Uncertainty Aversion, Robust Control and Asset Holdings with Stochastic Investment Opportunity Set

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Abstract

In this paper we formulate the portfolio choice problem as a robust control problem. Extending our previous work [32] , by considering a stochastic investment opportunity set, we derive optimal portfolio rules under uncertainty aversion, in the cases of one and two risky assets. In particular, with two risky assets and one risk-free asset, with the same level of ambiguity aversion for the two assets, we show that the robust portfolio rule could lead to an increase in the total holdings of risky assets as compared to the holdings under the Merton rule, which is the standard risk aversion case. Furthermore the investor is more likely to increase the holdings of the asset for which there is no ambiguity, and reduce the holdings of the asset for which there is ambiguity, a result that might provide an explanation for the home bias puzzle.

Key words Uncertainty Aversion, Model Misspecification, Robust Control, Portfolio Choice Models.
EFM classification codes 330, 370

1 Introduction

In finance uncertainty has been mainly related to the assumption of knowledge of a precise probability measure describing the evolution of an asset's price. Thus the expected utility maximization criterion can be used as a methodological framework. This assumption has come under some criticism because of its failure to explain certain "puzzles" such as the observed equity premium puzzle, or the investors home-bias puzzle. In trying to explain these puzzles, but also to extend the results of traditional portfolio choise theory, the concept of

Knightian uncertainty has been introduced. Under Knightian uncertainty the state space of outcomes is known but information is too imprecise in order to assign probabilities to outcomes.

Two main approaches have emerged recently for analyzing the problem of choice when the decision maker faces pure uncertainty in the Knightian sense (or ambiguity) and whose preference relationship is characterized by uncertainty aversion (Gilboa and Schmeidler [12]). In the first, the multiple priors model, the decision maker may formulate his/her objective by attaching a probability, say $(1-e)$, to a baseline prior and a probability e to the infimum of a family of the disturbed priors around the baseline one. This is the so-called e-contamination approach (Epstein and Wang [9]), which is consistent with uncertainty or ambiguity aversion.[1] The other, the robust dynamic control approach, provides another tractable way to incorporate uncertainty aversion (e.g. Hansen and Sargent, [19], [20], [21], [23], Hansen et al. [24]). This methodology models an agent who has not enough confidence in the initial benchmark model which has been estimated given a set of finite data. The agent has concerns about speciffiction errors, in the sense that there is a set of approximate models that are also consistent with the data and any one of them could be regarderd as possible true. To put it differently the decision maker is unable to make a reliable estimation of a probability law and so is unsure about what measure to use in order to form mathematical expectations. Disturbing a benchmark model generates approximating models, where the admissible disturbances reflect the set of possible probability measures that the decision maker is willing to accept. The objective of the resulting robust control problem is to design a rule that performs well across a variety of approximating models.

Portfolio choice theory has been a prominent area of application of the above approaches[2] (e.g. Dow and Werlang [5], Epstein and Wang [9], Chen and Epstein [7], Epstein and Miao [8], Uppal and Wang [31], Maenhout [25], Pathak [26], Liu [13], [14]). The idea behind the use of robust control methods in optimal portfolio choice is that the consumer-investor believes that the initial model is misspecified regarding the assets' price processes. In this set-up, the investor tries to find a portfolio rule that will work well, in the sense of maximizing utility, under a suitably restricted range of different model specifications. The concerns about model uncertainty is parametrized by the so called robustness parameter θ. When the desicion maker shows no concerns about model misspecification,

[1] Chen and Epstein [7] introduce ambiguity aversion to recursive multiple-prior models of utility by considering $\kappa - Ignorance$ which is a concept that allows differentiation between ambiguous and pure risk cases.

[2] Monetary policy can be regarded as the initial area of application of these approaches (e.g., Brainard, [1] Hansen and Sargent [23], Onatski and Stock [17], Onatski and Williams [18], Soderstrom [29]). See also Brock and Durlauf [2], Brock, Durlauf and West [3] for similar approaches to policy design and policy evaluation under uncertainty. Another area of interest is environmental and resource management where uncertainty aversion can be used to formulate the concept of the Precautionary Principle (Brock and Xepapadeas [4], Roseta-Palma and Xepapadeas [27])

then the robustness parameter $\theta \to \infty$.[3], [4]

A central result underlying the recent robust control literature in the portfolio selection context (Maenhout [25], Uppal and Wang [31]) suggests that model uncertainty implies cautiousness in the sense that the investor, under uncertainty aversion, will invest a smaller share of his/her wealth in the risky assets relative to the share implied by the standard Merton rule under risk aversion. In more general terms, it seems that uncertainty has been associated in the earlier literature with some kind of cautious or conservative behavior,[5] although more recent results in the area of monetary policy analysis under uncertainty seem to provide mixed findings, that is aggressiveness or conservatism depending on the structure of the model.[6]

Following Hansen and Sargent's approach, the present paper attempts to derive optimal portfolio rules parametrized by the robustness parameter θ, by formulating the portfolio choice problem as a robust control problem. In modelling the problem we consider a stochastic investment opportunity set where not only the evolution of asset prices is stochastic, but in addition the drift and the volatility rate of the prices processes could be stochastic too. We associate the intertemporal consumption-investement problem under standard risk aversion, that is the standard Merton's problem, with $\theta \to \infty$, and the intertemporal consumption-investement problem under uncertainty aversion (or ambiguity aversion, or concerns about model misspesification) with $\theta < \infty$. We show that as $\theta \to \infty$ the robust portfolio rule tends to Merton's rule.[7] The associated robust portfolio rule indicates that the holdings of risky assets as a proportion of the investor's wealth is not always smaller as compared to the holdings under the Merton rule, a result that comes in contradiction with the general beleif that uncertainty aversion is mainly associated with a conservative behaviour regarding portfolio choices. The derived conditions under which such an increase in the holdings of risky assets takes place, are independent of the value of the robustness parameter θ.

The rest of the paper is organized as follows. In the next section, we consider the case of one risky asset, with stochastic investment opportunity set allowing for ambiguity both with respect to the evolution of the asset's price process and

[3]The robustness parameter θ is a fixed exogenous parameter and can be interpreted as the Lagrangian multiplier associated with an entropy constraint, which determines the maximum specification error in the asset price equation that the investor is willing to accept (Hansen and Sargent [21]).

[4]In recent attempts to study the dynamic portfolio rules using robust control methodology, (Maenhout [25], and Uppal and Wang [31]) use certain transformations to eliminate θ from the portfolio rule. As shown by Pathak [26] these transformations break down the consistency of preferences with Gilboa and Schmeidler's axiomatization of uncertainty aversion. It seems that since the exogeneity of θ is required in order for the problem to be consistent with uncertainty aversion, robust portfolios are parametrized by θ. To estimate θ in order to fully characterize the robust portfolio, Hansen and Sargent [19] suggest the use of detection error probabilities.

[5]For example Brainard's [1] results suggest caution in the face of model uncertainty in a Bayesian framework.

[6]See for example Onatski and Williams [18] and the papers cited by them.

[7]This is in agreement to Meanhout's results.

the evolution of the mean rate of return or/and the volatility rate. We derive conditions under which the investor never increases the holdings of the risky asset relative to the standrd Merton rule which is the risk aversion case. Then we examine the case of two risky assets. In this case the robust portfolio rule indicates that it is possible to increase the holdings of the risky assets relative to risk aversion case. Finally considering no ambiguity for the one of the two assets, we show that the investor is more likely to increase the holdings of the asset for which there is no uncertainty aversion associated with the evolution of its price process, than the holdings of the other asset for which the investor has concerns regarding misspecification errors in the evolution of its price process. If we associate the asset for which there is no ambiguity aversion (but only risk aversion) with a home assets and the asset for which ambiguity aversion exists with a foreign asset, our results could be regarded as an additional explanation for the home bias puzzle.[8]

2 Robust Portfolio Choices With One Risky Asset

2.1 One risky asset with stochastic drift or volatility rate

We consider a market which consists of one riskless asset whose price evolves according to:

$$dS(t) = rS(t)dt \qquad S(0) = S_0, \quad t \geq 0,$$

where r denotes the risk free rate of return, and one risky asset. Denoting by α_1 the drift rate, or mean rate of return, and by σ_1 the volatility rate the evolution of the price P_1 of the asset, is given by:

$$\frac{dP_1}{P_1} = \alpha_1 dt + \sigma_1 dB_1, \tag{1}$$

where B_1 is a standard Brownian process defined on a probability space $(\Omega, \mathcal{F})$, with measure $\mathcal{P}_1$. We consider initially that the mean rate of return evolves

[8] There have been a number of arguments attempting to explain the home bias puzzle. Strong and Xu [30] explain the puzzle on the basis of optimism of fund managers towards their home equity market. Serrat [28] considers nontraded goods to operate as factors that shift the marginal utility of traded goods. This entails dynamic hedging policies which in turn are consistent with the home bias puzzle, while French and Poterba [11] consider information costs as an explanation of the puzzle. Pathak [26] also provides an explanation of the home bias puzzle using a two-asset model and a $\kappa - Ignorance$ framework, where the worst-case scenario is used to reduce the mean return of the asset price process. There is a subtle difference between our result and the $\kappa - Ignorance$, worst case scenario approach. In the latter approach the worst case scenario means that the reduction in the mean return of the asset price process is determined at the level where the entropy constraint $\mathcal{Q}(\tau) = \{Q \in \mathcal{Q} : R_t(\mathcal{Q} \parallel \mathcal{P}) \leq \tau \ \ \forall t\}$ is binding. In the robust control model developed in this paper, the robustness parameter associated with the penalty terms is the Lagrangian multiplier associated with the entropy constraint.

