\overline{V_2}}}s\right)}\right)^{\frac{3}{2}}$$

Göing-Jaeschke and Yor (2003) point out that the distribution given in Leblanc et al. (2000) is only true if the boundary level coincides with the long-run mean,[15] which in our case leads to $L=\dfrac{\mu-\gamma\overline{V_1}}{\alpha}$ in Equation (32). A digital up-in cash-or-nothing credit spread option pays off a certain amount $K$ at maturity if the credit spread $e^X$ crosses the barrier $e^L$ from below during the option life. Its pseudo-price is then given by:

[15] Linetsky (2003) derives an approximation to this density for any fixed boundary level in terms of Sturm-Liouville eigenfunction expansion, while Alili et al. (2004) provide an integral representation.

$$Dig_{UI}\left(X_t,\overline{V_1},\overline{V_2},L,r,t,T\right)=e^{-r(T-t)}K\int_t^T \chi\left(\overline{V_1},\overline{V_2};L,s\right)ds \tag{33}$$

In order to apply the series expansion in Equation (11), the derivatives of the pseudo-price w.r.t. $\left(\overline{V_1},\overline{V_2}\right)$ are computed by differentiating under the integral sign.

### 3.3 Option on average credit spread

Denoting the average credit spread by $Y_T=\int_t^T X_s ds$, the pseudo-price of a call on average credit spread with strike *K* can be obtained by:

$$\begin{aligned} C_{Ave}(\overline{V_1},\overline{V_2}) &\equiv e^{-r(T-t)}E\left(\left(\exp\left(\frac{Y_T}{T-t}\right)-K\right)^+\right) \\ &= e^{-r(T-t)}\int_{(T-t)\ln(K)}^{+\infty}\left(\exp(\frac{y}{T-t})-K\right)Q(Y_T\in dy) \end{aligned} \tag{34}$$

where the density of $Y_T$ can be computed by inverting its characteristic function[16] and then the integration in Equation (34) can be done numerically using Gaussian quadrature [see Tahani (2004)].

## 4 Numerical results

In order to assess the efficiency and the accuracy of the proposed methodology, we price some plain vanilla and exotic options in Heston (1993) and Tahani (2004) square-root stochastic volatility frameworks. For standard options, Heston (1993) and Tahani (2004) option prices will be considered as the true prices towards which the series expansion must converge. Tables 1 to 4 show the results for standard call options for different parameters settings. It is found that the numerical prices converge rapidly to the true prices; at most, we need the 4th order to achieve a good accuracy. Notice that, due to

[16] The moment generating function of $Y_T=\int_t^T X_s ds$ is given by $E_t\left(e^{\phi Y_T}\right)=\int_{-\infty}^{+\infty}\exp(\phi\, y)Q(Y_T\in dy)$ and can be computed as a zero-coupon bond price in Vasicek (1977) model where the instantaneous rate is $(-\phi X_s)$.

mean reversion, relative errors are much smaller in Tables 1 and 2 compared to Tables 3 and 4. For exotic options, since there are no closed-form formulas under stochastic volatility, we compute a Monte Carlo price with $10^5$ paths (repeated 50 times) using the antithetic variate method and consider it as the true price. Tables 5 to 10 show the results for different barrier and digital barrier options. To compare approximated prices with Monte Carlo ones, barrier levels are adjusted to account for the discrete sampling as suggested in Broadie et al. (1997). Numerical prices are shown to converge rapidly at low expansion orders, which proves that the analytic approximation is very accurate and very efficient even for complex derivatives such as exotic options.

## 5 Conclusion

We propose an analytic approximation using Taylor series expansion to price exotic options on stocks and credit spreads when the volatility is stochastic. The main purpose of the expansion method is the computation of the moments of the average variances in a square-root volatility model, which is done using either a closed-form formula if there is no mean reversion; or using a Frobenius series solution in the case of a mean reverting process. The series expansion method is found to be very accurate and very efficient when pricing both standard and some exotic options on stocks and credit spreads, such as barrier and digital barrier options. The main contribution of this article is the derivation of an analytic approximation for exotic options within a stochastic volatility framework. The proposed methodology can be easily adapted to price derivatives on other underlying assets, such as interest and currency rates or commodities. It can also be extended to other stochastic volatility models.

