Information-based trade[1]

Philip Bond
Finance Department
The Wharton School
University of Pennsylvania
Philadelphia, PA 19104-6367
pbond@wharton.upenn.edu

Hülya Eraslan
Finance Department
The Wharton School
University of Pennsylvania
Philadelphia, PA 19104-6367
eraslan@wharton.upenn.edu

First draft: November 2005
This draft: May 5, 2006

[1]This paper previously circulated under the title "Information, trade and common knowledge with endogenous asset values." We thank Franklin Allen, Lawrence Glosten, Gary Gorton, Richard Kihlstrom, Andrew McLennan, An Yan, and Alexandre Ziegler for stimulating conversations, along with seminar participants at the University of Chicago, the University of Pennsylvania, the Frontiers of Finance Conference, and the European Winter Finance Conference. Any errors are our own.

Abstract

We study the possibility of trade for purely informational reasons. We depart from previous analyses (notably Grossman and Stiglitz 1980 and Milgrom and Stokey 1982) by allowing the final payoff of the object being traded to depend on an action taken by its eventual owner. A leading example is the trade of a controlling stake in a corporation. We characterize conditions under which equilibria with trade exist. We discuss implications for when trade occurs, the correlation of actions with trade, and the efficiency of equilibrium actions.

1 Introduction

Following Grossman and Stiglitz (1980) and Milgrom and Stokey (1982), economists have reached a consensus that under many circumstances it is impossible for an individual to profit from superior information. This result is often described as the "no trade" or "no speculation" theorem. The underlying argument is, at heart, straightforward. If a buyer is prepared to buy an asset from a seller for price p, then the buyer must believe that conditional on the seller agreeing to the trade, the asset value must exceed p in expectation. But conversely, knowing this the seller is at least as well off keeping the asset.

This insight has had enormous consequences for financial economics. Almost all observers of financial markets regard trade for informational reasons — *information-based trade* — as a key motive for trade. It is, after all, implausible that all trade is driven by pure risk-sharing motivations (the only alternative to information-based trade under standard assumptions). In particular, one would need to posit that risk-sharing needs evolve rapidly to account for the large high-frequency fluctuations observed in trading volume; and market participants appear to devote substantial resources to acquiring information. To generate information-based trade, the vast majority of papers studying financial markets introduce "noise traders" who trade for (typically exogenous) non-informational reasons.[1] Provided strategic agents are unable to observe the volume of noise trader activity information-based trade is possible.

In this paper we develop a distinct and hitherto neglected reason for trade between differentially informed parties: if information allows superior productive decisions to be made, then the information released in trade is socially valuable. This possibility, which is implicitly ruled out in Milgrom and Stokey's otherwise general framework, is enough to generate trade under some circumstances. We provide a comprehensive

[1] See, for example, Kyle (1985), and Glosten and Milgrom (1985).

characterization of the conditions under which trade can occur.

An example

The intuition for our results is best illustrated by an example. A risk neutral agent owns an asset, the value of which depends on two factors: an underlying but currently unobservable state variable $\omega \in \{\alpha, \beta\}$, and what the eventual asset owner chooses to do with the asset. For specificity, one can think of the asset in question as a controlling interest in a firm.

The best action for the asset owner to take depends on ω. If $\omega = \alpha$ the best action is A, and the asset is worth 2 if this action is taken. If $\omega = \beta$ the best action is B, and the asset is worth 1 if it is taken. The asset is valueless if any action other than the (state-contingent) best action is taken.

The unconditional probability of state α is $1/2$. Both the initial asset owner (the *seller*) and a second party (the *buyer*) privately receive signals that are partially informative about the true state ω. The buyer and seller have the same "skill" in taking actions A and B, so that the state and action contingent asset payoffs for *both* parties are as given above. Conditional on the state the signals are distributed independently and identically. Specifically, if the true state is α (respectively, β) then each party observes signal a (respectively, b) with probability $3/4$.

Consider the following trading game: after observing his signal, the buyer decides whether or not to offer to buy the asset, and if so, the price p at which he offers to buy. The seller either accepts or rejects the offer. We claim the following is an equilibrium: the buyer offers to buy the asset for $p = 0.8$ independent of his signal, and the seller accepts if and only if he observes signal b.

First, consider the situation faced by the seller. If he ends up with the asset, he must decide what to do using only his own information. As such, if he sees signal a and does not sell, his expected payoff is $3/2$, while if he sees signal b and does not sell his expected payoff is $3/4$.[2] Consequently, after signal b the seller prefers to sell at a price $p = 0.8$ rather than keep the asset; and after signal a, prefers to keep the asset rather than sell at this price.