stochastically over time and that satisfies the stochastic differential equation:

$$\frac{d\alpha_1}{\alpha_1} = f_1 dt + g_1 dZ_1, \tag{2}$$

where f_1, g_1, are constants and Z_1 is a Brownian process which is corelated with B_1. Let ρ_1 denote the corelation coefficient between them [9] [10]. Merton's solution [11] of the optimal portfolio allocation problem for an infinite time horizon and one risky asset determines the optimal porfolio weight, w_1, that is the fraction of the investor's total wealth, W allocated to the risky asset as:

$$w_1 W = \frac{A(\alpha_1 - r)}{\sigma_1^2} + H_1 \frac{g_1 \rho \alpha_1}{\sigma_1} \tag{3}$$

$$A = -\frac{V_W}{V_{WW}} \tag{4}$$

$$H_1 = -\frac{V_{\alpha_1 w}}{V_{ww}} \tag{5}$$

where V is the value function of the problem, V_W, V_{WW} its first and second partial derivatives with respect to the wealth W and $V_W\alpha_1$the second partial derivative with respect to W and α_1.

Writting dZ_1 as $\rho dB_1 + \sqrt{1-\rho^2} dB_2$ the equation (2) takes the form:

$$\frac{d\alpha_1}{\alpha_1} = f_1 dt + g_1(\rho dB_1 + \sqrt{1-\rho^2} dB_2), \tag{6}$$

where B_1, B_2 are two independent Brownian processes, defined on an underlying probability space $(\Omega, \mathcal{F})$, with measure $\mathcal{P} = \mathcal{P}_1 \otimes \mathcal{P}_2$.

Following Hansen and Sargent (2002) [22], Hansen et al. (2002) [24], model $(1), (6)$ is regarded as a benchmark model. If the consumer-investor is not sure about the benchmark model then there would be concerns about robustness to model misspecification. Concerns for robustness to model misspesification can be reflected by a family of stochastic perturbations. Because there are two independed Brownian motions we are able to perturb each one separately [12] so that:

$$B_i(t) = \hat{B}_i(t) + \int_0^t h_i(s) ds, \quad i = 1, 2, \tag{7}$$

where $\{\hat{B}_i(t) : t \geq 0\}$ are Brownian motions and $\{h_i(t) : t \geq 0\}$ measurable drift distortions. Therefore the probabilities implied by $(1), (6)$ are distorted. The measure $\mathcal{P}$ is replaced by another probability measure $\mathcal{Q} = \mathcal{Q}_1 \otimes \mathcal{Q}_2$. As shown

[9] $\mathbb{E} dB_1 dZ_1 = \rho dt$

[10] Following Merton we provide our proofs by considering that the mean rate of return follows the above general equation. The results also hold if we consider the more realistic case of a mean reverse process $d\alpha_1 = (\alpha_1 f_1 - f_2)\, dt + \alpha_1 g_1 dZ_1$. To model the mean reverse process we need to add only a term related with f_2 which does not affect the results. The same also holds for the stochastic volatility case.

[11] see Merton (1971)[15], (1973)[16]

[12] This is the reason for the use of the specific form of equation (6).

by Hansen et al. (2002) the discrepancy between the distribution $\mathcal{P}$ and $\mathcal{Q}$ is measured as the relative entropy $R(\mathcal{Q} \parallel \mathcal{P})$. At this stage we consider distortions to the joint distribution of the asset and of the drift rate so we impose an overall entropy constraint for them. Based on Corollary $C3.3$ of Dupuis and Ellis [6], the entropy constraint becomes:

$$R(\mathcal{Q} \parallel \mathcal{P}) = \sum_{i=1}^{2} \int_0^\infty e^{-\delta u} \mathbb{E}_{\mathcal{Q}} \left(\frac{h_i^2}{2} \right) du. \tag{8}$$

The above equation allows to consider two separate distortion terms one for the asset and the other for the mean rate of return. However in order to reduce the complexity of the model, we assume symmetric distorted measures $\mathcal{Q}_1, \mathcal{Q}_2$ and examine the case with the same distortion terms h_i. In this specific case, the equations for wealth dynamics and the mean rate of return become:

$$\begin{bmatrix} d\alpha_1(t) \\ dW(t) \end{bmatrix} = \begin{bmatrix} \alpha_1[f_1 + g_1 h(\rho + \sqrt{1-\rho^2})] \\ w_1(\alpha_1 + \sigma_1 h - r) + (rW - c) \end{bmatrix} dt + \begin{bmatrix} \alpha_1 g_1 \rho & \alpha_1 g_1 \sqrt{1-\rho^2} \\ W\sigma_1 w_1 & 0 \end{bmatrix} \begin{bmatrix} d\hat{B}_1 \\ d\hat{B}_2 \end{bmatrix} \tag{9}$$

Under model misspecification a multiplier robust control problems can be associated with the problem, of maximizing the present value of lifetime expected utility,or:

$$\max_{w_1, C} \mathbb{E}_0 \int_0^\infty e^{-\delta t} U(C) dt \tag{10}$$

In this case the multiplier robust control problem becomes:

$$J(\theta) = \sup_{w_i, C} \inf_h \mathbb{E}_{\mathcal{Q}} \int_0^\infty e^{-\delta t} \big[U(C) + \theta_2 \frac{h^2}{2} \big] dt \tag{11}$$

subject to (9).

In the above equation because of (8) , $\theta_2 = 2\theta$ where θ denotes the robustness parameter which takes values greater or equal to zero. Thus it is assumed that concerns about model misspecification are the same for the price processes of the asset and of the mean rate of return. As shown by Hansen and Sargent (2002) θ is the Lagrangean mutiplier at the optimum, associated with the entropy constraint $\mathcal{Q}(\tau) = \{Q \in \mathcal{Q} : R_t(\mathcal{Q} \parallel \mathcal{P}) \le \tau \;\; \forall t\}$. A value of $\theta = \infty$, indicates that we are sure about the measure $\mathcal{P}$, with no preference for robustness. This case can be regarded as the risk aversion case and the problem is reduced to the standard Merton problem with objective given by (10). Lower values for θ indicate preference for robusness under model misspecification, or uncertainty aversion, where a value of $\theta = 0$ indicates that we have no knowledge about the measure $\mathcal{P}$.

Using Fleming and Souganidis (1989) [10], on the existence of a recursive solution to the multiplier problem, Hansen et al. (2002) show that problem (11) can be transformed into a stochastic infinite horizon two-player game between the investor and the Nature. Nature plays here the role of a ”mean agent” and chooses a reduction h in the mean return of assets to reduce the investors' life time utility. The Bellman-Isaacs conditions for this game imply that the value function $V(W, \alpha_1, \theta)$ satisfies the following equation:

$$\begin{aligned} \delta V &= \max_{w_1, C} \min_{h} \Big\{ U(C) + \theta_2 \frac{h^2}{2} + V_W [w_1(\alpha_1 + \sigma_1 h - r)W + (rW - c)] \quad (12) \\ &+ V_{\alpha_1} \alpha_1 [f_1 + g_1 h(\rho + \sqrt{1-\rho^2})] + \alpha_1 g_1 \rho W \sigma_1 w_1 V_{\alpha_1 W} + \\ & \frac{1}{2} V_{WW} \sigma_1^2 w_1^2 W^2 + \frac{1}{2} V_{\alpha_1 \alpha_1} \alpha_1^2 g_1^2 \Big\}. \end{aligned}$$

The first order conditions which describe the solution of the above two player-game are:

$$U'(C) = V_W, \quad (13)$$

$$h = -\frac{V_W W w_1 \sigma_1 + V_{\alpha_1} \alpha_1 g_1 (\rho + \sqrt{1-\rho^2})}{\theta_2}, \quad (14)$$

$$0 = V_W w_1 (\alpha_1 + \sigma_1 h - r) + V_{WW} \sigma_1^2 W w_1 + \alpha_1 g_1 \rho \sigma_1 V_{\alpha_1 W} \quad (15)$$

¿From the above system of equations it can be seen that as $\theta \to \infty$ the solution reduces to the solution of the standard Merton's problem given by (3). Using (14) to elminate h from (15) we obtain the robust portfolio weight, or equivalently the fraction of the wealth invested on the risky asset as:

$$w_1^* W \left(1 - \frac{V_W^2}{\theta_2 V_{WW}}\right) = \frac{A(\alpha_1 - r)}{\sigma_1^2} + H_1 \frac{g_1 \rho \alpha_1}{\sigma_1} + \frac{V_W V_{\alpha_1}}{\theta_2 V_{WW}} \frac{g_1 \alpha_1}{\sigma_1} (\rho + \sqrt{1-\rho^2}) \quad (16)$$

In order to determine the change in portfolio weights induced by uncertainty aversion relative to the risk aversion weights, we compare the relationships (3) and (16). It follows from the comparison that the term into the brackets at the left hand side of the above equation is always a number greater than one. Furthermore, the first two terms at the right hand side are exactly the same as the two corresponding terms appearing it equation (3). Thereffore when $\begin{array}{l} V_{\alpha_1} > 0 \quad if \quad \rho + \sqrt{1-\rho^2} < 0 \\ V_{\alpha_1} = 0 \quad if \quad \rho + \sqrt{1-\rho^2} = 0 \\ V_{\alpha_1}, < 0 \quad if \quad \rho + \sqrt{1-\rho^2} > 0 \end{array}$ then $w_1^* W$ is always less than $w_1 W$ and therefore an uncertainty averse investor always reduces the holdings of risky asset relative to the risk aversion case.

