Generalized Analytical Upper Bounds for American Option Prices*

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Abstract

This paper generalizes and tightens the analytical upper bounds of Chen and Yeh (2002) for American options under stochastic interest rates, stochastic volatility, and jumps where American option prices are difficult to compute with accuracy. We first generalize Theorem 1 of Chen and Yeh (2002) and apply it to derive a tighter upper bound for American calls when the interest rate is greater than the dividend yield. Our upper bounds are not only tight, but also converging to the accurate American call option prices when dividend yield or strike price is small or when volatility is large. We then propose a general theorem which can be applied to derive upper bounds for American options whose payoffs depend on several risky assets. As a demonstration, we apply our general theorem to derive upper bounds for American exchange options and American maximum options on multiple risky assets.

I. Introduction

American options require numerical methods, such as lattice methods, to provide accurate price estimates. The valuation problem is very time-consuming and difficult when multiple state variables are involved. For example, the options under stochastic interest rates, stochastic volatility, and jumps involve four random factors and require expensive lattice models. Another complex example is pricing an American option whose payoff depends on more than one underlying asset. For these situations, tight analytical upper bounds can provide useful benchmark values and control variates for the correction of numerical errors.

Option pricing bounds are useful, because (1) they provide qualitative properties of options, (2) they can be used to screen market data for empirical research, (3) they shed light on hedging, and (4) they are generally obtained with the least assumptions on the investor's preferences and the distributions of the underlying asset prices.

Option pricing bounds may be derived by (1) eliminating simple dominance among different portfolios, (see Merton (1973)) (2) applying a linear programming approach, (see Garman (1976), Ritchken (1985), and Ritchken and Kuo (1988)) (3) using some mathematical inequalities such as Jensen's inequality and Cauchy's inequality, (see Lo (1987), Boyle and Lin (1997), and Chen and Yeh (2002)) and (4) using second-order stochastic dominance (see Levy (1985) and Constantinides and Perrakis (2002)).

Probably due to the difficulty in dealing with the early exercise problem, the work on American option pricing bounds is limited. Carr, Jarrow, and Myneni (1992) derived an upper bound for American put options under the Black-Scholes economy, while Broadie and Detemple (1996) developed upper and lower bounds using the capped call option pricing technique (i.e. an American call option is a simple dominant portfolio of the capped call). Although the bounds provided by Carr, Jarrow, and Myneni (1992) and Broadie and Detemple (1996) are generally tight, their upper bounds are not in analytical form (except under the Black-Scholes economy) and require numerical techniques. Chen and Yeh (2002) provided analytical form upper bounds that are applicable to general American options, e.g. American calls on dividend paying stocks, American calls on futures, American puts on dividend paying stocks, and American puts on futures. Moreover, their upper bounds rely neither on the distribution of the state variable, nor do they rely on continuous time trading.

Although Chen and Yeh's (2002) analytical form upper bounds are very general, they can be applied only in the case where the interest rate is greater than the dividend yield. Their upper bounds may be inadequate for options on several underlying assets, because it is likely to happen that some underlying asset have a dividend yield larger than the risk-free rate. In contrast, this article provides two general theorems which can be used to derive upper bounds for American options under general situations, including the case where the dividend yield is larger than the risk-free rate. As a demonstration, we apply our general theorems to derive upper bounds for American calls when the interest rate is smaller than the dividend yield, for American exchange options, and for American maximum options on multiple risky assets.

We contribute to the literature on option pricing bounds in several ways. First, previous papers on option bounds concentrated on European options with a single underlying asset or a single state variable. In contrast, we provide upper bounds for American options whose pricing involves several risky assets and/or several risk factors (e.g. stochastic interest rates, stochastic volatility, and jumps) for each asset price process. Secondly, our upper bounds are not only tight, but also converging to the accurate American call option prices when dividend yield or strike price is small or when volatility is large. Thirdly, we correct typos in Chen and Yeh (2002) and provide numerical results to investigate the tightness of their upper bounds and the tightness of ours. The numerical results indicate that our upper bounds are generally tighter than those of Chen and Yeh (2002).

The rest of this article proceeds as follows. Section II provides a general analysis for obtaining upper bounds of American options. In this section two general theorems for developing American upper bounds are introduced. Section III discusses upper bounds under stochastic interest rates, stochastic volatility, and jumps using the inversion Fourier method. This method was used by Heston (1993), Scott (1997), Bakshi, Cao, and Chen (1997), and Chen and Yeh (2002), etc. We also derive upper bounds for American exchange options and American maximum options under the Black-Scholes economy in this section. Section IV provides numerical results to analyze the tightness of our upper bounds. Section V concludes the paper.

II. General Analysis

Theorem 1 of Chen and Yeh (2002) shows that an American option is bounded from above by the risk-neutral expectation of its maturity payoff if this expectation is greater than the intrinsic value at all times.[1] This theorem is very general and the only

[1] Following Chen and Yeh (2002), all expectations are taken under the risk-neutral measure throughout

assumptions required are that (i) the risk-neutral measure exists and (ii) the nominal risk-free rate is strictly positive. We restate Theorem 1 of Chen and Yeh (2002) as follows:

Theorem 1 of Chen and Yeh (2002)

An American option is bounded from above by the risk-neutral expectation of its maturity payoff if this expectation is greater than the intrinsic value at all times.

Theorem 1 of Chen and Yeh (2002) can be presented in formal mathematics as follows:

Let T be the maturity date of the American option, and $X(t)$ *be the intrinsic value at time t. If* $E_t[X(T)] > X(t)$ *for all t, where* $E_t[\cdot]$ *represents taking the expectation in the risk-neutral world at time t, then* $E_t[X(T)]$ *is an upper bound of the American option value.*

This article will extend the idea of Chen and Yeh in a way that it is not necessary to use the maturity payoff ($X(T)$) of the American option to derive the upper bound. Instead, we replace $X(T)$ with other functions in our generalized Theorem 1, which can be applied to derive tighter upper bounds. Note that both their theorem and our theorem are proved by a discrete approximation similar to the lattice approach. The results will hold in continuous time as Δt reaches a limit.

A. The Generalized Theorem 1

The Generalized Theorem 1 of Chen and Yeh (2002)

Let T be the maturity of the option contract. Define $Y(t,T) = h(t,T)X(T)$*, where* $X(t)$ *is the intrinsic value of the option at time t and* $h(t,T)$ *is any function which satisfies*

- *a.* $h(t,s) \geq \delta(t,s)$ *for any* $t < s$*, where* $\delta(t,s)$ *is the discount factor from time* t *to time* s*,*
- *b.* $h(t,T) = h(t,s)h(s,T)$ *for any* $s \in (t,T)$*,*
- *c.* $h(t,t) = 1$.

If $E_t[Y(t,T)] > X(t)$ *for all t, then* $E_t[Y(t,T)]$ *is an upper bound of the American option value at time t.*

Proof:

the paper.

Following Chen and Yeh (2002), we will prove this theorem using a discrete approximation similar to the lattice approach. At time $T-\Delta t$, consider the function $Y(T-\Delta t,T)$ defined in the generalized Theorem 1. It is true that $E_{T-\Delta t}[Y(T-\Delta t,T)]$ is larger than the discounted terminal value:

$$\begin{aligned} E_{T-\Delta t}[Y(T-\Delta t,T)] &= E_{T-\Delta t}[h(T-\Delta t,T)X(T)] \\ &> E_{T-\Delta t}[\delta(T-\Delta t,T)X(T)], \end{aligned}$$

where the second inequality comes from condition a. of the generalized Theorem 1. By constraint, $E_{T-\Delta t}[Y(T-\Delta t,T)]$ is also larger than the intrinsic value ($X(T-\Delta t)$) and thus is an upper bound of the American option price at time $T-\Delta t$.

Since $E_{T-\Delta t}[Y(T-\Delta t,T)] \geq X(T-\Delta t)$ is true, it is true that $E_{T-2\Delta t}[Y(T-2\Delta t,T)]$ is greater than the continuation value of the American option at time $T-2\Delta t$:

$$\begin{aligned} E_{T-2\Delta t}[Y(T-2\Delta t,T)] &= E_{T-2\Delta t}[h(T-2\Delta t,T)X(T)] \\ &= E_{T-2\Delta t}[h(T-2\Delta t,T-\Delta t)h(T-\Delta t,T)X(T)] \\ &= E_{T-2\Delta t}[h(T-2\Delta t,T-\Delta t)E_{T-\Delta t}[h(T-\Delta t,T)X(T)]] \\ &= E_{T-2\Delta t}[h(T-2\Delta t,T-\Delta t)E_{T-\Delta t}[Y(T-\Delta t,T)]] \\ &> E_{T-2\Delta t}[\delta(T-2\Delta t,T-\Delta t)E_{T-\Delta t}[Y(T-\Delta t,T)]] \\ &> E_{T-2\Delta t}[\delta(T-2\Delta t,T-\Delta t)\max\{E_{T-\Delta t}[\delta(T-\Delta t,T)X(T)], X(T-\Delta t)\}]. \end{aligned}$$

The second line follows from condition b. and the fifth line follows from condition a. of the generalized Theorem 1. By constraint, $E_{T-2\Delta t}[Y(T-2\Delta t,T)]$ is also greater than the intrinsic value of the American option and thus is an upper bound of the American option value at time $T-2\Delta t$.

By mathematical induction, it is straightforward to show that $E_t[Y(t,T)]$ is an upper bound of the American option value. The result will hold in continuous time when Δt approaches zero. (**Q.E.D**)

Note that all expectations are taken under the risk-neutral world. The main difference between Chen and Yeh's (2002) Theorem 1 and our generalized Theorem 1 is that we multiply the maturity payoff function by a function $h(t,T)$. Therefore, their upper bound is a special case of ours where $h(t,T)=1$. As long as we can find an appropriate function $h(t,T)$ which is smaller than one and satisfies the criteria in the generalized Theorem 1, then $E_t[Y(t,T)]$ is an upper bound which is tighter than Chen and Yeh's.

It should be noted that when the function $h(t,T)$ is always smaller than one, $E_t[Y(t,T)] \leq E_t[E_s[Y(s,T)]]$, for any $s \in (t,T)$. In this case, our upper bound is a sub-martingale process. In contrast, the upper bound of Chen and Yeh is a martingale process. Nevertheless, as with the discounted American option prices, the discounted processes of both upper bounds are super-martingale processes, i.e.

$$E_t[Y(t,T)] \geq E_t[\delta(t,s)E_s[Y(s,T)]],$$
$$E_t[X(T)] \geq E_t[\delta(t,s)E_s[X(T)]].$$

B. A Further Extension: Theorem 2

The generalized Theorem 1 is actually still restrictive in the sense that the upper bound is related to the maturity payoff of the American option. If we extend our concept to allow $Y(t,T)$ to be any random variables which satisfy similar (or same) criteria in the generalized Theorem 1, then it is possible to derive upper bounds for general types of American options. Next, we will first establish our Theorem 2 and give three applications later on.

