Firm Valuation with Endogenous Growth Opportunities

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Abstract

This paper provides a valuation framework for a firm with endogenous growth opportunities. Three interesting results are obtained. First, it is demonstrated that the firm's cost of capital is a weighted average of two components; the average cost of distributions and the average cost of investments, where the latter is shown to be the higher between the two. Second, we find that in optimum the marginal rate of return on investment is equal to the average cost of distributions. Third, we suggest a compact formula for the value of a firm with endogenous growth opportunities. Our framework can be used to check the efficiency of investments at the firm and macroeconomic levels.

JEL Classification: D21, D24, D92, E22, E23, G12, G31.

Introduction

The purpose of this paper is to provide a theoretical framework for the valuation of a firm that has endogenous growth opportunities. We assume that in every period the firm faces the same growth function. This growth function endogenously describes the relation between the retention ratio and the growth rate of the firm. We assume that the growth function is concave in the retention ratio. This assumption is consistent with diminishing returns to scale and reflects the fact that the firm faces returns that diminish with the size of the investment. We apply the discounted cash flow model with constant growth rates to the value of the firm. However, instead of using a constant given growth rate we assume that the growth rate endogenously depends on the retention ratio. The objective function of the firm is to maximize its value.

There are two alternative approaches to equity and debt valuation. The first is direct valuation that does not require the valuator to find the value of the firm, and the second uses derivative models that require the value of the firm as an input. Textbooks provide prescriptions for direct valuation of equity based on the discounted cash flow method, and some also describe equity valuation based on the price/earnings ratio. Most textbooks provide several widely used formulas for discounted cash flow models, including models without growth, models with constant growth (Gordon, 1962), and models with varying growth rates. Methods of direct valuation of default-risky corporate bonds that do not require valuation of the firm have been also suggested. The literature on this topic is extensive, and methods vary according to the type and characteristics of the bond. (Nandi, 1998, provides a literature review of this topic.) The second approach to equity and debt valuation starts with valuation of the firm and then follows to find the value of equity and debt using derivative models, such as the Black and Scholes model (1973). This paper concentrates on the first stage of the second approach and develops a theoretical framework for firm valuation.

We start with a simple framework with no taxes and no external financing. Three interesting results are obtained for this case. First, Modigliani and Miller (1958) demonstrate that the firm's cost of capital can be represented as a weighted average of the cost of equity and the cost of capital. This paper demonstrates that another decomposition is possible; a one that is related to the firm's investment strategy. We demonstrate that the firm's cost of capital can be represented as a weighted average of the cost of distributions and the cost of investments, where the latter is shown to be the higher between the two.

Second, we find an interesting optimality condition for the case of a firm with endogenous growth opportunities. Fisher (1930) is the first to present an equilibrium concept for optimal investment. Fisher demonstrates in a two-period setting that investment should be increased until the marginal rate of return is equal to the interest rate in the economy (Fisher's model does not incorporate risk).[1] Hirshleifer (1961) extends Fisher's argument to risky investments using an adjusted for risk discount rate. The results of Fisher (1930) and Hirshleifer (1958, 1961) sum up to the following conclusion: a firm should increase its investment in a project until the marginal rate of return is equal to the discount rate. This result is a common practice in capital budgeting (See, for example, Kolb and Rodriguez, 1995, chapter 2). Note that the solutions of Fisher and Hirshleifer are in a two period setting. Do their conclusions still hold in a perpetual setting with growth opportunities? Our results demonstrate that the answer to this question is negative. We find that in optimum the firm should invest until the marginal rate of return on investment is equal to the cost of distributions (not the cost of capital).

Third, we find simple formulas for the value of the firm and the value of future growth opportunities in the context of the discounted cash flow model with endogenous growth opportunities. The value of the firm (for the case without taxes) is equal to the expected profit of the firm at time 1 divided by the cost of distributions. In contrast, it is well known that the value of a firm without growth opportunities is equal to the expected profit of the firm at time 1 divided by the discount rate. We demonstrate that the cost of distributions is lower than the cost of capital, and therefore the value of a firm with growth opportunities is larger than the value of a firm without growth opportunities. The difference between the two is, of course, the value of future growth opportunities.

In section 5 of the paper we add taxes to the framework. We demonstrate that all the basic results for the case without taxes hold also here. In particular, the firm continues to retain the same proportion of profits. In section 6, we extend the

[1] Chapter 11 in Fisher (1930) describes the optimal solution (in graphic terms) to the investment problem.

framework to account for both taxes and external financing. The basic results and in particular the investment strategy remain the same also for this case.

This paper is organised as follows. Sections 1 through 4 develop a framework for firm valuation in a world without taxes and without external financing. Section 1 develops the objective function of the firm and solves for the optimal investment strategy. In section 2, we demonstrate that the cost of capital consists of the average cost of investments and the average cost of distributions, and we develop an appealing alternative representation of the firm's optimality criterion. In section 3, we suggest a compact and useful formula for the firm's value. In section 4, we demonstrate that the average cost of investments is higher than the average cost of distributions and we suggest a simple formula that captures the value of future investment opportunities. Section 5 relaxes the assumption of no taxes. Section 6 extends the results of section 5 for the case of taxes and external financing. Section 7 provides a simple example. Section 8 provides simulation results for the price/earnings ratio, the retention ratio, the cost of investments and the cost of distributions under different specifications of a simple power growth function. We conclude with a summary.

1. Objective function and optimal investment strategy

The decision makers in a firm are usually equity holders (or an agent assigned to the job by equity holders). Naturally, equity holders are interested in maximizing the value of equity. However, for any given level of debt (regardless of how it is determined), maximizing the value of equity is equivalent to maximizing the value of the firm because equity is just the residual from the firm's value after paying the debt. Maximizing the value of equity and the value of the firm do not always coincide, however. Agency problems can lead to non optimal firm value. For example, Myers (1976) demonstrates that in the presence of a large debt, equity holders might prefer a high-variance negative-NPV project to a positive-NPV project. In general, however, it is accepted that for a firm without agency problems the objective function of the firm is to maximize its value (see, for example, Kolb and Rodriguez, 1995, chapter 8). We therefore embrace this approach and consider a firm that is interested in maximizing its value.

We now follow to construct the objective function of the firm. Throughout the paper we assume that there are no transaction costs and no bankruptcy costs. In this section, we also assume that there are no taxes and no external financing. These two last assumptions are relaxed in the following sections.

