ASSET PRICING WITHOUT PROBABILITY

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Abstract. In this paper we propose a model of financial markets in which agents have limited ability to trade and no probability measure is given from the outset. In the absence of arbitrage opportunities, assets are priced according to a probability measure that lacks countable additivity. Pricing bubbles are shown to exist and a clear characterization is given in conditional terms. Despite finite additivity, we obtain an explicit representation of the expected value with respect to the pricing measure, based on some new results on finitely additive conditional expectation and finitely additive martingales. From this representation we derive a weak version of the Capital Asset Pricing Model according to which an appropriate linear transformation of assets returns is turned into a local martingale by multiplication by a stochastic discount factor. In general this conclusion need not be true for original returns and this is shown to imply deviations from the CAPM that may potentially contribute to explain the equity premium puzzle. We also discuss special cases in which the above results can be improved.

1. Introduction.

Continuous time financial models adopt a wide definition of the trading activity of agents, no matter the degree of market imperfections considered. The ground for such definition is laid by two basic assumptions: that a probability measure is given from the outset (and known to agents) and that with reference to this gains from trade may be modeled as semimartingales. The powerful mathematical construction of stochastic integration becomes thus available. We shall henceforth refer to this approach as the traditional setting. It is its merit to have permitted to rewrite under full generality the model of financial markets proposed by Arrow [5] for a finite state space and to which, by its nature, such assumptions are extraneous. This point emerges very clearly in Duffie and Huang [27]. Furthermore, this modeling choice has fostered a large number of important results in the theory of asset pricing and portfolio selection. In all such developments is therefore implicit the view of investors as agents of considerably refined ability, both in the assessment of uncertainty and in the trading of assets. The delta hedging strategy of Black and Scholes [10], a standard textbook case regardless of its overwhelming complexity, is a good case in point.

We present in this paper a theory of financial prices in continuous time and a with general state space but based on a more realistic picture of individual capabilities. To this end we introduce in section 2 a model with two distinctive features: no probability measure is taken as given and the trading of assets is considerably restricted. More precisely, we will only consider trading strategies which (i) extend over a finite time horizon, (ii) prescribe rebalancing positions a finite number of times and (iii) are contingent on a finite number of possible scenarios. This is, we believe, close in spirit to Arrow's original set up which still survives in binomial models but is here made entirely general. On the other hand, given our focus on the constraints to financial activity coming from the subjective side, we assume that markets are free of any imperfection and

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that investment (discounted) returns are bounded. In sections 3 to 8 we analyze the implications arising in this framework from the basic economic principle of absence of arbitrage opportunities. We obtain versions of the Fundamental Theorem of Asset Pricing and of the Capital Asset Pricing Model that compare with the corresponding results developed in the traditional setting but present at the same time differences which are significant in economic terms.

An explicit motivation for the assumption of a given probability measure may hardly be found in contributions to the theory of finance. However two main arguments underlying it refer to either the possibility of retrieving such a measure from historical data or to the circumstance that this be embodied in individual preferences over uncertain outcomes, i.e. expected utility. The first argument, helps explaining how is it that all agents have the same starting probability, given ease of accessing past data. Nevertheless, averaging data is not be free of troubles whenever time series are not sufficiently stationary, as it may either lead to unstable estimates or to severe undervaluation of rarely occurring events. Non stationarity of financial time series is one of the of stylized facts in financial analysis and it contributes significantly to exaplining well known puzzles, such as the equity premium (see [19] and [8] for alternative attempts to explain the equity premium puzzle based on non stationarity).

As for the second argument, a vast stream of literature, taking its moves from paradoxes of expected utility, has questioned the idea that a probabilistic assessment of uncertainty be implicit in preferences both on theoretical and empirical ground. Although in experimental psychology, subadditivity is a long-standing evidence (see [61] and [62] for pioneering work), more recent theoretical work has laid ground for models in which choice may not be based on probability measures but rather on set functions with a considerably poorer stricture. Examples are Choquet expected utility [57], case-based decision making [31], prospect theory ([45] and [63]) and support theory ([64]). On the other side, much experimental evidence has been obtained (see [20] and [21] for comprehensive reviews) showing how deeply individual choice is influenced by psychological elements such as the framing of decisions. These elements may lead investors to attach importance to events in a selective way and be responsible of market phenomena such as over- or under-reactions.

This brief discussion motivated the choice to abandon the familiar assumption of a given probability measure and to treat as the primitive of our model the collection $\mathcal{N}$ of events that do not affect individual decisions. Such events will be called *negligible*. Of course, the traditional setting may be easily reconciled with our framework when $\mathcal{N}$ amounts to the null sets generated by some probability prior Q. We want to stress, however, that in our approach this is only an important special case – another being $\mathcal{N} = \varnothing$. Our stance is that negligible events need not stem from a probabilistic assessment but, for example, from some sort of bounded rationality making individuals unable to make decisions contingent on some specific events. This part of the model is presented in section 2.2.

Most of this paper is dedicated to investigating the implications of the absence of arbitrage opportunities. The first conclusion we reach is that there exists a *pricing measure*, m, (a "martingale measure", in the terminology of traditional modeling) that does not charge negligible events and that will in general only be finitely additive. Second, we show that associated to m is a countably additive probability measure, P – which we denominate the *representing measure* – the role of which compares to that of the "physical" or "objective" measure in traditional models. The interplay between P and m is a distinguished feature of our model and most of what follows is based on it. The measure P, in particular, permits an explicit and analytically tractable representation of the pricing rule arising from m, described in Proposition 2. This is the core result of the paper and essentially it allows to overcome some of the difficulties involved in finitely additive expectation. The representing measure will also be relevant in establishing implicit mathematical properties of the return processes. It is worth highlighting that the measure P is endogenous in our approach (and typically non unique) and that it is generated by the pricing measure m, rather than the other way round as in traditional models.

The existence of a representing measure P induced by m relies on a new decomposition result for finitely additive measures (proved in [12] but restated in Lemma 1 below) which, to some extent, translates the celebrated result of Yosida and Hewitt [67] in the framework of filtered probability spaces. Some additional new tools for handling finitely additive measures also play an important role in our analysis. We prove in Proposition 1 the existence of a conditional expectation operator for finitely additive probabilities which possesses several important properties of ordinary conditional expectation and actually coincides with it in the case in which countable additivity obtains (a different proof with additional results is in [13]). This operator, of which we provide an explicit and familiar example in section 3.2, is employed to show that the martingale pricing of assets gets along with the existence of pricing bubbles, of which we offer a conditional version.

We give now a brief account of the other results obtained, which follow from the assumption that arbitrage opportunities are ruled out. First of all, we show that upon stopping at a given stopping time T, financial returns are P semimartingales – with $T = \infty$ in the case of complete financial markets. Second, we derive a formula partially analogous to the continuous time version developed by Merton [52] of the *CAPM* of Sharpe [59] and Lintner [48]. The two formulas actually coincide in the case of continuous return processes, although the difference between them is significant in the general case. The evolution of the decomposants of m with respect to time may be represented as a P positive supermartingale $X = M - A$. In our modified *CAPM*, M and A act as two distinct factors, the latter associated to the discontinuous part of the return process. Under the assumption of predictability (suitably defined for the finitely additive context in section 7) this result can be further refined and the analogy with the *CAPM* made more stringent. Several authors have extended the *CAPM* to the case of discontinuous asset returns (see [6], [42] and [58], among others). However, in the traditional setting there cannot be but one risk factor unless ad hoc structure of individuals preferences are invoked as in [26] or [24]. On the other hand, it has long been recognized that the existence of more than one factor could be responsible for the poor performance of the *CAPM* in explaining the equity premium.

The preceding result may be reformulated as follows: asset returns, conveniently transformed, are turned into local martingales if a positive local martingale Z (a *martingale density* in the terminology of [58]) is adopted as a discount factor. Apart from the intervening transformation of returns (which does not apply in important special cases), this situation is typical of most financial models, in which the existence of a martingale density is a convenient assumption. The local martingale nature of Z reflects the lack of countable additivity of the pricing measure m. The case in which Z is strictly positive and of class D has a clear characterization in terms of absence of free lunches, as illustrated by Delbaen and Schachermayer [23] in a highly influential paper (see also [43]). While uniform integrability requires substantial restrictions on the volatility process, the existence of a martingale density is not clearly related to absence of arbitrage. In section 8 we prove that given our assumptions on trading strategies if there are no free lunches on the market, then the martingale density is strictly positive provided a probability measure is assumed to be given.

On the relationship between this paper and other contributions to this literature, we will remark in due course. However, we cannot help mentioning the strong connection between our set up and the one proposed by Bättig and Jarrow in [7], a paper we came across only when the present one was almost complete. Indeed these authors introduce assumptions very similar to ours for what concerns trading strategies and the collection $\mathcal{N}$ of negligible sets (*null sets*, in their terminology) and must therefore be credited priority (for some relevant differences emerging in the treatment of $\mathcal{N}$, see the comments in section 2.2). Nevertheless, in [7] the absence of arbitrage opportunities is not considered, as the focus is on the second fundamental theorem of asset pricing rather than on the first one. Furthermore, in [7] the authors almost invariably revert to the standard case in which $\mathcal{N}$ is generated by some given probability measure.

The present paper is organized as follows. After describing the model, in section 2, we prove in section 3 the existence of the pricing measure m and discuss some of its properties. In particular, we obtain a characterization of asset bubbles in conditional terms, of which we provide an explicit example too. In section 4 we show the existence of a full probability measure P associated to m. In the following section 5 we obtain the main result of the paper, namely an explicit characterization of the expected value of asset returns with respect to the pricing measure. This crucial result, which heavily exploits the characterization of the structure of the separating measure over a filtered probability space studied in [12], allows to establish, in section 6, the conclusion that asset returns, conveniently transformed, are P semimartingales which may be turned into local martingales via a stochastic discount factor. In section 7 we restrict attention to predictable return processes, a class for which the preceding results may be significantly enhanced. In section 8 we replace the requirement that there be no arbitrage opportunities with the stronger notion of absence of free lunches, borrowed from [23]. Eventually, in section 9 we discuss the implication of our setting for empirical research and, in particular, we characterize the distribution function of assets returns with respect to the pricing measure m.

2. The Model.

2.1. **The Set-up.** The state space is represented, as customary, by an arbitrary set Ω relatively to which information evolves according to a right continuous filtration $(\mathcal{F}_t : t \in \mathbb{R}_+)$ satisfying $\mathcal{F}_0 = \{\varnothing, \Omega\}$. This assumption, not uncommon in the literature, is equivalent to interpreting $t = 0$ as time present and it will play an important role in what follows. By $\mathcal{F}$ we denote the smallest algebra on Ω containing $\mathcal{N} \cup \bigcup_{t\in\mathbb{R}_+} \mathcal{F}_t$, where $\mathcal{N}$ is the collection to be discussed below. Although, for the reasons addressed in the introduction, we will not refer to any probability measure, it will be important to know that a probability may be constructed on $\mathcal{F}$.

Assumption 1. *The set $\mathbb{P}(\mathcal{F})$ of probability measures on $\mathcal{F}$ is not empty.*

$\mathcal{T}$ will denote the set of stopping times of the filtration $(\mathcal{F}_t : t \in \mathbb{R}_+)$; $\mathcal{T}_0 = \{\tau \in \mathcal{T} : \tau < \infty\}$. If $X = (X_t : t \in \mathbb{R}_+)$ and $\tau \in \mathcal{T}$, by X^τ we indicate the "stopped" process $(X_{t\wedge\tau} : t \in \mathbb{R}_+)$. $\tilde{\mathcal{F}}$ is the product σ algebra $\mathcal{F} \otimes \mathcal{B}(\mathbb{R}_+)$ on $\tilde{\Omega} = \Omega \times \mathbb{R}_+$ – where $\mathcal{B}(\mathbb{R}_+)$ is the Borel σ algebra over $\mathbb{R}_+$ – and $\mathcal{P}$ is the predictable σ algebras of subsets of $\tilde{\Omega}$ (for standard terminology of the theory of stochastic processes we refer to [41] or [54]). The stochastic integral of θ with respect to X, whenever well defined, is indicated at will by $\int \theta dX$ or $\theta.X$. As a matter of notation, we do not distinguish between a set and its indicator (so that by FG we may denote the sets $F \cap G$ or $F \times G$ as well as their indicators); if $\mathcal{G}$ is a collection of subsets of Ω, by $\mathfrak{L}(\mathcal{G})$ we indicate the linear space spanned by the indicators of sets in $\mathcal{G}$; $\mathfrak{B}(\mathbb{X})$ denotes the space of all bounded, real valued functions on some set $\mathbb{X}$ as defined in [28]. By $ba(\mathcal{F})$ and $ca(\mathcal{F})$ we denote, as usual, the spaces of additive and countably additive set functions on $\mathcal{F}$ of bounded variation.

2.2. **Negligible Events.** Preferences are not the focus of this work and will therefore not be modelled explicitly. However, we introduce a weak notion of indifference that we denominate *negligibility.* This is defined with reference to a collection $\mathcal{N}$ of subsets of Ω, the class of *negligible events*, which is given *a priori.* Letting $\mathcal{N}$ take different forms, we can cover several situations of interest to financial modelling. We make the following assumption on $\mathcal{N}$.

Assumption 2. *The collection $\mathcal{N}$ satisfies the following properties:*

(*i*) $\Omega \notin \mathcal{N}$;
(*ii*) $A, B \in \mathcal{N}$ *implies* $A \cup B \in \mathcal{N}$;
(*iii*) $A \in \mathcal{N}$ *and* $B \subset A$ *imply* $B \in \mathcal{N}$.

As suggested in the introduction, $\mathcal{N}$ should be considered from the point of view of a decision maker and interpreted as describing the events that do not affect his choice. Several examples may be given. The most familiar and intuitive one is the class $\mathcal{N}_Q$ of null sets generated by some $Q \in \mathbb{P}(\mathcal{F})$ describing the agent's prior. Alternatively, the agent's attitude towards uncertainty may be associated with a capacity or a multiplicity of priors and $\mathcal{N}$ may thus amount to the collection of sets which are null with respect to the capacity or to all priors. In either one of these examples negligibility stems from an explicit, though possibly unconventional, assessment of the relative likelihood of events. We may however also consider situations in which agents are simply unable to carry out a proper assessment of the likelihood of events as they may feel, e.g., that the information available to them is too poor or that its processing cost is too high. Being required to consider the likelihood of some scientific discovery without being in the field is perhaps a case in point. The source of negligibility may in other words lie in some form of bounded rationality.

While property (*i*) only helps avoiding trivial cases, the brief discussion that precedes supports property (*iii*). As for (*ii*), although $\mathcal{N}$ need not be closed with respect to *countable* unions (as assumed in [7]), it is essential for what follows that it is so for finite unions. In fact (*ii*) may fail in special cases such as that of prior beliefs represented by a superadditive capacity, an expression of the propensity to uncertainty[1].

Definition 1. *$X : \Omega \to \mathbb{R}$ is negligible if $\{|X| > \eta\} \in \mathcal{N}$ for any $\eta > 0$.*

It is fairly clear that, due to property (*iii*), X is negligible if and only if $1 \wedge |X|$ belongs to the closure $\overline{\mathfrak{L}(\mathcal{N})}$ in $\mathfrak{B}\left(2^\Omega\right)$ of the linear space $\mathfrak{L}(\mathcal{N})$. This definition induces an equivalence relationship defined by saying that $X \sim_{\mathcal{N}} Y$ whenever $X - Y$ is negligible (we also say $X = Y$ up to an negligible set) as well as the quotient spaces $\mathfrak{B}(\mathcal{F}, \mathcal{N}) = \mathfrak{B}(\mathcal{F}) \Big\backslash \overline{\mathfrak{L}(\mathcal{N})}$ and $\mathfrak{B}\left(\tilde{\mathcal{F}}, \mathcal{N}\right) = \mathfrak{B}\left(\tilde{\mathcal{F}}\right) \Big\backslash \overline{\mathfrak{L}(\mathcal{N})}$. If $\mathcal{N} = \mathcal{N}_Q$ for some $Q \in \mathbb{P}(\mathcal{F})$ then $\mathfrak{B}(\mathcal{F}, \mathcal{N}_Q) = L^\infty(\mathcal{F}, Q)$ while $\mathfrak{B}(\mathcal{F}, \mathcal{N}) = \mathfrak{B}(\mathcal{F})$ whenever $\mathcal{N} = \{\varnothing\}$. In Lemma 4 in the Appendix we prove, not surprisingly, that bounded linear functionals on $\mathfrak{B}(\mathcal{F}, \mathcal{N})$ may be identified with finitely additive measures on $\mathcal{F}$ vanishing on $\mathcal{N}$ – i.e. elements of $ba(\mathcal{F}, \mathcal{N})$. We shall write $X \in \mathfrak{B}(\mathcal{F}, \mathcal{N})_+$ whenever $\{X < -\eta\} \in \mathcal{N}$ for any $\eta > 0$; $X \in \mathfrak{B}(\mathcal{F}, \mathcal{N})_{++}$ whenever $X \in \mathfrak{B}(\mathcal{F}, \mathcal{N})_+$ and there is some $\eta > 0$ such that $\{X > \eta\} \notin \mathcal{N}$.

The content of the next sections is compatible with any system $\succ$ of strict preferences such that $X \succ Y$ whenever $X - Y \in \mathfrak{B}(\mathcal{F}, \mathcal{N})_{++}$; if, moreover, $X \succsim Y$ whenever $X - Y \in \mathfrak{B}(\mathcal{F}, \mathcal{N})_+$ then negligibility implies

[1]Assumption 2 may be reconciled with more general situations if we interpret $\mathcal{N}$ as the subcollection of negligible sets possessing the listed properties.

indifference. Preferences of this sort may indeed be considered as an exemplification of over confidence, as $X - Y \in \mathfrak{B}(\mathcal{F},\mathcal{N})_{++}$ does not clearly rule out the event $\{X < Y - \eta\}$ even for η large but simply implies that it are not considered by the decision maker.

A natural question is whether negligibility, stemming from bounded rationality or not, may be reconciled with probability. An answer is provided by the following result, to which we shall refer later on.

Theorem 1. *Let $\mathcal{N}$ satisfy Assumption 2. Then:*

(1) *there exists $m \in ba(\mathcal{F},\mathcal{N})_+$ with $m(\Omega) = 1$;*
(2) *the following two statements are equivalent:*
   (a) *there exists $P \in \mathbb{P}(\mathcal{F})$ vanishing on $\mathcal{N}$;*
   (b) *there exists $Q \in \mathbb{P}(\mathcal{F})$ such that for any increasing sequence $\langle F_n \rangle_{n\in\mathbb{N}}$ of sets in $\mathcal{N}$*

$$\lim_n Q(F_n^c) > 0 \tag{2.1}$$

It is always possible to find some *finitely additive* probability which is compatible with $\mathcal{N}$ in the above sense – given Assumption 2 – but this may no longer be the case if countable additivity is required. Consider the case of a sequence $\langle F_n \rangle_{n\in\mathbb{N}}$ in $\mathcal{N}$ such that $\Omega = \bigcup_n F_n$: any $m \in ba(\mathcal{F},\mathcal{N})$ is then purely finitely additive. To see that this situation is not a pure mathematical *curiosum*, imagine an individual confronted with a real valued experiment (the example is taken from [35, p. 74]). With no informative prior over the experiment, he may reasonably assess two events of equal "size" to be equally likely, thus necessarily falling into the above case whenever the values of the experiment are rational numbers between 0 and 1. More generally, the uniform distribution over the natural numbers is a useful model both in probability (see [9, pp. 38-41] and [44]) and in economics (see [66] where measures of this family are employed to represent perfectly diversified portfolios in an APT framework) in which countable additivity cannot hold. Condition (2.1) clearly rules these cases out and all it requires is essentially that there exists a measure which is not in blatant contrast with the interpretation of $\mathcal{N}$ as a collection of null sets. Failure of (2.1) will bring several complications to the analysis that follows[2].

We let $\hat{\mathcal{F}}_t = \bigcap_{u>t} \sigma(\mathcal{F}_t \cup \mathcal{N})$.

2.3. **Asset Returns and Trading Strategies.** We shall make the following assumption concerning admissible returns:

Assumption 3. *$\mathbb{K}$ is a linear subspace of $\mathfrak{B}\left(\tilde{\mathcal{F}}\right)$ such that if $K \in \mathbb{K}$*

(*i*) *K is adapted to $(\mathcal{F}_t : t \in \mathbb{R}_+)$ – i.e. K_t is $\mathcal{F}_t$ measurable – and $K_0 = 0$;*
(*ii*) *there exists $T \in \mathbb{R}_+$ such that $K = K^T$;*
(*iii*) *$\theta.K \in \mathbb{K}$ whenever θ belongs to the set Θ of all processes of the form*

$$\theta(\omega, t) = \sum_{m=1}^{M} \theta_m]]\tau_m, \tau_{m+1}]] \tag{2.2}$$

where $\tau_m \in \mathcal{T}$ and $\theta_m \in \mathfrak{L}(\mathcal{F}_{\tau_m})$, $m = 1, \ldots, M$.

We also define

$$\mathcal{K} = \left\{K_\infty + \overline{\mathfrak{L}(\mathcal{N})} : K \in \mathbb{K}\right\} \tag{2.3}$$

[2]Let us just mention that omitting to impose condition (2.1) makes the whole construction in [7] potentially vacuous.