Theorem 2

Let T be the maturity of the option contract. Define $Y(t,T)$ as a random variable at time t which satisfies

a. $Y(T,T) \geq X(T)$,
b. $E_t[Y(t,T)] \geq E_t[\delta(t,t+\Delta t)Y(t+\Delta t,T)]$ *for any* $t \in [0, T-\Delta t]$,
c. $E_t[Y(t,T)] \geq X(t)$ *for all* $t \in [0,T]$,

where $X(t)$ *is the intrinsic value of the option at time t. Thus,* $E_t[Y(t,T)]$ *is an upper bound of the American value.*

Proof.

At time $T-\Delta t$, it is true that $E_{T-\Delta t}[Y(T-\Delta t,T)]$ is larger than the discounted terminal value:

$$\begin{aligned} E_{T-\Delta t}[Y(T-\Delta t,T)] &\geq E_{T-\Delta t}[\delta(T-\Delta t,T)Y(T,T)] \\ &= E_{T-\Delta t}[\delta(T-\Delta t,T)X(T)], \end{aligned}$$

where the first inequality comes from condition b. of Theorem 2. From condition c., $E_{T-\Delta t}[Y(T-\Delta t,T)]$ is also larger than the intrinsic value and hence is an upper bound of the American option price at time $T-\Delta t$. Since $E_{T-\Delta t}[Y(T-\Delta t,T)] \geq X(T-\Delta t)$ is true, it is true that $E_{T-2\Delta t}[Y(T-2\Delta t,T)]$ is greater than the continuation value of the American option at time $T-2\Delta t$:

$$
\begin{aligned}
E_{T-2\Delta t}\left[Y(T-2\Delta t,T)\right] &\geq E_{T-2\Delta t}\left[\delta(T-2\Delta t,T-\Delta t)Y(T-\Delta t,T)\right] \\
&= E_{T-2\Delta t}\left[\delta(T-2\Delta t,T-\Delta t)E_{T-\Delta t}\left[Y(T-\Delta t,T)\right]\right] \\
&\geq E_{T-2\Delta t}\left[\delta(T-2\Delta t,T-\Delta t)\max\left\{E_{T-\Delta t}\left[\delta(T-\Delta t,T)X(T)\right], X(T-\Delta t)\right\}\right]
\end{aligned}
$$

By constraint, $E_{T-2\Delta t}\left[Y(T-2\Delta t,T)\right]$ is also greater than the intrinsic value of the American option and thus is an upper bound of the American option value at time $T-2\Delta t$. By mathematical induction, it is straightforward to show that $E_t\left[Y(t,T)\right]$ is an upper bound of the American option value at time t. The result will hold in continuous time when Δt approaches zero. (**Q.E.D**)

The idea of our theorem 2 is quite intuitive and can be reasoned as follows. Condition a. is the terminal condition which has to be fulfilled by any upper bound. Condition b. implies that the upper bound for this period is larger than the discounted value of the upper bound in the next period. Condition c. is a necessary condition for an upper bound, i.e. an upper bound must be greater than the intrinsic value of the option at all times. Combining all three conditions will guarantee that $E_t\left[Y(t,T)\right]$ is an upper bound of the American option price.

It is worth noting that Theorem 1 of Chen and Yeh (2002), the generalized Theorem 1, and Theorem 2 are sustained even if the payoff function $X(T)$ or our general function $Y(t,T)$ depends on prices of multiple underlying assets. However, the Theorem 1 of Chen and Yeh is applicable only when dividend yields of all assets are smaller than the risk-free rate.[2] Now, we will show some applications of the generalized Theorem 1 and Theorem 2 in the following subsections.

C. Applications of the Generalized Theorem 1 and Theorem 2

1. American Calls on Dividend Paying Stocks (when $r > q$)

According to the generalized Theorem 1, if one can find a suitable $Y(t,T)$ that satisfies our criteria, then we say $E_t\left[Y(t,T)\right]$ is an upper bound of the American option price. Let $Y(t,T) = e^{\int_t^T (q_u - r_u)du} \max\{S_T - K\ , 0\}$, where S is the stock price, K is the strike price, r is the interest rate, and q is the dividend yield of the stock.[3] It is

[2] Chen and Yeh's upper bound may be very restrictive for certain types of options on multiple underlying assets. For example, $E_t\left[X(T)\right]$ is an upper bound of the American exchange option price only when $r \geq q_2 \geq q_1$, where q_1 and q_2 are the dividend yields of the first and second underlying assets, respectively.

[3] Note that when there can be no possible confusion, subscripts are sometimes omitted for simplicity.

easy to show that $h(t,T)=e^{\int_t^T (q_u - r_u)du}$ satisfies three criteria in the generalized Theorem 1. Moreover, we can verify that $E_t[Y(t,T)] > X(t)$ for all t:

$$E_t\left[e^{\int_t^T (q_u-r_u)du}\max\{S_T-K\ ,0\}\right] \geq \max\left\{E_t\left[S_T e^{\int_t^T (q_u-r_u)du} - Ke^{\int_t^T (q_u-r_u)du}\right],0\right\}$$
$$> \max\{S_t - K, 0\}. \quad (1)$$

The first line follows from Jensen's inequality and the second line holds since $E_t\left[S_T e^{\int_t^T (q_u-r_u)du}\right] = S_t$ and $r > q$. Therefore, $E_t[Y(t,T)]$ is an upper bound of the American call option.

Since $r > q$ is true, our upper bound ($E_t\left[e^{\int_t^T (q_u-r_u)du}\max\{S_T-K\ ,0\}\right]$) is tighter than Chen and Yeh's ($E_t[\max\{S_T-K\ ,0\}]$). Furthermore, our upper bound converges to the accurate American call option price under some circumstances. For instance, it is well known that when the dividend yield is zero, the American option price equals the price of its European counterpart (see Merton (1973)). Our upper bound also converges to the European option price when the dividend yield approaches zero. Moreover, when the strike price is very small or when volatility is very large, both the accurate American call option price and our upper bound will converge to the current stock price.

From our Theorem 2 we can propose another upper bound for American call options where $Y(t,T)$ follows:

$$Y(t,T)=\max\left\{e^{\int_t^T [q_u-r_u]du} S_T - Ke^{-\int_t^T r_u du},0\right\}.$$

However, this upper bound is not tighter than the above one and thus is not used in the numerical analysis later on.

2. American Call Options on Dividend Paying Stocks (when $r < q$)

Chen and Yeh's Theorem 1 can be applied to American options only when the interest rate is larger than the dividend yield. On the other hand, our Theorem 2 is applicable no matter whether the interest rate is larger or smaller than the dividend yield. When the interest rate is smaller than the dividend yield, we define a function $Y(t,T)$ for American call options as follows:

$$Y(t,T) = \max\left\{ e^{\int_t^T [q_u - r_u]du} S_T - K \ , 0 \right\}. \tag{2}$$

First of all, $Y(T,T) = \max\{S_T - K, 0\} = X(T)$. This satisfies condition a. of Theorem 2. Secondly, $Y(t,T)$ also fulfills condition b. of Theorem 2 as follows:

$$\begin{aligned} E_t[Y(t,T)] &= E_t\left[\max\left\{ S_T e^{\int_t^T (q_u - r_u)du} - K \ , 0 \right\} \right] \\ &> E_t\left[\max\left\{ S_T e^{\int_{t+\Delta t}^T (q_u - r_u)du} - K \ , 0 \right\} \right] \\ &> E_t\left[\delta(t, t+\Delta t) \max\left\{ S_T e^{\int_{t+\Delta t}^T (q_u - r_u)du} - K \ , 0 \right\} \right] \\ &= E_t[\delta(t, t+\Delta t) Y(t+\Delta t, T)]. \end{aligned} \tag{3}$$

We finally will show that $E_t[Y(t,T)]$ are always greater than the intrinsic value at any time *t*:

$$\begin{aligned} E_t\left[\max\left\{ e^{\int_t^T (q_u - r_u)du} S_T - K \ , 0 \right\} \right] &\geq \max\left\{ E_t\left[S_T e^{\int_t^T (q_u - r_u)du} - K \right], 0 \right\} \\ &= \max\{S_t - K \ , 0\}, \end{aligned} \tag{4}$$

where the first line follows from Jensen's inequality. Thus, from Theorem 2 we know that $E_t\left[\max\left\{ e^{\int_t^T (q_u - r_u)du} S_T - K \ , 0 \right\} \right]$ is indeed an upper bound for American call options when $r < q$.

3. American Exchange Options

An exchange option is an option to exchange one asset for another. The payoff from this option is

$$X(T) = \max\{S_{1T} - S_{2T}, 0\},$$

where S_{1T} and S_{2T} are values of asset one and asset two at time *T*, respectively. A closed-form solution for valuing European exchange option was first produced by

Margrabe (1978) under the Black-Scholes economy.

From Chen and Yeh's Theorem 1, it is easy to verify that $E_t[X(T)]$ is an upper bound of the American exchange option when $r \geq q_2 \geq q_1$, where q_1 and q_2 are the dividend yields of asset one and asset two, respectively. Now, we will derive another upper bound of the American exchange option.

Consider a function $Y(t,T)$ as follows:

(5) $$Y(t,T) = \max\left\{ e^{\int_t^T [q_{1u} - r_u] du} S_{1T} - e^{\int_t^T [\min(q_{1u}, q_{2u}) - r_u] du} S_{2T} \ , 0 \right\}.$$

Using a similar procedure we can easily show that conditions a. to c. of Theorem 2 are satisfied by $Y(t,T)$. Thus, $E_t[Y(t,T)]$ is an upper bound of the American exchange option price. It is not difficult to show that our upper bound is tighter than Chen and Yeh's, especially when both q_1 and q_2 are small or when q_1 is large and q_2 is small. Furthermore, it is true that our upper bound is applicable for any r, q_1, and q_2.

4. American Maximun Options on Multiple Risky Assets

Options on the maximum or minimum of two risky assets were first introduced by Stulz (1982). Stulz (1982) showed that many contingent claims, for example option-bonds, compensation plans, risk sharing contracts, etc., have a payoff function which includes the payoff function of a put or a call option on the maximum or minimum of two risky assets. The payoff of a European call option on the maximum of two risky assets is

$$X(T) = \max\{\max\{S_{1T}, S_{2T}\} - K, 0\},$$

where S_{1T} and S_{2T} are values of asset one and asset two at time *T*, respectively. A closed-form solution, which involves the bivariate cumulative standard normal distribution functions, for valuing this option was derived in Stulz (1982) under the Black-Scholes economy.

Following Theorem 1 of Chen and Yeh (2002), it is straightforward to show that the expected value of the maturity payoff at any arbitrary time t is an upper bound of the American maximum option, because its value is always greater than the early exercise value, i.e.,

$$E_t\left[\max\{\max\{S_{1T},S_{2T}\}-K,0\}\right]\geq \max\{E_t[\max\{S_{1T},S_{2T}\}]-K,0\}$$
$$\geq \max\{\max\{E_t[S_{1T}],E_t[S_{2T}]\}-K\ ,0\}$$
$$>\max\{\max\{S_{1t},S_{2t}\}-K\ ,0\}.$$

However, the above upper bound is sustained only when the dividend yields of all risky assets are smaller than the risk-free rate.