Let $E(X_t)$ denote the expected profit of a firm at time t. The expected profits of the firm are used for expected interest rate payments, $E(IN_t)$, expected dividend payments, $E(DV_t)$, expected share repurchases, $E(SR_t)$, and expected retained earnings, $E(RE_t)$. In terms of resources and uses, we can write

$$E(X_t) = E(IN_t) + E(DV_t) + E(SR_t) + E(RE_t) \tag{1}$$

where the right hand side is the uses and the left hand side is the resources. The total expected distributions of the firm to its security holders are denoted by $E(\psi_t)$, where

$$E(\psi_t) = E(IN_t) + E(DV_t) + E(SR_t) \tag{2}$$

and the expected investments of the firm, $E(K_t)$, are equal to the expected retained earnings

$$E(K_t) = E(RE_t)$$

We assume that, starting at time 1, expected profits are related by the following equation

$$E(X_t) = [1 + g_t(\alpha_{t-1})]E(X_{t-1}) \tag{3}$$

where $g_t(\alpha_{t-1})$ is a growth function that describes the growth rate at time t as a function of α_{t-1}, the retention ratio at time $t-1$. Note that the growth function is in terms of percentage and the retention ratio is a proportion so they are both scale free, and therefore the growth function is also scale free. Assuming now that the firm faces the same growth function every period, it implies that the same retention ratio is selected. It follows that we can remove the time subscripts from $g_{t+1}(\alpha_t)$ and use $g(\alpha)$ instead. This assumption reduces equation (3) to

$$E(X_t) = [1 + g(\alpha)]E(X_{t-1}) \tag{4}$$

We consider α that is in the range $0 \le \alpha \le 1$. Note that $\alpha > 1$ in the whole economy is impossible because it implies, by equations (1) and (7), that the uses are higher than the resources.[2]

Four assumptions regarding the properties of $g(\alpha)$ can be made in this setting. The first is $g''(\alpha) < 0$. This assumption implies concavity of the growth rate in the retention ratio. It is consistent with diminishing returns to scale and reflects the fact that the firm faces returns that diminish with the size of the investment. It is consistent with diminishing returns to scale and reflects the fact that the firm faces returns that diminish with the size of the investment.[3] The second assumption is $g(0) = 0$. This assumption implies that if there is no investment, then there is no growth. The assumption is necessary to ensure that the discounted cash flow model with growth opportunities converges to the value without growth opportunities when $\alpha = 0$. The third assumption is $g'(0) = \infty$. This assumption implies an infinite marginal growth rate for infinitesimally small investment levels. The assumption is necessary to ensure that α larger than zero is selected. A forth optional assumption is $g'(\alpha) > 0$. This assumption implies that the growth rate is monotonically increasing in the retention ratio. This assumption can be challenged on the grounds that the growth function might have a strict maximum at some α^*, $\alpha^* < 1$. This would imply a positive $g'(\alpha)$ up to α^* and a negative afterwards. Intuitively, such a peaking function implies that the solution will be in the range $0 \le \alpha \le \alpha^*$.

Assume now that the (post payment) value of the firm today (time 0) is given by the discounted value of all its future distributions

$$V_0 = E_0\left[\sum_{t=1}^{\infty}\left(\frac{1}{1+r_a}\right)^t \psi_t\right] \tag{5}$$

where r_a is the appropriate constant discount rate for the firm given its risk, and ψ_t is the distribution of the firm at time t.

Equation (5) can be also written as

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2 At the single firm level, $\alpha > 1$ is possible only for a limited horizon of time. If the firm is characterized by $\alpha > 1$, it implies that it has better growth opportunities than the economy in general and better growth rates. This can last only for a limited horizon of time. Otherwise, the weight of the firm in the economy will go to 1, and then $\alpha > 1$ will imply that the uses (at the macroeconomic level) are larger than the resources.

3 Models that employ convexity of the growth function result with corner solution for α; either $\alpha = 1$ or $\alpha = 0$. Growth functions that are convex at low levels of the retention ratio and concave for high levels of the retention ratio are a possible alternative.

$$V_0 = \sum_{t=1}^{\infty} \frac{E(\psi_t)}{(1+r_a)^t} \tag{6}$$

(To prevent cumbersome notations, we remove the 0 subscript from the expectation notation.)

Using the assumption of a constant retention ratio, it follows that the retention ratio can be computed from

$$\alpha = \frac{E(RE_t)}{E(X_t)} \tag{7}$$

It also follows that the distribution ratio of the firm is just

$$1 - \alpha = \frac{E(\psi_t)}{E(X_t)} \tag{8}$$

Using equation (8), equation (6) can be represented as

$$V_0(\alpha) = \sum_{t=1}^{\infty} \frac{(1-\alpha)E(X_t)}{(1+r_a)^t} \tag{9}$$

(Note that α is a decision variable, so we use $V_0(\alpha)$ instead of just V_0)

From (4) it follows that

$$E(X_t) = [1+g(\alpha)]^{t-1} E(X_1) \tag{10}$$

Substituting equation (10) in (9), we get

$$V_0(\alpha) = \sum_{t=1}^{\infty} \frac{(1-\alpha)[1+g(\alpha)]^{t-1} E(X_1)}{(1+r_a)^t} \tag{11}$$

From series-of-constants convergence-rules, we know that (11) can be represented as

$$V_0(\alpha) = \frac{(1-\alpha)E(X_1)}{r_a - g(\alpha)} \tag{12}$$

Equation (12) is very similar to the dividend discount model with constant growth rates (Gordon, 1962), but instead of applying the formula to equity, we apply it to the firm's value and instead of using dividends in the numerator we use the total distributions of the firm.[4]

Assume now that the firm is interested in maximizing its value. The problem of the firm is just

[4] Our model does not follow the assumption $E(X_t) = X_0(1+g(\alpha))^{t-1}$, because it is overly restrictive. For example, assume that the firm had a loss at time 0 (i.e. $X_0 < 0$). If we use $E(X_t) = X_0(1+g(\alpha))^{t-1}$ it implies that the firm will continue to have loses indefinitely.

$$\max_{\alpha} \; V_0(\alpha) = \frac{(1-\alpha)E(X_1)}{r_a - g(\alpha)} \tag{13}$$

Appendix A demonstrates that the optimal solution to the firm value-maximization problem in (13) is given by

$$g'(\alpha) = \frac{r_a - g(\alpha)}{(1-\alpha)} \tag{14}$$

The optimal retention ratio can be solved from equation (14).

2. A more appealing representation of the optimality criterion

We now follow to present the optimality condition, (14), in a more appealing and intuitive way. The expected return of a firm must be equal to its discount rate. The expected return is also equal to the sum of the distribution yield (consisting of dividends and interest rate payments) and the growth rate of the firm's value. This is given by

$$r_a = dy(\alpha) + g(\alpha) \tag{15}$$

where dy is the distribution yield.

Because r_a is the return on the value, $g(\alpha)$ is the growth rate of the value. Note, therefore, that in writing (15) we assume that the growth rate of the firm's value is identical to the growth rate of the firm's distributions. Violation of this assumption will lead to time varying expected returns.

Figure 1 presents a simple example of the relation in (15) for the case $g(\alpha) = 0.08\sqrt{\alpha}$ and $r_a = 0.1$. Figure 1 demonstrates that because $g(\alpha)$ changes with α, dy must also changes with α, because r_a is constant. This is why dy in equation (15) is presented as $dy(\alpha)$.