(where, as usual, $K_\infty + \overline{\mathfrak{L}(\mathcal{N})} = \left\{K_\infty + f : f \in \overline{\mathfrak{L}(\mathcal{N})}\right\}$) and

$$\mathcal{C} = \mathcal{K} - \mathfrak{B}(\mathcal{F}, \mathcal{N})_+ \tag{2.4}$$

In other words, the elements of $\mathcal{K}$ coincide, up to negligibility, with final returns from admissible investments.

By assumption, then, $0 \in \mathbb{K}$ which may be interpreted as the existence of an asset bearing no yield, such as money or, more generally, the *numéraire* asset with respect to which the processes in $\mathbb{K}$ have been normalized. Observe that, given (ii) above, we could equivalently require that each $\theta \in \Theta$ vanishes on $\Omega \times]T, \infty[$ for some $T \in \mathbb{R}_+$.

Assumption 3 seems to us a reasonable approximation to the way real markets actually work on three grounds. First, the strategies considered do not imply a life commitment on the side of investors. Second, trading only involves a finite number of transactions: the cost of trading – which may either consist of explicit transaction fees or be simply implicit in information processing – is then certain and reasonable. Eventually, each transaction is contingent on a finite number of scenarios, a feature making the actual implementation of the investment strategy realistically simple; it also captures the increasing importance of scenario analysis in the investment industry (see [50]). Observe that pathological situations which are of concern in the traditional approach – like so called “doubling strategies” – do not arise here, as our definition of stochastic integration is entirely trivial.

The boundedness property, although important in the following developments, may raise discussion. The existence of a lower bound on returns may be seen as the result of some form of financial regulation aiming at preventing the possibility of Ponzi schemes. The restriction of an upper bound is far less obvious and will therefore be relaxed in section 9, when dealing with applications.

A market in which the diversification of risk is to no extent restricted would allow investors to diversify their portfolios at will across admissible investment projects, provided their resulting position does not imply the possibility of unbounded losses. Portfolio returns would be described for such a market by the set

$$\mathbb{K}_\sigma = \left\{\sum_n K^n : K^n \in \mathbb{K},\ n \geq 1,\ \sum_n \left\|K^{n-}\right\| < \infty\right\} \tag{2.5}$$

from which

$$\mathcal{K}_\sigma = \left\{K_\infty + \overline{\mathfrak{L}(\mathcal{N})} : K \in \mathbb{K}_\sigma\right\} \tag{2.6}$$

and $\mathcal{C}_\sigma = \left(\mathcal{K}_\sigma - \mathbb{R}_+^\Omega\right) \cap \mathfrak{B}(\mathcal{F}, \mathcal{N})$.

A time honored issue in the theory of finance is that of completeness of markets, introduced in [36], [37] and [38] (see also [40] for a more recent treatment and [7] for a different approach). In our model completeness is defined as follows:

Definition 2. *Financial markets are complete if for every $f \in \mathfrak{B}(\mathcal{F}, \mathcal{N})$ there exists $\gamma(f) \in \mathbb{R}$ such that $f - \gamma(f) \in \mathcal{K}$.*

Remark that our definition slightly differs from the familiar one as the claims f with respect to which completeness is defined must have bounded maturity, in accordance to the view on financial markets described above. Of course, given the current set of assumptions, market completeness is not expected to prevail. In fact this condition will only be used as a reference, in Theorem 5.

2.4. **A Preliminary Result.** Starting from section 3, we shall be concerned with bounded, finitely additive measures over $\mathcal{F}$. Useful results on finitely additive measures are decomposition theorems, among which the one of Yosida and Hewitt [67] is probably the best known. In the sequel we shall heavily exploit the following variant on such theorem (proved in [13]):

Lemma 1. *Let $\mathcal{G}$ be a sub algebra of $\mathcal{F}$ and $\xi \in ba(\mathcal{G})$. There exists a unique way of writing*

$$\xi = \xi^e + \xi^p \tag{2.7}$$

with ξ^e, $\xi^p \in ba(\mathcal{G})$, where ξ^e admits a countably additive extension to $\mathcal{F}$ and any norm preserving extension of ξ^p to $\mathcal{F}$ is purely finitely additive. Furthermore,

(1) *if $\xi \geq 0$ then $\xi^e, \xi^p \geq 0$;*
(2) *if $\mathcal{G}$ is a σ algebra, $\epsilon > 0$ and $P \in ca(\mathcal{F})_+$ there exists $G \in \mathcal{G}$ such that $|\xi^p|(G) = 0$ and $P(G^c) < \epsilon$;*
(3) *if $\mathcal{H}$ is a sub σ algebra of $\mathcal{G}$, $\xi|\mathcal{H}$ the restriction of ξ to $\mathcal{H}$, and $\xi^e_{\mathcal{H}} + \xi^p_{\mathcal{H}}$ the decomposition of $\xi|\mathcal{H}$ in accordance to (2.7), then $\xi^e_{\mathcal{H}} \geq \xi^e|\mathcal{H}$ and $\xi^p_{\mathcal{H}} \leq \xi^p|\mathcal{H}$.*

It is clear that without Assumption 1, the statement of this lemma would be vacuous, as $\xi^e = 0$. In the case $\mathcal{G} = \mathcal{F}$ this decomposition coincides with that of Yosida and Hewitt – $\xi = \xi^c + \xi^\perp$, with $\xi^c \in ca(\mathcal{G})$ and $\xi^\perp$ purely finitely additive. This result illustrates how a probability assessment ξ on $\mathcal{G}$ contains in itself a completely additive probabilistic model on $\mathcal{F}$, namely that element $\bar{\xi}^e$ of $ca(\mathcal{F})$ such that $\bar{\xi}^e \big| \mathcal{G} = \xi^e$. Uniqueness of ξ^e does not imply in general that $\bar{\xi}^e$ is itself unique (unless of course $\mathcal{F} = \sigma(\mathcal{G})$). However, when $\mathcal{G} = \{\varnothing, \Omega\}$, any $P \in ca(\mathcal{F})$ represents an extension of ξ provided $P(\Omega) = \xi(\Omega)$. In section 4 the relationship between ξ and $\bar{\xi}^e$ will be viewed as the outcome of an inferential process.

In the context of a filtered probability space when $m \in ba(\mathcal{F})_+$ and $\tau \in \mathcal{T}$ we shall denote by m_τ the restriction of m to $\mathcal{F}_\tau$. Letting $\mathcal{G} = \mathcal{F}_\tau$ in the above Lemma 1, we obtain for each $\tau \in \mathcal{T}$ a decomposition $m_\tau = m^e_\tau + m^p_\tau$ in accordance with (2.7). Then, if $s < t$ we have

$$(m^e_s - m^e_t)|\mathcal{F}_s = (m_s - m_t)|\mathcal{F}_s + (m^p_t - m^p_s)|\mathcal{F}_s = (m^p_t - m^p_s)|\mathcal{F}_s \tag{2.8}$$

Although the decomposants m^e_t and m^p_t are orthogonal (and therefore as different as possible), (2.8) illustrates how the "processes" $\tilde{m}^e = (m^e_t : t \in \mathbb{R}_+)$ and $\tilde{m}^p = (m^p_t : t \in \mathbb{R}_+)$ (finitely additive processes, in the terminology of [4]) exhibit mirroring behaviour, analogously to the Poisson process, a purely discontinuous process admitting a predictable compensator with continuous paths. This suggests that the expectation with respect to the m^p component may be characterized to some extent by $\tilde{m}^e$, a much more treatable object. Much of section 5 builds on this remark.

Denote by $\hat{\mathcal{F}}_t$ the σ algebra $\bigcap_{u>t} \sigma(\mathcal{F}_u \cup \mathcal{N})$, by $\hat{m}_\tau$ the restriction of $\hat{m}_\tau$ to $\hat{\mathcal{F}}_\tau$ for $\tau \in \mathcal{T}$ and by $\hat{m}^e_\tau$ and $\hat{m}^p_\tau$ its decomposants.

3. Arbitrage, Martingales and Bubbles: The Pricing Measure

Any sensible model of financial markets should exclude the existence of free money, defined as an element k_0 of $\mathcal{K} \cap \mathfrak{B}(\mathcal{F}, \mathcal{N})_{++}$, i.e. an admissible return which is strictly positive (up to negligibility) in discounted terms. To understand this definition, remark that the initial cost of k_0 is null while it provides a strict improvement of welfare, i.e. $k_0 \succ 0$, for any agent with preferences as above. The existence on the market of an asset with these features contrasts therefore with the existence of equilibrium. In our setting the absence of arbitrage opportunities takes then the form:

$$\mathcal{K} \cap \mathfrak{B}(\mathcal{F}, \mathcal{N})_+ = \{0\} \tag{3.1}$$

Many versions of the above condition appear in the literature, all considerably more restrictive than (3.1). Further to assuming a richer structure of asset returns, the concept of an arbitrage opportunity is often conveniently reinforced into that of a free lunch (see [17], [18] and the seminal paper by Kreps [46], for a discussion). In our setting the absence of free lunches may be defined via the condition

$$\overline{\mathcal{C}} \cap \mathfrak{B}(\mathcal{F}, \mathcal{N})_+ = \{0\} \tag{3.2}$$

(the upper bar denotes the closure in the norm topology).

Let us introduce the following quantities, where $k \in \mathcal{K}$ and $f \in \mathbb{R}^\Omega \setminus \overline{\mathcal{L}(\mathcal{N})}$

$$\bar{\alpha}_k(f) = \inf_{N \in \mathcal{N}} \sup_{\omega \in N^c} (k+f)(\omega) \qquad \underline{\alpha}_k(f) = \sup_{N \in \mathcal{N}} \inf_{\omega \in N^c} (k+f)(\omega) \tag{3.3}$$

and

$$\bar{\alpha}_{\mathcal{K}}(f) = \inf_{k \in \mathcal{K}} \bar{\alpha}_k(f) \qquad \underline{\alpha}_{\mathcal{K}}(f) = \sup_{k \in \mathcal{K}} \underline{\alpha}_k(f) \tag{3.4}$$

We shall see in later sections that imposing condition (3.1) has several important implications, hinging on the following

Theorem 2. *Let (3.1) and Assumption 3 hold. Then*

(1) *there exists $m \in ba(\mathcal{F}, \mathcal{N})_+$ such that $m(\Omega) = 1$ and $m[\mathcal{K}] = 0$, i.e. a pricing measure;*
(2) *if markets are complete the pricing measure m is strictly positive, i.e. $m(F) = 0$ and $F \in \mathcal{F}$ imply $F \in \mathcal{N}$;*
(3) *there exists a countably additive separating measure whenever*
   (i). for any sequence $\langle f_n \rangle_{n \in \mathbb{N}}$ in $\mathfrak{B}(\mathcal{F}, \mathcal{N})_+$ with $\sum_n f_n \in \mathfrak{B}(\mathcal{F}, \mathcal{N})_+$,

$$\inf \left\{ \sum_n \bar{\alpha}_{k_n}(f_n) : k_n \in \mathcal{K}, n \in \mathbb{N} \right\} < \infty \tag{3.5}$$

   (ii). $\mathcal{N}$ is closed under countable unions and
   (iii). $\mathcal{C}_\sigma \cap \mathfrak{B}(\mathcal{F}, \mathcal{N})_+ = \{0\}$.

The last claim provides evidence that the countable additivity property of m is related to the degree to which portfolio diversification is allowed. The set $\mathcal{K}_\sigma$ will not be considered further in the paper but it contributes here to the view that countable additivity of the pricing measure is more an artifact of the theory than a property of actual markets. In particular, (3.5) requires that the cost incurred into by hedging each of the countable components of f separately is limited – as would clearly be the case if financial assets in $\mathcal{K}$ are sufficient to complete the markets.

Completeness of financial markets is not likely to prevail in general (particularly so under Assumption 3). Thus uniqueness of the pricing measure cannot be claimed and we denote

$$\mathcal{M}(\mathcal{K}) = \left\{ m \in ba(\mathcal{F}, \mathcal{N})_+ : m(\Omega) = 1, m[\mathcal{K}] = 0 \right\} \tag{3.6}$$

Alternative selection mechanisms out, such as the minimization of the expected value of some random element or of other functionals, raise issues of existence whenever countable additivity is required. The compactness of $\mathcal{M}(\mathcal{K})$ in the weak* topology guarantees that, for any $f \in \mathfrak{B}(\mathcal{F}, \mathcal{N})$, we can always find $m_f \in \mathcal{M}(\mathcal{K})$ such that, for example, $m_f(f) = \sup\{m(f) : m \in \mathcal{M}(\mathcal{K})\}$.

3.1. **Conditional Expectation, Asset Pricing and Bubbles.** A straightforward implication of the existence of a pricing measure in the traditional setting, as in [37] for example, is that investment returns obey a martingale restriction with respect to this measure. This is also of fundamental importance in order to establish a clear, backward pricing rule. It is not straightforward that these conclusions carry through to our model due to the fact that conditional expectation is not available with respect to finitely additive probability. The construction of conditional expectation in the finitely additive setting has received due attention in the subjective approach to probability theory in which it is obtained under the requirement of conditional coherence (see, among others, [22], [39] and [55]). In the following proposition we introduce a new operator acting on finitely additive probabilities and possessing some of the properties of ordinary conditional expectation (a different proof is in [13] which also contains a brief comparison of this operator with the subjective approach to conditional expectation).

Proposition 1. *Let $\mathcal{H}$ be an algebra of subsets of some set Ω, $\mathcal{G} \subset \mathcal{H}$ a σ algebra and $\xi \in ba(\mathcal{H})_+$. Denote by $\xi_{\mathcal{G}}$ the restriction of ξ to $\mathcal{G}$, let $\xi_{\mathcal{G}} = \gamma + \eta$ be an orthogonal decomposition of $\xi_{\mathcal{G}}$ with $\gamma \in ca(\mathcal{G})_+$ and $\eta \in ba(\mathcal{G})_+$ and define*

$$\mathcal{I}_\eta = \{F \in \mathcal{G} : \eta(F) = 0\} \tag{3.7}$$

Then, for each $f \in L^1(\mathcal{H}, \xi)$ there exists a unique $\xi(f|\mathcal{I}_\eta) \in L^1(\mathcal{G}, \gamma)$ such that

$$\xi(fI) = \xi(\xi(f|\mathcal{I}_\eta) I) = \gamma(\xi(f|\mathcal{I}_\eta) I) \tag{3.8}$$

for each $I \in \mathcal{I}_\eta$ and that for any $G \in \mathcal{G}$

$$\xi(fG|\mathcal{I}_\eta) = \xi(f|\mathcal{I}_\eta) G \tag{3.9}$$

The mapping $\xi(\cdot|\mathcal{I}_\eta) : L^1(\mathcal{H}, \xi) \to L^1(\mathcal{G}, \gamma)$ is a positive, unitary and linear operator.

We find it convenient to call the operator $\xi(\cdot|\mathcal{I}_\eta)$ "conditional expectation" (in [13] it is called *compensated conditional expectation*) for purely terminological reasons, although it is evident from (3.8) that it does not satisfy the law of iterated expectation but locally, i.e. with respect to sets in $\mathcal{I}_\eta$. From the point of view of a statistician it is perhaps regrettable that the forecast of a forecast may differ from the direct forecast. Although we have no explicit interest here for the otherwise important statistical interpretation of conditional expectation, it should be remarked that in the subjective approach following from de Finetti's work (see [22]), conditioning events are determined by admissible bets the family of which, therefore, need not be an algebra.

To illustrate the use we shall make of the preceding Proposition in the current and the following sections, let $\sigma \in \mathcal{T}$. Remark that in view of (2.8)

$$m^p|\mathcal{F}_\sigma = (m^e_\sigma - m^e)|\mathcal{F}_\sigma + m^p_\sigma$$

and $(m^e_\sigma - m^e)|\mathcal{F}_\sigma \geq 0$, by Lemma 1.3. Proposition 1 then applies with $\mathcal{G} = \mathcal{F}_\sigma$, $\xi = m^p$, $\gamma = (m^e_\sigma - m^e)|\mathcal{F}_\sigma$ and $\eta = m^p_\sigma$. Write $\mathcal{I}_\sigma = \mathcal{I}_{m^p_\sigma}$. If $I \in \mathcal{I}_\sigma$ and $f \in \mathfrak{B}(\mathcal{F})$

$$\begin{aligned} m^p(fI) &= m^p(m^p(f|\mathcal{I}_\sigma) I) \\ &= (m^p - m^p_\sigma)(m^p(f|\mathcal{I}_\sigma) I) \\ &= (m^e_\sigma - m^e)(m^p(f|\mathcal{I}_\sigma) I) \end{aligned}$$

When $f = G(K_\tau - K_\sigma)$ with $K \in \mathbb{K}$, $\tau \in \mathcal{T}$ $\tau \geq \sigma$ and $G \in \mathcal{F}_\sigma$, then by Assumption 3 and (3.1),

$$\begin{aligned} 0 &= m(GI(K_\tau - K_\sigma)) \\ &= m^e(GI(K_\tau - K_\sigma)) + m^p(GI(K_\tau - K_\sigma)) \\ &= m^e(m^e(K_\tau - K_\sigma | \mathcal{F}_\sigma) GI) + (m^e_\sigma - m^e)(m^p(K_\tau - K_\sigma | \mathcal{I}_\sigma) GI) \end{aligned}$$

i.e.

$$m^e_\sigma(K_\sigma GI) = m^e(m^e(K_\tau | \mathcal{F}_\sigma) GI) + (m^e_\sigma - m^e)(m^p(K_\tau | \mathcal{I}_\sigma) GI) \tag{3.10}$$

This expression can be further developed to obtain the following

Theorem 3. *Let Assumption 3 hold and $K \in \mathbb{K}$. If (3.1) is satisfied, there exists a stochastic process $h = (h_t : t \in \mathbb{R}_+)$ such that $h_0 = m^e(\Omega)$ and that for any $\sigma \in \mathcal{T}$ there exists a m^e_σ null set outside of which, $0 \le h_\sigma \le 1$, $h_\sigma \le m^e_\tau(h_\tau | \mathcal{F}_\sigma)$ and*

$$K_\sigma = h_\sigma m^e(K_\infty | \mathcal{F}_\sigma) + (1 - h_\sigma) m^p(K_\infty | \mathcal{I}_\sigma) = m(K_\infty | \mathcal{I}_\sigma) \tag{3.11}$$

Furthermore $h_\sigma = 1$ m^e_σ a.s. for each $\sigma \in \mathcal{T}$ if and only if m is countably additive.

The second equality in (3.11) establishes that, relatively to the conditioning operator introduced in Proposition 1, the pricing measure is indeed a "martingale" measure – although many analytical properties of ordinary martingales (such as convergence theorems) do not apply here. Pricing is therefore an intrinsically forward looking exercise and, provided the structure of the conditioning operator $m(\cdot | \mathcal{I}_\sigma)$ is explicit enough (as in the example that follows), then a clear and useful relationship exists between $\mathbb{K}$ and $\mathcal{K}$.

Let W represent the wealth process out of some admissible investment, so that $W - W_0 \in \mathbb{K}$: (3.11) clearly translates into

$$W_\sigma = h_\sigma \phi(W)_\sigma + (1 - h_\sigma)\beta(W)_\sigma \tag{3.12}$$

where $\phi(W)_\sigma = m^e(W_\infty | \mathcal{F}_\sigma)$ and $\beta(W)_\sigma = m^p(W_\infty | \mathcal{I}_\sigma)$ – so that

$$\phi(W)_0 = m^e(\Omega)^{-1} m^e(W_\infty) \text{ and } \beta(W)_0 = m^p(\Omega)^{-1} m^p(W_\infty)$$

Indeed (3.12) establishes that the pricing rule just described differs considerably from the traditional one. In fact one may remark that the conditioning operator $m^p(\cdot | \mathcal{I}_\sigma)$ inherits, through (3.8), the property that for any $P \in ca(\mathcal{F})_+$ and $\epsilon > 0$ there exists $F \in \mathcal{F}$ such that $P(F^c) < \epsilon$ and $m^p(F | \mathcal{I}_\sigma) = 0$. In other words, the component $\beta(W)$ of W only charges the remote behavior of the wealth process W, both with respect to time and randomness. It is therefore quite natural, after the seminal work of Gilles and Leroy [32], to interpret $\phi(W)$ as the fundamental value of the investment and $\beta(W)$ as its bubble part. The noteworthy properties of (3.12) is that such decomposition is established here in conditional terms, that it applies to bounded processes over a finite horizon and that, as the returns need not be positive throughout, bubbles may assume either sign.

3.2. **An Example.** Consider a traditional financial model with underlying probability Q in which (discounted) asset returns are turned into a martingale by means of multiplication by positive martingale Z with $Z_0 = 1$ but non necessarily uniformly integrable[3]. As we shall see in the following section 6 this situation is quite general. Let Z_∞ be the Q a.s. limit of Z. We can associate to Z the finitely additive probability

[3] Some aspects of this example were treated in [15].

measure μ defined as

$$\mu(F) = \operatorname*{LIM}_n Q(Z_n F) \tag{3.13}$$

for $F \in \mathcal{F}$ – where LIM denotes here the Banach limit introduced in [1] (but see also [56, p. 367])[4]. It is easy to conclude that μ is a pricing measure as for $k \in \mathcal{K}$

$$\mu(k) = \mu(K_t) = \lim_n Q(Z_n K_t) = Q(Z_t K_t) = 0$$

Furthermore, $\mu_t^e(F) = Q(Z_t F)$ for $F \in \mathcal{F}_t$ and $\mu_t^P(F) = \mathrm{LIM}_n\, Q((Z_n - Z_t) F)$ for $t \in \mathbb{R}_+$: in fact $Q(Z_t F)$ clearly extends to a countably additive measure on $\mathcal{F}$ while $\mathrm{LIM}_n\, Q((Z_n - Z_t) F)$ vanishes on $\left\{\sup_{s \leq t} Z_s < 2^n\right\}$ for each n so that it is purely finitely additive [9, theorem 10.3.3, p. 244] as well as any of its extensions.