As a demonstration, this paper will derive another upper bound for the American maximum options using our Theorem 2. If one follows our Theorem 2, it is easy to prove that the expected value of the following function is also an upper bound of the American maximum option:

$$Y(t,T)=\max\left\{\max\{e^{\int_t^T[q_{1u}-r_u]du}S_{1T},\ e^{\int_t^T[q_{2u}-r_u]du}S_{2T}\}-Ke^{\int_t^T[\min(q_{1u},q_{2u})-r_u]du},\ 0\right\}, \tag{6}$$

because

$$E_t\left[\max\left\{\max\{e^{\int_t^T[q_{1u}-r_u]du}S_{1T},\ e^{\int_t^T[q_{2u}-r_u]du}S_{2T}\}-Ke^{\int_t^T[\min(q_{1u},q_{2u})-r_u]du},\ 0\right\}\right]$$
$$\geq \max\left\{E_t\left[\max\left\{e^{\int_t^T[q_{1u}-r_u]du}S_{1T},e^{\int_t^T[q_{2u}-r_u]du}S_{2T}\right\}\right]-E_t\left[Ke^{\int_t^T[\min(q_{1u},q_{2u})-r_u]du}\right],0\right\}$$
$$\geq \max\left\{\max\left\{E_t[e^{\int_t^T[q_{1u}-r_u]du}S_{1T}],E_t[e^{\int_t^T[q_{2u}-r_u]du}S_{2T}]\right\}-E_t\left[K\ e^{\int_t^T[\min(q_{1u},q_{2u})-r_u]du}\right],0\right\}$$
$$>\max\{\max\{S_{1t},S_{2t}\}-K\ ,0\}.$$

There are many points worth discussing. First of all, although the following function satisfies conditions a. and c. of Theorem 2, it does not satisfy condition b. of Theorem 2, and thus it is not an upper bound of the American maximum call option:

$$Y'(t,T)=\max\left\{\max\{e^{\int_t^T[q_{1u}-r_u]du}S_{1T},\ e^{\int_t^T[q_{2u}-r_u]du}S_{2T}\}-K,0\right\}.$$

Secondly, our upper bound is valid for any r, q_1, and q_2 as long as $r>\min(q_1,q_2)$, e.g. $q_1\geq r\geq q_2$. Thirdly, our upper bound is not necessarily smaller than that derived from Theorem 1 of Chen and Yeh (2002). The reason is due to the fact that the strike price in our upper bound ($Ke^{\int_t^T[\min(q_{1u},q_{2u})-r_u]du}$) is smaller than the strike price in Chen and Yeh's upper bound (K). However, our upper bound is tighter than Chen and Yeh's upper bound when the dividend yields of all risky assets are small. When both upper bounds are applicable, one can take the minimum of both upper bounds as the upper bound of the American maximum option price. Finally, our upper bound will converge to the accurate American call option price under some circumstances, e.g. when the dividend yield or strike price is very small.

Johnson (1987) and Boyle and Tse (1990) further extended the analysis of Stulz to the pricing of European maximum options on *n* risky assets under the Black-Scholes economy. The payoff of a European call option on the maximum of *n* risky assets is

$$X(T) = \max\{\ \max\{S_{1T},\ S_{2T},\ \cdots,\ S_{nT}\} - K, 0\}.$$

Similarly, one can apply our Theorem 2 to show that the expected value of the following function is an upper bound of the American maximum option on *n* risky assets:

$$Y(t,T) = \max\left\{\max\{e^{\int_t^T [q_{1u}-r_u]du} S_{1T},\ e^{\int_t^T [q_{2u}-r_u]du} S_{2T}, \cdots, e^{\int_t^T [q_{nu}-r_u]du} S_{nT}\} - Ke^{\int_t^T [\min(q_{1u},\ldots,q_{nu})-r_u]du}, 0\right\}.$$

Note that under the Black-Scholes economy where interest rates and dividend yields are constant, the above upper bound is actually the price of a European call option on the maximum of *n* risky assets with adjusted initial prices $S_{1t}e^{q_1(T-t)}$, $S_{2t}e^{q_2(T-t)}$, …, and $S_{nt}e^{q_n(T-t)}$, and adjusted strike price $Ke^{\int_t^T \min(q_{1u},\ldots,q_{nu})du}$. Therefore, the analytical solutions of Johnson (1987) and the approximate solutions of Boyle and Tse (1990) are directly applicable to our upper bounds.

III. Modeling

Both Chen and Yen's upper bounds and our upper bounds have analytical solutions under many asset price models. In fact, as long as a European option has an analytical solution under a model, both upper bounds also have analytical solutions under the same model, because both upper bounds can be regarded as European options with an adjusted maturity payoff.[4] For example, a European option has an analytical solution under the stochastic volatility model of Heston (1993) and so does our upper bound.

In order to compare with Chen and Yeh (2002), we will derive analytical solutions for our upper bounds under stochastic interest rates, stochastic volatility, and jumps (SVSIJ model) in the single asset cases. However, we apply the Black-Scholes model in the multiple asset cases for simplicity.

[4] For example, Chen and Yeh's upper bound $E_t[X(T)]$ is actually the price of a European option with an adjusted payoff $e^{\int_t^T r_u du} X(T)$.

A. Single Asset Cases

1. American Calls on Dividend Paying Stocks when $r > q$

The Black-Scholes model has been extended to an environment under the stochastic interest rate, stochastic volatility, and jumps; see for example, Heston (1993), Scott (1977), and Bakshi, Cao, and Chen (1997). Following Chen and Yeh (2002), we assume that the stock price process is log normal and the drift and diffusion of the stock price process follow the square root processes,

$$
\begin{aligned}
dS_t &= y_t S_t dt + S_t \sqrt{v_t}\, dW_t^S ,\\
dv_t &= \alpha(\beta - v_t)dt + \gamma \sqrt{v_t}\, dW_t^v ,\\
dy_t &= a(b - y_t)dt + g \sqrt{y_t}\, dW_t^r ,
\end{aligned}
\tag{7}
$$

where $y_t = r_t - d_t$, S is the stock price, r is the interest rate, d is the continuous dividend yield, v is the stock return variance, and α, β, γ, a, b, and g are parameters associated with the processes. Finally, $dW_t^S dW_t^v = \rho dt$ and the interest rate process is assumed to be independent of the stock and the variance processes.

Note that the processes v_t and y_t are strictly positive, because they follow the square root processes. The upper bound of the American call options is

$$
\begin{aligned}
U^C &= E_t\left[e^{\int_t^T (-y_u)du} \max\{(S_T - K), 0\} \right]\\
&= E_t\left[S_T e^{\int_t^T (-y_u)du} \right]\Pi_1 - KE\left[e^{\int_t^T (-y_u)du} \right]\Pi_2\\
&= S_t \Pi_1 - KM(t,T)\Pi_2
\end{aligned}
\tag{8}
$$

where

$$
M(t,T) = E\left[e^{\int_t^T (-y_u)du} \right].
$$

To solve for the two probabilities and $M(t,T)$, we first identify the PDE where the upper bound has to satisfy:

$$
(9) \quad \begin{aligned} &\frac{1}{2}v\left(U_{xx}^{C}-U_{x}^{C}\right)+\frac{1}{2}\gamma^{2}vU_{vv}^{C}+\frac{1}{2}g^{2}yU_{yy}^{C}+\rho\gamma vU_{xv}^{C} \\ &\quad +yU_{x}^{C}+\left(\alpha\beta-(\alpha+\lambda)v\right)U_{v}^{C} \\ &\quad\quad +\left(ab-(a+l)y\right)U_{y}^{C}-U_{\tau}^{C}-yU^{C}=0, \end{aligned}
$$

where $x=\ln(S)$, $\tau=T-t$, and λ and l are market prices of risk associated with v and y. Plugging (8) into (9), we obtain the following PDEs for the probabilities and $M(t,T)$,

$$
(10) \quad \begin{aligned} &\frac{1}{2}v\Pi_{1x}+\frac{1}{2}v\Pi_{1xx}+\frac{1}{2}\gamma^{2}v\Pi_{1vv}+\frac{1}{2}g^{2}y\Pi_{1yy}+y\Pi_{1x} \\ &\quad +\left[\alpha\beta-(\alpha+\lambda)v\right]\Pi_{1v}+\left[ab-(a+l)y\right]\Pi_{1y}+\rho\gamma v\Pi_{1v}+\rho\gamma v\Pi_{1xv}-\Pi_{1\tau}=0, \end{aligned}
$$

$$
(11) \quad \begin{aligned} &\frac{1}{2}v\Pi_{2xx}-\frac{1}{2}v\Pi_{2x}+\frac{1}{2}\gamma^{2}v\Pi_{2vv}+\frac{M_{y}}{M}g^{2}y\Pi_{2y}+\frac{1}{2}g^{2}y\Pi_{2yy} \\ &\quad +y\Pi_{2x}+\left[\alpha\beta-(\alpha+\lambda)v\right]\Pi_{2v}+\left[ab-(a+l)y\right]\Pi_{2y}+\rho\gamma v\Pi_{2xv}-\Pi_{2\tau}=0, \end{aligned}
$$

and

$$
(12) \quad \frac{1}{2}g^{2}yM_{yy}+\left[ab-(a+l)y\right]M_{y}-M_{\tau}-yM=0.
$$

First, we will solve $M(t,T)$ from equation (12). We derive its closed-form solution as follows:

$$
M(t,T)=e^{A(t,T)+B(t,T)y_{t}},
$$

where $A(t,T)$ and $B(t,T)$ are shown in Appendix B. It can be easily shown that the characteristic functions f_1 and f_2 for solving Π_1 and Π_2 satisfy the same PDEs with the boundary condition at $t=T$ being $f=e^{iux}$. With this boundary condition, we can derive the characteristic functions as follows,

$$
(13) \quad \begin{aligned} f_{1}(u)&=e^{C_{1}(t,T)+D_{1}(t,T)v_{t}+E_{1}(t,T)y_{t}+iux}, \\ f_{2}(u)&=e^{C_{2}(t,T)+D_{2}(t,T)v_{t}+E_{2}(t,T)y_{t}+iux-\ln(M(t,T))}, \end{aligned}
$$

where $C_j(t,T)$, $D_j(t,T)$, and $E_j(t,T)$ ($j=1,2$) are shown in Appendix B. According the inversion theorem, probabilities and characteristic functions have the following relationship

$$
(14) \quad \Pi_{j}=\frac{1}{2}+\frac{1}{\pi}\int_{0}^{\infty}\mathrm{Re}\left[\frac{e^{-iu\ln K}f_{j}(u)}{iu}\right]du \qquad j=1,2.
$$

2. American Calls on Dividend Paying Stocks when $r<q$

To derive the upper bound of American calls when $r < q$, we assume the same asset price model as in the previous subsection except that

(15) $$dy'_t = a(b - y'_t)dt + g\sqrt{y'_t}\,dW_t^r,$$

where $y'_t = q_t - r_t > 0$. Following the similar procedure one can derive the upper bound of the American call options as:

(16)
$$\begin{aligned} U'^C &= E_t\left[e^{\int_t^T (y'_u)du} \max\{(S_T - K), 0\}\right] \\ &= E_t\left[S_T e^{\int_t^T (y'_u)du}\right]\Pi'_1 - K\Pi'_2 \\ &= S_t\Pi'_1 - K\Pi'_2. \end{aligned}$$

The two probabilities Π'_1 and Π'_2 in equation (16) follow the same formula as equation (14), but with two different characteristic functions as follows:

(17) $$f'_j(u) = e^{C'_j(t,T) + D'_j(t,T)v_t + E'_j(t,T)y'_t + iux},$$

where $C'_j(t,T)$, $D'_j(t,T)$, and $E'_j(t,T)$ (where $j = 1, 2$) are shown in Appendix C.

