An Economic Motivation for Variance Contracts

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Abstract

Variance contracts permit the trading of 'variance risk', i.e. the risk that the (squared) volatility of stock returns changes randomly over time. We discuss why investors might want to trade this type of risk, and why they might prefer a variance contract to standard calls and puts for this purpose.

Our main argument is that the variance contract is superior to a dynamic replication strategy due to discrete trading, parameter risk, and model risk. To show this we analyze the local hedging errors for the variance contract under different scenarios, namely under pure estimation risk (or parameter risk) in a stochastic volatility and in a jump-diffusion model, under model risk when the wrong type of risk factor is assumed to be present (stochastic volatility instead of jumps or vice versa), and under model risk when risk factors are omitted (e.g. when the true model contains jumps which are not present in the model assumed by the investor). The results confirm that the variance contract is exposed to model risk to an economically significant degree, and that it is much harder to hedge than, e.g., deep OTM puts. We thus conclude that the improvement provided by the introduction of a variance contract is greater than the one offered by the introduction of additional standard options.

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1 Introduction and Motivation

There is ample evidence from empirical studies of option markets that there are additional priced factors beyond plain stock price risk (see, e.g., Bakshi, Cao, and Chen (1997), Bates (2000), Pan (2002), and Eraker (2004)). Therefore, extensions of the seminal model developed by Black and Scholes (1973) contain factors like stochastic volatility (SV) or stochastic jumps (SJ). The presence of these additional factors makes markets incomplete, so that trading in the stock and the money market account is not enough to replicate an arbitrary derivative payoff. For this purpose more contracts are needed. This insight is probably the most important motivation for the introduction of innovative derivatives like variance contracts, which recently took place on major exchanges like the CBOE, as described in Bondarenko (2004). A variance contract permits the trading of the stochastic variation of stock return volatility by basically paying off the sum of squared price changes. The larger the realized variance of the stock due to increases in stochastic volatility or jumps in the stock price, the higher the payoff of the variance contract, which therefore provides insurance against high volatility. One might argue that risk-averse investors are willing to accept a negative excess return for this kind of insurance, and there is indeed empirical evidence, e.g. in Bondarenko (2004), that the risk premium of the variance contract is significantly negative.

The first question we address is why investors would want to trade volatility risk. The standard answer lists two main motives: speculation and hedging. Liu and Pan (2003) solve a portfolio planning problem in a model with stochastic volatility and jumps (of deterministic size) in the stock price. They consider a complete market and derive the investor's optimal exposure to stock price risk, volatility risk, and jump risk. Their results show that the investor wants to trade both stochastic volatility and jump risk, and their results also allow to see under what circumstances different investors would want to hold long and short positions in volatility risk. This confirms that there is indeed a demand for positions in variance risk.

The main part of our paper is dedicated to the question why investors could prefer the introduction of the variance contract to an augmentation of the set of available options, in which they could then trade to generate the desired payoff patterns. A natural first step seems to be to distinguish between a complete and an incomplete market. In standard option pricing models like Heston (1993) the market is complete as soon as one convex claim (like a standard call option) is traded in addition to the stock and the money market account. In an SJ model with an infinite number of possible jump sizes, like the one suggested by Merton (1976), the market is complete if a continuum of options is traded. If the market is incomplete, one reason for the introduction of a variance contract could be that it is market-completing. However, the intuition is that the introduction of additional options should be enough to complete the market. Of course, we could replace any of these options by the variance contract. Still, if too few contracts are traded, why should one introduce a rather complicated derivative asset like a variance contract instead of additional European options? On the other hand, if the market is complete, the variance contract can be replicated by a dynamic trading strategy, and this also holds for other claims. There is no need to introduce further redundant claims, which in addition would only reduce the liquidity of existing contracts. Summing up: In an incomplete market the variance contract does not seem to be the most natural contract to introduce, and in a complete market, there is no need to introduce it.

Up to this point we have assumed that the investor does not face model risk and can trade in continuous time. If one or both conditions are not met, the variance contract or any other claim will be better than the associated 'replicating' strategy. The extent of this improvement should not simply be measured by transaction costs, since the amount of implicit and explicit transaction costs for such a new contract cannot be determined reliably before its actual introduction. We rather argue that the replicating strategy may fail due to discrete trading and — more importantly — due to model risk which makes it impossible for the investor to know the correct composition of the hedge portfolio in the first place.

Our ultimate argument for the variance contract then consists of three steps: First, a new contract should only be introduced if investors want to trade the payoff profile or risk exposure provided by this contingent claim. We argue that this condition holds for the variance contract, which provides insurance against increasing market uncertainty. Second, the introduction makes sense only if the replicating strategy for the new contract suffers from model risk and discrete trading to an economically significant extent, and we show that this is the case for the variance contract. Third, if there is still more than one contract that could be introduced, the choice should be the one which is hardest to replicate. The main result of our analysis is indeed that the variance contract is much harder to replicate than standard derivatives like a deep OTM put option.

To concentrate on the main issues we make several simplifying assumptions. First, we restrict our analysis to the case where only the stock, the money market account, and one further option are already traded, and we choose the models such that the market is complete with this set of basis assets. We thus ignore incompleteness of the true model. Since incompleteness would make it even harder to replicate a given claim, our analysis can be considered conservative. Second, we also ignore static strategies and strategies involving more than one option. Certainly, hedges containing a larger number of instruments would reduce hedging errors. However, our choice of simpler strategies avoids any dependence of the results on the choice of strategy and can be regarded as an analysis under a magnifying glass. We also do not use the semi-static replication strategies for variance contracts considered, e.g., by Carr and Wu (2004). They show that for an underlying without jumps the variance contract is equal to a log-contract (which can be statically replicated via a continuum of options) and some dynamic strategy in the underlying. In practice, this approach suffers from at least two problems. The first is the assumption that a continuum of options is available. Even if enough options are traded to *price* the variance contract, this does not necessarily imply that the static hedge of the log-contract is of similar accuracy. Second, the hedge would anyway be perfect only for a certain class of models with deterministic interest rates and without jumps. Especially the latter represents a quite restrictive assumption.

In the case of discrete trading the main question is how a replication strategy performs if it is not adjusted continuously. The objective is to trade volatility risk and to receive a profit whenever volatility increases or when there are jumps in the stock price. Additionally the strategy should not have linear exposure to stock price risk, i.e. it should be delta-neutral. The first possibility is to use a variance contract which fulfills these requirements by construction without the need to continuously adjust the position. An alternative would be to use a straddle, which is often considered an ideal instrument for volatility trading. However, the straddle would have to be delta-neutral, so the choice of the strike price is model-dependent, and the hedge would have to be adjusted continuously, in particular after large jumps in the stock price. Finally, one could employ a dynamic replicating strategy for the variance contract. However, we would also have to adjust the position continuously, so the problems would be quite similar to those described for the delta-neutral straddle.

Model risk means that the data generating process is not known so that the replication strategy will be determined in some model (the hedge model), which is in general not equal to the true model. The dynamic replication strategy will thus be more risky than trading the contract itself. Model risk can occur in different degrees. We consider the impact of parameter uncertainty and model mis-specification. Parameter risk means that the correct type of model is used, but with incorrect parameters (possibly derived from some estimation, so that parameter risk is basically equivalent to estimation risk). A second form of model risk is when the wrong risk factors are included in the hedge model. The most prominent example is to use a hedge model which contains a stochastic volatility component, while the true model exhibits stochastic jumps (or vice versa). Finally, risk factors present in the true model may be omitted in the hedge model, so that instead of stochastic volatility and stochastic jumps together only one of these risk factors is included in the hedge model.

To analyze the impact of model risk on the replicating strategy we derive analytical expressions for local hedging errors over an infinitesimal time interval. The global hedging error over some interval from t to $t+\tau$ is then simply given by the integral over these local errors, so that an analysis of local errors seems sufficient. We show that for risk factors contained in the true and in the hedge model the hedging error depends on the error in the sensitivities of the claim with respect to the risk factors as well as on the error in the sensitivities of the hedge instruments. We also derive a robustness condition, under which the strategy would have a zero exposure to model risk, and which basically says that the two errors mentioned above should exactly offset each other. A similar decomposition of the hedging error and similar robustness conditions can be derived for risk factors contained in the true model, but not in the hedge model.

Intuitively, we expect model risk to be only a moderate problem when the claim to be hedged and the hedge instrument are very similar and to be the more severe the more the two contracts differ. We rely on numerical examples to assess the impact of model risk and to compare the exposures of different claims to model risk. In these examples we use the true model to generate prices. We then calibrate the hedge model to the prices of selected options and analyze the hedging errors over the next infinitesimal interval for given changes in the state variables. To assess the impact of model risk we have to compare the hedging error in the presence of model risk to the error when the true model is used to compute hedge ratios. We assume that the true model is complete so that in principle a perfect hedge with zero error would be possible.

We will now give a brief summary of our results. In our hedging experiments the stock, the money market account, and standard European options with varying strike prices are used as hedge instruments. The hedge of a deep out-of-the-money (OTM) put option serves as the benchmark case. We have chosen this instrument, since it represents an example for a standard claim which may compete with the variance contract for introduction. While the motivation for the latter is an insurance against variance risk, the argument for the put is that it provides crash protection. It is not surprising that the hedging error for this OTM put is the smaller the lower the difference between its strike and the strike of the option we use as hedge instrument, which means that we can actually choose some best hedge instrument among the available contracts. For the variance contract the hedging errors are comparable in size to those observed for the benchmark put. However, the variance contract is more difficult to hedge for two reasons. First, there is no ideal hedge instrument for which the hedging error due to model risk would vanish. Second, a hedge instrument which provides a good hedge for the variance contract in one scenario may perform rather badly in other scenarios, so there is no dominant choice of hedge instruments. The main conclusion thus is that the variance contract is exposed to model risk much more than a put option deep OTM. That is why the actual introduction of the variance contract is an improvement over the situation when investors have to replicate its payoff using traded options only, and it offers a more significant improvement than a deep OTM put. In a nutshell, the variance contract provides the easiest way to generate a positive exposure to increasing realized stock variation without sensitivity with respect to the stock price and irrespective of the true model.

Several strands of the literature are related to our paper. First of all, to identify the risk factors present in options markets some researchers performed tests based on either the spanning properties of primitive basis assets or on the properties of hedging errors for presumably delta-neutral positions. Examples for the first kind of analyses are the work by Buraschi and Jackwerth (2001) and by Coval and Shumway (2001), while Bakshi and Kapadia (2003) is the best-known representative for the second class of approaches. The key findings are in all cases that there are additional risk factors beyond stock price risk which influence the prices of traded options. The variance contract was investigated, among others, by Carr and Wu (2004) and by Bondarenko (2004). Whereas the first paper mostly deals with issues of pricing and replication of variance contracts, the second paper explicitly discusses the opportunity to use such a contract as a vehicle for an investment into volatility changes. Finally, Liu and Pan (2003) provide an in-depth analysis of portfolio selection problems in SV and SJ models, when the investor can also hold derivative assets.

The remainder of the paper is structured as follows. In Section 2 we analyze the variance contract with respect to its pricing and the risk factors its holder is exposed to. In Section 3 we discuss why investors want to trade variance risk. Section 4 contains the main results of the paper. Section 5 concludes.

2 Risk Factors Affecting Variance Contracts

The first step in analyzing a new contract is to investigate its exposure to the different risk factors in the given model. We show that, possibly in contrast to intuition, the 'variance risk' captured by the variance contract is not just stochastic volatility, and that the risk premium earned is not just a premium for stochastic volatility.

2.1 Model Setup

We use a model with stochastic volatility and jumps in both the stock price and in volatility. This type of model has been investigated by Duffie, Pan, and Singleton (2000), Eraker (2004), and Broadie, Chernov, and Johannes (2004). The stochastic processes for the state variables under the true measure P are given by the stochastic differential equations

$$\begin{aligned} dS_t &= \mu S_t dt + \sqrt{V_t} S_t dW_t^{(S)} + S_{t-} \left[\left(e^{X_t} - 1 \right) dN_t - E^P \left[e^X - 1 \right] k^P \, dt \right] \qquad (1) \\ dV_t &= \kappa^P \left(\theta^P - V_t \right) dt + \sigma_V \sqrt{V_t} \left(\rho dW_t^{(S)} + \sqrt{1-\rho^2} dW_t^{(V)} \right) + Y_t dN_t. \qquad (2) \end{aligned}$$

The drift of the stock μ depends on the market prices of risk as explained below. The intensity of the jump process under P is given by $k^P = k_0^P + k_1^P V_t$, the jump size X for the log of the stock price is assumed to be normally distributed with mean $\ln(1+\mu_X^P) - 0.5\sigma_X^2$ and variance σ_X^2. The volatility jump size Y follows an exponential distribution with mean μ_Y^P. We make the simplifying assumptions that jumps in volatility occur simultaneously with jumps in the stock price, and that the jump sizes are uncorrelated. Furthermore, we assume that the distributions of X and Y do not depend on time t.