Next, consider the buyer. Is he prepared to buy at the price $p = 0.8$? The key point to note is that in equilibrium the seller only accepts his offer when he observes signal b. The buyer can use this information to make a better decision.

Specifically, if the buyer observes signal a and knows the seller saw signal b, the buyer regards $\omega = \alpha$ and $\omega = \beta$ as equally likely. Consequently he will choose action A, giving an expected payoff of $2 \times 1/2 = 1$. On the other hand, if the buyer observes signal b, then given the seller also observed signal b the buyer's probability assessment that $\omega = \beta$ is $9/10$.[3] Given this, he chooses action B, yielding an expected payoff of $1 \times 9/10 = 9/10$. In both cases, the buyer's expected payoff exceeds the price $p = 0.8$. As such, the behavior described is indeed an equilibrium.[4]

In this example both parties are strictly better off under the trade. Moreover, they are both better off even after conditioning on any information they acquire in equilibrium. The reason this is possible is that the information revealed by the agents' equilibrium actions enhances the asset's value for its eventual owner. In contrast, in Grossman and Stiglitz (1980) and Milgrom and Stokey (1982) the final asset payoffs are exogenous.

In this paper we analyze the degree to which efficiency gains arising from additional information make information-based trade possible. Before proceeding to the details

[2]Note that since $3/4 \times 1 > 1/4 \times 2$, action B is the better action to take if the only information available is that one of the signals is b.

[3]Specifically, the buyer's posterior belief is given by:

$$\Pr(\beta|bb) = \frac{\Pr(\beta)\Pr(b|\beta)^2}{\Pr(\alpha)\Pr(b|\alpha)^2 + \Pr(\beta)\Pr(b|\beta)^2} = \frac{\left(\frac{3}{4}\right)^2}{\left(\frac{1}{4}\right)^2 + \left(\frac{3}{4}\right)^2} = \frac{9}{10}.$$

[4]In Proposition 5 below we show that there exist out-of-equilibrium beliefs under which the buyer cannot profitably deviate and offer $\tilde{p} \neq 0.8$.

of our analysis, however, we wish to make the following clear: we are *not* arguing that trade is a superior mechanism relative to other alternatives. Instead, we view trade as one information-sharing mechanism among many — albeit a mechanism that is widely observed,has long interested economists, and has many appealing features.

Paper outline

In Section 2 we present our main model, which is a generalization of the example above. In Section 3 we establish necessary conditions for trade to take place. In particular, we show that the buyer must learn something about the seller's information. In order to provide a complete characterization of when trade can occur, in Section 4 we focus on the special case of our model in which the signals of both the buyer and seller are drawn from binary distributions. We provide a succinct condition that is both necessary and sufficient for trade. We then discuss various properties of the resulting equilibria. In particular, we show the following. Trade only occurs when the seller sees what is from his perspective a bad signal. Thus when the buyer acquires the asset he learns the seller's signal in equilibrium. This extra information allows the buyer to make better use of the asset — 0the source of the gains from trade. However, when the seller sees the "good" signal, he keeps the asset, and does not learn the buyer's information. Trade is correlated with the action taken, in the sense that there exists an action that is taken *only* after trade occurs.

In Section 5 we depart from the binary signal assumption, and verify that trade is still possible even when agents' signal sets are very rich. Finally, in Section 6 we explore the possibility of trade in a setting where agents cannot directly control asset payoffs. However, they still face a significant economic decision, namely the optimal allocation of their portfolios. As in our basic model, this decision is enough to generate trade. In the same section we also discuss our model's implications for the price response to trade announcements.

Related literature

The key assumption of our model is that the economic agent who decides how to use an asset is able to infer useful information from the trading process. The notion that prices reveal information that is useful for real decisions is an old one in economics. Nonetheless, it is only comparatively recently that researchers have constructed formal models in which, for example, managers learn from the share price. The key difficulty, of course, is that if share prices affect decisions, those decisions in turn affect share prices. Contributions to this so-called "feedback effect" literature include Khanna, Slezak, and Bradley (1994), Dow and Gorton (1997), Subrahmanyam and Titman (1999), Dye and Sridhar (2002), Dow and Rahi (2003), Goldstein and Guembel (2005), and Dow, Goldstein and Guembel (2006). Chen, Goldstein and Jiang (2005) and Durnev, Mork and Yeung (2004) both present empirical evidence that managers are indeed able to make better decisions as a result of information obtained from stock prices. In more general terms, our paper belongs to a growing literature that seeks to combine insights from corporate finance with those from the distinct market microstructure and asset pricing literatures.