3. Jumps

It has been well documented that the jump component is important for pricing stock and stock index options. The jump-diffusion model was first introduced by Merton (1976) and then used by Bates (1991), Bakshi, Cao, and Chen (1997), and Scott (1997), etc. Following Bakshi, Cao, and Chen (1997), we assume the following jump-diffusion process:

$$dS_t = (y_t S_t - \lambda_J \mu_J S_t)dt + S_t\sqrt{v_t}\,dW_t^S + J_t S_t d\xi_t,$$

$$\ln[1 + J_t] \sim N\left(\ln[1 + \mu_J] - \tfrac{1}{2}\sigma_J^2, \sigma_J^2\right),$$

where:

λ_J *is the frequency of jumps per year;*

J_t *is the percentage jump size (conditional on a jump occurring) that is lognormally, identically, and independently distributed over time, with unconditional mean* μ_J. *The standard deviation of* $\ln[1 + J_t]$ *is* σ_J;

ξ_t *is a Poisson jump counter with intensity* λ_J; *that is,* $\Pr\{d\xi_t = 1\} = \lambda_J dt$ *and* $\Pr\{d\xi_t = 0\} = 1 - \lambda dt$;

ξ_t *and* J_t *are uncorrelated with each other or with* W_t^S, W_t^v, *and* W_t^r.

The characteristic functions for the jump component are shown by Bakshi, Cao, and Chen (1997) and Scott (1997) as:

$$\text{(18)} \qquad \begin{aligned} f_{J1}(u) &= \exp\left(\lambda_J (1+\mu_J)(T-t)\left[(1+\mu_J)^{iu} e^{(iu/2)(1+iu)\sigma_J^2} - 1\right] - \lambda_J iu\mu_J (T-t)\right), \\ f_{J2}(u) &= \exp\left(\lambda_J (T-t)\left[(1+\mu_J)^{iu} e^{(iu/2)(iu-1)\sigma_J^2} - 1\right] - \lambda_J iu\mu_J (T-t)\right). \end{aligned}$$

Bakshi, Cao, and Chen (1997) and Scott (1997) show that if jumps occur independently with the stock price level and interest rates, then the characteristic function of the jump component can be combined with the characteristic function of the diffusion component. Therefore, the characteristic functions are respectively multiplied by the original functions of (14) or (17) to calculate upper bounds.

B. Multiple Asset Cases

1. American Exchange Options on Dividend Paying Stocks

Margrabe (1978) valued an option to exchange one asset for another, which is called an exchange option. We state his model in this subsection and use his setup to derive a closed-form solution for the upper bound of the American exchange option. Following Margrabe (1978) we assume that the two asset prices follow:[5]

$$dS_{it} = y_i S_{it} dt + \sigma_i S_{it} dW_{it}, \qquad i = 1, 2$$

where $y_i = r - q_i$, q_i is the dividend yield of asset i, σ_i is the standard deviation of return of asset i, and dW_{it} is the Brownian motion of asset i.

For simplicity, we follow Margrabe (1978) to assume that dividend yields and the risk-free rate are constant. Under this simplified assumption, Margrabe (1978) shows that the price of the European exchange option is:

$$\text{(19)} \qquad w_t = S_{1t} e^{-q_1\tau} N(z_1) - S_{2t} e^{-q_2\tau} N(z_2),$$

where

$$\begin{aligned} z_1 &= \frac{\ln\left(S_{1t} e^{-q_1\tau} \big/ S_{2t} e^{-q_2\tau}\right) + \frac{1}{2}\sigma^2\tau}{\sigma\sqrt{\tau}} \\ &= \frac{\ln\left(S_{1t} \big/ S_{2t}\right) + \left(q_2 - q_1 + \frac{1}{2}\sigma^2\right)\tau}{\sigma\sqrt{\tau}}, \end{aligned}$$

[5] For simplicity we assume that interest rate and dividend yield are constant in the multiple asset case.

$$z_2 = z_1 - \sigma\sqrt{\tau},$$

$$\tau = T - t,$$
$$\sigma^2 = \sigma_1^2 + \sigma_2^2 - 2\rho_{12}\sigma_1\sigma_2,$$

$$\rho_{12} = Corr\left(\frac{dS_1}{S_1}, \frac{dS_2}{S_2}\right),$$

$N(\cdot)$ = the cumulative density function (c.d.f) of the standard normal distribution.

Similar to the proof of Margrabe (1978), we can derive upper bounds for American exchange options under the Black-Scholes economy. According to the results in the previous section, one upper bound of an American exchange option is:

$$E_t[Y(t,T)] = E_t\left[\max\left\{e^{\int_t^T [q_1 - r]ds} S_{1T} - e^{\int_t^T [\min(q_1,q_2) - r]ds} S_{2T}\ ,\ 0\right\}\right]$$
$$= E_t\left[e^{-r\tau}\max\{\zeta_{1T} - \zeta_{2T},\ 0\}\right]$$

where $\zeta_{1T} \equiv S_{1T}e^{q_1\tau}$ and $\zeta_{2T} \equiv S_{2T}e^{\min(q_1,q_2)\times\tau}$. The upper bound in the above equation can be regarded as the value of an European exchange option where the initial value of asset one is $\zeta_{1t} = S_{1t}e^{q_1\tau}$ and the value initial of asset two is $\zeta_{2t} = S_{2t}e^{\min(q_1,q_2)\times\tau}$. Therefore, our upper bound has the following closed-form solution:

(20) $\quad S_{1t}N(z_1^*) - S_{2t}e^{(\min(q_1,q_2)-q_2)\tau}N(z_2^*),$

where

$$z_1^* = \frac{\ln\left(S_{1t}/S_{2t}\right) + \left(q_2 - \min(q_1,q_2) + \frac{\sigma^2}{2}\right)\tau}{\sigma\sqrt{\tau}},$$
$$z_2^* = z_1^* - \sigma\sqrt{\tau}.$$

2. American Options on Maximum of Two Risky Assets

Under the Black-Scholes economy, Stulz (1982) derived the closed-form solution for a European call option on the maximum of two risky assets as follows:

$$C_{\max}(S_{1t}, S_{2t}, K, r, q_1, q_2, \sigma_1, \sigma_2, \rho\sigma_1\sigma_2, \tau)$$
$$= C_{BS}(S_{1t}, K, r, q_1, \sigma_1, \tau) + C_{BS}(S_{2t}, K, r, q_2, \sigma_2, \tau) - C_{\min}(S_{1t}, S_{2t}, K, r, q_1, q_2, \sigma_1, \sigma_2, \rho\sigma_1\sigma_2, \tau),$$

where $C_{BS}(.)$ are Black-Scholes formulae for European call options and $C_{\min}(.)$ is the price of a European call option on the minimum of two risky assets which follows:

$$
\begin{aligned}
&C_{\min}(S_{1t}, S_{2t}, K, r, q_1, q_2, \sigma_1, \sigma_2, \rho_{12}\sigma_1\sigma_2, \tau) \\
&= S_{1t}e^{-q_1\tau} N_2\left(d_1 + \sigma_1\sqrt{\tau},\ \left. \left(\ln\frac{S_{1t}}{S_{2t}} + (q_1 - q_2 - \frac{\sigma^2}{2})\tau \right) \middle/ \sigma\sqrt{\tau} \right.,\ \frac{\rho_{12}\sigma_2 - \sigma_1}{\sigma} \right) \\
&+ S_{2t}e^{-q_1\tau} N_2\left(d_2 + \sigma_2\sqrt{\tau},\ \left. \left(\ln\frac{S_{2t}}{S_{1t}} + (q_2 - q_1 - \frac{\sigma^2}{2})\tau \right) \middle/ \sigma\sqrt{\tau} \right.,\ \frac{\rho_{12}\sigma_1 - \sigma_2}{\sigma} \right) \\
&- Ke^{-r\tau} N_2\left(d_1,\ d_2,\ \frac{\rho_{12}\sigma_1 - \sigma_2}{\sigma} \right),
\end{aligned}
\tag{21}
$$

where

$$
\begin{aligned}
d_1 &= \frac{\ln\left(S_{1t} / K \right) + \left(r - q_1 - \frac{\sigma_1^2}{2} \right)\tau}{\sigma_1\sqrt{\tau}}, \\
d_2 &= \frac{\ln\left(S_{2t} / K \right) + \left(r - q_2 - \frac{\sigma_2^2}{2} \right)\tau}{\sigma_2\sqrt{\tau}}, \\
\sigma^2 &= \sigma_1^2 + \sigma_2^2 - 2\rho_{12}\sigma_1\sigma_2.
\end{aligned}
$$

As stated in the previous section, our upper bound for the American maximum call option is actually the price of a European maximum call option with adjusted initial stock prices and adjusted strike price. Therefore, it follows a similar formula, i.e. $C_{\max}(S_{1t}e^{q_1\tau}, S_{2t}e^{q_2\tau}, Ke^{\min(q_1,q_2)\times\tau}, r, q_1, q_2, \sigma_1, \sigma_2, \rho_{12}\sigma_1\sigma_2, \tau)$.

IV. Numerical Results

A. Correct Numerical Results in Chen and Yeh (2002)

The pricing formulae for Chen and Yeh's (2002) upper bounds under the stochastic interest rates, volatility, and jumps model are not correct. The correct formulae are shown in Appendix A. The numerical results in Chen and Yeh (2002) are thus wrong. We report the correct numbers in their Table 1 and Table 2 to show the tightness of their upper bounds.

Table 1 shows their upper bounds of calls and puts under a stochastic volatility environment (SV model). The risk-free rate and the dividend yield remain constant and no jumps occur in the stock price. Their parameters are $S_t = 100$, $v_t = 0.04$, $r = 0.05$, $q = 0.03$, $\alpha = 1.5$, $\beta = 0.04$, $\gamma = 0.1$, $\lambda = 0$, $\rho = -0.5$, and $\tau = 1$. Table 1 shows that the original upper bounds are even smaller than the European option values, thus confirming that their formulae are doubtful.

Table 2 shows the upper bounds of American calls on dividend paying stocks, puts on non-dividend paying stocks, calls on futures, puts on futures, and their counterpart European option values under the stochastic volatility, stochastic interest rates, and jumps (SVSIJ) model. Following Chen and Yeh (2002), we also compute the European option values using formulae in Bakshi, Cao, and Chen (1997) since these numbers serve as lower bounds of American option values. The parameter values used in Chen and Yeh (2002) are close to those estimated by Bakshi, Cao, and Chen (1997) using S&P 500 index option data. Their parameters are $S_t = 100$, $F_t = 100$, $v_t = 0.04$, $r_t = 0.05$, $q = 0.03$, $\alpha = 1.5$, $\beta = 0.04$, $\gamma = 0.1$, $\lambda = 0$, $\rho = -0.5$, $a = 0.6$, $b = 0.02$ (in Panels B and F, $b = 0.05$),[6] $g = 0.05$, $l = 0$, $\mu_J = 0$, $\sigma_J^2 = 0.1$, and $\lambda_J = 0.6$.