Under the risk-neutral measure Q the processes are

$$\begin{aligned} dS_t &= r S_t dt + \sqrt{V_t} S_t d\widetilde{W}_t^{(S)} + S_{t-} \left[\left(e^{X_t} - 1 \right) dN_t - E^Q \left[e^X - 1 \right] k^Q \, dt \right] \\ dV_t &= \underbrace{(\kappa^P + \sigma_V \lambda_V)}_{\kappa^Q} \Bigg(\underbrace{\frac{\kappa^P \theta^P}{\kappa^P + \sigma_V \lambda_V}}_{\theta^Q} - V_t \Bigg) dt + \sigma_V \sqrt{V_t} \left(\rho d\widetilde{W}_t^{(S)} + \sqrt{1-\rho^2} d\widetilde{W}_t^{(V)} \right) + Y_t dN_t. \end{aligned}$$

The change in κ and θ depends on the market price λ^V of volatility diffusion risk. The Q-intensity of the jump process is $k^Q = k_0^Q + k_1^Q V_t$, the distribution of the jump size for the stock changes to $X \sim N(\ln(1+\mu_X^Q) - 0.5\sigma_X^2, \sigma_X^2)$, and the jump size in volatility is exponentially distributed with mean μ_Y^Q. We now discuss our assumptions concerning the market prices of risk. For an asset exposed to stock price risk there is a premium for diffusion risk equal to $\lambda^S \sqrt{V_t}$ times the amount of risk. Furthermore, there is a jump risk premium which is determined by the differences in the jump intensity and the jump size distribution between P and Q. The stock price has a drift of μ given by

$$\mu \equiv r + \lambda^S V_t + E^P \left[\left(e^X - 1 \right) k^P - E^Q \left(e^X - 1 \right) k^Q \right].$$

For a claim exposed to dV there is also a compensation for the diffusion risk in V, given by $\lambda_V \sqrt{V_t}$ times the amount of risk, and a premium for jump risk in volatility, which again depends on the difference in the intensity of the jump and the jump size distribution.

2.2 Pricing the Variance Contract

The payoff C_T of a variance contract at its maturity date T is equal to the realized variance $RV(0,T)$ of the stock over the time interval $[0,T]$. When discrete returns are used we have

$$RV(0,T) = \int_0^T V_u du + \int_0^T \left(e^{X_u} - 1\right)^2 dN_u,$$

while in the case of log-returns the payoff is given by

$$RV(0,T) = \int_0^T V_u du + \int_0^T X_u^2 dN_u.$$

The first integral is the accumulated variance of the diffusion component in stock returns (the extension to multiple diffusions is straightforward), i.e. the larger V, the larger the payoff. The second integral represents the sum of squared jumps in the stock price (note that the sign of the jumps does not matter). If there are sudden large changes in the stock price, the payoff of the variance contract also increases. In what follows we assume continuous monitoring and do not discuss problems related to discretization error or measurement error, as it is done, e.g., in Bondarenko (2004) and Carr and Wu (2004). Furthermore, we focus on the case of log returns.

The price at time t of the variance contract for continuously monitored log returns is given by

$$C_t = E^Q\left[e^{-r(T-t)}\left(\int_0^T V_u du + \int_0^T X_u^2 dN_u\right) \Bigg| \mathcal{F}_t\right],$$

which is equal to

$$C_t = e^{-r(T-t)}\Bigg\{RV(0,t) + E^Q[X^2]k_0^Q(T-t) + \left(1 + k_1^Q E^Q[X^2]\right)\left[\widetilde{\theta}^Q\ (T-t) + \frac{1 - e^{-\widetilde{\kappa}^Q(T-t)}}{\widetilde{\kappa}^Q}\left(V_t - \widetilde{\theta}^Q\right)\right]\Bigg\}, \quad (3)$$

where $\widetilde{\kappa}^Q = \kappa^Q - k_1^Q E^Q[Y]$ and $\widetilde{\theta}^Q = \frac{\kappa^Q\theta^Q + k_0^Q E^Q[Y]}{\widetilde{\kappa}^Q}$. A proof of the pricing formula is given in Appendix A.1. We write the price as a function of the state variables i.e. $C_t = c(t, V_t, RV(0,t), \ldots)$. For discrete returns, one can simply replace X by $e^X - 1$ in the above formula.

We assume that $RV(0,t)$ is observable, so, at time t it is easy to decompose the variance contract over the period $[0;T]$ into an investment in the money market account and an investment in a variance contract over $[t;T]$. The price of the future realized variance, $RV(t,T)$, depends on the distribution of jumps in the stock price (irrespective of their signs), on their intensity, and on the expected volatility over the life of the contract. The mixed term, where both jumps and volatility show up, is due to a volatility dependent jump intensity.

2.3 Risk Premia for the Variance Contract

The sensitivity of the variance contract with respect to the stock price is zero, so it is delta-neutral by construction. The partial derivative with respect to volatility is given by

$$\frac{\partial c}{\partial v} \;=\; e^{-r(T-t)}\left(1+k_1^Q E^Q[X^2]\right)\frac{1-e^{-\widetilde{\kappa}^Q(T-t)}}{\widetilde{\kappa}^Q} \;>\; 0$$

where we assume that $k_1^Q > 0$. This assumption is not too restrictive, since otherwise the jump intensity could become negative for sufficiently large values of V. Jumps in the stock price and in volatility also have an impact on C. The price change due to a simultaneous jump in the stock price and volatility is given by

$$\Delta C_t = e^{-r(T-t)}X_t^2 dN_t + e^{-r(T-t)}\left(1+k_1^Q E^Q[X^2]\right)\frac{1-e^{-\widetilde{\kappa}^Q(T-t)}}{\widetilde{\kappa}^Q} Y_t dN_t \geq 0.$$

The first term on the right hand side represents the impact of a jump in the stock price, increasing the accumulated payoff $RV(0,t)$, while the second term captures the impact of a jump in volatility, changing the price of future realized variance $RV(t,T)$.

One of the main characteristics of the variance contract is that, by construction, it is delta-neutral and always provides a positive exposure to the realized variance of the stock. We do not have to rely on a dynamic (and therefore model-dependent) strategy.

The difference between the drifts of C under P and Q is the local risk premium on the variance contract:

$$\begin{aligned}
&E^P[dC_t|\mathcal{F}_t] - E^Q[dC_t|\mathcal{F}_t] \\
&\quad= e^{-r(T-t)}\Big\{ \left(E^P[X^2]k^P - E^Q[X^2]k^Q\right) \\
&\qquad + \left(1+k_1^Q E^Q[X^2]\right)\frac{1-e^{-\widetilde{\kappa}^Q(T-t)}}{\widetilde{\kappa}^Q}\left[\lambda_V \sigma_V V_{t-} + \left(E^P[Y]k^P - E^Q[Y]k^Q\right)\right]\Big\}dt.
\end{aligned}$$

A proof is given in Appendix A.2. We can decompose the premium into the compensation for jump risk in the stock price (depending on the expectation of squared jumps and on the jump intensity), for volatility diffusion risk (depending on λ^V), and for volatility jump risk (depending on the expected jump size and on the jump intensity). Note that there is a premium for quadratic, but not for linear stock price risk, a consequence of the fact that the variance contract is delta-neutral. Furthermore, note that the excess return does not depend on the *sign* of the jump, because jumps increase the volatility of the underlying, no matter whether they are upward or downward.

This decomposition of the risk premium shows that there are several explanations for the empirically observed negative premium on the varaince contract (as found in Bondarenko (2004)). It can either be attributed to jump risk in the stock price (when jumps are more severe and more frequent under Q than under P), to volatility diffusion risk (with a negative market price of risk, given $\sigma_V > 0$ and $V > 0$, as in Heston (1993)), or to jump risk in volatility (again when jumps are larger and more frequent under Q than under P). Hence, for empirical studies it is important to keep in mind that the risk premium for the variance contract is not equal to the volatility risk premium, and that a negative risk premium does not necessarily imply a negative market price of volatility risk. Rather, it can be negative even when the volatility risk premium is equal to zero.

3 Motives for Trading Variance Contracts

If a contract is to be introduced, an important condition for its success that it provides a payoff profile or risk exposure that investors actually want to trade. For the variance contract the question is thus whether the investor is interested in an exposure in the second moment of stock returns. As the theoretical basis for our analyses we use the model suggested by Liu and Pan (2003), which is characterized by SV and stock price jumps of random occurrence, but of deterministic size. The dynamics of the stock price and the SV component given in Equations (1) and (2) simplify to

$$\begin{aligned} dS_t &= \left(r + \eta V_t + \mu_X k_1^P V_t - \mu_X k_1^Q V_t\right) S_t dt + \sqrt{V_t} S_t dW_t^{(S)} + \mu_X S_{t-}\left(dN_t - k_1^P V_t\, dt\right) \\ dV_t &= \kappa^P(\theta^P - V_t)dt + \sigma_V \sqrt{V_t}\left(\rho dW_t^{(S)} + \sqrt{1-\rho^2} dW_t^{(V)}\right). \end{aligned}$$

The intensity of the jump process is now equal to $k_1^P V_t$, the constant term k_0^P is set to zero, and there are no jumps in volatility. The risk premium for one unit of $dW_t^{(S)}$ is given by $\eta\sqrt{V_t}$, the compensation for $dW_t^{(V)}$ is $\xi\sqrt{V_t}$. The market prices of risk introduced in Section 2.1 are given by

$$\begin{aligned} \lambda^S &= \eta \\ \lambda^V &= \rho\eta + \sqrt{1-\rho^2}\xi. \end{aligned}$$

In the above setup the market is complete with two additional instruments besides the stock and the money market account. Liu and Pan (2003) solve the portfolio planning problem for an investor with constant relative risk aversion (power utility). The dynamics of wealth W are

$$\begin{aligned} dW_t = W_t \Big\{ & r dt + \theta_t^{(S)}\left(\sqrt{V_t} dW_t^{(S)} + \eta V_t dt\right) + \theta_t^{(V)}\left(\sqrt{V_t} dW_t^{(V)} + \xi V_t dt\right) \\ & + \theta_t^{(N)}\left[\mu_X dN_t - \mu_X k_1^P V_t\, dt + \mu_X\left(k_1^P - k_1^Q\right) V_t\, dt\right]\Big\}. \end{aligned}$$

As part of the solution the optimal exposure to the three fundamental sources of risk $W^{(S)}$, $W^{(V)}$, and N is determined as

$$\begin{aligned} \theta_t^{(S)} &= \frac{\eta}{\gamma} + \sigma_V \rho H(T-t) \\ \theta_t^{(V)} &= \frac{\xi}{\gamma} + \sigma_V \sqrt{1-\rho^2} H(T-t) \\ \theta_t^{(N)} &= \frac{1}{\mu_X} \left[\left(\frac{k_1^P}{k_1^Q} \right)^{\frac{1}{\gamma}} - 1 \right] \end{aligned}$$

with γ as the investor's coefficient of risk aversion, T as the investment horizon, and

$$\begin{aligned} H(\tau) &= \frac{\left(1-e^{-k_2\tau}\right)\delta}{2k_2 + (k_1-k_2)\left(1-e^{-k_2\tau}\right)} \\ \delta &= \frac{1-\gamma}{\gamma^2}\left(\eta^2+\xi^2\right) + 2k_1^Q \left[\left(\frac{k_1^P}{k_1^Q} \right)^{\frac{1}{\gamma}} - 1 + \frac{1-\frac{k_1^P}{k_1^Q}}{\gamma} \right] \\ k_1 &= \kappa^P - \frac{1-\gamma}{\gamma}\sigma_V\left(\rho\,\eta + \sqrt{1-\rho^2}\xi\right) \\ k_2 &= \sqrt{k_1^2 - \delta\sigma_V^2}. \end{aligned}$$

The optimal exposure can be decomposed into a speculative demand and a hedging demand. For the diffusion components the speculative demand depends on the ratio of the risk premium to the coefficient of risk aversion. The hedging demands are mainly driven by the correlation between stock returns and volatility changes. Depending on the level of risk aversion and the amount of the risk premia, the sign of the optimal exposure can vary, i.e. both long and short positions in the risk factors can basically be optimal. For the jump component the structure of the optimal demand is more involved, and the reader is referred to Liu and Pan (2003) for details.

4 Replication Strategies

As mentioned in the introduction we consider the degree of sensitivity to model risk as a key parameter in measuring the economic value of a newly introduced derivative contract. Even if the contract was basically replicable in a scenario with perfect knowledge about the true data-generating process, the hedge errors occurring in a mis-specified hedge model may nevertheless be substantial. In this case investors prefer trading the variance contract itself to the supposedly replicating strategy which turns out to be more risky. However, this argument not only holds for the variance contract, but also for other derivatives. Which contract should ultimately be introduced then depends on the amount of model risk exposure.

In this section we will analyze the hedging results for the variance contract under various types of model risk. Our results show that the variance contract is indeed significantly exposed to model risk. We furthermore analyze the hedging errors for a deep OTM put option with a strike price equal to 85% of the current stock price. This option is one example for a contract that competes with the variance contract for introduction. It also completes the market in the sense that it enlargens the set statically replicable payoffs. From an economic point of view, it provides crash protection and would thus be of interest to investors. For these two reasons we take the deep OTM put as a benchmark. The results show that the variance contract is exposed to model risk much more than the put. Taken together, if there is a need for an enlargement of the set of traded contracts, the variance contract offers a more significant improvement than just another put option.

4.1 Basic Setup

The objective is to replicate some claim H, and in the following this H will either be an OTM-put or the variance contract. The number of shares of the stock in the hedge portfolio at time t is denoted by $\phi_t^{(S)}$, the number of units of the hedge instrument i at time t is denoted by $\phi_t^{(i)}$ for $i = 1, \ldots, n$. S_t is the current stock price, $C_t^{(i)} = c^{(i)}(t, S_t, V_t, \ldots)$ stands for the price of the i-th hedge instrument written as a function of the state variables, and the price of the claim to be hedged is $H_t = h(t, S_t, V_t, \ldots)$. If there is only one instrument we denote its price by C and the associated number of units by $\phi_t^{(C)}$. Except for time, partial derivatives are denoted by subscripts. There is a true model, and there is a model assumed by the investor to run the hedge (hedge model). Whenever the calculation of the portfolio composition or the prices is done in the hedge model, we denote the variables by a tilde. For example, $\widetilde{\phi}^{(S)}$ is the number of shares of the stock in the hedge portfolio as calculated in the hedge model.