Let $\sigma \in \mathcal{T}$. Recalling Assumption 3 above we easily get that ZW is a martingale too and that, therefore,

$$\begin{aligned} Z_\sigma W_\sigma &= \lim_n Z_\sigma^n W_\sigma^n \\ &= \lim_n Q(Z_n W_n | \mathcal{F}_\sigma) \\ &= Q(Z_\infty W_\infty | \mathcal{F}_\sigma) + \lim_n Q((Z_n - Z_\infty) W_n | \mathcal{F}_\sigma)^5 \end{aligned} \tag{3.14}$$

If we define (with the convention $\frac{0}{0} = 0$) $h_\sigma = Z_\sigma^{-1} Q(Z_\infty | \mathcal{F}_\sigma)$, $\phi(W)_\sigma = Q(Z_\infty | \mathcal{F}_\sigma)^{-1} Q(Z_\infty W_\infty | \mathcal{F}_\sigma)$ and $\beta(W)_\sigma = (Z_\sigma - Q(Z_\infty | \mathcal{F}_\sigma))^{-1} \lim_n Q((Z_n - Z_\infty) W_{\tau_n} | \mathcal{F}_\sigma)$, then (3.14) is the exact translation of (3.12) to the present setting. It should be remarked that $h_t = 1$ Q a.s. is equivalent to the case in which Z is a uniformly integrable martingale i.e. μ is countably additive. In models of optimal consumption and portfolio selection (such as those treated e.g. in [6] and [29], for example) Z emerges as the process describing marginal utility of consumption along the optimal path. In these models it cannot usually be established that Z is a uniformly integrable martingale nor is it clear which economically meaningful conditions could be imposed in order to obtain such property. This remark is a point in case for the finitely additive model we propose here. Of course, it would be important to see if a partial converse could be established, i.e. if in the full generality of our model the separating measure could be associated to a martingale density. An answer to this problem will be offered in section 6.

4. Probability from Asset Returns: The Representing Measure.

In this section we shall show that the absence of arbitrage opportunities induces the existence of a full probability measure P on $\mathcal{F}$. The role played by P may be compared to that of the objective measure in the traditional setting and we will refer to it as the representing measure generated by m. Although pricing is performed via the finitely additive measure m, it could still be of practical as well as of theoretical worth to establish statistical properties of the return process for which countable additivity matters, as will clearly emerge from the following section 5. Thus m and P play entirely different roles and may in principle be far apart not only for what concerns additivity. Our treatment in section 2.2 has shown, for example, that P may not vanish on negligible sets, making the connection with preferences remote. An important issue is that of the difference between the collections of null sets of m and P – the issue of *consistency* between m and P, in the terminology of section 8. This has an immediate answer in the context of complete markets (by the second claim of Theorem 2) while it is more delicate in the general case.

[4] Definition (3.13) may be given in terms of ordinary limits if and only if Z is a uniformly integrable martingale (see [15]).

Let $t \in \mathbb{R}_+$. Given m and the information $\mathcal{F}_t$ available at time t, the component m_t^e, by its same definition, allows to infer a completely additive measure $\bar{m}_t^e$ over the whole of $\mathcal{F}$: in other words, agents may extract from the restriction of m to $\mathcal{F}_t$ a fully additive view concerning randomness, i.e. on $\mathcal{F}$. However, the probabilistic view implicit in $\bar{m}_t^e$ has only a local meaning and is bound to change considerably as time passes by, as the effect of the arrival of new information. In particular, the last claim of Lemma 1 implies

$$\bar{m}_t^e|\mathcal{F}_s = m_t^e|\mathcal{F}_s \leq m_s^e = \bar{m}_s^e|\mathcal{F}_s \tag{4.1}$$

for $s \leq t$. This ilustrates how deeply the decomposition (2.7) depends on the underlying information structure. The question therefore arises whether it is possible to extract from the collection $\{\bar{m}_t^e : t \in \mathbb{R}_+\}$ a global perspective P on $\mathcal{F}$ not contradicting the inference made at each point in time, m_t^e. Although different, sensible criteria could be considered in order to judge whether P contrasts with m_t^e or not, a clear contradiction definitely exists between these two measures whenever, for some $F \in \mathcal{F}_t$, $m_t^e(F) > 0$ but $P(F) = 0$: it may well be that events that were first assessed to be null are later deemed likely, as new information becomes available, but the opposite would indeed imply that the global assessment expressed by P implicitly disproves the one embodied in m_t^e. In the context of a model in which agents form their beliefs based on past experience, Kurz [47, axiom 2, p. 13] suggests the above criterion as a definition of individual beliefs not contradicting observable data (so called *rational beliefs*). The following result (proved in [12]) provides a positive answer to the above question.

Theorem 4. *There exist $P, \hat{P} \in \mathbb{P}(\mathcal{F})$ such that $m_\tau^e \ll P|\mathcal{F}_\tau$ and $\hat{m}_\tau^e \ll \hat{P}\Big|\hat{\mathcal{F}}_\tau$ for each $\tau \in \mathcal{T}_0$. If the condition (2.1) is satisfied, then $\hat{P}$ may be chosen such that $\hat{P}[\mathcal{N}] = 0$.*

It should be remarked that P is partly influenced by subjective elements – namely the collection $\mathcal{N}$ – and partly by the structure of markets. When confronted with a richer structure either of negligible events or of admissible trading strategies the resulting set $\mathcal{K}'$ of marketed claims would be strictly larger than $\mathcal{K}$ and both the separating measure and the probability associated to $\mathcal{K}'$ will differ from the ones arising from $\mathcal{K}$. Denote

$$\mathbb{P}(m) = \{P \in \mathbb{P}(\mathcal{F}) : m_\tau^e \ll P|\mathcal{F}_\tau,\ \tau \in \mathcal{T}_0\} \tag{4.2}$$

An almost immediate consequence of Theorem 4 and (4.1) is the following

Corollary 1. *Let $P \in \mathbb{P}(m)$, $\tau \in \mathcal{T}_0$ and $dm_\tau^e/dP_\tau = X_\tau$. The stochastic process $X = (X_t; t \in \mathbb{R}_+)$ is a P right continuous, positive supermartingale, decomposing as*

$$X = M - A \tag{4.3}$$

where M is a positive local martingale and A an increasing, predictable process (see [41]*) with $A_0 = 0$ and $P(A_\infty) < \infty$ (see* [49]*).*[6]

Given right continuity of A, we can define $\lambda \in ca\left(\tilde{\mathcal{F}}\right)_+$ implicitly through the equation $\lambda(F) = P\int F dA$.

It is clear from (4.2) that if $P \in \mathbb{P}(m)$ and $P' \gg P$ then $P' \in \mathbb{P}(m)$. If beliefs are formed in accordance with the inferential or learning mechanism described above, then, in principle, they will be strongly heterogeneous among agents. This notwithstanding, in the following sections we will treat $P \in \mathbb{P}(m)$ as fixed.

[6] With the aid of Corollary 1 and the convention $\frac{0}{0} = 0$, we may identify explicitely the process h in (3.11) as $h_\sigma = P(X_\infty|\mathcal{F}_\sigma)X_\sigma^{-1}$.

5. An Explicit Representation.

Due to finite additivity, the expected value of asset returns with respect to m has a limited analytical tractability and this may represent a major drawback of the present approach, both in theory and in applications. For what concerns applications, this issue will be addressed in section 9. In this section we shall prove that the expectation $m(k)$ may receive an explicit and convenient representation whenever $k \in \mathcal{K}$, obtained by mapping each element in $\mathcal{K}$ onto its past history. The advantage of doing so is that, on the space of processes, countable additivity of the integral is partly restored. In the following, it will be clear that it makes no difference wether we adopt as filtration the original one rather than its completion so that we can take it to be complete. Condition (2.1) plays quite a role and will be assumed.

The structure of the m_t^e component has been characterized in Corollary 1 through the supermartingale X. For what concerns the m_τ^p component we shall take advantage of the following result:

Lemma 2. *There exists a collection* $\left\{\bar{m}_\tau^p \in ba(\mathcal{F})_+ : \tau \in \mathcal{T}_0\right\}$ *such that* $\bar{m}_\tau^p$ *is an extension of* m_τ^p *to* $\mathcal{F}$ *and* $\bar{m}_\tau^p \geq \bar{m}_\sigma^p$ *whenever* $\sigma \in \mathcal{T}_0$ *and* $\sigma \leq \tau$.

We shall now investigate more deeply the properties of the pricing kernel. Define to this end the following quantities:

- The collection $\mathcal{H}$ of all pairs $H = \left(\langle t_i\rangle_{i=0}^I, \langle F_i\rangle_{i=0}^I\right)$ of finite sequences such that
  - (i) $t_i \in \mathcal{T}_0$ and $0 = t_0 \leq t_1 \leq \ldots \leq t_I$, P a.s.,
  - (ii) $F_i \in \mathcal{I}_{t_i}$, $F_i \subset F_{i-1}$ for $i = 1, \ldots, I$, $F_I = \varnothing$ and
  - (iii) M^{t_I} is a uniformly integrable martingale.
- $D_i^H(K) = F_i\left(K_{t_{i+1}} - K_{t_i}\right)$ and
- $K^H = \left(K_t^H : t \in \mathbb{R}_+\right)$ where

$$K_t^H = \sum_{i=0}^{I-1} D_i^H\left(K^t\right) \tag{5.1}$$

K_t^H is an "approximation" of K_t obtained according to $H \in \mathcal{H}$. Later on it will be useful to rewrite (5.1) as $K_t^H = \sum_{i=0}^{I-1} F_i F_{i+1}^c K_{t_{i+1}\wedge t}$. The trading strategy behind (5.1) prescribes to stop at time t_i whenever F_i does not occur. Choosing F_i appropriately, this criterion will apply very rarely, if we judge likelihood by $m_{t_i}^e$, although with certainty if we evaluate it under $m_{t_i}^p$. In this way the role of the "irregular" component of m at the start of each investing period (t_i, t_{i+1}) may be entirely neglected and, if the behavior of m_t^p is sufficiently regular with respect to time and the length of the interval sufficiently short, then it may be conjectured that $m_{t_{i+1}}^p$ will play a minor role. Clearly, this argument hinges on the behavior of K^H when passing to the limit, provided convergence obtains in some suitable sense. It is crucial to our aims that if $K_t \in \mathcal{K}$ then $K_t^H \in \mathcal{K}$ as well. Observe that $D_i^H(K^t)$ and K_t^H are $\mathcal{F}_{t_{i+1}\wedge t}$ and $\mathcal{F}_{t_I\wedge t}$ measurable respectively and that $\bar{m}_{t_{i+1}\wedge t}^p(F_i; t_i \geq t) = 0$ (see Lemma 7 in the Appendix). We shall write $\mathcal{I}_{t_i}^t$ as short for $\mathcal{I}_{m_{t_i\wedge t}^p}$ (see (3.7)).

Let us define the following key terms:

$$J_H(K)_t = \sum_{i=0}^{I-1}\left(\bar{m}_{t_I\wedge t}^p - \bar{m}_{t_{i+1}\wedge t}^p\right)\left(D_i^H\left(K^t\right)\right) - \sum_{i=1}^{I-1}\bar{m}_{t_{i+1}\wedge t}^p\left(F_i K_{t_i}^t\right) \tag{5.2}$$

and

$$I_H(K)_t = \sum_{i=0}^{I-1}\bar{m}_{t_{i+1}\wedge t}^p\left(F_i K_{t_{i+1}}^t\right) \tag{5.3}$$

From (5.1) – (5.3) it clearly follows the decomposition

$$m^p_{t_I\wedge t}\left(K^H_t\right)=J_H\left(K\right)_t+I_H\left(K\right)_t \tag{5.4}$$

Exploiting (2.8) and Proposition 1, we show in Proposition 2 below that the terms $J_H\left(K\right)$ and $I_H\left(K\right)$ can be described explicitly. This result is based on the following intuition. First, since, as we have seen, $\bar{m}^p_{t_{i+1}\wedge t}\left(F_i\right)=m^p_{t_{i+1}\wedge t}\left(F_i;t_i<t\right)$ and $F_i\left\{t_i<t\right\}\in\mathcal{I}^t_{t_i}$, then

$$\begin{aligned}
I_H\left(K\right)_t &= \sum_{i=0}^{I-1} m^p_{t_{i+1}\wedge t}\left(F_i\left\{t_i<t\right\}K^t_{t_{i+1}}\right)\\
&= \sum_{i=0}^{I-1} m^p_{t_{i+1}\wedge t}\left(m^p_{t_{i+1}\wedge t}\left(K^t_{t_{i+1}}\middle|\mathcal{I}^t_{t_i}\right)F_i\left\{t_i<t\right\}\right)\\
&= \sum_{i=0}^{I-1}\left(m^p_{t_{i+1}\wedge t}-m^p_{t_i\wedge t}\right)\left(m^p_{t_{i+1}\wedge t}\left(K^t_{t_{i+1}}\middle|\mathcal{I}^t_{t_i}\right)F_i\left\{t_i<t\right\}\right)\\
&= P\sum_{i=0}^{I-1}\left(A^t_{t_{i+1}}-A^t_{t_i}\right)m^p_{t_{i+1}\wedge t}\left(K^t_{t_{i+1}}\middle|\mathcal{I}^t_{t_i}\right)F_i\left\{t_i<t\right\}
\end{aligned}$$

Observe that this can be rewritten more concisely as

$$I_H\left(K\right)_t=P\int_0^t f^t_H\left(K^t\right)dA=\int_0^t f^t_H\left(K^t\right)d\lambda \tag{5.5}$$

where for $Y\in\mathfrak{B}\left(2^{\tilde{\Omega}}\right)$,

$$f^u_H\left(Y\right)=\sum_{i=0}^{I-1}\bar{m}^p_{t_{i+1}\wedge u}\left(Y_{t_{i+1}}\middle|\mathcal{I}^u_{t_i}\right)F_i\left\{t_i<u\right\}]]t_i,t_{i+1}]] \tag{5.6}$$

We obtain then from (5.5) that I behaves like an ordinary stochastic integral: several nice properties become thus available.

For what concerns the J term, by (2.8)

$$\begin{aligned}
\sum_{i=0}^{I-1}\left(m^p_{t_I\wedge t}-m^p_{t_{i+1}\wedge t}\right)\left(D^H_i\left(K^t\right)\right) &= P\sum_{i=0}^{I-1}\left(A^t_{t_I}-A^t_{t_{i+1}}\right)D^H_i\left(K^t\right)\\
&= P\sum_{i=0}^{I-1}\sum_{j=i+1}^{I-1}\left(A^t_{t_{j+1}}-A^t_{t_j}\right)D^H_i\left(K^t\right)\\
&= P\sum_{j=1}^{I-1}\left(A^t_{t_{j+1}}-A^t_{t_j}\right)\sum_{i=0}^{j-1}D^H_i\left(K^t\right)\\
&= P\sum_{j=1}^{I-1}\left(A^t_{t_{j+1}}-A^t_{t_j}\right)K^H_{t\wedge t_j}
\end{aligned}$$

Then from (5.2) and the fact that $K_0 = 0$ (by definition) it follows that

$$\begin{aligned} J_H(K)_t &= P\sum_{j=1}^{I-1}\left(A^t_{t_{j+1}} - A^t_{t_j}\right)K^H_{t\wedge t_j} - P\sum_{i=1}^{I-1}\left(A^t_{t_{i+1}} - A^t_{t_i}\right)F_i K^t_{t_i} \\ &= P\sum_{i=1}^{I-1}\left(A^t_{t_{i+1}} - A^t_{t_i}\right)\left(K^H_{t_i} - F_i K_{t_i}\right) \\ &= P\sum_{i=1}^{I-1}\left(A^t_{t_{i+1}} - A^t_{t_i}\right)\sum_{j=0}^{i-1} F_j F^c_{j+1} K_{t_{j+1}} \end{aligned}$$

so that we obtain the bound

$$|J_H(K)_t| \leq \|K\|\, P\left(A_t F^c_{I-1}\right) \tag{5.7}$$

It is natural to conjecture from (5.7) that the J term may be set so to converge to 0; a more delicate issue is that of existence of the limit for the "stochastic integral" I and of its representation. This is solved in the following

Proposition 2. *Let $\tau \in \mathcal{T}_0$ be such that X^τ is uniformly integrable. If $K \in \mathbb{K}$ and (3.1) holds, then under Assumption 3*

$$m^p_\tau(K_\tau) = \int_0^\tau f(K)\, d\lambda \quad i.e. \quad m(K_\tau) = P\left(X_\tau K_\tau + \int_0^\tau f(K)\, dA\right) \tag{5.8}$$

where $f : \mathfrak{B}\left(2^{\tilde{\Omega}}, \mathcal{N}\right) \to L^\infty(\mathcal{P}, \lambda)$ is a positive, linear operator of unitary norm and such that $f(YZ) = f(Y)Z$ whenever $Y, Z \in \mathfrak{B}\left(\tilde{\mathcal{F}}, \mathcal{N}\right)$ and Z is càglàd. Therefore, if $K \in \mathfrak{B}\left(\tilde{\mathcal{F}}\right)$ is càdlàg, $f(K) = K_- + f(\Delta K)$.

The representation (5.8) following from the no arbitrage principle has a number of implications that will be developed in the present and in the following sections. The operator f defined in Proposition 2 is to some extent similar to the P predictable projection – denoted in the sequel by $\mathcal{P}(X)$. However, f is not invariant with respect to predictable processes but to *càglàd* processes only: once again the difference amounts to lack of continuity. In section 7 we will consider an extension of the notion of predictability suitable for the present finitely additive context.

It is worth noticing that (5.8) establishes, in restriction to $\mathcal{K}$ and $\mathbb{K}$ respectively, a correspondence between m^p and λ which, so to speak, restores countable additivity by translating expectation of random elements into expectation of random processes. In fact, if $T_f : \mathfrak{B}(\mathcal{F}) \to L^\infty(\mathcal{P}, \lambda)$ is the linear and continuous mapping defined implicitly via $T_f(K_\infty) = f(K)$, then $m^p = \lambda T_f$. The lack of countable additivity of m is therefore a consequence of the discontinuity of T_f: $f(K^n)$ may not converge to 0 although the sequence $\langle K^n_\infty\rangle_{n\in\mathbb{N}}$ may be such that $K^n_\infty \downarrow 0$, a situation which contrasts with the usual setting of a countably additive pricing measure. The pricing rule (5.8) is therefore intrinsically path dependent since, regardless of the actual structure of the asset return, it is based on the whole process $f(K)$ rather than just on K_∞.

6. The Martingale Property

The representation obtained in Proposition 2 will be shown in this section to deliver a number of noteworthy implications concerning the nature of the return process K. To make explicit the financial content of the representation obtained in Proposition 2 we need the following assumption

Assumption 4. *Every $K \in \mathbb{K}$ is càdlàg.*

The following stopping time has a key role in our analysis.[7]

$$T = \inf \{t \in \mathbb{R}_+ : X_{t-} = 0 \text{ or } X_t = 0\} \tag{6.1}$$

Theorem 5. *Let $K \in \mathbb{K}$ and (3.1) hold. Then under Assumptions 3 and 4:*

- (i) *XK is a P special semimartingale;*
- (ii) *the stochastic process K stopped at T, i.e. K^T, is a P semimartingale;*
- (iii) *if financial markets are complete and (2.1) holds, then P may be chosen such that $P(T < \infty) = 0$ so that K is a P semimartingale.*

Theorem 5 establishes that, in some appropriate form, the absence of arbitrage opportunities implies the semimartingale nature of asset returns, a pervasive assumption in all financial models. It should be highlighted that there are predecessors to this result, particularly Ansel and Stricker [3, theorem 8, p. 383] and Stricker [60, theorem 3 p. 456 and theorem 5, p. 458] (but see also [23, theorem 7.2, p. 504]). The noticeable fact is that this property, which crucially depends on the underlying probability measure, is obtained here without explicit reference to any preassigned probability: it is therefore entirely endogenous. Of course, the behavior of K after the random time T is totally unrestricted. In fact our model contains no prediction over $]]T, \infty[[$, as the behavior of the separating measure becomes purely finitely additive over that domain. The last claim, which anticipates in its proof the main ideas of Proposition 3 below, establishes that this cannot be the case if markets are complete. Once again the implicit probability model of returns turns out to depend in a crucial way on the structure of markets. The issue of the positivity of X outside the special case of complete markets will be addressed in section 8.

Define $\Delta K^{\sim} = f(\Delta K) - \mathcal{P}(\Delta K)$, $D_K = \{\Delta K^{\sim} = 0\}$. Denoting by $\mathcal{E}$ the exponential semimartingale of Doléans-Dade (and $\mathcal{L}$ its inverse, the stochastic logarithm) we may then define

$$\hat{K}_t = \int_0^t D_K dK^T \tag{6.2}$$

$$Z = \mathcal{E}\left(\int X_-^{-1} dM\right) \tag{6.3}$$

– clearly a positive local martingale – and

$$B = \mathcal{E}\left(\int X_-^{-1} dA\right) \tag{6.4}$$

– clearly a predictable process of locally integrable variation (in the proof of the theorem that follows it is shown that indeed Z and B are well defined, as assumed here). By Theorem 5, K^T admits a unique decomposition $K^T = M^K + V^K$ where M^K is a local martingale and V^K a predictable process of locally integrable variation.

