An Application of Statistical Bootstrapping in Option Pricing

Abstract. This paper explores the use of statistical bootstrapping in inferring the finite-sample distribution of option prices when an estimate is used in place of the true but unknown variance. This approach may have advantages over conventional methods, especially in small samples, such as: bias reduction, increased efficiency, computational simplicity and making fewer assumptions. The procedure can be easily adapted to infer the distribution of many useful nonlinear estimators, such as hedge ratios, even in the absence of analytical option pricing formulas. In an empirical application using S&P 500 data, we study the distribution of European and American option prices and deltas using bootstrapping and compare the results to those given by Lo's (1986) asymptotic approach. We also explore behaviour of these distributions when we shift volatility levels, maturities, risk-free rates and dividend yields.

Keywords : Option Pricing, Bootstrapping, Asymptotic Theory, Estimation Risk, American Options

EFM Classification : 410, 450, 760

1. Introduction

Inference on the distribution of option prices has been a long-standing problem in the financial literature. In particular, the use of estimates in place of the true but unknown volatility in the Black-Scholes formula (hereafter, BSF), gives rise to an interesting statistical problem since the estimate of the variance rate affects the estimate of the corresponding option price. This problem involves estimation risk whereby the model is valid but the input parameters are uncertain (see Gibson *et al*, 1997; Derman, 1998). The only simple case is when volatility is a deterministic function of time and can be replaced by its average value over the life of the option (Merton, 1973). A significant complication arises from the fact that even an unbiased estimate of the variance does not produce an unbiased estimate of the option price since the BSF is nonlinear with respect to volatility (see Ingersoll, 1976; Merton, 1976; Boyle and Anathanarayanan, 1977). Moreover, it is well known that reducing estimation risk by increasing sample size, either by sampling frequency or time horizon, may not be always appropriate in view of microstructures and nonstationarities (e.g., see Boyle and Anathanarayanan, 1977; Campbell *et al.*, 1997, Ch. 3).

Boyle and Anathanarayanan (1977) derived nonsymmetric option price confidence intervals on the basis of a chi-square distribution for historical variance and proposed a Bayesian approach for reducing variance estimation error. By numerically integrating the BSF over the chi-squared probability density function (*pdf*) of the variance estimate, assuming that the true variance is known, they concluded that, in general, the average Black-Scholes value does not equal the true value. However, they also showed that differences become negligible even for moderate samples. Ball and Torous (1984) attempted to overcome the problem that resulted from the BSF non-linear transformation, by constructing an asymptotic maximum likelihood estimator based on high, low and closing prices. The attractive feature of this estimator was its invariance under non-linear transformations. Butler and Schachter (1986) used a Taylor expansion of the BSF and the moments of the estimated variance rate in developing a minimum variance unbiased price estimator. The authors presented some evidence that their approach had reasonably good performance in small samples. Lo (1986) developed the most general framework based on asymptotic statistical theory for estimation and testing contingent-claim asset pricing models, such as the BSF. A computational advantage of this approach is that the limiting distribution of the contingent claim can be derived regardless of whether the claim is priced in closed form or numerically. Knight and Satchell (1997) proved that the uniformly minimum unbiased estimator for the BSF proposed by Butler and Schachter (1986) exists if and only if the option is at-the-money. Ncube and Satchell (1997) demonstrated that the asymptotic approach of Lo (1986) is inappropriate if the constant volatility model is to be regarded as an approximation to a slowly changing time varying volatility model. They also emphasised that the true confidence intervals are non-symmetric and that the asymptotic approach may yield negative option prices.

The problem of statistical bias due to the use of historical variance has not received much attention in the empirical option pricing literature.[1] Knight and Satchell (1997) argued that this could be due to the relatively small magnitude of biases reported in simulation studies, the development of more general formulae than the BSF and the widespread adoption of implied volatilities instead of historical variances. However, as discussed by the authors, the reported biases may appear so small due to the conservative volatility levels and the short expirations adopted in the simulation studies. Finally, the authors stress that implied volatilities are not always readily available, as for example, in real option pricing.

[1] For the impact of estimation risk within the context of modern portfolio theory see, for example, Jones (1999) and Lo (2003).

Recently Phillips and Yu (2004) proposed the use of the jackknife as a general method of bias reduction for pricing bond options and other derivative securities. They concentrated on estimation bias of continuous time models and, in particular, on the mean reversion parameter of interest rate diffusion processes. Their simulation results suggest that the jackknife provides substantial improvements in pricing bond options over maximum likelihood methods.

This paper proposes the use of a bootstrap methodology in inferring the properties of the finite sample distribution of option prices and hedge parameters when estimates of variance are used. Despite its computational intensity, it is argued that this methodology may have considerable advantages over the alternatives discussed previously, in terms of: small sample statistical properties, data requirements, computational simplicity and consistency against no-arbitrage bounds. As reported by the numerous applications of the bootstrap methodology in statistics and econometrics, it is particularly well suited for the problem underhand, which involves a highly nonlinear estimation with a limited amount of data. In an empirical application, we explore the merit of the proposed methodology in comparison to Lo's asymptotic approach using spot data from the S&P 500 index. While the empirical literature in this area has been concerned with the distribution of European option prices, we also look into the distribution of a hedge parameter, delta, which is of great concern to investors, particularly to those selling options. Moreover, we examine the distribution of American options, something that also has not received attention in the empirical literature.[2] Although American options present significant computational problems, since no analytical pricing model exists, they are by far the most widely traded derivatives. Finally, motivated by Knight and Satchell (1997), we also look into the effect of high volatility levels, long maturities, and shifts in the risk-free rate and the dividend yield, respectively. In general, our conclusions are comparable to those drawn by previous researchers. More

[2] Ncube and Satchell (1997) studied the problem of estimation risk for the specific case of perpetual American put options, where a closed form solution is available.

specifically, we also find that biases in the BSF option prices due to the use of estimates in historical variance are likely to be small, and that asymptotic approaches may produce confidence intervals for prices that violate no-arbitrage bounds. We extend these results to the hedge parameter delta and to American option prices, where we obtain similar results. However, we find that an increase in volatility and time to maturity, contrary to what has been conjectured in the literature, does not necessarily increase biases in option prices. Shifting volatility, time to maturity, risk-free rates and dividend yields appears to have significant, yet mixed, effects on the shape of the distribution of option prices and deltas, respectively.