The value of the hedge portfolio at time t is denoted by Π_t. We assume that the hedge portfolio is self-financing, i.e. any proceeds are invested into the money market account, earning the constant risk-free rate r. At time t the hedging error is $D_t = H_t - \Pi_t$. If D_t is positive, the claim is worth more than the hedge portfolio, and vice versa.

The hedge portfolio must be chosen in such a way that its sensitivities with respect to the different risk factors are equal to those of the claim to be hedged. For the diffusion risk of the stock this means

$$\phi^{(S)}1 + \phi^{(1)}c_s^{(1)} + \ldots + \phi^{(n)}c_s^{(n)} \quad = \quad h_s, \tag{4}$$

while for state variable X_i ($i = 1, \ldots, m$) we must have that

$$\phi^{(1)}c_{X_i}^{(1)} + \ldots + \phi^{(n)}c_{X_i}^{(n)} \quad = \quad h_{X_i}. \tag{5}$$

Note that in our case $m = 1$ with SV as the only state variable, i.e. $X_1 = V$. The analogous condition for jump risk is

$$\phi^{(S)}\Delta S + \phi^{(1)}\Delta c + \ldots + \phi^{(n)}\Delta c^{(n)} \quad = \quad \Delta h, \tag{6}$$

where Δh and $\Delta c^{(i)}$ denote the change in the prices of the claim and the i-th hedge instrument, respectively. Of course, we would have a condition like this for any possible jump size.

The above equations are given for the true model. In the hedge model analogous conditions are used to determine the hedge portfolio, but the number of risk factors and the type of risk factors need not be the same as in the true model.

The hedge model is calibrated to the prices of certain options, i.e. the parameters are chosen such that the hedge model is as close as possible to observable market prices. We permit a non-perfect fit of the hedge model to the data, i.e. a hedge model is considered acceptable as long as the maximum deviation of model prices from given market prices is not too large (the maximum deviations are always in the range of 1% of the option price). The reason for this is that the investor might consider the hedge model as a simple approximation to the much more complicated true model. Furthermore, real world market frictions like bid-ask spreads could make it almost impossible to infer the exact model and/or the exact parameters anyway, as shown by Dennis and Mayhew (2004). Given the cross section of noisy prices, the investor thus cannot avoid parameter risk and model risk.

We consider simple extensions of the Black-Scholes model. To be specific, we work with the jump-diffusion model developed by Merton (1976), the SV model of Heston (1993), and the very general model suggested by Bakshi, Cao, and Chen (1997) (assuming a constant interest rate), which are the most prominent models that include stochastic volatility and/or jump risk. The analysis would only become more involved in more complicated models. We calibrate the hedge models to the prices of European options with moneyness (strike-to-spot ratio) of 0.95, 1, 1.05, and (in case of the SV model) also 0.9, and with a time to maturity of half a year. The time to maturity of the claim to be hedged is also equal to six months.

We assume that both the true model and the hedge model are complete with one or two additional options. For the SV model one additional option is enough to complete the market. For the jump-diffusion model we make the simplifying assumption of a deterministic jump size, so that the market is again complete with only one additional option. This setup makes it possible to focus on the impact of parameter risk and model risk, but to eliminate model incompleteness. The latter would just add another term to the hedging error, but it would not change our main results.

4.2 Parameter Risk

We start our analysis by considering the case of parameter risk. Here, the assumption is that the correct model type is used by the investor (e.g., Heston (1993) or Merton (1976)), but with an incorrect parametrization. The problem is realistic, since even if an investor knew the true model type with certainty, he or she would have to estimate the parameters and thus could not avoid estimation risk (parameter risk). To compare the hedge based on the hedge model to the perfect hedge, we calculate the local hedging error, i.e. we derive the stochastic differential equation (SDE) for the hedging error.

4.2.1 Stochastic Volatility Model

In the model suggested by Heston (1993), the local variance of the stock follows a mean-reverting square-root process. The dynamics under the true measure are given by

$$
\begin{aligned}
dS_t &= (r + \lambda^S V_t)\, dt + \sqrt{V_t} S_t dW_t^S \\
dV_t &= \kappa^P(\theta^P - V_t)\, dt + \sigma_V \sqrt{V_t} \left(\rho dW_t^S + \sqrt{1-\rho^2} dW_t^V \right)
\end{aligned}
$$

The hedging error under parameter risk is derived for a hedge portfolio consisting of the stock, the hedge instrument and the money market account. This portfolio is chosen such that it would replicate the claim in the hedge model. The SDE for the hedging error D is stated in

Proposition 1 (SV under Parameter Risk)

$$
\begin{aligned}
dD_t &= (H_t - \Pi_t) r dt + \left\{ h_s - \widetilde{h}_s - \widetilde{\phi}_t^{(C)} (c_s - \widetilde{c}_s) \right\} (dS_t - rS_t dt) \\
&\quad + \left\{ h_v - \widetilde{h}_v - \widetilde{\phi}_t^{(C)} (c_v - \widetilde{c}_v) \right\} \left(dV_t - \kappa^Q(\theta^Q - V_t) dt \right), \qquad (7)
\end{aligned}
$$

where the number of units of the hedge instrument is given by

$$
\widetilde{\phi}_t^{(C)} = \frac{\widetilde{h}_v}{\widetilde{c}_v}.
$$

A proof can be found in Appendix A.3.

The first term in brackets on the right hand side of Equation (7) is the interest earned on the hedging error accumulated up to time t. It is not relevant for our analysis, since it does not depend on the choice of the hedge portfolio at time t. We rather focus on those components of the local hedging error that could still be avoided if we knew the correct model. This error due to the use of an incorrect hedge model is captured by the second and the third term. The expressions in curly brackets are the remaining exposure of the hedge portfolio to stock price risk and volatility risk. This exposure is multiplied by the risk factors and their risk premia. For stock price risk, the expression $dS_t - rS_t dt \equiv \sqrt{V_t} S_t \left(dW_t^{(S)} + \lambda^{(S)} \sqrt{V_t} dt \right)$ represents the diffusion component of the stock price and the associated risk premium. Analogously, the term $dV_t - \kappa^Q(\theta^Q - V_t)dt \equiv \sigma_V \sqrt{V_t} \left(\rho dW_t^{(S)} + \sqrt{1-\rho^2} dW_t^{(V)} + \lambda^{(V)} \sqrt{V_t} dt \right)$ is equal to the diffusion component of stochastic volatility and its risk premium.

The remaining exposure to stock price risk and volatility risk arises due to model risk. It can be explained by two kinds of errors. First, the sensitivities of the claim are calculated in the hedge model and therefore deviate from the true partials. We thus hedge the wrong exposure to stock price risk and volatility risk, which results in the terms $h_s - \widetilde{h}_s$ and $h_v - \widetilde{h}_v$. The second error is caused by the fact that the sensitivities of the hedge instrument are computed assuming an incorrect model. The hedge instrument C is used to hedge volatility risk. The first step is thus to eliminate stock price risk by a delta hedge. However, the delta in the hedge model usually deviates from the delta in the true model. Consequently, some stock price risk will remain, represented by the difference $c_s - \widetilde{c}_s$. The sensitivity of the hedge instrument with respect to volatility is also calculated in the wrong model, leaving a (generally) non-zero difference $c_v - \widetilde{c}_v$ so that the wrong number of units of the hedge instruments is employed to eliminate a given volatility risk exposure.

Despite parameter risk there is still a chance for the hedge to produce a zero error. When the errors in the sensitivities of the claim and the hedge instrument exactly offset each other, i.e. when

$$h_s - \widetilde{h}_s - \widetilde{\phi}_t^{(C)} (c_s - \widetilde{c}_s) = 0$$

and

$$h_v - \widetilde{h}_v - \widetilde{\phi}_t^{(1)} (c_v - \widetilde{c}_v) = 0,$$

the hedging error vanishes. These conditions can be rewritten in a more compact fashion as

$$\frac{\widetilde{h}_v}{\widetilde{c}_v} = \frac{h_v}{c_v} = \frac{\widetilde{h}_s - h_s}{\widetilde{c}_s - c_s} = \frac{\widetilde{h}_v - h_v}{\widetilde{c}_v - c_v}.$$

The robustness conditions certainly hold if the sensitivities in the hedge model are equal to those in the true model. However, they will also hold if the errors for H and the hedge instrument C offset each other.

Our conjecture is that the more similar the claim to be hedged to the hedge instrument, the more similar the partial derivatives and also the associated errors, and thus the lower the replication error. This implies that when we use options as our hedge instruments, other options with a different strike should be easier to hedge than the variance contract. Furthermore, the hedge for a put should be the better the smaller the difference of the strike prices between the put and the option used in the hedge.

The calibration of the hedge model was done as described in Section 4.1, the parameters of the true model are taken from Bates (2000) (except for rounding). In Figure 1 we plot the relative hedging error (i.e. the hedging error divided by the price of the contract) for a change in the stock price by $\sqrt{V_t} S_t$, i.e. by one (local) standard deviation. The same is done for the stochastic volatility component, which is shocked by $\sigma_V \sqrt{V_t}$.

As we can see from the left column of graphs in Figure 1, the hedge for the deep OTM-put with a moneyness of 0.85 is the better the lower the difference between the strike prices of the put to be hedged and the hedge instrument. Trivially, the 'ideal' hedge instrument is an option with identical strike. However, the important issue here is that the two curves for the hedging errors generated by a one standard deviation shock in the state variables are still close to zero for strikes in the vicinity of 0.85. We can thus identify the best hedge instrument with a high degree of accuracy. For the variance contract (right column of graphs) the overall size of the relative hedging errors is the same as for the put. Nevertheless, model risk is a much more severe problem for the variance contract than for the put. First, there is no ideal hedge instrument for which the remaining exposure to both risk factors would vanish simultaneously. Second, and more importantly, it does not seem possible to determine a strike, i.e to choose a hedge instrument, for which the hedge is robust with respect to parameter risk. For the two different calibrated parameter sets, which fit the given prices of options with strike-to-spot ratios of 0.9, 0.95, 1.0, and 1.05 correctly, the optimal strike for the option used as the hedge instrument varies considerably, and an option that appears very good under one parameter set is very bad for another. It is thus not possible to decide on a 'best' hedge instrument.

Finally, we analyze the hedging error if an ATM option is used as the hedge instrument. It is often argued that an ATM straddle is a good instrument to trade volatility. If that was actually true, the ATM put should be a reasonable hedge instrument for the variance contract irrespective of the parametrization of the model. Looking at the graphs it becomes clear that the ATM option indeed provides an acceptable hedge for the variance contract under the first set of parameters, where hedging errors for a strike around 100 seem rather small. However, for the second parameter set this is no longer true, since especially variance shocks can cause considerable hedging errors.

4.2.2 Jump-Diffusion Model

Now the true model is the jump-diffusion (JD) model suggested by Merton (1976), with the slight variation that we assume a deterministic jump size. The SDE for the hedging error D is given in the next proposition:

Proposition 2 (JD under Parameter Risk)

$$\begin{aligned} dD_t &= (H_t - \Pi_t) r dt + \left\{ h_s - \widetilde{h}_s - \widetilde{\phi}_t^{(C)} \left(c_s - \widetilde{c}_s \right) \right\} \sigma S_t dW_t \\ &\quad + \left\{ \Delta h - \widetilde{h}_s \Delta S - \left(\Delta \widetilde{h} - \widetilde{h}_s \Delta \widetilde{S} \right) - \widetilde{\phi}_t^{(C)} \left[\Delta c - \widetilde{c}_s \Delta S - \left(\Delta \widetilde{c} - \widetilde{c}_s \Delta \widetilde{S} \right) \right] \right\} \\ &\quad + \ldots dt \end{aligned} \tag{8}$$

where the omitted dt-terms capture the risk premia for the risk remaining in the portfolio and where the number of claims is

$$\widetilde{\phi}_t^{(C)} = \frac{\Delta \widetilde{h} - \widetilde{h}_s \Delta \widetilde{S}}{\Delta \widetilde{c} - \widetilde{c}_s \Delta \widetilde{S}}.$$

The proof can be found in Appendix A.4. Note that the jump size in the true model ΔS can well be different from the assumed jump size $\Delta \widetilde{S}$ in the hedge model.

The interpretation of the SDE for the hedging error is similar to the case of the SV model discussed in the previous subsection. The first term in curly brackets is the remaining exposure to the diffusion risk of the stock, caused by an incorrect assessment of the diffusion risk exposure of the claim to be hegded, (h_s versus $\widetilde{h}_s$), and by the fact that the sensitivity of the hedge instrument is also mis-calculated (c_s versus $\widetilde{c}_s$). The second term in curly brackets represents the remaining exposure to jump risk. To interpret it first assume that the correct model is used for hedging. In this case, we can use the stock to hedge the exposure of the claim to stock diffusion risk and calculate the additional exposure of the claim to stock jump risk after the position has been made delta-neutral ($\Delta h - h_s \Delta S$). This exposure then has to be hedged using the additional exposure of the hedge instrument, after it has also been made delta-neutral ($\Delta c - c_s \Delta S$). In case of model risk the errors made in assessing the additional jump risk of the claim H can be caused by an incorrect calculation of either the jump risk exposure of the claim (Δh versus $\Delta \widetilde{h}$), or the jump risk of the stock (ΔS versus $\Delta \widetilde{S}$), or the use of an incorrect hedge ratio to eliminate diffusion risk of stock (h_s versus $\widetilde{h}_s$), or a combination of all three factors. The same kinds of errors can, of course, also be made in the case of the hedge instrument.