From Table 2, we find that Chen and Yeh's upper bounds are generally quite tight, because their values are very close to European option values. The differences between the upper bounds and the counterpart European option values are within 5% of the counterpart European option values for most cases. The only exception is for American puts on non-dividend paying stocks where the differences may be larger than 30% (see Panel B and Panel F of Table 2).

B. American Calls on Dividend Paying Stocks (when $r > q$)

We now compare the tightness of our upper bound with that of Chen and Yeh's for American calls on dividend paying stocks when $r > q$. We use the SVSIJ model and adopt the same parameter values as Table 2. We also calculate European option values as the benchmark values to investigate the tightness of both upper bounds. From Table 3, we can see that our upper bound of calls is indeed tighter than Chen and Yeh's upper bound when $r > q$. On average, our upper bounds are 3.04% larger than European option values while Chen and Yeh's upper bounds are 5.13% larger.

[6] Note that the underlying asset does not pay a dividend in Panels B and F. Thus, we set $b = 0.05$ to match the initial risk-free rate.

C. American Calls on Dividend Paying Stocks (when $r < q$)

Chen and Yeh's upper bounds are not available for American calls on dividend paying stocks when $r < q$. On the other hand, our upper bounds are still available in this case. To demonstrate the tightness of our upper bounds in this case, we also use the SVSIJ model and adopt the same parameter values as Table 2 except that we set $r_t = 0.03$ and $q_t = 0.05$. The results are presented in Table 4. We find that when the dividend yield is larger than the risk-free rate our upper bounds are not as tight as the case where the dividend yield is smaller than the risk-free rate. The results are expected, because our upper bounds work best (i.e. no error) for American calls when the dividend yield is zero.

D. American Exchange Options

Pricing American exchange options is a two-dimensional stochastic problem under the Black-Scholes economy. Bjerksund and Stensland (1993) proved that the above two-dimensional stochastic problem can be simplified to a one-dimensional stochastic problem. Let $F(S_1, S_2, r, b_1, b_2, \sigma_1, \sigma_2, \rho\sigma_1\sigma_2, \tau)$ denote the price of an American exchange option on two assets with initial stock prices S_1 and S_2, risk-free rate r, risk-adjusted drift terms b_1 and b_2,[7] volatilities σ_1 and σ_2, correlation coefficient ρ_{12}, and time to maturity τ. Bjerksund and Stensland (1993) showed that the following relationship holds:

(22) $$F(S_{1t}, S_{2t}, r, b_1, b_2, \sigma_1, \sigma_2, \rho_{12}\sigma_1\sigma_2, \tau) = F(S_{1t}, S_{2t}, r - b_2, b_1 - b_2, 0, \sigma_{12}, 0, 0, \tau),$$

where $\sigma_{12}^2 = \sigma_1^2 + \sigma_2^2 - 2\rho_{12}\sigma_1\sigma_2$.

Note that both the drift term and the volatility parameter related to the second asset is zero. Therefore, the left-hand side of the above equation corresponds to the price of an American call where the underlying asset has current value S_{1t}, risk-adjusted drift $b_1 - b_2$, and volatility σ_{12}, the exercise price and the maturity of the call are S_{2t} and τ, respectively, and where the risk-free rate is $r - b_2$.

To analyze the tightness of our upper bound, we need benchmark values of American exchange options. We apply the adaptive mesh model of Figlewski and Gao (1999) to calculate the American exchange option price using the one-dimensional solution of

[7] The risk-adjusted terms b_1 (b_2) are equal to $r - q_1$ ($r - q_2$) in the Black and Scholes economy.

Bjerksund and Stensland (1993). The number of time steps in the adaptive mesh model is 10,000. The parameter values are adopted from Chen, Chung, and Yang (2002) as follows: $S_1 = 40$, $\sigma_1 = 0.2$, $\sigma_2 = 0.3$, $\rho_{12} = 0.5$, $r = 0.05$, $\tau = 0.5833$, and S_2, q_1, and q_2 are varied.

From Panel A of Table 5, it is clear that our upper bounds are very tight, i.e. very close to the accurate prices of the American exchange options. Our upper bounds are about 1% larger than the accurate prices in general. It is also true that our upper bounds are tighter than Chen and Yeh's (2002). In Panel B where the dividend yield of asset one is larger than the risk-free rate, Chen and Yeh's upper bound is not available while ours is still workable. However, our upper bounds are not as tight as the case in Panel A. The results are consistent with our argument that our upper bounds are tighter when dividend yields of the underlying assets are small.

E. American Maximum Call Options

We finally investigate the tightness of our upper bounds for American maximum call options on two dividend paying stocks. We use the lattice model of Chen, Chung, and Yang (2002) to calculate the accurate price of American maximum call options. The number of time steps in the lattice is limited to 1,000, because it is a two-dimensional lattice. To correct the numerical errors due to the chosen medium number of time steps, we employ the control variate technique of Hull and White (1988) to obtain the accurate price of American maximum call options as follows:

(23) $$P_{\max}^{A} = \tilde{P}_{\max}^{A} + \left(P_{\max}^{E} - \tilde{P}_{\max}^{E}\right),$$

where $\tilde{P}_{\max}^{A}$ and $\tilde{P}_{\max}^{E}$ are prices of American and European maximum call options calculated from the lattice model of Chen, Chung, and Yang (2002) with 1,000 time steps, respectively, and $P_{\max}^{E}$ is the closed-form solution of the European maximum call option. The parameter values used here are also from Chen, Chung, and Yang (2002) as follows: $S_1 = S_2 = 40$, $\sigma_1 = 0.2$, $\sigma_2 = 0.3$, $\rho_{12} = 0.5$, $r = 0.05$, $\tau = 0.5833$, and K, q_1, and q_2 are varied.

From Panel A of Table 6, we find that our upper bounds are also quite tight for in-the-money American maximum options. Our upper bounds are about 2% larger than the accurate prices in this case. As the options become out-of-the-money, our upper bounds are looser than the previous case, but the error is still smaller than 6%.

It is also interesting to know that our upper bounds work well for in-the-money cases while Chen and Yeh's (2002) work well for out-of-the-money cases. Moreover, Chen and Yeh's upper bound is not available while ours is still workable in Panel B where the dividend yield of asset one is larger than the risk-free rate. Generally speaking, the results in Table 6 confirm our previous claim that our upper bounds are tighter when dividend yields of the underlying assets are small.

V. Conclusion

Following the framework of Chen and Yeh (2002), we derive upper bounds of American option prices. These upper bounds are especially useful when there are several state variables involved in the pricing model. Our upper bounds are closed form when the counterpart European option has a closed-form solution. Our upper bounds are very general in the sense that they do not rely on distribution assumptions or continuous trading. Moreover, our upper bounds are not only tight, but also converging to the accurate American call option prices when dividend yield or strike price are small or when volatility is large.

The only required inputs to implement our upper bounds are the risk-neutral processes of the state variables. This is not a problem, because of the recent advances in empirical derivatives research. For example, one can apply the implied binomial tree approach of Rubinstein (1994) and its many extensions, such as Derman, Kani, and Chriss (1996), Jackwerth (1997), Britten-Jones and Neuberger (2000), etc., to obtain the risk-neutral process of the stock price. See Jackwerth (1999) for an excellent review on option-implied risk-neutral distributions and processes.

Our upper bound is still feasible even in the extremely complicated case where the pricing of American options depends on multiple risky assets and multiple risk factors (e.g. stochastic interest rates, stochastic volatility, and jumps) for each asset price process. In this case one can apply standard Monte Carlo simulations to calculate the expected value of our upper bound, which is computationally more efficient (and may be tighter) than other upper bounds generated by other complicated Monte Carlo methods.[8] This issue is left for future research.

In future research, we would like to empirically compare the tightness of our upper

[8] Monte Carlo methods can provide biased high estimates of the American option prices using the *foresight bias* (see Broadie and Glasserman (1997)) approach or the duality approach (see Haugh and Kogan (2001) and Rogers (2001)). The computation is generally time consuming, because it requires the simultaneous determination of the optimal exercise boundary.

bound with those generated by other approaches, e.g. Carr, Jarrow, and Myneni (1992) and Broadie and Detemple (1996). Although the upper bounds developed by them are based on the Black-Scholes economy, they can be extended to general distributions with slight modifications. Finally, we like to know if the risk-neutral processes of state variables are implied by the European option prices, and how often the American option prices may violate our upper bounds. The results will shed light on the efficiency of option markets.

Appendix

A. Correct the Typos of Chen and Yeh (2002)

A1. Derivation of the Futures Price

Guess the futures price as follows:

$$F_{t,T} = S\,e^{A(t,T)+yB(t,T)+vC(t,T)}.$$

By Ito's lemma, we obtain the following PDE for *F*:

$$\begin{aligned}&\frac{1}{2}vS^2F_{SS}+\frac{1}{2}v\gamma^2F_{vv}+\frac{1}{2}yg^2F_{yy}+\rho\gamma vSF_{Sv}+ySF_S\\&+\left[\alpha\beta-(\alpha+\lambda)v\right]F_v+\left[ab-(a+l)y\right]F_y+F_t=0\end{aligned}$$

Plug in the guessed solution for the futures price and obtain a system of three ODEs:

$$C_t=-\frac{1}{2}\gamma^2C^2+(\alpha+\lambda-\rho\gamma)C,$$

$$B_t=-\frac{1}{2}g^2B^2+(a+l)B-1,$$

$$A_t=-\alpha\beta C-abB.$$

With boundary conditions $A(T)=B(T)=C(T)=0$, we can show that $C=0$,

$$B=\frac{d_B}{g^2}\left[\frac{\left(\frac{d_B-(a+l)}{d_B+(a+l)}\right)e^{-d_B(T-t)}-1}{\left(\frac{d_B-(a+l)}{d_B+(a+l)}\right)e^{-d_B(T-t)}+1}+\frac{a+l}{d_B}\right],$$

where $d_B=\sqrt{(a+l)^2-2g^2}$, and

$$\begin{aligned}A=-\frac{ab}{g^2}\{&(T-t)\left[d_B-(a+l)\right]\\&+2\ln\left[1+\frac{d_B-(a+l)}{d_B+(a+l)}e^{-d_B(T-t)}\right]\\&-2\ln\left[1+\frac{d_B-(a+l)}{d_B+(a+l)}\right]\}.\end{aligned}$$

A2. Derivation of the Characteristic Functions of Calls

Guess the upper bound of calls as follows:

$$\begin{aligned}U^C&=E_t\left[\max\{S_T-K,0\}\right]\\&=E_t\left[S_T\right]\Pi_1-K\Pi_2\\&=F_{t,T}\Pi_1-K\Pi_2.\end{aligned}$$