Theorem 6. *Let $K \in \mathbb{K}$ and define $\hat{K}$, Z and B as in (6.2), (6.3) and (6.4) respectively. If (3.1) holds then:*

1. *$V^K + \left[\mathcal{L}(Z), M^K\right] + \int \Delta K^{\sim} d\mathcal{L}(B)$ is a P local martingale, i.e.*

$$V^K + \mathcal{P}\left(\left[\mathcal{L}(Z), M^K\right]\right) + \int \Delta K^{\sim} d\mathcal{L}(B) = 0 \tag{6.5}$$

2. *$Z\hat{K}$ is a P local martingale.*

[7] In [41], lemma III.3.6 it is shown that T is a stopping time. Remark that, since $X_0 = 1$ and X is right continuous, $P(T > 0) = 1$. In the following we will refer to T assuming that $m \in \mathcal{M}(\mathcal{K})$ and $P \in \mathbb{P}(m)$ is given.

The Intertemporal Capital Asset Pricing Model, a core result of modern asset pricing that dates back to [52], is often stated in one of these two equivalent formulations: (i) there exists a stochastic discount factor transforming asset returns into local martingales, (ii) the expected (excess) return of assets equals the (negative) of the quadratic covariation of the return process with the "market price for risk", often identified with the marginal utility from consumption of a representative agent in equilibrium. These formulations find their translation into the present context in the above statements which, however, consider a linear transformation of asset returns. Theorem 6 differs then from the traditional *CAPM* if either $D_K^c \neq \varnothing$ or $P(T<\infty)>0$. We will discuss these two conditions in some detail in sections 7 and 8 (although easy special cases are that of continuous return processes – when $D_K^c=\varnothing$ – and of complete financial markets). Apart from such issues, the stochastic discount factor Z will in general only be a positive local martingale rather than a uniformly integrable one: following [58] we will refer to it as a *martingale density*. According to the celebrated result of Delbaen and Schachermayer [23], the existence of a maritngale density is neither necessary nor sufficient to exclude the existence of free lunches, unless the stochastic factor is strictly positive and of class D. The latter condition is usually obtained by imposing considerable constraints on the volatility of returns (particularly in the form of some lower bound) in contrast, though, with the fact that volatility is usually the primary focus of most models and with the evidence that periods of high and of low volatility often alternate randomly on the market.

(6.5) suggests that the original intuition underlying the *CAPM* should actually be adapted to keep into account the correlation with an additional factor represented by B i.e., ultimately, by A. M and A act in this model as separate market factors and the latter plays a specific role in the pricing of the jump part of K. It is then indeed tempting to consider A as a market price of the risk implicit in the discontinuities of asset returns, an exemple of which are unexpected, large falls of the market index. (6.5) contributes to the view that correlation with a unique discount factor is not enough to explain excess returns, an issue addressed repeatedly by the literature on the equity premium puzzle. It is remarkable that this conclusion does not hinge on any special assumption, as on preferences or beliefs, which are usually invoked to explain such stylized fact[8].

An equivalent reformulation of (6.5) is the joint condition:

$$V^{K,c}+\left\langle \mathcal{L}(Z), M^K\right\rangle+\int \Delta K^{\sim} d\mathcal{L}(B)^c = 0 \tag{6.6}$$

$$V^{K,d}+\sum p\left(\Delta\mathcal{L}(Z)\,\Delta M^K\right)+\sum \Delta K^{\sim}\Delta\mathcal{L}(B) = 0 \tag{6.7}$$

Although the term $\int \Delta K^{\sim} d\mathcal{L}(B)$ is actually a function of the jumps of the return process, we cannot conclude that it is itself a jump process. In other words, (6.6) suggests that the risk originating from the discontinuities of the return process may affect its continuous part too. One may expect that exceptional corporate actions, such as dividend payouts or mergers, or events influencing the prospects of default of firms will influence the path of the corresponding stock more substantially than in the narrow proximity of the

[8] The existence of multiple factors in the *CAPM* may also be obtained from a non additive specification of preference. In a model with habit formation, Detemple and Zapatero [24, equation (6.5), p. 1647] characterize the second factor as covariance with disutility of future standards of living. In the context of stochastic differential utility, Duffie and Epstein [26, equation (18), p. 422] recover two additional factors further to equilibrium consumption, one of which being related to market portfolio. In a model with differential information, Ziegler [68, equation (24), p. 9] obtains additional factors ensuing from the updating process. It should be stressed that all these papers consider a model of general equilibrium while our analysis has only a partial equilibrium flavour.

event considered. This marks quite a difference with the more traditional version of the *CAPM* extended to include possible jumps[9].

Of course in some of the previous examples of discontinuities (such as dividends announcements), the timing of discontinuities in asset returns may be announced with due notice. This influences our result as follows

Corollary 2. *Let $K \in \mathbb{K}$ and $[[\tau]] \subset \{\Delta K \neq 0\}$. If τ is predictable, then*

$$P \int f(\Delta K_\tau)\, dA = P f(\Delta K_\tau)\, \Delta A_\tau \tag{6.8}$$

Therefore, if $\{\Delta K \neq 0\}$ is exhausted by the sequence $\langle \tau_n \rangle_{n \in \mathbb{N}}$ of predictable times, then $\int \Delta K^{\sim} d\mathcal{L}(B) = \sum_n \Delta K^{\sim}_{\tau_n} \Delta \mathcal{L}(B)_{\tau_n}$ and $V^{K,c} + \langle \mathcal{L}(Z), M^K \rangle = 0$.

In the special case considered in Corollary 2, the finitely additive nature of m only bears consequences on the pricing of discontinuities but as far as the continuous part of asset returns is concerned the model is indistinguishable from the traditional one. It is surprising that, as will be shown in section 9, this same conclusion emerges relatively to the statistical properties of returns with respect to the separating measure. It remains true that exceptional events taking place at times that do not admit being announced will have a deeper impact on returns. Examples of these may be firm specific – such as changes in credit ratings – or market driven. Our conclusion is therefore that events such as October 87 may have a long lasting influence on the pricing of assets, a view which has received some consense.

7. Predictable Returns.

A vast majority of financial models are written under the assumption that the price process is *càdlàg* and predictable – or even that it has continuous sample paths. In this section we will comply with predictability, a property that allows for a representation of the pricing kernel more explicit than (5.8). The notion of predictability has though to be partly adapted to our finitely additive set up. In fact let σ be a stopping time predictable with respect to some $Q \in \mathbb{P}(\mathcal{F})$ and $\langle \sigma^r \rangle_{r \in \mathbb{N}}$ its announcing sequence. Then, for each n and r there exists $\delta > 0$ such that $Q(\sigma_r > \sigma - \delta; \sigma > 0) < 2^{-n} Q(\sigma > 0)$: in other words, most of σ can be anticipated with fixed notice. This same property may not hold whenever Q is only finitely additive, a situation that deprives the announcing sequence of much of its economic content. Denote by $m_{\tau-}$ the restriction of m to $\mathcal{F}_{\tau-}$ when $\tau \in \mathcal{T}$ and by $m^p_{\tau-}$ and $m^e_{\tau-}$ its components.

Definition 3. *A stopping time σ is (m, P) predictable if it admits a sequence $\langle \sigma^r \rangle_{r \in \mathbb{N}}$ of stopping times such that:*

(i). $\sigma^r \uparrow \sigma$, P a.s. and $P(\sigma^r < \sigma) = P(0 < \sigma)$,
(ii). $\lim_n m(\sigma - \sigma^r \leq 2^{-n}) = 0$ and
(iii). $\lim_r \left(m^p_{\sigma-} - m^p_{\sigma^r} \right)(\Omega) = 0$.

Definition 4. *A càdlàg, adapted process K is (m, P) predictable if*

(i) *there exists a sequence $\langle \upsilon_r \rangle_{r \in \mathbb{N}}$ of (m, P) predictable stopping times such that $\{\Delta K \neq 0\} = \bigcup_r [[\upsilon_r]]$ and*
(ii) *for each r, K_{υ_r} is $\mathcal{F}_{\upsilon_r -}$ measurable.*

[9] Jarrow and Rosenfeld [42] consider an extension of the *CAPM* to include jumps. See also [6].

It is clear that continuous processes are (m,P) predictable and that the above condition coincides with the usual definition of a predictable stopping time in the case in which m is countably additive.

Theorem 7. *Let $K \in \mathbb{K}$ be càdlàg with $\{\Delta K \neq 0\} = \bigcup_r [[\upsilon_r]]$, let (3.1) hold and let $\tau \in \mathcal{T}$ be such that X^τ is uniformly integrable. If K is (m,P) predictable*

$$m^p_\tau(K_\tau) = \int_0^\tau K d\lambda \quad i.e. \quad m(K_\tau) = P\left\{M_\tau K_\tau - \int_0^\tau A_- dK\right\} \tag{7.1}$$

It follows that ZK^T – see (6.3) – is a P local martingale.

Essentially, in the first part of this theorem it is established a sufficient condition under which $f(K) = K$.

8. Consistent Pricing Measures

It is commonly believed that financial markets are incomplete. However, it is as widely shared the view that any contingent claim may be introduced and traded on the market provided its price is set fairly. The pricing measure should then not only be considered as a tool to evaluate currently traded assets, as in the preceding sections, but it should also provide reliable indications for the pricing of claims that do not yet exist on the market but that it may sensible to introduce at some later stage. Viewing the current market setting as the outcome of some equilibrium process (and borrowing from game theoretic terminology) we conclude that the pricing measure may partly depend on out of equilibrium elements.

Consider a situation in which f is a bounded, strictly positive random element and agents investigate what would the fair price of an asset paying f at maturity be in a market free of arbitrage opportunities. Of course the answer is trivial if such asset is actually traded, as its price cannot be but $m(f)$. For the more general case we can only conclude that its price to be should be positive, in the absence of arbitrage opportunities.

Definition 5. *Let m be a pricing measure and $P \in \mathbb{P}(m)$. The pair (m,P) is consistent if $f \in \mathfrak{B}(\mathcal{F},\mathcal{N})_+$ and $P(f>0)>0$ imply $m(f)>0$. $m \in \mathcal{M}(\mathcal{K})$ is consistent if there exists $P \in \mathbb{P}(m)$ such that (m,P) is consistent.*

To understand better the economic content of the preceding definition, imagine that condition (2.1) holds. Then $P(f>0)>0$ implies that f is not negligible and, as such, it would be reasonable to write a claim contingent on it. However, pricing such claim by m would result in a violation of the no arbitrage principle, given that $m(f)=0$. Therefore consistency demands that m may be extended as a pricing functional to a larger collection of claims than those actually traded.

To illustrate a situation in which the pair (m,P) is not consistent, imagine that $P(T \le t) > 0$ for some $t \in \mathbb{R}_+$ (where T is defined as in (6.1) with reference to m and P). Then, by Lemma 1, for each ϵ there exists a set $F \in \mathcal{F}_t$ such that $F \subset \{T \le t\}$, $P(F) \ge (1-\epsilon)P(T \le t)$ and $m^p_t(F) = 0$. But then

$$m(F) = m^e_t(F) = P(X_t F\{T \le t\}) = 0$$

It is then clear that the consistency of m requires that $P(T<\infty)=0$. This situation is more general than it appears at a first glance

Lemma 3. *Let (2.1) hold, $m \in \mathcal{M}(\mathcal{K})$, $P \in \mathbb{P}(m)$ and T be defined as in (6.1) with reference to m and P. (m,P) is consistent if and only if $P(T<\infty)=0$.*

We introduce this additional definition:

Definition 6. *Let π be a real valued function on $\mathfrak{B}(\mathcal{F},\mathcal{N})$, $f\in\mathfrak{B}(\mathcal{F},\mathcal{N})$ and define*

$$\mathcal{K}(f;\pi)=\{k+d(f-\pi(f)):k\in\mathcal{K},\ d\in\mathbb{R}\}$$

If

$$\mathcal{K}(f;\pi)\cap\mathfrak{B}(\mathcal{F},\mathcal{N})_{+}=\{0\} \tag{8.1}$$

then we write $\pi(f)\in\mathcal{A}(f,\mathcal{K})$ and we say that π is an admissible pricing rule for f and that $\mathcal{K}$ possesses the extension property with respect to f. If $\mathcal{A}(f,\mathcal{K})\neq\varnothing$ for any $f\in\mathfrak{B}(\mathcal{F},\mathcal{N})$ then $\mathcal{K}$ is said to possess the extension property.

It is clear from the definition that the extension property reinforces that of absence of arbitrage opportunities. In Theorem 8 we provide a useful characterization of the extension property (in the Appendix a more general result is proved, see Theorem 10).

Theorem 8. *$\mathcal{K}$ has the extension property if and only if it admits no free lunches, i.e. (3.2) holds.*

The abstract *NFL* condition translates thus into the practical issue of whether markets may or not be extended consistently with the no arbitrage principle. This characterization helps providing economic content to the mathematical notion of free lunch, often criticized for not having a clear market interpretation (see especially [17] and [18]). In their seminal paper Harrison and Kreps [36, theorem 1, p. 386-7.] have already pointed out the relationship between the extension property and viability, i.e. the property that asset prices may support the optimal choice of an agent with regular preferences (see also [46]).

The property introduced, however, is not sufficient to guarantee that the market could be extended to any arbitrary set of new contracts in respect of the no arbitrage principle. In general, for example, it will not be possible to stretch the given pricing measure to a consistent price system for the completed financial market. According to Theorem 5 (*iii*) this would guarantee the semimartingale nature of assets returns and the existence of a positive martingale density. In the next result we show, however, that such an extension is possible under the additional assumption of a given probability measure Q generating $\mathcal{N}$.

Proposition 3. *Let (3.2) be satisfied and assume that $\mathcal{N}=\mathcal{N}_Q$ for some $Q\in\mathbb{P}(\mathcal{F})$. Then there exists a consistent pricing measure.*

This result, even if cast in the traditional setting of an existing probability measure, is new and provides some firm ground for much of the existing financial literature.The statement of the proposition will however not hold in the general case, i.e. without the assumption that negligible sets originate from a probability measure. A closer look at the proof, based on a fixed point argument, reveals that this may be due to the impossibility to select for each $m\in\mathcal{M}(\mathcal{K})$ a measure $P_m\in\mathbb{P}(m)$ in such a way that the function $m\to P_m$ be continuous. Once again the point is that the pricing measure and the representing measure may be considerably different mathematical objects.

9. Applications.

A deep objection to the finitely additive model in asset pricing comes from the need for a tractable description of the distribution function of asset returns under the risk neutral measure. Statistical properties of asset returns under the pricing measure are in fact important. A major area of empirical research, originating from [11], suggests for example that the prices of derivatives may be useful to extract an estimate of such function (see [2] and [16] for samples of more recent work). It would be doubtful in the finitely additive setting that such a distribution is analytically tractable and, even more, that it may be inferred from current prices of derivative contracts. Particularly so in a model of financial markets less restrictive than the one considered under Assumption 3.

A more realistic assumption on the process of investment returns may be obtained by relaxing the condition of the existence of an upper bound.

Assumption 5. $\mathbb{K} \subset \mathbb{R}^{\tilde{\Omega}} \Big\backslash \overline{\mathfrak{L}(\mathcal{N})}$ *is such that if* $K \in \mathbb{K}$

(*i'*) *K is adapted to* $(\mathcal{F}_t : t \in \mathbb{R}_+)$, $K_0 = 0$ *and* $K^- \in \mathfrak{B}\left(\tilde{\mathcal{F}}, \mathcal{N}\right)$;

(*ii*) *there exists* $T \in \mathbb{R}_+$ *such that* $K = K^T$;

(*iii'*) *if* $\theta \in \Theta$, $a, b \in \mathbb{R}$, $K_1, K_2 \in \mathbb{K}$ *then* $\theta.K$,$aK_1+bK_2 \in \mathbb{K}$ *provided* $(\theta.K)^-$,$(aK_1 + bK_2)^- \in \mathfrak{B}\left(\tilde{\mathcal{F}}, \mathcal{N}\right)$.

Furthermore, we assume that there are no arbitrage opportunities – so that (3.1) is in place. We easily deduce the analogous of Theorem 2 for the present context:

Theorem 9. *Let Assumption 5 hold. Then if there are no arbitrage opportunities there exists* $m \in ba(\mathcal{F}, \mathcal{N})_+$ *such that* $m(\Omega) = 1$ *and* $m[\mathcal{K}] \leq 0$. *I*

It is implicit in the statement that if k is the overall, discounted return from an admissible trading strategy then necessarily k is m integrable. From this it readily follows that $m(|k| > 2^n)$ converges to 0 as n increases. Let then μ be the measure on $(\mathbb{R}_+, \mathcal{B}(\mathbb{R}_+))$ induced by k – i.e. $\mu = m \circ k^{-1}$. Then for each ε there exists a η such that $\mu([-\eta, \eta]) < \varepsilon$ or, in other words, the measure μ is tight. Then, Dubins and Savage have proved [25, see pp. 190-191] that one may associate to μ a countably additive measure μ^* on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ – the *conventional companion* in their terminology – which is unique and has the following properties:

(1) $\int h d\mu = \int h d\mu^*$ for each $h : \mathbb{R} \to \mathbb{R}$ continuous and bounded;

(2) $\mu([x - \varepsilon, \infty[) \leq \mu^*([x, \infty))$ and $\mu^*([x - \varepsilon, \infty[) \leq \mu([x, \infty[)$ for each $x \in \mathbb{R}$ and $\varepsilon > 0$.

The first property ensures that μ^* may be employed to compute all moments of K_∞ along with other important quantities. The second property implies that the two distribution functions have exactly the same points of continuity and on these they agree with each other. It also follows that $\int h d\mu = \int h d\mu^*$ for any function $h : \mathbb{R} \to \mathbb{R}$ for which either integral is well defined [25, lemma 3, p. 191].

In particular let $K \in \mathbb{K}$ be the underlying asset in discounted terms at maturity T and let $c_t(s, T)$ be the time t price of a call option maturing at T and with discounted strike price equal to s. Then, if there are no arbitrage opportunities and Assumption 5 holds asset payoff functions are integrable with respect to the separating measure m, by Theorem 9. By the preceding remarks, then,

$$\begin{aligned} c_t(s, T) &= \int (K_T - s)^+ \, dm(\omega) \\ &= \int (x - s)^+ \, d\mu(x) \\ &= \int_s^\infty (x - s) \, d\mu^*(x) \end{aligned}$$

Then, as in the case of a countably additive risk neutral measure, we deduce the inequalities

$$\varepsilon\mu^*(]s + \varepsilon, \infty[) \leq c_t(s, T) - c_t(s + \varepsilon, T) \leq \varepsilon\mu^*(]s, \infty[)$$

from which it follows that the right derivative of the call price with respect to the strike price, i.e.

$$\lim_n \varepsilon^{-1}\left[c_t\left(s,T\right)-c_t\left(s+\varepsilon,T\right)\right]$$

exists and coincides with $\mu^*\left(]s,\infty[\right)$. Of course this quantity may differ from $\mu\left(]s,\infty[\right)$ and, more precisely, $\mu\left(]s,\infty[\right)\leq\mu^*\left(]s,\infty[\right)$ unless s is a point of continuity.

This analysis suggests that the significance of derivative prices for evaluating the risk neutral measure carries over to the case in which such measure lacks countable additivity. Nevertheless it is implicit that allowing for the existence of points of discontinuity receives now greater significance since these are the only points in which the risk neutral measure may differ from its conventional companion. In particular, we conclude that the standard approach may induce an over-estimate of the mass assigned by the risk neutral measure to the right hand tail. The importance of discontinuities partly contrasts with many empirical works in which it is common to assume that μ is absolutely continuous and to estimate therefore the density function. Although this choice has the advantage of guaranteeing that the resulting distribution function is increasing, it has, though, the additional meaning of ruling out the role of finitely additive measures.

Appendix A. Proofs from Section 2.

We recall that each $X \in \mathfrak{B}(\mathcal{F}, \mathcal{N})$ is be of the form $\varkappa(f) = \left\{ f + h : h \in \overline{\mathfrak{L}(\mathcal{N})} \right\}$ for some $f \in \mathfrak{B}(\mathcal{F})$ and that a norm is defined by letting $\|X\|_{\mathfrak{B}(\mathcal{F},\mathcal{N})} = \inf\left\{ \|f\|_{\mathfrak{B}(\mathcal{F})} : X = \varkappa(f) \right\}$. It is clear from the definition that $f \in X$ and $\|X\|_{\mathfrak{B}(\mathcal{F},\mathcal{N})} = 0$ if and only if $f \in \overline{\mathfrak{L}(\mathcal{N})}$ i.e. if and only if f is negligible. $\mathfrak{B}(\mathcal{F}, \mathcal{N})$ is complete [56, p. 219], hence a Banach space and the embedding $\varkappa$ of $\mathfrak{B}(\mathcal{F})$ into $\mathfrak{B}(\mathcal{F}, \mathcal{N})$ is a linear homeomorphism with $\|\varkappa\| \leq 1$. We shall exploit the following result which is fairly obvious and in which $ba(\mathcal{F}, \mathcal{N})$ denotes the set of bounded, finitely additive set functions vanishing on $\mathcal{N}$.