Similar to our analysis of the SV model we can derive robustness conditions under which the hedge will produce a zero error, despite the fact that incorrect parameters are used:

$$\frac{h_s - \widetilde{h}_s}{c_s - \widetilde{c}_s} = \frac{\Delta \widetilde{h} - \widetilde{h}_s \Delta \widetilde{S}}{\Delta \widetilde{c} - \widetilde{c}_s \Delta \widetilde{S}} = \frac{\Delta h - \widetilde{h}_s \Delta S}{\Delta c - \widetilde{c}_s \Delta S}.$$

Figure 2 shows the relative hedging error for a one standard deviation change in the stock price equal to $\sqrt{V_t} S_t$, and for the case when a jump occurs, i.e. for a change of the stock price by $\mu_X S_{t-}$. The results are in general similar to those found for the SV model. For the OTM put the hedge is the better the smaller the difference between its strike and the strike of the option used as the hedge instrument. The variance contract is again more difficult to hedge than the put. As in the SV case there is no choice of the hedge instrument for which the hedge would be insensitive to model risk, and the ATM option is once again not the ideal hedge instrument. Furthermore, an OTM put which is often considered as a reliable hedge against jump risk does not perform well either. The pictures also show that jump risk is in general more difficult to hedge than stock price risk or volatility risk. This cannot be explained by a generic model incompleteness due to jumps or by the fact that a local delta hedge could possibly not control for large changes in the stock price due to a jump. Our model is complete by construction, so that a perfect hedge is basically feasible. However, especially the estimation of the jump size turns out to be a quite severe problem.

4.3 Mis-Specification of Risk Factors

Model mis-specification describes a situation where the wrong model is used for the hedge, and not just a model of the correct type with incorrect parameter values. We would expect that this has much more severe consequences for the general structure of hedging errors than an incorrect parametrization.

In this section we focus on the case of a mis-specification of risk factors. The investor assumes the JD model, although the true model is SV, and vice versa. In Section 4.4 we will then analyze the case of omitted risk factors, where the hedge model is a restricted version of the true model.

4.3.1 Stochastic Volatility Model

First we discuss the case when SV is the true model, but JD is used as the hedge model. Since under certain parameter scenarios both models can, for example, produce a downward sloping smile, this case can certainly not be excluded from the analysis. The following proposition gives the SDE for the hedging error D in the true (SV) model:

Proposition 3 (SV under Model Error (Hedge: JD))

$$\begin{aligned} dD_t &= (H_t - \Pi_t) r dt + \left\{ h_s - \widetilde{h}_s - \widetilde{\phi}_t^{(C)} (c_s - \widetilde{c}_s) \right\} \sqrt{V_t} S_t dW_t^{(S)} \\ &\quad + \left\{ h_v - \widetilde{\phi}_t^{(C)} c_v \right\} \sigma_V \sqrt{V_t} \left(\rho dW_t^{(S)} + \sqrt{1-\rho^2} dW_t^{(V)} \right) + \ldots dt \end{aligned} \tag{9}$$

where the omitted dt-terms capture the risk premia for the risk remaining in the hedge portfolio and the number of units of the hedge instrument is given as

$$\widetilde{\phi}_t^{(C)} = \frac{\Delta \widetilde{h} - \widetilde{h}_s \Delta \widetilde{S}}{\Delta \widetilde{c} - \widetilde{c}_s \Delta \widetilde{S}}.$$

The proof of the proposition is analogous to the one for Proposition 1 in Appendix A.3 and is omitted to save space.

The structure of the remaining exposure to diffusion risk of the stock (after the delta hedge) is well-known by now, see Proposition 1 for an interpretation. The remaining exposure to volatility risk is different under model risk, as considered here, and parameter risk, which was the issue in Proposition 1. When there is parameter risk, the remaining exposure depends on the error in the sensitivities with respect to volatility risk for both the claim and the hedge instrument. In Proposition 3, however, there is no stochastic volatility in the hedge model. The error now is not that sensitivities are computed *incorrectly*, but that this risk factor is *completely ignored*. Instead, the investor computes an additional exposure to jump risk in the hedge model and chooses the position in the hedge instrument to eliminate this exposure, which is not even present in the true model.

This error made by the mis-specification of risk factors also becomes clear from the robustness conditions:

$$\frac{h_s - \widetilde{h}_s}{c_s - \widetilde{c}_s} = \frac{\Delta \widetilde{h} - \widetilde{h}_s \Delta \widetilde{S}}{\Delta \widetilde{c} - \widetilde{c}_s \Delta \widetilde{S}} = \frac{h_v}{c_v}.$$

The position in the hedge instrument should depend on the ratio of the sensitivities with respect to volatility risk. However, it is calculated based on the assumed exposures to jump risk. Only if these two ratios coincide by chance the hedge will be correct. The mixing up of models and of risk factors seems a much more fundamental error than the use of wrong parameters, and the hedge will only be correct by chance. This will also become obvious when we look at numerical examples for hedging errors.

Figure 3 shows the relative hedging errors for the OTM put and the variance contract. Again, these errors are based on parameter vectors for the hedge model which are compatible with observed market prices for a certain set of options. The general result is that hedging errors are much larger than under parameter risk, which confirms our intuition that the use of an incorrect *model* is a much greater problem than the use of a wrong *parameter vector*. This not only holds for volatility risk, which is hedged in a fundamentally wrong manner, but also for stock price risk.

A comparison of the contracts again shows that the OTM put is less exposed to model risk than the variance contract. The option can be hedged rather well using some other put with a similar strike, so that the rule for selecting its optimal hedge instrument still applies under model risk. Also similar to previous results we cannot find a robust hedge for the variance contract, but a hedge instrument which is 'good' under one set of parameters can be quite 'bad' under a different parametrization, as can be seen from the graphs in the figure. In particular, an OTM put with strike-to-spot ratio of 0.95, which might be recommended as a hedge against downward jump risk, performs quite well for the second parameter set, but rather poorly for the first. The ATM put, on the other hand, provides a pretty good hedge under the first calibrated parameter set, while it performs rather badly under the second.

4.3.2 Jump-Diffusion Model

Now the situation will be reversed, and JD with deterministic jump size will be the true model, while SV will be used as the hedge model. The SDE for the hedging error D under the true model is the content of the next proposition.

Proposition 4 (JD under Model Error (Hedge: SV))

$$\begin{aligned} dD_t &= (H_t - \Pi_t) r dt + \left\{ h_s - \widetilde{h}_s - \widetilde{\phi}_t^{(C)} \left(c_s - \widetilde{c}_s \right) \right\} \sigma S_t dW_t \\ &\quad + \left\{ \Delta h - \widetilde{h}_s \Delta S - \widetilde{\phi}_t^{(C)} \left(\Delta c - \widetilde{c}_s \Delta S \right) \right\} + \ldots dt, \end{aligned} \tag{10}$$

where the omitted dt-terms capture the risk premia for the risk remaining in the hedge portfolio and where the number of claims is determined as

$$\widetilde{\phi}_t^{(C)} = \frac{\widetilde{h}_v}{\widetilde{c}_v}.$$

The proof of this proposition is analogous to that for Proposition 2.

The structure of the remaining exposure to diffusion risk of the stock has already been discussed extensively. The remaining exposure to the jump risk of the stock is much more interesting. Like for the SV model we now compare the remaining exposure under model risk in Equation (10) to its counterpart under parameter risk in Equation (8). Under parameter risk the hedging error depends on the difference between the exposures to jump risk in the true and in the hedge model for both the claim and the hedge instrument. Here the hedge model does not even contain a jump component, so that the exposure with respect to this risk factor is basically set equal to zero. Instead the investor aims at hedging volatility risk not present in the true model. Again the hedge is only correct, if, by chance, the ratio of exposure to volatility in the hedge model is equal to the ratio of the exposures to additional jump risk in the true model.

Figure 4 shows the familiar result that the hedging error for the OTM put can be kept small by choosing a put with similar strike price as the hedge instrument. Also there is no ideal hedge instrument for the variance contract, which further underlines that this derivative asset is harder to replicate in a world with model risk than the simple put. Especially the impact of a jump is quite pronounced.

4.4 Model Risk: Missing Risk Factors

Another variant of model mis-specification is that the hedge model is less general than the true model, i.e. some of the risk factors included in the true model are omitted from the hedge model. For example, the true model could contain a multi-factor specification for stochastic volatility, as in Bates (2000), whereas the hedge model is a one-factor model like the one suggested by Heston (1993). It could also be the case that the general model developed by Bakshi, Cao, and Chen (1997) generates the data, while the hedge model is a restricted variant, like Heston (1993) or Merton (1976), where either stochastic jumps or stochastic volatility are missing.

Again, this kind of model risk is quite likely to strike when a hedge is implemented. The true model will be able to explain every observable phenomenon correctly, and it will usually be quite sophisticated with a large number of state variables and parameters. Even if we can find the correct type of model its parameters will be difficult to identify. We thus assume that the investor uses a simpler model which fits the data 'sufficiently' well. Once such a simpler model has been found, it would be hard to justify a more complex approach.

We will discuss two cases of omitted risk factors, where the true model is always given by a version of Bakshi, Cao, and Chen (1997) with a deterministic jump size for the stock. First, the jump component is omitted in the hedge model, and second, we analyze the consequences of leaving out the stochastic volatility part. The number of hedge instruments is chosen such that the hedge model is complete. For the SV model and the JD model with a deterministic jump size we have to take one additional option into the hedge portfolio so that both risk factors can be spanned.

4.4.1 Missing Jump Component

We start our analysis with the case where the jump component is missing from the hedge model. As usual, we first derive the dynamics of the hedging error in the true model:

Proposition 5 (BCC Model: Hedge under Model Risk (SV))

$$
\begin{aligned}
dD_t &= (H_t - \Pi_t) r dt + \left\{ h_s - \widetilde{h}_s - \widetilde{\phi}_t^{(C)} (c_s - \widetilde{c}_s) \right\} \sqrt{V_t} dW_t^{(S)} \\
&\quad + \left\{ h_v - \widetilde{h}_v - \widetilde{\phi}_t^{(C)} (c_v - \widetilde{c}_v) \right\} \sigma_V \sqrt{V_t} \left(\rho dW_t^{(S)} + \sqrt{1-\rho^2} dW_t^{(V)} \right) \\
&\quad + \left\{ \Delta h - \widetilde{h}_s \Delta S - \widetilde{\phi}_t^{(C)} (\Delta c - \widetilde{c}_s \Delta S) \right\} + \ldots dt \qquad (11)
\end{aligned}
$$

where the omitted dt-terms capture the risk premia for the risk remaining in the hedge portfolio and where the number of claims is

$$\widetilde{\phi}_t^{(C)} = \frac{\widetilde{h}_v}{\widetilde{c}_v}.$$

The proof is similar to that for Proposition 1 in Appendix A.3.

The remaining exposure to stock price risk and to volatility risk has the same structure as in the case of parameter risk in the SV model and has already been discussed in Proposition 1. The most interesting part of Equation (11) is the one that relates to the jump risk exposure left in the hedge portfolio. An interpretation for the components of this term is offered in the discussion of Proposition 4. Both in Proposition 4 and here jump risk has been incorrectly interpreted as stochastic volatility. However, now the problem is much more severe. In the situation when the wrong type of risk factor is included in the hedge model, there would still be a chance to set up the correct hedge. Here, one component is missing entirely, so that there are not enough instruments in the hedge portfolio from the start, and the bad hedge for jump risk is unavoidable. Jump risk could now be eliminated only by chance, if the ratio of the additional exposure to jump risk for H and the hedge instrument coincides with the analogous ratio with respect to volatility risk. If this condition is not met, jump risk cannot be eliminated.

Figure 5 shows the result of the analysis of the local relative hedging error. For the put the results are the same as in the cases studied before. One additional point to note is that the sensitivity of the hedge to shocks in the risk factors looks rather large even for this simple instrument, and that, depending on the model, volatility risk (present in the hedge model) may be as hard to hedge as jump risk (missing from the hedge model). Furthermore, looking at the left column of graphs we can see that the quality of the hedge deteriorates especially when the stock price has jumped. For the variance contract there is again no optimal hedge instrument which provides robustness with respect to model risk. Additionally, we have to keep in mind that the hedge model (Heston (1993)) is exposed to a kind of parameter risk. As discussed in Section 4.3.1, there may be more than one parametrization which fits the given prices. In summary these two problems add up in the hedging error.

4.4.2 Missing Stochastic Volatility

As the last case we analyze the hedging error when the hedge model only contains a jump component, but no stochastic volatility.

Proposition 6 (BCC Model: Hedge under Model Risk (JD))

$$\begin{aligned} dD_t &= (H_t - \Pi_t) r dt + \left\{ h_s - \widetilde{h}_s - \widetilde{\phi}_t^{(C)} \left(c_s - \widetilde{c}_s \right) \right\} \sqrt{V_t} S_t dW_t^{(S)} \\ &+ \left\{ h_v - \widetilde{\phi}_t^{(C)} c_v \right\} \sigma_V \sqrt{V_t} \left(\rho dW_t^{(S)} + \sqrt{1-\rho^2} dW_t^{(V)} \right) \\ &+ \left\{ \Delta h - \widetilde{h}_s \Delta S - \left(\Delta \widetilde{h} - \widetilde{h}_s \Delta \widetilde{S} \right) - \widetilde{\phi}_t^{(C)} \left[\left(\Delta c - \widetilde{c}_s \Delta S \right) - \left(\Delta \widetilde{c} - \widetilde{c}_s \Delta \widetilde{S} \right) \right] \right\} \\ &+ \ldots dt \end{aligned} \tag{12}$$

where the omitted dt-terms capture the risk premia for the risk remaining in the hedge portfolio and where the number of claims is

$$\widetilde{\phi}_t^{(C)} = \frac{\Delta \widetilde{h} - \widetilde{h}_s \Delta \widetilde{S}}{\Delta \widetilde{c} - \widetilde{c}_s \Delta \widetilde{S}}.$$

The proof is analogous to that for Proposition 2 in Appendix A.4.