The PDEs for the probabilities become:

$$\frac{v}{2}\Pi_{1xx}+\frac{\gamma^2 v}{2}\Pi_{1vv}+\frac{g^2 y}{2}\Pi_{1yy}+\rho\gamma v\Pi_{1xv}$$
$$+\left[\alpha\beta-(\alpha+\lambda-\rho\gamma)v\right]\Pi_{1v}+\left(y+\frac{v}{2}\right)\Pi_{1x}$$
$$+\left[ab-(a+l)y+g^2By\right]\Pi_{1y}+\Pi_{1t}=0$$

and

$$\frac{v}{2}\Pi_{2xx}+\frac{\gamma^2 v}{2}\Pi_{2vv}+\frac{g^2 y}{2}\Pi_{2yy}+\rho\gamma v\Pi_{2xv}$$
$$+\left[\alpha\beta-(\alpha+\lambda)v\right]\Pi_{2v}+\left(y-\frac{v}{2}\right)\Pi_{2x}$$
$$+\left[ab-(a+l)y\right]\Pi_{2y}+\Pi_{2t}=0.$$

Guess the following form for the characteristic functions:

$$f_1(u)=e^{C_1(t,T)+D_1(t,T)v+E_1(t,T)y+iux+\ln(S/F)},$$
$$f_2(u)=e^{C_2(t,T)+D_2(t,T)v+E_2(t,T)y+iux}.$$

Plug in the guessed solution for the characteristic functions and obtain a series of ODEs as follows:

$$D_{1t}=-\frac{1}{2}\gamma^2 D_1^2+\left[(\alpha+\lambda)-\rho\gamma(iu+1)\right]D_1-\frac{1}{2}\left(iu-u^2\right),$$

$$E_{1t}=-\frac{1}{2}g^2E_1^2+(a+l)E_1-(iu+1),$$

$$C_{1t}=-\alpha\beta D_1-abE_1,$$

and

$$D_{2t}=-\frac{1}{2}\gamma^2 D_2^2+\left[(\alpha+\lambda)-\rho\gamma iu\right]D_2+\frac{1}{2}\left(u^2+iu\right),$$

$$E_{2t}=-\frac{1}{2}g^2E_2^2+(a+l)E_2-iu,$$

$$C_{2t}=-\alpha\beta D_2-abE_2.$$

The solutions to C_j, D_j, and E_j $(j=1,2)$ are:

$$D_j=\frac{d_{Dj}}{\gamma^2}\left[\frac{\left(\frac{d_{Dj}-b_{Dj}}{d_{Dj}+b_{Dj}}\right)e^{-d_{Dj}(T-t)}-1}{\left(\frac{d_{Dj}-b_{Dj}}{d_{Dj}+b_{Dj}}\right)e^{-d_{Dj}(T-t)}+1}+\frac{b_{Dj}}{d_{Dj}}\right],$$

$$E_j = \frac{d_{Ej}}{g^2}\left[\frac{\left(\dfrac{d_{Ej}-b_{Ej}}{d_{Ej}+b_{Ej}}\right)e^{-d_{Ej}(T-t)}-1}{\left(\dfrac{d_{Ej}-b_{Ej}}{d_{Ej}+b_{Ej}}\right)e^{-d_{Ej}(T-t)}+1}+\frac{b_{Ej}}{d_{Ej}}\right],$$

$$C_j = -\frac{\alpha\beta}{\gamma^2}\left\{(d_{Dj}-b_{Dj})(T-t)+2\ln\left[1+\left(\frac{d_{Dj}-b_{Dj}}{d_{Dj}+b_{Dj}}\right)e^{-d_{Dj}(T-t)}\right]-2\ln\left(1+\frac{d_{Dj}-b_{Dj}}{d_{Dj}+b_{Dj}}\right)\right\}$$
$$-\frac{ab}{g^2}\left\{(d_{Ej}-b_{Ej})(T-t)+2\ln\left[1+\left(\frac{d_{Ej}-b_{Ej}}{d_{Ej}+b_{Ej}}\right)e^{-d_{Ej}(T-t)}\right]-2\ln\left(1+\frac{d_{Ej}-b_{Ej}}{d_{Ej}+b_{Ej}}\right)\right\},$$

where

$$d_{D1} = \sqrt{[(\alpha+\lambda)-\rho\gamma(iu+1)]^2-\gamma^2(iu-u^2)},$$
$$b_{D1} = (\alpha+\lambda)-\rho\gamma(iu+1),$$
$$d_{E1} = \sqrt{(a+l)^2-2g^2(iu+1)},$$
$$b_{E1} = b_{E2} = a+l,$$
$$d_{D2} = \sqrt{[(\alpha+\lambda)-\rho\gamma iu]^2+\gamma^2(iu+u^2)},$$
$$b_{D2} = (\alpha+\lambda)-\rho\gamma iu,$$
$$d_{E2} = \sqrt{(a+l)^2-2g^2 iu}.$$

A3. Derivation of the Characteristic Functions of Puts

Guess the upper bound of puts as follows:

$$U^P = E_t\left[K-e^{-\int_t^T y_u du}S_T\right]$$
$$= K(1-\Pi_2)-S_t(1-\Pi_1^*),$$

where Π_2 is the same as calls and Π_1^* is the probability obtained in the forward measure. The PDE for the Π_1^* becomes:

$$\frac{v}{2}\Pi^*_{1xx}+\frac{\gamma^2 v}{2}\Pi^*_{1vv}+\frac{g^2 y}{2}\Pi^*_{1yy}+\rho\gamma v\Pi^*_{1xv}$$
$$[\alpha\beta-(\alpha+\lambda-\rho\gamma)v]\Pi^*_{1v}$$
$$+\left(y+\frac{v}{2}\right)\Pi^*_{1x}+[ab-(a+l)y]\Pi^*_{1y}+\Pi^*_{1t}=0.$$

Guess the following form for the characteristic function:

$$f_1^*(u) = e^{C_1^*(t,T)+D_1^*(t,T)v+E_1^*(t,T)y+iux}.$$

Plug in the guessed solution for the characteristic functions and obtain a series of ODEs as follows:

$$D^*_{1t} = -\frac{1}{2}\gamma^2 D^{*2}_1 + [(\alpha+\lambda)-\rho\gamma(iu+1)]D^*_1 - \frac{1}{2}(iu-u^2),$$

$$E^*_{1t} = -\frac{1}{2}g^2 E^{*2}_1 + (a+l)E^*_1 - iu,$$

$$C^*_{1t} = -\alpha\beta D^*_1 - abE^*_1.$$

The solutions to C_1^*, D_1^*, and E_1^* are:

$$D_1^* = D_1,$$

$$E_1^* = \frac{d_{E_1^*}}{g^2}\left[\frac{\left(\frac{d_{E_1^*}-b_{E_1^*}}{d_{E_1^*}+b_{E_1^*}}\right)e^{-d_{E_1^*}(T-t)}-1}{\left(\frac{d_{E_1^*}-b_{E_1^*}}{d_{E_1^*}+b_{E_1^*}}\right)e^{-d_{E_1^*}(T-t)}+1}+\frac{b_{E_1^*}}{d_{E_1^*}}\right],$$

$$C_1^* = -\frac{\alpha\beta}{\gamma^2}\left\{(d_{D1}-b_{D1})(T-t)+2\ln\left[1+\left(\frac{d_{D1}-b_{D1}}{d_{D1}+b_{D1}}\right)e^{-d_{D1}(T-t)}\right]-2\ln\left(1+\frac{d_{D1}-b_{D1}}{d_{D1}+b_{D1}}\right)\right\}$$
$$-\frac{ab}{g^2}\left\{(d_{E_1^*}-b_{E_1^*})(T-t)+2\ln\left[1+\left(\frac{d_{E_1^*}-b_{E_1^*}}{d_{E_1^*}+b_{E_1^*}}\right)e^{-d_{E_1^*}(T-t)}\right]-2\ln\left(1+\frac{d_{E_1^*}-b_{E_1^*}}{d_{E_1^*}+b_{E_1^*}}\right),\right\}$$

where

$$d_{D1} = \sqrt{[(\alpha+\lambda)-\rho\gamma(iu+1)]^2-\gamma^2(iu-u^2)},$$

$$b_{D1} = (\alpha+\lambda)-\rho\gamma(iu+1),$$

$$d_{E_1^*} = \sqrt{(a+l)^2-2g^2 iu},$$

$$b_{E_1^*} = a+l.$$

A4. Derivation of the Characteristic Functions of Futures Options

Guess the upper bound of futures calls as follows,

$$U^{FC} = E_t[\max\{F_{T,s}-K,0\}]$$
$$= F_{t,s}\Pi_1^F - K\Pi_2^F.$$

The upper bound of futures puts is as follows:

$$U^{FP} = E_t[\max\{K-F_{T,s},0\}]$$
$$= K(1-\Pi_2^F) - F_{t,s}(1-\Pi_1^F).$$

The PDEs for the probabilities become:

$$\frac{v}{2}\Pi^F_{1xx}+\frac{\gamma^2 v}{2}\Pi^F_{1vv}+\rho\gamma v\Pi^F_{1xv}$$
$$+\left[\alpha\beta-(\alpha+\lambda-\rho\gamma)v\right]\Pi^F_{1v}+\frac{v}{2}\Pi^F_{1x}+\Pi^F_{1t}=0,$$

and

$$\frac{v}{2}\Pi^F_{2xx}+\frac{\gamma^2 v}{2}\Pi^F_{2vv}+\rho\gamma v\Pi^F_{2xv}$$
$$+\left[\alpha\beta-(\alpha+\lambda)v\right]\Pi^F_{2v}-\frac{v}{2}\Pi^F_{2x}+\Pi^F_{2t}=0,$$

where $x=\ln(F)$. Guess the following form for the characteristic functions:

$$f^F_j(u)=e^{C^F_j(t,T)+D^F_j(t,T)v+iux},\quad j=1\,,2\,.$$

Plug in the guessed solution for the characteristic functions and obtain a series of ODEs as follows:

$$D^F_{1t}=-\frac{1}{2}\gamma^2 D^{F\,2}_1+\left[(\alpha+\lambda)-\rho\gamma(iu+1)\right]D^F_1-\frac{1}{2}\left(iu-u^2\right),$$

$$C^F_{1t}=-\alpha\beta D^F_1\,,$$

$$D^F_{2t}=-\frac{1}{2}\gamma^2 D^{F\,2}_2+\left[(\alpha+\lambda)-\rho\gamma iu\right]D^F_2+\frac{1}{2}\left(u^2+iu\right),$$

$$C^F_{2t}=-\alpha\beta D^F_2\,.$$

The solutions to C^F_j and D^F_j ($j=1\,,2$) are:

$$D^F_j=D_j\,,$$

$$C^F_j=-\frac{\alpha\beta}{\gamma^2}\left\{\left(d_{Dj}-b_{Dj}\right)(T-t)+2\ln\left[1+\left(\frac{d_{Dj}-b_{Dj}}{d_{Dj}+b_{Dj}}\right)e^{-d_{Dj}(T-t)}\right]-2\ln\left(1+\frac{d_{Dj}-b_{Dj}}{d_{Dj}+b_{Dj}}\right)\right\},$$

where

$$d_{D1}=\sqrt{\left[(\alpha+\lambda)-\rho\gamma(iu+1)\right]^2-\gamma^2\left(iu-u^2\right)}\,,$$
$$b_{D1}=(\alpha+\lambda)-\rho\gamma(iu+1),$$
$$d_{D2}=\sqrt{\left[(\alpha+\lambda)-\rho\gamma iu\right]^2+\gamma^2\left(iu+u^2\right)},$$
$$b_{D2}=(\alpha+\lambda)-\rho\gamma iu\,.$$