Lemma 4. *There exists an isometric isomorphism between $\mathfrak{B}(\mathcal{F}, \mathcal{N})^*$ and $ba(\mathcal{F}, \mathcal{N})$ defined implicitly via the equation*

$$\phi(\varkappa(f)) = \int f d\mu \tag{A.1}$$

Proof. If $\mu \in ba(\mathcal{F}, \mathcal{N})$ the right hand side of (A.1) defines a functional over $\mathfrak{B}(\mathcal{F}, \mathcal{N})$ which is linear by the linearity of $\varkappa$. Let $\varkappa^* : \mathfrak{B}(\mathcal{F}, \mathcal{N})^* \to \mathfrak{B}(\mathcal{F})^*$ be the adjoint of $\varkappa$ and $\phi \in \mathfrak{B}(\mathcal{F}, \mathcal{N})^*$. Then $\varkappa^*\phi \in \mathfrak{B}(\mathcal{F})^*$ and is therefore isometrically isomorphic to some $\mu \in ba(\mathcal{F})$ via $\int f d\mu = (\varkappa^*\phi)(f) = \phi(\varkappa(f))$. Given that $\overline{\mathfrak{L}(\mathcal{N})} \in \varkappa(0)$, if $N \in \mathcal{N}$ then $\mu(N) = \phi(\varkappa(0)) = 0$ and, as $\mathcal{N}$ is closed with respect to intersection, it follows that $|\mu|(N) = 0$ for each $N \in \mathcal{N}$: in other words, $\mu \in ba(\mathcal{F}, \mathcal{N})$. (A.1) establishes then an isomorphism. Given $\|\varkappa^*\| \leq 1$, $\|\mu\| \leq \|\phi\|$; however if $X = \varkappa(f)$, then $|\phi(X)| = \left|\int f d\mu\right| \leq \|\mu\| \|f\|$ so that $|\phi(X)| \leq \|\mu\| \inf\left\{ \|f\|_{\mathfrak{B}(\mathcal{F})} : X = \varkappa(f) \right\} = \|\mu\| \|X\|$, i.e. $\|\phi\| \leq \|\mu\|$. □

Let $\mathcal{R}(\mathcal{N})$ be the σ ring generated by the collection $\mathcal{N}$ and $\mathcal{R}(\mathcal{N})^\perp = \{F \subset \Omega : F^c \in \mathcal{R}(\mathcal{N})\}$.

Lemma 5. *If $m, n \in ba(\sigma(\mathcal{N}), \mathcal{R}(\mathcal{N}))_+$, then $m = n$ if and only if $m(\Omega) = n(\Omega)$. Furthermore, m is countably additive.*

Proof. If $m \in ba(\sigma(\mathcal{N}), \mathcal{R}(\mathcal{N}))_+$ then for $F \in \mathcal{R}(\mathcal{N})^\perp$, $m(F) = m(\Omega) - m(F^c) = m(\Omega)$ and the first claim follows from the well known fact $\sigma(\mathcal{N}) = \mathcal{R}(\mathcal{N}) \cup \mathcal{R}(\mathcal{N})^\perp$. If $F, G \in \mathcal{R}(\mathcal{N})^\perp$ and $FG = \varnothing$, then $\Omega \in \mathcal{R}(\mathcal{N})$ and $ba(\sigma(\mathcal{N}), \mathcal{R}(\mathcal{N}))_+$ only contains the null measure which is trivially countably additive. Let $\langle F_n \rangle_{n\in\mathbb{N}}$ be a disjoint sequence of $\sigma(\mathcal{N})$ measurable sets, then with at most one element in $\mathcal{R}(\mathcal{N})^\perp$, say F_1, and let $F = \bigcup_n F_n$ If $F_1 \in \mathcal{R}(\mathcal{N})$ then $F \in \mathcal{R}(\mathcal{N})$ so that $m(F) = 0 = \sum_n m(F_n)$; if $F_1 \notin \mathcal{R}(\mathcal{N})$, $m(F) = m(F_1) = \sum_n m(F_n)$. □

Proof of Theorem 1. (first claim). $\mathfrak{B}(\mathcal{F}, \mathcal{N})$ is a Banach space and $\varkappa(\Omega)$ an inner point for $\mathcal{B}(\mathcal{F}, \mathcal{N})_+$ as $\|\varkappa(f) - \varkappa(\Omega)\| < \eta$ implies $\{f < 1 - 2\eta\} \in \mathcal{N}$. The linear functional ϕ separating the convex sets $\mathcal{B}(\mathcal{F}, \mathcal{N})_+$ and $\{0\}$ will therefore be bounded and non trivial, i.e. $\phi(\varkappa(\Omega)) > 0$ and such that $\phi\left[\mathcal{B}(\mathcal{F}, \mathcal{N})_+\right] \geq \phi(0) = 0$. By Lemma 4 ϕ is associated to some $m \in ba(\mathcal{F}, \mathcal{N})_+$ that can be normalized so that $m(\Omega) = 1$.

(second claim). Assume that (b) holds and let $\mathcal{N}' = \{F \cup G : F \in \mathcal{N}, G \in \mathcal{N}_Q\}$. $\mathcal{N}'$ satisfies Assumption 2 and moreover $\Omega \notin \mathcal{R}(\mathcal{N}')$. According to the first claim of this Lemma and Lemma 5 the latter condition is necessary and sufficient for the existence of $Q_0 \in \mathbb{P}(\sigma(\mathcal{N}'))$ which vanishes on $\mathcal{N}'$. Let $F \in \mathcal{F}$. Each two versions of the conditional expectation $Q(F|\sigma(\mathcal{N}))$ coincide outside some set in $\mathcal{N}_Q \subset \mathcal{N}'$, i.e. they coincide Q_0 a.s.. Let $P(F) = Q_0(Q(F|\sigma(\mathcal{N})))$. Then P is unambiguously defined, positive and $P(\Omega) = 1$; furthermore, P vanishes on $\mathcal{N}$. If $\langle F_n \rangle_{n\in\mathbb{N}}$ is a disjoint sequence of $\mathcal{F}$ measurable sets, then

$Q\left(\bigcup_n F_n \middle| \sigma(\mathcal{N})\right) = \sum_n Q\left(F_n \middle| \sigma(\mathcal{N})\right)$ up to a Q_0 null set and since $Q_0 \in \mathbb{P}(\sigma(\mathcal{N}'))$,

$$\begin{aligned} P\left(\bigcup_n F_n\right) &= Q_0\left(Q\left(\bigcup_n F_n \middle| \sigma(\mathcal{N})\right)\right) \\ &= Q_0\left(\sum_n Q\left(F_n \middle| \sigma(\mathcal{N})\right)\right) \\ &= \sum_n Q_0\left(Q\left(F_n \middle| \sigma(\mathcal{N})\right)\right) \\ &= \sum_n P(F_n) \end{aligned}$$

and (a) follows. The reverse implication is obvious.

Lemma 6. *Let $\hat{\mathcal{F}}_t$ be defined as in the text (see p. 6). Then $\hat{\mathcal{F}}_0 = \sigma(\mathcal{N})$.*

Proof. It is easy to see that $\sigma(\mathcal{F}_t \cup \mathcal{N}) = \{F \Delta N : F \in \mathcal{F}_t,\ N \in \mathcal{R}(\mathcal{N})\}$. Let $t_n < 2^{-n}$. If $F \in \hat{\mathcal{F}}_0$ then $F \in \sigma(\mathcal{F}_{t_n} \cup \mathcal{N})$ for each n and we may therefore write $F = F_n \Delta N_n$ or even

$$\begin{aligned} F &= \bigcup_k \bigcap_{n>k} (F_n \Delta N_n) \\ &= \bigcup_k \left\{ \left(\bigcap_{n>k} F_n N_n^c \right) \cup N_k^* \right\} \\ &= \bigcup_k \left\{ \left(\bigcap_{n>k} F_n \cap \bigcap_{n>k} N_n^c \right) \right\} \cup \bigcup_k N_k^* \\ &= \left\{ \bigcup_k \bigcap_{n>k} F_n \cap \bigcup_k \bigcap_{n>k} N_n^c \right\} \cup \bigcup_k N_k^* \end{aligned}$$

It is clear that $N_k^* \subset \bigcup_{n>k} N_n$ so that $N_k^*, \bigcup_k N_k^* \in \mathcal{R}(\mathcal{N})$ by Assumption 2. On the other hand, $\bigcup_k \bigcap_{n>k} F_n \in \mathcal{F}_0$ by right continuity so that $\bigcup_k \bigcap_{n>k} F_n$ is either Ω or $\varnothing$: in the former case, $F = \left(\bigcup_k \bigcap_{n>k} N_n^c\right) \cup \left(\bigcup_k N_k^*\right)$; in the latter $F = \bigcup_k N_k^*$. In either case, $F \in \sigma(\mathcal{N})$. □

Appendix B. Proofs from Section 3.

Proof of Theorem 2. Since $\varkappa(\Omega)$ is an internal point of $\mathfrak{B}(\mathcal{F}, \mathcal{N})_+$, there exists a non trivial, continuous linear functional ϕ that separates $\mathfrak{B}(\mathcal{F}, \mathcal{N})_+ \setminus \{0\}$ and $\mathcal{K}$. Since $\phi[\mathcal{K}]$ and $\phi\left[\mathfrak{B}(\mathcal{F}, \mathcal{N})_+\right]$ are a linear subspace and a convex cone in $\mathbb{R}$, respectively, and since $\phi[\mathcal{K}] \cap \phi\left[\mathfrak{B}(\mathcal{F}, \mathcal{N})_+\right] \subset \{0\}$, it must be that $\phi[\mathcal{K}] = 0 \leq \phi\left[\mathfrak{B}(\mathcal{F}, \mathcal{N})_+\right]$ and $\phi(\varkappa(\Omega)) > 0$. By Lemma 4 and normalization we may represent ϕ via $m \in ba(\mathcal{F}, \mathcal{N})_+$ with $m(\Omega) = 1$.

Let $k = -\sum_n k_n \in \mathbb{R}^\Omega$, $f_n \in \mathbb{R}^\Omega_+$ and $f = \sum_n f_n$; let also $N_n \in \mathcal{N}$, $n \geq 1$ and $N_0 = \bigcup_n N_n$. Then,

$$N_0^c (f - k) \leq \sup_{\omega \in N_0^c} (f - k)(\omega) = \sup_{\omega \in N_0^c} \sum_n (f_n + k_n)(\omega) \leq \sum_n \sup_{\omega \in N_n^c} (f_n + k_n)(\omega)$$

Suppose that $f \in \mathfrak{B}(\mathcal{F}, \mathcal{N})_+$, that N_n is such that $\bar{\alpha}_{k_n}(f_n) \geq \sup_{\omega \in N_n^c} (f_n + k_n)(\omega) - \eta 2^{-n}$ and that $\sum_n \bar{\alpha}_{k_n}(f_n) < \infty$ then for $\omega \in N_0^c$,

$$\bar{\alpha}_{k_n}(f_n) + \eta 2^{-n} > k_n(\omega)$$

Let $K^n \in \mathbb{K}$ be such that $k_n = K^n_\infty$ and $F \in \mathcal{F}_t$. Since $F\,]]t, \infty[[\, \in \Theta$, then $F\left(K^n_\infty + h - K^n_t\right) \in \mathcal{K}$ whenever $h \in \overline{\mathcal{L}(\mathcal{N})}$. Given that

$$\left(K^n_\infty + h - K^n_t\right) F \geq -\left(\|(K^n_\infty + h) \wedge 0\| + K^n_t\right) F$$

the absence of arbitrage opportunities implies that for all $h \in \overline{\mathcal{L}(\mathcal{N})}$ and $F \in \mathcal{F}_t$ with $F \neq \varnothing$ there must be $\omega \in F$ such that $-\|(K^n_\infty + h) \wedge 0\| \leq K^n_t$ i.e. $\sup_{\omega \in F} K^n_t(\omega) \geq -\|k^-_n\|$ so that $\{K^n_t < -\|k^-_n\|\} = \varnothing$ and $\{K^n < -\|k^-_n\|\} = \varnothing$: therefore $\|K^{n-}\| \leq \|k^-_n\|$ and $k \in \mathcal{K}_\sigma$ (see the definition (2.5)). Moreover, the absence of arbitrage opportunities implies $\sup_{\omega \in N^c_n}(f_n + k_n)(\omega) \geq 0$ for each n so that

$$N^c_0 f \leq \sum_n \sup_{\omega \in N^c_n} (f_n + k_n)(\omega) + N^c_0 (k \wedge \|f\|) \leq \sum_n \bar{\alpha}_{k_n}(f_n) + \eta + N^c_0 (k \wedge \|f\|)$$

Remark that $k \wedge \|f\| \in \mathcal{C}_\sigma$ and choose $m \in ba(\mathcal{F}, \mathcal{N})_+$ with $m(\Omega) = 1$ such that $m[\mathcal{C}_\sigma] \leq 0 \leq m\left[\mathfrak{B}(\mathcal{F}, \mathcal{N})_+\right]$. Then, $m(f) \leq \sum_n \bar{\alpha}_{k_n}(f_n) + \eta$ for each $\eta \geq 0$ i.e. $m(f) \leq \sum_n \bar{\alpha}_{k_n}(f_n)$. Replacing f by $\sum_{n>N} f_n$ we obtain likewise the inequality $m\left(\sum_{n>N} f_n\right) \leq \sum_{n>N} \bar{\alpha}_{k_n}(f_n)$ from which we deduce that

$$\lim_N m\left(\sum_{n>N} f_n\right) \leq \lim_N \sum_{n>N} \bar{\alpha}_{k_n}(f_n) = 0$$

i.e. $m(f) = \sum_n m(f_n)$.

Appendix C. Proofs from Section 3.1.

Proof of Proposition 1. By proving the statement separately for f^+ and f^- we can reduce to the case where $f \in L^1(\mathcal{H}, \xi)_+$. Let $I \in \mathcal{I}_\eta$ and $G \in \mathcal{G}$. The set function $\phi_{fI}(G) = \xi(fIG)$ defined on $\mathcal{G}$ is positive and additive. Let $\langle G_n \rangle_{n \in \mathbb{N}}$ be a sequence of $\mathcal{G}$ measurable sets such that $\lim_n \gamma(G_n) = 0$ and fix $\varepsilon, \delta > 0$. As f is ξ integrable, then letting $h_n = fIG_n$

$$\begin{aligned}
\xi(h_n > \varepsilon) &\leq \xi\left(f \leq \delta^{-1}; h_n > \varepsilon\right) + \delta\xi(f) \\
&\leq \xi(IG_n > \varepsilon\delta) + \delta\xi(f) \\
&\leq \xi(IG_n) + \delta\xi(f) \\
&\leq \gamma(G_n) + \delta\xi(f)
\end{aligned}$$

as $IG_n \in \mathcal{I}_\eta$. Since δ was chosen arbitrarily, we deduce that h_n converges to 0 in ξ measure and, since $|h_n| \leq f$, this implies [28, theorem III.3.7, p. 124] that

$$\lim_n \phi_{fI}(G_n) = \lim_n \xi(h_n) = 0$$

ϕ_{fI} is then absolutely continuous with respect to γ and therefore countably additive. Denote then by $\xi(fI|\mathcal{I}_\eta)$ the corresponding Radon Nikodym derivative. If $G \in \mathcal{G}$, $I \subset I'$ and $I, I' \in \mathcal{I}_\eta$ then

$$\gamma(\xi(fI|\mathcal{I}_\eta)G) = \xi(fIG) \leq \xi(fI'G) = \gamma(\xi(fI'|\mathcal{I}_\eta)G)$$

i.e. $0 \leq \xi(fI|\mathcal{I}_\eta) \leq \xi(fI'|\mathcal{I}_\eta)$ up to a γ null set. Let $\langle I_n \rangle_{n \in \mathbb{N}}$ be an increasing sequence of sets in $\mathcal{I}_\eta$ with the property that $\lim_n \xi(fI_n) = \sup_{I \in \mathcal{I}_\eta} \xi(fI)$: then the sequence $\langle \xi(fI_n|\mathcal{I}_\eta) \rangle_{n \in \mathbb{N}}$ is γ a.s. increasing and we may thus define $\xi(f|\mathcal{I}_\eta) = \lim_n \xi(fI_n|\mathcal{I}_\eta)$ outside some γ null set. Monotone convergence and the inequality $\gamma(\xi(fI_n|\mathcal{I}_\eta)) \leq \xi(f)$ imply that $\xi(f|\mathcal{I}_\eta) \in L^1(\mathcal{G}, \gamma)_+$. To prove (3.8) let $I \in \mathcal{I}_\eta$.

$$\gamma(\xi(f|\mathcal{I}_\eta)I) = \lim_n \gamma(\xi(fI_n|\mathcal{I}_\eta)I) = \lim_n \xi(fI_nI) \leq \xi(fI)$$

However, if $\lim_n \xi(fI_nI) < \xi(fI)$, then,

$$\begin{aligned}\sup_{I' \in \mathcal{I}_\eta} \xi(fI') &= \lim_n \xi(fI_n) \\ &= \lim_n \xi(fI_nI) + \lim_n \xi(fI_nI^c) \\ &< \xi(fI) + \lim_n \xi(fI_nI^c) \\ &= \lim_n \xi(f(I_nI^c \cup I))\end{aligned}$$

which is contradictory since $I^cI_n \cup I \in \mathcal{I}_\eta$.

To prove uniqueness remark that, since $\mathcal{G}$ is a σ algebra and γ is countably additive, for any k there exists a set $I_k \in \mathcal{I}_\eta$ such that $\gamma(I_k^c) < 2^{-k}$. Let $y \in L^1(\mathcal{G}, \gamma)_+$ satisfy (3.8) and $G \in \mathcal{G}$. Then, $y = \xi(f|\mathcal{I}_\eta)$ up to a γ null set as

$$\gamma(yG) = \lim_k \gamma(yGI_k) = \lim_k \xi(fGI_k) = \lim_k \gamma(\xi(f|\mathcal{I}_\eta)GI_k) = \gamma(\xi(f|\mathcal{I}_\eta)G)$$

Given uniqueness and additivity of ξ, $\xi(f+g|\mathcal{I}_\eta) = \xi(f|\mathcal{I}_\eta) + \xi(g|\mathcal{I}_\eta)$; (3.9) is a consequence of the fact that $IG \in \mathcal{I}_\eta$ whenever $I \in \mathcal{I}_\eta$ and $G \in \mathcal{G}$.

Proof of Theorem 3. Observe that, by Lemma 1 claim 3, we have for $\sigma \le \tau$ and $\sigma, \tau \in \mathcal{T}$, $0 \le m^e|\mathcal{F}_\sigma \le m_\tau^e|\mathcal{F}_\sigma \le m_\sigma^e$. From the theorem of Radon Nikodym, we can therefore write $m^e(F) = m_\sigma^e(h_\sigma F)$ for each $F \in \mathcal{F}_\sigma$, where $0 \le h_\sigma \le 1$, m_σ^e a.s.. Of course, if $F \in \mathcal{F}_\sigma$ it is also true that

$$m_\sigma^e(h_\sigma F) = m^e(F) = m_\tau^e(h_\tau F) = m_\tau^e(m_\tau^e(h_\tau|\mathcal{F}_\sigma)F) \le m_\sigma^e(m_\tau^e(h_\tau|\mathcal{F}_\sigma)F)$$

i.e. $h_\sigma \le m_\tau^e(h_\tau|\mathcal{F}_\sigma)$, m_σ^e a.s.. Therefore, (3.10) becomes

$$m_\sigma^e(K_\sigma GI) = m_\sigma^e(h_\sigma m^e(K_\tau|\mathcal{F}_\sigma)GI) + m_\sigma^e((1-h_\sigma)m^p(K_\tau|\mathcal{I}_\sigma)GI)$$

Since $G \in \mathcal{F}_\sigma$ is arbitrary and I may be so chosen that $m_\sigma^e(I^c) < \epsilon$ for each ϵ, we obtain that

$$K_\sigma = h_\sigma m^e(K_\tau|\mathcal{F}_\sigma) + (1-h_\sigma)m^p(K_\tau|\mathcal{I}_\sigma)$$

up to a m_σ^e null set. By Assumption 3 $K_\infty = K_T$ for some $T \in \mathbb{R}_+$: choosing $\tau = T$ proves (3.11). $h_\sigma = 1$ up to a m_σ^e null set is equivalent to $m^e|\mathcal{F}_\sigma = m_\sigma^e$ i.e. $m^p(\Omega) = m_0^p(\Omega)$. However, since $\mathcal{F}_0$ is trivial and given Assumption 1, m_0^p clearly admits a countably additive extension to $\mathcal{F}$ or, in other terms, $m_0 = m_0^e$ i.e. $m_0^p = 0$: it follows that $m^p = 0$ too. By the remarks following Lemma 1, $m^p = m^\perp$ and m is countably additive.

Appendix D. Proofs from Section 4.

Proof of Theorem 4. The first claim is essentially Theorem 2 in [12], in which it is proved that a countably additive, positive *measure* P' exists with this property: essentially, $P' = \sum_n 2^{-n}\bar{m}_{t_n}^e$ where $\langle t_n \rangle_{n\in\mathbb{N}}$ is a judiciously chosen sequence in $\mathbb{R}_+$ and $\bar{m}_{t_n}^e$ is the countably additive extension of $m_{t_n}^e$ to $\mathcal{F}$. It remains to prove that $P'(\Omega) > 0$ and that P' can thus be normalized to be a probability, P. This is immediate by noting that $P' = 0$ is equivalent to $m_t^e = 0$ for each $t \in \mathbb{R}_+$. However, as $\mathcal{F}_0$ is trivial and given Assumption 1, then any probability measure on $\mathcal{F}$ is an extension of m_0 so that $m(\Omega) = m_0^e(\Omega)$ and, since m is the separating measure, the non triviality of m follows by standard arguments. Replacing $m_{t_n}^e$ by $\hat{m}_{t_n}^e$ would still give a countably additive, positive measure $\hat{P}'$ such that $\hat{P}'\Big|\hat{\mathcal{F}}_\tau \gg \hat{m}_\tau^e$ and that, being generated by the extensions of $\hat{m}_{t_n}^e$ to $\mathcal{F}$ and since $\mathcal{N} \subset \hat{\mathcal{F}}_{t_n}$, necessarily vanishes on $\mathcal{N}$. If Assumption 1 holds there exists a probability on $\mathcal{F}$ while, by lemmas 5 and 6, the restriction to $\hat{\mathcal{F}}_0$ coincides necessarily with $\hat{m}_0$. In other words, $\hat{m}_0^e(\Omega) = m(\Omega)$ so that necessarily $\hat{P}'(\Omega) > 0$. $\hat{P}$ is then obtained, as before, by normalization.