Assume now that the mean rate of return is constant and consider the case where the investor-consumer is uncertainty averse due to the stochastic evolution of the volatility rate, or

$$\frac{d\sigma_1}{\sigma_1} = f_2 dt + g_2 dZ_2, \quad (17)$$

where f_2, g_2 are constants and Z_2 is a Brownian process. If by ρ we denote again the corelation coefficient between dZ_2, dB_1, then following the previous approach the following Proposition can be stated

In a market with one risky and one riskless asset an uncertainty averse investor with respect either to the stochastic evolution of the mean rate of return, or to the evolution of the volatility rate, always reduces the total holdings of the risky asset relative to a risk averse investor.

$$\begin{array}{llll} & \rho+\sqrt{1-\rho^2}<0 & & V_{\alpha_1}, V_{\sigma_1}>0 \\ if & \rho+\sqrt{1-\rho^2}=0 & when & V_{\alpha_1}, V_{\sigma_1}=0 \quad respectively. \\ & \rho+\sqrt{1-\rho^2}>0 & & V_{\alpha_1}, V_{\sigma_1}<0 \end{array}$$ [13]

2.2 The case of stochastic mean rate of return and stochastic volatility rate

In this section we examine the case where the investor is uncertainty averse with respect to both, the stochastic evolution of the volatility rate, and the mean rate of return. If

$$System=\begin{cases} \rho_2+\tau_1+\tau_2 \gtreqless 0 \ \ when\ V_{\alpha_1} \gtreqless 0\ respectively \\ \rho_1+\sqrt{1-\rho_1^2} \gtreqless 0\ when\ V_{\sigma_1} \gtreqless 0\ respectively \\ (1-\rho_2^2)(1-\rho_1^2)-(\rho_{3-}\rho_1\rho_2)^2>0 \\ 1 \leq \rho_2 \leq 1 \\ -1 \leq \rho_3 \leq 1 \end{cases} \tag{18}$$

then the following proposition can be stated.

If the above System (18), of inequalities is satisfied an uncertainty averse investor, always reduces the holdings of the risky asset relative to the risk aversion case.

For the proof see Appendix.

3 Robust Portfolio Choices With Two Risky Assets

Suppose now that the market consists of two risky and one risk free asset. Equation (1) along with:

$$\frac{dP_2}{P_2}=\alpha_2 dt+\sigma_2 dB_2, \tag{19}$$

describe the evolution of the two risky assets, while (2) refers to the mean rate of return of the first asset which we assume that evolves stochastically over time. All the parameters of the above relationships are assumed to be constants and it is furthermore assumed that the three Brownian motions are corelated, with ρ_1, ρ_2, ρ_3 the corelation coefficients between $(dB_1, dB_2), (dB_1, dZ_1)$,

[13] If this condition is not satisfied we are not able to determine the direction of change in the assets' holdings.

and (dB_2, dZ_1) respectively. Merton's solution for the maximization problem (10) in the two risky assets case is:

$$\begin{aligned} w_1 W &= \frac{A(\alpha_1 - r)\sigma_2^2}{\sigma_1^2\sigma_2^2(1-\rho_1^2)} - \frac{A(\alpha_2 - r)\sigma_{12}}{\sigma_1^2\sigma_2^2(1-\rho_1^2)} + \frac{H_1\alpha_1 g_1\sigma_1\sigma_2^2\,(\rho_2 - \rho_1\rho_3)}{\sigma_1^2\sigma_2^2(1-\rho_1^2)}, \quad (20)\\ w_2 W &= -\frac{A(\alpha_1 - r)\sigma_{12}}{\sigma_1^2\sigma_2^2(1-\rho_1^2)} + \frac{A(\alpha_2 - r)\sigma_2^2}{\sigma_1^2\sigma_2^2(1-\rho_1^2)} + \frac{H_1\alpha_1 g_1\sigma_1^2\sigma_2\,(-\rho_2\rho_1 + \rho_3)}{\sigma_1^2\sigma_2^2(1-\rho_1^2)} \quad (21)\\ A &= -\frac{V_W}{V_{WW}},\ H_1 = -\frac{V_{\alpha_1 w}}{V_{ww}}. \end{aligned}$$

In the following we consider the optimal robust portfolio allocation problem, for the maximization of the lifetime expected utility from consumption and we derive condition under which the holdings invested in the risky assets increase relative to the risk averson case.[14] For this specific case the equations for wealth dynamics and α_1 can be written as:

$$\begin{aligned} \begin{bmatrix} dW(t) \\ d\alpha_1(t) \end{bmatrix} &= \begin{bmatrix} w_1(\alpha_1 - r) + w_2(\alpha_2 - r) + (rW - c) \\ \alpha_1 f_1 \end{bmatrix} dt + \quad (22)\\ & \begin{bmatrix} W\sigma_1 w_1 + W\sigma_2 w_2\rho_1 & W\sigma_2 w_2\sqrt{1-\rho_1^2} & 0 \\ \alpha_1 g_1 \rho_2 & \alpha_1 g_1 \tau_1 & \alpha_1 g_1 \tau_2 \end{bmatrix} \begin{bmatrix} d\widetilde{B}_1 \\ d\widetilde{B}_2 \\ d\widetilde{B}_3 \end{bmatrix} \end{aligned}$$

where $\widetilde{B}_i$ $i = 1, 2, 3$ denote three independent Brownian motions. Considering again distortions to the joint distribution of the two assets and the drift rate, and imposing an overall entropy constraint for them, the above equation becomes:

$$\begin{bmatrix} dW(t) \\ d\alpha_1(t) \end{bmatrix} = \begin{bmatrix} F_1 \\ F_2 \end{bmatrix} dt + G \begin{bmatrix} d\hat{B}_1 \\ d\hat{B}_2 \\ d\hat{B}_3 \end{bmatrix} \quad (23)$$

where

$$\begin{aligned} F_1 &= w_1(\alpha_1 - r + \sigma_1 h) + (rW - c) \\ &\quad + w_2\left(\alpha_2 - r + \sigma_2 h\left(\rho_1 + \sqrt{1-\rho_1^2}\right)\right) \\ F_2 &= \alpha_1 f_1 + h\alpha_1 g_1\,(\rho_2 + \tau_1 + \tau_2) \\ G &= \begin{bmatrix} W\sigma_1 w_1 + W\sigma_2 w_2\rho_1 & W\sigma_2 w_2\sqrt{1-\rho_1^2} & 0 \\ \alpha_1 g_1 \rho_2 & \alpha_1 g_1 \tau_1 & \alpha_1 g_1 \tau_2 \end{bmatrix} \end{aligned}$$

The Bellman-Isaacs conditions for this game imply that the value function $V(W, \alpha_1, \theta)$ satisfies the following equation:

$$\delta V = \max_{w_i, C} \min_{h} \Big\{ U(C) + \theta_3 \frac{h^2}{2} + V_W F_1 + V_{\alpha_1} F_2 + \frac{1}{2} trace(G^T \partial^2 V G) \Big\}. \quad (24)$$

[14] We will provide the proof only for the stochastic mean rate. Similar conditions can be derived for the stochastic volatility case

where now $\partial^2 V = \begin{bmatrix} V_{WW} & V_{W\alpha_1} \\ V_{W\alpha_1} & V_{\alpha_1\alpha_1} \end{bmatrix}$. The first order conditions for the above two players game are:

$$\begin{aligned}
U'(C) &= V_W, \\
h &= \frac{-V_W W\big(\sigma_1 w_1^* - \sigma_2(\rho + \sqrt{1-\rho^2})w_2^*\big) - \alpha_1 g_1 q V_{\alpha_1}}{\theta_3} \\
q &= \rho_2 + \tau_1 + \tau_2 \\
\sum_{j=1}^{2} w_j^* W \sigma_{1j} &= A(\alpha_1 - r) + A\sigma_1 h + H_1 \alpha_1 g_1 \rho_2 \sigma_1 \\
\sum_{j=1}^{2} w_j^* W \sigma_{2j} &= A(\alpha_2 - r) + A\sigma_2(\rho_1 + \sqrt{1-\rho_1^2})h + H_1 \alpha_1 g_1 \rho_3 \sigma_2. \\
A &= -\frac{V_W}{V_{WW}}, \quad H_1 = -\frac{V_{\alpha_1 w}}{V_{ww}}
\end{aligned}$$