The interpretation of Equation (12) is very similar to those for the previous propositions. The structure of the remaining exposure to jump risk and to stock diffusion risk is identical to the case of parameter risk which has been given in Proposition 2. For volatility risk the structure of the remaining exposure had already been discussed in Proposition 4, where stochastic volatility was also not included in the hedge model. However, in the case where stochastic volatility had incorrectly been interpreted as jump risk, there would at least have been the chance to construct the correct hedge, since the right set of instruments was available. Here, this is basically impossible, since there are not enough instruments in the hedge portfolio to achieve completeness and volatility risk is only hedged by chance.

Figure 6 compares the hedging errors for a deep OTM put and the variance contract. It shows that a hedge for the variance contract based on a mis-specified model can generate substantial hedging errors and that there is no robust choice of hedge instruments. For example, the upper graph in the right column seems to suggest that including a put with a strike price of roughly 106 generates relatively small errors. However, the lower graph shows that this does not hold in general. The hedging errors for this hedge instrument can be rather large under a different parameter scenario, which nevertheless prices the set of given contracts with acceptable precision. In particular, the remaining exposure to stochastic volatility is rather large.

5 Conclusion

Variance contracts are innovative derivative assets. They provide exposure to the variation risk of a stock with the two components stochastic volatility and jumps. Empirically there is evidence for a negative risk premium on the variance contract, which can be explained in several ways. First, there is the well-documented negative market price of risk for stochastic volatility. A second explanation, however, is that jumps can be perceived to be more severe and more frequent under the risk-neutral than under the physical measure.

The main motivation for trading the variance contract is that the investor wants an instrument providing him with exposure to variation risk of the stock. A formal motivation can be derived from the literature on portfolio planning, which shows that investors can have a demand for portfolios providing a hedge against fluctuations in the state variables. The question is then why investors would prefer the variance contract to a replicating strategy using standard options.

In our opinion the main economic motivation for the introduction of variance contracts is that the variance contract is 'better' than its replicating strategy and that it provides a more significant improvement relative to its replicating strategy than a standard option. This implies that there is a stronger motivation to introduce the variance contract than the option. There are at least two arguments for the superiority of the variance contract compared to its dynamic replication strategy. First, in an economy with discrete trading, holding the variance contract (long or short) eliminates the need to adjust the hedge position continuously to keep it delta-neutral. Second, the dynamic replication strategy has to be based on some assumed model, and a hedging error will result if the hedge model is not equal to the true model. We focus on model risk and consider the cases of parameter uncertainty and of mis-specified and omitted risk factors.

Under all of these scenarios we derive analytical expressions for the local hedging errors. A graphical analysis shows that the hedging error for the variance contract is in general slightly larger than that for a deep OTM put, which was chosen as the benchmark asset and as an alternative candidate for the new derivative contract to be introduced. However, the variance contract is exposed to model risk to a much higher degree. For the put, the hedge is the more robust against model risk the smaller the difference between its strike price and the strike price of the hedge instrument. For the variance contract, there is no option for which the hedge is robust against model risk. Dynamic hedges for the variance contract are thus much riskier than those for put options.

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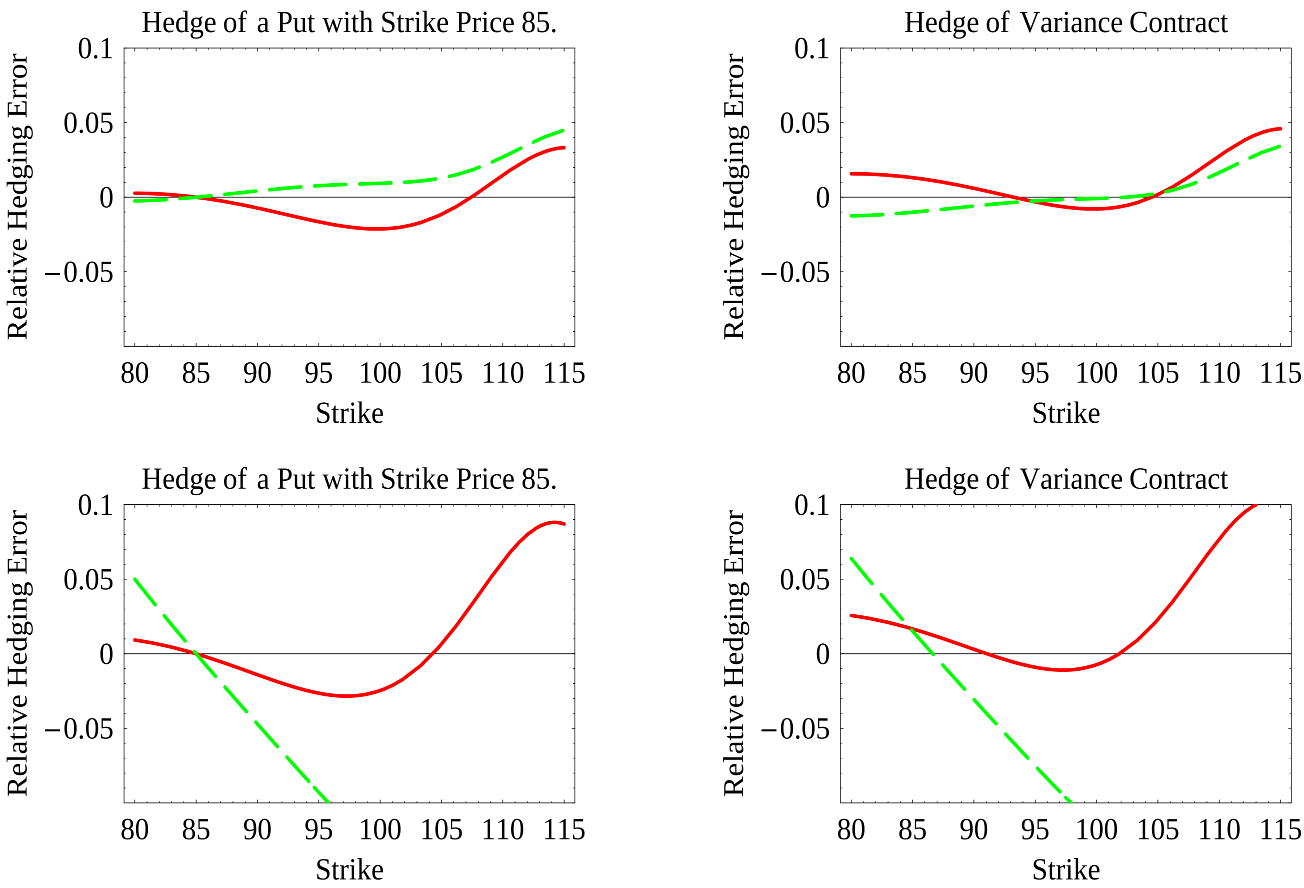

Figure 1: SV Model: Relative Hedging Error under Parameter Risk

The local relative hedging errors for a deep OTM put and for the variance contract are shown as a function of the strike price of the option that is used as hedge instrument in addition to the stock and the money market account. The hedging errors are given for a change in the stock price S by one standard deviation (solid line) and for a change in squared volatility V by one standard deviation (dashed line).

The figures in the two rows are based on two sets of parameters which price options with moneyness K/S of 0.90, 0.95, 1.0 and 1.05 with a relative error of less than 0.3%. The current stock price is equal to 100, the time to maturity of all contracts is six months.

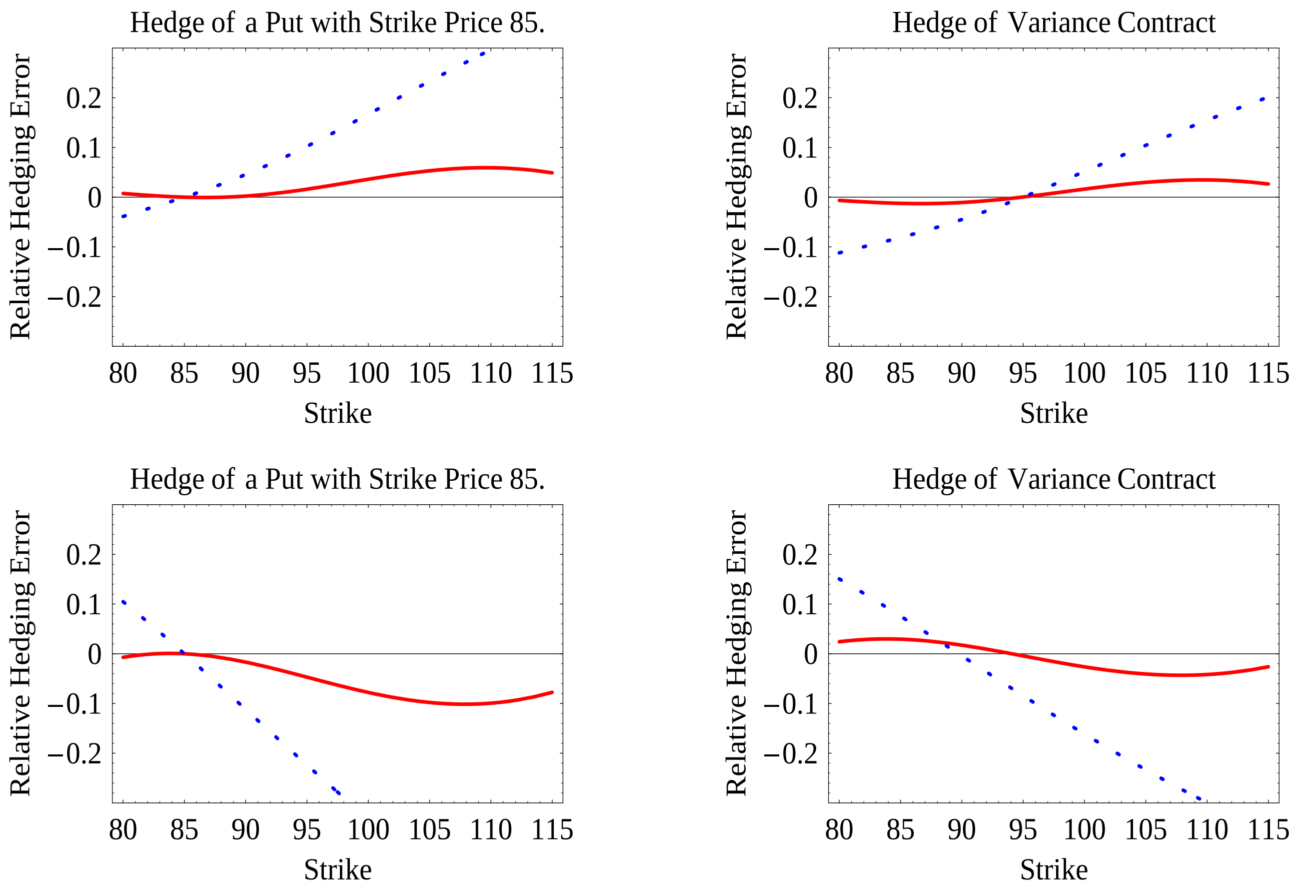

Figure 2: JD Model: Relative Hedging Error under Parameter Risk

The local relative hedging errors for a deep OTM put and for the variance quadratic contract are shown as a function of the strike price of the option that is used as hedge instrument in addition to the stock and the money market account. The hedging errors are given for a change in the stock price S by one standard deviation (solid line) and for a jump in the stock price (dotted line) where the jump size is assumed to be deterministic.

The figures in the two rows are based on two hedge models which price options with moneyness K/S of 0.95, 1.0, and 1.05 with a relative error of less than 0.5%. The current stock price is equal to 100, the time to maturity of all contracts is six months.

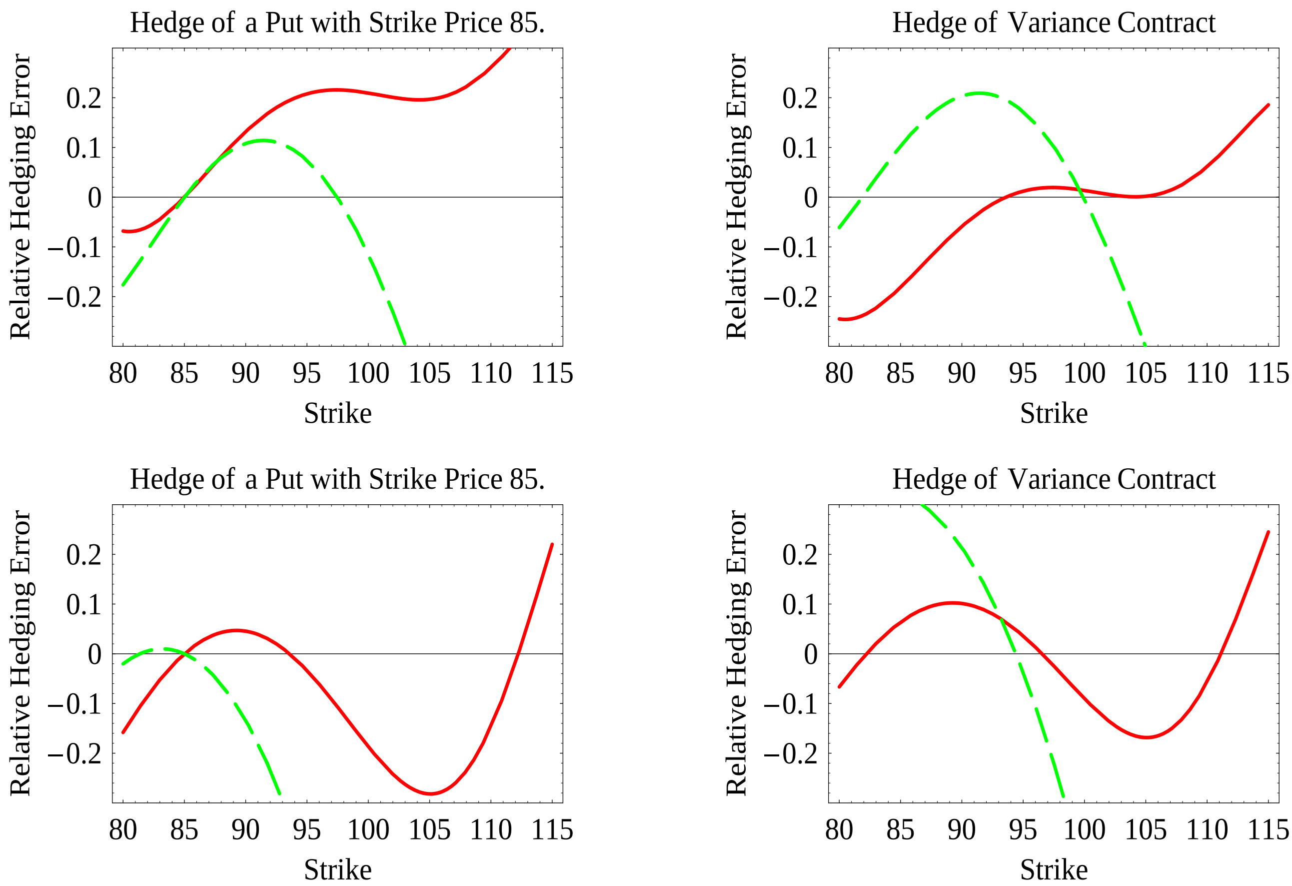

Figure 3: SV Model: Relative Hedging Error under Model Risk (JD)

The local relative hedging errors for a deep OTM put and for the variance contract are shown as a function of the strike price of the option that is used as hedge instrument in addition to the stock and the money market account. The hedging errors are given for a change in the stock price S by one standard deviation (solid line) and for a change in squared volatility V by one standard deviation (dashed line).