**Table 1**

**Plain vanilla call price under Heston (1993) model**

| Order | Price | Absolute error | Relative error |
|---|---|---|---|
| 0 | 8.1802 | 1.27E-02 | 1.55E-03 |
| 2 | 8.1674 | 1.53E-04 | 1.88E-05 |
| 3 | 8.1677 | 2.03E-04 | 2.49E-05 |
| 4 | 8.1675 | 1.70E-05 | 2.08E-06 |
| 5 | 8.1675 | 9.43E-06 | 1.16E-06 |
| 6 | 8.1675 | 1.84E-06 | 2.25E-07 |
| 7 | 8.1675 | 7.31E-07 | 8.94E-08 |
| 8 | 8.1675 | 1.83E-07 | 2.25E-08 |
| 9 | 8.1675 | 1.82E-07 | 2.23E-08 |
| 10 | 8.1675 | 4.26E-08 | 5.21E-09 |
| **True price** | 8.1675 | | |

Note: Table 1 presents the results of the valuation of a plain vanilla call within Heston (1993) model for different expansion orders. The true price is given by Heston closed-form formula. The option's parameters are $X_0 = \ln(100)$; $V_0 = 0.04$; $K = 100$; $r = 0.05$; $T = 1$. The model's parameters are $\mu = r$; $\alpha = 0$; $\gamma = 0.5$; $\sigma = 0.1$; $\lambda = 4$; $\kappa = 1$ and $\theta = 0.05$.

**Table 2**

**Plain vanilla call price under Heston (1993) model**

| Order | Price | Absolute error | Relative error |
|---|---|---|---|
| 0 | 15.2350 | 1.95E-03 | 1.28E-04 |
| 2 | 15.2369 | 2.60E-05 | 1.71E-06 |
| 3 | 15.2367 | 2.28E-04 | 1.49E-05 |
| 4 | 15.2369 | 2.29E-05 | 1.50E-06 |
| 5 | 15.2369 | 1.53E-05 | 1.00E-06 |
| 6 | 15.2369 | 3.21E-06 | 2.11E-07 |
| 7 | 15.2369 | 1.19E-06 | 7.82E-08 |
| 8 | 15.2369 | 2.98E-07 | 1.95E-08 |
| 9 | 15.2369 | 4.88E-07 | 3.20E-08 |
| 10 | 15.2369 | 2.77E-08 | 1.82E-09 |
| **True price** | 15.2369 | | |

Note: Table 2 presents the results of the valuation of a plain vanilla call within Heston (1993) model for different expansion orders. The true price is given by Heston closed-form formula. The option's parameters are $X_0 = \ln(100)$; $V_0 = 0.04$; $K = 90$; $r = 0.05$; $T = 1$. The model's parameters are $\mu = r$; $\alpha = 0$; $\gamma = 0.5$; $\sigma = 0.1$; $\lambda = 4$; $\kappa = 1$ and $\theta = 0.05$.

**Table 3**

**Plain vanilla call price under Tahani (2004) model**

| Order | Price | Absolute error | Relative error |
|---|---|---|---|
| 0 | 0.0019759 | 2.08E-06 | 1.05E-03 |
| 2 | 0.0019738 | 4.71E-08 | 2.39E-05 |
| 3 | 0.0019738 | 2.17E-09 | 1.10E-06 |
| 4 | 0.0019738 | 2.04E-08 | 1.03E-05 |
| 5 | 0.0019738 | 1.89E-08 | 9.58E-06 |
| 6 | 0.0019738 | 1.88E-08 | 9.53E-06 |
| 7 | 0.0019738 | 1.91E-08 | 9.67E-06 |
| 8 | 0.0019738 | 1.88E-08 | 9.54E-06 |
| 9 | 0.0019738 | 1.90E-08 | 9.62E-06 |
| 10 | 0.0019738 | 1.89E-08 | 9.56E-06 |
| **True price** | 0.0019738 | | |

Note: Table 3 presents the results of the valuation of a plain vanilla call within Tahani (2004) square-root model for different expansion orders. The true price is given by Tahani semi-closed-form formula. The option's parameters are $X_0 = \ln(0.02)$; $V_0 = 0.04$; $K = 0.02$; $r = 0.05$; $T = 1$. The model's parameters are $\mu = 0.03$; $\alpha = 0.01$; $\gamma = 0.2$; $\sigma = 0.1$; $\lambda = 4$; $\kappa = 1$ and $\theta = 0.05$.