A number of classic papers (notably, Hirshleifer 1971) note the distinction between information in an exchange economy and information in a production economy. However, the subsequent literature on the possibility of trade between differentially and privately informed parties has focused almost exclusively on information in an exchange economy. In particular, the seminal papers of Grossman and Stiglitz (1980) and Milgrom and Stokey (1982) show that under many circumstances trade is impossible in such an environment. Milgrom and Stokey's "no trade" or "no speculation" result (see also Holmström and Myerson 1983) rests on two assumptions: Pareto optimality of the initial allocation, and concordancy of beliefs, in the sense that agents agree on how to interpret future information. A large subsequent literature has explored conditions under which the "no trade" conclusion does not hold. The literature is too large to adequately survey. Representative approaches include departing from the common prior assumption, as in Morris (1994) and Biais and Bossaerts (1998), and thus breaking belief concordancy; departing from Pareto optimality, as in Dow and Gorton (1995), who assume that some agents can trade only a subset of assets; and introducing multiple trading rounds, as Grundy and McNichols (1998) do when they show that both belief concordancy and Pareto optimality may fail at the intermediate date of a three-period model.[5]

None of the above papers study the possibility of trade for purely informational reasons in an economy in which asset owners must decide how to use their assets. To the best of our knowledge the only previous consideration of this case is a chapter of Diamond's (1980) dissertation. He derives conditions under which a rational expectations equilibrium (REE) with trade exists when there are two types of agents: one type is uninformed, while the other type observes a noisy signal. The main differences between our paper and his are that (i) we study trade between agents who both possess information, (ii) we show that as a consequence, information is never fully revealed, and (iii) instead of restricting attention to the competitive (REE) outcome, in the spirit of Milgrom and Stokey (1982) we allow for all possible trading mechanisms. Moreover, Diamond's assumption that one side of the trade is completely uninformed means that assets always flow from the less to the more informed party.[6] In contrast, when both parties to the trade have some information, assets can flow to the party with lower quality information.

[5] One can also avoid the no-trade conclusion by using non-standard preferences: see, e.g., Halevy (2004).

[6] Diamond does consider an equilibrium in which the uninformed party ends up holding the asset. However, to support the equilibrium he must assume that the uninformed agent learns only from the price at which the trade takes place, and not from the volume of trade.

2 The model

Our model is a generalization of the example above. There are two risk neutral agents, who we refer to as a seller (agent 1) and a buyer (agent 2). The seller owns an asset. To rule out trades in which both parties are exactly indifferent between trading and not trading the asset, we assume that a small cost is associated with transferring the asset from agent 1 to agent 2. Specifically, we assume that whenever the asset changes hands its final value is reduced by $\delta > 0$.

As in the example, the eventual asset owner must decide what action to take. Regardless of whether the asset-owner is agent 1 or 2, the range of available actions is given by a compact set $\mathcal{X}$, with a typical element denoted by X.

The payoff from the asset depends on the combination of the action taken by the asset-owner and the realization of an unobserved state variable $\omega \in \{\alpha, \beta\}$.[7] We write the payoff when action X is taken and the state is ω as $v(X, \omega)$. We emphasize that the asset payoff is independent of the identity of the asset-owner — both agents 1 and 2 are equally capable of executing all actions in $\mathcal{X}$.

Examples

Many applications fall within this framework. For example, the asset in question might be a large block of shares in a troubled firm, one action is the decision to restructure the firm, and another action is the decision to liquidate it by selling off all its divisions. A second example is that in which the asset is a distressed debt claim. The owner of the claim must engage in restructuring negotiations with the issuer. One action is a refusal to accept any write-down of the debt, while another action is an agreement to postpone some the payments.

[7] The assumption that the underlying state space is binary implies that uncertainty is unidimensional.

Informational assumptions

Before meeting, both agents $i = 1, 2$ receive noisy and informative (with respect to ω) signals $s_i \in S_i$, where S_i is finite.[8] We assume that conditional on ω signals s_1 and s_2 are independent.

Signals are important in our model because they provide information about whether the state is α or β. The asset-owner's action choice depends on his assessment of the relative probabilities of states α and β. In general, an agent's information is represented by a partition $\mathcal{P}$ of the signal space $S_1 \times S_2$.[9] The information he has available after signal realizations $s_1 s_2$ is the element of the partition that contains $s_1 s_2$. For example, if the seller (agent 1) learns nothing in equilibrium and so knows only his own signal, his information partition is $\{\{s_1\} \times S_2 : s_1 \in S_1\}$, the set of signal realizations he believes possible following signal realization $s_1 s_2$ is $\{s_1\} \times S_2$.