The figures in the two rows are based on two sets of parameters which price options with moneyness K/S of 0.95, 1.0 and 1.05 with a relative error of less than 1.0%. The current stock price is equal to 100, the time to maturity of all contracts is six months.

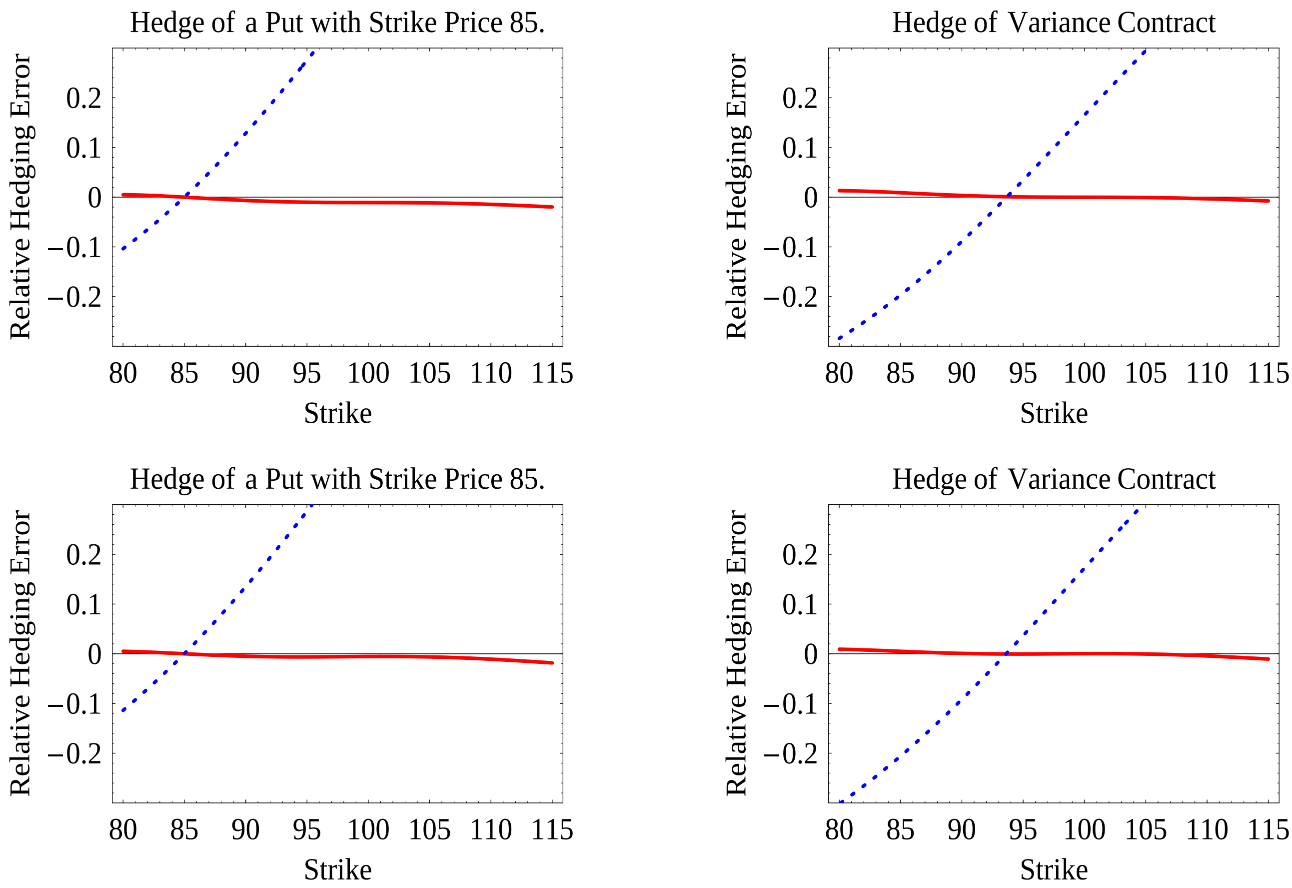

Figure 4: JD Model: Relative Hedging Error under Model Risk (SV)

The local relative hedging errors for a deep OTM put and for the variance contract are shown as a function of the strike price of the option that is used as hedge instrument in addition to the stock and the money market account. The hedging errors are given for a change in the stock price S by one standard deviation (solid line) and for a jump in the stock price (dotted line) where the jump size is assumed to be deterministic.

The figures in the two rows are based on two parameter sets which price options with moneyness K/S of 0.90, 0.95, 1.0 and 1.05 with a relative error of less than 0.1%. The current stock price is equal to 100, the time to maturity of all contracts is six months.

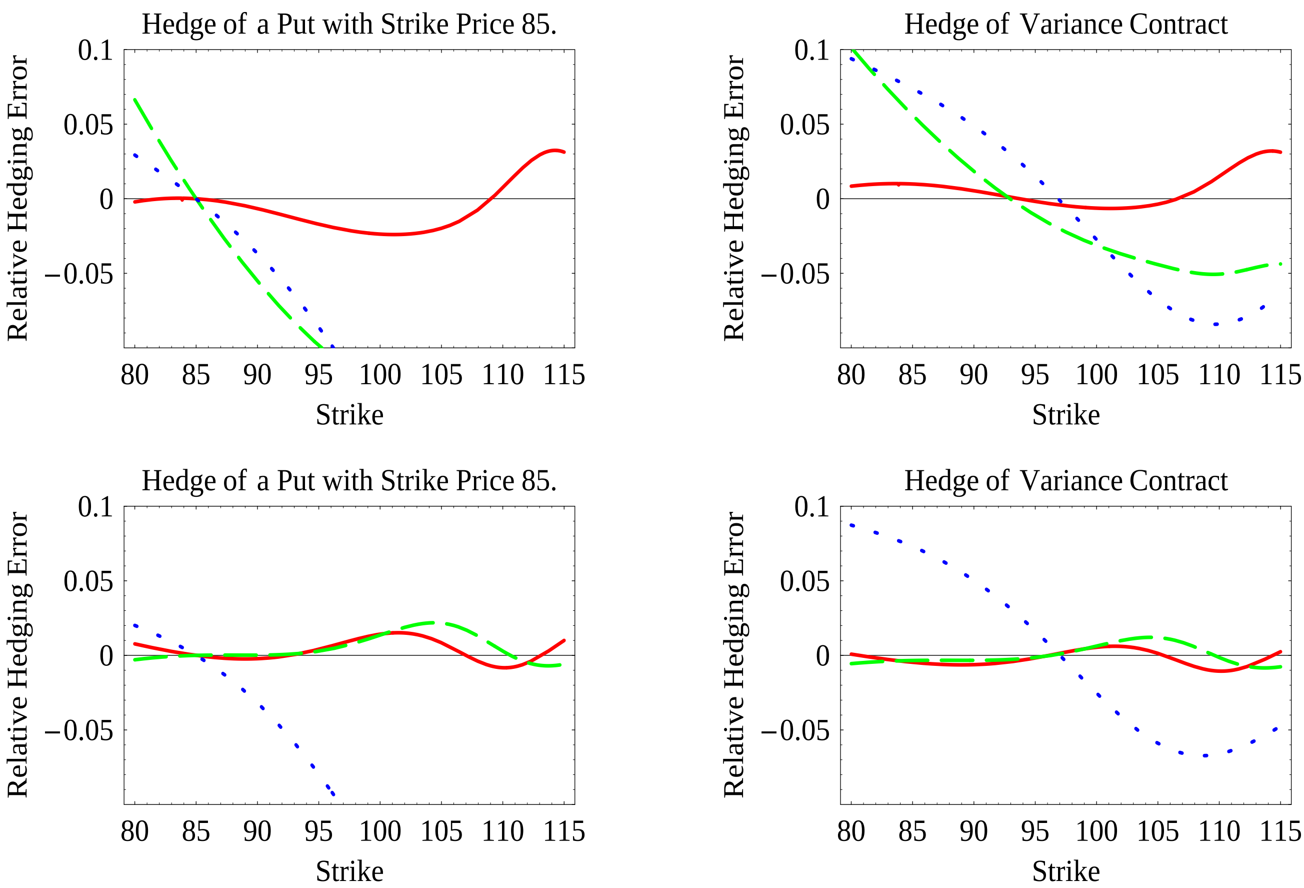

Figure 5: BCC Model: Relative Hedging Error under Model Risk (SV)

The local relative hedging errors for a deep OTM put and for the variance contract are shown as a function of the strike price of the option that is used as hedge instrument in addition to the stock and the money market account. The hedging errors are given for a change in the stock price S by one standard deviation (solid line), for a change in squared volatility V by one standard deviation (dashed line) and for a jump in the stock price (dotted line) where the jump size is assumed to be deterministic.

The figures in the two rows are based on two sets of parameters which price options with moneyness K/S of 0.90, 0.95, 1.0 and 1.05 with a relative error of less than 0.3%. The current stock price is equal to 100, the time to maturity of all contracts is six months.

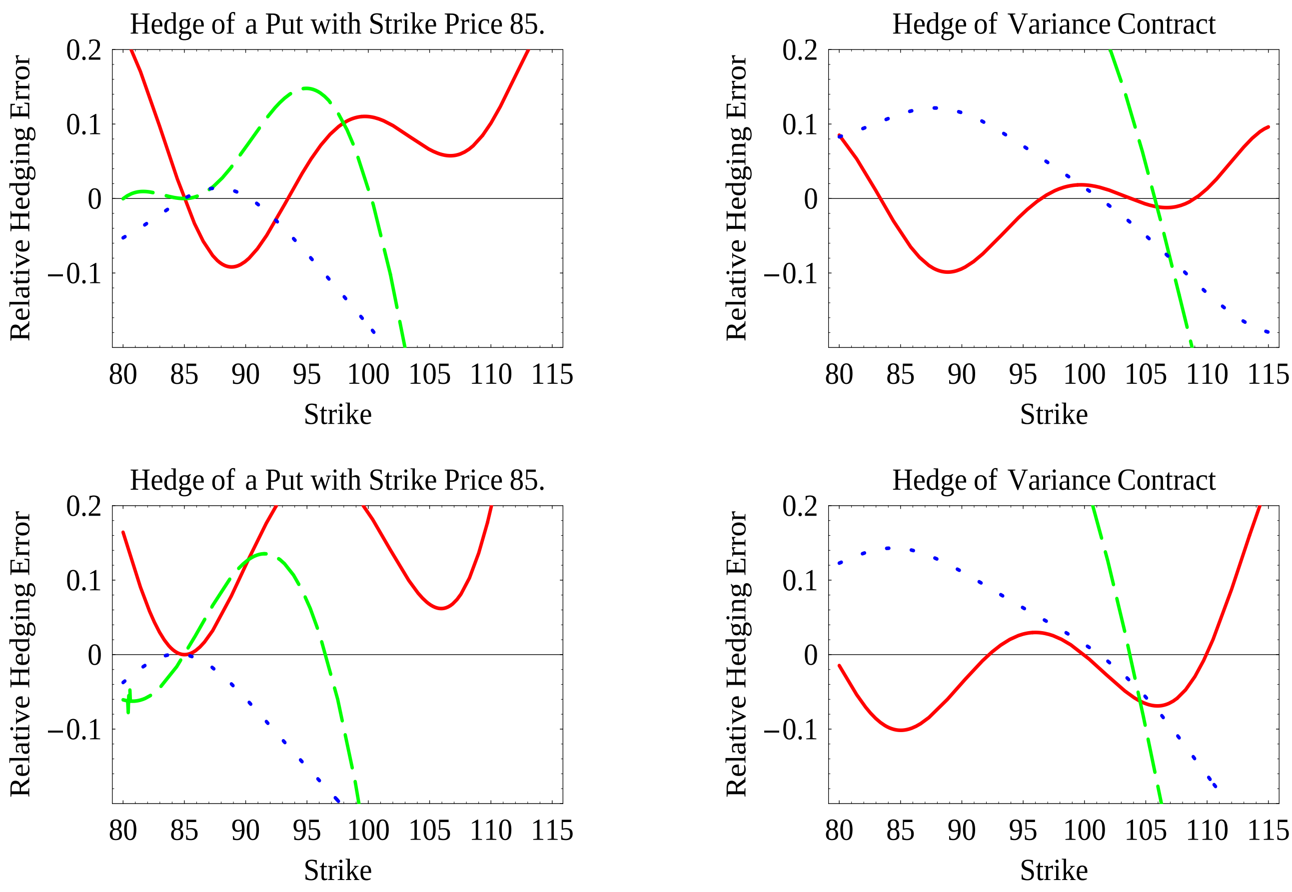

Figure 6: BCC Model: Relative Hedging Error under Model Risk (JD)

The local relative hedging errors for a deep OTM put and for the variance contract are shown as a function of the strike price of the option that is used as hedge instrument in addition to the stock and the money market account. The hedging errors are given for a change in the stock price S by one standard deviation (solid line), for a change in squared volatility V by one standard deviation (dashed line) and for a jump in the stock price (dotted line) where the jump size is assumed to be deterministic.