Using matrix notation the solution of the above problem can be described by the following equation:

$$\begin{bmatrix} w_1^* W & w_2^* W \end{bmatrix} \Lambda = \begin{bmatrix} A(\alpha_1 - r) & A(\alpha_2 - r) \end{bmatrix} + \begin{bmatrix} H_1 \alpha_1 g_1 \rho_2 \sigma_1 & H_1 \alpha_1 g_1 \rho_3 \sigma_2 \end{bmatrix}$$

$$+ \begin{bmatrix} -A\sigma_1 \frac{\alpha_1 g_1(\rho_2+\tau_1+\tau_2)V_{\alpha_1}}{\theta_3} & -A\sigma_2(\rho_1 + \sqrt{1-\rho_1^2})\frac{\alpha_1 g_1(\rho_2+\tau_1+\tau_2)V_{\alpha_1}}{\theta_3} \end{bmatrix}$$

where:

$$\Lambda = \begin{bmatrix} \sigma_{11}(1 - \frac{V_W^2}{\theta_3 V_{WW}}) & \sigma_{12}\big(1 - \frac{V_W^2}{\theta_3 V_{WW}} \frac{\rho_1 + \sqrt{1-\rho_1^2}}{\rho}\big) \\ \sigma_{12}\big(1 - \frac{V_W^2}{\theta_3 V_{WW}} \frac{\rho_1 + \sqrt{1-\rho_1^2}}{\rho_1}\big) & \sigma_{22}\big(1 - \frac{V_W^2}{\theta_3 V_{WW}}(1 + 2\rho_1\sqrt{1-\rho_1^2})\big) \end{bmatrix}. \tag{25}$$

If Σ denotes the diagonal matrix with elements σ_1, σ_2 then:

$$\Lambda = \Sigma \begin{bmatrix} (1 - \frac{V_W^2}{\theta_2 V_{WW}}) & \big(\rho - \frac{V_W^2}{\theta_3 V_{WW}}(\rho_1 + \sqrt{1-\rho_1^2})\big) \\ \big(\rho_1 - \frac{V_W^2}{\theta_3 V_{WW}}(\rho_1 + \sqrt{1-\rho_1^2})\big) & \big(1 - \frac{V_W^2}{\theta_3 V_{WW}}(1 + 2\rho_1\sqrt{1-\rho_1^2})\big) \end{bmatrix} \Sigma. \tag{26}$$

Solving the above system we determine the fraction of the wealth invested on the first and second asset under robust portfolio choices as:

$$\begin{bmatrix} w_1^* W & w_2^* W \end{bmatrix} = \frac{1}{(1-\rho_1^2)\big(1 - 2\frac{V_W^2}{\theta_3 V_{WW}}\big)} M \Sigma^{-1} \tag{27}$$

$$\begin{bmatrix} 1 - \frac{V_W^2}{\theta_3 V_{WW}}(1 + 2\rho_1\sqrt{1-\rho_1^2}) & -\rho_1 + \frac{V_W^2}{\theta_3 V_{WW}}(\rho_1 + \sqrt{1-\rho_1^2}) \\ -\rho_1 + \frac{V_W^2}{\theta_3 V_{WW}}(\rho_1 + \sqrt{1-\rho_1^2}) & 1 - \frac{V_W^2}{\theta_3 V_{WW}} \end{bmatrix} \Sigma^{-1} \tag{28}$$

where M is the matrix:

$$
\begin{aligned}
M &= \left[\begin{array}{cc} A(\alpha_1 - r) & A(\alpha_2 - r) \end{array}\right] + \left[\begin{array}{cc} H_1\alpha_1 g_1\rho_2\sigma_1 & H_1\alpha_1 g_1\rho_3\sigma_2 \end{array}\right] + \\
&\left[\begin{array}{cc} -A\sigma_1\frac{\alpha_1 g_1(\rho_2+\tau_1+\tau_2)V_{\alpha_1}}{\theta_3} & -A\sigma_2(\rho_1+\sqrt{1-\rho_1^2})\frac{\alpha_1 g_1(\rho_2+\tau_1+\tau_2)V_{\alpha_1}}{\theta_3} \end{array}\right]
\end{aligned}
$$

Next we examine, as in the previous section, the changes in the robust portfolio weights $\Delta w_i = w_i - w_i^*,\ \ i = 1, 2$ between the risk aversion case ($\theta \to \infty$) and the uncertainty aversion case ($\theta < \infty$) .Using (20),(21),(27) we obtain:[15]

$$
\left[\begin{array}{cc} W\Delta w_1 & W\Delta w_2 \end{array}\right] = \frac{1}{(1-\rho_1^2)} M_1 \Sigma^{-1}\Xi\Sigma^{-1} - TM_2, \tag{29}
$$

where:

$$
\begin{aligned}
\Xi &= \left[\begin{array}{cc} 1 & -\rho_1 \\ -\rho_1 & 1 \end{array}\right] - \frac{1}{\left(1-2\frac{V_W^2}{\theta_3 V_{WW}}\right)} \qquad (30) \\
&\left[\begin{array}{cc} 1-\frac{V_W^2}{\theta_3 V_{WW}}(1+2\rho_1\sqrt{1-\rho_1^2}) & -\rho_1+\frac{V_W^2}{\theta_3 V_{WW}}(\rho_1+\sqrt{1-\rho_1^2}) \\ -\rho_1+\frac{V_W^2}{\theta_3 V_{WW}}(\rho_1+\sqrt{1-\rho_1^2}) & 1-\frac{V_W^2}{\theta_3 V_{WW}} \end{array}\right]
\end{aligned}
$$

$$
\begin{aligned}
M_1 &= \left[\begin{array}{cc} A(\alpha_1 - r) & A(\alpha_2 - r) \end{array}\right] + \left[\begin{array}{cc} H_1\alpha_1 g_1\rho_2\sigma_1 & H_1\alpha_1 g_1\rho_3\sigma_2 \end{array}\right] \\
M_2 &= [M_{21}\ M_{22}] \\
M_{21} &= -A\sigma_1\frac{\alpha_1 g_1(\rho_2+\tau_1+\tau_2)V_{\alpha_1}}{\theta_3} \qquad . \\
M_{22} &= -A\sigma_2(\rho_1+\sqrt{1-\rho_1^2})\frac{\alpha_1 g_1\left(\rho_2+\tau_1+\tau_2\right)V_{\alpha_1}}{\theta_3}
\end{aligned}
$$

and

$$
\begin{aligned}
T &= \frac{1}{(1-\rho_1^2)\left(1-2\frac{V_W^2}{\theta_3 V_{WW}}\right)}\Sigma^{-1}Q\Sigma^{-1} \\
Q &= \left[\begin{array}{cc} 1-\frac{V_W^2}{\theta_3 V_{WW}}(1+2\rho_1\sqrt{1-\rho_1^2}) & -\rho_1+\frac{V_W^2}{\theta_3 V_{WW}}(\rho_1+\sqrt{1-\rho_1^2}) \\ -\rho_1+\frac{V_W^2}{\theta_3 V_{WW}}(\rho_1+\sqrt{1-\rho_1^2}) & 1-\frac{V_W^2}{\theta_3 V_{WW}} \end{array}\right]
\end{aligned}
$$

After some manipulations we obtain:

$$
\begin{aligned}
&\left[\begin{array}{cc} W\Delta w_1 & W\Delta w_2 \end{array}\right] = -TM_2 + \qquad (31) \\
&\frac{AV_W^2}{(1-\rho_1^2)(\theta_3 V_{WW}-V_W^2)} M_1\Sigma^{-1}\left[\begin{array}{cc} 2\rho_1\sqrt{1-\rho_1^2}-1 & \rho_1-\sqrt{1-\rho_1^2} \\ \rho_1-\sqrt{1-\rho_1^2} & -1 \end{array}\right]\Sigma^{-1}.
\end{aligned}
$$

[15] For infinitesimal changes in θ, this is basically a comparative statics exercise that characterizes the derivative $\partial w_i^*/\partial\theta$.

From the above equation it can be shown that the changes in the robust portfolio weights are affected by two componets. One which is related to M_1 and one wich is related to M_2. If the effect of each one of them induces an increase of total holdings invested on each risky asset then we will have that the portfolio weights will be increased relative to risk aversion case. Matrix M_1 includes two sub matrices, where the first one appears also in the case where the mean rate of return of the price process is not stochastic. (see [32]). If by $(\cdot)_1$ we denote the change of the invested wealth due to the M_1 component then:

$$\begin{aligned}
(W\Delta w_1)_1 &= \frac{\kappa}{\sigma_1}\left[\frac{\alpha_1 - r}{\sigma_1}(2\rho_1\sqrt{1-\rho_1^2} - 1) + \frac{\alpha_2 - r}{\sigma_2}(\rho_1 - \sqrt{1-\rho_1^2})\right] && (32)\\
(W\Delta w_2)_1 &= \frac{\kappa}{\sigma_2}\left[\frac{\alpha_1 - r}{\sigma_1}(\rho_1 - \sqrt{1-\rho_1^2}) - \frac{\alpha_2 - r}{\sigma_2}\right] && (33)\\
\kappa &= \frac{AV_W^2}{(1-\rho_1^2)(\theta_3 V_{WW} - V_W^2)}. && (34)
\end{aligned}$$