The next section describes Lo's asymptotic approach along with the proposed bootstrap methodology. Section 3 presents an empirical application of these methodologies for inferring the distribution of European and American option prices and hedging parameters using variance estimated from S&P 500 data. The final section summarises our contribution and concludes the paper.

2. Methodology: Inferring the Distribution of Option Prices and Hedge Parameters

2.1 Asymptotic Approach

Following the analysis by Lo (1986), the maximum likelihood estimator of σ^2 is given by:

$$\hat{\sigma}_{ML}^2 = \frac{1}{T}\sum_{k=1}^{n}\left(X_k - \frac{1}{n}\sum_{j=1}^{n} X_j \right)^2 \tag{1}$$

where X_k is the log difference of asset prices $S(kh)/S((k-1)h)$ and $h=T/n$ for $n+1$ are equally spaced observations of $S(t)$ in the time interval [0, T]. This can be used to obtain a point estimate for option prices and hedge parameters, respectively. Standard results show that $\hat{\sigma}^2_{ML}$ has the following asymptotic distribution:

$$\sqrt{n}(\hat{\sigma}^2_{ML} - \hat{\sigma}^2) \xrightarrow{A} N(0, 2\sigma^4) \tag{2}$$

Taking a first order Taylor approximation, the asymptotic distribution of a non-linear function, such as an option price, $\hat{F}_{ML}$, that depends on $\hat{\sigma}^2_{ML}$, is given by:

$$\sqrt{n}\left(\hat{F}_{ML} - F\right) \xrightarrow{A} N\left(0, 2\sigma^4 \left(\frac{\partial F\left(\sigma^2\right)}{\partial \sigma^2}\right)\right) \tag{3}$$

The same formula applies in the case we choose to study the distribution of hedge parameters (see Campbell *et al.*, 1997). As shown by Lo (1986), in the case of BSF, the asymptotic distribution is:

$$\sqrt{n}\left(B\hat{S}F - BSF\right) \xrightarrow{A} N\left(0, \frac{1}{2} S^2 \sigma^2 \tau \phi^2\left(d_1\right)\right) \tag{4}$$

where $\varphi(\cdot)$ is the standard normal *pdf*. Once the asymptotic distribution is known, confidence intervals (hereafter *asymptotic confidence intervals*, ACI) and hypothesis tests can be constructed using the statistic $\frac{\hat{F}_{ML} - F}{\sqrt{V_F}}$, where $\sqrt{V_F}$ is the asymptotic standard error (ASE). In the case where closed form solutions are not available, such as in American options, the asymptotic variance can still be obtained via numerical differentiation with respect to the variance input. The (1-*a*)100% confidence interval is given by:

$$\text{Prob}\left(F(\hat{\sigma}^2_{ml}) - z_{a/2}\sqrt{V_F} \le F(\sigma^2_{ml}) \le F(\hat{\sigma}^2_{ml}) + z_{a/2}\sqrt{V_F} \right) = 1 - a \tag{5}$$

2.2 Bootstrap Approach

The bootstrap is a resampling method of simulation for inferring the distribution of a statistic derived from a sample by treating the sample as the population. It is nonparametric in the sense that unlike alternatives such as Monte Carlo simulation methods, it does not draw repeated samples from assumed distributions. The bootstrap carries out conventional statistical calculations in an unconventional way: by purely computational means, rather than through the use of mathematical formulas. Bootstrap can be thought as a nonparametric maximum likelihood theory applied via the computer to a more complicated class of estimation problems. The development of this approach is relatively new (Efron, 1979) and has become increasingly popular with the widespread availability of powerful computers. Although bootstrapping is computationally intensive, it is intuitively appealing and particularly easy to implement. A variety of resampling schemes, standard error estimation algorithms and hypothesis testing procedures have been developed within the bootstrap methodology.[3] It has been widely demonstrated that under mild conditions, boostrapping provides more accurate approximations in small samples to the distribution of many statistics than classical large sample approximations (eg, see Singh, 1981; Babu, 1986). As demonstrated by Horowitz (2001), under mild regularity conditions, the bootstrap provides approximations to distributions of statistics, coverage probabilities of confidence intervals, and rejection probabilities of hypothesis tests that are at least accurate as the approximations of 1st order asymptotic distribution theory, without entailing the algebraic complexity of higher order expansions. Even if asymptotic and bootstrap standard errors are the

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[3] An informal yet comprehensive treatment is Efron and Tibshirani (1993). Applications in econometrics and finance are reviewed by Horowitz (2001) and Maddala and Li (1996), respectively. For a more mathematical treatment, including Edgeworth expansions, see Hall (1992).

same, the confidence intervals can be different if the bootstrap distribution is skewed. Although bootstrap estimates are asymptotically efficient, they are not necessarily unbiased, but tend to have small biases compared to the magnitude of their standard errors. The bootstrap is especially well suited for cases where it is difficult to calculate the asymptotic distribution of an estimator or statistic. In the case of finite variance the bootstrap distribution converges weakly to normality. Singh (1981) has showed that under the existence of third moments, the bootstrap is asymptotically a better approximation to the true distribution than the normal on the basis of Edgeworth expansion. In the case where third moments do not exist then the bootstrap approximation is asymptotically equivalent to the normal (Hall, 1988). One of the major drawbacks of the bootstrap distribution is that it cannot provide consistent estimates in the presence of infinite variance (see LePage and Billard, 1992).

Assume that we observe *n iid* data points of log difference of asset prices and that we want to price an option on the underlying asset. The bootstrap algorithm[4] starts by generating a large number of independent bootstrap samples $\mathbf{x}^{*1}, \mathbf{x}^{*2}, \ldots, \mathbf{x}^{*B}$, each of size *n*, The samples are generated by uniformly sampling from the original sample with replacement. The star notation indicates that $\mathbf{x}^*$ is not the actual data set $\mathbf{x}$ but rather a randomised or resampled version of it. For each bootstrap sample we estimate the variance of returns, that is we obtain a vector of variances $(\sigma^2)^{*1}, (\sigma^2)^{*2}, \ldots, (\sigma^2)^{*B}$. We then evaluate the bootstrap replication, that is, the option price corresponding to each replication sample:

$$\hat{u}^* (b) = F((\sigma^2)^{*b}, \cdot) \qquad b = 1, 2, 3, \ldots, B \qquad (6)$$

[4] See Efron and Tibshirani (1993).