For arbitrary subsets Y and Y' of $S_1 \times S_2$, we will say that Y is more pro-α than Y' (or equivalently, Y' is more pro-β than Y) whenever

$$\Pr(\alpha|Y) \geq \Pr(\alpha|Y').$$

It is straightforward to verify that Y is more pro-α than Y' if and only if the likelihood condition

$$\frac{\Pr(Y|\alpha)}{\Pr(Y|\beta)} \geq \frac{\Pr(Y'|\alpha)}{\Pr(Y'|\beta)}$$

holds. Notationally, we write $Y \succeq Y'$ whenever Y is more pro-α than Y'.

In many cases we are interested in what an agent i concludes about the probability of state α after observing only his own signal. Formally, he knows $\{s_i\} \times S_j$: that is, he knows nothing about the signal observed by agent $j \neq i$. With slight abuse of notation, we say that s_i is more pro-α than s_i' (or $s_i \succeq s_i'$) if $\{s_i\} \times S_j$ is more pro-α than $\{s_i'\} \times S_j$.

[8] In Section 5 we allow for continuous signals.

[9] For an overview of partition representations of information, see, e.g., Chapter 5 of Osborne and Rubinstein (1994).

The informativeness of the signals observed by the two agents potentially differs. We assume throughout that the seller's information is weakly better, in the sense detailed below. Focusing on this case makes trade harder to obtain compared to the opposite case. Moreover, for most applications it is natural to assume that the existing owner knows more about the asset than does a potential buyer.

We formalize this assumption as follows. For agent $i \in \{1,2\}$, let s_i^α and s_i^β respectively be the most pro-α and pro-β signals in S_i. We assume that the quality of seller's (agent 1's) information is weakly better than the buyer's (agent 2) in the following sense:

$$\frac{\Pr\left(s_1^\alpha s_2^\beta|\alpha\right)}{\Pr\left(s_1^\alpha s_2^\beta|\beta\right)} \geq \frac{\Pr\left(s_1^\beta s_2^\alpha|\alpha\right)}{\Pr\left(s_1^\beta s_2^\alpha|\beta\right)}. \tag{1}$$

That is, if agents receive conflicting and extreme signals, the seller's signal is weakly more indicative of the true state. Equal information quality is, of course, a special case.[10]

[10]Note that it is quite possible for the expected value of the asset to the buyer to exceed the expected value of the asset to the seller, even if both observe only their own signals s_2 and s_1 respectively. Consider the following. As in the opening example, the unconditional probability of state α is 1/2; the action set is $\mathcal{X} = \{A, B\}$; the asset payoffs are $v(A,\alpha) = 2$, $v(B,\beta) = 1$, $v(A,\beta) = v(B,\alpha) = 0$; and the signal sets for both agents are binary: $S_i = \{a_i, b_i\}$. For the seller, $\Pr(a_1|\alpha) = 0.97$ and $\Pr(b_1|\beta) = 0.73$. For the buyer, $\Pr(a_2|\alpha) = \Pr(b_2|\beta) = 0.9$. It is easily verified that agent i takes action A if $s_i = a_i$ and action B if $s_i = b_i$, for $i = 1, 2$. Note that

$$\frac{\Pr(a_1 b_2|\alpha)}{\Pr(a_1 b_2|\beta)} = \frac{.97}{.27}\frac{.1}{.9} > \frac{.03}{.73}\frac{.9}{.1} = \frac{\Pr(b_1 a_2|\alpha)}{\Pr(b_1 a_2|\beta)},$$

so that the seller's signal is more informative than the buyer's, in the sense that condition (1) is satisfied. Observe that while the seller's signal b_1 is more pro-β than the buyer's signal b_2, the seller's signal a_1 is *less* pro-α than the buyer's signal a_2. Concretely, the seller's signal a_1 is not very informative because it is observed often when the state is β. As such, the seller often makes the wrong decision in state β. In contrast, the buyer is less likely to observe a_2 in state β, and so makes the wrong decision in state β less often. Conversely, he makes the wrong decision in state α more often than the seller does. However the absolute cost of the seller's mistakes exceeds that of

Finally, we assume that the signals of both agents are at least somewhat informative: at a minimum, $\Pr(s_i^\alpha|\alpha) > \Pr(s_i^\alpha|\beta)$.

The eventual asset owner must select an action $X \in \mathcal{X}$ conditional on knowing that the signal realization $s_1 s_2$ falls in subset $Y \subset S_1 \times S_2$. Based on information Y, he can conclude that the probability of state α is $\Pr(\alpha|Y)$. We define $V(p; X)$ to be the expected payoff of the asset when the asset owner takes action X and the probability of state α is p. Let $V(p)$ denote the expected value of the asset, given that the asset owner takes the best action and the probability of state α is p, i.e.,

$$V(p) = \max_{X \in \mathcal{X}} V(p; X).$$

Abusing the notation, we will often write $V(Y)$ for $V(\Pr(\alpha|Y))$. Figure 2 plots $V(p; X)$ and $V(p)$ for the opening example.