In the above equation κ is always a negative number, so by setting:

$$\lambda = \frac{\frac{\alpha_2 - r}{\sigma_2}}{\frac{\alpha_1 - r}{\sigma_1}}, \tag{35}$$

we obtain

$$\begin{aligned}
(W\Delta w_1)_1 &< 0 \quad \text{if} \quad \frac{\rho_1 - \sqrt{1-\rho_1^2}}{1 - 2\rho_1\sqrt{1-\rho_1^2}} > \frac{1}{\lambda}, && (36)\\
(W\Delta w_2)_1 &< 0 \quad \text{if} \quad \rho_1 - \sqrt{1-\rho_1^2} > \lambda. && (37)
\end{aligned}$$

Similar if $(.)_2$ refers to the change in the invested wealth in each asset, associated with the second submatrix of the M_1 component, that is
$\begin{bmatrix} H_1\alpha_1 g_1\rho_2\sigma_1 & H_1\alpha_1 g_1\rho_3\sigma_2 \end{bmatrix}$ then :

$$\begin{aligned}
(W\Delta w_1)_2 &= \frac{\widetilde{\kappa}}{\sigma_1}\left[\rho_2(2\rho_1\sqrt{1-\rho_1^2} - 1) + \rho_3(\rho_1 - \sqrt{1-\rho_1^2})\right] && (38)\\
(W\Delta w_2)_2 &= \frac{\widetilde{\kappa}}{\sigma_2}\left[\rho_2(\rho_1 - \sqrt{1-\rho_1^2}) - \rho_3\right] && (39)\\
\widetilde{\kappa} &= \frac{H_1\alpha_1 g_1 V_W^2}{(1-\rho_1^2)(\theta_3 V_{WW} - V_W^2)} \gtrless 0 \text{ if } H_1 \lessgtr 0.
\end{aligned}$$

So

$$\begin{aligned}
(W\Delta w_1)_2 &< 0 \text{ if } \rho_2(2\rho_1\sqrt{1-\rho_1^2} - 1) + \rho_3(\rho_1 - \sqrt{1-\rho_1^2}) \lessgtr 0, && (40)\\
(W\Delta w_2)_2 &< 0 \text{ if } \rho_2(\rho_1 - \sqrt{1-\rho_1^2}) - \rho_3 \lessgtr 0. && (41)
\end{aligned}$$

Furthermore if by $(\cdot)_3$ we denote similar effects due to the M_2 component we obtain, for $V_{\alpha_1} > 0$, that:

$$
\begin{aligned}
(W\Delta w_1)_3 &< 0 \text{ if } \quad M_{21} \gtrless 0 \Leftrightarrow \rho_2 + \tau_1 + \tau_2 \gtrless 0 && (42)\\
(W\Delta w_2)_3 &< 0 \text{ if } \quad M_{22} \gtrless 0 \Leftrightarrow (\rho_1 + \sqrt{1-\rho_1^2})\,(\rho_2 + \tau_1 + \tau_2) \gtrless 0 && (43)
\end{aligned}
$$

If we combine (32) – (33), we obtain that:

$$
\begin{aligned}
(W\Delta W)_1 = (W\Delta w_1)_1 + (W\Delta w_2)_1 = && (44)\\
\kappa\frac{\alpha_1 - r}{\sigma_1}\frac{1}{\sigma_2}\big[(2\rho_1\sqrt{1-\rho_1^2} - 1)\sigma + \lambda\sigma(\rho_1 - \sqrt{1-\rho_1^2}) + && (45)\\
(\rho_1 - \sqrt{1-\rho_1^2}) - \lambda\big] < 0 \quad if && (46)\\
(\lambda\sigma + 1)(\rho_1 - \sqrt{1-\rho_1^2}) > \lambda + \sigma(1 - 2\rho_1\sqrt{1-\rho_1^2}) \qquad or &&\\
\hat{\lambda}(\rho_1 - \sqrt{1-\rho_1^2} - \frac{1}{\sigma}) > \sigma(1 - 2\rho_1\sqrt{1-\rho_1^2}) - (\rho_1 - \sqrt{1-\rho_1^2}) && (47)\\
\text{with } \hat{\lambda} = \frac{a_2 - r}{a_1 - r},\ \sigma = \frac{\sigma_2}{\sigma_1} &&
\end{aligned}
$$

Combining (38) – (39) for the corresponding term $(\cdot)_2$, we obtain that:

$$
\begin{aligned}
(W\Delta W)_2 &= (W\Delta w_1)_2 + (W\Delta w_2)_2 && (48)\\
&= \frac{\widetilde{\kappa}}{\sigma_1}\big[\sigma\left(\rho_2(2\rho_1\sqrt{1-\rho_1^2} - 1) + \rho_3(\rho_1 - \sqrt{1-\rho_1^2})\right) &&\\
&\quad +\rho_2(\rho_1 - \sqrt{1-\rho_1^2}) - \rho_3\big] &&\\
&< 0 \quad if \quad P \lessgtr 0 \text{ where} &&\\
P &= \big[\sigma\left(\rho_2(2\rho_1\sqrt{1-\rho_1^2} - 1) + \rho_3(\rho_1 - \sqrt{1-\rho_1^2})\right) && (49)\\
&\quad +\rho_2(\rho_1 - \sqrt{1-\rho_1^2}) - \rho_3\big] &&
\end{aligned}
$$

Therefore for admissible values of ρ_i , $i = 1, 2, 3$ and independent the value of the robustness parameter θ, we can state the following proposition:[16]

Assuming that the mean rate of return or the volatility rate evolves stochastically over time, robust portfolio choices under uncertainty aversion imply for a market consisting of one riskless and two risky assets the following:

1. *If (36), (40), (42) hold at the same time there is an increase in the holdings invested in the first risky asset, relative to risk aversion, or* $\Delta w_1 = (W\Delta w_1)_1 + (W\Delta w_1)_2 + (W\Delta w_1)_3 < 0$.

[16] Using the Mathematica software package we are able to verify that there exist values of ρ_i satisfying the sufficient conditions provided in the following proposition.

2. *If (37), (41), (43) hold at the same time there is an increase in the holdings invested in the second risky asset, relative to risk aversion, or* $\Delta w_2 = (W\Delta w_2)_1 + (W\Delta w_2)_2 + (W\Delta w_2)_3 < 0$.

3. If $(42)-(43)$ and $(47), (48)$ hold at the same time then *there is an increase in the total holdings invested in the risky assets, relative to risk aversion, or* $\Delta W < 0$.

4. *When concerns about model misspecification do not exist, or* $\theta \to \infty$, *then the difference in portfolio choices between uncertainty aversion and risk aversion vanishes* $\Delta W = \Delta w_1 + \Delta w_2 \to 0$, $(\Delta w_1, \Delta w_2) \to 0$

4 Robust Portfolio Rules and Differences in Ambiguity

We consider now a similar problem with two risky assets, but we assume that the first one is a foreign asset for which concerns about misspecification of its price processes exist, while the second one is a home asset for which the investor believes that she/he knows the true evolution of its price process over time through the benchmark model. Thus, both assets are risky but there is uncertainty (or ambiguity) aversion regarding the price processes of the foreign asset. The investor is risk averse, in the standard way, regarding the price process of the home asset.

In this case the derivation of robust portfolio desicion rules requires distorting only the Brownian motions which are related to the evolution of the first asset and of its mean rate of return α_1. Relationship (22) also holds in this case and the respective componets of the equation (23) become*:*

$$\begin{aligned} F_1 &= w_1(\alpha_1 - r + \sigma_1 h) + (rW - c) + w_2\left(\alpha_2 - r + \sigma_2 h\rho_1 + \sigma_2\sqrt{1-\rho_1^2}\right) \\ F_2 &= \alpha_1 f_1 + \tau_1\alpha_1 g_1 + h\alpha_1 g_1(\rho_2 + \tau_2) \qquad (50) \end{aligned}$$

while G remains the same. The Bellman-Isaacs conditions for this game implies that the value function $V(W, \alpha_1, \theta)$ satisfies again equation (24) where the first order conditions for the above two players game are:

$$\begin{aligned} U'(C) &= V_W, \qquad (51) \\ h &= -\frac{V_W W(\sigma_1 w_1^* + \sigma_2\rho_1 w_2^*) + (\alpha_1 g_1(\rho_2 + \tau_2)V_{\alpha_1})}{\theta_3} \\ \sum_{j=1}^{2} w_j^* W\sigma_{1j} &= A(\alpha_1 - r) + A\sigma_1 h + H_1\alpha_1 g_1\rho_2\sigma_1 \\ \sum_{j=1}^{2} w_j^* W\sigma_{2j} &= A(\alpha_2 - r) + A\sigma_2(\rho_1 h + \sqrt{1-\rho_1^2}) + H_1\alpha_1 g_1\rho_3\sigma_2. \end{aligned}$$

$$A = -\frac{V_W}{V_{WW}}, \quad H_1 = -\frac{V_{\alpha_1 w}}{V_{ww}}$$

Using matrix notation the solution of the above problem can be described by the following equation:

$$\left[\begin{array}{cc} w_1^* W & w_2^* W \end{array}\right] \Lambda = \left[\begin{array}{cc} A(\alpha_1 - r) & A(\alpha_2 - r + \sigma_2\sqrt{1-\rho_1^2}) \end{array}\right] +$$

$$\left[\begin{array}{cc} H_1\alpha_1 g_1\rho_2\sigma_1 & H_1\alpha_1 g_1\rho_3\sigma_2 \end{array}\right] + \left[\begin{array}{cc} -A\sigma_1\frac{\alpha_1 g_1(\rho_2+\tau_2)V_{\alpha_1}}{\theta_3} & -A\sigma_2\rho_1\frac{\alpha_1 g_1(\rho_2+\tau_2)V_{\alpha_1}}{\theta_3} \end{array}\right]$$

where

$$\Lambda = \left[\begin{array}{cc} \sigma_{11}(1 - \frac{V_W^2}{\theta_3 V_{WW}}) & \sigma_{12}(1 - \frac{V_W^2}{\theta_3 V_{WW}}) \\ \sigma_{12}(1 - \frac{V_W^2}{\theta_3 V_{WW}}) & \sigma_{22}\big(1 - \frac{V_W^2}{\theta_3 V_{WW}}\rho_1^2\big) \end{array}\right].$$