**Table 4**

**Plain vanilla call price under Tahani (2004) model**

| Order | Price | Absolute error | Relative error |
|---|---|---|---|
| 0 | 0.0066518 | 4.02E-06 | 6.04E-04 |
| 2 | 0.0066561 | 2.89E-07 | 4.34E-05 |
| 3 | 0.0066559 | 3.74E-08 | 5.62E-06 |
| 4 | 0.0066557 | 1.75E-07 | 2.62E-05 |
| 5 | 0.0066559 | 7.57E-08 | 1.14E-05 |
| 6 | 0.0066557 | 1.57E-07 | 2.36E-05 |
| 7 | 0.0066558 | 2.72E-08 | 4.08E-06 |
| 8 | 0.0066558 | 2.45E-08 | 3.69E-06 |
| 9 | 0.0066556 | 2.16E-07 | 3.25E-05 |
| 10 | 0.0066561 | 2.48E-07 | 3.72E-05 |
| **True price** | 0.0066558 | | |

Note: Table 4 presents the results of the valuation of a plain vanilla call within Tahani (2004) square-root model for different expansion orders. The true price is given by Tahani semi-closed-form formula. The option's parameters are $X_0 = \ln(0.025)$; $V_0 = 0.04$; $K = 0.02$; $r = 0.05$; $T = 1$. The model's parameters are $\mu = 0.03$; $\alpha = 0.01$; $\gamma = 0$; $\sigma = 0.2$; $\lambda = 4$; $\kappa = 1$ and $\theta = 0.05$.

**Table 5**

**Down-out barrier call price under Heston (1993) model**

| Order | Price | Absolute error | Relative error |
|---|---|---|---|
| 0 | 8.4046 | 1.79E-02 | 2.12E-03 |
| 2 | 8.4226 | 1.40E-04 | 1.66E-05 |
| 3 | 8.4221 | 4.09E-04 | 4.85E-05 |
| 4 | 8.4226 | 1.54E-04 | 1.83E-05 |
| 5 | 8.4226 | 9.69E-05 | 1.15E-05 |
| 6 | 8.4226 | 1.29E-04 | 1.54E-05 |
| 7 | 8.4226 | 1.23E-04 | 1.46E-05 |
| 8 | 8.4226 | 1.26E-04 | 1.49E-05 |
| 9 | 8.4226 | 1.25E-04 | 1.49E-05 |
| 10 | 8.4226 | 1.25E-04 | 1.49E-05 |
| **Monte Carlo** | 8.4225 | | |

Note: Table 5 presents the results of the valuation of a down-out barrier call within Heston (1993) model for different expansion orders. The Monte Carlo price is given with $10^5$ paths (repeated 50 times) using the antithetic variate method. The option's parameters are $X_0 = \ln(100)$; $V_0 = 0.04$; $K = 90$; $L = 95$; $r = 0.05$; $T = 0.36$. The model's parameters are $\mu = r$; $\alpha = 0$; $\gamma = 0.5$; $\sigma = 0.1$; $\lambda = 4$; $\kappa = 1$ and $\theta = 0.05$.

**Table 6**

**Down-out barrier call price under Heston (1993) model**

| Order | Price | Absolute error | Relative error |
|---|---|---|---|
| 0 | 9.3847 | 3.17E-02 | 3.37E-03 |
| 2 | 9.4168 | 4.23E-04 | 4.49E-05 |
| 3 | 9.4153 | 1.10E-03 | 1.17E-04 |
| 4 | 9.4166 | 2.35E-04 | 2.50E-05 |
| 5 | 9.4164 | 2.67E-05 | 2.83E-06 |
| 6 | 9.4165 | 1.37E-04 | 1.46E-05 |
| 7 | 9.4165 | 1.07E-04 | 1.13E-05 |
| 8 | 9.4165 | 1.20E-04 | 1.27E-05 |
| 9 | 9.4165 | 1.15E-04 | 1.23E-05 |
| 10 | 9.4165 | 1.17E-04 | 1.24E-05 |
| **Monte Carlo** | 9.4164 | | |

Note: Table 6 presents the results of the valuation of a down-out barrier call within Heston (1993) model for different expansion orders. The Monte Carlo is given with $10^5$ paths (repeated 50 times) using the antithetic variate method. The option's parameters are $X_0 = \ln(100)$; $V_0 = 0.04$; $K = 90$; $L = 95$; $r = 0.05$; $T = 1$. The model's parameters are $\mu = r$; $\alpha = 0$; $\gamma = 0.5$; $\sigma = 0.1$; $\lambda = 4$; $\kappa = 1$ and $\theta = 0.05$.