The average of the bootstrap samples can be used as an estimate of the option price. The same procedure can be applied in studying the distribution of hedge parameters. The standard error se(û) (BSE) can be estimated by the sample standard deviation of the B replications:

$$se_B = \{ \sum[\hat{u}^*(b) - \hat{u}^*(\cdot)]^2 / (B - 1) \}^{1/2} \qquad b = 1, 2, 3, \ldots, B \qquad (7)$$

$$\text{where } \hat{u}^*(\cdot) = \sum \hat{u}^*(b) / B \qquad b = 1, 2, 3, \ldots, B$$

In this way the standard deviation of the estimated option price is the empirical standard deviation of the replications. The result is termed the bootstrap estimate of standard error se_B with B being the number of bootstrap samples used. Obviously, the limit of se_B as $B \to \infty$ is the ideal bootstrap estimate.

Confidence intervals and hypothesis tests can then be constructed either on the basis of the empirical percentiles (hereafter *bootstrap nonparametric confidence intervals*, BNCI) or using the normality assumption (hereafter *bootstrap parametric confidence intervals*, BPCI). Let $\hat{G}$ be the cumulative function of û*. The 1-2α percentile interval is defined by the α and the 1-α percentiles respectively of $\hat{G}$:

$$\left[\hat{u}^*_{\%lo}, \hat{u}^*_{\%up} \right] = \left[\hat{G}^{-1}(a), \hat{G}^{-1}(1-a) \right] \qquad (8)$$

Since by definition $\hat{G}^{-1}(a) = \hat{u}^{*(a)}$, the 100th percentile of the bootstrap distribution can also be written as

$$\left[\hat{u}^*_{\%lo}, \hat{u}^*_{\%up} \right] = \left[\hat{u}^{*(a)}, \hat{u}^{*(1-a)} \right] \qquad (9)$$

In addition to the statistical advantages in moderate samples mentioned above, the bootstrap distributions of option prices and hedge parameters will have a shape that is consistent with the true distributions and will not violate no-arbitrage bounds. The standard errors in option prices estimated by both the asymptotic and bootstrap approach will be identical for both calls and puts only in the case of European options. Put-call parity can then be used to infer the price of put options, for example, from the price of call options for constructing confidence intervals. However, in wide variety of options, including American, no generic put-call parity relationship exists.

4. An Empirical Application

In this example we study European and American option prices and hedge parameters, respectively. The data employed consist of daily closing prices for the Standard and Poors 500 Index (S&P500) from 6/10/03 to 31/12/03, a total of 60 observations. The relatively short sample length was determined according to standard practice (e.g. see Hull, 2004), in order to avoid nonstationarities. Daily returns were calculated as usual via logarithmic differencing. A maximum likelihood estimate of variance is used to calculate volatility levels, and, subsequently, point estimates of option prices using the BSF and the Barrone-Adesi (1989) approximation to American option prices, respectively. Lo's (1986) asymptotic standard errors (ASE) of prices are then calculated using the methodology described previously. For the European options we also study the asymptotic distribution of the partial derivative of option price with respect to the underlying asset, the so-called hedge parameter delta. We then compare the results for European and American options with those obtained via bootstrapping, respectively.

4.1 European Options

Table 1 presents the results for European call option prices and deltas under different levels of moneyness. A first observation is that price differences between BSF point estimates and bootstrap estimates are marginal across different levels of moneyness, something that is consistent with previous results (eg., see Boyle and Anathanarayanan, 1977; Lo, 1986). An interesting conclusion is that biases are small also for deltas, something that has also been observed by Figlewski and Green (1999). As indicated by the skewness and kurtosis coefficients, the nonparametric bootstrap distributions of prices are normal only for at-the-money options, whereas the bootstrap distributions of the hedge parameter delta are non-normal across all the different levels of moneyness.

Table 1. European Call Option Prices and Deltas for different levels of moneyness

	S/X = 110%		S/X = 105%		S/X = 100%		S/X = 95%		S/X = 90%	
	Price	*Delta*	*Price*	*Delta*	*Price*	*Delta*	*Price*	*Delta*	*Price*	*Delta*
Point Estimate	**108.542**	**0.976**	57.472	0.836	20.864	0.488	4.577	0.159	0.574	0.027
Average	**108.572**	**0.976**	57.459	0.838	20.756	0.487	4.569	0.157	0.609	0.028
Difference	-0.03%	0.04%	0.02%	-0.25%	0.52%	0.06%	0.18%	**1.36%**	**-5.83%**	-1.12%
ASE	0.233	0.008	1.211	0.020	**1.987**	0.003	1.212	**0.023**	0.315	0.011
BSE	0.241	0.008	1.200	0.021	**1.990**	0.003	1.201	**0.023**	0.322	0.011
CV	0.22%	0.82%	2.09%	2.51%	9.59%	0.62%	26.29%	14.65%	**52.87%**	**39.29%**
Skewness	**1.030**	**-0.536**	0.248	0.270	0.003	-0.358	0.273	-0.217	0.934	0.457
Kurtosis	**1.534**	0.163	-0.054	-0.017	-0.095	**0.179**	-0.037	-0.064	1.216	0.051
JB	**1,373.86**	**244.827**	51.915	60.765	1.885	113.387	62.418	40.156	1,034.75	174.378
BPCI Left	**108.100**	**0.959**	55.106	0.798	16.855	0.482	2.215	0.112	-0.021	0.006
BPCI Right	**109.044**	**0.992**	59.811	0.879	24.657	0.493	6.922	0.202	1.240	0.049
BNCI Left	**108.235**	**0.958**	55.289	0.801	16.907	0.482	2.415	0.111	0.144	0.009
BNCI Right	**109.139**	**0.989**	59.900	0.881	24.585	0.492	7.024	0.198	1.355	0.051
ACI Left	**108.085**	**0.960**	55.098	0.797	16.969	0.482	2.201	0.114	-0.043	0.005
ACI Right	**108.999**	**0.992**	59.846	0.875	24.759	0.494	6.953	0.204	1.191	0.049
BPCI/BNCI-1 (L)	-0.12%	0.17%	-0.33%	-0.44%	-0.30%	0.08%	-8.27%	1.63%	**-114.46%**	**-32.44%**
BPCI/BNCI-1 (R)	-0.09%	0.27%	-0.15%	-0.23%	0.29%	0.11%	-1.45%	1.67%	**-8.53%**	**-3.52%**
BPCI/ACI-1 (L)	0.01%	-0.14%	0.01%	0.15%	-0.67%	-0.02%	0.61%	-1.69%	**-51.61%**	**10.29%**
BPCI/ACI-1 (R)	0.04%	0.03%	-0.06%	0.43%	-0.41%	-0.18%	-0.44%	**-1.02%**	**4.08%**	0.91%