If Σ denotes the diagonal matrix with elements σ_1, σ_2 then:

$$\Lambda = \Sigma \left[\begin{array}{cc} (1 - \frac{V_W^2}{\theta_3 V_{WW}}) & (1 - \frac{V_W^2}{\theta_3 V_{WW}}) \\ (1 - \frac{V_W^2}{\theta_3 V_{WW}}) & \big(1 - \frac{V_W^2}{\theta_3 V_{WW}}\rho_1^2\big) \end{array}\right] \Sigma.$$

Solving the above system we determine the fraction of the wealth invested on the first and second risky asset under robust portfolio choices as:

$$\left[\begin{array}{cc} w_1^* W & w_2^* W \end{array}\right] = \frac{1}{(1-\rho_1^2)\big(1 - \frac{V_W^2}{\theta_3 V_{WW}}\big)\frac{V_W^2}{\theta_3 V_{WW}}} M\Sigma^{-1} \tag{52}$$

$$\left[\begin{array}{cc} \big(1 - \frac{V_W^2}{\theta_3 V_{WW}}\rho_1^2\big) & -(1 - \frac{V_W^2}{\theta_3 V_{WW}}) \\ -(1 - \frac{V_W^2}{\theta_3 V_{WW}}) & (1 - \frac{V_W^2}{\theta_3 V_{WW}}) \end{array}\right] \Sigma^{-1}.$$

where now M is the matrix:

$$M = \left[\begin{array}{cc} A(\alpha_1 - r) & A(\alpha_2 - r + \sigma_2\sqrt{1-\rho_1^2}) \end{array}\right] + \left[\begin{array}{cc} H_1\alpha_1 g_1\rho_2\sigma_1 & H_1\alpha_1 g_1\rho_3\sigma_2 \end{array}\right]$$

$$+ \left[\begin{array}{cc} -A\sigma_1\frac{\alpha_1 g_1(\rho_2+\tau_2)V_{\alpha_1}}{\theta_3} & -A\sigma_2\rho_1\frac{\alpha_1 g_1(\rho_2+\tau_2)V_{\alpha_1}}{\theta_3} \end{array}\right]$$

Next we examine the changes in the robust portfolio weights $\Delta w_i = w_i - w_i^*, \; i = 1, 2$ between risk aversion $(\theta \to \infty)$ and uncertainty aversion $(\theta < \infty)$.

Using, (20),(21),(52) we obtain:

$$\left[\begin{array}{cc} W\Delta w_1 & W\Delta w_2 \end{array}\right] = \frac{1}{(1-\rho_1^2)} M_1\Sigma^{-1}\Xi\Sigma^{-1} - TM_2, \tag{53}$$

where in this specific case:

$$\Xi = \left[\begin{array}{cc} 1 & -\rho_1 \\ -\rho_1 & 1 \end{array}\right] - \frac{1}{\big(1 - \frac{V_W^2}{\theta_3 V_{WW}}\big)\frac{V_W^2}{\theta_3 V_{WW}}} \left[\begin{array}{cc} \big(1 - \frac{V_W^2}{\theta_3 V_{WW}}\rho_1^2\big) & -(1 - \frac{V_W^2}{\theta_3 V_{WW}}) \\ -(1 - \frac{V_W^2}{\theta_3 V_{WW}}) & (1 - \frac{V_W^2}{\theta_3 V_{WW}}) \end{array}\right]$$

$$
\begin{aligned}
M_1 &= \left[\begin{array}{cc} A(\alpha_1 - r) & A(\alpha_2 - r + \sigma_2\sqrt{1-\rho_1^2}) \end{array}\right] + \\
&\quad \left[\begin{array}{cc} H_1\alpha_1 g_1 \rho_2 \sigma_1 & H_1 \alpha_1 g_1 \rho_3 \sigma_2 \end{array}\right] \\
M_2 &= [M_{21}\ M_{22}] \\
M_{21} &= -A\sigma_1 \frac{\alpha_1 g_1 (\rho_2 + \tau_2) V_{\alpha_1}}{\theta_3}, M_{22} = -A\sigma_2\rho_1 \frac{\alpha_1 g_1 (\rho_2 + \tau_2) V_{\alpha_1}}{\theta_3}
\end{aligned}
$$

and

$$
\begin{aligned}
T &= \frac{1}{(1-\rho_1^2)\left(1 - \frac{V_W^2}{\theta_3 V_{WW}}\right)\frac{V_W^2}{\theta_3 V_{WW}}} \Sigma^{-1} Q \Sigma^{-1} \\
Q &= \left[\begin{array}{cc} \left(1 - \frac{V_W^2}{\theta_3 V_{WW}}\rho_1^2\right) & -(1 - \frac{V_W^2}{\theta_3 V_{WW}}) \\ -(1 - \frac{V_W^2}{\theta_3 V_{WW}}) & (1 - \frac{V_W^2}{\theta_3 V_{WW}}) \end{array}\right]
\end{aligned}
$$

After some manipulations we obtain:

$$
\left[\begin{array}{cc} W\Delta w_1 & W\Delta w_2 \end{array}\right] = -TM_2 + \frac{1}{(1-\rho_1^2)} M_1 \Sigma^{-1} \left[\begin{array}{cc} \xi & x - \rho_1 \\ x - \rho_1 & 1 - x \end{array}\right] \Sigma^{-1}. \tag{54}
$$

$$
where\ \ \xi = 1 + \frac{x\left(\rho_1^2 - x\right)}{x-1} > 0, x = \frac{\theta_3 V_{WW}}{V_W^2}.
$$

From the above equation we can see that the changes in the robust portfolio weights are affected by two componets M_1 and M_2, where matrix M_1 includes two submatrices matrices. Working as in the previous section we obtain:

$$
(W\Delta w_1)_1 = \frac{\kappa}{\sigma_1}\left[\frac{\alpha_1 - r}{\sigma_1}\xi + \frac{\alpha_2 - r + \sigma_2\sqrt{1-\rho_1^2}}{\sigma_2}(x - \rho_1)\right] \tag{55}
$$

$$
(W\Delta w_2)_1 = \frac{\kappa}{\sigma_2}\left[\frac{\alpha_1 - r}{\sigma_1}(x - \rho_1) - \frac{\alpha_2 - r + \sigma_2\sqrt{1-\rho_1^2}}{\sigma_2}(1 - x)\right] \tag{56}
$$

$$
where\ now\ \kappa = \frac{1}{(1-\rho_1^2)}.
$$

In the above equations κ is always a positive number, so by setting:

$$
\widehat{\lambda} = \frac{\frac{\alpha_2 - r + \sigma_2\sqrt{1-\rho_1^2}}{\sigma_2}}{\frac{\alpha_1 - r}{\sigma_1}}, \tag{57}
$$

we obtain

$$
(W\Delta w_1)_1 < 0 \quad \text{if} \quad \xi < \widehat{\lambda}(x - \rho_1) \tag{58}
$$

$$
(W\Delta w_2)_1 < 0 \quad \text{if} \quad \frac{x - \rho_1}{1 - x} < \widehat{\lambda}. \tag{59}
$$

The above equations indicate that the effect is to always reduce the holdings of the second asset *if* $\rho_1 > 0$ and $\widehat{\lambda} > 0$, or otherwise when $\widehat{\lambda} > 1$.