B. Derivation of $M(t,T)$ and the Characteristic Functions of Our Upper Bound of Calls When $r > d$

Guess the following form for $M(t,T)$:

$$M(t,T) = e^{A(t,T)+B(t,T)y_t}.$$

Plug in the guessed solution for (12) and obtain a series of ODEs:

$$B_t = -\frac{1}{2}g^2 B^2 + (a+l)B + 1,$$

$$A_t = -abB.$$

With boundary condition $A(T\ ,0) = B(T\ ,0) = 0$, we can show that:

$$B = \frac{d_B}{g^2}\left\{\frac{\left[\dfrac{d_B - b_B}{d_B + b_B}\right]e^{-d_B(T-t)} - 1}{\left[\dfrac{d_B - b_B}{d_B + b_B}\right]e^{-d_B(T-t)} + 1} + \frac{b_B}{d_B}\right\},$$

where $d_B = \sqrt{(a+l)^2 + 2g^2}$, $b_B = a + l$, and

$$A = -\frac{ab}{g^2}\{(T-t)[d_B - b_B] + 2\ln\left[1 + \left[\frac{d_B - b_B}{d_B + b_B}\right]e^{-d_B(T-t)}\right] - 2\ln\left[1 + \frac{d_B - b_B}{d_B + b_B}\right]\}.$$

Next, we will show the characteristic functions of our upper bound. Guess the following form for the characteristic functions:

$$f_1(u) = e^{C_1(t,T)+D_1(t,T)v_t+E_1(t,T)y_t+iux},$$

$$f_2(u) = e^{C_2(t,T)+D_2(t,T)v_t+E_2(t,T)y_t+iux-\ln(M(t,T))}.$$

Plug in the guessed solution for the characteristic functions and obtain a series of ODEs:

$$D_{1t} = -\frac{1}{2}\gamma^2 D_1^2 + [(\alpha+\lambda) - \rho\gamma(1+iu)]D_1 - \frac{1}{2}(iu - u^2),$$

$$E_{1t} = -\frac{1}{2}g^2 E_1^2 + (a+l)E_1 - iu,$$

$$C_{1t} = -\alpha\beta D_1 - abE_1,$$

and

$$D_{2t} = -\frac{1}{2}\gamma^2 D_2^2 + [(\alpha+\lambda) - \rho\gamma iu]D_2 + \frac{1}{2}(iu + u^2),$$

$$E_{2t} = -\frac{1}{2}g^2 E_2^2 + (a+l)E_2 - (iu - 1),$$

$$C_{2t} = -\alpha\beta D_2 - abE_2.$$

The solutions to C_j, D_j, and E_j ($j = 1,2$) are:

$$D_j = \frac{d_{Dj}}{\gamma^2}\left[\frac{\left(\frac{d_{Dj}-b_{Dj}}{d_{Dj}+b_{Dj}}\right)e^{-d_{Dj}(T-t)}-1}{\left(\frac{d_{Dj}-b_{Dj}}{d_{Dj}+b_{Dj}}\right)e^{-d_{Dj}(T-t)}+1}+\frac{b_{Dj}}{d_{Dj}}\right],$$

$$E_j = \frac{d_{Ej}}{g^2}\left[\frac{\left(\frac{d_{Ej}-b_{Ej}}{d_{Ej}+b_{Ej}}\right)e^{-d_{Ej}(T-t)}-1}{\left(\frac{d_{Ej}-b_{Ej}}{d_{Ej}+b_{Ej}}\right)e^{-d_{Ej}(T-t)}+1}+\frac{b_{Ej}}{d_{Ej}}\right],$$

$$\begin{aligned} C_j = &-\frac{\alpha\beta}{\gamma^2}\left\{(d_{Dj}-b_{Dj})(T-t)+2\ln\left[1+\left(\frac{d_{Dj}-b_{Dj}}{d_{Dj}+b_{Dj}}\right)e^{-d_{Dj}(T-t)}\right]-2\ln\left(1+\frac{d_{Dj}-b_{Dj}}{d_{Dj}+b_{Dj}}\right)\right\} \\ &-\frac{ab}{g^2}\left\{(d_{Ej}-b_{Ej})(T-t)+2\ln\left[1+\left(\frac{d_{Ej}-b_{Ej}}{d_{Ej}+b_{Ej}}\right)e^{-d_{Ej}(T-t)}\right]-2\ln\left(1+\frac{d_{Ej}-b_{Ej}}{d_{Ej}+b_{Ej}}\right)\right\}, \end{aligned}$$

where

$$d_{D1} = \sqrt{[(\alpha+\lambda)-\rho\gamma(iu+1)]^2-\gamma^2(iu-u^2)},$$

$$b_{D1} = (\alpha+\lambda)-\rho\gamma(iu+1),$$

$$d_{E1} = \sqrt{(a+l)^2-2g^2 iu},$$

$$b_{E1} = b_{E2} = a+l,$$

$$d_{D2} = \sqrt{[(\alpha+\lambda)-\rho\gamma iu]^2+\gamma^2(iu+u^2)},$$

$$b_{D2} = (\alpha+\lambda)-\rho\gamma iu,$$

$$d_{E2} = \sqrt{(a+l)^2-2g^2(iu-1)}.$$

C. Derivation of the Characteristic Functions of Upper Bound of Calls When $r < d$

Guess the following form for the characteristic functions:

$$f_j'(u) = e^{C_j'(t,T)+D_j'(t,T)v_t+E_j'(t,T)y_t'+iux}.$$

Plug in the guessed solution for the characteristic functions and obtain a series of ODEs:

$$D_{1t}' = -\frac{1}{2}\gamma^2 D_1'^2+[(\alpha+\lambda)-\rho\gamma(1+iu)]D_1'-\frac{1}{2}(iu-u^2),$$

$$E_{1t}' = -\frac{1}{2}g^2E_1'^2+(a+l)E_1'+iu,$$

$$C_{1t}' = -\alpha\beta D_1'-abE_1',$$

and

$$D'_{2t} = -\frac{1}{2}\gamma^2 D'^2_2 + [(\alpha+\lambda) - \rho\gamma iu]D'_2 + \frac{1}{2}(iu + u^2),$$

$$E'_{2t} = -\frac{1}{2} g^2 E'^2_2 + (a+l)E'_2 + iu\ ,$$

$$C'_{2t} = -\alpha\beta D'_2 - abE'_2 .$$

The solutions to C'_j, D'_j, and E'_j ($j = 1,2$) are:

$$D'_j = \frac{d_{D'_j}}{\gamma^2}\left[\frac{\left(\frac{d_{D'_j} - b_{D'_j}}{d_{D'_j} + b_{D'_j}}\right)e^{-d_{D'_j}(T-t)} - 1}{\left(\frac{d_{D'_j} - b_{D'_j}}{d_{D'_j} + b_{D'_j}}\right)e^{-d_{D'_j}(T-t)} + 1} + \frac{b_{D'_j}}{d_{D'_j}}\right],$$

$$E'_j = \frac{d_{E'_j}}{g^2}\left[\frac{\left(\frac{d_{E'_j} - b_{E'_j}}{d_{E'_j} + b_{E'_j}}\right)e^{-d_{E'_j}(T-t)} - 1}{\left(\frac{d_{E'_j} - b_{E'_j}}{d_{E'_j} + b_{E'_j}}\right)e^{-d_{E'_j}(T-t)} + 1} + \frac{b_{E'_j}}{d_{E'_j}}\right],$$

$$\begin{aligned} C'_j = &-\frac{\alpha\beta}{\gamma^2}\left\{\left(d_{D'_j} - b_{D'_j}\right)(T-t) + 2\ln\left[1 + \left(\frac{d_{D'_j} - b_{D'_j}}{d_{D'_j} + b_{D'_j}}\right)e^{-d_{D'_j}(T-t)}\right] - 2\ln\left(1 + \frac{d_{D'_j} - b_{D'_j}}{d_{D'_j} + b_{D'_j}}\right)\right\} \\ &-\frac{ab}{g^2}\left\{\left(d_{E'_j} - b_{E'_j}\right)(T-t) + 2\ln\left[1 + \left(\frac{d_{E'_j} - b_{E'_j}}{d_{E'_j} + b_{E'_j}}\right)e^{-d_{E'_j}(T-t)}\right] - 2\ln\left(1 + \frac{d_{E'_j} - b_{E'_j}}{d_{E'_j} + b_{E'_j}}\right)\right\}, \end{aligned}$$

where

$$d_{D'_1} = \sqrt{[(\alpha+\lambda) - \rho\gamma(iu+1)]^2 - \gamma^2(iu - u^2)},$$

$$b_{D'_1} = (\alpha+\lambda) - \rho\gamma(iu+1),$$

$$d_{E'_1} = \sqrt{(a+l)^2 - 2g^2 iu}\ ,$$

$$b_{E'_1} = b_{E'_2} = a + l\ ,$$

$$d_{D'_2} = \sqrt{[(\alpha+\lambda) - \rho\gamma iu]^2 + \gamma^2(iu + u^2)},$$

$$b_{D'_2} = (\alpha+\lambda) - \rho\gamma iu\ ,$$

$$d_{E'_2} = \sqrt{(a+l)^2 - 2g^2(iu-1)}\ .$$

TABLE 1

Correct numerical results in Chen and Yeh's Table 1

K	Call Option Original Upper bound	Call Option Correct Upper bound	Call Option European Option Value	Put Option Original Upper bound	Put Option Correct Upper bound	Put Option European Option Value
110	2.73	4.86	4.63	12.05	14.05	12.22
105	4.47	6.73	6.40	8.53	10.72	9.23
100	6.88	9.06	8.62	5.66	7.86	6.70
95	9.96	11.90	11.32	3.48	5.50	4.64
90	13.65	15.22	14.47	1.97	3.65	3.04
CPU(sec)	0.01	0.01	0.01	0.01	0.01	0.01

The stock price is 100, the initial volatility is 0.04, the risk-free rate is 0.05, the dividend is 0.03, the time to maturity is one year, and other parameters are: $\alpha = 1.5$, $\beta = 0.04$, $\gamma = 0.1$, $\lambda = 0$, and $\rho = -0.5$.