The function $V(p; X)$ is linear for each action $X \in \mathcal{X}$. Since $V(p)$ is the upper envelope of linear functions, it is a convex function. As such, it cannot achieve a strict maximum in the interior of a set. These observations deliver the following simple result, which we use heavily throughout the paper:

Lemma 1. *For all* $Y, Y', Y'' \subset S_1 \times S_2$, *if* $Y \succeq Y' \succeq Y''$, *then* $V(Y') \leq \max\{V(Y), V(Y'')\}$.

Information-based trade and Pareto optimality of the original allocation

Milgrom and Stokey's "no speculation" theorem establishes that trade cannot occur purely for information-based reasons. Of course, this in no way affects the possibility of trade for risk-sharing reasons. As such, Milgrom and Stokey's result is predicated on the Pareto optimality of the pre-trade state-contingent allocation.

In our setting, both agents are risk neutral, and are equally capable of executing any action $X \in \mathcal{X}$. As such, the *only* possible motivation for trade is the differential

the buyer's (even though mistakes are more costly in state α than β).

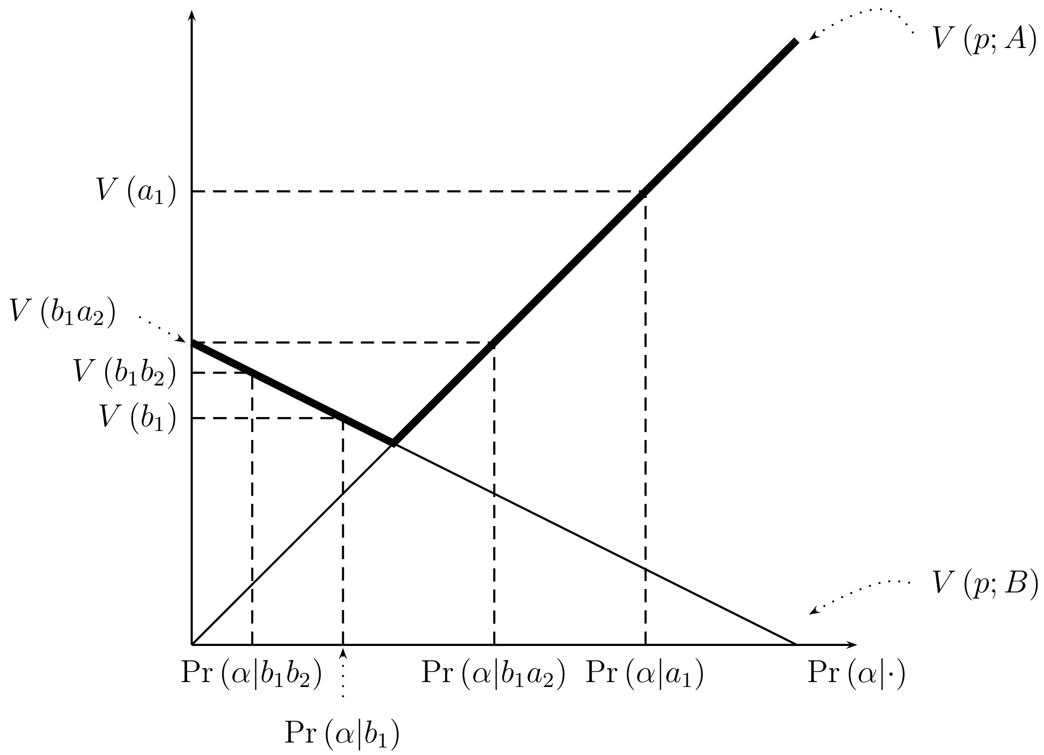

Figure 1: The graph displays $V(p; X)$ for the opening example: the action set is $\mathcal{X} = \{A, B\}$ and the signal sets are $S_i = \{a_i, b_i\}$ for $i = 1, 2$. The bold line is the upper envelope of these two functions, and corresponds to the function $V(p)$.

information of the two parties. Formally, since risk-sharing motivations are absent, any state-contingent allocation is Pareto optimal. Of course, this ignores the fact that agents 1 and 2 potentially have different information, and so take different actions. However, trade motivated by such considerations is precisely information-based trade, and is the main object of our analysis.

3 Necessary conditions for trade

The main goal of our analysis is to characterize when trade can — and cannot — occur for purely informational reasons. The answer to this question clearly depends to some extent on the institutional environment. However, it is also clear that we want our results to be as independent as possible of *a priori* assumptions about the trading environment.

