Application of Fourier Inversion Methods to Credit Portfolio Models with Integrated Interest Rate and Credit Spread Risk

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Abstract:

Most credit portfolio models currently used by the banking industry rely on Monte Carlo simulations for calculating the probability distribution of the future credit portfolio value, which can be quite computer time consuming. Adding market risk factors, such as stochastic interest rates or credit spreads, as additional ingredients of a credit portfolio model, the computational burden of full Monte Carlo simulations even increases and the need for efficient methods for calculating credit risk measures becomes even more obvious. In this study, based on a version of the well-known credit portfolio model CreditMetrics extended by correlated interest rate and credit spread risk, it is analyzed whether the use of characteristic functions and inverse Fourier transformation can be an efficient tool for calculating risk measures in the context of integrated credit portfolio models. Unfortunately, the characteristic function of the credit portfolio value at the risk horizon can not be calculated in closed-form, but has to be computed by Monte Carlo simulations. However, this method can be much faster than a full Monte Carlo simulation of the future credit portfolio distribution. The accuracy of the method depends on the composition of the portfolio.

Keywords:
credit risk, interest rate risk, credit spread risk, credit portfolio model, Value at Risk, characteristic function, inverse Fourier transforms

JEL classification: C 63, G 21

I. Introduction

For calculating risk measures of credit portfolios, such as Value-at-Risk or expected shortfall, a range of models have been developed. Prominent examples are CreditMetrics by J.P. Morgan Chase, CreditPortfolioView by McKinsey, Portfolio Manager by KMV or CreditRisk$^+$ by CSFP. With the exception of CreditRisk$^+$ and one version of the Portfolio Manager all these models rely on Monte Carlo simulations for calculating the probability distribution of the future credit portfolio value, which can be quite computer time consuming, especially for portfolios with many obligors and when percentiles corresponding to high confidence levels have to be estimated with sufficient accuracy.

A typical shortcoming of most credit portfolio models currently used in the banking industry is that relevant risk factors, such as interest rates or credit spreads, are not modeled as stochastic terms and hence are ignored during the revaluation of the credit sensitive instruments at the risk horizon. For example, fixed income instruments, such as bonds or loans, are revalued at the risk horizon using the current forward rates and (rating class specific) forward credit spreads for discounting future cash flows. Thus, the stochastic nature of the instrument's value in the future which results from changes in factors other than credit quality is ignored, and the riskiness of the credit portfolio at the risk horizon can be underestimated. An additional consequence is that correlations between changes of the credit quality of the debtors and changes of market risk factors and hence the exposure at default cannot be integrated into the credit portfolio model. This is especially a problem for market-driven instruments, such as interest rate derivatives. Finally, correlations between the exposures at default of different instruments, which depend on the same or correlated market risk factors, cannot be modeled, too. Various studies[1] showed that the missing stochastic modeling of market risk factors or credit spread risk causes a severe underestimation of economic capital, especially for high grade credit portfolios with a low stochastic dependence between the obligors' credit quality changes.

Adding market risk factors, such as stochastic interest rates or credit spreads, as additional ingredients of a credit portfolio model, the computational burden of full Monte Carlo simulations increases and the need for efficient methods for calculating credit risk measures becomes even more obvious. The aim of this study is to analyze whether the use of characteristic functions and inverse Fourier transformation can be an efficient tool for calculating risk measures in the context of a credit portfolio model with integrated market risk. This technique has already been successfully applied to market risk portfolio models.[2] But there are only a few papers which are concerned with the application of this method to credit risk portfolio models, especially with integrated market or credit spread risk.[3] However, it is certainly beyond the scope of this paper to present a full comparison of *all* methods which might improve the efficiency of risk measure calculations in credit portfolio models with integrated market risk factors. Possible further candidates would be saddle-point methods[4], granularity adjustment techniques[5] or Monte Carlo simulations combined with suitable variance reduction techniques[6].

This paper is structured as follows: In section II a short overview of the computational approach is given. Then, in section III a general version of an integrated market and credit portfolio model is presented. Afterwards it is shown how industry standards fit into this general model and, finally, as a special case of the general framework the CreditMetrics model is extended by correlated interest rate and credit spread risk. The use of characteristic functions and inverse Fourier transformation is explained when this specific modeling framework is applied to a portfolio of defaultable zero coupon bonds. Section IV contains a discussion of necessary changes of the approach when some of the assumptions of the previous section are modified. In particular, the case that the portfolio is composed of European call options with counterparty risk is analyzed. The differences between the percentiles of the future credit portfolio distribution when calculated either by a full Monte Carlo simulation or by the method proposed in sections III and IV are presented within a numerical example in section V. Finally, in section VI the main results are summarized.

II. General Computation Approach

The characteristic function of a continuous random variable X with density function $f(x)$ is a complex-valued function defined as:[7]

$$\varphi_X(s) := E\left[e^{i\cdot s\cdot X}\right] = \int_{-\infty}^{\infty} e^{i\cdot s\cdot x}\cdot f(x)dx = \int_{-\infty}^{\infty}\cos(s\cdot x)\cdot f(x)dx + i\int_{-\infty}^{\infty}\sin(s\cdot x)\cdot f(x)dx\,, \tag{2.1}$$

where $s\in\mathbb{R}$ and $i=\sqrt{-1}$ is the imaginary unit. Up to the real argument s which is replaced by $i\cdot s$ the characteristic function equals the moment generating function, but it has the advantage that, as a consequence of the boundedness of $e^{i\cdot s\cdot x}$, it always exists. In a non-probabilistic context the characteristic function $\varphi_X(s)$ (up to a factor $1\big/\sqrt{2\pi}$) is called the *Fourier transform* of the (density) function $f(x)$. Two fundamental properties of characteristic functions are used in the following. First, the characteristic function of a sum of independent random variables equals the product of the characteristic functions of the individual random variables. Second, the characteristic function of a random variable uniquely determines its probability distribution, which can be recovered from the characteristic function for example by the following inversion formulas:[8]

$$P(X < x) \equiv F(x) = \frac{1}{2} + \frac{1}{2\pi}\int_0^{\infty}\frac{e^{i\cdot s\cdot x}\cdot\varphi_X(-s) - e^{-i\cdot s\cdot x}\cdot\varphi_X(s)}{i\cdot s}\,ds\,, \tag{2.2}$$

or[9]

$$P(X < x) \equiv F(x) = \frac{1}{2} - \frac{1}{\pi}\int_0^\infty \mathrm{Im}\left(\frac{e^{-i\cdot s\cdot x}\cdot\varphi_X(s)}{s}\right) ds\,, \tag{2.3}$$

and, supposed $|\varphi_X(s)|$ is integrable, the density function of the random variable X is given by

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\cdot t\cdot x}\cdot\varphi_X(t)\, dt\,, \tag{2.4}$$

which is also called (up to a factor $1/\sqrt{2\pi}$) the *inverse Fourier transform*.

III. The Integrated Market and Credit Portfolio Model

III.1 General Approach

It is assumed that the credit portfolio consists of N market and credit risk sensitive instruments issued by N different corporates. The risk horizon H of the credit portfolio model is one year and P denotes the real world probability measure. The number of possible credit qualities at the risk horizon is K, where for default mode credit portfolio models we have $K = 2$ and for mark-to-market credit portfolio models $K > 2$.

The central part of most credit portfolio models currently used is the definition of the obligors' conditional default and transition probabilities. Denoting by $\eta_H^n \in \{1,\ldots,K\}$ the credit quality of obligor n at the risk horizon H and by η_0^n the respective rating at $t = 0$, the conditional default (transition) probabilities are formally defined as:

$$P\left(\eta_H^n = k \middle| \eta_0^n = i, Z_1 = z_1,\ldots,Z_C = z_C\right) := f_{n,i,k}(z_1,\ldots,z_C) \text{ with } f_{n,i,k} : \mathbb{R}^C \to [0,1] \tag{3.1}$$

($k \in \{1,\ldots,K\}$, $i \in \{1,\ldots,K-1\}$, $n \in \{1,\ldots,N\}$).

The set of variables $\mathrm{Z} = (Z_1, \ldots, Z_C) \sim F^C$ are systematic credit risk factors that might be thought of as changes in equity indices or macro-economic variables within the risk horizon, and which influence credit quality changes of all obligors within the risk horizon. This vector is assumed to evolve according to the multivariate distribution F^C. Given the realization $(Z_1 = z_1, \ldots, Z_C = z_C)$ of the systematic credit risk factors and hence of the conditional default (transition) probabilities, credit quality changes of all obligors are assumed to be stochastically independent. Thus, this is a classical 'conditional independence'-framework for describing joint credit quality changes within a credit portfolio. Sampling from the N discrete distributions (3.1), the credit quality of all obligors at the risk horizon can be simulated for a specific scenario $(Z_1 = z_1, \ldots, Z_C = z_C)$.

The price of the instrument i_n (e.g. defaultable (zero) coupon bonds or options with counterparty risk) at the risk horizon H, whose issuer n has not already defaulted before H and exhibits the rating $\eta_H^n \in \{1, \ldots, K-1\}$, is denoted by

$$p_n(\eta_H^n; X_1, \ldots, X_M; \mathrm{P_n}), \tag{3.2}$$

where the stochastic vector $\mathrm{X} = (X_1, \ldots, X_M) \sim F^M$ represents the value of relevant market risk factors, such as e.g. risk-free interest rates, at the risk horizon. This vector is assumed to evolve according to the multivariate distribution F^M. $\mathrm{P_n}$ denotes a vector of additional parameters relevant for the pricing of the respective instrument i_n at the risk horizon. Note that the set of systematic credit risk factors $Z_1, \ldots, Z_C$ and the set of market risk factors $X_1, \ldots, X_M$ can overlap, e.g. if a risk-free interest rate is also a relevant credit risk driver. The joint distribution of the stochastic vector $(Z_1, \ldots, Z_C; X_1, \ldots, X_M)$ is denoted by F.

If the issuer n of the instrument i_n has already defaulted ($\eta_H^n = K$) before the risk horizon H, its value, in the case this value is positive, is set equal to a fraction δ of the value the instrument would have at the risk horizon, when its issuer would be free of default risk. If the market value of this instrument is negative, nothing is changed, because the bank whose credit portfolio is considered is a debtor of the defaulted issuer. The shape of the distribution of the recovery rate can vary with the seniority of a claim and the value of individual collaterals. For all defaulted issuers the recovery rate is drawn individually which ensures independence of the recovery rates across the different exposures. Usually, it is assumed that the recovery rate is beta-distributed and independent from all other stochastic variables of the respective model, such as the systematic credit risk drivers or the market risk factors, but it could also be a function of these risk factors.[10]

Finally, the value $\Pi(H)$ of the entire portfolio at the risk horizon H is just the sum over the individual values:

$$\Pi(H) = \sum_{n=1}^{N} p_n(\eta_H^n; X_1, \ldots, X_M; \mathrm{P_n}) . \tag{3.3}$$

III.2 Industry Standards as Special Cases

Industry standards, such as the well-known credit portfolio models CreditMetrics by J.P. Morgan, CreditPortfolioView by McKinsey or CreditRisk$^+$ by CSFP, can be seen as a special case of the general modeling approach described above. All of these models have in common that stochastic fluctuations of market risk factors are not considered for the re-pricing of the instruments at the risk horizon:

$$p_n(\eta_H^n; X_1, \ldots, X_M; \mathrm{P_n}) = p_n(\eta_H^n; \mathrm{P_n}) . \tag{3.4}$$

For example, for pricing a corporate bond random interest rates are ignored and instead risk-adjusted forward rates are employed for discounting future cash flows of the bond which are due beyond the risk horizon. However, these models also differ in the assumptions concerning the functional form $f_{n,i,k}(\cdot)$ of the conditional default (transition) probabilities and the distribution of the systematic credit risk factors $Z_1,\ldots,Z_C$.