TABLE 2
Correct numerical results in Chen and Yeh's Table 2

			ρ		
K	- 0.8	- 0.4	0.0	0.4	0.8
Panel A. Calls on Dividend Paying Stocks					
110	8.70	8.79	8.86	8.94	9.01
	8.28	8.36	8.43	8.50	8.57
105	10.59	10.64	10.68	10.72	10.76
	10.07	10.12	10.16	10.20	10.24
100	12.84	12.85	12.85	12.86	12.86
	12.21	12.22	12.23	12.23	12.23
95	15.47	15.44	15.41	15.38	15.34
	14.72	14.69	14.66	14.63	14.59
90	18.48	18.43	18.37	18.30	18.23
	17.58	17.53	17.47	17.41	17.34
Panel B. Puts on Non-Dividend Paying Stocks					
110	17.61	17.71	17.80	17.89	17.98
	14.33	14.39	14.45	14.50	14.55
105	14.35	14.41	14.47	14.53	14.59
	11.59	11.61	11.63	11.65	11.66
100	11.46	11.48	11.50	11.52	11.53
	9.19	9.18	9.16	9.14	9.12
95	8.96	8.94	8.92	8.90	8.87
	7.14	7.10	7.05	7.00	6.97
90	6.86	6.81	6.75	6.69	6.63
	5.44	5.38	5.31	5.24	5.16
Panel C. Calls on Futures					
110	7.82	7.91	8.00	8.09	8.17
	7.43	7.52	7.61	7.69	7.77
105	9.55	9.61	9.67	9.73	9.78
	9.09	9.14	9.20	9.25	9.30
100	11.65	11.67	11.69	11.71	11.73
	11.08	11.10	11.12	11.14	11.16
95	14.13	14.11	14.10	14.08	14.05
	13.44	13.43	13.41	13.39	13.37
90	17.00	16.95	16.90	16.85	16.79
	16.17	16.13	16.08	16.03	15.97
Panel D. Puts on Futures					
110	17.82	17.91	18.00	18.09	18.17
	16.95	17.04	17.12	17.20	17.28
105	14.55	14.61	14.67	14.73	14.78
	13.84	13.90	13.96	14.01	14.06
100	11.65	11.67	11.69	11.71	11.73
	11.08	11.10	11.12	11.14	11.16
95	9.13	9.11	9.10	9.08	9.05
	8.68	8.67	8.65	8.63	8.61
90	7.00	6.95	6.90	6.85	6.79
	6.66	6.61	6.57	6.52	6.46

(continued on next page)

TABLE 2 (continued)
Correct numerical results in Chen and Yeh's Table 2

	λ_J				
K	0.0	0.2	0.4	0.6	0.8
Panel E. Calls on Dividend Paying Stocks					
110	4.87	6.25	7.55	8.77	9.92
	4.63	5.94	7.18	8.34	9.43
105	6.73	8.11	9.41	10.62	11.77
	6.40	7.72	8.95	10.11	11.20
100	9.07	10.41	11.66	12.85	13.96
	8.62	9.90	11.10	12.22	13.28
95	11.90	13.15	14.34	15.45	16.50
	11.32	12.51	13.64	14.70	15.70
90	15.22	16.35	17.42	18.44	19.41
	14.48	15.55	16.57	17.54	18.46
Panel F. Puts on Non-Dividend Paying Stocks					
110	13.83	15.20	16.48	17.69	18.82
	10.54	11.90	13.18	14.38	15.50
105	10.50	11.89	13.18	14.40	15.54
	7.84	9.18	10.43	11.60	12.71
100	7.66	9.02	10.29	11.48	12.60
	5.59	6.86	8.06	9.18	10.24
95	5.32	6.61	7.81	8.95	10.02
	3.80	4.97	6.07	7.11	8.10
90	3.50	4.67	5.78	6.82	7.81
	2.44	3.48	4.46	5.39	6.29
Panel G. Calls on Futures					
110	4.09	5.44	6.70	7.89	9.01
	3.89	5.17	6.37	7.50	8.57
105	5.77	7.13	8.40	9.60	10.73
	5.48	6.78	7.99	9.13	10.21
100	7.90	9.24	10.49	11.67	12.78
	7.52	8.79	9.98	11.10	12.15
95	10.54	11.81	13.00	14.12	15.18
	10.02	11.23	12.36	13.43	14.44
90	13.68	14.84	15.93	16.96	17.95
	13.01	14.11	15.15	16.14	17.07
Panel H. Puts on Futures					
110	14.09	15.44	16.70	17.89	19.01
	13.41	14.68	15.88	17.01	18.08
105	10.77	12.13	13.40	14.60	15.73
	10.24	11.53	12.75	13.89	14.96
100	7.90	9.24	10.49	11.67	12.78
	7.52	8.79	9.98	11.10	12.15
95	5.54	6.81	8.00	9.12	10.18
	5.27	6.47	7.61	8.67	9.68
90	3.68	4.84	5.93	6.96	7.95
	3.50	4.60	5.64	6.63	7.56

The top numbers are upper bound values and the bottom numbers are European values. The initial values for the state variables are 100 for the stock price, 0.04 for the initial volatility value, and 0.05 for the interest rate level. The parameters for the variance process are $\alpha = 1.5$, $\beta = 0.04$, $\gamma = 0.01$, $\rho = -0.5$, and $\lambda = 0$. The parameters for the interest rate process are $a = 0.6$, $b = 0.02$ (in Panels B and F, $b = 0.05$), $g = 0.05$, and $l = 0$. The parameters for the jump process are $\mu_J = 0$, $\sigma_J^2 = 0.1$, and $\lambda_J = 0.6$. The dividend yield is assumed to be a constant of 0.03.

TABLE 3
Upper bound of the American calls on dividend paying stocks when $r > q$

	ρ				
K	- 0.8	- 0.4	0.0	0.4	0.8
110	8.70	8.79	8.86	8.94	9.01
	8.53	8.61	8.69	8.76	8.83
	8.28	8.36	8.43	8.50	8.57
105	10.59	10.64	10.68	10.72	10.76
	10.38	10.43	10.47	10.51	10.55
	10.07	10.12	10.16	10.20	10.24
100	12.84	12.85	12.85	12.86	12.86
	12.59	12.59	12.60	12.60	12.60
	12.21	12.22	12.23	12.23	12.23
95	15.47	15.44	15.41	15.38	15.34
	15.16	15.14	15.11	15.07	15.03
	14.72	14.69	14.66	14.63	14.59
90	18.48	18.43	18.37	18.30	18.23
	18.11	18.06	18.00	17.94	17.87
	17.58	17.53	17.47	17.41	17.34

	λ_J				
K	0.0	0.2	0.4	0.6	0.8
110	4.87	6.25	7.55	8.77	9.92
	4.77	6.12	7.40	8.59	9.72
	4.63	5.94	7.18	8.34	9.43
105	6.73	8.11	9.41	10.62	11.77
	6.60	7.95	9.22	10.41	11.54
	6.40	7.72	8.95	10.11	11.20
100	9.07	10.41	11.66	12.85	13.96
	8.89	10.20	11.43	12.59	13.68
	8.62	9.90	11.10	12.22	13.28
95	11.90	13.15	14.34	15.45	16.50
	11.66	12.89	14.05	15.14	16.18
	11.32	12.51	13.64	14.70	15.70
90	15.22	16.35	17.42	18.44	19.41
	14.92	16.03	17.08	18.07	19.02
	14.48	15.55	16.57	17.54	18.46

The top numbers are Chen and Yeh's upper bound values. The second numbers are our upper bound values. The bottom numbers are European option values. The parameter values are the same as Table 2.

TABLE 4
Upper bound of the American calls on dividend paying stocks when $r < q$

			ρ		
K	- 0.8	- 0.4	0.0	0.4	0.8
110	7.78	7.88	7.97	8.06	8.14
	6.80	6.90	7.00	7.09	7.18
105	9.52	9.58	9.64	9.70	9.75
	8.35	8.42	8.49	8.55	8.62
100	11.62	11.64	11.66	11.68	11.70
	10.23	10.27	10.30	10.34	10.37
95	14.10	14.08	14.07	14.05	14.02
	12.49	12.49	12.49	12.48	12.47
90	16.97	16.93	16.88	16.82	16.76
	15.13	15.10	15.06	15.02	14.98

			λ_J		
K	0.0	0.2	0.4	0.6	0.8
110	4.05	5.40	6.67	7.86	8.98
	3.32	4.58	5.76	6.88	7.94
105	5.72	7.09	8.37	9.57	10.70
	4.76	6.05	7.26	8.40	9.48
100	7.86	9.20	10.46	11.63	12.75
	6.64	7.93	9.13	10.26	11.33
95	10.50	11.77	12.96	14.09	15.15
	9.00	10.24	11.40	12.49	13.52
90	13.64	14.81	15.90	16.94	17.92
	11.87	13.02	14.09	15.11	16.07

The top numbers are upper bound values and the bottom numbers are European values. The parameter values are the same as Table2 except $r = 0.03$ and $q = 0.05$.

TABLE 5

Upper bounds of the American exchange options on dividend paying stocks

Panel A: $r > q_2 > q_1$

S_2	Chen and Yeh	Our upper bound	American values	European values
30	10.5850	10.4014	10.2882	10.2807
35	6.4077	6.2965	6.2254	6.2235
40	3.3862	3.3274	3.2904	3.2888
45	1.5759	1.5486	1.5330	1.5306
50	0.6582	0.6468	0.6411	0.6393

The parameter values are $S_1 = 40$, $\sigma_1 = 0.2$, $\sigma_2 = 0.3$, $\rho_{12} = 0.5$, $r = 0.05$, $\tau = 0.5833$, $q_1 = 0.02$, and $q_2 = 0.03$.

Panel B: $q_1 > r > q_2$

S_2	Chen and Yeh	Our upper bound	American values	European values
30	N.A.	10.2427	10.0000	8.8666
35	N.A.	6.1506	5.4948	5.0592
40	N.A.	3.2191	2.6432	2.4938
45	N.A.	1.4827	1.1259	1.0776
50	N.A.	0.6128	0.4328	0.4177

The parameter values are $S_1 = 40$, $\sigma_1 = 0.2$, $\sigma_2 = 0.3$, $\rho_{12} = 0.5$, $r = 0.05$, $\tau = 0.5833$, $q_1 = 0.08$, and $q_2 = 0.03$.

TABLE 6

Upper bounds of the American maximum call options on two dividend paying stocks

Panel A: $r > q_2 > q_1$

K	Chen and Yeh	Our upper bound	American values	European values
30	13.8760	13.7591	13.4800	13.4771
35	9.1213	9.0819	8.8598	8.8591
40	5.1847	5.1969	5.0358	5.0357
45	2.5682	2.5982	2.4944	2.4944
50	1.1553	1.1817	1.1221	1.1221

The parameter values are $S_1 = S_2 = 40$, $\sigma_1 = 0.2$, $\sigma_2 = 0.3$, $\rho_{12} = 0.5$, $r = 0.05$, $\tau = 0.5833$, $q_1 = 0.02$, and $q_2 = 0.03$.

Panel B: $q_1 > r > q_2$

K	Chen and Yeh	Our upper bound	American values	European values
30	N.A.	13.5890	12.8706	12.7852
35	N.A.	8.9025	8.3002	8.2494
40	N.A.	5.0450	4.6415	4.6207
45	N.A.	2.4975	2.2960	2.2899
50	N.A.	1.1263	1.0488	1.0474

The parameter values are $S_1 = S_2 = 40$, $\sigma_1 = 0.2$, $\sigma_2 = 0.3$, $\rho_{12} = 0.5$, $r = 0.05$, $\tau = 0.5833$, $q_1 = 0.08$, and $q_2 = 0.03$.

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