To meet these objectives, we begin by establishing necessary conditions for trade to occur in a very wide class of trading mechanisms. The only condition we impose that trades must be *ex post individually rational.* That is, both agents 1 and 2 must prefer the post-trade outcome to the original allocation (in which agent 1 owns the asset), even after conditioning on any information they acquire in equilibrium. (We give a formal statement in (2) below.) This condition must be met state-by-state. We adopt this requirement for two reasons. First, it is a demanding condition to satisfy, and so biases our analysis against generating trade. Second, it is used in many prior analyses. In particular, it is equivalent to Milgrom and Stokey's (1982) requirement of common knowledge of gains from trade;[11] and is part of the definition

[11] In Milgrom and Stokey, agent i evaluates the trade according to the partition P_i, "his information at the time of trading, including whatever he can infer from prices or from the behavior of other traders" (page 19). We take this information to include at least the information revealed by the post-trade allocation. In Milgrom and Stokey's framework, there would still be no trade even if one instead assumed that agent i possessed coarser information. In contrast, in our model coarsening

of a rational expectations equilibrium.

The main result that we establish in this section is:

Proposition 1. *There is no* ex post *individually rational trade in which the buyer learns nothing in all states in which he acquires the asset.*

Preliminaries

As a preliminary, we start by introducing some more formal notation to describe information, allocations, and payoffs. As discussed above, we represent agents' information by partitions of the signal space $S_1 \times S_2$. If an agent i receives only his own signal, and does not learn anything about the signal of agent $j \neq i$, his information partition is

$$\hat{\mathcal{P}}_i \equiv \{\{s_i\} \times S_j : s_i \in S_i\}.$$

An allocation in our economy is a pair of mappings $\kappa : S_1 \times S_2 \to \{1, 2\}$ and $\tau : S_1 \times S_2 \to \Re$ where κ specifies which agent owns the asset, and τ specifies a transfer from agent 2 to agent 1. Let $(\hat{\kappa}, \hat{\tau})$ denote the initial allocation, in which agent 1 owns the asset and no transfer takes place: $(\hat{\kappa}, \hat{\tau}) \equiv (1, 0)$.

A trade is an allocation (κ, τ) distinct from $(\hat{\kappa}, \hat{\tau})$. Trades potentially reveal information. Formally, let $\mathcal{Q}_i^{\kappa,\tau}$ be the information revealed through trade to agent i. Note that $\mathcal{Q}_i^{\kappa,\tau}$ certainly contains the information revealed directly by the trade; depending on the trading mechanism, it may also include additional information. The information available to agents 1 and 2 respectively after conditioning on information revealed by the trade is thus $\mathcal{P}_1^{\kappa,\tau} \equiv \hat{\mathcal{P}}_1 \vee \mathcal{Q}_1^{\kappa,\tau}$ and $\mathcal{P}_2^{\kappa,\tau} \equiv \hat{\mathcal{P}}_2 \vee \mathcal{Q}_2^{\kappa,\tau}$, where $\vee$ denotes the coarsest common refinement.

For any partition $\mathcal{P}$ of $S_1 \times S_2$ let $\mathcal{P}(s_1 s_2)$ denote the partition element containing the signal realization $s_1 s_2$. The value of the asset for an agent who knows the signal

the information that agent i uses to evaluate the trade will generally enhance trade opportunities, since it weakens the *ex post* individual rationality condition.

realization lies in $\mathcal{P}(s_1 s_2)$ is $V(\mathcal{P}(s_1 s_2))$. For an arbitrary allocation (κ, τ) and arbitrary information partitions $\mathcal{P}_1$ and $\mathcal{P}_2$, agents' payoffs for the signal realization $s_1 s_2$ are thus

$$u_1(s_1 s_2; \kappa, \tau, \mathcal{P}_1) \equiv \begin{cases} V(\mathcal{P}_1(s_1 s_2)) + \tau(s_1 s_2) & \text{if } \kappa(s_1 s_2) = 1 \\ \tau(s_1 s_2) & \text{if } \kappa(s_1 s_2) = 2 \end{cases}$$

$$u_2(s_1 s_2; \kappa, \tau, \mathcal{P}_2) \equiv \begin{cases} -\tau(s_1 s_2) & \text{if } \kappa(s_1 s_2) = 1 \\ V(\mathcal{P}_2(s_1 s_2)) - \delta - \tau(s_1 s_2) & \text{if } \kappa(s_1 s_2) = 2 \end{cases}.$$