The "parameters" price and delta correspond to that of a call option with spot price S, strike price X, risk free rate 1%, dividend yield 2% and time to maturity 3 months. Variance is calculated using 3 months worth of data (6.10.03 – 31.12.03) either via maximum likelihood or via bootstrapping simulation using 5,000 simulations. "Point estimates" of the parameters are calculated from the ML-variance based volatilities. "Average values" of the parameters are calculated from the bootstrapping sampling distribution. The % difference between the point estimates and average values is given across the difference line. St. Dev, Skewness, Kurtosis and the Jarque-Berra (JB) normality test statistic are calculated from the bootstrap distribution. The JB is distributed under the null as χ^2 with 2 degrees of freedom and for a=5% the critical value is 5.99. CV is a coefficient of variation derived as BSE/Average. Left and right confidence intervals are calculated via parametric bootstrapping (BPCI), nonparametric bootstrapping (BNCI) and Lo's 1st order asymptotic approximation (ACI), respectively. ASE and BSE are the asymptotic and bootstrap standard errors, respectively. Confidence intervals under BPCI and ACI are symmetric and assume normality. BPCI/BNCI-1 and BPCI/BNCI-1 give % differences between left (L) and right (R) confidence intervals, respectively. Bold letters express the maximum absolute value across each line for option prices and deltas, respectively.

These results are consistent with the findings by Knight and Satchell (1997) who proved that a uniformly minimum unbiased estimator for the BSF exists if and only if the option is at-the-money. This can be explained using the second order Taylor expansion of the BSF with respect to variance. The bias depends on the term $0.5F''(\sigma^2)\text{var}(\hat{\sigma}^2)$, where F'' is the second partial derivative with respect to variance, and is zero only for at-the-money options. Moreover, the variance of the estimates, as suggested by both the nonparametric bootstrap standard errors, BSE, and asymptotic standard errors, ASE, is largest (smallest) for at-the-money options prices (deltas). More intuitively, the variance of the option price distribution will be at its highest for at-the-money options, since at that point the sensitivity of the price with respect to variance, the so-called vega, has a maximum.[5] Conversely, the variance of the delta will be at its lowest for at-the-money options, since at that point the sensitivity of the delta with respect to volatility, the so-called delvar, is zero. This information is important for option writers since the level of uncertainty due to variance varies across moneyness. In practice, option traders treat very cautiously at-the-money options, since they know that they exhibit high sensitivity to changes in volatility. Here we show that the risk related to volatility is relatively higher even if volatility is assumed to be constant. Graphical depictions of the vega and delvar functions are given in Figure 1.

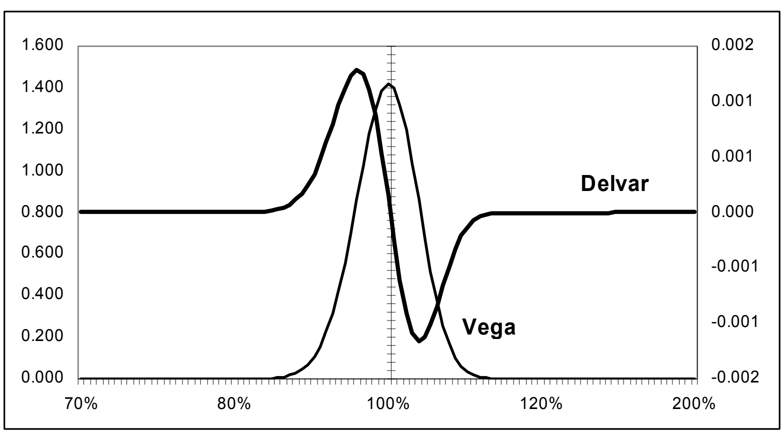

Figure 1. Sensitivities of delta and option price with respect to variance. Delvar is the sensitivity of delta with respect to variance. Vega here is defined as the sensitivity of the option price with respect to variance. These are calculated from a call European option priced on the Black Scholes formula using different moneyness levels S/X = 70%…200%, risk free rate 1%, dividend yield 2% and time to maturity 3 months.

[5] The standard practice is to define vega as the sensitivity of the option price with respect to volatility. Although here we define it as the sensitivity to variance, we retain the same term for reasons of simplicity.

If we take into account the magnitude of the option price and delta, as suggested by the coefficients of variation (CV) given in Table 1, standard errors are higher in relative terms for the out-of-the money options. This is reasonable since the option price and delta level decrease significantly as the option becomes out-of-the money.