The figures in the two rows are based on two sets of parameters which price options with moneyness K/S of 0.95, 1.0 and 1.05 with a relative error of less than 1.0%. The current stock price is equal to 100, the time to maturity of all contracts is six months.

A Appendix

A.1 Pricing of Variance Contract

The payoff of the variance contract at time T is

$$C_T = RV(0,T) = \int_0^T V_u du + \int_0^T X_u^2 dN_u.$$

Risk-neutral pricing then gives

$$\begin{aligned}
C_t &= E^Q\left[e^{-r(T-t)}\left(\int_0^T V_u du + \int_0^T X_u^2 dN_u\right)|\mathcal{F}_t\right] \\
&= e^{-r(T-t)}\left\{\int_0^t V_u du + \int_0^t X_u^2 dN_u + \int_t^T E^Q\left[V_u|\mathcal{F}_t\right]du + \int_t^T E^Q\left[X_u^2 dN_u|\mathcal{F}_t\right]\right\} \\
&= e^{-r(T-t)}\left\{RV(0,t) + \int_t^T E^Q\left[V_u|\mathcal{F}_t\right]du + \int_t^T E^Q\left[E^Q\left[X_u^2|\mathcal{F}_{u-}\right]\left(k_0^Q + k_1^Q V_{u-}\right)|\mathcal{F}_t\right]du\right\} \\
&= e^{-r(T-t)}\left\{RV(0,t) + \int_t^T E^Q\left[V_u|\mathcal{F}_t\right]du + \int_t^T E^Q\left[E^Q\left[X^2\right]\left(k_0^Q + k_1^Q V_{u-}\right)|\mathcal{F}_t\right]du\right\}
\end{aligned}$$

where the last equality follows from the assumption that the jump size X of the log return neither depends on time u nor on the other state variables. By rearranging the equation, we get

$$\begin{aligned}
C_t &= e^{-r(T-t)}\left\{RV(0,t) + \left(1 + k_1^Q E^Q\left[X^2\right]\right)\int_t^T E^Q\left[V_u|\mathcal{F}_t\right]du + \int_t^T k_0^Q E^Q\left[X^2\right]du\right\} \\
&= e^{-r(T-t)}\left\{RV(0,t) + \left(1 + k_1^Q E^Q\left[X^2\right]\right)\int_t^T E^Q\left[V_u|\mathcal{F}_t\right]du + (T-t)k_0^Q E^Q\left[X^2\right]\right\}.
\end{aligned}$$

To calculate the expectation of the variance, we start from the SDE

$$dV_t = \kappa^Q\left(\theta^Q - V_t\right)dt + \sigma_V\sqrt{V_t}\left(\rho d\widetilde{W}_t^{(1)} + \sqrt{1-\rho^2}d\widetilde{W}_t^{(2)}\right) + Y_t dN_t.$$

Taking expectations gives

$$\begin{aligned}
dE^Q\left[V_u|\mathcal{F}_t\right] &= \kappa^Q\left(\theta^Q - E^Q\left[V_u|\mathcal{F}_t\right]\right)du + E^Q\left[Y_u|\mathcal{F}_t\right]\left(k_0^Q + k_1^Q E^Q\left[V_u|\mathcal{F}_t\right]\right)du \\
&= \Big(\underbrace{\kappa^Q\theta^Q + E^Q\left[Y_u|\mathcal{F}_t\right]k_0^Q}_{\widetilde{\kappa}^Q\widetilde{\theta}^Q}\Big)du - \Big(\underbrace{\kappa^Q - E^Q\left[Y_u|\mathcal{F}_t\right]k_1^Q}_{\widetilde{\kappa}^Q}\Big)E^Q\left[V_u|\mathcal{F}_t\right]du.
\end{aligned}$$

The solution to the ordinary differential equation is

$$E^Q\left[V_u|\mathcal{F}_t\right] = e^{-\widetilde{\kappa}^Q(u-t)}V_t + \left(1 - e^{-\widetilde{\kappa}^Q(u-t)}\right)\widetilde{\theta}^Q.$$

Plugging this into the pricing equation, we get

$$\begin{aligned} C_t &= e^{-r(T-t)} \left\{ \int_0^t V_u du + \int_0^t X_u^2 dN_u + (T-t) k_0^Q E^Q \left[X^2\right] \right. \\ &\qquad \left. + \left(1 + k_1^Q E^Q \left[X^2\right]\right) \int_t^T \left[e^{-\widetilde{\kappa}^Q (u-t)} V_t + \left(1 - e^{-\widetilde{\kappa}^Q (u-t)}\right) \widetilde{\theta}^Q \right] du \right\} \\ &= e^{-r(T-t)} \left\{ \int_0^t V_u du + \int_0^t X_u^2 dN_u + (T-t) k_0^Q E^Q \left[X^2\right] \right. \\ &\qquad \left. + \left(1 + k_1^Q E^Q \left[X^2\right]\right) \left(\frac{1 - e^{-\widetilde{\kappa}^Q (T-t)}}{\widetilde{\kappa}^Q} (V_t - \widetilde{\theta}^Q) + (T-t) \widetilde{\theta}^Q \right) \right\}. \end{aligned}$$

A.2 Expected Return of the Variance Contract

To calculate the expected return of the variance contract, we first derive the dynamics of the claim price. From the pricing equation for the variance contract, we get (after some simple, but time-consuming manipulations of the equations) the SDE

$$\begin{aligned} dC_t &= rC_t dt + e^{-r(T-t)} \left\{ V_t dt + X_t^2 dN_t - E^Q[X^2] k_0^Q dt \right. \\ &\qquad - \left(1 + k_1^Q E^Q[X^2]\right) \left[e^{-\widetilde{\kappa}^Q (T-t)} V_t + \left(1 - e^{-\widetilde{\kappa}^Q (T-t)}\right) \widetilde{\theta}^Q \right] dt \\ &\qquad \left. + \left(1 + k_1^Q E^Q[X^2]\right) \frac{1 - e^{-\widetilde{\kappa}^Q (T-t)}}{\widetilde{\kappa}^Q} dV_t \right\}. \end{aligned}$$

To obtain the risk premium, we compare the drift of the price under the physical measure P and under the risk-neutral measure Q. With a slight abuse of notation, we get

$$\begin{aligned} & E^P \left[dC_t | \mathcal{F}_t\right] - E^Q \left[dC_t | \mathcal{F}_t\right] \\ &= e^{-r(T-t)} \left\{ E^P \left[X^2\right] k^P - E^Q \left[X^2\right] k^Q \right. \\ &\qquad + \left(1 + k_1^Q E^Q[X^2]\right) \frac{1 - e^{-\widetilde{\kappa}^Q (T-t)}}{\widetilde{\kappa}^Q} \left(\kappa^P \left(\theta^P - V_t\right) + E^P[Y_t] k^P \right. \\ &\qquad \left. \left. - \kappa^Q \left(\theta^Q - V_t\right) - E^Q[Y_t] k^Q \right) \right\} dt \\ &= e^{-r(T-t)} \left\{ E^P \left[X^2\right] k^P - E^Q \left[X^2\right] k^Q \right. \\ &\qquad \left. + \left(1 + k_1^Q E^Q[X^2]\right) \frac{1 - e^{-\widetilde{\kappa}^Q (T-t)}}{\widetilde{\kappa}^Q} \left(\lambda^V \sigma_V V_t + E^P[Y_t] k^P - E^Q[Y_t] k^Q \right) \right\} dt. \end{aligned}$$

A.3 Proof of Proposition 1

From the definition of the hedging error, we know that

$$dD_t = dH_t - d\Pi_t.$$

For the claim price $H = h(t, S_t, V_t, \ldots)$, we can derive the SDE by using Ito

$$dH_t = h_t dt + h_s dS_t + h_v dV_t + \frac{1}{2} h_{ss} V_t S_t^2 dt + \frac{1}{2} h_{vv} \sigma_V^2 V_t dt + h_{sv} \rho \sigma_V V_t S_t dt.$$

Applying the fundamental partial differential equation then gives

$$dH_t = H_t r dt + h_s (dS_t - r S_t dt) + h_v \left[dV_t - \kappa^Q (\theta^Q - V_t) dt \right].$$

The same equation holds for the price of the claim C.

The hedge portfolio consists of $\widetilde{\phi}_t^{(S)}$ units of the stock, $\widetilde{\phi}_t^{(C)}$ units of the claim C, and an investment of $\Pi_t - \widetilde{\phi}_t^{(S)} S_t - \widetilde{\phi}_t^{(C)} C_t^{(1)}$ in the money market account, which is chosen such that the portfolio is self-financing. The SDE for the value of the hedge portfolio is then

$$\begin{aligned} d\Pi_t &= \widetilde{\phi}_t^{(S)} dS_t + \widetilde{\phi}_t^{(C)} dC_t + \left(\Pi_t - \widetilde{\phi}_t^{(S)} S_t - \widetilde{\phi}_t^{(C)} C_t \right) r dt \\ &= \Pi_t r dt + \widetilde{\phi}_t^{(S)} (dS_t - r S_t dt) + \widetilde{\phi}_t^{(C)} (dC_t - r C_t dt). \end{aligned}$$

Plugging the expressions for dH_t and $d\Pi_t$ into the definition of dD_t and sorting the terms by the risk factors, that is by stock price risk and volatility risk, gives

$$\begin{aligned} dD_t &= H_t r dt + h_s (dS_t - r S_t dt) + h_v \left[dV_t - \kappa^Q (\theta^Q - V_t) dt \right] \\ &\quad - \Pi_t r dt - \widetilde{\phi}_t^{(S)} (dS_t - r S_t dt) - \widetilde{\phi}_t^{(C)} (dC_t - r C_t dt) \\ &= H_t r dt + h_s (dS_t - r S_t dt) + h_v \left[dV_t - \kappa^Q (\theta^Q - V_t) dt \right] \\ &\quad - \Pi_t r dt - \widetilde{\phi}_t^{(S)} (dS_t - r S_t dt) - \widetilde{\phi}_t^{(C)} \left\{ c_s (dS_t - r S_t dt) + c_v \left[dV_t - \kappa^Q (\theta^Q - V_t) dt \right] \right\} \\ &= (H_t - \Pi_t) r dt + \left\{ h_s - \widetilde{\phi}_t^{(S)} - \widetilde{\phi}_t^{(C)} c_s \right\} (dS_t - r S_t dt) \\ &\quad + \left\{ h_v - \widetilde{\phi}_t^{(C)} c_v \right\} \left[dV_t - \kappa^Q (\theta^Q - V_t) dt \right]. \end{aligned} \tag{13}$$

The number of claims in the hedge portfolio follows from the conditions

$$\begin{aligned} \widetilde{\phi}_t^{(S)} + \widetilde{\phi}_t^{(C)} \widetilde{c}_s &= \widetilde{h}_s \\ \widetilde{\phi}_t^{(C)} \widetilde{c}_v &= \widetilde{h}_v \end{aligned}$$

which yield

$$\begin{aligned} \widetilde{\phi}_t^{(S)} &= \widetilde{h}_s - \widetilde{\phi}_t^{(C)} \widetilde{c}_s \\ \widetilde{\phi}_t^{(C)} &= \frac{\widetilde{h}_v}{\widetilde{c}_v}. \end{aligned}$$

Plugging the number of stocks $\widetilde{\phi}_t^{(S)}$ into Equation (13) gives

$$
\begin{aligned}
dD_t &= (H_t - \Pi_t) r dt + \left\{ h_s - \widetilde{h}_s + \widetilde{\phi}_t^{(C)} \widetilde{c}_s - \widetilde{\phi}_t^{(C)} c_s \right\} (dS_t - rS_t dt) \\
&\quad + \left\{ h_v - \widetilde{\phi}_t^{(C)} c_v \right\} \left[dV_t - \kappa^Q (\theta^Q - V_t) dt \right] \\
&= (H_t - \Pi_t) r dt + \left\{ h_s - \widetilde{h}_s - \widetilde{\phi}_t^{(C)} \left[c_s - \widetilde{c}_s \right] \right\} (dS_t - rS_t dt) \\
&\quad + \left\{ h_v - \widetilde{\phi}_t^{(C)} c_v \right\} \left[dV_t - \kappa^Q (\theta^Q - V_t) dt \right] \\
&= (H_t - \Pi_t) r dt + \left\{ h_s - \widetilde{h}_s - \widetilde{\phi}_t^{(C)} \left[c_s - \widetilde{c}_s \right] \right\} (dS_t - rS_t dt) \\
&\quad + \left\{ h_v - \widetilde{h}_v + \widetilde{\phi}_t^{(C)} \widetilde{c}_v - \widetilde{\phi}_t^{(C)} c_v \right\} \left[dV_t - \kappa^Q (\theta^Q - V_t) dt \right] \\
&= (H_t - \Pi_t) r dt + \left\{ h_s - \widetilde{h}_s - \widetilde{\phi}_t^{(C)} \left[c_s - \widetilde{c}_s \right] \right\} (dS_t - rS_t dt) \\
&\quad + \left\{ h_v - \widetilde{h}_v - \widetilde{\phi}_t^{(C)} \left[c_v - \widetilde{c}_v \right] \right\} \left[dV_t - \kappa^Q (\theta^Q - V_t) dt \right].
\end{aligned}
$$

A.4 Proof of Proposition 2

In the model of Merton (1976), the SDE for the stock price is

$$
\begin{aligned}
dS_t &= \left(r + \sigma \lambda^{(S)} - E^Q[e^X - 1] k^Q \right) S_t dt + \sigma S_t dW_t^{(S)} + \left(e^{X_t} - 1 \right) S_{t-} dN_t \\
&= r S_t dt + \sigma S_t \left(dW_t^{(S)} + \lambda^{(S)} dt \right) + S_{t-} \left[\left(e^{X_t} - 1 \right) dN_t - E^Q[e^X - 1] k^Q dt \right].
\end{aligned}
$$

The first term captures the risk-free rate on the current price, the second term captures the diffusion risk of the stock and the risk premium for diffusion risk, and the last term describes the jump risk and the premium paid for jump risk.