Figure 1 about here

Let $r_g(\alpha)$ denote the average return on investments, defined as

$$r_g(\alpha) = \frac{g(\alpha)}{\alpha} \tag{16}$$

Let $r_\psi(\alpha)$ denote the average return on distributions, defined as

$$r_\psi(\alpha) = \frac{dy(\alpha)}{1-\alpha} \tag{17}$$

From (17), (16) and (15) it follows that

$$r_a = (1-\alpha)r_\psi(\alpha) + \alpha \cdot r_g(\alpha) \tag{18}$$

Appendix B demonstrates that $r_g(\alpha)$ is monotonically decreasing with α. An opposite argument cannot be said about $r_\psi(\alpha)$. Figure 2 presents $r_g(\alpha)$ and $r_\psi(\alpha)$ for the case $g(\alpha) = 0.08\sqrt{\alpha}$ and $r_a = 0.1$, and one can see that $r_\psi(\alpha)$ is not monotonic in α.

Figure 2 about here

Equation (18) demonstrates that the firm's cost of capital is equal to a weighted average of the average cost of investments and the average cost of distributions.
From (15) it follows that

$$r_a - g(\alpha) = dy(\alpha) \tag{19}$$

Substituting (19) in (14), the optimality condition becomes

$$g'(\alpha) = \frac{dy(\alpha)}{(1-\alpha)} \tag{20}$$

Substituting now (17) in (20), it follows that the optimality condition is

$$g'(\alpha) = r_\psi(\alpha) \tag{21}$$

To gain some intuition about this optimality condition recall that

$$r_g(\alpha) = \frac{g(\alpha)}{\alpha} \tag{22}$$

Recall also that

$$g'(\alpha) = \frac{d(g(\alpha))}{d\alpha} \tag{23}$$

The similarity between (22) and (23) and the fact that (22) is the average return on investment suggests that $g'(\alpha)$ can be interpreted as the marginal rate of return on investment. Intuitively, one can think of two streams going out of the firm; one for investments and the second for distributions. It is very popular to find in optimization problems of this type that the marginal output of the two streams is equal. Note, however, that for distributions there is no marginal return, only average return, and therefore the condition in (21) implies that in equilibrium the marginal rate of return on investment should be equal to the average cost of distributions. Figure 3 presents $g'(\alpha)$ and $r_{\psi}(\alpha)$ for the case $g(\alpha)=0.08\sqrt{\alpha}$ and $r_a=0.1$.

3. An alternative valuation formula

From (12) and (19) it follows that

$$V_0(\alpha)=\frac{(1-\alpha)E(X_1)}{dy(\alpha)}$$

Using (17) we get

$$V_0(\alpha)=\frac{(1-\alpha)E(X_1)}{(1-\alpha)r_{\psi}(\alpha)}$$

Implying that

$$V_0(\alpha)=\frac{E(X_1)}{r_{\psi}(\alpha)} \tag{24}$$

This valuation formula demonstrates a very simple way to find the value of a firm that has endogenous growth opportunities. The value of the firm is just the expected profit of the firm divided by the cost of distributions.

Figure 4 presents the value of the firm for the case $g(\alpha)=0.08\sqrt{\alpha}$, $r_a=0.1$, and $E(X_1)=\$1M$.

Figure 4 about here

4. The cost of capital and the value of future growth opportunities

To find the value of a firm without growth opportunities, we substitute $\alpha=0$ and $g(0)=0$ in equation (12) and get

$$V_0=\frac{E(X_1)}{r_a} \tag{25}$$

The difference between (24) and (25) looks minor, but in fact it is crucial because it captures the value of future investment opportunities. To show this argument, denote the optimal retention ration as α^* (this is the optimal retention ratio from solving equation (14)). By the definition of α^* as the optimal solution

$$V_0(\alpha^*) > V_0(\alpha) \tag{26}$$

for any α in the range $0 \le \alpha \le 1$, $\alpha \ne \alpha^*$. In particular, equation (26) is correct for $\alpha = 0$ (assuming $\alpha^* \ne 0$), so

$$V_0(\alpha = \alpha^*) > V_0(\alpha = 0) \tag{27}$$

and it follows immediately that

$$\frac{X_1}{r_\psi(\alpha^*)} > \frac{X_1}{r_a} \tag{28}$$

and, clearly, it must follow that

$$r_\psi(\alpha^*) < r_a \tag{29}$$

Let VG_0 denote the value of growth opportunities. It follows from (27) and (28) that

$$VG_0 = V_0(\alpha = \alpha^*) - V_0(\alpha = 0) = \frac{E(X_1)}{r_\psi(\alpha^*)} - \frac{E(X_1)}{r_a} \tag{30}$$

Because we find in equation (29) that $r_\psi(\alpha^*) < r_a$, it follows immediately from (18) that $r_g(\alpha^*) > r_a$. The following ranking of costs thus applies: $r_g(\alpha^*) > r_a > r_\psi(\alpha^*)$. This demonstrates that in equilibrium the firm's cost of investments is higher than the firm's cost of distributions. This condition is necessary for investments to exist. If there is no α such that $r_g(\alpha) > r_a > r_\psi(\alpha)$, then no investments are made and all profits are distributed to security holders.

To ensure that the value of the firm does not go to infinity one must consider a bound from below on the value of the discount rate. This bound is given by $r_a > \max_{0 \le \alpha \le 1} g(\alpha)$.

5. Taxes

Let T_C denote the corporate tax rate, T_B the tax rate on interest rate payments, T_{DIV} the tax rate on dividends, T_{CG} the tax rate on capital gains, and T_{SR} the tax rate on share repurchases. Miller (1977) uses a single tax rate for all sorts of income from shares and demonstrates (in a macroeconomic setting) that in equilibrium

$$(1 - T_B) = (1 - T_C)(1 - T_S) \tag{31}$$

where T_S is the tax rate on income from shares. Miller argument is correct as long as the tax on all sorts of income from shares is the same[5]

$$T_S = T_{DIV} = T_{CG} = T_{SR} \tag{32}$$

Assume now that both (31) and (32) hold. Let

$$(1 - T_B) = (1 - T_C)(1 - T_S) = (1 - T)$$

where T is a homogenous tax rate on all types of distributions.

In terms of resources and uses, we can write the following

$$E(X_t) = E(IN_t) + E(DV_t) + E(SR_t) + E(RE_t) + E(TX_t) \tag{33}$$

where $E(X_t)$ is the profit before taxes and distributions, and $E(TX_t)$ is the expected amount of taxes paid at time t.