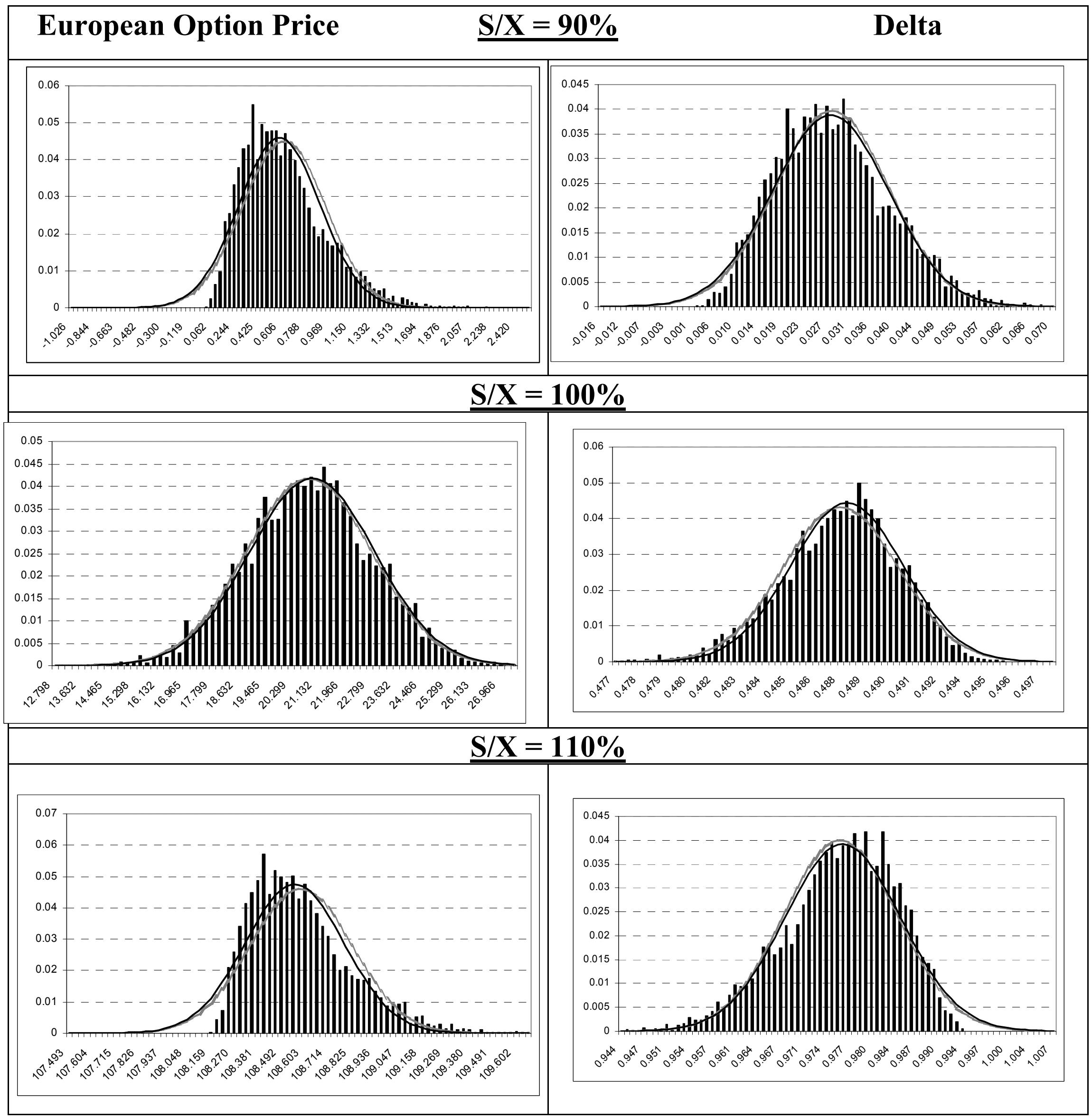

Figure 2. Distributions of European Call Option Prices and Deltas for different levels of moneyness. The gray (solid) curve is the normal probability curve corresponding to the average and standard error of the bootstrap (asymptotic) distribution. The histogram corresponds to the actual bootstrap distribution.

Figure 2, depicts the density functions implied by the first two moments of the asymptotic and parametric bootstrap distribution, respectively. These are superimposed over the bootstrap distribution. It is interesting to note that asymptotic and parametric bootstrap distributions are almost identical. The bootstrap distribution of the option price shifts to the right for in-the-money and out-of-the-money options, while the distribution of the hedge parameter shifts to the right (left) for out-of-the-money (in-the-money) options. This is to be expected since the delvar alters signs, whereas vega remains always positive, as depicted in Figure 1.

The final step is to construct and compare confidence intervals based on the three distributions: parametric bootstrapping (BPCI), nonparametric bootstrapping (BNCI) and asymptotic (ACI). Numerical values and differences of the 95% confidence intervals are also given in Table 1. The differences between the three confidence intervals can also be inferred by the graphical depiction of the three underlying distributions, given in Figure 2. Consistent with the findings of Ncube and Satchell (1997), the parametric and asymptotic confidence intervals violate the no-arbitrage option price bounds by giving negative prices for 10% out-of-the money options. However, non-parametric bootstrap confidence intervals always remain within the no-arbitrage limits. As discussed previously, we see clearly now that although the asymptotic and bootstrap standard errors may be very close, the confidence intervals can be different when the bootstrap distribution is skewed. Moneyness influences the differences between the three confidence intervals calculated. The greatest differences emerge at the lower bound of the confidence intervals, an effect that is magnified as the option moves out-of-the-money. The difference is largest at the point where the two parametric confidence intervals violate the no-arbitrage bound by yielding negative prices at the left points. Concerning the hedge parameter, the confidence interval behaviour for out-of-the-money options is consistent with Figlewski (1989), who shows that estimation risk is more pronounced for the deltas of out-of-the-money options.