For example, in the CreditMetrics model all risk factors are normally distributed, $\mathrm{Z}=(Z_1,\ldots,Z_C)\sim N(0,\Sigma_Z)$ with the diagonal elements of $\Sigma_Z\in\mathbb{R}^{C\times C}$ equal to one, and the conditional transition probabilities for migrating from credit quality i to one of the K credit qualities, which correspond to rating grades, within the risk horizon are given by:

$$
\begin{aligned}
&P\left(\eta_H^n=k \middle| \eta_0^n=i, Z_1=z_1,\ldots,Z_C=z_C\right)\\
&=\begin{cases}
\Phi\left(\dfrac{R_K^i-\sum_{i=1}^{C}\omega_{n,i}\cdot z_i}{\omega_{n,C+1}}\right) & \text{for } k=K\\
\Phi\left(\dfrac{R_k^i-\sum_{i=1}^{C}\omega_{n,i}\cdot z_i}{\omega_{n,C+1}}\right)-\Phi\left(\dfrac{R_{k+1}^i-\sum_{i=1}^{C}\omega_{n,i}\cdot z_i}{\omega_{n,C+1}}\right) & \text{for } k\in\{2,\ldots,K-1\}\\
1-\Phi\left(\dfrac{R_2^i-\sum_{i=1}^{C}\omega_{n,i}\cdot z_i}{\omega_{n,C+1}}\right) & \text{for } k=1,
\end{cases}
\end{aligned}
\tag{3.5}
$$

where $\Phi(\cdot)$ denotes the cumulative distribution function of the standard normal distribution and 1 is the best and K the worst credit quality, namely the default state. The thresholds R_k^i appearing in the above formula for the conditional transition probabilities are derived from an one-year transition matrix $Q=(q_{ik})_{1\le i\le K-1,1\le k\le K}$, whose elements q_{ik} specify the unconditional probability that an obligor migrates from the rating grade i to the rating grade k within one year (see table 1). The thresholds R_k^i ($1\le i\le K-1,\ 2\le k\le K$) are computed by ensuring that the probability for the realization of a standardized normally distributed random variable R_n, a latent variable, which is usually interpreted as the (standardized) return on firm n's assets within the risk horizon, to be in the interval $[R_{k+1}^i, R_k^i]$ coincides with the probability q_{ik} from the migration matrix:[11]

$$R_k^i = \Phi^{-1}\left(\sum_{l=k}^{K} q_{il}\right), \tag{3.6}$$

where $\Phi^{-1}(\cdot)$ denotes the inverse of the cumulative density function of the standard normal distribution. The weights $\omega_{n,1},\ldots,\omega_{n,C}$ are from a representation of the latent variable R_n as a linear combination of the systematic credit risk factors $Z_1,\ldots,Z_C$ and a standard normally distributed idiosyncratic risk factor ε_n specific to firm n. The size of these weights corresponds to the importance of the respective systematic credit risk factors for explaining the standardized volatility of the latent variable R_n. The weight $\omega_{n,C+1}$ for the idiosyncratic risk factor is chosen as a residual term in order to guarantee that the standard deviation of the normally distributed latent variable is one.

- insert table 1 about here -

In the MACRO-version of the model CreditPortfolioView the conditional default probability of a speculative grade obligor n belonging to a segment $s \in \{1,\ldots,S\}$ is assumed to be:

$$P\left(\eta_H^n = K \middle| \eta_0^n \in \{BB,\ldots,C\}, Z_{1,H} = z_{1,H},\ldots,Z_{C,H} = z_{C,H}\right) := \frac{1}{1+e^{b_{s,0}+b_{s,1}\cdot z_{1,H}+b_{s,2}\cdot z_{2,H}+\ldots+b_{s,C}\cdot z_{C,H}+\upsilon_{s,H}}}, \tag{3.7}$$

where the correlated systematic credit risk factors $Z_1,\ldots,Z_C$ are modeled by AR(2)-processes and $\upsilon_{s,H}$ ($s \in \{1,\ldots,S\}$) are normally distributed noise terms with mean zero, which are correlated across sectors. All obligors who belong to the same sector (e.g. a specific country and industry combination) are assumed to be statistically identical. In CreditPortfolioView the systematic credit risk factors $Z_1,\ldots,Z_C$ are usually macroeconomic variables, such as the GDP growth rate, the unemployment rate or the inflation rate. The factor weights have to be estimated by conducting a logit regression. The conditional default probability of speculative grade obligors is used for shifting the transition probabilities into the non-default states: If the conditional default probability is below its unconditional counterpart, downgrades become more probable and vice versa. These conditioned transition probabilities are then used for simulating the credit quality changes of the obligors in the portfolio within the risk horizon.

Finally, in the actuarial default mode model CreditRisk$^+$ the conditional default probability of obligor n is defined by:

$$P\left(\eta_H^n = K \middle| \eta_0^n \neq K, Z_{1,H} = z_{1,H}, \ldots, Z_{C,H} = z_{C,H}\right) := q_{n,K} \cdot \left(\omega_{n,1} \cdot \frac{z_1}{E[z_1]} + \ldots + \omega_{n,C} \cdot \frac{z_C}{E[z_C]}\right), \quad (3.8)$$

where the systematic credit risk factors $Z_1, \ldots, Z_C$ are independent gamma distributed, $q_{n,K}$ is obligor n's unconditional default probability and $\sum_{s=1}^{S} \omega_{n,s} = 1$, $\omega_{n,s} \geq 0 \quad \forall\, n \in \{1, \ldots, N\}$, $s \in \{1, \ldots, S\}$. The CreditRisk$^+$ model works with the additional assumption that, conditional on a scenario $Z_1, \ldots, Z_C$, the number of defaults in the portfolio can be approximated by a Poisson distribution (instead of employing the exact binomial distribution), whose intensity parameter equals the sum of the above individual conditional default probabilities. As a consequence, the unconditional distribution of the total number of defaults until the risk horizon can be calculated in closed-form and equals the negative binomial distribution.

III.3 CreditMetrics with Integrated Correlated Interest Rate and Credit Spread Risk

As a special case of the general integrated model described in section III.1, in this section the usual CreditMetrics framework is extended by correlated interest rate and credit spread risk and applied to a credit portfolio consisting of N zero coupon bonds with identical face value F and maturity date T issued by N different corporates. This specification of the general integrated model will be used in the following as a working example in order to demonstrate the use of the Fourier inversion method.

It is assumed that the return R_n on firm n's assets can be described by a normally distributed random variable, which is – without loss of generality – standardized:

$$R_n = \sqrt{\rho_V - \rho_{r,V}^2} \cdot Z + \rho_{r,V} \cdot X_r + \sqrt{1-\rho_V} \cdot \varepsilon_n \qquad (\rho_{r,V}^2 \le \rho_V,\ n \in \{1,\ldots,N\}),^{12} \tag{3.9}$$

where $Z, X_r, \varepsilon_1,\ldots,\varepsilon_N$ are mutually independent standard normally distributed stochastic variables. The stochastic variables Z and X_r represent systematic credit risk, by which all firms are affected, whereas the ε_n's stand for idiosyncratic credit risk.

The risk-free short rate is modeled for simplicity as a mean-reverting Ornstein-Uhlenbeck process introduced already by *Vasicek* (1977):

$$dr(t) = \kappa \cdot (\theta - r(t)) \cdot dt + \sigma_r \cdot dW_r(t)\,, \tag{3.10}$$

where $\kappa, \theta, \sigma \in \mathbb{R}_+$ are constants and $W_r(t)$ is a standard Brownian motion under P. The process $(r(t))_{t \in \mathbb{R}_+}$ always tends back to the mean level θ; the higher the value κ the more unlikely are deviations from this level. The solution of the stochastic differential equation (3.10) is:

$$r(t) = \underbrace{\theta + (r(0) - \theta) \cdot e^{-\kappa \cdot t}}_{=E^P[r(t)]} + \sqrt{\frac{\sigma_r^2}{2\kappa} \cdot \left(1 - e^{-2 \cdot \kappa \cdot t}\right)} \cdot X_r\,, \tag{3.11}$$

where $X_r \sim N(0,1)$ enters the definition (3.9) of the firms' asset returns. As it can be easily seen, the definition (3.9) of the asset returns implies that all pairs of asset returns exhibit a correlation parameter of ρ_V and that the asset returns R_n and the interest rate factor X_r (and hence the short rate $r(H)$) are correlated with parameter $\rho_{r,V}$. In this section, it is assumed that the correlation ρ_V between each pair of asset returns as well as the correlation $\rho_{r,V}$ between each asset return and the risk-free short rate are identical.

As in the CreditMetrics methodology, the rating η_H^n of the N obligors at the risk horizon $t = H$ is simulated by the N-variate normally distributed random vector $R = (R_1, \ldots, R_N)$, whose components exhibit means zero, variances one and equal pairwise correlations ρ_V. An obligor n with current rating i is assumed to be in rating class k at the risk horizon if the realization of R_n lies between two thresholds R_{k+1}^i and R_k^i with $R_{k+1}^i < R_k^i$.[13]

The price of a zero coupon bond at the risk horizon H, whose issuer n has not already defaulted until H and exhibits the rating $\eta_H^n \in \{1, \ldots, K-1\}$, is given by:

$$v(X_r, \eta_H^n, H, T) = F \cdot e^{-\left(R(X_r, H, T) + S_{\eta_H^n}(H,T)\right) \cdot (T-H)}, \tag{3.12}$$

where $R(X_r, H, T)$ denotes the stochastic risk-free spot yield for the time interval $[H, T]$, and $S_{\eta_H^n}(H,T)$ is the stochastic credit spread of rating class η_H^n for the time interval $[H, T]$. In the *Vasicek* model the stochastic risk-free spot yield $R(X_r, H, T)$ can easily be calculated in closed-form[14] and is a linear function of the risk factor X_r appearing in (3.11). The rating specific credit spreads $S_{\eta_H^n}(H,T)$ ($\eta_H^n \in \{1, \ldots, K-1\}$) are assumed to be multivariate normally distributed random variables.[15] Furthermore, it is assumed that the random variable X_r, which drives the term structure of risk-free interest rates, and the systematic credit risk factor Z respectively are both correlated with the credit spreads. For the sake of simplicity, these correlation parameters are set equal to constants $\rho_{X_r,S}$ and $\rho_{Z,S}$ respectively, regardless of the rating grade. Besides, it is assumed that the idiosyncratic credit risk factors ε_n ($n \in \{1, \ldots, N\}$) are independent of the credit spreads $S_k(H,T)$ ($k \in \{1, \ldots, K-1\}$).

If the issuer n of a zero coupon bond has already defaulted ($\eta_H^n = K$) until the risk horizon H, the value of the bond is set equal to a beta-distributed fraction δ_n of the value $p(X_r, H, T)$ of a risk-free but otherwise identical zero coupon bond:[16]

$$v_n(X_r, K, H, T) = \delta_n \cdot p(X_r, H, T), \tag{3.13}$$

where $E\left[\delta_n\right] = \mu_\delta$ and $Var(\delta_n) = \sigma_\delta^2$. These first two moments of the distribution of the recovery rate can vary with the seniority of a claim and the value of individual collaterals. For simplicity, we use a uniform recovery rate distribution for all issuers, but for each defaulted issuer a beta-distributed recovery rate is drawn individually which ensures independence of the recovery rates across the different exposures. The recovery rate is assumed to be independent from all other stochastic variables of the model, e.g. the systematic credit risk factors Z and X_r, the idiosyncratic risk factors ε_n and the credit spreads $S_k(H,T)$ ($k \in \{1, \ldots, K-1\}$).

The value $\Pi(H)$ of the entire portfolio of defaultable zero coupon bonds at the risk horizon H is:

$$\Pi(H) = \sum_{n=1}^{N} \sum_{k=1}^{K-1} v(X_r, k, H, T) \cdot 1_{\{\eta_H^n = k\}} + \delta_n \cdot p(X_r, H, T) \cdot 1_{\{\eta_H^n = K\}}$$

$$= \sum_{n=1}^{N} \sum_{k=1}^{K-1} F \cdot e^{-R(X_r, H, T)\cdot(T-H)} \cdot \left(e^{-S_k(H,T)\cdot(T-H)} \cdot 1_{\{\eta_H^n = k\}} + \delta_n \cdot 1_{\{\eta_H^n = K\}} \right), \tag{3.14}$$

where the indicator function $1_{\{\eta_H^n = k\}}$ is one if obligor n is in the rating class k at $t = H$ and zero otherwise.

The probability of migrating from rating class i to $k \in \{2, \ldots, K-1\}$ until the risk horizon H, conditional on the realizations of the systematic credit risk factors Z and X_r, is given by

$$f_{i,k}(z,x_r) := P\left(\eta_H^n = k \middle| \eta_0^n = i, Z = z, X_r = x_r\right) = P\left(R_{k+1}^i < R_n \le R_k^i \middle| Z = z, X_r = x_r\right)$$

$$= \Phi\underbrace{\left(\frac{R_k^i - \sqrt{\rho_V - \rho_{r,V}^2}\cdot z - \rho_{r,V}\cdot x_r}{\sqrt{1-\rho_V}}\right)}_{=:t_{i,k}(z,x_r)} - \Phi\underbrace{\left(\frac{R_{k+1}^i - \sqrt{\rho_V - \rho_{r,V}^2}\cdot z - \rho_{r,V}\cdot x_r}{\sqrt{1-\rho_V}}\right)}_{=:\ t_{i,k+1}(z,x_r)}. \tag{3.15}$$

The conditional default probability is

$$f_{i,K}(z,x_r) := P\left(R_n \le R_K^i \middle| Z = z, X_r = x_r\right) = \Phi\underbrace{\left(\frac{R_K^i - \sqrt{\rho_V - \rho_{r,V}^2}\cdot z - \rho_{r,V}\cdot x_r}{\sqrt{1-\rho_V}}\right)}_{=:\ t_{i,K}(z,x_r)}, \tag{3.16}$$

and the conditional probability of being in the best rating class 1 equals

$$f_{i,1}(z,x_r) := P\left(R_n > R_2^i \middle| Z = z, X_r = x_r\right) = 1 - \Phi\underbrace{\left(\frac{R_2^i - \sqrt{\rho_V - \rho_{r,V}^2}\cdot z - \rho_{r,V}\cdot x_r}{\sqrt{1-\rho_V}}\right)}_{=:\ t_{i,2}(z,x_r)}. \tag{3.17}$$

As (3.15), (3.16) and (3.17) show, the specification (3.9) of the multi-factor model for the individual asset returns implies that the transition process and the term structure of risk-free interest rates are correlated. The degree of correlation is determined by the value of the sensitivity $\rho_{r,V}$. As it is assumed that the random variable X_r and the systematic credit risk factor Z respectively are correlated with the credit spreads, we even have a model, in which the transition process, the risk-free interest rates and the credit spreads are all pairwise correlated.[17]