The outflows of cash from the firm at time t (including taxes) are given by

$$E(X_t) - E(K_t) = E(IN_t) + E(DV_t) + E(SR_t) + E(TX_t)$$

We define the retention ratio as

$$\alpha = \frac{E(RE_t)}{E(X_t)}$$

The complementary of the retention ratio is given by

$$1 - \alpha = \frac{E(X_t) - E(RE_t)}{E(X_t)} \tag{34}$$

Note now that the tax rate is given by

$$T = \frac{E(TX_t)}{E(X_t) - E(K_t)}$$

and therefore

$$1 - T = \frac{E(IN_t) + E(DV_t) + E(SR_t)}{E(X_t) - E(K_t)} \tag{35}$$

From (35) it follows that the periodical distribution to security holders is given by

$$E(\psi_t) = E(IN_t) + E(DV_t) + E(SR_t) = (1 - T)[E(X_t) - E(K_t)] \tag{36}$$

Using (34), equation (36) can be represented as

$$E(\psi_t) = E(IN_t) + E(DV_t) + E(SR_t) = (1 - T)(1 - \alpha)E(X_t) \tag{37}$$

The value of the firm to security holders is just

[5] The Jobs and Growth Tax Relief Reconciliation Act of 2003 equates the tax rate paid on capital gains and dividends in the US. Previous to that, capital gains and share repurchases typically had different tax rates compared to dividends.

$$V_{0,TX} = E_0\left[\sum_{t=1}^{\infty}\left(\frac{1}{1+r_a}\right)^t \psi_t\right]$$

where $V_{0,TX}$ denotes the value of the firm for security holders for the case with taxes. Replicating steps (6) through (13) it follows that the optimization problem of the firm is given by

$$\max_{\alpha} \ V_{0,TX}(\alpha) = \frac{(1-T)(1-\alpha)E(X_1)}{r_a - g(\alpha)} \tag{38}$$

Appendix C demonstrates that the optimality criterion is given by

$$g'(\alpha) = \frac{r_a - g(\alpha)}{(1-\alpha)} \tag{39}$$

which is identical to the optimality criterion for the case without taxes.
The optimality criterion in (39) implies that the investment strategy of the firm for the case with taxes is the same as for the case without taxes. In other words, the optimal level of investment for the case with taxes is the same as for the case without taxes. What changes is the distribution to security holders. In the case without taxes all the outflows of cash went to security holders, but with taxes the outflows are divided between distributions to security holders and taxes.

Note that the retention ratio is the same for the cases with and without taxes and therefore the growth rate is identical in the two case. Therefore, equation (15) holds also here in exactly the same manner it did for the case without taxes. Following the same analysis in equations (15) through (21), the optimality criterion for the case with taxes is exactly the same

$$g'(\alpha) = r_{\psi}(\alpha)$$

6. External financing

Let $E(\Delta B_t)$ be the expected dollar change in the amount of debt taken by the firm at time t. Let $E(\Delta E_t)$ be the expected dollar change in the amount of equity sold by the firm at time t. Raising new funds by issuing new debt or new equity increases the resources of the firm, however, these new funds also increase investments. (Assuming no agency problems, it is useless to issue new debt or new equity just in order to distribute it back.) Therefore, the resources and uses equation in (33) can be adjusted to new equity and new debt issues in the following way:

$$E(X_t)+E(\Delta B_t)+E(\Delta E_t)=E(IN_t)+E(DV_t)+E(SR_t)+E(RE_t)+E(\Delta B_t)+E(\Delta E_t)+E(TX_t) \tag{40}$$

Rearranging, we can write this equation as

$$E(X_t)=\left[E(IN_t)+E(DV_t)+E(SR_t)-E(\Delta B_t)-E(\Delta E_t)\right]+\left[E(RE_t)+E(\Delta B_t)+E(\Delta E_t)\right]+E(TX_t) \tag{41}$$

where the elements on the right hand side of this equation can be divided into three main components; net distributions, investments, and taxes. The net distributions are given by

$$E(\psi_t)=E(IN_t)+E(DV_t)+E(SR_t)-E(\Delta B_t)-E(\Delta E_t)$$

and investments are given by

$$E(K_t)=E(RE_t)+E(\Delta B_t)+E(\Delta E_t)$$

Equation (41) demonstrates why capital structure decisions do not matter; if financial markets are perfect, the firm can always play with issues of new debt and new equity in a way that the investment strategy of the firm is implemented.

The retention ratio for the case with external financing is given by

$$\alpha=\frac{E(K_t)}{E(X_t)}$$

and the complementary of the retention ratio by

$$1-\alpha=\frac{E(X_t)-E(K_t)}{E(X_t)} \tag{42}$$

The tax rate as before is given by

$$T=\frac{E(TX_t)}{E(X_t)-E(K_t)}$$

and

$$1-T=\frac{E(\psi_t)}{E(X_t)-E(K_t)} \tag{43}$$

From (42) and (43), the value of the firm is given by

$$\max_{\alpha}\ V_{0,TX,EF}(\alpha)=\frac{(1-T)(1-\alpha)E(X_1)}{r_a-g(\alpha)} \tag{44}$$

where $V_{0,TX,EF}(\alpha)$ is the value of the firm with taxes and external financing (EF).

Note that the optimization problem in (44) is identical to that in (38) and therefore, again, the optimality condition is the same as for the case without taxes and without external financing

$$g'(\alpha) = r_\psi(\alpha)$$

7. A simple example

Because the basic solution is the same for the above three discussed cases, we provide an example without taxes and without external financing. We assume a simple power growth function. We demonstrate how the value of the firm is found for one special case and then proceed in section 8 to conduct sensitivity analysis of the price/profit ratio, $r_g(\alpha)$, $r_\psi(\alpha)$, and the retention ratio under different scenarios of the power growth function. Assume that the periodical growth function of the firm is given by

$$g(\alpha) = b \cdot \alpha^{1-\theta} \tag{45}$$

where $0 < \theta < 1$ in order to satisfy the requirement of concavity of the growth function. The function in (45) is clearly monotonically increasing in the retention ratio, α. By substituting $\alpha = 1$, we find a lower bound on the value of the discount rate, which is given by

$$g(\alpha = 1) = b$$

The derivative of (45) with respect to α is given by

$$g'(\alpha) = b(1-\theta)\alpha^{-\theta}$$

To find the optimal retention ratio, we use equation (14), which implies that

$$r_a - b\alpha^{1-\theta} = (1-\alpha)b(1-\theta)\alpha^{-\theta} \tag{46}$$

To demonstrate the solution, we need to make some more specific assumptions. Assume thus that $b = 0.08$, $\theta = 0.5$ and $r_a = 0.1$. Substituting these numbers into equation (46), we get

$$0.1 - 0.08\sqrt{\alpha} = (1-\alpha)\frac{0.08}{2\sqrt{\alpha}}$$

With some algebraic manipulations, we get the following quadratic equation

$$4\alpha^2 - 17\alpha + 4 = 0$$

The two roots of this equation are $\alpha = 4$ and $\alpha = 0.25$. The first root is outside the possible range, so we are left with $\alpha = 0.25$ as a possible solution. We verify numerically that $\alpha = 0.25$ is a local maximum.[6] It follows that $\alpha^* = 0.25$.

To find the value of the firm we need to compute $g(\alpha^*)$, $g'(\alpha^*)$ and $r_\psi(\alpha^*)$. Starting with $g(\alpha)$, it is equal to

$$g(\alpha^*) = 0.08\sqrt{\alpha^*} = 0.04$$

The marginal growth rate is

$$g'(\alpha^*) = \frac{0.08}{2\sqrt{\alpha}} = 0.08$$

Equation (21) states that in equilibrium the average cost of distributions is equal to the marginal growth rate. To verify that, we use equation (17), which states that

$$r_\psi(\alpha^*) = \frac{dy(\alpha^*)}{1-\alpha^*}$$

But $dy(\alpha^*) = r_a - g(\alpha^*) = 0.1 - 0.04 = 0.06$ and $1-\alpha^* = 0.75$, so

$$r_\psi(\alpha^*) = 0.08$$

Assume now that $E(X_1)$ is equal to \$1M. The firm's value is

$$V_0(\alpha^*) = \frac{E(X_1)}{r_\psi(\alpha^*)} = \frac{1}{0.08} = \$12.5\text{M}$$

If the firm has no investment opportunities, its value should be

$$V_0 = \frac{E(X_1)}{r_a} = \frac{1}{0.1} = \$10\text{M}$$

The value of growth opportunities in this case is therefore \$2.5M.