Table 2. European Call Option Prices and Deltas under different pricing assumptions

	Benchmark		Time		Volatility		Risk free		Dividend	
10% out of the money calls and puts	**T-t = 0.25 σ = 10% r=1% q = 2%**		**T-t = 0.5 σ = 10% r=1% q = 2%**		**T-t = 0.25 σ = 20% r=1% q = 2%**		**T-t = 0.25 σ = 10% r=5% q = 2%**		**T-t = 0.25 σ = 10% r=1% q = 5%**	
	Price	***Delta***	***Price***	***Delta***	***Price***	***Delta***	***Price***	***Delta***	***Price***	***Delta***
Point Estimate	0.574	0.027	2.948	0.084	**10.247**	**0.177**	0.949	0.042	0.382	0.019
Average	0.609	0.028	2.986	0.083	**10.210**	**0.175**	0.986	0.042	0.415	0.020
Difference	-5.83%	-1.12%	-1.28%	1.20%	0.36%	1.36%	-3.73%	0.13%	**-7.77%**	**-2.41%**
ASE	0.315	0.011	1.092	0.020	**2.598**	**0.024**	0.452	0.014	0.234	0.009
BSE	0.322	0.011	1.078	0.020	**2.573**	**0.024**	0.455	0.014	0.241	0.009
CV	52.87%	39.29%	36.10%	24.10%	25.20%	13.71%	46.15%	33.33%	**58.07%**	**45.00%**
Skewness	0.934	0.457	0.506	0.033	0.237	-0.286	0.759	0.287	**1.078**	**0.594**
Kurtosis	1.216	0.051	0.222	-0.206	0.089	0.155	0.728	-0.128	**1.708**	**0.267**
JB	1,034.750	174.378	223.716	9.794	48.262	73.108	590.897	71.939	**1,577.226**	**308.410**
BPCI Left	-0.021	0.006	0.872	0.044	**5.167**	**0.128**	0.095	0.015	-0.058	0.003
BPCI Right	1.240	0.049	5.100	0.121	**15.253**	**0.222**	1.877	0.070	0.887	0.036
BNCI Left	0.144	0.009	1.187	0.045	**5.465**	**0.125**	0.288	0.018	0.082	0.006
BNCI Right	1.355	0.051	5.302	0.121	**15.489**	**0.219**	2.010	0.071	0.986	0.038
ACI Left	-0.043	0.005	0.808	0.045	**5.155**	**0.130**	0.063	0.015	-0.077	0.001
ACI Right	1.191	0.049	5.088	0.123	**15.339**	**0.224**	1.835	0.069	0.841	0.037
BPCI/BNCI-1 (L)	-114.46%	-32.44%	-26.50%	-2.10%	-5.45%	2.40%	-67.11%	-14.90%	**-170.23%**	**-54.55%**
BPCI/BNCI-1 (R)	-8.53%	-3.52%	-3.81%	0.51%	-1.52%	1.37%	-6.62%	-1.70%	**-10.04%**	**-5.07%**
BPCI/ACI-1 (L)	**-51.61%**	10.29%	7.96%	-1.79%	0.23%	-1.51%	50.60%	3.02%	-24.32%	**120.59%**
BPCI/ACI-1 (R)	4.08%	0.91%	0.23%	**-1.79%**	-0.56%	-0.91%	2.29%	0.81%	**5.51%**	-1.75%

Variance is calculated using 3 months worth of data (6.10.03 – 31.12.03) either via maximum likelihood or via bootstrapping simulation using 5,000 simulations. "Point estimates" of the parameters are calculated from the ML-variance based volatilities. "Average values" of the parameters are calculated from the bootstrapping sampling distribution. The % difference between the point estimates and average values is given across the difference line. St. Dev, Skewness, Kurtosis and the Jarque-Berra (JB) normality test statistic are calculated from the bootstrap distribution. The JB is distributed under the null as χ^2 with 2 degrees of freedom, for a=5% the critical value is 5.99. CV is a coefficient of variation derived as BSE/Average. Left and right confidence intervals are calculated via parametric bootstrapping (BPCI), nonparametric bootstrapping (BNCI) and Lo's 1st order asymptotic approximation (ACI), respectively. ASE and BSE are the asymptotic and bootstrap standard errors, respectively. Confidence intervals under BPCI and ACI are symmetric and assume normality. BPCI/BNCI-1 and BPCI/BNCI-1 give % differences between left (L) and right (R) confidence intervals, respectively. Bold letters express the maximum absolute value across each line for option prices and deltas, respectively.

Table 2 summarises the results of changes in the option pricing parameters with respect to: time to maturity, volatility[6], risk-free rate and dividend yield. Moneyness is retained constant at 10% out-of-the-money while for ease of comparison we repeat the last two columns of results obtained for the "benchmark" case from Table 1. Contrary to the conjecture of Satchell and Knight (1997), biases in option prices due to variance uncertainty are less pronounced as time to maturity and volatility increase. This is due to the fact that the delvar and vega functions, as depicted in Figure 1, become flatter as time to maturity and volatility increases and the second partial derivative with respect to variance decreases in absolute value. While the same effect seems to hold for the risk free rate, the opposite can be observed for increases in the dividend yield. Again, it is the underlying second partial derivative behaviour with respect to variance that explains these effects. These effects influence also the shape of the option price distributions and deltas, as seen by the differences in the skewness and kurtosis coefficients, and normality test statistics. Likewise, standard errors for both option prices and hedge parameters increase in all parameter scenarios except for the dividend yield. In general, changes in all parameters, except for the dividend yield, appear to narrow the confidence intervals. Both the parametric bootstrap and asymptotic approach violate the no-arbitrage bounds by yielding negative prices when dividends increase. Shifts in interest rates (dividend yields) improve (deteriorate) parametric and asymptotic confidence intervals relative to the nonparametric bootstrap. Additional experiments indicate that further rising time to maturity and volatility makes all confidence intervals to converge. For example, for options with 6 months time to maturity and volatility at 30%, the bootstrap distributions become normal for all levels of moneyness. The distribution of the hedge parameter remains non-normal for all scenarios, but differences in the confidence intervals become negligible. The above results suggest that the problems that are likely to arise from the asymptotic approach with respect to variance uncertainty are less pronounced as time to maturity, volatility and risk-free rates

[6] Volatility is shifted by linearly scaling returns, which does not affect the higher moments of the distribution.

increase. Moreover, it appears that dividend yields are also important and will affect the results in a significant, yet opposite, direction.