III.4 Application of Inverse Fourier Transformation

Next, the method of characteristic functions and inverse Fourier transformation are applied to the model and the portfolio described in section III.3 in order to accelerate the computation of the probability distribution of $\Pi(H)$.[18] Conditional on the realizations of the stochastic variables Z, X_r and $S = (S_1(H,T),\ldots,\ S_{K-1}(H,T))$ all N summands of the outer sum in (3.14) are independent because the only remaining stochastic variables are the independent idiosyncratic risk factors ε_n and the independent recovery rates δ_n ($n \in \{1,\ldots,N\}$). Hence, at first the conditional characteristic function of the value of a single instrument is computed, where the initial rating of all obligors is assumed to be $j \in \{1,\ldots,K-1\}$:

$$\varphi_{\sum_{k=1}^{K-1} F\cdot e^{-R(X_r,H,T)\cdot(T-H)}\cdot\left(e^{-S_k(H,T)\cdot(T-H)}\cdot 1_{\{\eta_H^n=k\}}+\delta_n\cdot 1_{\{\eta_H^n=K\}}\right)\Big|Z,X_r,S}(s)$$

$$= E\left[e^{i\cdot s\cdot\left(\sum_{k=1}^{K-1} F\cdot e^{-R(x_r,H,T)\cdot(T-H)}\cdot\left(e^{-s_k(H,T)\cdot(T-H)}\cdot 1_{\{\eta_H^n=k\}}+\delta_n\cdot 1_{\{\eta_H^n=K\}}\right)\right)} \middle| Z=z, X_r=x_r, S=\left(s_1(H,T),\ldots,s_{K-1}(H,T)\right)\right]$$

$$= \int_0^1 \int_{-\infty}^{\infty} e^{i\cdot s\cdot\left(F\cdot e^{-R(x_r,H,T)\cdot(T-H)}\cdot\left(e^{-s_1(H,T)\cdot(T-H)}\cdot 1_{\{\varepsilon_n>t_{j,2}\}}+\sum_{k=2}^{K-1} e^{-s_k(H,T)\cdot(T-H)}\cdot 1_{\{t_{j,k+1}<\varepsilon_n\le t_{j,k}\}}+\delta_n\cdot 1_{\{\varepsilon_n\le t_{j,K}\}}\right)\right)}\cdot\phi(\varepsilon_n)\cdot\beta(\delta_n)\,d\varepsilon_n d\delta_n,$$ [19]

where $\phi(\varepsilon_n)$ and $\beta(\delta_n)$ denote the density functions of the random variables ε_n and δ_n respectively. Splitting up the integration path of ε_n yields:

$$\int_0^1 \left\{ \int_{-\infty}^{t_{j,K}} e^{i\cdot s\cdot\delta_n\cdot F\cdot e^{-R(x_r,H,T)\cdot(T-H)}}\cdot\phi(\varepsilon_n)\,d\varepsilon_n + \sum_{k=2}^{K-1}\int_{t_{j,k+1}}^{t_{j,k}} e^{i\cdot s\cdot F\cdot e^{-(R(x_r,H,T)+s_k(H,T))\cdot(T-H)}}\cdot\phi(\varepsilon_n)\,d\varepsilon_n \right.$$
$$\left. + \int_{t_{j,2}}^{\infty} e^{i\cdot s\cdot F\cdot e^{-(R(x_r,H,T)+s_1(H,T))\cdot(T-H)}}\cdot\phi(\varepsilon_n)\,d\varepsilon_n \right\}\cdot\beta(\delta_n)\,d\delta_n$$

$$= \int_0^1 \left\{ e^{i\cdot s\cdot\delta_n\cdot F\cdot e^{-R(x_r,H,T)\cdot(T-H)}}\cdot\Phi(t_{j,K}) + \sum_{k=2}^{K-1} e^{i\cdot s\cdot F\cdot e^{-(R(x_r,H,T)+s_k(H,T))\cdot(T-H)}}\cdot\left(\Phi(t_{j,k})-\Phi(t_{j,k+1})\right) \right.$$
$$\left. + e^{i\cdot s\cdot F\cdot e^{-(R(x_r,H,T)+s_1(H,T))\cdot(T-H)}}\cdot\left(1-\Phi(t_{j,2})\right)\right\}\cdot\beta(\delta_n)\,d\delta_n$$

$$= \varphi_\delta\left(s\cdot F\cdot e^{-R(x_r,H,T)\cdot(T-H)}\right)\cdot\Phi(t_{j,K}) + \sum_{k=2}^{K-1} e^{i\cdot s\cdot F\cdot e^{-(R(x_r,H,T)+s_k(H,T))\cdot(T-H)}}\cdot\left(\Phi(t_{j,k})-\Phi(t_{j,k+1})\right)$$
$$+ e^{i\cdot s\cdot F\cdot e^{-(R(x_r,H,T)+s_1(H,T))\cdot(T-H)}}\cdot\left(1-\Phi(t_{j,2})\right), \tag{3.18}$$

where $\varphi_\delta(\cdot)$ denotes the characteristic function of the beta-distributed recovery rate δ_n.[20] Note that as a consequence of the homogeneity assumption the term (3.18) does not depend on the identity of the obligor. Because of the conditional independence of the summands in the outer sum of (3.9) the conditional characteristic function of $\Pi(H)$ can be written as:

$$\varphi_{\Pi(H)|Z,X_r,S}(s)=\prod_{n=1}^{N}\varphi_{\sum_{k=1}^{K-1}F\cdot e^{-R(X_r,H,T)\cdot(T-H)}\cdot\left(e^{-S_k(H,T)\cdot(T-H)}\cdot 1_{\{\eta_H^n=k\}}+\delta_n\cdot 1_{\{\eta_H^n=K\}}\right)\Big|Z,X_r,S}(s)$$

$$\underset{Homog.}{=}\left(\varphi_\delta\left(s\cdot F\cdot e^{-R(X_r,H,T)\cdot(T-H)}\right)\cdot\Phi(t_{j,K})+\sum_{k=2}^{K-1}e^{i\cdot s\cdot F\cdot e^{-(R(X_r,H,T)+S_k(H,T))\cdot(T-H)}}\cdot\left(\Phi(t_{j,k})-\Phi(t_{j,k+1})\right)\right.$$
$$\left.+e^{i\cdot s\cdot F\cdot e^{-(R(X_r,H,T)+S_1(H,T))\cdot(T-H)}}\cdot\left(1-\Phi(t_{j,2})\right)\right)^N. \tag{3.19}$$

Finally, the unconditional characteristic function of $\Pi(H)$ is the expectation of the conditional characteristic function $\varphi_{\Pi(H)|Z,X_r,S}(s)$:

$$\varphi_{\Pi(H)}(s)=E\left[\varphi_{\Pi(H)|Z,X_r,S}(s)\right]. \tag{3.20}$$

Unfortunately, the above expectation can not be calculated in closed-form, but has to be computed by Monte Carlo simulations. Of course, with one drawn sample of Z, X_r and $S=(S_1(H,T),\ldots,S_{K-1}(H,T))$ the conditional characteristic function $\varphi_{\Pi(H)|Z,X_r,S}(s)$ can be computed for several values of s. Finally, having calculated $\varphi_{\Pi(H)}(s)$, we get the distribution function of the credit portfolio value $\Pi(H)$ via formula (2.3) and numerical integration. Of course, it is not completely satisfying that we still have to employ simulation methods in order to compute the characteristic function (3.20), but, as the numerical example in section V will show, this method can be much faster than a full Monte Carlo simulation of the future credit portfolio value. However, the speed gain depends on the number of grid points s needed for the numerical integration in (2.3) because with a large number of grid points also a large number of expectations has to be calculated and hence a large number of function evaluations has to be done, which is a potential drawback of the method described above. Thus, the use of an numerical integration rule which only needs a moderate number of grid points for a sufficient accuracy is essential.

IV. Extensions

The purpose of this section is to discuss the computational consequences which result from changes of some of the assumptions made above. First, we want to analyze the consequences of giving up the homogeneity assumptions concerning the composition of the credit portfolio, i.e. we want to deal with inhomogeneous exposures, inhomogeneous initial ratings and inhomogeneous asset return correlations.

Beginning with inhomogeneous exposures and assuming that there are E different exposure buckets with respective face values $F_1,\ldots,F_E$, formula (3.19) would have to be altered as follows:

$$\begin{aligned}&\varphi_{\Pi(H)|Z,X_r,S}(s)\\ &=\prod_{e=1}^{E}\Bigg(\varphi_{\delta}\left(s\cdot F_e\cdot e^{-R(X_r,H,T)\cdot(T-H)}\right)\cdot\Phi(t_{j,K})+\sum_{k=2}^{K-1}e^{i\cdot s\cdot F_e\cdot e^{-(R(X_r,H,T)+S_k(H,T))(T-H)}}\cdot\left(\Phi(t_{j,k})-\Phi(t_{j,k+1})\right)\\ &\qquad+e^{i\cdot s\cdot F_e\cdot e^{-(R(X_r,H,T)+S_1(H,T))(T-H)}}\cdot\left(1-\Phi(t_{j,2})\right)\Bigg)^{n_e},\end{aligned} \tag{4.1}$$

where n_e ($e\in\{1,\ldots,E\}$) denotes the number of obligors whose zero coupon bond has a face value of F_e. Thus, within each simulation run of the conditional characteristic function $\varphi_{\Pi(H)|Z,X_r,S}(s)$ E conditional characteristic functions (instead of 1) have to be calculated for each grid point s so that the computational burden increases with the number of exposure buckets rising. However, the probability terms, which include the calculation of the cumulative distribution function of the standard normal distribution, have only to be computed once for each simulation run.

For inhomogeneous initial ratings the adapted formula (3.19) is:

$$\varphi_{\Pi(H)|Z,X_r,S}(s)$$

$$= \prod_{j=1}^{K-1} \Bigg(\varphi_\delta \left(s \cdot F \cdot e^{-R(X_r,H,T)\cdot(T-H)} \right) \cdot \Phi(t_{j,K}) + \sum_{k=2}^{K-1} e^{i \cdot s \cdot F \cdot e^{-(R(X_r,H,T)+S_k(H,T))(T-H)}} \cdot \left(\Phi(t_{j,k}) - \Phi(t_{j,k+1}) \right)$$
$$+ e^{i \cdot s \cdot F \cdot e^{-(R(X_r,H,T)+S_1(H,T))(T-H)}} \cdot \left(1 - \Phi(t_{j,2})\right) \Bigg)^{n_j}, \tag{4.2}$$

where n_j ($j \in \{1, \ldots, K-1\}$) now denotes the number of obligors whose initial rating is j. Note that only the thresholds $t_{j,k}$ depend on the initial rating index j, which implies that within each simulation run the exp-terms, which depend on s, have not to be recalculated for each rating grade j.

Finally, assuming G different groups of obligors, in which each pair of asset returns exhibits a correlation parameter ρ_{V_g} ($g \in \{1, \ldots, G\}$), yields an adapted formula (3.19) which resembles (4.2), but with $K-1$ replaced by G in the upper index of the product. Again, only the thresholds $t_{j,k}^g$ depend on the asset return correlation index g.

Another kind of modification could concern the probability distribution for the risk factors. However, using e.g. a multivariate t–distribution for the asset returns, is not problematic. We only would have to use the inverse of the cumulative density function of the respective probability distribution when calculating the thresholds $t_{j,k}$ and use its cumulative density function instead of $\Phi(\cdot)$ in (3.19).

Furthermore, correlated recovery rates depending on the systematic credit risk factors as well as on the individual asset returns could easily be introduced, for example by the following representation:[21]

$$\delta(Z, X_r, \varepsilon_n, \eta_n) := \min \left\{ e^{\mu + \sigma \cdot \left(\alpha \cdot Z + \beta \cdot X_r + \gamma \cdot R_n + \sqrt{1-\alpha^2-\beta^2-\gamma^2} \cdot \eta_n \right)}; 1 \right\}$$

$$= \min\left\{ e^{\mu+\sigma\cdot\left(\left(\gamma\cdot\sqrt{\rho_V-\rho_{r,V}^2}+\alpha\right)\right)\cdot Z+\left(\gamma\cdot\rho_{r,V}+\beta\right)\cdot X_r+\gamma\cdot\sqrt{1-\rho_V}\cdot\varepsilon_n+\sqrt{1-\alpha^2-\beta^2-\gamma^2}\cdot\eta_n\right)};1\right\}, \tag{4.3}$$

where $\alpha,\gamma\in\mathbb{R}_+$, $\beta\in\mathbb{R}$, $\alpha^2+\beta^2+\gamma^2\leq 1$ and the $\eta_n \sim N(0,1)$ ($n\in\{1,\ldots,N\}$) are independent of each other as well as from all other random variables in the model, especially Z, X_r and the ε_n. As the recovery rates $\delta(Z,X_r,\varepsilon_n,\eta_n)$ are independent conditional on the realizations of Z and X_r, the conditional characteristic function of the credit portfolio value equals (3.19) with $\varphi_\delta\left(s\cdot F\cdot e^{-R(x_r,H,T)\cdot(T-H)}\right)\cdot\Phi(t_{j,K})$ replaced by the integral

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{t_{j,K}} e^{i\cdot s\cdot\delta(z,x_r,\varepsilon,\eta)\cdot F\cdot e^{-R(x_r,H,T)(T-H)}}\cdot\phi(\varepsilon)\cdot\phi(\eta)\,d\varepsilon\,d\eta\,, \tag{4.4}$$

which has to be solved numerically.