**Table 7**

**Up-out barrier put price under Heston (1993) model**

| Order | Price | Absolute error | Relative error |
|---|---|---|---|
| 0 | 1.8159 | 1.30E-02 | 7.19E-03 |
| 2 | 1.7992 | 3.80E-03 | 2.11E-03 |
| 3 | 1.7994 | 3.55E-03 | 1.97E-03 |
| 4 | 1.7993 | 3.70E-03 | 2.05E-03 |
| 5 | 1.7993 | 3.70E-03 | 2.05E-03 |
| 6 | 1.7993 | 3.69E-03 | 2.05E-03 |
| 7 | 1.7993 | 3.69E-03 | 2.05E-03 |
| 8 | 1.7993 | 3.69E-03 | 2.05E-03 |
| 9 | 1.7993 | 3.69E-03 | 2.05E-03 |
| 10 | 1.7993 | 3.69E-03 | 2.05E-03 |
| **Monte Carlo** | 1.8030 | | |

Note: Table 7 presents the results of the valuation of an up-out barrier put within Heston (1993) model for different expansion orders. The Monte Carlo price is given with $10^5$ paths (repeated 50 times) using the antithetic variate method. The option's parameters are $X_0 = \ln(100)$; $V_0 = 0.04$; $K = 100$; $L = 110$; $r = 0.05$; $T = 0.24$. The model's parameters are $\mu = r$; $\alpha = 0$; $\gamma = 0.5$; $\sigma = 0.1$; $\lambda = 4$; $\kappa = 1$ and $\theta = 0.05$.

**Table 8**

**Up-out barrier put price under Heston (1993) model**

| Order | Price | Absolute error | Relative error |
|---|---|---|---|
| 0 | 6.6753 | 6.20E-02 | 9.38E-03 |
| 2 | 6.6119 | 1.41E-03 | 2.14E-04 |
| 3 | 6.6143 | 9.68E-04 | 1.46E-04 |
| 4 | 6.6116 | 1.70E-03 | 2.57E-04 |
| 5 | 6.6120 | 1.34E-03 | 2.02E-04 |
| 6 | 6.6117 | 1.58E-03 | 2.39E-04 |
| 7 | 6.6118 | 1.51E-03 | 2.29E-04 |
| 8 | 6.6118 | 1.55E-03 | 2.35E-04 |
| 9 | 6.6118 | 1.54E-03 | 2.32E-04 |
| 10 | 6.6118 | 1.55E-03 | 2.34E-04 |
| **Monte Carlo** | 6.6133 | | |

Note: Table 8 presents the results of the valuation of an up-out barrier put within Heston (1993) model for different expansion orders. The Monte Carlo price is given with $10^5$ paths (repeated 50 times) using the antithetic variate method. The option's parameters are $X_0 = \ln(100)$; $V_0 = 0.09$; $K = 100$; $L = 120$; $r = 0.05$; $T = 1$. The model's parameters are $\mu = r$; $\alpha = 0$; $\gamma = 0.5$; $\sigma = 0.1$; $\lambda = 1$; $\kappa = 2$ and $\theta = 0.05$.

**Table 9**

**Digital down-out asset-or-nothing option price under Heston (1993) model**

| Order | Price | Absolute error | Relative error |
|---|---|---|---|
| 0 | 78.7675 | 1.58E-01 | 2.00E-03 |
| 2 | 78.9361 | 1.09E-02 | 1.38E-04 |
| 3 | 78.9344 | 9.17E-03 | 1.16E-04 |
| 4 | 78.9329 | 7.68E-03 | 9.73E-05 |
| 5 | 78.9334 | 8.15E-03 | 1.03E-04 |
| 6 | 78.9329 | 7.70E-03 | 9.76E-05 |
| 7 | 78.9330 | 7.83E-03 | 9.92E-05 |
| 8 | 78.9330 | 7.76E-03 | 9.83E-05 |
| 9 | 78.9330 | 7.78E-03 | 9.86E-05 |
| 10 | 78.9330 | 7.77E-03 | 9.84E-05 |
| **Monte Carlo** | 78.9252 | | |

Note: Table 9 presents the results of the valuation of a digital down-out asset-or-nothing option within Heston (1993) model for different expansion orders. The Monte Carlo price is given with $10^5$ paths (repeated 50 times) using the antithetic variate method. The option's parameters are $X_0 = \ln(100)$; $V_0 = 0.04$; $L = 90$; $r = 0.05$; $T = 0.36$. The model's parameters are $\mu = r$; $\alpha = 0$; $\gamma = 0.5$; $\sigma = 0.1$; $\lambda = 4$; $\kappa = 1$ and $\theta = 0.05$.