Again, we first derive the SDE for the price $H_t = h(t, S_t, \ldots)$ of a contingent claim. From Ito, we get

$$
\begin{aligned}
dH_t &= h_t dt + h_s \left\{ dS_t - \left(e^{X_t} - 1 \right) S_{t-} dN_t \right\} + \frac{1}{2} h_{ss} \sigma^2 S_t^2 dt \\
&\quad + \left[h(t, s e^{X_t}, \ldots) - h(t, s, \ldots) \right] dN_t.
\end{aligned}
$$

The price of the claim has to fulfill the fundamental partial differential equation, so that we get

$$
\begin{aligned}
dH_t &= r H_t dt + h_s \left\{ dS_t - r S_t dt - \left(e^{X_t} - 1 \right) S_{t-} dN_t + E^Q[e^X - 1] k^Q S_{t-} dt \right\} \\
&\quad + \left[h(t, s e^{X_t}, \ldots) - h(t, s, \ldots) \right] dN_t - E^Q \left[h(t, s e^X, \ldots) - h(t, s, \ldots) \right] k^Q dt \\
&= r H_t dt + h_s \sigma S_t \left(dW_t^{(S)} + \lambda^{(S)} dt \right) \\
&\quad + \left[h(t, s e^{X_t}, \ldots) - h(t, s, \ldots) \right] dN_t - E^Q \left[h(t, s e^X, \ldots) - h(t, s, \ldots) \right] k^Q dt.
\end{aligned}
$$

The same equation holds for the price of the claim C.

The hedge portfolio consists of $\widetilde{\phi}_t^{(S)}$ units of the stock, $\widetilde{\phi}_t^{(C)}$ units of the claim C, and an investment of $\Pi_t - \widetilde{\phi}_t^{(S)} S_t - \widetilde{\phi}_t^{(C)} C_t^{(1)}$ in the money market account, which is chosen such that the portfolio is self-financing. The SDE for the value of the hedge portfolio is then

$$\begin{aligned} d\Pi_t &= \widetilde{\phi}_t^{(S)} dS_t + \widetilde{\phi}_t^{(C)} dC_t + \left(\Pi_t - \widetilde{\phi}_t^{(S)} S_t - \widetilde{\phi}_t^{(C)} C_t\right) r dt \\ &= \Pi_t r dt + \widetilde{\phi}_t^{(S)} (dS_t - rS_t dt) + \widetilde{\phi}_t^{(C)} (dC_t - rC_t dt). \end{aligned}$$

The number of stocks in the hedge portfolio follows from

$$\widetilde{\phi}_t^{(S)} + \widetilde{\phi}_t^{(C)} \widetilde{c}_s = \widetilde{h}_s$$

so that

$$\widetilde{\phi}_t^{(S)} = \widetilde{h}_s - \widetilde{\phi}_t^{(C)} \widetilde{c}_s.$$

Plugging this expression for the number of stocks and the SDE for the claim price C into the SDE for the value of the hedge portfolio, the latter becomes

$$\begin{aligned} d\Pi_t &= \Pi_t r dt + \left\{\widetilde{h}_s - \widetilde{\phi}_t^{(C)} \widetilde{c}_s\right\} (dS_t - rS_t dt) + \widetilde{\phi}_t^{(C)} (dC_t - rC_t dt) \\ &= \Pi_t r dt + \left\{\widetilde{h}_s - \widetilde{\phi}_t^{(C)} \widetilde{c}_s + \widetilde{\phi}_t^{(C)} c_s\right\} \sigma S_t \left(dW_t^{(S)} + \lambda^{(S)} dt\right) \\ &\quad + \left\{\widetilde{h}_s - \widetilde{\phi}_t^{(C)} \widetilde{c}_s\right\} \left[\left(e^{X_t} - 1\right) S_{t-} dN_t - E^Q[e^X - 1] S_{t-} k^Q dt\right] \\ &\quad + \widetilde{\phi}_t^{(C)} \left\{\left[c(t, se^{X_t}, \ldots) - c(t, s, \ldots)\right] dN_t - E^Q\left[c(t, se^X, \ldots) - c(t, s, \ldots)\right] k^Q dt\right\}. \end{aligned}$$

Now we plug the SDEs for H and Π into the definition of dD_t:

$$\begin{aligned} dD_t &= rH_t dt + h_s \sigma S_t \left(dW_t^{(S)} + \lambda^{(S)} dt\right) \\ &\quad + \left[h(t, se^{X_t}, \ldots) - h(t, s, \ldots)\right] dN_t - E^Q\left[h(t, se^X, \ldots) - h(t, s, \ldots)\right] k^Q dt \\ &\quad - \Pi_t r dt - \left\{\widetilde{h}_s - \widetilde{\phi}_t^{(C)} \widetilde{c}_s + \widetilde{\phi}_t^{(C)} c_s\right\} \sigma S_t \left(dW_t^{(S)} + \lambda^{(S)} dt\right) \\ &\quad - \left\{\widetilde{h}_s - \widetilde{\phi}_t^{(C)} \widetilde{c}_s\right\} \left[S_{t-}\left(e^{X_t} - 1\right) dN_t - S_{t-} E^Q[e^{X_t} - 1] k^Q dt\right] \\ &\quad - \widetilde{\phi}_t^{(C)} \Big\{ \left[c(t, se^{X_t}, \ldots) - c(t, s, \ldots)\right] dN_t - E^Q\left[c(t, se^{X_t}, \ldots) - c(t, s, \ldots)\right] k^Q dt \Big\} \\ &= (H_t - \Pi_t) r dt + \left\{h_s - \widetilde{h}_s + \widetilde{\phi}_t^{(C)} \widetilde{c}_s - \widetilde{\phi}_t^{(C)} c_s\right\} \sigma S_t \left(dW_t^{(S)} + \lambda^{(S)} dt\right) \\ &\quad + \left[h(t, se^{X_t}, \ldots) - h(t, s, \ldots)\right] dN_t - E^Q\left[h(t, se^{X_t}, \ldots) - h(t, s, \ldots)\right] k^Q dt \\ &\qquad\qquad - \widetilde{h}_s \left[S_{t-}\left(e^{X_t} - 1\right) dN_t - S_{t-} E^Q[e^{X_t} - 1] k^Q dt\right] \\ &\quad - \widetilde{\phi}_t^{(C)} \Big\{ \left[c(t, se^{X_t}, \ldots) - c(t, s, \ldots)\right] dN_t - E^Q\left[c(t, se^{X_t}, \ldots) - c(t, s, \ldots)\right] k^Q dt \\ &\qquad\qquad - \widetilde{c}_s \left[S_{t-}\left(e^{X_t} - 1\right) dN_t - S_{t-} E^Q[e^{X_t} - 1] k^Q dt\right] \Big\}. \end{aligned}$$

If the jump size is deterministic, then $e^{X_t} - 1 = \mu_X$, and the SDE for the hedging error becomes

$$
\begin{aligned}
dD_t &= (H_t - \Pi_t) r dt + \left\{ h_s - \widetilde{h}_s + \widetilde{\phi}_t^{(C)} \widetilde{c}_s - \widetilde{\phi}_t^{(C)} c_s \right\} \sigma S_t \left(dW_t^{(S)} + \lambda^{(S)} dt \right) \\
&\quad + \left[h(t, s(1+\mu_X), \ldots) - h(t, s, \ldots) \right] dN_t - \left[h(t, s(1+\mu_X), \ldots) - h(t, s, \ldots) \right] k^Q dt \\
&\qquad - \widetilde{h}_s \left[S_{t-} \mu_X dN_t - S_{t-} \mu_X k^Q dt \right] \\
&\quad - \widetilde{\phi}_t^{(C)} \left\{ \left[c(t, s(1+\mu_X), \ldots) - c(t, s, \ldots) \right] dN_t - \left[c(t, s(1+\mu_X), \ldots) - c(t, s, \ldots) \right] k^Q dt \right. \\
&\qquad \left. - \widetilde{c}_s \left[S_{t-} \mu_X dN_t - S_{t-} \mu_X k^Q dt \right] \right\}.
\end{aligned}
$$

With the abbreviations

$$
\begin{aligned}
\Delta S &= S_{t-} \mu_X \\
\Delta \widetilde{S} &= S_{t-} \widetilde{\mu}_X \\
\Delta h &= h(t, s(1+\mu_X), \ldots) - h(t, s, \ldots) \\
\Delta \widetilde{h} &= \widetilde{h}(t, s(1+\widetilde{\mu}_X), \ldots) - \widetilde{h}(t, s, \ldots)
\end{aligned}
$$

and the analogous terms for the claim C, we can rewrite the SDE for the hedging error as

$$
\begin{aligned}
dD_t &= (H_t - \Pi_t) r dt + \left\{ h_s - \widetilde{h}_s + \widetilde{\phi}_t^{(C)} \widetilde{c}_s - \widetilde{\phi}_t^{(C)} c_s \right\} \sigma S_t \left(dW_t^{(S)} + \lambda^{(S)} dt \right) \\
&\quad + \Delta h dN_t - \Delta h k^Q dt - \widetilde{h}_s \left[\Delta S dN_t - \Delta S k^Q dt \right] \\
&\quad - \widetilde{\phi}_t^{(C)} \left\{ \Delta c dN_t - \Delta c k^Q dt - \widetilde{c}_s \left[\Delta S dN_t - \Delta S k^Q dt \right] \right\} \\
&= (H_t - \Pi_t) r dt + \left\{ h_s - \widetilde{h}_s + \widetilde{\phi}_t^{(C)} \widetilde{c}_s - \widetilde{\phi}_t^{(C)} c_s \right\} \sigma S_t \left(dW_t^{(S)} + \lambda^{(S)} dt \right) \\
&\quad + \Delta h dN_t - \Delta h k^Q dt - \widetilde{h}_s \left[\Delta S dN_t - \Delta S k^Q dt \right] \\
&\quad - \Delta \widetilde{h} dN_t + \Delta \widetilde{h} k^Q dt + \widetilde{h}_s \left[\Delta \widetilde{S} dN_t - \Delta \widetilde{S} k^Q dt \right] \\
&\quad + \Delta \widetilde{h} dN_t - \Delta \widetilde{h} k^Q dt - \widetilde{h}_s \left[\Delta \widetilde{S} dN_t - \Delta \widetilde{S} k^Q dt \right] \\
&\quad - \widetilde{\phi}_t^{(C)} \left\{ \Delta c dN_t - \Delta c k^Q dt - \widetilde{c}_s \left[\Delta S dN_t - \Delta S k^Q dt \right] \right\}.
\end{aligned}
$$

From the conditions on the hedge ratios, we know that

$$
\widetilde{\phi}_t^{(C)} = \frac{\Delta \widetilde{h} - \widetilde{h}_s \Delta \widetilde{S}}{\Delta \widetilde{c} - \widetilde{c}_s \Delta \widetilde{S}}
$$

With this expression for the hedge ratio $\widetilde{\phi}_t^{(C)}$, we can rewrite the SDE as

$$
\begin{aligned}
dD_t &= (H_t - \Pi_t) r dt + \left\{ h_s - \widetilde{h}_s + \widetilde{\phi}_t^{(C)} \widetilde{c}_s - \widetilde{\phi}_t^{(C)} c_s \right\} \sigma S_t \left(dW_t^{(S)} + \lambda^{(S)} dt \right) \\
&\quad + \Delta h dN_t - \Delta h k^Q dt - \widetilde{h}_s \left[\Delta S dN_t - \Delta S k^Q dt \right] \\
&\qquad - \Delta \widetilde{h} dN_t + \Delta \widetilde{h} k^Q dt + \widetilde{h}_s \left[\Delta \widetilde{S} dN_t - \Delta \widetilde{S} k^Q dt \right] \\
&\quad - \widetilde{\phi}_t^{(C)} \left\{ \Delta c dN_t - \Delta c k^Q dt - \widetilde{c}_s \left[\Delta S dN_t - \Delta S k^Q dt \right] \right. \\
&\qquad \left. - \Delta \widetilde{c} dN_t + \Delta \widetilde{c} k^Q dt + \widetilde{c}_s \left[\Delta \widetilde{S} dN_t - \Delta \widetilde{S} k^Q dt \right] \right\}.
\end{aligned}
$$