We compute now $r_g(\alpha^*)$

$$r_g(\alpha^*) = \frac{g(\alpha^*)}{\alpha^*} = \frac{0.04}{0.25} = 0.16$$

Note that substituting $r_\psi(\alpha^*) = 0.08$ and $r_g(\alpha^*) = 0.16$ in (18), we get the cost of capital $r_a = 0.1$. Also note that the results of this example satisfy the condition $r_g(\alpha^*) > r_a > r_\psi(\alpha^*)$.

[6] To verify that $\alpha = 0.25$ is a local maximum we check 1001 values between $\alpha = 0$ and $\alpha = 1$ with steps of 0.001, and we find that the maximum is obtained at $\alpha = 0.25$.

8. Simulating the cost of investments, the cost of distributions, the retention ratio, and the price/earnings ratio for different growth functions

The power growth function in (45) is used in this section for simulation studies of $r_g(\alpha)$, $r_\psi(\alpha)$, the retention ratio and the price/profit ratio, under different specifications of the parameters, b and θ. Table 1 presents results for $r_g(\alpha^*)$ and $r_\psi(\alpha^*)$, table 2 for the optimal retention ratio, and table 3 for the price/profit ratio.

Summary

The purpose of this paper is to provide a theoretical framework for the valuation of a firm that has endogenous growth opportunities. We assume that in every period the firm faces the same growth function. This growth function endogenously describes the relation between the retention ratio and the growth rate of the firm. We assume that the growth function is concave in the retention ratio. This assumption is consistent with diminishing returns to scale and reflects the fact that the firm faces returns that diminish with the size of the investment. We apply the discounted cash flow model with constant growth rates to the value of the firm. However, instead of using a constant given growth rate we assume that the growth rate endogenously depends on the retention ratio. The objective function of the firm is to maximize its value.

We begin with the simple case of no taxes and no external financing. Three interesting results are obtained. First, Modigliani and Miller (1958) demonstrate that the firm's cost of capital can be represented as a weighted average of the cost of equity and the cost of debt. This paper demonstrates that another decomposition is possible; a one that is related to the firm's investment strategy. We demonstrate that the firm's cost of capital can be represented as a weighted average of the cost of distributions and the cost of investments, where the latter is shown to be the higher between the two. Second, we find an interesting optimality condition that maximizes the value of the firm. This condition implies that the firm should increase its investment until the marginal rate of return on investment is equal to the average cost of distributions (and not the cost of capital). Third we offer simple formulas for the value of a firm with endogenous growth opportunities and for the value of growth opportunities. We follow to examine the case of taxes and external financing. We find that the same basic results, and in particular the optimal investment criterion, continue to hold.

Appendix A

Proposition

The extremum points of

$$V_0 = \frac{(1-\alpha)E(X_1)}{r_a - g(\alpha)} \tag{47}$$

Are given by the following condition:

$$g'(\alpha) = \frac{r_a - g(\alpha)}{1-\alpha}$$

Proof

Differentiating equation (47) with respect to α and equating to zero, we get

$$\frac{\partial V_0}{\partial \alpha} = \left[\frac{(-1)}{r_a - g(\alpha)} + (-1)\frac{(1-\alpha)}{(r_a - g(\alpha))^2}(-g'(\alpha))\right]E(X_1) = 0$$

Dividing by $E(X_1)$ and rearranging we get

$$\left[\frac{-1}{r_a - g(\alpha)} + \frac{(1-\alpha)g'(\alpha)}{(r_a - g(\alpha))^2}\right] = 0 \tag{48}$$

Multiplying (48) by $r_a - g(\alpha)$

$$\left[\frac{(1-\alpha)g'(\alpha)}{(r_a - g(\alpha))}\right] = 1$$

Rearranging, we get the optimality condition

$$g'(\alpha) = \frac{r_a - g(\alpha)}{(1-\alpha)}$$

Q.E.D.

Appendix B

Proposition

$r_g(\alpha)$ declines with α.

Proof

We need to show that

$$\frac{d(r_g(\alpha))}{d\alpha} < 0 \tag{49}$$

Substituting the definition of $r_g(\alpha)$ in (49), we need to show that

$$\frac{d\left(\frac{g(\alpha)}{\alpha}\right)}{d\alpha} < 0$$

Applying the general differentiation rule

$$\frac{d\left(\frac{u}{v}\right)}{d\alpha} = \frac{v\frac{du}{d\alpha} - u\frac{dv}{d\alpha}}{v^2}$$

we get

$$\frac{d\left(\frac{g(\alpha)}{\alpha}\right)}{d\alpha} = \frac{\alpha\frac{d(g(\alpha))}{d\alpha} - g(\alpha)\frac{d\alpha}{d\alpha}}{\alpha^2}$$

and with few more algebraic manipulations, we get

$$\frac{d\left(\frac{g(\alpha)}{\alpha}\right)}{d\alpha} = \frac{\frac{d(g(\alpha))}{d\alpha} - \frac{g(\alpha)}{\alpha}}{\alpha} \tag{50}$$

Select now a specific α, and denote it by $\alpha^\dagger$. The tangent line that goes through $\alpha^\dagger$ can be written as

$$g(\alpha) = \lambda + g'(\alpha^\dagger)\alpha \tag{51}$$

Note that because $g(\alpha)$ is defined in the first quadrant and because it starts from the origin (due to the assumption $g(0) = 0$) and because it is concave, it follows that λ must be positive. The slope of (51), $g'(\alpha^\dagger)$, can be computed from any two points on this linear line, particularly from $(0, \lambda)$ and $(\alpha^\dagger, g(\alpha^\dagger))$, implying that

$$g'(\alpha^\dagger) = \frac{g(\alpha^\dagger) - \lambda}{\alpha^\dagger}$$

But note that because $g(\alpha)$, α and λ are all positive, then

$$g'(\alpha^{\dagger}) = \frac{g(\alpha^{\dagger}) - \lambda}{\alpha^{\dagger}} < \frac{g(\alpha^{\dagger})}{\alpha^{\dagger}}$$

Therefore, the numerator in (50) must be negative, implying that $r_g(\alpha)$ declines with α.

Q.E.D.