Recall that we have assumed that $\mathcal{P}_i^{\kappa,\tau}$ contains any information directly revealed by the allocation (κ, τ). As such, both κ and τ are $\mathcal{P}_i^{\kappa,\tau}$-measurable, and so $u_i(\cdot; \kappa, \tau, \mathcal{P}_i^{\kappa,\tau})$ is also. We are now in a position to give a precise statement of the *ex post* individual rationality requirement. Consider an arbitrary allocation (κ, τ), along with associated information $\mathcal{P}_1^{\kappa,\tau}$ and $\mathcal{P}_2^{\kappa,\tau}$. *Ex post* individual rationality is satisfied when in every state $s_1 s_2$, agent i's utility under the allocation (κ, τ) exceeds his utility under the original allocation $(\hat{\kappa}, \hat{\tau})$. The information used by agent i to value the asset is $\mathcal{P}_i^{\kappa,\tau}$. Formally, the condition is: for agents $i = 1, 2$ and all signal realizations $s_1 s_2 \in S_1 \times S_2$,

$$u_i\left(s_1 s_2; \kappa, \tau, \mathcal{P}_i^{\kappa,\tau}(s_1 s_2)\right) \geq u_i\left(s_1 s_2; \hat{\kappa}, \hat{\tau}, \mathcal{P}_i^{\kappa,\tau}(s_1 s_2)\right). \tag{2}$$

The proof of Proposition 1

Proposition 1 is established by contradiction. Suppose to the contrary that an *ex post* individually rational trade (κ, τ) exists in which the buyer learns nothing whenever he acquires the asset. Let $s_1 s_2$ be a signal realization at which the buyer acquires the assets, and $p = \tau(s_1 s_2)$ the price paid at that realization. Since the buyer learns nothing, trade must occur at the same terms over $S_1 \times \{s_2\}$. It follows that the subset of the signal space in which trade occurs at price p is of the form $S_1 \times S_2^T$, where S_2^T is a subset of S_2. (Of course, trade may occur at a different price in some other subset of the signal space.)

Since the buyer does not learn anything, individual rationality implies that for all $s_2 \in S_2^T$

$$p < V(S_1 \times \{s_2\}).$$

The seller's information partition after trade is $\mathcal{P}_1^{\kappa,\tau}$. Note that $S_1 \times S_2^T$ is $\mathcal{P}_1^{\kappa,\tau}$-measurable since the seller learns at least the information conveyed by the trade. The seller's individual rationality condition implies

$$p \geq V(Q)$$

for all elements $Q \in \mathcal{P}_1^{\kappa,\tau}$ such that $Q \subset S_1 \times S_2^T$.

Lemma 1 gives the required contradiction provided that we can find $Q, Q' \in \mathcal{P}_1^{\kappa,\tau}$ such that $Q, Q' \subset S_1 \times S_2^T$ and $s_2 \in S_2^T$ such that

$$Q \succeq S_1 \times \{s_2\} \succeq Q'. \tag{3}$$

That is, trade in which the buyer learns nothing is impossible if there is a signal realization s_2 for the buyer such that sometimes the seller's information is more pro-α, and sometimes it is more pro-β. (All three pieces of information here are associated with trade.) Roughly speaking, the reason trade is impossible in this case is that the asset is less valuable to an agent who is unsure about the true state than to one who is relatively confident about the true state. (Formally, this follows from the the convexity of the function V.)

The existence of $Q, Q' \in \mathcal{P}_1^{\kappa,\tau}$ and $s_2 \in S_2^T$ satisfying (3) follows from the next result, which is direct consequence of our assumption that the seller's signal is weakly more informative than the buyer's (condition (1)).

Lemma 2. *Let the seller's and buyer's information partitions be $\mathcal{P}_1$ and $\mathcal{P}_2$ respectively, and suppose there exists a subset S_2^T of the buyer's signal set S_2 such that (i) the buyer learns only his own signal when $s_2 \in S_2^T$, that is, $S_1 \times \{s_2\} \in \mathcal{P}_2$ for all*

$s_2 \in S_2^T$; and (ii) $S_1 \times S_2^T$ is measurable with respect to the seller's information $\mathcal{P}_1$. Then there exist $Q, Q' \in \mathcal{P}_1$ and $s_2 \in S_2^T$ such that $Q, Q' \subset S_1 \times S_2^T$ and condition (3) holds.

Proof of Lemma 2: The general proof of Lemma 2 is relegated to the Appendix. Here, we establish it for the special case of binary signal sets: $S_i = \{a_i, b_i\}$ for $i = 1, 2$. Without loss we assume that $\Pr(\alpha|a_i) \geq \Pr(\alpha) \geq \Pr(\alpha|b_i)$ for $i = 1, 2$.