Proof of Corollary 1. The supermartingale nature of X is obvious given 4.1. For right continuity, see [12, lemma 2.].

Appendix E. Proofs from Section 5.

Proof of Lemma 2. Let $n_t = (m^p - m_t^p)|\mathcal{F}_t$ and $\tilde{n} = (n_t : t \in \mathbb{R}_+)$. As $n_s \geq n_t|\mathcal{F}_s \geq 0$ (by Lemma 1, claim 3), $\tilde{n}$ describes a positive, bounded finitely additive supermartingale and generates therefore a positive, bounded measure ϕ on the collection of all sets of the form $(F_0 \times \{0\}) \cup]]\sigma, \infty[[$ where $\sigma \in \mathcal{T}$ and $F_0 \in \mathcal{F}_0$ defined by letting

$$\phi\left((F_0 \times \{0\}) \cup]]\sigma, \infty[[\right) = n_0(F_0) + n_\sigma(\sigma < \infty)$$

By lemma 2 in [12], ϕ admits a positive, bounded extension $\bar{\phi}$ to the whole of $2^{\tilde{\Omega}}$. For $\tau \in \mathcal{T}$, define $\bar{m}_\tau^p(F) = m^p(F) - \bar{\phi}(F \times \mathbb{R}_+]]\tau, \infty[[)$ for $F \subset \Omega$. This is clearly an extension of m_τ^p satisfying the claim.

We shall repeatedly exploit the following lemma.

Lemma 7. *If $\sigma, \tau \in \mathcal{T}_0$ and $F \in \mathcal{I}_\tau$, then $\bar{m}_{\sigma\vee\tau}^p(F; \tau \geq \sigma) = 0$.*

Proof. Observe that $F\{\tau \geq \sigma\} \in \mathcal{F}_\tau$ and that

$$\begin{aligned} m_{\sigma\vee\tau}^p(F\{\tau \geq \sigma\}) &= (m_{\sigma\vee\tau}^p - m_\tau^p)(F; \tau \geq \sigma) \\ &= (m_\tau^e - m_{\sigma\vee\tau}^e)(F; \tau \geq \sigma) \\ &= P((X_\tau - X_{\sigma\vee\tau}) F\{\tau \geq \sigma\}) \\ &= 0 \end{aligned}$$

□

We start the proof of Proposition 2 by defining a net in $\mathcal{H}$. Let $\mathbb{D}$ be the set of all *càdlàg* processes with respect to P; $\mathcal{D}$ the collection of all finite sets D in $\mathbb{D}$ which include the processes Z where $Z_t = t$; $\mathbf{A} = \mathcal{D} \times \mathbb{R}_+^3$. Despite the potential incompleteness of the filtration [41, lemma I.1.28, p. 7], to any $\alpha \in \mathbf{A}$, with $\alpha = (D_\alpha; (t_\alpha, \eta_\alpha, \epsilon_\alpha))$, we can associate the sequence $\langle t_i^\alpha \rangle_{i\in\mathbb{N}}$ in $\mathcal{T}_0$ defined recursively as follows (with $\inf \varnothing = \infty$): $t_0^\alpha = 0$;

$$t_i^\alpha = \inf\left\{t > t_{i-1}^\alpha : \bigvee_{X\in D_\alpha} \left|X_t - X_{t_{i-1}^\alpha}\right| > \eta_\alpha\right\} \wedge \inf\{t : M_t^* > i\eta_\alpha\} \tag{E.1}$$

By construction, $t_i^\alpha \leq t_{i-1}^\alpha + \eta_\alpha \leq i\eta_\alpha$ and $M_{t\wedge t_i^\alpha} \leq 2M_{t_i^\alpha -} + M_{t_i^\alpha} \leq 2i\eta_\alpha + M_{t_i^\alpha}$. It follows that $M^{t_i^\alpha}$ is uniformly integrable. On the set $\{\lim_i t_i^\alpha < t_\alpha\}$ there either exist one process $X \in D_\alpha$ which has infinitely many oscillations larger than η_α or $\sup_{t\leq t_\alpha} M_t = \infty$: since both are P null events $0 = P(\lim_i t_i^\alpha < t_\alpha) = \lim_i P(t_i^\alpha < t_\alpha)$. Define then

$$I_\alpha = \min\{i \in \mathbb{N} : P(t_i^\alpha < t_\alpha) \leq \epsilon_\alpha\}$$

Let $\mathbf{A}$ be directed with respect to the partial order defined implicitly by letting $\alpha \geq \beta$ whenever $D_\beta \subset D_\alpha$, $t_\beta \leq t_\alpha$, $\eta_\beta \geq \eta_\alpha$ and $\epsilon_\beta \geq \epsilon_\alpha$. For each $\alpha \in \mathbf{A}$ the set of elements $\left(\langle t_i^\alpha\rangle_{i=0}^{I^\alpha}, \langle F_i^\alpha\rangle_{i=0}^{I^\alpha}\right)$ in $\mathcal{H}$ such that $P\left(F_{I_\alpha - 1}^{\alpha c}\right) \leq \epsilon_\alpha$ is non empty. Invoking the axiom of choice, we can select for each $\alpha \in \mathbf{A}$ an element

$H_\alpha = \left(\langle t_i^\alpha \rangle_{i=0}^{I^\alpha}, \langle F_i^\alpha \rangle_{i=0}^{I^\alpha}\right) \in \mathcal{H}$ with the above properties. $\langle H_\alpha \rangle_{\alpha \in \mathbf{A}}$ will be a fixed net throughout this section. Let $I_\alpha(Y)_t = I_{H_\alpha}(Y)_t$, $J_\alpha(Y)_t = J_{H_\alpha}(Y)_t$ and $K^\alpha = K^{H_\alpha}$.

We start with a preliminary result. Let $\tilde{\Omega}_0 = [[0]]$, $\tilde{\Omega}_j^\alpha =]]t_{j-1}^\alpha, t_j^\alpha]]$ for $0 < j \le I_\alpha$ and $\tilde{\Omega}_{I_\alpha+1}^\alpha =]]t_{I_\alpha}^\alpha, \infty[[$ and define

$$\mathcal{P}_j^\alpha = \left\{(F_j \times \mathbb{R}_+)\,\tilde{\Omega}_j^\alpha : F_j \in \mathcal{F}_{t^\alpha_{(j-1)\vee 0}}\right\}$$

for $j = 0, \ldots, I_\alpha + 1$. It is clear that $\mathcal{P}_j^\alpha$ is a σ algebra of subsets of $\tilde{\Omega}_j^\alpha$.

Lemma 8. *For each $\alpha \in \mathbf{A}$ define*

$$\mathcal{P}^\alpha = \left\{\bigcup_{j=0}^{I_\alpha+1} \tilde{E}_j : \tilde{E}_j \in \mathcal{P}_j^\alpha, 0 \le j \le I_\alpha + 1\right\} \tag{E.2}$$

$\mathcal{P}^\alpha$ is a σ algebra of subsets of $\tilde{\Omega}$ and any $g : \tilde{\Omega} \to \mathbb{R}$ is $\mathcal{P}^\alpha$ measurable if and only if it is of the form

$$g = g_0\,[[0]] + \sum_{i=0}^{I_\alpha-1} g_i\,]]t_i^\alpha, t_{i+1}^\alpha]] + g_{I_\alpha}\,]]t_{I_\alpha}^\alpha, \infty[[\tag{E.3}$$

with g_i measurable with respect to $\mathcal{F}_{t_i^\alpha}$.

Proof. Since $\mathcal{P}^\alpha = \left\{\tilde{E} \subset \tilde{\Omega} : \tilde{E}\tilde{\Omega}_j^\alpha \in \mathcal{P}_j^\alpha,\ 0 \le j \le I_\alpha + 1\right\}$ and $\left\{\tilde{\Omega}_j^\alpha : j = 0, \ldots, I_\alpha + 1\right\}$ is a partition of $\tilde{\Omega}$ then $\mathcal{P}^\alpha$ is a σ algebra [56, problem 2, p. 257]. If g is of the form (E.3) it is clearly $\mathcal{P}^\alpha$ measurable. On the other hand the class of all bounded processes of the form (E.3) is a vector space which contains the indicators of all $\mathcal{P}^\alpha$ measurable sets and is closed with respect to increasing, bounded sequences. The claim then follows by a monotone class argument. □

Lemma 9. *There exists a mapping $f : \mathfrak{B}\left(2^{\tilde{\Omega}}\right) \to L^\infty(\mathcal{P}, \lambda)$ such that for each $Y \in \mathfrak{B}\left(2^{\tilde{\Omega}}\right)$ and $\upsilon \in \mathcal{T}$,*

$$\operatorname*{LIM}_\alpha I_\alpha(Y)_\upsilon = P\int_0^\upsilon f(Y)\,dA$$

f is a positive, linear operator such that $\|f\| = 1$ and $f(YZ) = f(Y)Z$ whenever $Y, YZ \in \mathfrak{B}\left(2^{\tilde{\Omega}}\right)$ and $Z \in \mathfrak{B}\left(\tilde{\mathcal{F}}, \mathcal{N}\right)$ is càglàd.

Proof. Adapting (5.6) to the present setting we have, for $\upsilon \in \mathcal{T}$ and $Y \in \mathfrak{B}\left(2^{\tilde{\Omega}}\right)$,

$$f_\alpha^\upsilon(Y) = \sum_{i=0}^{I_\alpha-1} \bar{m}^p_{t^\alpha_{i+1}\wedge\upsilon}\left(Y_{t^\alpha_{i+1}} \middle| \mathcal{I}^\upsilon_{t_i^\alpha}\right) F_i^\alpha \{t_i^\alpha < \upsilon\}\,]]t_i^\alpha, t_{i+1}^\alpha]] \tag{E.4}$$

Remark that $f_\alpha^\upsilon(Y) \in L^\infty(\mathcal{P}^\alpha, \lambda)$ by Lemma 8 with $\|f_\alpha^\upsilon(Y)\| \le \|Y\|$ and that $\{t_i^\alpha < \upsilon\} \subset \left\{t_{i+1}^\alpha < \upsilon + \eta_\alpha\right\}$ by (E.1) so that $f_\alpha^\upsilon(Y)$ vanishes then on $]]\upsilon + \eta_\alpha, \infty[[$. If $g \in L^1(\mathcal{P}, \lambda)$ then $\left|\int g f_\alpha^\upsilon(Y)\,d\lambda\right| \le \|Y\|\,\|g\|$ and the quantity $\phi_Y^\upsilon(g) = \mathrm{LIM}_\alpha \int g f_\alpha^\upsilon(Y)\,d\lambda$ is well defined and finite. The functional $\phi_Y^\upsilon : L^1(\mathcal{P}, \lambda) \to \mathbb{R}$, is linear and $\|\phi_Y^\upsilon\| \le \|Y\|$: by standard representation theorems [28, IV.8.5], we can associate to ϕ_Y^υ an element $f^\upsilon(Y) \in L^\infty(\mathcal{P}, \lambda)$ such that

$$\operatorname*{LIM}_\alpha \int g f_\alpha^\upsilon(Y)\,d\lambda = \int g f^\upsilon(Y)\,d\lambda$$

holds for any $g \in L^1(\mathcal{P}, \lambda)$. Furthermore, by the remark following (E.4), $f^\upsilon(Y)$ vanishes outside $[[0, \upsilon]]$; eventually, f^υ is additive since f_α^υ is.

By Lemma 8 we can define $g^\alpha = \lambda\left(g|\,\mathcal{P}_\alpha\right)$ and we deduce from (E.4) and (E.3) that $f^v_\alpha(Y)\,g^\alpha = f^v_\alpha(Yg^\alpha)$. Therefore,

$$\int_0^v f^v_\alpha(Y)\,gd\lambda = \int_0^v f^v_\alpha(Y)\,g^\alpha d\lambda = \int_0^v f^v_\alpha(Yg^\alpha)\,d\lambda = \sum_{i=0}^{I_\alpha-1}\bar{m}^p_{t^\alpha_{i+1}\wedge v}\left(Y_{t^\alpha_{i+1}}g^\alpha_{t^\alpha_{i+1}}F^\alpha_i\right)$$

and consequently

$$\int f^v_\alpha(Y)\,gd\lambda = \sum_{i=0}^{I_\alpha-1}\bar{m}^p_{t^\alpha_{i+1}\wedge v}\left(Y_{t^\alpha_{i+1}}g^\alpha_{t^\alpha_{i+1}}F^\alpha_i\right) + \int_v^{v+\eta_\alpha} f^v_\alpha(Yg^\alpha)\,d\lambda$$

Let now $\sigma\in\mathcal{T}$ be such that $v\geq\sigma$. Since $\mathrm{LIM}_\alpha\int_v^{v+\eta_\alpha}gd\lambda = 0$ as λ is countably additive,

$$\operatorname*{LIM}_\alpha\left|\int\left[f^v_\alpha(Y)-f^\sigma_\alpha(Y)\right]gd\lambda\right| \leq 2\,\|Y\|\operatorname*{LIM}_\alpha\sum_{i=0}^{I_\alpha-1}\left|\left(\bar{m}^p_{t^\alpha_{i+1}\wedge v}-\bar{m}^p_{t^\alpha_{i+1}\wedge\sigma}\right)\left(g^\alpha_{t^\alpha_{i+1}}F^\alpha_i\right)\right| \tag{E.5}$$

Remark now that

$$\begin{aligned}\sum_{i=0}^{I_\alpha-1}\left|\left(\bar{m}^p_{t^\alpha_{i+1}\wedge v}-\bar{m}^p_{t^\alpha_{i+1}\wedge\sigma}\right)\left(F^\alpha_i\left\{t^\alpha_i<\sigma\right\}\right)\right| &\leq P\sum_{i=0}^{I_\alpha-1}\left(A^v_{t^\alpha_{i+1}}-A^\sigma_{t^\alpha_{i+1}}\right)\left\{t^\alpha_i<\sigma\leq t^\alpha_{i+1}\right\}\\ &\leq P\sum_{i=0}^{I_\alpha-1}\left(A_{\sigma+\eta_\alpha}-A_\sigma\right)\left\{t^\alpha_i<\sigma\leq t^\alpha_{i+1}\right\} \qquad\text{(E.6)}\\ &\leq P\left(A_{\sigma+\eta_\alpha}-A_\sigma\right)\end{aligned}$$

Let $h_\alpha = \sum_{i=0}^{I_\alpha-1}\{\sigma\leq t^\alpha_i<v\}\,]]t^\alpha_i,t^\alpha_{i+1}]]$, recall (from Lemma 7) that $\bar{m}^p_{t^\alpha_{i+1}\wedge\sigma}\left(F^\alpha_i\{t^\alpha_i\geq\sigma\}\right)=0$ and that $g^\alpha_{t^\alpha_{i+1}}$ is $\mathcal{F}_{t^\alpha_i}$ measurable. Then,

$$\begin{aligned}\sum_{i=0}^{I_\alpha-1}\left|\left(\bar{m}^p_{t^\alpha_{i+1}\wedge v}-\bar{m}^p_{t^\alpha_{i+1}\wedge\sigma}\right)\left(F^\alpha_i g^\alpha_{t^\alpha_{i+1}}\{t^\alpha_i\geq\sigma\}\right)\right| &\leq \sum_{i=0}^{I_\alpha-1}m^p_{t^\alpha_{i+1}\wedge v}\left(F^\alpha_i\left|g^\alpha_{t^\alpha_{i+1}}\right|\{\sigma\leq t^\alpha_i<v\}\right)\\ &\leq P\sum_{i=0}^{I_\alpha-1}\left(A^v_{t^\alpha_{i+1}}-A^v_{t^\alpha_i}\right)\left|g^\alpha_{t^\alpha_{i+1}}\right|\{\sigma\leq t^\alpha_i<v\} \qquad\text{(E.7)}\\ &= P\int_0^v|g^\alpha|\,h_\alpha d\lambda\end{aligned}$$

It is clear that h^α is $\mathcal{P}_\alpha$ measurable and vanishes on $[[0,\sigma]]$. Joining (E.5), (E.6) and (E.7) we obtain that if $g\in L^\infty(\mathcal{P},\lambda)_+$ vanishes on $[[0,\sigma]]^c$, then

$$\begin{aligned}\left|\int\left[f^v(Y)-f^\sigma(Y)\right]gd\lambda\right| &\leq \operatorname*{LIM}_\alpha\left|\int\left[f^v_\alpha(Y)-f^\sigma_\alpha(Y)\right]gd\lambda\right|\\ &\leq 2\,\|Y\|\operatorname*{LIM}_\alpha\sum_{i=0}^{I_\alpha-1}\left(\bar{m}^p_{t^\alpha_{i+1}\wedge v}-\bar{m}^p_{t^\alpha_{i+1}\wedge\sigma}\right)\left(g^\alpha_{t^\alpha_{i+1}}F^\alpha_i\right)\\ &\leq 2\,\|Y\|\operatorname*{LIM}_\alpha P\int_0^v g^\alpha h^\alpha d\lambda\\ &= 2\,\|Y\|\operatorname*{LIM}_\alpha P\int_0^v gh^\alpha d\lambda\\ &= 0\end{aligned}$$

i.e. that $f^v(Y) = f^\sigma(Y)$ on $[[0,\sigma]]$ up to a λ null set. We can then define $f:\mathfrak{B}\left(2^{\tilde{\Omega}}\right)\to L^\infty(\mathcal{P},\lambda)$ by setting

$$f(Y) = \sum_n f^n(Y)\,]n-1,n] \tag{E.8}$$

f is positive and linear since $f^n(Y)$ is for each n; furthermore, up to a λ null set, $f(Y) = f^\upsilon(Y)$ on $[[0,\upsilon]]$ for each $\upsilon \in \mathcal{T}$.

Let now U be *càdlàg* and bounded and define $\pi_\alpha(U) = U_0\{0\} + \sum_{i=0}^{I_\alpha-1} U_{t_i^\alpha}\,]]t_i^\alpha, t_{i+1}^\alpha]]$. Since $U \in \mathbb{D}$, then for $\alpha \in \mathbf{A}$ sufficiently large, $|U_- - \pi_\alpha(U)| \le \eta_\alpha$ on $[[0,t_{I_\alpha}^\alpha]]$. Therefore, since $f_\alpha^T(YU)[[0,t_{I_\alpha}^\alpha]] = f_\alpha^T\left(YU[[0,t_{I_\alpha}^\alpha]]\right)$ and $\pi_\alpha(U)$ is $\mathcal{P}_\alpha$ measurable, we obtain for $Y \in \mathfrak{B}\left(2^{\tilde{\Omega}}\right)$

$$
\begin{aligned}
\int |f(YU_-) - f(Y)U_-|\,d\lambda &= \lim_n \int_0^n |f^n(YU_-) - f^n(Y)U_-|\,d\lambda \\
&= \lim_n \operatorname*{LIM}_\alpha \int_0^{n\wedge t_{I_\alpha}^\alpha} |f_\alpha^n(YU_-) - f_\alpha^n(Y)U_-|\,d\lambda \\
&= \lim_n \operatorname*{LIM}_\alpha \int_0^{n\wedge t_{I_\alpha}^\alpha} |f_\alpha^n(YU_-) - f_\alpha^n(Y)\pi_\alpha(U)|\,d\lambda \\
&= \lim_n \operatorname*{LIM}_\alpha \int_0^{n\wedge t_{I_\alpha}^\alpha} |f_\alpha^n(Y(U_- - \pi_\alpha(U)))|\,d\lambda \\
&\le \lim_\alpha 2\eta_\alpha \|Y\| P(A_\infty) \\
&= 0
\end{aligned}
$$

so that, by linearity,

$$
\int f(Y)\,d\lambda = \int f(\Delta Y)\,d\lambda + \int f(Y_-)\,d\lambda = \int f(\Delta Y)\,d\lambda + \int Y_- d\lambda
$$

It is clear that $\|f\| \le 1$ but we have just considered a case in which $f(Y) = Y$. $\square$

Proof of Proposition 2. For any $\tau \in \mathcal{T}$ and $K \in \mathbb{K}$

$$
\begin{aligned}
m_\tau^p(K_\tau) &= m_\tau^p(K_\tau - K_\tau^\alpha) + m_{\tau\wedge t_{I_\alpha}^\alpha}^p(K_\tau^\alpha) + \left(m_\tau^p - m_{\tau\wedge t_{I_\alpha}^\alpha}^p\right)(K_\tau^\alpha) \\
&= m_\tau^p(K_\tau - K_\tau^\alpha) + J_\alpha(K)_\tau + \left(m_\tau^p - m_{\tau\wedge t_{I_\alpha}^\alpha}^p\right)(K_\tau^\alpha) + I_\alpha(K)_\tau
\end{aligned}
$$

As for the first term,

$$
\begin{aligned}
|K_\tau - K_\tau^\alpha| &\le \left|K_{\tau\wedge t_{I_\alpha}^\alpha} - K_\tau^\alpha\right| + \left|K_\tau - K_{\tau\wedge t_{I_\alpha}^\alpha}\right| \\
&\le \sum_{i=0}^{I_\alpha-1} F_i^{\alpha c}\left|K_{\tau\wedge t_{i+1}^\alpha} - K_{\tau\wedge t_i^\alpha}\right| + \left|K_\tau - K_{\tau\wedge t_{I_\alpha}^\alpha}\right| \\
&\le 2\|K\|\left\{F_{I_\alpha-1}^{\alpha c} \cup \left\{\tau > t_{I_\alpha}^\alpha\right\}\right\}
\end{aligned}
$$

and given that $K_\tau^\alpha, K_\tau \in \mathcal{K}$

$$
\begin{aligned}
|m_\tau^p(K_\tau^\alpha - K_\tau)| &\le |m_\tau(K_\tau^\alpha - K_\tau)| + |m_\tau^e(K_\tau^\alpha - K_\tau)| \\
&\le m_\tau^e(|K_\tau^\alpha - K_\tau|) \\
&\le 2\|K\| P\left(X_\tau\left(F_{I_\alpha-1}^{\alpha c} + \left\{\tau > t_{I_\alpha}^\alpha\right\}\right)\right)
\end{aligned}
\tag{E.9}
$$

i.e. (i) $\mathrm{LIM}_\alpha |m_\tau^p(K_\tau^\alpha - K_\tau)| = 0$. On the other hand, from (5.7) we conclude that (ii) $\lim_\alpha J_\alpha(K)_\tau = 0$. Eventually, by (2.8),

$$
\left|\left(m_\tau^p - m_{\tau\wedge t_{I_\alpha}^\alpha}^p\right)(K_\tau^\alpha)\right| \le \|K\| P\left(X_{\tau\wedge t_{I_\alpha}^\alpha} - X_\tau\right)
$$

so that (iii) $\lim_\alpha \left(m_\tau^p - m_{\tau\wedge t_{I_\alpha}^\alpha}^p\right)(K_\tau^\alpha) = 0$ whenever τ is such that X^τ is uniformly integrable. We then conclude that, when $\tau \in \mathcal{T}$ and X^τ is uniformly integrable, $m_\tau^p(K_\tau) = \lim_\alpha I_\alpha(K)_\tau$. The claim then follows from Lemma 9.