Considering the case where $V_{\alpha_1} > 0, H_1 < 0$, then if $(.)_2$ refers to the term related to the matrix
$\begin{bmatrix} H_1\alpha_1 g_1\rho_2\sigma_1 & H_1\alpha_1 g_1\rho_3\sigma_2 \end{bmatrix}$, we have:

$$
\begin{aligned}
(W\Delta w_1)_2 &= \frac{\widetilde{\kappa}}{\sigma_1}\Big[\rho_2\xi + \rho_3(x-\rho_1)\Big] &(60)\\
(W\Delta w_2)_2 &= \frac{\widetilde{\kappa}}{\sigma_2}\Big[\rho_2(x-\rho_1) + \rho_3(1-x)\Big] &(61)\\
\textit{where now } \widetilde{\kappa} &= \frac{H_1\alpha_1 g_1}{(1-\rho_1^2)} < 0
\end{aligned}
$$

So

$$
\begin{aligned}
(W\Delta w_1)_2 &< 0 \text{ if } \quad \rho_2 > 0, \rho_1 > 0, \rho_3 < 0. &(62)\\
(W\Delta w_2)_2 &< 0 \text{ if } \quad 0 < \rho_2 < \rho_3,\ \rho_1 > 0 \text{ or}\\
\rho_1 &> 0,\ \rho_2 < 0\ ,\ \rho_3 > 0. &(63)
\end{aligned}
$$

Finally if $(.)_3$ refers to the term $-TM_2$ related to the matrix $\begin{bmatrix} M_{21} & M_{22} \end{bmatrix}$ then:

$$
\begin{aligned}
(W\Delta w_1)_3 &= \frac{\widetilde{\kappa'}}{\sigma_1}\Big[(\rho_2+\tau_2)\frac{(\rho_1^2-x)\,x}{1-x} - \rho_1(\rho_2+\tau_2)\,x\Big]\\
(W\Delta w_2)_3 &= \frac{\widetilde{\kappa'}}{\sigma_2}\Big[-(\rho_2+\tau_2)\,x + \rho_1(\rho_2+\tau_2)\,x\Big]\\
\textit{where } \widetilde{\kappa'} &= -\frac{-A\frac{\alpha_1 g_1 V_{\alpha_1}}{\theta_3}}{(1-\rho_1^2)} > 0
\end{aligned}
$$

Therefore

$$
\begin{aligned}
(W\Delta w_1)_3 &< 0 \text{ if } \quad \rho_1 < 0,\ (\rho_2+\tau_2) > 0. &(64)\\
(W\Delta w_2)_3 &< 0 \text{ if } \quad \rho_1(\rho_2+\tau_2) > (\rho_2+\tau_2) &(65)
\end{aligned}
$$

Equations (58) with (62) and (64), can be never satisfied simultaneously. Therefore in this particular case we are not able to derive a general rule regarding the increase in holdings of the first "ambiguous" asset relative to the risk aversion case. Thus, when we consider the case where a consumer-investor is ambiguity averse regarding the evolution of the first asset's price process, we have shown that, when $V_{\alpha_1} > 0, H_1 > 0$ the following proposition holds.

For a market consisting of one riskless and two risky assets, when ambiguity for the price process equation of one of them is considered, robust portfolio choices under uncertainty aversion imply the following:

1. *If (59), (63), (65) hold at the same time then there is an increase in the holdings of the second "no ambiguous asset", relative to risk aversion, or* $\Delta w_2 = (W\Delta w_2)_1 + (W\Delta w_2)_2 + (W\Delta w_2)_3 < 0$.

2. *When concerns about model misspecification do not exist, or* $\theta \to \infty$, *then the difference in portfolio choices between uncertainty aversion and risk aversion vanishes.* $\Delta W = \Delta w_1 + \Delta w_2 \to 0$, $(\Delta w_1, \Delta w_2) \to 0$.

Thus in this case uncertainty aversion for one asset only, implies that the holdings of the other, "no ambiguous" asset might increase relative to the case where the investor is risk averse for both assets.

Applying the same approach to the other three cases associated with the relationship of the signs between H_1 and V_{a_1},[17] when the mean rate of the price process is uncertain we obtain the following result:[18].

For a market consisting of one riskless and two risky assets, with ambiguity associated only with the price process of one of them, robust portfolio choices under uncertainty aversion imply the following:

1. *It is more likely to have an increase in the holdings of the second "no ambiguous asset," relative to risk aversion, or* $\Delta w_2 = (W\Delta w_2)_1 + (W\Delta w_2)_2 + (W\Delta w_2)_3 < 0$.

2. *When concerns about model misspecification do not exist, or* $\theta \to \infty$, *then the difference in portfolio choices between uncertainty aversion and risk aversion vanishes.* $\Delta W = \Delta w_1 + \Delta w_2 \to 0$, $(\Delta w_1, \Delta w_2) \to 0$.

The proof is given in the Appendix.

This proposition suggests that the consumer investor is more likely to increase the holdings of the second "home" or "no ambiguous" asset and reduce the holdings of the first "foreign", "ambiguous" asset, in a situation where she/he believes that the benchmark price process for the home asset is an adequate representation, but has concerns regarding model misspecification associated to the price process of the foreign asset. The result holds both for the case of uncertain mean return or uncertain volatility of the price proces. This is a result that can be regarded as providing an additional explanation for the home bias puzzle.

5 Concluding Remarks

By considering a stochastic investment opportunity set, where not only the assets' price processes, but also the drift and the volatility of the price processes

[17] The three cases are: $H_1 > 0, V_{\alpha_1} < 0$; $H_1 > 0, V_{\alpha_1} > 0$; $H_1 < 0, V_{\alpha_1} < 0$

[18] If we assume that uncertainty is associated with the volatility of the price process, instead of the rate of return, then the result remains the same. For the proofs we need only to substitute H_2 for H_1 and V_{σ_1} for V_{a_1}.

are driven by stochastic processes themselves, we derive optimal robust portfolio rules and provide another explanation of the home bias puzzle based on the optimizing behavior of a consumer - investor exhibiting uncertainly aversion for the foreign asset and risk aversion for the home asset. Our robust portfolio rules are parametrized using the exogenous parameter θ, and not eliminating it, in order to preserve the consistency of preferences with Gilboa and Schmeidler's axiomatization of uncertainty aversion. Furthermore the derived robust rules suggest that total holdings of risky assets may increase, for certain parameter values, under uncertainty aversion relative to the risk aversion case, which is a result that can be contrasted to results suggesting that robust methods in portfolio selection imply a reduction in the total holdings of risky assets. The fact that changes could go either way depending on the structure of the model parameters suggests that uncertainty aversion and adoption of robust portfolio rules should not be associated with conservative behavior regarding the holdings of risky assets.

A Appendix

Proof of proposition 2: Equations (1), (2), (17), which describe the dynamics of P_1, α_1, σ_1 can be written as:

$$
\begin{aligned}
\frac{dP_1}{P_1} &= \alpha_1 dt + \sigma_1 dB_1, &(66)\\
\frac{d\alpha_1}{\alpha_1} &= f_1 dt + g_1(\rho_1 dB_1 + \sqrt{1-\rho_1^2} dB_2),\\
\frac{d\sigma_1}{\sigma_1} &= f_2 dt + g_2(\rho_2 dB_1 + \tau_1 dB_2 + \tau_2 dB_3),\\
\tau_1 &= \frac{\rho_{3-}\rho_1\rho_2}{\sqrt{1-\rho_1^2}}, &(67)\\
\tau_2 &= (i)\sqrt{1-\rho_2^2-\tau_1^2} \quad if \quad (1-\rho_2^2-\tau_1^2) > (<) 0, &(68)
\end{aligned}
$$

where in the above system of equations B_1, B_2, B_3 are independent Brownian motions and ρ_1, ρ_2, ρ_3 denote the corelation coefficients between $(dB_1, dZ_1), (dB_1, dZ_2)$, and (dZ_1, dZ_2) respectively[19 20].

For the problem (10) of maximizing expected lifetime utility of consumption Merton's solution determines the optimal portfolio weight w_1, for the risky asset as:

$$
\begin{aligned}
w_1 W &= \frac{A(\alpha_1 - r)}{\sigma_1^2} + H_1 \frac{g_1 \rho \alpha_1}{\sigma_1} + H_2 \rho g_2, &(69)\\
A &= -\frac{V_W}{V_{WW}},\\
H_1 &= -\frac{V_{\alpha_1 w}}{V_{ww}},\\
H_2 &= -\frac{V_{\sigma_1 w}}{V_{ww}}.
\end{aligned}
$$

We face again the similar problem of a consumer-investor who is not sure about the benchmark model (66) and seeks to find robust desicion rules. Aplying the same technique as before the probabilities implied by the above model are

[19] We use the fact that the corelation matrix is $RR^T = \begin{bmatrix} 1 & \rho_1 & \rho_2 \\ \rho_1 & 1 & \rho_3 \\ \rho_2 & \rho_3 & 1 \end{bmatrix}$ where $R = \begin{bmatrix} 1 & 0 & 0 \\ \rho_1 & \sqrt{1-\rho_1^2} & 0 \\ \rho_2 & \tau_1 & \tau_2 \end{bmatrix}$ .We solve our prblem for the case where the matrix R is a real matrix which happens when $(1-\rho_2^2-\tau_1^2) > 0 \Leftrightarrow (1-\rho_2^2)(1-\rho_1^2) - (\rho_3 - \rho_1\rho_2)^2 > 0$. We are able to undertake the above analysis when all the main determinates of the initial matrix are strictly positive numbers and the method we use in order to achieve this is the orthogonormalization method of Grant-Smith.