A third kind of modification could concern the type of credit-sensitive instrument the portfolio is composed of. Here, we want to consider the two examples of coupon bonds and European call options with counterparty risk on (default) risk-free zero coupon bonds. Let us first assume that the portfolio consists of N coupon bonds with identical face value F, maturity date T, coupon c and coupon dates $H\leq t_1<\ldots<t_M=T$ issued by N different corporates. The vector of stochastic credit spreads

$$S=\left(S_1(H,t_1),\ldots,S_{K-1}(H,t_1),S_1(H,t_2),\quad\ldots\quad,S_{K-1}(H,t_{M-1}),S_1(H,t_M),\ldots,S_{K-1}(H,t_M)\right)$$

now consists of $(K-1)\cdot M$ components where M denotes the number of coupon dates. Thus, now we also need to know the intertemporal correlations of credit spreads of different rating grades. Assuming a recovery payment of $\delta_n\cdot(F+c)$ in $t=H$ in the case of a default until the risk horizon, the conditional characteristic function of the credit portfolio value is

$$\varphi_{\Pi(H)|Z,X_r,S}(s)=\prod_{n=1}^{N}\varphi_{\sum_{k=1}^{K-1}\left(\sum_{m=1}^{M}c\cdot e^{-(R(X_r,H,t_m)+S_k(H,t_m))\cdot(t_m-H)}+F\cdot e^{-(R(X_r,H,T)+S_k(H,T))\cdot(T-H)}\right)\cdot 1_{\{\eta_H^n=k\}}+\delta_n\cdot(c+F)\cdot 1_{\{\eta_H^n=K\}}\middle|X_r,Z,S}(s)$$

$$= \left(\varphi_\delta\left(s\cdot(F+c)\right)\cdot\Phi(t_{j,K}) + \sum_{k=2}^{K-1} e^{i\cdot s\cdot\left(\sum_{m=1}^{M} c\cdot e^{-(R(X_r,H,t_m)+S_k(H,t_m))\cdot(t_m-H)} + F\cdot e^{-(R(X_r,H,T)+S_k(H,T))\cdot(T-H)}\right)} \cdot \left(\Phi(t_{j,k}) - \Phi(t_{j,k+1})\right) \right.$$

$$\left. + e^{i\cdot s\cdot\left(\sum_{m=1}^{M} c\cdot e^{-(R(X_r,H,t_m)+S_1(H,t_m))\cdot(t_m-H)} + F\cdot e^{-(R(X_r,H,T)+S_1(H,T))\cdot(T-H)}\right)} \cdot \left(1-\Phi(t_{j,2})\right) \right)^N . \tag{4.5}$$

As now each exp-term has M (instead of 1) exp-terms in its exponent, the computational burden increases with the number of coupon dates, but these additional exp-terms in the exponents have only to be calculated once (and not for each value of s) within a simulation run.

Next, let us assume that the portfolio consists of N European call options issued by N different corporates with identical expiration date T^C, identical exercise price X and identical underlying (default) risk-free zero coupon bond $p(r(t),t,T)$ with face value F and maturity date $T \geq T^C$. Working within the term structure model of *Vasicek* (1977), the $t = H$-price $C(r(H),H,X,T^C,T)$ of a European call option on a risk-free zero coupon bond *without any counterparty risk* is given by:[22]

$$C(r(H),H,X,T^C,T) = E^{\tilde{P}}\left[e^{-\int_H^{T^C} r(s)ds} \cdot \max\left\{p(r(T^C),T^C,T) - X, 0\right\} \right]$$

$$= p(r(H),H,T)\cdot\Phi(d_1) - X\cdot p(r(H),H,T^C)\cdot\Phi(d_2) \tag{4.6}$$

with

$$d_1 = \frac{1}{\upsilon}\cdot\ln\left(\frac{p(r(H),H,T)}{X\cdot p(r(H),H,T^C)}\right) + \frac{1}{2}\cdot\upsilon,$$

$$d_2 = d_1 - \upsilon,$$

$$\upsilon = \sqrt{\frac{1}{2}\cdot\frac{\sigma^2}{\kappa^3}\cdot\left(\left(1-e^{-\kappa\cdot(T-T^C)}\right)^2 - \left(e^{-\kappa\cdot(T-H)} - e^{-\kappa\cdot(T^C-H)}\right)^2\right)},$$

where $\tilde{P}$ denotes the risk-neutral probability measure relevant for pricing purposes. In order to price a European call option on a (default) risk-free zero coupon bond *with counterparty risk*, we assume that a default is only possible at the maturity date T^C of the option,[23] and that in this case the recovery payment is a fixed exogenous fraction δ of the options regular pay off. Furthermore, for simplicity we assume for the pricing of the options independence between the movements of the risk-free interest rates and the credit quality changes of the counterparties.[24] With these assumptions the price of a call written by counterparty n, whose rating at the risk horizon is $\eta_H^n \in \{1,\dots,K-1\}$, is given by:

$$C^{def}(r(H),\eta_H^n,H,X,T^C,T)$$

$$= \delta \cdot C(r(H),H,X,T^C,T) + (1-\delta)\cdot C(r(H),H,X,T^C,T)\cdot \tilde{P}\left(\tau_n > T^C \middle| \eta_H^n\right), \qquad (4.7)$$

where τ_n denotes the default time of counterparty n. Assuming that a default is an absorbing state under $\tilde{P}$, the event $\{\tau_n > T^C\}$ is equivalent to the event $\{\eta_{T^C}^n \neq K\}$, whose probability can simply be calculated by summing up all individual risk-neutral probabilities for a rating change from η_H^n to a non-default state within the time interval $[H,T^C]$. Given the – for pricing purposes – assumed independence between the risk-free interest rates and the rating transitions, the transition probabilities under $\tilde{P}$ can easily calculated out of the prices of defaultable bonds of the respective counterparty.[25] However, for the ease of exposition, we do not differ between the real-world probability measure P and the risk-neutral probability measure $\tilde{P}$, but instead assume also for pricing purposes that the transition processes of all counterparties can be modeled by a time-homogeneous Markov chain with (real-world) one year transition matrix Q (see table 1), which is also used for modeling the rating transitions in the time interval $[0,H]$. Of course, it has to be stressed that this approach does not reflect reality because the two measures will typically differ, especially over longer risk horizons used for credit risk management. The probabilities of rating changes within a T year horizon are then simply given by the matrix product

$$Q^T = \underbrace{Q \cdot \ldots \cdot Q}_{T\ times} \tag{4.8}$$

of the one year transition matrix Q. The value $\Pi(H)$ of the entire portfolio of long positions in European call options with counterparty risk at the risk horizon H is:

$$\Pi(H) = \sum_{n=1}^{N} \sum_{k=1}^{K-1} C^{def}(r(H), k, H, X, T^C, T) \cdot 1_{\{\eta_H^n = k\}} + \delta \cdot C(r(H), H, X, T^C, T) \cdot 1_{\{\eta_H^n = K\}}, \tag{4.9}$$

where the second summand in the inner sum of (4.9) is the $t = H$-value of the recovery payment due at $t = T^C$. Conditional on the realizations of the stochastic variables X_r and Z all N summands of the outer sum in (4.9) are independent because the only remaining stochastic variables are again the independent idiosyncratic risk factors ε_n ($n \in \{1, \ldots, N\}$). Hence, we proceed as before. The initial rating of all obligors is assumed to be $j \in \{1, \ldots, K\}$:[26]

$$\varphi_{\sum_{k=1}^{K-1} C^{def}(X_r, k, H, X, T^C, T) \cdot 1_{\{\eta_H^n = k\}} + \delta \cdot C(X_r, H, X, T^C, T) \cdot 1_{\{\eta_H^n = K\}} \big| Z, X_r}(s)$$

$$= E\left[e^{i \cdot s \cdot \left(\sum_{k=1}^{K-1} C^{def}(x_r, k, H, X, T^C, T) \cdot 1_{\{\eta_H^n = k\}} + \delta \cdot C(x_r, H, X, T^C, T) \cdot 1_{\{\eta_H^n = K\}} \right)} \middle| Z = z, X_r = x_r \right]$$

$$= \int_{-\infty}^{\infty} e^{i \cdot s \cdot \left(C^{def}(x_r, 1, H, X, T^C, T) \cdot 1_{\{\varepsilon_n > t_{j,2}\}} + \sum_{k=2}^{K-1} C^{def}(x_r, k, H, X, T^C, T) \cdot 1_{\{t_{j,k+1} < \varepsilon_n \le t_{j,k}\}} + \delta \cdot C(x_r, H, X, T^C, T) \cdot 1_{\{\varepsilon_n \le t_{j,K}\}} \right)} \cdot \phi(\varepsilon_n)\, d\varepsilon_n,$$

where $\phi(\varepsilon_n)$ again denotes the density function of a standard normal distribution. Splitting up the integration path of ε_n yields:

$$\int_{-\infty}^{t_{j,K}} e^{i \cdot s \cdot \delta \cdot C(x_r, H, X, T^C, T)} \cdot \phi(\varepsilon_n)\, d\varepsilon_n + \sum_{k=2}^{K-1} \int_{t_{j,k+1}}^{t_{j,k}} e^{i \cdot s \cdot C^{def}(x_r, k, H, X, T^C, T)} \cdot \phi(\varepsilon_n)\, d\varepsilon_n + \int_{t_{j,2}}^{\infty} e^{i \cdot s \cdot C^{def}(x_r, 1, H, X, T^C, T)} \cdot \phi(\varepsilon_n)\, d\varepsilon_n$$

$$= e^{i \cdot s \cdot \delta \cdot C(x_r, H, X, T^C, T)} \cdot \Phi(t_{j,K}) + \sum_{k=2}^{K-1} e^{i \cdot s \cdot C^{def}(x_r, k, H, X, T^C, T)} \cdot \left(\Phi(t_{j,k}) - \Phi(t_{j,k+1}) \right)$$

$$+ e^{i \cdot s \cdot C^{def}(x_r, 1, H, X, T^C, T)} \cdot \left(1 - \Phi(t_{j,2}) \right).$$

Because of the conditional independence of the summands in the outer sum of (4.9) the conditional characteristic function of $\Pi(H)$ can be written as:

$$\varphi_{\Pi(H)|Z,X_r}(s) = \prod_{n=1}^{N} \varphi_{\sum_{k=1}^{K-1} C^{def}(X_r,k,H,X,T^C,T)\cdot 1_{\{\eta_H^n=k\}} + \delta\cdot C(X_r,H,X,T^C,T)\cdot 1_{\{\eta_H^n=K\}}\Big|Z,X_r}(s)$$

$$\underset{Homog.}{=} \left(e^{i\cdot s\cdot\delta\cdot C(X_r,H,X,T^C,T)}\cdot\Phi(t_{j,K}) + \sum_{k=2}^{K-1} e^{i\cdot s\cdot C^{def}(X_r,k,H,X,T^C,T)}\cdot\left(\Phi(t_{j,k})-\Phi(t_{j,k+1})\right) + e^{i\cdot s\cdot C^{def}(X_r,1,H,X,T^C,T)}\cdot\left(1-\Phi(t_{j,2})\right)\right)^N . \tag{4.10}$$

Finally, the unconditional characteristic function $\varphi_{\Pi(H)}(s)$ of $\Pi(H)$ is again the expectation $E^P\left[\varphi_{\Pi(H)|Z,X_r}(s)\right]$ of the conditional characteristic function $\varphi_{\Pi(H)|Z,X_r}(s)$. Unfortunately, this expectation again can not be calculated in closed-form, but has to be computed by Monte Carlo simulations.