Appendix C

Proposition

The extremum points of

$$\max_{\alpha} \; V_{0,TX}(\alpha) = \frac{(1-T)(1-\alpha)E(X_1)}{r_a - g(\alpha)} \tag{52}$$

are given by the following condition:

$$g'(\alpha) = \frac{r_a - g(\alpha)}{1-\alpha}$$

Proof

Differentiating equation (47) with respect to α and equating to zero, we get

$$\frac{\partial V_{0,TX}}{\partial \alpha} = \left[\frac{(-1)}{r_a - g(\alpha)} + (-1)\frac{(1-\alpha)}{(r_a - g(\alpha))^2}(-g'(\alpha)) \right](1-T)E(X_1) = 0$$

Dividing by $(1-T)E(X_1)$ and rearranging, we get

$$\left[\frac{-1}{r_a - g(\alpha)} + \frac{(1-\alpha)g'(\alpha)}{(r_a - g(\alpha))^2} \right] = 0 \tag{53}$$

Multiplying (53) by $r_a - g(\alpha)$, we have

$$\left[\frac{(1-\alpha)g'(\alpha)}{(r_a - g(\alpha))} \right] = 1$$

Rearranging, we get the optimality criterion

$$g'(\alpha) = \frac{r_a - g(\alpha)}{(1-\alpha)}$$

Q.E.D.

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Table 1: Simulation of the average cost of investments and the average cost of distributions

Table 1 presents simulation results for the cost of investments, $r_g(\alpha^*)$, and the cost of distributions, $r_\psi(\alpha^*)$. We use different specifications of the parameters (b and θ) in the power growth function

$$g(\alpha) = b \cdot \alpha^{1-\theta}$$

where α in this function is the retention ratio.

Panel A: $b = 0.08$									
	$\theta = 0.1$	$\theta = 0.2$	$\theta = 0.3$	$\theta = 0.4$	$\theta = 0.5$	$\theta = 0.6$	$\theta = 0.7$	$\theta = 0.8$	$\theta = 0.9$
$r_a = 9\%$	$r_g = 9.9\%$ $r_\psi = 8.9\%$	$r_g = 10.6\%$ $r_\psi = 8.5\%$	$r_g = 11.3\%$ $r_\psi = 7.9\%$	$r_g = 12.1\%$ $r_\psi = 7.3\%$	$r_g = 13.1\%$ $r_\psi = 6.6\%$	$r_g = 14.4\%$ $r_\psi = 5.8\%$	$r_g = 16.2\%$ $r_\psi = 4.9\%$	$r_g = 19.3\%$ $r_\psi = 3.9\%$	$r_g = 27.2\%$ $r_\psi = 2.7\%$
$r_a = 10\%$	$r_g = 11.1\%$ $r_\psi = 10.0\%$	$r_g = 12.1\%$ $r_\psi = 9.7\%$	$r_g = 13.2\%$ $r_\psi = 9.3\%$	$r_g = 14.5\%$ $r_\psi = 8.7\%$	$r_g = 16.0\%$ $r_\psi = 8.0\%$	$r_g = 18.0\%$ $r_\psi = 7.2\%$	$r_g = 21.0\%$ $r_\psi = 6.3\%$	$r_g = 26.2\%$ $r_\psi = 5.2\%$	$r_g = 39.8\%$ $r_\psi = 4.0\%$
$r_a = 11\%$	$r_g = 12.2\%$ $r_\psi = 11\%$	$r_g = 13.5\%$ $r_\psi = 10.8\%$	$r_g = 14.9\%$ $r_\psi = 10.4\%$	$r_g = 16.5\%$ $r_\psi = 9.9\%$	$r_g = 18.5\%$ $r_\psi = 9.3\%$	$r_g = 21.3\%$ $r_\psi = 8.5\%$	$r_g = 25.3\%$ $r_\psi = 7.6\%$	$r_g = 32.4\%$ $r_\psi = 6.5\%$	$r_g = 51.6\%$ $r_\psi = 5.1\%$
$r_a = 12\%$	$r_g = 13.3\%$ $r_\psi = 12.0\%$	$r_g = 14.8\%$ $r_\psi = 11.9\%$	$r_g = 16.5\%$ $r_\psi = 11.6\%$	$r_g = 18.5\%$ $r_\psi = 11.1\%$	$r_g = 20.9\%$ $r_\psi = 10.5\%$	$r_g = 24.3\%$ $r_\psi = 9.7\%$	$r_g = 29.2\%$ $r_\psi = 8.8\%$	$r_g = 38.3\%$ $r_\psi = 7.7\%$	$r_g = 63.0\%$ $r_\psi = 6.3\%$
$r_a = 13\%$	$r_g = 14.3\%$ $r_\psi = 13.0\%$	$r_g = 16.1\%$ $r_\psi = 12.9\%$	$r_g = 18.1\%$ $r_\psi = 12.6\%$	$r_g = 20.3\%$ $r_\psi = 12.2\%$	$r_g = 23.3\%$ $r_\psi = 11.6\%$	$r_g = 27.2\%$ $r_\psi = 10.9\%$	$r_g = 33.2\%$ $r_\psi = 10.0\%$	$r_g = 44.2\%$ $r_\psi = 8.8\%$	$r_g = 73.6\%$ $r_\psi = 7.4\%$
Panel B: $b = 0.1$									
	$\theta = 0.1$	$\theta = 0.2$	$\theta = 0.3$	$\theta = 0.4$	$\theta = 0.5$	$\theta = 0.6$	$\theta = 0.7$	$\theta = 0.8$	$\theta = 0.9$
$r_a = 11\%$	$r_g = 12.0\%$ $r_\psi = 10.8\%$	$r_g = 12.8\%$ $r_\psi = 10.3\%$	$r_g = 13.6\%$ $r_\psi = 9.5\%$	$r_g = 14.5\%$ $r_\psi = 8.7\%$	$r_g = 15.6\%$ $r_\psi = 7.8\%$	$r_g = 17.0\%$ $r_\psi = 6.8\%$	$r_g = 18.9\%$ $r_\psi = 5.7\%$	$r_g = 22.2\%$ $r_\psi = 4.4\%$	$r_g = 30.5\%$ $r_\psi = 3.0\%$
$r_a = 12\%$	$r_g = 13.2\%$ $r_\psi = 11.9\%$	$r_g = 14.4\%$ $r_\psi = 11.5\%$	$r_g = 15.6\%$ $r_\psi = 10.9\%$	$r_g = 17.0\%$ $r_\psi = 10.2\%$	$r_g = 18.6\%$ $r_\psi = 9.3\%$	$r_g = 20.8\%$ $r_\psi = 8.3\%$	$r_g = 24.0\%$ $r_\psi = 7.2\%$	$r_g = 29.5\%$ $r_\psi = 5.9\%$	$r_g = 43.5\%$ $r_\psi = 4.4\%$
$r_a = 13\%$	$r_g = 14.4\%$ $r_\psi = 13.0\%$	$r_g = 15.8\%$ $r_\psi = 12.7\%$	$r_g = 17.4\%$ $r_\psi = 12.2\%$	$r_g = 19.2\%$ $r_\psi = 11.5\%$	$r_g = 21.3\%$ $r_\psi = 10.7\%$	$r_g = 24.2\%$ $r_\psi = 9.7\%$	$r_g = 28.4\%$ $r_\psi = 8.5\%$	$r_g = 36.0\%$ $r_\psi = 7.2\%$	$r_g = 55.5\%$ $r_\psi = 5.6\%$
$r_a = 14\%$	$r_g = 15.6\%$ $r_\psi = 14.0\%$	$r_g = 17.2\%$ $r_\psi = 13.8\%$	$r_g = 19.0\%$ $r_\psi = 13.3\%$	$r_g = 21.2\%$ $r_\psi = 12.7\%$	$r_g = 23.8\%$ $r_\psi = 11.9\%$	$r_g = 27.3\%$ $r_\psi = 10.9\%$	$r_g = 32.6\%$ $r_\psi = 9.8\%$	$r_g = 42.1\%$ $r_\psi = 8.4\%$	$r_g = 67.4\%$ $r_\psi = 6.7\%$
$r_a = 15\%$	$r_g = 16.7\%$ $r_\psi = 15.0\%$	$r_g = 18.5\%$ $r_\psi = 14.8\%$	$r_g = 20.7\%$ $r_\psi = 14.4\%$	$r_g = 23.1\%$ $r_\psi = 13.9\%$	$r_g = 26.2\%$ $r_\psi = 13.1\%$	$r_g = 30.4\%$ $r_\psi = 12.1\%$	$r_g = 36.5\%$ $r_\psi = 11.0\%$	$r_g = 47.9\%$ $r_\psi = 9.6\%$	$r_g = 78.7\%$ $r_\psi = 7.8\%$