[20] i is the imagine unit of a complex number: $i^2 = -1$

distorted. After some manipulations we can write the dynamics of the system for the distorted model as:

$$
\begin{aligned}
dS &= Adt + \Sigma d\hat{B} \qquad (70)\\
S &= \begin{bmatrix} W(t) \\ \alpha_1(t) \\ \sigma_1(t) \end{bmatrix}
\end{aligned}
$$

$$
\begin{aligned}
A &= \begin{bmatrix} w_1(\alpha_1 + \sigma_1 h - r) + (rW - c) \\ \alpha_1[f_1 + g_1 h(\rho_1 + \sqrt{1-\rho_1^2})] \\ \sigma_1[f_2 + g_2 h(\rho_2 + \tau_1 + \tau_2)] \end{bmatrix}\\
\Sigma &= \begin{bmatrix} W\sigma_1 w_1 & 0 & 0 \\ \alpha_1 g_1 \rho_1 & \alpha_1 g_1(\rho_1 + \sqrt{1-\rho_1^2}) & 0 \\ \sigma_1 g_2 \rho_2 & \sigma_1 g_2 \tau_1 & \sigma_1 g_2 \tau_2 \end{bmatrix}\\
\hat{B} &= \begin{bmatrix} \hat{B}_1 \\ \hat{B}_2 \\ \hat{B}_3 \end{bmatrix}
\end{aligned}
$$

In this case the associated multiplier robust control problem becomes:

$$
J(\theta) = \sup_{w_i, C} \inf_{h} \mathbb{E}_{\mathcal{Q}} \int_0^{\infty} e^{-\delta t} \big[U(C) + \theta_3 \frac{h^2}{2}\big] dt \qquad (71)
$$

subject to (70),

where $\theta_3 = 3\theta$, and θ denotes the robustness parameter which takes values greater or equal to zero. The Bellman-Isaacs conditions for this game implies that the value function $V(W, \alpha_1, \sigma_1, \theta)$ satisfies the following equation:

$$
\begin{aligned}
\delta V &= \max_{w_1, C} \min_{h} \Big\{ U(C) + \theta_3 \frac{h^2}{2} + V_W[w_1(\alpha_1 + \sigma_1 h - r)W + (rW - c)] \\
&\quad + V_{\alpha_1}\alpha_1[f_1 + g_1 h(\rho_1 + \sqrt{1-\rho_1^2})] + V_{\sigma_1}\sigma_1[f_2 + g_2 h(\rho_2 + \tau_1 + \tau_2)] \\
&\quad + \frac{1}{2} trace(\Sigma^T \partial^2 V \Sigma) \Big\}.
\end{aligned}
$$

where by $\partial^2 V$[21] denotes the matrix of second partial derivatives with respect W, α_1, σ_1[22]. The first order conditions for the above two players game are:

$$
\begin{aligned}
U'(C) &= V_W, \\
h &= -\frac{V_W W w_1 \sigma_1 + V_{\alpha_1}\alpha_1 g_1(\rho_1 + \sqrt{1-\rho_1^2}) + V_{\sigma_1}\sigma_1 g_2(\rho_2 + \tau_1 + \tau_2)}{\theta_2}, \\
0 &= V_W w_1(\alpha_1 + \sigma_1 h - r) + V_{WW}\sigma_1^2 W w_1 + \alpha_1 g_1 \rho_1 \sigma_1 V_{\alpha_1 W} + V_{\sigma_1 W} g_2 \rho_2 \sigma_1^2
\end{aligned}
$$

[21] $\partial^2 V = \begin{bmatrix} V_{WW} & V_{W\alpha_1} & V_{W\sigma_1} \\ V_{W\alpha_1} & V_{\alpha_1\alpha_1} & V_{W\sigma_1} \\ V_{W\sigma_1} & V_{W\sigma_1} & V_{\sigma_1\sigma_1} \end{bmatrix}$

[22] Superscript T denotes the transpose of a matrix.

Solving the above system of equations we obtain the fraction of the wealth w_1^* invested in the risky asset for the case of uncertainty aversion with respect both, to the stochastic evolution of the volatility rate and of the mean rate of return of the price processes.

$$\begin{aligned} w_1^* W\left(1 \quad - \quad \frac{V_W^2}{\theta_2 V_{WW}}\right) &= \frac{A(\alpha_1 - r)}{\sigma_1^2} + H_1 \frac{g_1 \rho_1 \alpha_1}{\sigma_1} + H_2 \rho g_2 \qquad (72) \\ &+ \quad \frac{V_W V_{\alpha_1}}{\theta_3 V_{WW}} \frac{g_1 \alpha_1 (\rho_1 + \sqrt{1-\rho_1^2})}{\sigma_1} + \frac{V_W V_{\sigma_1}}{\theta_3 V_{WW}} g_2(\rho_2 + \tau_1 + \tau_2) \end{aligned}$$

The first three terms on the right hand side of the above equation are the same as in (69), therefore based on the usuall argument as in the previous cases we have to distinguish four cases depending on the signs of the partials derivatives of the value function with respect to mean rate and the volatility rate. Particuralrly when $V_{\alpha_1} \gtreqless 0$ and $V_{\sigma_1} \gtreqless 0$ if $\rho_1 + \sqrt{1-\rho_1^2} \gtreqless 0$ and $\rho_2 + \tau_1 + \tau_2 \gtreqless 0$ then an uncertainty averse investor always reduces the total holdings of the risky asset relative to a risk averse.Therefore if

$$System = \begin{cases} \rho_2 + \tau_1 + \tau_2 \gtreqless 0 \;\; when \; V_{\alpha_1} \gtreqless 0 \; respectively \\ \rho_1 + \sqrt{1-\rho_1^2} \gtreqless 0 \; when \; V_{\sigma_1} \gtreqless 0 \; respectively \\ (1-\rho_2^2)(1-\rho_1^2) - (\rho_{3-}\rho_1\rho_2)^2 > 0 \\ 1 \leq \rho_2 \leq 1 \\ -1 \leq \rho_3 \leq 1 \end{cases}$$

the proposition has been proved.

Proof of proposition 5: We will present the proof for the stochastic volatility case when $H_2 > 0, V_{\sigma_1} < 0$. Equations (50) – (54) also hold in this specific case with the difference that $H_2, \sigma_1, g_2, f_2, V_{\alpha_1}$ have replaced H_1, α_1, g_1, f_1, V_{σ_1} respectively. Following this proof we obtain that equations (58), (59) also hold this time.

Similar if $(.)_2$ refers to the corresponding term related with the matrix $\left[\begin{array}{cc} H_2\sigma_1 g_2 \rho_2 \sigma_1 & H_2 \sigma_1 g_1 \rho_3 \sigma_2 \end{array}\right]$ then :

$$\begin{aligned} (W\Delta w_1)_2 &= \frac{\widetilde{\kappa}}{\sigma_1}\left[\rho_2 \xi + \rho_3(x - \rho_1)\right] \qquad (73) \\ (W\Delta w_2)_2 &= \frac{\widetilde{\kappa}}{\sigma_2}\left[\rho_2(x-\rho_1) + \rho_3(1-x)\right] \qquad (74) \\ \widetilde{\kappa} &= \frac{H_2\sigma_1 g_2}{(1-\rho_1^2)} > 0. \end{aligned}$$

So

$$\begin{aligned} (W\Delta w_1)_2 &< 0 \text{ if } \;\; \rho_2 < 0, \rho_1 > 0, \rho_3 > 0. \qquad (75) \\ (W\Delta w_2)_2 &< 0 \text{ if } \;\; 0 > \rho_2 > \rho_3, \; \rho_1 > 0 \text{ or} \qquad (76) \\ \rho_1 &> 0, \; \rho_2 > 0 \; , \; \rho_3 < 0. \qquad (77) \end{aligned}$$

Moreover if $(.)_3$ refers to the corresponding term $-TM_2$ related with the matrix $\left[\begin{array}{cc} M_{21} & M_{22} \end{array}\right]$ then :

$$
\begin{aligned}
(W\Delta w_1)_3 &= \frac{\widetilde{\kappa'}}{\sigma_1}\left[(\rho_2+\tau_2)\frac{(\rho_1^2-x)\,x}{1-x}-\rho_1\,(\rho_2+\tau_2)\,x\right] \\
(W\Delta w_2)_3 &= \frac{\widetilde{\kappa'}}{\sigma_2}\left[-(\rho_2+\tau_2)\,x+\rho_1\,(\rho_2+\tau_2)\,x\right] \\
\widetilde{\kappa'} &= -\frac{-A\frac{\sigma_1 g_2 V_{\alpha_1}}{\theta_3}}{(1-\rho_1^2)}<0.
\end{aligned}
$$

So

$$
\begin{aligned}
(W\Delta w_1)_3 &< 0 \text{ if } \quad \rho_1<0,\ (\rho_2+\tau_2)<0. && (78)\\
(W\Delta w_2)_3 &< 0 \text{ if } \quad \rho_1\,(\rho_2+\tau_2)<(\rho_2+\tau_2) && (79)
\end{aligned}
$$

Therefore we have shown that:

1. *If (59), (79) with (76) or (77), hold simultaneously then there is an increase in the holdings of the second risky asset, relative to risk aversion, or* $\Delta w_2 = (W\Delta w_2)_1 + (W\Delta w_2)_2 + (W\Delta w_2)_3 < 0$.

2. *When concerns about model misspecification do not exist, or* $\theta \to \infty$, *then the difference in portfolio choices between uncertainty aversion and risk aversion vanishes.* $\Delta W = \Delta w_1 + \Delta w_2 \to 0$, $(\Delta w_1, \Delta w_2) \to 0$.

The proof for the other two cases, either $\begin{matrix} H_1>0, V_{\alpha_1}>0 \\ H_1<0, V_{\alpha_1}<0 \end{matrix}$ when we refer to the case of a stochastic mean rate of return, or $\begin{matrix} H_2>0, V_{\sigma_1}>0 \\ H_2<0, V_{\sigma_1}<0 \end{matrix}$ when we examine the case of a stochastic volatility rate, is similar.

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