**Table 10**

**Digital up-in cash-or-nothing option price under Tahani (2004) model**

| Order | Price | Absolute error | Relative error |
|---|---|---|---|
| 0 | 0.9393 | 3.01E-04 | 3.21E-04 |
| 2 | 0.9397 | 9.27E-05 | 9.87E-05 |
| 3 | 0.9396 | 6.97E-05 | 7.41E-05 |
| 4 | 0.9397 | 9.36E-05 | 9.96E-05 |
| 5 | 0.9396 | 8.93E-05 | 9.50E-05 |
| 6 | 0.9397 | 9.19E-05 | 9.78E-05 |
| 7 | 0.9397 | 9.11E-05 | 9.69E-05 |
| 8 | 0.9397 | 9.15E-05 | 9.73E-05 |
| 9 | 0.9397 | 9.13E-05 | 9.72E-05 |
| 10 | 0.9397 | 9.14E-05 | 9.72E-05 |
| **Monte Carlo** | 0.9396 | | |

Note: Table 10 presents the results of the valuation of a digital up-in cash-or-nothing option within Tahani (2004) model for different expansion orders. The Monte Carlo price is given with $10^5$ paths (repeated 50 times) using the antithetic variate method. The option's parameters are $X_0 = \ln(0.045)$; $V_0 = 0.04$; $K = 1$; $L = 0.05$; $r = 0$; $T = 1$. The model's parameters are $\mu = -0.03$; $\alpha = 0.01$; $\gamma = 0$; $\sigma = 0.1$; $\lambda = 4$; $\kappa = 1$ and $\theta = 0.05$.

## Appendix A: Derivation of Equations (4)-(9)

Assuming that the diffusion processes for the state variable and its volatility are given by:

$$dX_u = \left(\mu - \alpha X_u - \gamma V_u\right)dt + \sqrt{V_u}\left(\rho dB_u + \sqrt{1-\rho^2}\, dW_u\right) \tag{A.1}$$

$$dV_u = \left(\kappa\theta - \lambda V_u\right)dt + \sigma\sqrt{V_u}\, dB_u \tag{A.2}$$

the SDE for $X$ can be solved as:

$$\begin{aligned} X_T &= e^{-\alpha(T-t)}X_t + \mu\int_t^T e^{-\alpha(T-s)}ds - \gamma\int_t^T e^{-\alpha(T-s)}V_s ds \\ &\quad + \rho\int_t^T e^{-\alpha(T-s)}\sqrt{V_s}\, dB_s + \sqrt{1-\rho^2}\int_t^T e^{-\alpha(T-s)}\sqrt{V_s}\, dW_s \end{aligned} \tag{A.3}$$

By defining the process $Y$:

$$Y_{t,T} = \rho\int_t^T e^{-\alpha(T-s)}\sqrt{V_s}\, dB_s - \frac{1}{2}\rho^2\int_t^T e^{-2\alpha(T-s)}V_s ds \tag{A.4}$$

this allows for the solution $X$ to be rewritten as:

$$\begin{aligned} X_T &= e^{-\alpha(T-t)}X_t + Y_{t,T} + \mu\int_t^T e^{-\alpha(T-s)}ds - \gamma\int_t^T e^{-\alpha(T-s)}V_s ds \\ &\quad + \frac{1}{2}\rho^2\int_t^T e^{-2\alpha(T-s)}V_s ds + \sqrt{1-\rho^2}\int_t^T e^{-\alpha(T-s)}\sqrt{V_s}\, dW_s \end{aligned} \tag{A.5}$$

In the zero correlation case, we then get:

$$\begin{aligned} X_T &= e^{-\alpha(T-t)}X_t + Y_{t,T} + \mu\int_t^T e^{-\alpha(T-s)}ds - \gamma\int_t^T e^{-\alpha(T-s)}V_s ds \\ &\quad + \int_t^T e^{-\alpha(T-s)}\sqrt{V_s}\, dW_s \end{aligned} \tag{A.6}$$