Table 2: Simulation of retention ratios

Table 2 presents simulation results for the optimal retention ratio. We use different specifications of the parameters (b and θ) in the power growth function

$$g(\alpha) = b \cdot \alpha^{1-\theta}$$

where α in this function is the retention ratio.

Panel A: $b = 0.08$									
	$\theta = 0.1$	$\theta = 0.2$	$\theta = 0.3$	$\theta = 0.4$	$\theta = 0.5$	$\theta = 0.6$	$\theta = 0.7$	$\theta = 0.8$	$\theta = 0.9$
$r_a = 9\%$	0.123	0.245	0.313	0.352	0.372	0.376	0.365	0.332	0.257
$r_a = 10\%$	0.039	0.125	0.187	0.227	0.250	0.258	0.252	0.227	0.168
$r_a = 11\%$	0.015	0.073	0.125	0.163	0.186	0.196	0.193	0.174	0.126
$r_a = 12\%$	0.006	0.046	0.089	0.123	0.146	0.157	0.157	0.141	0.101
$r_a = 13\%$	0.003	0.030	0.066	0.097	0.118	0.130	0.131	0.118	0.085
Panel B: $b = 0.1$									
	$\theta = 0.1$	$\theta = 0.2$	$\theta = 0.3$	$\theta = 0.4$	$\theta = 0.5$	$\theta = 0.6$	$\theta = 0.7$	$\theta = 0.8$	$\theta = 0.9$
$r_a = 11\%$	0.160	0.288	0.356	0.393	0.412	0.415	0.403	0.368	0.290
$r_a = 12\%$	0.060	0.160	0.226	0.266	0.288	0.295	0.287	0.259	0.195
$r_a = 13\%$	0.026	0.100	0.158	0.197	0.220	0.230	0.225	0.202	0.149
$r_a = 14\%$	0.012	0.066	0.117	0.153	0.177	0.187	0.185	0.166	0.120
$r_a = 15\%$	0.006	0.046	0.089	0.123	0.146	0.157	0.157	0.141	0.101

Table 3: Simulation of the price/profits ratio

Table 3 presents simulation results for the price/profit ratio. We use different specifications of the parameters (b and θ) in the power growth function

$$g(\alpha) = b \cdot \alpha^{1-\theta}$$

where α in this function is the retention ratio.

Note that from equation (24) the value of the firm is

$$V_0(\alpha) = \frac{E(X_1)}{r_\psi(\alpha)}$$

where $E(X_1)$ are the expected profits of the firm, and $r_\psi(\alpha)$ is the cost of distributions.

It follows immediately that in optimum the price/profit ratio is given by

$$\frac{V_0(\alpha^*)}{E(X_1)} = \frac{1}{r_\psi(\alpha^*)}$$

Panel A: $b = 0.08$									
	$\theta = 0.1$	$\theta = 0.2$	$\theta = 0.3$	$\theta = 0.4$	$\theta = 0.5$	$\theta = 0.6$	$\theta = 0.7$	$\theta = 0.8$	$\theta = 0.9$
$r_a = 9\%$	11.26	11.79	12.60	13.72	15.24	17.38	20.57	25.86	36.85
$r_a = 10\%$	10.04	10.31	10.80	11.51	12.50	13.88	15.88	19.07	25.16
$r_a = 11\%$	9.11	9.26	9.58	10.08	10.78	11.77	13.19	15.41	19.44
$r_a = 12\%$	8.34	8.43	8.65	9.02	9.55	10.30	11.38	13.03	15.94
$r_a = 13\%$	7.69	7.75	7.91	8.19	8.60	9.19	10.04	11.33	13.56
Panel B: $b = 0.1$									
	$\theta = 0.1$	$\theta = 0.2$	$\theta = 0.3$	$\theta = 0.4$	$\theta = 0.5$	$\theta = 0.6$	$\theta = 0.7$	$\theta = 0.8$	$\theta = 0.9$
$r_a = 11\%$	9.25	9.75	10.48	11.47	12.83	14.75	17.63	22.47	32.80
$r_a = 12\%$	8.39	8.67	9.14	9.81	10.73	12.02	13.92	16.97	22.95
$r_a = 13\%$	7.71	7.88	8.21	8.70	9.39	10.34	11.73	13.91	17.98
$r_a = 14\%$	7.15	7.26	7.50	7.87	8.40	9.15	10.22	11.88	14.89
$r_a = 15\%$	6.67	6.74	6.92	7.21	7.64	8.24	9.10	10.42	12.75

Figure 1: An example of the relation between the growth function the discount rate and the distribution yield

This figure demonstrates the relation between the growth function, the discount rate and the distribution yield. The discount rate is assumed to be equal to 0.1. The growth function is $g(\alpha) = 0.08\sqrt{\alpha}$. The distribution yield is the difference between the two.