Table 3. American Call/Put Option Prices for different levels of moneyness

	S/X=110%		S/X=100%		S/X = 90%	
	Call	***Put***	***Call***	***Put***	***Call***	***Put***
Point Estimate	**111.192**	0.397	21.027	23.633	0.585	**114.258**
Average	**111.192**	0.427	20.92	23.525	0.620	**114.293**
Difference	0.00%	**-7.10%**	0.51%	0.46%	**-5.72%**	-0.03%
ASE	0.117	0.369	**1.639**	**1.641**	0.468	0.466
BSE	0.008	0.241	**1.987**	**1.990**	0.324	0.322
CV	0.01%	**56.44%**	9.50%	8.46%	**52.26%**	0.28%
Skewness	**31.431**	**1.030**	0.004	0.003	0.928	0.934
Kurtosis	**1,172.78**	**1.534**	-0.095	-0.095	1.199	1.216
JB	**287,368,594**	**1,374**	1.894	1.888	1,017	1,035
BPCI Left	**111.177**	-0.045	17.026	19.625	-0.015	**113.663**
BPCI Right	**111.208**	0.899	24.814	27.426	1.255	**114.923**
BNCI Left	**111.192**	0.251	17.078	19.676	0.150	**113.828**
BNCI Right	**111.192**	0.179	24.743	27.355	1.371	**115.039**
ACI Left	**110.963**	-0.326	17.815	20.417	-0.332	**113.345**
ACI Right	**111.421**	1.120	24.239	26.849	1.502	**115.171**
BPCI/BNCI-1 (L)	-0.01%	**-117.93%**	-0.30%	-0.26%	**-110.00%**	-0.14%
BPCI/BNCI-1 (R)	0.01%	**402.23%**	0.29%	0.26%	**-8.46%**	-0.10%
BPCI/ACI-1 (L)	0.19%	**-86.21%**	-4.43%	-3.88%	**-95.49%**	0.28%
BPCI/ACI-1 (R)	-0.19%	**-19.75%**	2.37%	2.15%	**-16.46%**	-0.22%

The ”parameters” price and delta correspond to that of a call option with spot price S, strike price X, risk free rate 1%, dividend yield 2% and time to maturity 3 months. Variance is calculated using 3 months worth of data (6.10.03 – 31.12.03) either via maximum likelihood or via bootstrapping simulation using 5,000 simulations. “Point estimates” of the parameters are calculated from the ML-variance based volatilities. “Average values” of the parameters are calculated from the bootstrapping sampling distribution. The % difference between the point estimates and average values is given across the difference line. St. Dev, Skewness, Kurtosis and the Jarque-Berra (JB) normality test statistic are calculated from the bootstrap distribution. CV is a coefficient of variation derived as BSE/Average. JB test statistics are all highly significant, over 2,000, and are omitted. Left and right confidence intervals are calculated via parametric bootstrapping (BPCI), nonparametric bootstrapping (BNCI) and Lo’s 1st order asymptotic approximation (ACI), respectively. ASE and BSE are the asymptotic and bootstrap standard errors, respectively. Confidence intervals under BPCI and ACI are symmetric and assume normality. BPCI/BNCI-1 and BPCI/BNCI-1 give % differences between left (L) and right (R) confidence intervals, respectively. Bold letters express the maximum absolute value across each line for option prices and deltas, respectively.

4.2 American Options

The results with respect to the distribution of American option prices under different levels of moneyness are presented in Table 3. In order to ease presentation, we now present only three scenarios of moneyness levels. We analyse both call and put options since, as mentioned previously, no put-call parity relationship holds. However, we do not examine now the hedge parameter delta since it requires numerical differentiation that will induce additional approximation errors to the sampling distributions under examination.

We see that at-the-money calls/puts exhibit the largest standard errors, as for their European counterparts, but not the smallest biases. As before, this effect is induced by the behaviour of the partial derivative functions of the option price with respect to variance. Asymptotic and parametric confidence intervals fail once again in the out-of-the-money area in that they allow negative prices. Unsurprisingly, the confidence intervals of American and European option prices are very similar. The deep in-the-money American call option price is almost insensitive to variance uncertainty as suggested by the very small standard error. This is expected since it will reach its intrinsic value at this volatility level and time to maturity, and, since the dividend yield exceeds the risk free rate, the option should be exercised immediately. Similar results would have occurred for the put option should the risk free rate be greater than the dividend yield. As before, in Table 4 we summarise the results of altering the option pricing parameters for both out-of-the-money American calls and puts with respect to: time to maturity, volatility, risk-free rate and dividend yield. In general, results are comparable to those obtained for the European option case in Table 2. More specifically, both call and put American option price biases become smaller with increases in time to maturity and volatility. Mixed results are obtained for increases in the two remaining parameters under study with respect to biases in prices and confidence intervals. It appears that an increase in interest rates increases (decreases) the bias and confidence intervals of American call (put) options while the inverse effect holds for shifts in the dividend yield. This can be explained by the sensitivity of the second derivative of the option price, with respect to variance, to changes in interest rates and divided yields, respectively.