Duffie and *Pan* (2001) propose to use a delta-gamma approximation for the option values at the risk horizon, which – together with additional assumptions – allows them to calculate the unconditional characteristic function of the credit portfolio value in closed-form. In the numerical example of section V we want to test whether the delta-gamma approximation approach is also appropriate for a risk horizon of one year and percentile calculations corresponding to high confidence levels as they are usual in credit risk management. For this purpose the $t=H$-price $C(r(H),H,X,T^C,T)$ of the European call without counterparty risk and the $t=H$-price $C^{def}(X_r,k,H,X,T^C,T)$ of the corresponding option with counterparty risk respectively are approximated by a second order Taylor series expansion around the expected future value of the risk-free short rate $r(H)$ at $t=H$:[27]

$$\bar{r}(H) \equiv E^P[r(H)] \underset{(3.3)}{=} \theta + (r(0)-\theta)\cdot e^{-\kappa\cdot H} . \tag{4.11}$$

In order to shorten the notation the prices $C(r(H),H,X,T^C,T)$ and $C^{def}(r(H),\eta_H^n,H,X,T^C,T)$ are abbreviated by $C(r(H),H)$ and $C^{def}(r(H),\eta_H^n,H)$ respectively. This yields:

$$\begin{aligned} C(r(H),H) \approx C^{\Delta,\Gamma}(r(H),H) = C(\overline{r}(H),H) + \left.\frac{\partial C(r(H),H)}{\partial r(H)}\right|_{r(H)=\overline{r}(H)} \cdot \left(r(H)-\overline{r}(H)\right) \\ + \frac{1}{2}\cdot \left.\frac{\partial^2 C(r(H),H)}{(\partial r(H))^2}\right|_{r(H)=\overline{r}(H)} \cdot \left(r(H)-\overline{r}(H)\right)^2, \end{aligned} \tag{4.12}$$

and

$$\begin{aligned} C^{def}(r(H),\eta_H^n,H) \approx C^{def,\Delta,\Gamma}(r(H),\eta_H^n,H) \\ = \delta\cdot C^{\Delta,\Gamma}(r(H),H) + (1-\delta)\cdot C^{\Delta,\Gamma}(r(H),H)\cdot \tilde{P}\left(\tau_n > T^C \middle| \eta_H^n\right). \end{aligned} \tag{4.13}$$

The value $\Pi(H)$ of the entire portfolio of long positions in European call options with counterparty risk at the risk horizon H is now approximated by:

$$\Pi(H) \approx \Pi(H)^{\Delta,\Gamma} = \sum_{n=1}^{N}\sum_{k=1}^{K-1} C^{def,\Delta,\Gamma}(r(H),k,H)\cdot 1_{\{\eta_H^n=k\}} + \delta\cdot C^{\Delta,\Gamma}(r(H),H)\cdot 1_{\{\eta_H^n=K\}}, \tag{4.14}$$

and the conditional characteristic function of $\Pi(H)^{\Delta,\Gamma}$ is:

$$\begin{aligned} \varphi_{\Pi(H)^{\Delta,\Gamma}|Z,X_r}(s) = \Bigg(e^{i\cdot s\cdot\delta\cdot C^{\Delta,\Gamma}(X_r,H)}\cdot\Phi(t_{j,K}) + \sum_{k=2}^{K-1} e^{i\cdot s\cdot C^{def,\Delta,\Gamma}(X_r,k,H)}\cdot\left(\Phi(t_{j,k})-\Phi(t_{j,k+1})\right) \\ + e^{i\cdot s\cdot C^{def,\Delta,\Gamma}(X_r,1,H)}\cdot\left(1-\Phi(t_{j,2})\right)\Bigg)^N. \end{aligned} \tag{4.15}$$

In a pure market risk context, one advantage of the delta-gamma approximation in the case of multivariate normally distributed risk factors is that the portfolio value can be expressed by a linear polynomial of independent chi-squared and normally distributed random variables. Based on this representation the characteristic function of the portfolio value at the risk horizon can be calculated in closed-form and the inversion theorem (2.3) can directly be applied. Unfortunately, this advantage is lost in the credit portfolio context (at least in the extended CreditMetrics model described in section III.3) and the (unconditional) characteristic function of the credit portfolio value $\Pi(H)^{\Delta,\Gamma}$ has to be computed again by Monte Carlo simulations.

V. Numerical Example

V.1 Parameters

In this section a numerical example is presented which demonstrates the differences in accuracy and speed when the percentile values are calculated on the one hand with a full Monte Carlo simulation and on the other hand by an application of characteristic functions and the inversion theorem (2.3). For both methods we calculate the expectation of $\Pi(H)$ and the $p\%$-percentiles $\alpha_{p\%}(\Pi(H))$ of the credit portfolio distribution with $p \in \{0.1\%, 1\%, 5\%, 20\%, 40\%, 60\%\}$. The percentiles corresponding to the probabilities 20%, 40% and 60% are only computed in order to check each method's accuracy for the body of the probability distribution.

First, it is assumed that the portfolio consists of $N = 500$ defaultable zero coupon bonds, which are issued by N different obligors, but are otherwise identical. The face value is chosen to be $F = 1$. The simulations are done for the homogeneous initial ratings $\eta_0 \in \{\text{Aa, Baa, B}\}$. The parameters of the Ornstein-Uhlenbeck process (3.2) modeling the risk-free short rate are from *Lehrbass* (1997), who estimated these parameters using the DEM-LIBOR overnight rates within the period July 31, 1991 to May 31, 1995. The market price of interest rate risk λ, which is needed for calculating the price of a risk-free zero coupon bond, is the average of the values given by *Lehrbass* (1997). For simplicity, the recovery rate is set equal to a constant $\mu_\delta = 53.80\%$, which is Moody's mean recovery rate of senior unsecured bonds during 1970 to 1995.[28] The employed transition matrix (see table 1) is also from Moody's. The time to maturity of the zero coupon bonds is chosen as $T = 3$, implying a remaining time to maturity of two years at the risk horizon. The value of the correlation parameter ρ_V of the asset returns is chosen as 10%, which is within the range of values proposed by the Basle Committee on Banking Supervision for corporate exposures in the Internal Ratings-based approach[29], and 40% respectively. The parameter $\rho_{r,V}$, which determines the correlation between the firms' asset returns and the term structure of risk-free interest rates, is set equal to $\rho_{r,V} = -0.05$. Taking into consideration recent empirical studies of structural credit risk models[30] this value seems reasonable. The means and standard deviations of the multivariate normally distributed rating grade specific credit spreads $S_k(H,T)$ ($k \in \{1,\ldots,K\}$) as well as their correlation parameters, which are used for simulating the credit spreads, can be seen in table 2 for $T - H = 2$. These values are taken from *Kiesel*, *Perraudin* and *Taylor* (2003). The correlation coefficient $\rho_{X_r,S}$ between the credit spreads and the risk-free interest rate factor is set equal to -0.1. The correlation coefficient $\rho_{Z,S}$ between the systematic credit risk factor Z and the credit spreads, which is also independent of the rating grade, is assumed to be -0.1, too.

- insert table 2 about here -

Afterwards, it is assumed that the portfolio consists of $N = 500$ European call options with counterparty risk on (default) risk-free zero coupon bonds, which are written by N different counterparties. The parameters of the short rate process (3.2), the recovery rate, the transition matrix, the asset return correlation parameter as well as the correlation parameter between the asset returns and the risk-free interest rates are chosen as above. Again, the simulations are done for the homogeneous initial ratings $\eta_0 \in \{\text{Aa, Baa, B}\}$. The expiration date of the options is set equal to $T^C = 2$, and the exercise price is chosen as $X = 0.92190$, which is the $t = 2$-forward price of the underlying risk-free zero coupon bond. The numerical values of the first and the second derivative of the option pricing formula appearing in (4.12) are -0.09795 and 1.18128 respectively.[31]

V.2 Portfolio of Zero Coupon Bonds

In order to use the inversion formula (2.3), we have to calculate

$$\operatorname{Im}\left(\frac{e^{-i\cdot s\cdot x}\cdot\varphi_{\Pi(H)}(s)}{s}\right)=\operatorname{Im}\left(\frac{e^{-i\cdot s\cdot x}\cdot E\left[\varphi_{\Pi(H)|Z,X_r,S}(s)\right]}{s}\right)$$

$$\begin{aligned}=\operatorname{Im}\Bigg(\frac{e^{-i\cdot s\cdot x}}{s}\cdot E\Bigg[\Bigg(&e^{i\cdot s\cdot\delta\cdot F\cdot e^{-R(X_r,H,T)\cdot(T-H)}}\cdot\Phi(t_{j,K}(Z,X_r)\\&+\sum_{k=2}^{K-1}e^{i\cdot s\cdot F\cdot e^{-(R(X_r,H,T)+S_k(H,T))\cdot(T-H)}}\cdot\left(\Phi(t_{j,k}(Z,X_r))-\Phi(t_{j,k+1}(Z,X_r))\right)\\&+e^{i\cdot s\cdot F\cdot e^{-(R(X_r,H,T)+S_1(H,T))\cdot(T-H)}}\cdot\left(1-\Phi(t_{j,2}(Z,X_r))\right)\Bigg)^N\Bigg]\Bigg),\end{aligned}$$

where $x\in\mathbb{R}$ denotes the argument of the probability distribution of $\Pi(H)$. As the calculation of the expectation is done by Monte Carlo simulation, for each realization $(Z=z,X_r=x_r,S=(s_1(H,T),\ldots,s_{K-1}(H,T)))$ we have to compute for all grid points s of the numerical integration rule, which we employ later:

$$\begin{aligned}&e^{i\cdot s\cdot\delta\cdot F\cdot e^{-R(x_r,H,T)\cdot(T-H)}}\cdot\Phi(t_{j,K}(z,x_r)+\sum_{k=2}^{K-1}e^{i\cdot s\cdot F\cdot e^{-(R(x_r,H,T)+s_k(H,T))\cdot(T-H)}}\cdot\left(\Phi(t_{j,k}(z,x_r))-\Phi(t_{j,k+1}(z,x_r))\right)\\&+e^{i\cdot s\cdot F\cdot e^{-(R(x_r,H,T)+s_1(H,T))\cdot(T-H)}}\cdot\left(1-\Phi(t_{j,2}(z,x_r))\right)\end{aligned}$$

$$\begin{aligned}&=\cos\left(s\cdot\delta\cdot F\cdot e^{-R(x_r,H,T)\cdot(T-H)}\right)\cdot\Phi(t_{j,K}(z,x_r)\\&\quad+\sum_{k=2}^{K-1}\cos\left(s\cdot F\cdot e^{-(R(x_r,H,T)+s_k(H,T))\cdot(T-H)}\right)\cdot\left(\Phi(t_{j,k}(z,x_r))-\Phi(t_{j,k+1}(z,x_r))\right)\\&\quad+\cos\left(s\cdot F\cdot e^{-(R(x_r,H,T)+s_1(H,T))\cdot(T-H)}\right)\cdot\left(1-\Phi(t_{j,2}(z,x_r))\right)\\&\quad+i\cdot\Bigg(\sin\left(s\cdot\delta\cdot F\cdot e^{-R(x_r,H,T)\cdot(T-H)}\right)\cdot\Phi(t_{j,K}(z,x_r)\\&\qquad+\sum_{k=2}^{K-1}\sin\left(s\cdot F\cdot e^{-(R(x_r,H,T)+s_k(H,T))\cdot(T-H)}\right)\cdot\left(\Phi(t_{j,k}(z,x_r))-\Phi(t_{j,k+1}(z,x_r))\right)\\&\qquad+\sin\left(s\cdot F\cdot e^{-(R(x_r,H,T)+s_1(H,T))\cdot(T-H)}\right)\cdot\left(1-\Phi(t_{j,2}(z,x_r))\right)\Bigg)\end{aligned}$$

$$=:\operatorname{Re}\left(u(s)^{(w)}\right)+i\cdot\operatorname{Im}\left(u(s)^{(w)}\right)\equiv u(s)^{(w)},\tag{5.1}$$

where we have used the Euler formula $e^{i\cdot\vartheta} = \cos(\vartheta) + i \cdot \sin(\vartheta)$ and w denotes the w^{th} simulation run. Next, we represent the complex number $u(s)^{(w)} \in \mathbb{C}$ by $u(s)^{(w)} = \left|u(s)^{(w)}\right| \cdot e^{i\cdot\vartheta(s)}$, where the absolute value is

$$\left|u(s)^{(w)}\right| = \sqrt{\operatorname{Re}\left(u(s)^{(w)}\right)^2 + \operatorname{Im}\left(u(s)^{(w)}\right)^2} \tag{5.2}$$

and the argument $\vartheta(s) \in (-\pi, \pi]$ is

$$\vartheta(s) = \arctan\left(\frac{\operatorname{Im}\left(u(s)^{(w)}\right)}{\left|u(s)^{(w)}\right| + \operatorname{Re}\left(u(s)^{(w)}\right)}\right), \tag{5.3}$$

so that we can calculate a realization of the conditional characteristic function $\varphi_{\Pi(H)|Z,X_r,S}(s) = \left(u(s)^{(w)}\right)^N$ as follows:

$$\left(u(s)^{(w)}\right)^N = \left(\left|u(s)^{(w)}\right| \cdot e^{i\cdot\vartheta(s)}\right)^N = \left|u(s)^{(w)}\right|^N \cdot e^{i\cdot\vartheta(s)\cdot N}$$

$$= \underbrace{\left|u(s)^{(w)}\right|^N \cdot \cos(\vartheta(s)\cdot N)}_{=\operatorname{Re}\left(\varphi_{\Pi(H)|Z,X_r,S}(s)\right)} + i \cdot \underbrace{\left|u(s)^{(w)}\right|^N \cdot \sin(\vartheta(s)\cdot N)}_{=\operatorname{Im}\left(\varphi_{\Pi(H)|Z,X_r,S}(s)\right)}.$$