Appendix F. Proofs from Section 6.

Proof of Theorem 5. By localization, we can assume temporarily that (5.8) holds for every $\tau \in \mathcal{T}$. Observe that the process $Y_t = X_t K_t + \int_0^t f(K)\, dA$ is right continuous, admits a terminal variable and $Y_0 = 0$. Then, [41, lemma I.1.44], Y is a uniformly integrable martingale, i.e. $X_t K_t = Y_t - \int_0^t f(K)\, dA$ a special semimartingale, given that $\int f(K)\, dA$ is predictable. If W is a bounded process, then $W^T [[T,\infty[[$ consists of a bounded jump at time T and is therefore *càdlàg* and of integrable variation, i.e. a semimartingale. It follows that $K^T [[T,\infty[[$ and $[[T,\infty[[$ are semimartingales as well as the process

$$XK + K^T [[T,\infty[[= (X + [[T,\infty[[)\, K^T = UK^T$$

The process U is a strictly positive semimartingale, as $P(X_t = 0) = 0$ when $t < T$. Let

$$r_n = \inf\left\{ t \in \mathbb{R}_+ : \sup_{s \le t} X_s > 2^n \text{ or } X_t \le 2^{-n} \right\}$$

$D = [[T,\infty[[$ and let superscript n denote a process stopped before time R_n, i.e. $U^n = U^{r_n-}$. U^n takes its values in the compact set $[2^{-n}, 2^n]$ on which the inverse function h is well defined and, being convex, admits a Lipschitz constant c_n. Let $F \in \mathcal{F}_s$ and $s < t$. Then $|h(X_t^n + D_t^n) - h(X_t^n + D_s^n)| \le c_n (D_t^n - D_s^n)$ so that

$$\begin{aligned} P\left(h(X_t^n + D_t^n) + c_n (D_t^n - D_s^n) \middle| \mathcal{F}_s\right) &\ge P\left(h(X_t^n + D_s^n) \middle| \mathcal{F}_s\right) \\ &\ge h\left(P\left(X_t^n + D_s^n \middle| \mathcal{F}_s\right)\right) \\ &\ge h(X_s^n + D_s^n) \end{aligned}$$

In other words, $h(U^n) + c_n D^n$ is a submartingale therefore $h(U^n) = h(U)^{r_n-}$ is a semimartingale. As the sequence $\langle R_n \rangle_{n\in\mathbb{N}}$ increases to ∞, P a.s. it follows [54, theorem 6, p. 46] that $h(U) = U^{-1}$ is a semimartingale. But then K^T, being the product of two semimartingales, is itself a semimartingale by Ito's lemma.

If markets are complete and $F \in \mathcal{F}_t$ is such that $m_t^p(F) = 0$ then

$$m(F; T \le t) = m_t^e(F; T \le t) \le P(X_t \{T \le t\}) = 0$$

so that, by Theorem 2, $\{F; T \le t\} \in \mathcal{N}$. But under the current assumptions, by Theorem 4, we conclude that P vanishes on $\mathcal{N}$. Since it is possible to find a sequence $\langle F_n \rangle_{n\in\mathbb{N}}$ with $F_n \in \mathcal{I}_{2^n}$ and $P(F_n^c) < 2^{-n}$, we conclude that $P(T < \infty) = \lim_n P(T \le 2^n) = \lim_n P(F_n; T \le 2^n) = 0$.

Proof of Theorem 6. First of all, remark that $M = A$ on $\{X_- = 0\}$ up to indistinguishability and therefore, by Doob Meyer theorem, M and A remain constant over that stochastic interval so that the stochastic integrals $\int X_-^{-1} dM$ and $\int X_-^{-1} dA$ are well defined. The first statement is a fairly obvious consequence of integration by parts and (5.8) from which we obtain that the process Y, where $Y_t = X_t K_t + \int_0^t f(K)\, dA$, is a local martingale (see the proof of Theorem 5). Integration by parts implies

$$Y - \int X_- dM^K - \int K_-^T dM = \int X_- dV^K + \left[K^T, M\right] + \int_0^t \Delta K^{\sim} dA$$

where the left hand side is a local martingale while the right hand side is of finite variation: (6.5) follows from [41, lemmas I.3.11 and I.3.22]. We also deduce that the process

$$\int X_-^{-1} D_K d\left\{ Y - \int K_-^T dM \right\} = \hat{K} + \int X_-^{-1} d\left[M, \hat{K}\right] = \hat{K} + \int d\left[\mathcal{L}(Z), \hat{K}\right]$$

is a local martingale. The formula

$$Z\hat{K}=\int\hat{K}_-dZ+\int Z_-d\hat{K}+\left[Z,\hat{K}\right]=\int\hat{K}_-dZ+\int Z_-d\left(\hat{K}+\left[\mathcal{L}\left(Z\right),\hat{K}\right]\right)$$

proves the second claim.

Proof of Corollary 2. If θ is *càglàd* $\Delta\left(\theta.K\right)=\theta\Delta K$ and $f\left(\Delta\left(\theta.K\right)\right)=\theta f\left(\Delta K\right)$. Since the same invariance property holds for the ordinary predictable projection, we conclude that $\left(\Delta\left(\theta.K\right)\right)^{\sim}=\theta\left(\Delta K\right)^{\sim}$. Let τ_n be announced by the sequence $\langle\tau_n^r\rangle_{r\in\mathbb{N}}$. We obtain that

$$\lim_r\int\left(\Delta\left(]]\tau_n^r,\tau_n]]\,.K\right)\right)^{\sim}d\lambda=\lim_r\int_{\tau_n^r}^{\tau_n}\left(\Delta K^{\sim}\right)d\lambda=P\left(\left(\Delta K\right)_{\tau_n}^{\sim}\Delta A_{\tau_n}\right)$$

There is no loss of generality assuming $\tau_n<\tau_{n+1}$ P a.s. so that, replacing τ_n^r by $\tau_n^r\vee\tau_{n-1}$, we may assume that the sequence $\langle]]\tau_n^r,\tau_n]]\rangle_{n\in\mathbb{N}}$ is disjoint and therefore $\Delta K=\sum_n]]\tau_n^r,\tau_n]]\,\Delta K$. Let $\Delta^kK=\Delta K\left\{|\Delta K|>k\right\}$ and $\Delta_kK=\Delta K-\Delta^kK$ and exploiting the fact that *càdlàg* processes only admit finitely many jumps of width larger than k on each compact interval

$$\begin{aligned}\int_0^t\left(\Delta^kK\right)^{\sim}d\lambda &= \lim_r\int_0^t\left(\sum_n]]\tau_n^r,\tau_n]]\,\Delta^kK\right)^{\sim}d\lambda\\ &= \sum_n\lim_r\int_0^t\left(]]\tau_n^r,\tau_n]]\,\Delta^kK\right)^{\sim}d\lambda\\ &= \sum_nP\left(\left(\Delta^kK\right)_{\tau_n\wedge t}^{\sim}\Delta A_{\tau_n\wedge t}\right)\end{aligned}$$

while $\left|\int_0^t\left(\Delta_kK\right)^{\sim}d\lambda\right|+P\left|\sum_n\left(\Delta_kK\right)_{\tau_n\wedge t}^{\sim}\Delta A_{\tau_n\wedge t}\right|\le 6kP\left(A_\infty\right)$ so that

$$\begin{aligned}\int_0^t\left(\Delta K\right)^{\sim}d\lambda &= \lim_{k\downarrow0}\int_0^t\left(\Delta^kK\right)^{\sim}d\lambda\\ &= \lim_{k\downarrow0}P\sum_n\left(\Delta^kK\right)_{\tau_n\wedge t}^{\sim}\Delta A_{\tau_n\wedge t}\\ &= P\sum_n\left(\Delta K\right)_{\tau_n\wedge t}^{\sim}\Delta A_{\tau_n\wedge t}\end{aligned}$$

In other words, $\Delta K^{\sim}=\sum_n\left(\Delta K\right)_{\tau_n}^{\sim}[[\tau_n]]$ which proves the claim.

Appendix G. Proofs from Section 7.

Proof of Theorem 7. The same argument used in the proof of Corollary 2 may be employed to obtain that, under the current assumptions, $f\left(\Delta K\right)=\sum_nf\left(\Delta K_{\tau_n}\right)[[\tau_n]]$. Therefore, by definition of the operator f,

$$\begin{aligned}\int f\left(\Delta K_{\tau_n}\right)d\lambda &= \operatorname*{LIM}_\alpha\int f_\alpha^{\tau_n}\left(\Delta K_{\tau_n}\right)d\lambda\\ &= \operatorname*{LIM}_\alpha\sum_{i=0}^{I_\alpha-1}m^p_{t^\alpha_{i+1}\wedge\tau_n}\left(\Delta K_{\tau^n}\left\{t^\alpha_{i+1}=\tau_n\right\}F_i^\alpha\right)\\ &= \operatorname*{LIM}_\alpha m^p_{\tau_n}\left(\Delta K_{\tau^n}\bigcup_{i=0}^{I_\alpha-1}\left\{t^\alpha_{i+1}=\tau_n\right\}F_i^\alpha\right)\end{aligned}$$

Under the current assumptions, however,

$$\underset{\alpha}{\mathrm{LIM}} \sum_{i=0}^{I_\alpha - 1} m^p_{\tau_n}\left(\left\{t^\alpha_i < \tau^r_n < \tau_n \le t^\alpha_{i+1}\right\}\right) \le \lim_\alpha m\left(\tau_n - \tau^r_n < \eta_\alpha\right) = 0 \tag{G.1}$$

and, for r sufficiently large, $\left\|\bar{m}^p_{\tau^r_n} - \bar{m}^p_{\tau_n -}\right\| = \left(m^p_{\tau^r_n} - m^p_{\tau_n -}\right)(\Omega) < \epsilon$. For $F \in \bigcup \mathcal{F}_{\tau^r_n}$ we deduce from (2.8)

$$\begin{aligned} m^e_{\tau_n -}(F) &= \lim_r m^e_{\tau^r_n}(F) + \lim_r \left(m^p_{\tau^r_n} - m^p_{\tau_n -}\right)(F) \\ &= \lim_r P\left(A_{\tau^r_n} F\right) \\ &= P\left(A_{\tau_n -} F\right) \end{aligned}$$

and since $\mathcal{F}_{\tau_n -} = \sigma\left(\bigcup \mathcal{F}_{\tau^r_n}\right)$ we conclude that $dm^e_{\tau_n -} \big/ dP_{\mathcal{F}_{\tau_n -}} = A_{\tau_n -}$ by uniqueness of the Carathéodori extension. Moreover,

$$\begin{aligned} \sum_{i=0}^{I_\alpha - 1} m^p_{\tau_n -}\left(\left\{t^\alpha_{i+1} = \tau_n ; t^\alpha_i > \tau^r_n\right\} F^\alpha_i\right) &\le \sum_{i=0}^{I_\alpha - 1}\left(m^p_{\tau_n -} - \bar{m}^p_{\tau^r_n}\right)\left(t^\alpha_{i+1} = \tau_n\right) \\ &= \left(m^p_{\tau_n -} - m^p_{\tau^r_n}\right)\left(\bigcup_{i=1}^{I_\alpha - 1}\left\{t^\alpha_{i+1} = \tau_n\right\}\right) \\ &\le \epsilon \end{aligned} \tag{G.2}$$

We can now remark that $\Delta K_{\tau^n}\left\{t^\alpha_{i+1} = \tau_n\right\} F^\alpha_i$ is $\mathcal{F}_{\tau_n -}$ measurable by assumption, that $\left(m^p_{\tau_n} - m^p_{\tau_n -}\right)\big| \mathcal{F}_{\tau_n -} = \left(m^e_{\tau_n -} - m^e_{\tau_n}\right)\big| \mathcal{F}_{\tau_n -}$ and that $\left\{\Delta^k K^{t_{I_\alpha - 1}} \neq 0\right\} \subset \bigcup_{i=1}^{I_\alpha - 1}\left[\left[t^\alpha_i\right]\right]$ whenever α is large enough. Eventually we conclude that

$$\begin{aligned} \int f\left(\Delta K_{\tau_n}\right) d\lambda &= \underset{\alpha}{\mathrm{LIM}} \sum_{i=0}^{I_\alpha - 1}\left(m^p_{\tau_n} - m^p_{\tau_n -}\right)\left(\Delta K_{\tau^n}\left\{t^\alpha_{i+1} = \tau_n\right\} F^\alpha_i\right) \\ &= \underset{\alpha}{\mathrm{LIM}}\, P\left(\Delta K_{\tau^n} \bigcup_{i=1}^{I_\alpha - 1}\left\{t^\alpha_{i+1} = \tau_n\right\} \Delta A_{\tau_n}\right) \\ &= \lim_k \underset{\alpha}{\mathrm{LIM}}\, P\left(\Delta^k K^{t_{I_\alpha - 1}}_{\tau^n} \Delta A_{\tau_n}\right) \\ &= P\left(\Delta K_{\tau^n} \Delta A_{\tau_n}\right) \end{aligned}$$

i.e. that $\int f(\Delta K)\, d\lambda = P \sum \Delta K_{\tau^n} \Delta A_{\tau_n}$. The last claim is obvious given Theorem 6.2 and that fact that in the current context $D_K = \varnothing$.

Appendix H. Proofs from Section 8.

Proof of Lemma 3. Necessity is obvious given the remark preceding the lemma; the inequality, $m(F) \ge m^e_t(F) = P_m(X_t F) = P_m(X_t F\{T_m > t\})$ for $F \in \mathcal{F}_t$ implies that this is sufficient as well.

Lemma 10. *Let $f \in \mathfrak{B}(\mathcal{F}, \mathcal{N})$, $\upsilon \in ba(\mathcal{F}, \mathcal{N})_+$ with $\upsilon(\Omega) = 1$ and $k \in \mathcal{K}$, then*

$$\overline{\alpha}_k(f) \ge \upsilon(k + f) \ge \underline{\alpha}_k(f) \tag{H.1}$$

If NA holds

1. $\left(\underline{\alpha}_{\mathcal{K}}(f); \overline{\alpha}_{\mathcal{K}}(f)\right) \subset \mathcal{A}(f, \mathcal{K}) \subset \left[\underline{\alpha}_{\mathcal{K}}(f); \overline{\alpha}_{\mathcal{K}}(f)\right]$*;*
2. $\overline{\alpha}_{\mathcal{K}}(f) = \underline{\alpha}_{\mathcal{K}}(f)$ *if and only if* $f - \frac{1}{2}\left(\overline{\alpha}_{\mathcal{K}}(f) + \underline{\alpha}_{\mathcal{K}}(f)\right) \in \bar{\mathcal{C}}$.

Proof. Fix $k \in \mathcal{K}$ and let $N_+, N_- \in \mathcal{N}$ be such that $\sup_{\omega \in N_+^c} (k+f)(\omega) \leq \overline{\alpha}_k(f) + 2^{-n}$ and $\inf_{\omega \in N_-^c} (k+f)(\omega) + 2^{-n} \geq \underline{\alpha}_k(f)$. Then on $N^c = N_+^c N_-^c$

$$2^{-n} + \overline{\alpha}_k(f) \geq k + f \geq \underline{\alpha}_k(f) - 2^{-n} \tag{H.2}$$

The first claim follows from $N \in \mathcal{N}$.

(1). Suppose that $\pi(f) \notin \mathcal{A}(f, \mathcal{K})$: then for some $k \in \mathcal{K}$ and $d \in \mathbb{R}$ we would have $k + d(f - \pi(f)) \in \mathfrak{B}(\mathcal{F}, \mathcal{N})_+$ or, equivalently, $N_{k,\eta} = \{k + d(f - \pi(f)) < -\eta\} \in \mathcal{N}$ for all $\eta > 0$. Take the case $d > 0$, then

$$\underline{\alpha}_{d^{-1}k}(f) \geq \inf_{\omega \in N_{k,\eta}^c} \left(d^{-1}k + f\right)(\omega) = d^{-1} \inf_{\omega \in N_{k,\eta}^c} (k + df)(\omega) \geq \pi(f) - \eta$$

If $d < 0$ we likewise deduce $\bar{\alpha}_{-d^{-1}k}(f) \leq \pi(f) + \eta$. Of course, η being arbitrary, we conclude there exists $k \in \mathcal{K}$ such that either $\underline{\alpha}_k(f) \geq \pi(f)$ or $\bar{\alpha}_k(f) \leq \pi(f)$. We deduce that $(\underline{\alpha}_{\mathcal{K}}(f); \overline{\alpha}_{\mathcal{K}}(f)) \subset \mathcal{A}(f, \mathcal{K})$. If $\pi(f) \in \mathcal{A}(f, \mathcal{K})$, $\mathcal{K}(f; \pi)$ and $\mathfrak{B}(\mathcal{F}, \mathcal{N})_+$ are separated and there exists therefore $m_f \in \mathcal{M}(\mathcal{K})$ such that $m_f(f) = \pi(f)$. By (H.2), we conclude that $\pi(f) \in [\underline{\alpha}_{\mathcal{K}}(f); \overline{\alpha}_{\mathcal{K}}(f)]$.

(2). Let $\hat{f} = f - \frac{1}{2}(\overline{\alpha}_{\mathcal{K}}(f) + \underline{\alpha}_{\mathcal{K}}(f))$ and suppose that $\hat{f} \notin \bar{\mathcal{C}}$. Then $\left\{\hat{f}\right\}$ and $\bar{\mathcal{C}}$ may be separated by a finitely additive probability m_f vanishing on $\mathcal{N}$ and on $\mathcal{K}$ and such that $m_f\left(\hat{f}\right) > 0$. Since m_f is a separating measure for $\mathcal{K}$, by (H.2) it follows that $\overline{\alpha}_{\mathcal{K}}\left(\hat{f}\right) \geq m_f\left(\hat{f}\right) \geq \underline{\alpha}_{\mathcal{K}}\left(\hat{f}\right)$. Given that both functionals, $\overline{\alpha}_{\mathcal{K}}(\cdot)$ and $\underline{\alpha}_{\mathcal{K}}(\cdot)$, are linear with respect to constants the preceding double inequality translates into

$$\overline{\alpha}_{\mathcal{K}}(f) \geq m_f\left(\hat{f}\right) + \frac{1}{2}(\overline{\alpha}_{\mathcal{K}}(f) + \underline{\alpha}_{\mathcal{K}}(f)) > \frac{1}{2}(\overline{\alpha}_{\mathcal{K}}(f) + \underline{\alpha}_{\mathcal{K}}(f)) \geq \underline{\alpha}_{\mathcal{K}}(f)$$

i.e. $\overline{\alpha}_{\mathcal{K}}(f) > \underline{\alpha}_{\mathcal{K}}(f)$. On the other hand, if $\hat{f} \in \bar{\mathcal{C}}$ then $m\left(\hat{f}\right) \leq 0$ for each $m \in \mathcal{M}(\mathcal{K})$ so that $\mathcal{A}\left(\hat{f}, \mathcal{K}\right) \subset \mathbb{R}_-$ and therefore, by the first claim, $0 \geq \overline{\alpha}_{\mathcal{K}}\left(\hat{f}\right)$ i.e. $\overline{\alpha}_{\mathcal{K}}(f) \leq \underline{\alpha}_{\mathcal{K}}(f)$. □

We shall now prove a theorem more general that Theorem 8. Let us introduce the following definition.