Defining the average variances $\overline{V_1}$ and $\overline{V_2}$ by:

$$\begin{cases} \overline{V_1}\int_t^T e^{-\alpha(T-s)}ds = \int_t^T e^{-\alpha(T-s)}V_s ds \\ \overline{V_2}\int_t^T e^{-2\alpha(T-s)}ds = \int_t^T e^{-2\alpha(T-s)}V_s ds \end{cases} \tag{A.7}$$

shows that conditional on $\mathcal{G}_t$, the process $X_T$ in Equation (A.6) is Gaussian with mean $\left(e^{-\alpha(T-t)}X_t + \mu\int_t^T e^{-\alpha(T-s)}ds - \gamma\overline{V_1}\int_t^T e^{-\alpha(T-s)}ds\right)$ and variance $\overline{V_2}\int_t^T e^{-2\alpha(T-s)}ds$.

## Appendix B: Frobenius series solution for ODE (19)

The function $S$ given in Equation (21) by:

$$\begin{cases} S(z) \equiv z^{-\beta}U(\tau) \\ \beta = \dfrac{\lambda - \alpha}{2\alpha} \end{cases} \tag{B.1}$$

where $z = \exp(-\alpha\tau)$ must solve the following second-order ODE:

$$\begin{cases} z^2 S''(z) - \left(\beta(\beta+1) + \dfrac{a\sigma^2}{2\alpha^2}z + \dfrac{b\sigma^2}{2\alpha^2}z^2\right)S(z) = 0 \\ S(1) = 1 \quad , \quad S'(1) = -\beta \end{cases} \tag{B.2}$$

Developing a Frobenius series solution for ODE (B.2) consists in finding a special solution taking the form $\sum_{n=0}^{+\infty} k_n z^{n+\varepsilon}$. Substituting this series in ODE (B.2) gives:

$$z^2\sum_{n=0}^{+\infty} k_n(n+\varepsilon)(n+\varepsilon-1)z^{n+\varepsilon-2} - \left(\beta(\beta+1) + \frac{a\sigma^2}{2\alpha^2}z + \frac{b\sigma^2}{2\alpha^2}z^2\right)\sum_{n=0}^{+\infty} k_n z^{n+\varepsilon} = 0 \tag{B.3}$$

or equivalently:

$$\sum_{n=0}^{+\infty} k_n\big((n+\varepsilon)(n+\varepsilon-1) - \beta(\beta+1)\big)z^{n+\varepsilon} - \sum_{n=1}^{+\infty}\frac{a\sigma^2}{2\alpha^2}k_{n-1}z^{n+\varepsilon} - \sum_{n=2}^{+\infty}\frac{b\sigma^2}{2\alpha^2}k_{n-2}z^{n+\varepsilon} = 0 \tag{B.4}$$

Comparing the series coefficients yields to:

$$\begin{cases} k_0\big(\varepsilon(\varepsilon-1) - \beta(\beta+1)\big) = 0 \\ k_1\big((1+\varepsilon)\varepsilon - \beta(\beta+1)\big) - \dfrac{a\sigma^2}{2\alpha^2}k_0 = 0 \\ k_n\big((n+\varepsilon)(n+\varepsilon-1) - \beta(\beta+1)\big) - \dfrac{a\sigma^2}{2\alpha^2}k_{n-1} - \dfrac{b\sigma^2}{2\alpha^2}k_{n-2} = 0 \quad , n > 1 \end{cases} \tag{B.5}$$

Thus, ODE (B.2) has a Frobenius series solution only if $\varepsilon = \beta + 1$ or $\varepsilon = -\beta$, in which cases, the series coefficients are given by:

$$\begin{cases} k_0 \text{ arbitrary constant} \\ k_1(\varepsilon) = \dfrac{a\sigma^2}{4\alpha^2\varepsilon} k_0 \\ k_n(\varepsilon) = \dfrac{\sigma^2}{2\alpha^2} \dfrac{ak_{n-1}(\varepsilon) + bk_{n-2}(\varepsilon)}{(n^2 + (2\varepsilon - 1)n)} \quad , \ n \geq 2 \end{cases} \tag{B.6}$$

A general solution to ODE (B.2) is then given by:

$$S(z) = \Phi \sum_{n=0}^{+\infty} k_n(\beta + 1) z^{n+\beta+1} + \Psi \sum_{n=0}^{+\infty} k_n(-\beta) z^{n-\beta} \tag{B.7}$$

where the constants $\Phi$ and $\Psi$ are determined using the conditions $S(1) = 1$ and $S'(1) = -\beta$. The arbitrary constant $k_0$ can be taken equal to 1.

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