As we have

$$E\left[\varphi_{\Pi(H)|Z,X_r,S}(s)\right] = E\left[\operatorname{Re}\left(\varphi_{\Pi(H)|Z,X_r,S}(s)\right) + i\cdot\operatorname{Im}\left(\varphi_{\Pi(H)|Z,X_r,S}(s)\right)\right]$$

$$= E\left[\operatorname{Re}\left(\varphi_{\Pi(H)|Z,X_r,S}(s)\right)\right] + i\cdot E\left[\operatorname{Im}\left(\varphi_{\Pi(H)|Z,X_r,S}(s)\right)\right],$$

a Monte Carlo estimate of the real and the imaginary part of the unconditional characteristic function can be calculated by summing up all realizations of the real and the imaginary part of the conditional characteristic function respectively (for each grid point s) and dividing by the number of realizations, which is chosen as $50,000$. After having generated an estimate of the unconditional characteristic function $\varphi_{\Pi(H)}(s) \in \mathbb{C}$, which is the most computer time consuming part of the calculations, we represent – using (5.2) and (5.3) – this complex number also as $\varphi_{\Pi(H)}(s) = \left|\varphi_{\Pi(H)}(s)\right| \cdot e^{i\cdot\vartheta(s)}$ so that the integrand of the inversion formula (2.3) can be written as:

$$\operatorname{Im}\left(\frac{e^{-i\cdot s\cdot x}\cdot\varphi_{\Pi(H)}(s)}{s}\right) = \operatorname{Im}\left(\frac{e^{-i\cdot s\cdot x}\cdot\left|\varphi_{\Pi(H)}(s)\right|\cdot e^{i\cdot\vartheta(s)}}{s}\right)$$

$$= \frac{\left|\varphi_{\Pi(H)}(s)\right|}{s}\cdot\operatorname{Im}\left(e^{i\cdot(\vartheta(s)-s\cdot x)}\right) = \frac{\left|\varphi_{\Pi(H)}(s)\right|}{s}\cdot\sin\left(\vartheta(s)-s\cdot x\right).$$

Thus, we have to calculate numerically the following integral:

$$P\left(\Pi(H) < x\right) = \frac{1}{2} - \frac{1}{\pi}\int_0^\infty \frac{\left|\varphi_{\Pi(H)}(s)\right|}{s}\cdot\sin\left(\vartheta(s)-s\cdot x\right)ds, \tag{5.4}$$

which is done by employing first Gaussian integration with $n = 96$ grid points applied on each of the intervals $[0,1]$, $(1,3]$, $(3,10]$ and $(10,50]$[32] and second the trapezoidal rule with step size $h = 0.01$ applied on the interval $[0,50]$. Truncating the integration interval $[0,\infty)$ in (2.3) at $s = 50$ ensures – for the chosen parameter values – that the absolute value of the oscillating integrand is usually smaller than 10^{-4}. For applying the trapezoidal rule we need the value of the integrand at $s = 0$, which is obtained by l'Hôpital's rule:

$$\lim_{s\to 0}\left(\operatorname{Im}\left(\frac{e^{-i\cdot s\cdot x}\cdot E\left[\varphi_{\Pi(H)|Z,X_r,S}(s)\right]}{s}\right)\right)$$

$$\begin{aligned}
&= N\cdot E\Bigg[\left(\delta\cdot F\cdot e^{-R(X_r,H,T)\cdot(T-H)} - \frac{x}{N}\right)\cdot\Phi(t_{j,K}(Z,X_r) \\
&\quad + \sum_{k=2}^{K-1}\left(F\cdot e^{-(R(X_r,H,T)+S_k(H,T))\cdot(T-H)} - \frac{x}{N}\right)\cdot\left(\Phi(t_{j,k}(Z,X_r)) - \Phi(t_{j,k+1}(Z,X_r))\right) \\
&\quad + \left(F\cdot e^{-(R(X_r,H,T)+S_1(H,T))\cdot(T-H)} - \frac{x}{N}\right)\cdot\left(1-\Phi(t_{j,2}(Z,X_r))\right)\Bigg].
\end{aligned} \tag{5.5}$$

Finally, the various percentiles are calculated by using the bisection method, where the iteration is stopped when the difference between two following arguments of the probability distribution is smaller than 10^{-9}. Computing percentiles of the credit portfolio distribution the way described above, three types of errors are introduced: First, the simulation error affecting the unconditional characteristic function; second, the truncation error caused by cutting the integration interval in (2.3) at $s = 50$ and, third, the discretization error introduced by choosing a finite number of grid points for the numerical integration. In order to check for the magnitude of these possible error sources we compare the percentiles calculated by the method described above with those which result from a Monte Carlo simulation with a very high number of simulation runs.

Table 3 shows the percentiles resulting from Monte Carlo simulation with $1,000,000$ simulation runs and from application of the inversion formula (2.3) combined either with Gaussian integration or with the trapezoidal rule. For all three methods the (mean) percentile values are close together indicating that the discretization and the truncation error of the inversion method is, at least for the considered portfolio composition, not too large.

In order to roughly estimate the accuracy of the Monte Carlo simulation approach, the 1,000,000 simulations are split into 10 groups, with 100,000 for each group, and the respective percentiles are estimated for each group separately by sorting the generated realizations of $\Pi(H)$ in ascending order and taking for example the 1000$^{\text{th}}$ of these sorted values as an estimate of the 1%-percentile. Then, the mean and the standard deviation of these 10 percentile estimates (for each confidence level) are calculated. In order to estimate the standard error of the percentile estimators associated with 1,000,000 simulation runs, these standard deviations are scaled down by the factor $\sqrt{10}$. For the inversion method combined with Gaussian integration, the calculation of the mean and the standard error of the percentile estimators is based on 50 repetitions of the percentile computations. Due to the longer computation time for the inversion method combined with the trapezoidal rule, the mean and the standard error of the percentile estimators are calculated in this case based on only 10 repetitions. For all methods, the percentile values presented in table 3 equal the mean percentile estimates. Furthermore, table 3 shows the percentage confidence interval for the percentiles, which is two times the standard error of the percentile estimator divided by the (mean) percentile value. Given an assumed normality of the percentile estimator this is the maximum percentage error of the percentile estimator which is not exceeded with a probability of 95.4%. Of course, the standard errors and the confidence intervals of the percentiles are themselves estimates only, which would change with each new repeated simulation.

As table 3 shows, the standard error of the percentiles increases with decreasing probability p, worsening credit quality and rising asset return correlation (only for the initial rating Aa this latter observation is not unambiguous).

The standard error resulting from an application of the Monte Carlo simulation is smaller than that one resulting from an application of the inversion formula (2.3), but, as table 4 shows, the inversion method combined with Gaussian integration is, at least for the considered portfolio, seven times faster than the Monte Carlo simulation. Thus, as the standard error is in both cases within acceptable bounds, the inversion method seems to be superior.

With respect to the accuracy, both integration rules applied to the inversion formula (2.3), the Gaussian integration and the trapezoidal rule, perform equally well. But as the Gaussian integration employs (in our implementation) only 384 grid points in contrast to 5,000 for the trapezoidal rule, the Gaussian integration rule is preferred because much fewer time consuming calculations of the unconditional characteristic function as the expectation of the conditional counterpart are needed. This fact is also reflected by table 4: Applying the trapezoidal rule the computation is even slower than the Monte Carlo simulation. We also tested (not in the table) the application of the trapezoidal rule with a larger step size, for example with a step size $h = 0.01$ applied on the interval $[0,3]$ and with step size $h = 0.1$ applied on the interval $(3,50]$, where the integrand of (2.3) is already very small. In this case, the accuracy was comparable with the two other integration rules, but the computation time was still longer than that one resulting from an application of the Gaussian integration rule. Reducing the step size h further, for example using an uniform step size of $h = 0.1$ on the whole interval $[0,50]$, reduces the computation time, but leads to problems when computing the percentiles for portfolios with the low credit quality B.

– insert tables 3 and 4 about here –

V.3 Portfolio of European Call Options with Counterparty Risk

The inversion formula (2.3) is applied to the portfolio of European calls with counterparty risk analogously to the way described before for the portfolio of defaultable zero coupon bonds. Table 5 shows the percentiles resulting from a Monte Carlo simulation with $1,000,000$ simulation runs and from an application of the inversion formula (2.3) combined with Gaussian integration, both using either the exact option pricing formula (4.7) or the delta-gamma approximation (4.13).

Generally, the fit between the mean Monte Carlo based estimates of the percentiles and those estimates based on the inversion formula (2.3) is not as good as in the previous case of a portfolio of defaultable zero coupon bonds, but still acceptable; only for the low credit quality B the fit is bad. This indicates that the error induced by the usage of the numerical integration rule increases for a portfolio composed of options.

Unfortunately, also the simulation error gets larger: For both methods, the percentage confidence intervals of the percentile estimates increase compared to the portfolio of defaultable zero coupon bonds. Especially in the case of the inversion method, the absolute deterioration is unsatisfactorily high. At least, the inversion method combined with Gaussian integration is, as before, seven times faster than the Monte Carlo simulation (see table 6).

Employing a delta-gamma approximation for the option price at the risk horizon instead of the exact option price (4.7) leads for both methods to substantial differences in the percentile estimates. Thus, the use of this approximation can not be recommended, especially as this loss in precision is not rewarded by a significant reduction of the computation time (see table 6). Furthermore, these differences between the “true” percentiles and those calculated with the Monte Carlo simulation approach or the inversion formula (2.3) respectively combined with a delta-gamma approximation of the option price are expected to get larger with rising interest rate volatility σ_r or with the introduction of jumps in the interest rate process (3.2) as proposed for example by *Duffie* and *Pan* (2001).

– insert tables 5 and 6 about here –

VI. Conclusions

Most credit portfolio models currently used by the banking industry rely on Monte Carlo simulations for calculating the probability distribution of the future credit portfolio value, which can be quite computer time consuming, especially for portfolios with many obligors and when percentiles corresponding to high confidence levels are needed with sufficient accuracy. Adding market risk factors, such as interest rate or credit spread risk, as additional ingredients of a credit portfolio model, the computational burden of full Monte Carlo simulations even increases and the need for efficient methods for calculating credit risk measures becomes even more obvious.

In this study it is analyzed whether the use of characteristic functions and inverse Fourier transformation, which formerly have already been successfully applied to market risk portfolio models, can be an efficient tool for calculating risk measures in the context of a credit portfolio model with integrated market risk factors. Fur this purpose, based on a version of the well-known credit portfolio model CreditMetrics extended by correlated interest rate and credit spread risk, the percentiles corresponding to various confidence levels of the probability distribution of a portfolio of defaultable zero coupon bonds and European call options with counterparty risk respectively are calculated by this method. Unfortunately, the characteristic function of the credit portfolio value at the risk horizon can not be calculated in closed-form, but has to be computed by Monte Carlo simulations. However, depending on the integration rule applied, this method can be much faster than a full Monte Carlo simulation of the future credit portfolio distribution. For the portfolio of defaultable zero coupon bonds also the accuracy is satisfactory, but for the portfolio of European call options with counterparty risk the relatively high standard error of the percentile estimators is unsatisfactory. Perhaps, this problem can be resolved by an application of variance reduction techniques when simulating the characteristic function of the credit portfolio value.

TABLES

Table 1: Transition Matrix

initial rating	rating after one year							
	Aaa	**Aa**	**A**	**Baa**	**Ba**	**B**	**Caa-C**	**default**
Aaa	91,80	7,37	0,81	0,00	0,02	0,00	0,00	0,00
Aa	1,21	90,73	7,67	0,28	0,08	0,01	0,00	0,02
A	0,05	2,49	91,97	4,84	0,51	0,12	0,01	0,01
Baa	0,05	0,26	5,45	88,55	4,72	0,72	0,09	0,16
Ba	0,02	0,04	0,51	5,57	85,42	6,71	0,45	1,28
B	0,01	0,02	0,14	0,41	6,69	83,37	2,57	6,79
Caa-C	0,00	0,00	0,00	0,62	1,59	4,12	68,04	25,63
Default	0	0	0	0	0	0	0	100

The above transition matrix is based on *Moody's Investors Service* (2002, p. 31). The elements q_{ij} of the transition matrix specify the probability (in %) that an obligor migrates from the rating class i to the rating class j within one year. These probabilities are average values of all corporates in the period 1970-2001. The category 'rating withdrawn' has been eliminated by distributing its probability mass among all other categories, corresponding to their individual weights.

Table 2: Descriptive Statistics for the Multivariate Normally Distributed Credit Spreads (Maturity: 2 years)

rating	AAA	AA	A	BBB	BB	B	CCC
	means						
	35.6	41.0	58.2	86.0	189.6	331.2	1320
	standard deviations						
	14.3	14.8	21.5	30.6	74.0	117	480
	correlation matrix						
AAA	1.00	0.92	0.84	0.72	0.70	0.64	0.64
AA	0.92	1.00	0.86	0.70	0.75	0.61	0.64
A	0.84	0.86	1.00	0.89	0.81	0.67	0.61
BBB	0.72	0.70	0.89	1.00	0.77	0.69	0.67
BB	0.70	0.75	0.81	0.77	1.00	0.65	0.69
B	0.64	0.61	0.67	0.69	0.65	1.00	0.65
CCC	0.64	0.64	0.61	0.67	0.69	0.65	1.00

The means, volatilities and correlation parameters are from *Kiesel*, *Perraudin* and *Taylor* (2003, table 1, p. 10, and table 2, p. 18, the procedure to estimate the values for the rating class CCC is explained in their Appendix, pp. 32). The means are sample means of spread levels in basis points. The standard deviations are sample standard deviations of changes in 2 year maturity spreads over a one year horizon in basis points. The correlation coefficients are sample correlation coefficients for the different 2 year maturity spread changes over a one year horizon.