Definition 7. *Let $\mathcal{U} \subset ba(\mathcal{F}, \mathcal{N})$ and $\mathcal{J} \subset \mathfrak{B}(\mathcal{F}, \mathcal{N})$. $\mathcal{U}$ is norm attaining for $\mathcal{J}$ if for each $f \in \mathcal{J}$, $\|f\| = \sup_{\upsilon \in \mathcal{U}} \upsilon(f)$.*

Theorem 10. *The following properties are mutually equivalent:*

- (a) *there exists a subset $\mathcal{U}$ of finitely additive probabilities vanishing on $\mathcal{N}$ which is (i) norm attaining for $\mathfrak{B}(\mathcal{F}, \mathcal{N})_+$ and such that (ii) if $\upsilon \in \mathcal{U}$ and $\langle h_n \rangle_{n \in \mathbb{N}}$ is a sequence in $\mathcal{C}$ such that $\|h_n^-\| \to 0$ then h_n converges to 0 in υ measure;*
- (b) *for every $k \in \mathcal{K}$ and $f \in \mathfrak{B}(\mathcal{F}, \mathcal{N})$, $\bar{\alpha}_k(f) = \underline{\alpha}_{\mathcal{K}}(f)$ if and only if $\underline{\alpha}_k(f) = \bar{\alpha}_{\mathcal{K}}(f)$;*
- (c) *$\mathcal{K}$ has the extension property;*
- (d) *there are no free lunches, i.e. (3.2) holds.*

Proof. (a)→(b). Suppose that for some $k_0 \in \mathcal{K}$, say, $\underline{\alpha}_{k_0}(f) = \bar{\alpha}_{\mathcal{K}}(f)$ (so that $\underline{\alpha}_{\mathcal{K}}(f) = \bar{\alpha}_{\mathcal{K}}(f)$) and assume, without loss of generality, that $\bar{\alpha}_{\mathcal{K}}(f) \geq 0$ (if not, replace f by $f - \bar{\alpha}_{\mathcal{K}}(f)$). Then, for each n there exists $k_n \in \mathcal{K}$ such that $\underline{\alpha}_{k_0}(f) > \bar{\alpha}_{k_n}(f) - 2^{-n}$ from which one deduces easily that, letting $h_n = (1 + \|k_0 + f\|)^{-1}(k_0 - k_n)$, $h_n > -2^{-n}$ outside some negligible set N_n' and $h_n \in \mathcal{K}$. Choose $\langle k_n \rangle_{n \in \mathbb{N}}$ such that $\langle \bar{\alpha}_{k_n}(f) \rangle_{n \in \mathbb{N}}$ is monotonically decreasing to $\bar{\alpha}_{\mathcal{K}}(f)$. Let $N_n'' \in \mathcal{N}$ be such that $\bar{\alpha}_{k_n}(f) \geq$

$\sup_{\omega\in N_n''^c}(k_n+f)(\omega)-2^{-n}$ and $N_n=N_n'\cup N_n''$. Then, as $N_n\in\mathcal{N}$

$$\begin{aligned}(k_0+f)(\omega)N_n^c &= (k_0-k_n)(\omega)N_n^c+(k_n+f)(\omega)N_n^c\\ &= [(k_0-k_n)(\omega)N_n^c+(k_n+f)(\omega)N_n^c]\wedge\|k_0+f\|\\ &\le \left[(k_0-k_n)(\omega)N_n^c+\bar{\alpha}_{k_n}(f)+2^{-(n-1)}\right]\wedge\|k_0+f\|\\ &\le [(k_0-k_n)(\omega)N_n^c\wedge\|k_0+f\|]+\bar{\alpha}_{k_n}(f)+2^{-(n-1)}\end{aligned}$$

If $\upsilon\in\mathcal{U}$ then $\upsilon(h_n\wedge 1)$ converges to 0 and therefore

$$\begin{aligned}\upsilon(k_0+f) &\le \lim_n\left[(1+\|k_0+f\|)\upsilon(h_n\wedge 1)+\bar{\alpha}_{k_n}(f)+2^{-(n-1)}\right]\\ &= \lim_n\bar{\alpha}_{k_n}(f)\\ &= \bar{\alpha}_{\mathcal{K}}(f)\end{aligned}$$

By definition of $\mathcal{U}$ then $\bar{\alpha}_{k_0}(f)\le\sup_{\upsilon\in\mathcal{U}}\upsilon(k_0+f)\le\bar{\alpha}_{\mathcal{K}}(f)$, i.e.$\bar{\alpha}_{k_0}(f)=\bar{\alpha}_{\mathcal{K}}(f)$. If, on the other side, $\bar{\alpha}_{k_0}(f)=\underline{\alpha}_{\mathcal{K}}(f)$, then the same argument can be used to show that $\underline{\alpha}_{k_0}(f)=\pi(f)$. In other words, (b) holds.

$(b)\to(c)$. By (b) we have $\underline{\alpha}_{\mathcal{K}}(f)=\bar{\alpha}_{\mathcal{K}}(f)$. Let $\pi(f)=\frac{1}{2}(\underline{\alpha}_{\mathcal{K}}(f)+\bar{\alpha}_{\mathcal{K}}(f))$ and suppose that there exist $k\in\mathcal{K}$ and $d\in\mathbb{R}$ such that $y=k+d(f-\pi(f))\in\mathfrak{B}(\mathcal{F},\mathcal{N})_+$ i.e. such that $\{y<-\eta\}\in\mathcal{N}$ for each $\eta>0$. A shown in the proof of Lemma 10, this implies either $\underline{\alpha}_k(f)=\pi(f)$ (if $d>0$) or $\bar{\alpha}_k(f)=\pi(f)$ (if $d<0$), in any case $\pi(f)=\underline{\alpha}_{\mathcal{K}}(f)=\bar{\alpha}_{\mathcal{K}}(f)$. Say $d>0$. But then, since $\bar{\alpha}_k(f)=\pi(f)$ by (b), we conclude that for any $\eta>0$ there must exist a set $N\in\mathcal{N}$ such that $\sup_{\omega\in N^c}y(\omega)<\eta$ i.e. $\{y>\eta\}\in\mathcal{N}$ from which the implication $y=0$ follows.

$(c)\to(d)$. Let $f\in\bar{\mathcal{C}}$. Then, by assumption it has an admissible price $\pi(f)$ but by Lemma 10 it must be $\pi(f)=0$. Then, as $f-\pi(f)\in\mathfrak{B}(\mathcal{F},\mathcal{N})_+$ if and only if $f=\pi(f)$ we deduce that $\bar{\mathcal{C}}\cap\mathfrak{B}(\mathcal{F},\mathcal{N})_+=\{0\}$.

$(d)\to(a)$. Let $\eta,\varepsilon>0$ and $c\in\bar{\mathcal{C}}$ be such that $c>-\varepsilon$ up to negligibility. If $\upsilon\in\mathcal{M}(\mathcal{K})$ then $0\ge\upsilon(c)$ and from this we easily deduce that $\upsilon(c\ge\eta)\le\frac{\varepsilon}{\varepsilon+\eta}$. It follows that every sequence $\langle c_n\rangle_{n\in\mathbb{N}}$ in $\bar{\mathcal{C}}$ converges to 0 in υ measure whenever c_n^- converges to 0 in norm. The same property easily extends to the collection $\mathcal{M}(\mathcal{K})^*$ of all finitely additive probabilities absolutely continuous with respect to some $\upsilon\in\mathcal{M}(\mathcal{K})$. If (d) holds, then for any set $F\in\mathcal{F}$ not negligible there exists a corresponding $\upsilon_F\in\mathcal{M}(\mathcal{K})$ such that $\upsilon_F(F)>0$ while the finitely additive probability $\bar{\upsilon}_F=\upsilon_F(F)^{-1}Fd\upsilon_F$ is clearly absolutely continuous with respect to υ_F [28, theorem III.2.20, p. 114]. Letting $F=\{h>(1-\eta)\|h\|\}$ for $h\in\mathfrak{B}(\mathcal{F},\mathcal{N})_+$, then we have $\bar{\upsilon}_F(h)\ge(1-\eta)\|h\|$. But then the collection $\mathcal{M}(\mathcal{K})^*$ is norm attaining for $\mathfrak{B}(\mathcal{F},\mathcal{N})_+$ and (a) is therefore satisfied. □

The equivalence between (a) and (d) has a direct correspondence in a result of Delbaen and Schachermayer ([23, corollary 3.7, p. 477], [43, lemma 2.2, p. 193]) obtained under the assumption $\mathcal{N}=\mathcal{N}_Q$.

Proof of Proposition 3. We start by proving that $Q(T_m<\infty)>0$ for each $m\in\mathcal{M}(\mathcal{K})$ if and only if there exists η such that for any sequence $\langle m_n\rangle_{n\in\mathbb{N}}$ in $\mathcal{M}(\mathcal{K})$ $Q\left(\bigcap_n\{T_{m_n}<\infty\}\right)>\eta$. If this were not true it should be possible to find a sequence $\langle m_n\rangle_{n\in\mathbb{N}}$ in $\mathcal{M}(\mathcal{K})$ such that $Q\left(\bigcap_n\{T_{m_n}<\infty\}\right)=0$. Let $m=\sum_n 2^{-n}m_n$: then, $m\in\mathcal{M}(\mathcal{K})$ and for each $\tau\in\mathcal{T}$ and $m_\tau^e=\sum_n 2^{-n}m_{n,\tau}^e$ (by uniqueness of the decomposition (2.7)): therefore, $m_\tau^e\gg m_{n,\tau}^e$. As a consequence, $Q(T_m<T_{m_n})=0$ for each n: in other words, up to a null set $\bigcap_n\{T_{m_n}<\infty\}\subset\{T_m<\infty\}$ and $Q(T_m<\infty)=0$.

In the attempt to derive a contradiction, assume that $\eta > 0$. Fix $\epsilon > 0$ and consider the mapping F_ϵ that associates to each $m \in \mathcal{M}(\mathcal{K})$ the set

$$F_\epsilon(m) = \left\{ f \in \mathfrak{B}(\mathcal{F}, \mathcal{N})_+ : f \le 1,\ Q(f) \ge \eta(1-\epsilon),\ m(f) < \epsilon \right\}$$

It is clear from the definition that F_ϵ is convex valued and non empty if (in fact $m(T_m < \infty) = m^p(T_m < \infty)$ and m^p and Q are orthogonal). Letting $\mathfrak{X}$ be $\mathcal{M}(\mathcal{K})$ endowed with the weak* topology of $ba(\mathcal{F}, \mathcal{N})$ and $\mathfrak{Y} = \mathfrak{B}(\mathcal{F}, \mathcal{N})$ we easily establish that $\mathfrak{X}$ is Hausdorff and compact and that $\mathfrak{Y}$ is a Banach space. In order to show that F_ϵ is lower hemicontinuous, consider an open set $\mathcal{U}_{f_0} \subset \mathfrak{Y}$ containing $f_0 \in F_\epsilon(m_0)$. It is clear that $\mathcal{V}_f = \{m \in \mathcal{M}(\mathcal{K}) : m(f) < \epsilon\}$ is open and that

$$F_\epsilon^{-1}(\mathcal{U}_{f_0}) = \{m \in \mathcal{M}(\mathcal{K}) : F_\epsilon(m) \cap \mathcal{U}_{f_0} \neq \varnothing\} = \bigcup_{\left\{f \in \mathcal{U}_{f_0}:\ Q(f) \ge \eta(1-\epsilon)\right\}} \mathcal{V}_f$$

In other words, the lower inverse F_ϵ^{-1} of F_ϵ maps open sets into open sets, i.e. F_ϵ is lower hemicontinuous. By virtue of Michael selection theorem [51, footnote 7, p. 364.], F_ϵ admits therefore a continuous function ϕ_ϵ such that $\phi_\epsilon(m) \in \overline{F_\epsilon(m)}$ for each $m \in \mathcal{M}(\mathcal{K})$ so that (i) $0 < \phi_\epsilon(m) \le 1$, (ii) $Q(\phi_\epsilon(m)) \ge \eta(1-\epsilon)$ and (iii) $m(\phi_\epsilon(m)) \le \epsilon$.

Consider now the mapping M that associates to each $f \in \mathfrak{Y}$ the set

$$M(f) = \{m \in \mathcal{M}(\mathcal{K}) : m(f) = \bar{\alpha}_{\mathcal{K}}(f)\}$$

$M(f)$ is clearly a non empty, compact and convex subset of $\mathfrak{X}$. Let $\mathcal{V}$ be a closed subset of $\mathfrak{X}$ and $f_0 \in \overline{M^{-1}(\mathcal{V})}$: for each δ there exists then $f_\delta \in M^{-1}(\mathcal{V})$ such that $\|f_\delta - f_0\| < \delta$. By definition this implies that for some $m_\delta \in \mathcal{V}$, $m_\delta(f_\delta) = \bar{\alpha}_{\mathcal{K}}(f_\delta)$ so that

$$\begin{aligned} m_\delta(f_0) &\ge m_\delta(f_\delta) - \delta \\ &= \bar{\alpha}_{\mathcal{K}}(f_\delta) - \delta \\ &\ge \bar{\alpha}_{\mathcal{K}}(f_0) - 2\delta \end{aligned}$$

Put it differently, for each $\delta > 0$ the set $\mathcal{V}_{f_0,\delta} = \{m \in \mathcal{V} : m_\delta(f_0) \ge \bar{\alpha}_{\mathcal{K}}(f_0) - 2\delta\}$ is non empty. It then ensues from the finite intersection property that $\bigcap_{\delta>0} \mathcal{V}_{f_0,\delta} = \{m \in \mathcal{V} : m_\delta(f_0) = \bar{\alpha}_{\mathcal{K}}(f_0)\}$ is also non empty or, in other words, that $f_0 \in M^{-1}(\mathcal{V})$ and therefore $M^{-1}(\mathcal{V})$ is closed. We conclude that M is upper hemicontinuous and that so is the composite map $\Phi_\epsilon = M \circ \phi_\epsilon : \mathfrak{X} \to \mathfrak{X}$; further, Φ_ϵ is convex and compact valued. It follows that Φ_ϵ has closed graph and, $\mathfrak{X}$ being a Hausdorff, locally convex topological vector space, it admits a fixed point m^* as a result of a well known theorem of Glicksberg [33, p. 171]. Letting $f^* = \phi_\epsilon(m^*)$ we have that

$$\epsilon \ge m^*(f^*) = \bar{\alpha}_{\mathcal{K}}(f^*)$$

while $Q(f^*) \ge \eta(1-\epsilon)$. This can be considered as an orthogonality condition between Q and $\bar{\alpha}_{\mathcal{K}}$.

Given that ϵ was entirely arbitrary, we can establish the same conclusion replacing ϵ with $\epsilon_n = 2^{-n-1}$: let m_n be the fixed point of Φ_{ϵ_n} and $f_n = \phi_{\epsilon_n}(m_n)$ so that $Q(f_n) \ge \eta(1-\epsilon_n)$ and $\epsilon_n \ge \bar{\alpha}_{\mathcal{K}}(f_n)$. These same inequalities remain valid if we replace f_n by $g'_n = \sum_{i \ge n} a_{i,n} f_i$ where $a_{i,n} \ge 0$, the sequence $\langle a_{i,n} \rangle_{i \in \mathbb{N}}$ contains finitely many non null elements and $\sum_{i \ge n} a_{i,n} = 1$. In fact

$$\bar{\alpha}_{\mathcal{K}}(g'_n) \le \sum_{i \ge n} a_{i,n} \bar{\alpha}_{\mathcal{K}}(f_i) \le \sum_{i \ge n} a_{i,n} 2^{-i-1} \le 2^{-n-1}$$

Choosing weights conveniently, we obtain by Komlòs lemma, that the sequence $\langle g'_n\rangle_{n\in\mathbb{N}}$ converges P a.s.; by Egoroff theorem there will therefore exists a set $F\in\mathcal{F}$ such that $P(F^c)<\eta\epsilon$ and $g_n=g'_nF$ converges uniformly to some $g\geq 0$. But then, since $0\leq g'_n\leq 1$

$$\begin{aligned} P(g) &= \lim_n P(g_n) \\ &\geq \lim_n P(g'_n)-P(F^c) \\ &\geq \lim_n \eta(1-\epsilon_n-\epsilon) \\ &= \eta(1-\epsilon) \end{aligned}$$

so that $g\neq 0$. However

$$\begin{aligned} \bar{\alpha}_{\mathcal{K}}(g) &= \lim_n \bar{\alpha}_{\mathcal{K}}(g_n) \\ &\leq \lim_n \bar{\alpha}_{\mathcal{K}}(g'_n) \\ &= 0 \end{aligned}$$

By the second claim in Lemma 3, the last inequality implies $g\in\bar{\mathcal{C}}$ and the *NFL* property is therefore violated, a contradiction.

Appendix I. Proofs from Section 9.

Define the functional $\pi_{\mathcal{K}}:\mathfrak{B}(\mathcal{F},\mathcal{N})\to\mathbb{R}$ implicitly as $\pi_{\mathcal{K}}(f)=-\underline{\alpha}_{\mathcal{K}}(-f)$. The following is a fairly trivial lemma.

Lemma 11. *Let Assumption 5 hold.*

(1) *The functional $\pi_{\mathcal{K}}$ is positive, sub additive, positively homogeneous and such that $\pi_{\mathcal{K}}(1)\leq 1$;*
(2) *if (3.1) holds, then $\pi_{\mathcal{K}}(1)=1$ and $\pi_{\mathcal{K}}(k)\leq 0$ when $k\in\mathcal{K}$.*

Proof. (claim 1). Let $f,g\in\mathfrak{B}(\mathcal{F},\mathcal{N})$. By definition (3.4), $\underline{\alpha}_{\mathcal{K}}(f)\geq\underline{\alpha}_0(f)\geq\sup_{N\in\mathcal{N}}\inf_{\omega\in N^c}f(\omega)$ so that $\underline{\alpha}_{\mathcal{K}}$ is positive and $\pi_{\mathcal{K}}(1)\leq 1$. As $k\in\mathcal{K}$ if and only if $k=k_1+k_2$ with $k_1,k_2\in\mathcal{K}$

$$\begin{aligned} \underline{\alpha}_{\mathcal{K}}(f+g) &= \sup_{k\in\mathcal{K}}\sup_{N\in\mathcal{N}}\inf_{\omega\in N^c}(k+f+g)(\omega) \\ &= \sup_{k_1,k_2\in\mathcal{K}}\sup_{N\in\mathcal{N}}\inf_{\omega\in N^c}(k_1+k_2+f+g)(\omega) \\ &\geq \sup_{k_1\in\mathcal{K}}\sup_{N\in\mathcal{N}}\inf_{\omega\in N^c}(k_1+f)(\omega)+\sup_{k_2\in\mathcal{K}}\sup_{N\in\mathcal{N}}\inf_{\omega\in N^c}(k_1+f)(\omega) \\ &= \underline{\alpha}_{\mathcal{K}}(f)+\underline{\alpha}_{\mathcal{K}}(g) \end{aligned}$$

Since $\lambda^{-1}k\in\mathcal{K}$ whenever $\lambda>0$, then

$$\begin{aligned} \underline{\alpha}_{\mathcal{K}}(\lambda f) &= \lambda\sup_{k\in\mathcal{K}}\sup_{N\in\mathcal{N}}\inf_{\omega\in N^c}\left(\lambda^{-1}k+f\right)(\omega) \\ &= \lambda\sup_{k'\in\mathcal{K}}\sup_{N\in\mathcal{N}}\inf_{\omega\in N^c}(k'+f)(\omega) \\ &= \lambda\underline{\alpha}_{\mathcal{K}}(f) \end{aligned}$$

from which $\pi_{\mathcal{K}}(0)=0$ also follows.

(claim 2). If $\underline{\alpha}_k(-1)>-1$ then there exists $N\in\mathcal{N}$ such that $k>0$ on N^c, a contradiction of (3.1). If $k_0\in\mathcal{K}$ then $\pi_{\mathcal{K}}(k_0)=\inf_{k\in\mathcal{K}}\inf_{N\in\mathcal{N}}\sup_{\omega\in N^c}(-k+k_0)(\omega)\leq\underline{\alpha}_{k_0}(-k_0)=0$. $\pi_{\mathcal{K}}[\mathcal{K}]\leq 0$ follows from positivity of $\pi_{\mathcal{K}}$. $\square$

Proof of Theorem 9. Consider the functional $\pi_{\mathcal{K}}$ on $\mathfrak{B}(\mathcal{F},\mathcal{N})$ and, appealing to Hahn Banach theorem, construct a functional ϕ on $\mathfrak{B}(\mathcal{F},\mathcal{N})$ such that $\phi(\Omega)=\pi_{\mathcal{K}}(\Omega)$ and $\phi\leq\pi_{\mathcal{K}}$. By Lemmas 4 and 11 we may thus represent ϕ via a finitely additive probability m vanishing on $\mathcal{N}$ and such that $m[\mathcal{C}]\leq 0$. If $k\in\mathcal{K}$, then

$$m\left(k>2^n\right)\leq\pi_{\mathcal{K}}\left(k>2^n\right)\leq\pi_{\mathcal{K}}\left(\frac{k+\|k^-\|}{2^n}\right)\leq 2^{-n}\left\|k^-\right\|$$

Then k is m integrable and $m(k)=\lim_n m(k\wedge n)\leq\pi_{\mathcal{K}}(k)=0$.

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