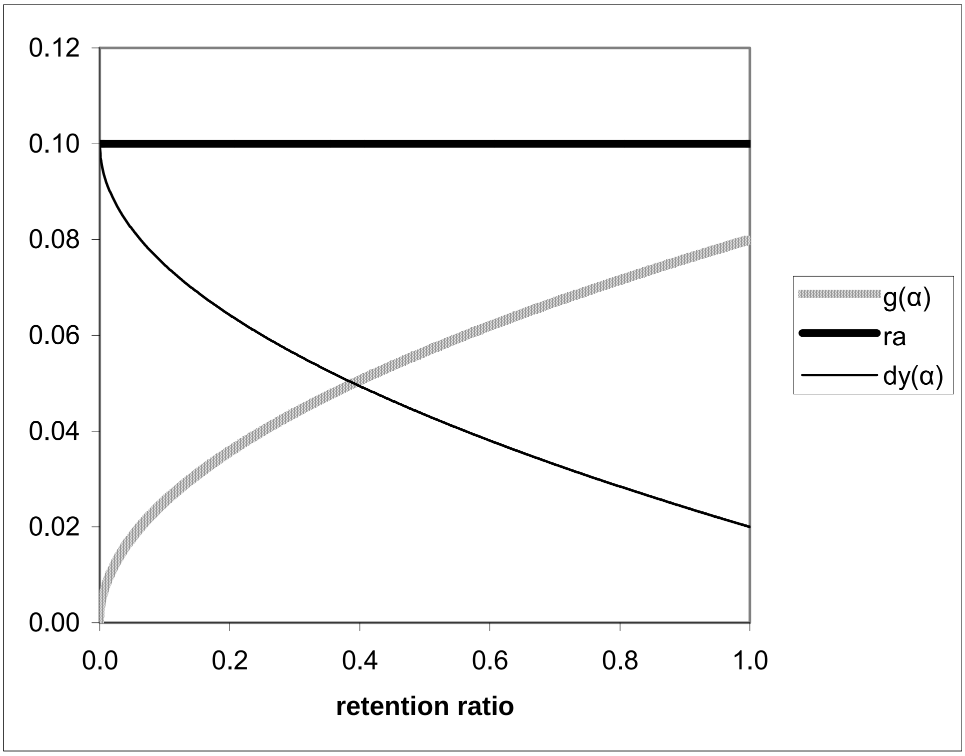

Figure 2: An example of the average cost of investments and the average cost of distributions

This figure demonstrates the average cost of investments and the average cost of distributions for $r_a = 0.1$ and a growth function given by $g(\alpha) = 0.08\sqrt{\alpha}$. The average cost of investments is given by $r_g(\alpha) = \dfrac{g(\alpha)}{\alpha}$. The average cost of distributions is given by $r_\psi(\alpha) = \dfrac{dy}{1-\alpha}$.

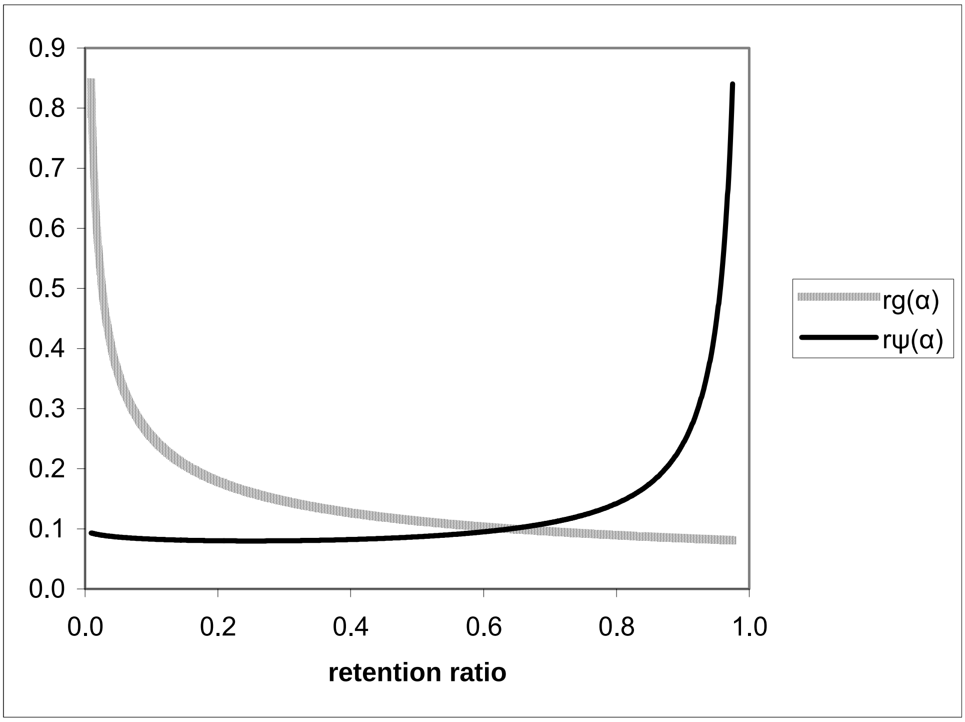

Figure 3: An example of the marginal growth rate vs. the average cost of distributions

Figure 3 reports the results for the average cost of distributions, $r_\psi(\alpha)$, and the marginal growth rate, $g'(\alpha)$, for $r_a = 0.1$ and a growth function given by $g(\alpha) = 0.08\sqrt{\alpha}$. The optimal retention ratio is generated from the equality between the two.

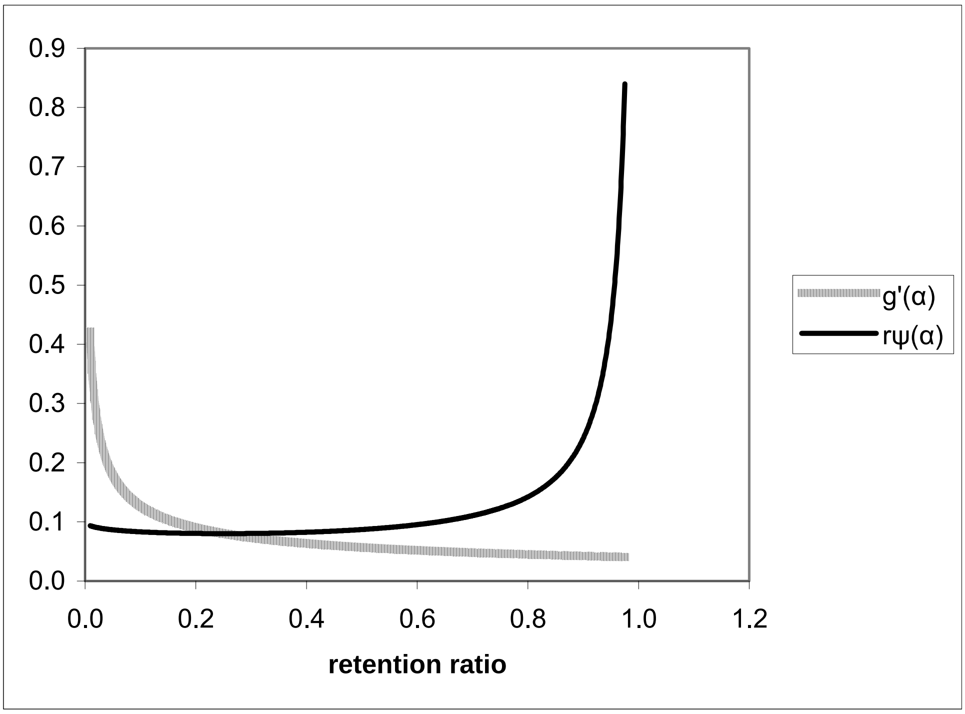

Figure 4: An example of the firm's value as a function of the retention ratio

Figure 4 presents the value of the firm as a function of the retention ratio for the case $r_a = 0.1$, a growth function given by $g(\alpha) = 0.08\sqrt{\alpha}$, and an expected cash flow at time 1 given by $X_1 = \$1M$.