Table 3: Percentiles of the Portfolio of Defaultable Zero Coupon Bonds

	$E^P[\Pi(H)]$	$\alpha_{0.1\%}(\Pi(H))$	$\alpha_{1\%}(\Pi(H))$	$\alpha_{5\%}(\Pi(H))$	$\alpha_{20\%}(\Pi(H))$	$\alpha_{40\%}(\Pi(H))$	$\alpha_{60\%}(\Pi(H))$
Aa, $\rho_V = 10\%$							
(1)	427.3366	409.0518	413.5736	417.5523	422.2835	425.7839	428.8157
	(0.5987%)	(7.1210%)	(1.9850%)	(1.0971%)	(1.0307%)	(0.7160%)	(0.9285%)
	*0.0028%	*0.0348%	*0.0096%	*0.0053%	*0.0049%	*0.0034%	*0.0043%
(2)	427.3429	409.1305	413.5760	417.5508	422.2914	425.7902	428.8265
	(2.9298%)	(27.0460%)	(9.5413%)	(5.8456%)	(3.9577%)	(3.3561%)	(2.8834%)
	*0.0137%	*0.1322%	*0.0461%	*0.0280%	*0.0187%	*0.0158%	*0.0134%
(3)	427.3314	409.0934	413.5225	417.5283	422.2656	425.7806	428.8024
	(1.9847%)	(20.8014%)	(8.6656%)	(4.2272%)	(3.0127%)	(2.6987%)	(3.4261%)
	*0.0093%	*0.1017%	*0.0419%	*0.0202%	*0.0143%	*0.0127%	*0.0160%
Aa, $\rho_V = 40\%$							
(1)	427.3431	408.7868	413.4926	417.5303	422.2797	425.7933	428.8357
	(0.6012%)	(4.7999%)	(1.6606%)	(0.8165%)	(0.5351%)	(0.5040%)	(0.4356%)
	*0.0028%	*0.0235%	*0.0080%	*0.0039%	*0.0025%	*0.0024%	*0.0020%
(2)	427.3422	408.7376	413.4679	417.5154	422.2790	425.7898	428.8327
	(3.1264%)	(37.2488%)	(10.3504%)	(5.8198%)	(4.2015%)	(3.8830%)	(3.7492%)
	*0.0146%	*0.1823%	*0.0501%	*0.0279%	*0.0199%	*0.0182%	*0.0175%
(3)	427.3438	408.7578	413.4893	417.5086	422.2844	425.8018	428.8425
	(2.9837%)	(24.4194%)	(10.3324%)	(5.8113%)	(4.2183%)	(3.7300%)	(2.7797%)
	*0.0140%	*0.1195%	*0.0500%	*0.0278%	*0.0200%	*0.0175%	*0.0130%
Baa, $\rho_V = 10\%$							
(1)	422.8689	402.5797	407.8331	412.2592	417.4714	421.2390	424.4950
	(0.6451%)	(4.8110%)	(4.3209%)	(1.1671%)	(0.6921%)	(0.5037%)	(0.6589%)
	*0.0031%	*0.0239%	*0.0212%	*0.0057%	*0.0033%	*0.0024%	*0.0031%
(2)	422.8704	402.6901	407.8096	412.2595	417.4537	421.2410	424.5017
	(2.5422%)	(28.0524%)	(10.3822%)	(5.7183%)	(3.8045%)	(2.6905%)	(3.3453%)
	*0.0120%	*0.1393%	*0.0509%	*0.0277%	*0.0182%	*0.0128%	*0.0158%
(3)	422.8956	402.6606	407.8412	412.3098	417.4810	421.2679	424.5184
	(1.9122%)	(32.9432%)	(9.2254%)	(4.1382%)	(2.2524%)	(2.8603%)	(3.4838%)
	*0.0090%	*0.1636%	*0.0452%	*0.0201%	*0.0108%	*0.0136%	*0.0164%
Baa, $\rho_V = 40\%$							
(1)	422.8549	388.9061	405.3018	411.6891	417.4392	421.3626	424.6793
	(0.6921%)	(20.8514%)	(4.5073%)	(1.8088%)	(0.8911%)	(0.8205%)	(0.8216%)
	*0.0033%	*0.1072%	*0.0222%	*0.0088%	*0.0043%	*0.0039%	*0.0039%
(2)	422.8508	388.5830	405.2514	411.6786	417.4327	421.3592	424.6760
	(2.9066%)	(131.8661%)	(22.6387%)	(5.4471%)	(3.7987%)	(3.8422%)	(3.2904%)
	*0.0137%	*0.6787%	*0.1117%	*0.0265%	*0.0182%	*0.0182%	*0.0155%
(3)	422.8636	388.7386	405.3431	411.6811	417.4551	421.3848	424.6873
	(2.5787%)	(123.8244%)	(20.8250%)	(6.8985%)	(4.4086%)	(3.2757%)	(2.8484%)
	*0.0122%	*0.6371%	*0.1028%	*0.0335%	*0.0211%	*0.0155%	*0.0134%
B, $\rho_V = 10\%$							
(1)	390.9386	337.8536	353.7773	366.2635	379.4226	388.1275	395.1686
	(1.4267%)	(16.2142%)	(7.7565%)	(2.5812%)	(1.3697%)	(1.3561%)	(1.8436%)
	*0.0073%	*0.0960%	*0.0438%	*0.0141%	*0.0072%	*0.0070%	*0.0093%

(2)	390.9304	337.9016	353.6273	366.2740	379.4219	388.1235	395.1524
	(6.3298%)	(78.9189%)	(24.2021%)	(14.2867%)	(9.0151%)	(7.3852%)	(7.6737%)
	*0.0324%	*0.4671%	*0.1369%	*0.0780%	*0.0475%	*0.0381%	*0.0388%
(3)	390.9295	337.8860	353.5980	366.2038	379.4167	388.1180	395.1556
	(4.2030%)	(68.7533%)	(25.9366%)	(5.8492%)	(9.6533%)	(7.0277%)	(4.5036%)
	*0.0215%	*0.4070%	*0.1467%	*0.0319%	*0.0509%	*0.0362%	*0.0228%
				B, $\rho_V = 40\%$			
(1)	390.9590	270.5614	309.0435	345.7822	377.6273	391.1795	399.7319
	(2.2866%)	(32.6563%)	(25.0781%)	(11.4919%)	(1.8110%)	(1.7789%)	(1.5502%)
	*0.0117%	*0.2414%	*0.1623%	*0.0665%	*0.0096%	*0.0091%	*0.0078%
(2)	390.9362	270.8610	309.3076	345.8114	377.5437	391.1714	399.7091
	(10.2512%)	(140.3692%)	(85.8495%)	(49.3674%)	(16.0783%)	(10.0392%)	(7.1406%)
	*0.0524%	*1.0365%	*0.5551%	*0.2855%	*0.0852%	*0.0513%	*0.0357%
(3)	390.9121	271.7079	309.0354	345.9979	377.5251	391.1150	399.6588
	(6.8077%)	(159.7397%)	(41.8969%)	(42.2782%)	(10.2027%)	(5.9831%)	(5.8092%)
	*0.0348%	*1.1758%	*0.2711%	*0.2444%	*0.0541%	*0.0306%	*0.0291%

(1) Monte Carlo simulation with 1,000,000 simulation runs

(2) Inversion formula (2.3) computed with Gaussian integration with $n = 96$ grid points applied on the intervals $[0,1]$, $(1,3]$, $(3,10]$ and $(10,50]$

(3) Inversion formula (2.3) computed with the trapezoidal rule with step size $h = 0.01$ applied on the interval $[0,50]$

For applying the inversion formula the unconditional characteristic function of the credit portfolio value has to be calculated by means of Monte Carlo simulations. For the above values 50,000 simulation runs have been used. The bisection method employed for finding the respective percentiles stopped when the difference between two arguments of the cumulative density function was smaller than 10^{-9}.

() Standard error of the mean credit portfolio value and the percentile estimators respectively. In order to estimate the accuracy of the Monte Carlo simulation approach (1), the 1,000,000 simulations are split into 10 groups, with 100,000 for each group, and the respective percentiles are estimated for each group separately by sorting the generated realizations of $\Pi(H)$ in ascending order and taking for example the 1000$^{\text{th}}$ of these sorted values as an estimate of the 1%-percentile. Then, the mean and the standard deviation of these 10 percentile estimates (for each confidence level) are calculated. In order to estimate the standard error of the percentile estimators associated with 1,000,000 simulation runs, these standard deviations are scaled down by the factor $\sqrt{10}$. For method (2), the calculation of the mean and the standard error of the percentile estimators is based on 50 repetitions of the percentile computations. Due to the longer computation time for method (3) caused by the use of the trapezoidal rule, the mean and the standard error of the percentile estimators are calculated in this case based on only 10 repetitions. For all methods, the presented percentile values equal the mean percentile estimates.

* Two times the standard error of the percentile estimator divided by the (mean) percentile value. Given an assumed normality of the percentile estimator this the maximum percentage error of the percentile estimator which is not exceeded with a probability of 95.4%.

Parameters:
$N = 500$, $F = 1$, $T = 3$, $H = 1$, $\rho_{r,V} = -0.05$, $\rho_{X_r,S} = -0.1$, $\rho_{Z,S} = -0.1$, $\delta = 0.538$, $\kappa = 1.169$, $\theta = 0.061$, $\sigma_r = 0.029$, $\lambda = 0.88$, $r(0) = 0.061$.

Table 4: Computation Times for the Portfolio of Defaultable Zero Coupon Bonds

Portfolio of Defaultable Zero Coupon Bonds		
(1)	**(2)**	**(3)**
1	7.21	0.65

Computation times for the expected credit portfolio value and the percentiles of the portfolio of defaultable zero coupon bonds calculated with various methods. The computation time for the Monte Carlo simulation (1) is taken as the base, which is divided by the computation times of the other methods. Values larger than one indicate speed gains, values smaller than one indicate speed losses compared to the Monte Carlo simulation. For the explanation of the methods see table 3. Parameters: initial rating Baa, $\rho_V = 0.1$, other parameters see table 3.

Table 5: Percentiles of the Portfolio of European Call Options with Counterparty Risk on Risk-Free Zero Coupon Bonds

	$E^P[\Pi(H)]$	$\alpha_{0.1\%}(\Pi(H))$	$\alpha_{1\%}(\Pi(H))$	$\alpha_{5\%}(\Pi(H))$	$\alpha_{20\%}(\Pi(H))$	$\alpha_{40\%}(\Pi(H))$	$\alpha_{60\%}(\Pi(H))$
				Aa, $\rho_V = 10\%$			
(1)	2.5563	0.6214	0.9219	1.2689	1.7864	2.2447	2.6926
	(0.0890%)	(0.2569%)	(0.1372%)	(0.1202%)	(0.1297%)	(0.1164%)	(0.1284%)
	*0.0696%	*0.8269%	*0.2977%	*0.1894%	*0.1452%	*0.1037%	*0.0954%
(2)	2.5548	0.6236	0.9197	1.2674	1.7866	2.2438	2.6904
	(0.3947%)	(1.3252%)	(0.8187%)	(0.6338%)	(0.4219%)	(0.4613%)	(0.5335%)
	*0.3089%	*4.2504%	*1.7802%	*1.0002%	*0.4723%	*0.4111%	*0.3966%
(3)	2.5577	0.6518	0.9297	1.2699	1.7870	2.2448	2.6913
	(0.0893%)	(0.2926%)	(0.0987%)	(0.1282%)	(0.0979%)	(0.0869%)	(0.1377%)
	*0.070%	*0.8978%	*0.2124%	*0.2019%	*0.1096%	*0.0775%	*0.1023%
(4)	2.5565	0.6503	0.9254	1.2667	1.7859	2.2435	2.6905
	(0.4206%)	(1.0943%)	(0.8043%)	(0.5704%)	(0.4595%)	(0.4829%)	(0.5834%)
	*0.3291%	*3.3656%	*1.7383%	*0.9007%	*0.5146%	*0.4305%	*0.4336%
				Aa, $\rho_V = 40\%$			
(1)	2.5548	0.6235	0.9229	1.2693	1.7869	2.2441	2.6906
	(0.0889%)	(0.2056%)	(0.1606%)	(0.0824%)	(0.1144%)	(0.0908%)	(0.1168%)
	*0.0696%	*0.6594%	*0.3481%	*0.1299%	*0.1280%	*0.0809%	*0.0868%
(2)	2.5554	0.6195	0.9212	1.2701	1.7874	2.2440	2.6907
	(0.3355%)	(1.5225%)	(0.7937%)	(0.5725%)	(0.4119%)	(0.4192%)	(0.5078%)
	*0.2626%	*4.9156%	*1.7232%	*0.9014%	*0.4608%	*0.3736%	*0.3775%
(3)	2.5572	0.6506	0.9275	1.2707	1.7861	2.2441	2.6904
	(0.0893%)	(0.2551%)	(0.1628%)	(0.1002%)	(0.1089%)	(0.0920%)	(0.1166%)
	*0.070%	*0.7844%	*0.3512%	*0.1576%	*0.1219%	*0.0820%	*0.0867%
(4)	2.5573	0.6484	0.9271	1.2703	1.7870	2.2442	2.6905
	(0.3409%)	(1.3121%)	(0.7789%)	(0.5630%)	(0.3947%)	(0.4511%)	(0.4816%)
	*0.2666%	*4.0468%	*1.6805%	*0.8864%	*0.4417%	*0.4020%	*0.3580%
				Baa, $\rho_V = 10\%$			
(1)	2.5505	0.6133	0.9185	1.2647	1.7824	2.2377	2.6862
	(0.0889%)	(0.2445%)	(0.1215%)	(0.1018%)	(0.0997%)	(0.1045%)	(0.1346%)
	*0.0697%	*0.7972%	*0.2645%	*0.1610%	*0.1118%	*0.0934%	*0.1002%
(2)	2.5513	0.6222	0.9199	1.2668	1.7835	2.2397	2.6867
	(0.3467%)	(1.2193%)	(0.7682%)	(0.4809%)	(0.3624%)	(0.3475%)	(0.4222%)
	*0.2718%	*3.9193%	*1.6701%	*0.7592%	*0.4064%	*0.3103%	*0.3143%
(3)	2.5519	0.6489	0.9253	1.2651	1.7817	2.2385	2.6863
	(0.0892%)	(0.3004%)	(0.1617%)	(0.0880%)	(0.1062%)	(0.1226%)	(0.1149%)
	*0.0699%	*0.9260%	*0.3496%	*0.1391%	*0.1192%	*0.1096%	*0.0855%
(4)	2.5532	0.6509	0.9263	1.2666	1.7829	2.2395	2.6871
	(0.3409%)	(0.9946%)	(0.7414%)	(0.4875%)	(0.3581%)	(0.3384%)	(0.4208%)
	*0.2670%	*3.0562%	*1.6009%	*0.7698%	*0.4017%	*0.3022%	*0.3132%
				Baa, $\rho_V = 40\%$			
(1)	2.5520	0.6229	0.9197	1.2676	1.7843	2.2406	2.6878
	(0.0888%)	(0.2491%)	(0.1141%)	(0.1117%)	(0.0962%)	(0.0900%)	(0.1050%)
	*0.0696%	*0.7999%	*0.2482%	*0.1763%	*0.1078%	*0.0803%	*0.0781%