Table 4. American Call/Put Option Prices under different pricing assumptions

	Benchmark		Time		Volatility		Interest		Dividend	
10% out of the money calls and puts	**T-t = 0.25 σ = 10% r=1% q = 2%**		**T-t = 0.5 σ = 10% r=1% q = 2%**		**T-t = 0.25 σ = 20% r=1% q = 2%**		**T-t = 0.25 σ = 10% r=5% q = 2%**		**T-t = 0.25 σ = 10% r=1% q = 5%**	
	Call	***Put***	***Call***	***Put***	***Call***	***Put***	***Call***	***Put***	***Call***	***Put***
Point Estimate	0.585	0.397	3.005	2.662	**10.287**	**8.424**	0.949	0.251	0.430	0.587
Average	0.620	0.427	3.043	2.697	**10.249**	**8.402**	0.986	0.277	0.459	0.620
Difference	-5.72%	-7.10%	-1.24%	-1.29%	0.37%	0.26%	-3.73%	**-9.14%**	**-6.45%**	-5.26%
ASE	0.468	0.369	1.217	1.093	**2.571**	**2.22**	0.595	0.283	0.392	0.447
BSE	0.324	0.241	1.088	0.975	**2.578**	**2.212**	0.455	0.168	0.252	0.317
CV	52.26%	56.44%	35.75%	36.15%	25.15%	26.33%	46.15%	**60.65%**	**54.90%**	51.13%
Skewness	0.928	**1.030**	0.501	0.507	0.236	0.264	0.759	1.194	**1.028**	0.889
Kurtosis	1.199	1.534	0.216	0.223	0.089	0.106	0.728	**2.181**	**1.515**	1.080
JB	1,017.15	1,374.32	218.89	224.57	47.962	60.375	590.48	**2,179.02**	**1,358.83**	901.60
BPCI Left	-0.015	-0.045	0.910	0.787	**5.197**	**4.067**	0.095	-0.053	-0.034	-0.001
BPCI Right	1.255	0.899	5.176	4.607	**15.301**	**12.738**	1.877	0.606	0.953	1.241
BNCI Left	0.150	0.251	1.224	1.071	**5.495**	**4.354**	0.288	0.053	0.105	0.155
BNCI Right	1.371	0.179	5.377	4.790	**15.536**	**12.967**	2.010	0.681	1.063	1.350
ACI Left	-0.332	-0.326	0.620	0.520	**5.248**	**4.073**	-0.217	-0.304	-0.339	-0.326
ACI Right	1.502	1.120	5.390	4.804	**15.326**	**12.775**	2.115	0.806	1.198	1.502
BPCI/BNCI-1 (L)	-110.04%	-117.75%	-25.69%	-26.55%	-5.42%	-6.59%	67.11%	**-200.1%**	**-132.90%**	-100.49%
BPCI/BNCI-1 (R)	-8.45%	**402.47%**	-3.75%	-3.82%	-1.51%	-1.77%	-6.62%	-11.05%	**-10.40%**	-8.05%
BPCI/ACI-1 (L)	-95.49%	**-86.21%**	46.85%	51.43%	-0.97%	-0.14%	**-143.74%**	-82.55%	-89.84%	51.35%
BPCI/ACI-1 (R)	-16.46%	-19.75%	-3.98%	-4.11%	-0.16%	-0.29%	-11.26%	**-24.78%**	**-20.47%**	-16.46%

Variance is calculated using 3 months worth of data (6.10.03 – 31.12.03) either via maximum likelihood or via bootstrapping simulation using 5,000 simulations. "Point estimates" of the parameters are calculated from the ML-variance based volatilities. "Average values" of the parameters are calculated from the bootstrapping sampling distribution. The % difference between the point estimates and average values is given across the difference line. St. Dev, Skewness and Kurtosis are calculated from the bootstrap distribution. CV is a coefficient of variation derived as BSE/Average. Left and right confidence intervals are calculated via parametric bootstrapping (BPCI), nonparametric bootstrapping (BNCI) and Lo's 1st order asymptotic approximation (ACI), respectively. ASE and BSE are the asymptotic and bootstrap standard errors, respectively. Confidence intervals under BPCI and ACI are symmetric and assume normality. BPCI/BNCI-1 and BPCI/BNCI-1 give % differences between left (L) and right (R) confidence intervals, respectively. Bold letters express the maximum absolute value across each line for option prices and deltas, respectively.

4. Conclusions

This paper proposed the use of statistical bootstrap simulation in inferring the finite sample distribution of option prices and hedge parameters when an estimate of variance is used as a proxy for the true unobserved variance. After a brief literature review, we discuss this methodology in comparison to the standard approach in the literature, which is based on asymptotic theory (Lo, 1986). Like the asymptotic approach, bootstrapping has the advantage that limiting distributions can be derived regardless of whether the option is priced in closed form or numerically. Despite high computational costs, the bootstrap has advantages in terms of excellent small sample performance and computational simplicity. It also has the advantage of providing sampling distributions for option prices in finite samples that are consistent with no-arbitrage bounds.

We demonstrate an application of the proposed approach using data on the S&P500 index. We explore differences between option prices obtained from the Black-Scholes formula, assuming a point estimate of variance, and, average Black-Scholes option prices, derived from the bootstrap distribution. We also study the confidence intervals of option prices obtained from three different approaches: parametric bootstrap, nonparametric bootstrap and asymptotic. In addition to what has been considered in the literature, we also investigate the behaviour of European option hedge parameters (deltas) and American call and put option prices under different levels of moneyness. Finally, we look into the effect of shifting time to maturity, volatility, interest rates and dividend yields. Extending previous findings (see for example, Knight and Satchell, 1997), we find that the deltas and option prices obtained from the bootstrap approach are marginally different than those obtained using a point-estimate (asymptotic) approach. In accordance to Ncube and Satchell (1997), we find that both the asymptotic and parametric bootstrap approach may yield European and American option price confidence intervals that include negative values. On the contrary, the nonparametric bootstrap gives confidence intervals that are consistent with no-arbitrage bounds. Contrary to the conjecture of Satchell and Knight (1997), we find that biases in option prices due to variance uncertainty are not necessarily more pronounced as time to maturity and volatility increase.

This paper does not suggest that the bootstrap approach has clear advantages over the asymptotic approach. The parameteric analysis of large sample theory offers a powerful, elegant and well researched framework. However, the bootstrap can supplement this analysis and offer highly intuitive insights in the problems underhand. We argued that any problems that may arise from the asymptotic approach with respect to variance uncertainty will become less pronounced as time to maturity, volatility and risk-free rates increase and dividends decrease, respectively. In purely practical terms, we think that the bootstrap is especially well suited for sampling distribution problems in option pricing since it is particularly difficult to calculate asymptotic distributions. We believe that further research in this area is justified. It would also make sense to apply the methodology to more complicated option pricing models than those studied here, involving say many parameters or multiple dimensions (eg, basket options, stochastic volatility, etc). Following the suggestion of Satchell and Knight (1997), our present efforts are concentrated on investigating sampling distributions of real option prices where implied volatilities are not available and historical variances are estimated with great risk. It would be interesting to explore the application of the bootstrap methodology in other sampling distribution problems in finance, for example, risk management and portfolio theory (eg, Lo, 2003). Although all this literature has concentrated on historical variance risk, it would be also useful to look into the effect of implied volatility estimation risk on the sampling distribution of option prices. It is important to emphasise that implied volatilities do not resolve the problem of estimation risk since they are also subject to considerable error when option characteristics are observed with plausible errors (see Hentschel, 2003). Finally, rather than concentrating only on issues of estimation and bias, one could look directly to decision problems underhand, for example, hedging, with respect to estimation risk.

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