(2)	2.5513 (0.4219%) *0.3307%	0.6194 (1.6086%) *5.1943%	0.9208 (0.7131%) *1.5490%	1.2668 (0.5717%) *0.9026%	1.7828 (0.4958%) *0.5562%	2.2399 (0.4975%) *0.4442%	2.6873 (0.5063%) *0.3768%
(3)	2.5528 (0.0892%) *0.0699%	0.6471 (0.1175%) *0.3633%	0.9243 (0.1167%) *0.2524%	1.2661 (0.0835%) *0.1320%	1.7835 (0.0769%) *0.0862%	2.2396 (0.0592%) *0.0528%	2.6861 (0.0498%) *0.0371%
(4)	2.5528 (0.4610%) *0.3611%	0.6479 (1.2713%) *3.9245%	0.9263 (0.6492%) *1.4018%	1.2670 (0.5552%) *0.8765%	1.7824 (0.5167%) *0.5797%	2.2397 (0.5451%) *0.4868%	2.6865 (0.5660%) *0.4214%
				B, $\rho_V = 10\%$			
(1)	2.3611 (0.0838%) *0.0709%	0.5605 (0.2053%) *0.7325%	0.8338 (0.2022%) *0.4851%	1.1550 (0.1021%) *0.1767%	1.6383 (0.0843%) *0.1030%	2.0655 (0.1018%) *0.0986%	2.4848 (0.1008%) *0.0811%
(2)	2.4036 (0.3942%) *0.3280%	0.5736 (1.3770%) *4.8011%	0.8569 (0.6341%) *1.4800%	1.1847 (0.3872%) *0.6537%	1.6726 (0.4979%) *0.5953%	2.1061 (0.4732%) *0.4494%	2.5306 (0.4992%) *0.3945%
(3)	2.3633 (0.0842%) *0.0713%	0.5838 (0.2240%) *0.7675%	0.8398 (0.0960%) *0.2287%	1.1553 (0.0865%) *0.1498%	1.6368 (0.0969%) *0.1184%	2.0651 (0.1113%) *0.1078%	2.4862 (0.1101%) *0.0885%
(4)	2.4055 (0.3662%) *0.3045%	0.6016 (1.1255%) *3.7414%	0.8626 (0.5849%) *1.3561%	1.1851 (0.3865%) *0.6523%	1.6726 (0.4777%) *0.5712%	2.1061 (0.4259%) *0.4045%	2.5307 (0.4696%) *0.3711%
				B, $\rho_V = 40\%$			
(1)	2.3610 (0.0855%) *0.0725%	0.5221 (0.2709%) *1.0378%	0.8017 (0.1782%) *0.4446%	1.1288 (0.0800%) *0.1417%	1.6226 (0.0599%) *0.0738%	2.0585 (0.0845%) *0.0821%	2.4877 (0.0942%) *0.0758%
(2)	2.4034 (0.3216%) *0.2676%	0.5667 (1.2493%) *4.4087%	0.8472 (0.6882%) *1.6248%	1.1748 (0.6678%) *1.1368%	1.6661 (0.4419%) *0.5304%	2.1021 (0.3846%) *0.3660%	2.5312 (0.4355%) *0.3441%
(3)	2.3640 (0.0857%) *0.0725%	0.5498 (0.1936%) *0.7042%	0.8120 (0.0982%) *0.2419%	1.1332 (0.1313%) *0.2316%	1.6244 (0.0823%) *0.1013%	2.0605 (0.0951%) *0.0923%	2.4900 (0.1204%) *0.0967%
(4)	2.4052 (0.3128%) *0.2601%	0.5920 (1.1983%) *4.0480%	0.8517 (0.6243%) *1.4659%	1.1749 (0.6278%) *1.0687%	1.6657 (0.4520%) *0.5427%	2.1022 (0.3829%) *0.3643%	2.5309 (0.4075%) *0.3220%

(1) Monte Carlo simulation with exact option pricing formula (4.7) and 1,000,000 simulation runs

(2) Inversion formula (2.3) with exact option pricing formula (4.7) computed with Gaussian integration with $n = 96$ grid points applied on the intervals $[0,1]$, $(1,3]$, $(3,10]$ and $(10,50]$

(3) Monte Carlo simulation with delta-gamma approximation (4.13) and 1,000,000 simulation runs

(4) Inversion formula (2.3) with delta-gamma approximation (4.13) computed with Gaussian integration with $n = 96$ grid points applied on the intervals $[0,1]$, $(1,3]$, $(3,10]$ and $(10,50]$
For applying the inversion formula the unconditional characteristic function of the credit portfolio value has to be calculated by means of Monte Carlo simulations. For the above values 50,000 simulation runs have been used. The bisection method employed for finding the respective percentiles stopped when the difference between two arguments of the cumulative density function was smaller than 10^{-9}.

(), * See table 3.

Parameters: $N = 500$, $F = 1$, $T = 3$, $T^C = 2$, $H = 1$, $X = 0.92190$, $\rho_{r,V} = -0.05$, $\delta = 0.538$, $\kappa = 1.169$, $\theta = 0.061$, $\sigma_r = 0.029$, $\lambda = 0.88$, $r(0) = 0.061$.

Table 6: Computation Times for the Portfolio of European Call Options with Counterparty Risk on Risk-Free Zero Coupon Bonds

Portfolio of European Call Options with Counterparty Risk on Risk-Free Zero Coupon Bonds

(1)	**(2)**	**(3)**	**(4)**
1	7.06	1.04	7.13

Computation times for the expected credit portfolio value and the percentiles of the portfolio of European call options with counterparty risk on risk-free zero coupon bonds calculated with various methods. The computation time for the Monte Carlo simulation (1) is taken as the base, which is divided by the computation times of the other methods. Values larger than one indicate speed gains, values smaller than one indicate speed losses compared to the Monte Carlo simulation. For the explanation of the methods see table 5. Parameters: initial rating Baa, $\rho_V = 0.1$, other parameters see table 5.

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ENDNOTES

1 Studies, which analyze the effect of integrating an additional risk factor, such as stochastic interest rates or stochastic credit spreads, into a credit portfolio model are from *Barth* (2000), *Kijima* and *Muromachi* (2000), *Barnhill* and *Maxwell* (2002), *Kiesel*, *Perraudin* and *Taylor* (2003) and *Grundke* (2004a, b). There are also first attempts to create an integrated market and credit risk portfolio framework for commercial credit portfolio models, for example that one developed by the risk management firm Algorithmics (see *Iscoe*, *Kreinin* and *Rosen* (1999)).

2 See e.g. *Rouvinez* (1997) or *El-Jahel*, *Perraudin* and *Sellin* (1999).

3 Exceptions are *Merino* and *Nyfeler* (2002), *Laurent* and *Gregory* (2003), *Duffie* and *Pan* (2001) and *Reiß* (2003) (the two latter ones with additionally integrated market risk factors).

4 See e.g. *Martin*, *Thompson* and *Browne* (2001) and *Barco* (2004).

5 See e.g. *Wilde* (2001), *Martin* and *Wilde* (2002), *Gordy* (2003) or *Pykhtin* (2004).

6 See e.g. *Glasserman* and *Li* (2003), *Kalkbrener*, *Lotter* and *Overbeck* (2004), *Merino* and *Nyfeler* (2004) or *Tchistiakov*, *de Smet* and *Hoogbruin* (2004).

7 See *Stuart* and *Ord* (1994, pp. 125).

8 See *Gil-Pelaez* (1951, p. 481), *Stuart* and *Ord* (1994, p. 126).

9 See *Davies* (1973, p. 415). In the above representation the symmetry of the integrand has been used. $\mathrm{Im}(\cdot)$ denotes the imaginary part of the complex argument.

10 See e.g. *Frye* (2000, 2003) and *Pykhtin* (2003).

11 For details concerning this procedure see *Gupton*, *Finger* and *Bhatia* (1997, pp. 85).

12 It will be assumed that the interest rate factor X_r and the systematic credit risk factor Z are correlated with the credit spreads, which implies a non-zero correlation between the transition process and the credit spreads. *Kiesel*, *Perraudin* and *Taylor* (2003, p. 7, fn 9) already hint at the possibility of introducing the latter correlation by allowing the latent variables driving the transitions to be correlated with the credit spreads.

13 See the explanation of the CreditMetrics model in section III.2.

14 See *de Munnik* (1996, p. 71); *Vasicek* (1977, pp. 185).

15 *Kiesel*, *Perraudin* and *Taylor* (2003) show that the joint distribution of credit spread changes is approximately normal, at least for longer horizons such as one year, which are usually employed in the context of credit portfolio modeling.

16 This is the so-called Recovery-of-Treasury assumption used for example in the credit risk pricing models of *Jarrow* and *Turnbull* (1995), *Jarrow*, *Lando* and *Turnbull* (1997) or *Longstaff* and *Schwartz* (1995). See *Duffie* and *Singleton* (1999) for a discussion of various recovery assumptions.

17 This is the modeling framework recently used by *Grundke* (2004b). Beside this base case specification, *Grundke* (2004b) also studies the influence of various inhomogeneities in the portfolio composition, different distributional assumptions for the risk factors, and a recovery model, in which the recovery rate depends on the realization of the common systematic credit risk factors Z and X_r and the individual asset return realizations R_n. In each case, the effect that an integration of interest rate and credit spread risk into the model has on the credit portfolio distribution is analyzed. However, the focus of this paper is another one so that we deal only with the base case model described before.

18 Using moment generating functions, *Finger* (1999) describes a similar approach for the original CreditMetrics™ framework without integrated correlated interest rate and credit spread risk.

19 In order to simplify the notation, the dependence of the conditional thresholds $t_{j,k}$ on the realizations of Z and X_r is suppressed.

20 See *Johnson*, *Kotz* and *Balakrishnan* (1995, p. 218) for the characteristic function of a beta-distributed random variable, which can be expressed as a confluent hypergeometric function.

21 See *Pykhtin* (2003).

22 See *de Munnik* (1996, pp. 74).

23 This assumption can also be found for example in *Klein* (1996) and *Klein* and *Inglis* (2001). See the latter paper (pp. 997) also for an attempt to justify this at first sight rather restrictive assumption.

24 There are also – much more lengthy - analytical pricing formulas for European options with counterparty risk available when the credit quality of the counterparty and the underlying of the option are correlated (see e.g. *Klein* (1996)).

25 See for example *Jarrow*, *Lando* and *Turnbull* (1997).

26 The dependence of the price of the European call with counterparty risk on the risk-free interest rate $r(H)$ at the risk horizon is replaced by X_r because the realization of this random variable determines $r(H)$.

27 For other kinds of quadratic approximations of portfolio values (e.g. interpolation or least squares approach) see e.g. *Holton* (2003, pp. 334).

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[28] See *Moody's Investors Service* (1996). However, using Moody's estimates of the mean ignores the fact that the rating agency defines the recovery rate as a percentage of par and not as a percentage of a risk-free but otherwise identical zero coupon bond.

[29] See *Basel Committee on Banking Supervision* (2004).

[30] See *Eom*, *Helwege* and *Huang* (2004, table 1, p. 505) and *Lyden* and *Saraniti* (2000, table 6, p. 38).

[31] The numerical values of the derivatives are calculated with MAPLE.

[32] The grid points and weights of the Gaussian integration for $n = 96$ are taken from *Abramowitz* and *Stegun* (1984, p. 397). The length of the intervals on which the Gaussian integration is applied increases because for rising values of s the absolute value of the oscillating integrand decreases rapidly.