There are three possibilities for the set S_2^T: $\{a_2, b_2\}$, $\{a_2\}$ and $\{b_2\}$. We consider each in turn. The key reason why Lemma 2 holds is condition (1). In the binary signal case, this says that $a_1 b_2 \succeq b_1 a_2$, which in turn implies that either $a_1 \succeq a_2$ or $b_2 \succeq b_1$. That is, either the seller's most pro-α signal is more pro-α than any of the buyer's signals; or the seller's most pro-β signal is more pro-β than any of the buyer's signals.

First, suppose that $S_2^T = \{a_2, b_2\}$. One possibility is that the seller learns only his own signal, $\mathcal{P}_1 = \{\{a_1\} \times S_2, \{b_1\} \times S_2\}$. Consider the case in which $a_1 \succeq a_2$. By setting $Q = \{a_1\} \times S_2$ and $s_2 = a_2$ we obtain seller information Q that is more pro-α than buyer information $S_1 \times \{a_2\}$. It is easy to find seller information Q' that is more pro-β than the buyer information $S_1 \times \{a_2\}$ — simply set $Q' = \{b_1\} \times S_2$. Together, Q, Q' and $s_2 = a_2$ satisfy condition (3). The case $b_2 \succeq b_1$ follows similarly.

If instead of learning only his own signal the seller learns more, his most pro-α signal becomes weakly more pro-α and his most pro-β signal becomes weakly more pro-β. As such, it becomes easier to choose Q, Q' and s_2 so as to satisfy condition (3).

The remaining cases in which $S_2^T = \{a_2\}$ and $S_2^T = \{b_2\}$ are straightforward. In the former case, since $S_1 \times \{a_2\}$ must be measurable with respect to the seller's information partition, $\mathcal{P}_1 = \{\{a_1 a_2\}, \{b_1 a_2\}, \{a_1 b_2\}, \{b_1 b_2\}\}$. So setting $Q = \{a_1 a_2\}$, $Q' = \{b_1 a_2\}$ and $s_2 = a_2$ satisfies condition (3). The case $S_2^T = \{b_2\}$ follows in an identical manner. ∎

As noted, Lemma 2 delivers the contradiction required to complete the proof of Proposition 1.[12]

Trade at extreme buyer signals

Proposition 1 says that no trade is possible unless some useful information is conveyed to the buyer. This conclusion is very much in line with those reached in the existing no-trade literature. At the same time, and as our opening example makes clear, trade is at least sometimes possible if it enables the buyer to learn the seller's signal.

Information is valuable to the buyer because it allows him to make a better decision with what to do with the asset, which in turn raises the asset's value to him. Loosely speaking, the value of the asset is increasing in the extent to which its owner is confident that he knows the realization of the state ω — regardless of whether the realization is α or β. As such — and again loosely speaking — if the buyer is happy to acquire the asset after seeing a pro-α signal s_2, he is happy to acquire it after seeing even more pro-α signals. Formally, we establish:

Proposition 2. *For any $s_2, s_2', s_2'' \in S_2$ such that $s_2 \succeq s_2' \succeq s_2''$, if there is trade in $S_1^T \times \{s_2'\}$ for some $S_1^T \subset S_1$, then there exists $\tilde{S}_1 \subset S_1$ such that there is trade either in $\tilde{S}_1 \times \{s_2\}$ or $\tilde{S}_1 \times \{s_2''\}$.*

Proof of Proposition 2: The proof is in the Appendix. It uses a mechanism design approach to show that if the buyer acquires the asset at s_2' but not at either of signals s_2, s_2'', then the buyer has an incentive to deviate at at least one of signals s_2, s_2''. ∎

[12] Given our assumption that the seller's signal is weakly better than the buyer's, some readers may conjecture that a more direct proof is available: if the seller knows weakly more, the asset is on average more valuable to him than to the buyer, and so no trade is possible. However, this conclusion is not valid. The example of footnote 10 provides a counterexample.

An immediate implication of Proposition 2 is that the signal realizations at which the buyer *never* acquires the asset are contiguous:

Corollary 1. *Enumerate S_2 as $s_2^1, \ldots, s_2^m$, where $s_2^1 \succeq \ldots \succeq s_2^m$. Let S_2^T be the subset of signals at which the buyer sometimes acquires the asset in an* ex post *incentive compatible trade. Then S_2^T is of the form $S_2 \setminus \left\{s_2^k, s_2^{k+1}, \ldots, s_2^{j-1}, s_2^j\right\}$, some k, j.*

4 Necessary and sufficient conditions for trade

So far we have characterized several conditions that are necessary for trade. In particular, Proposition 1 establishes that trade is only possible if the buyer learns something from the seller.

In this section we study a special case of our economy in which the action set $\mathcal{X}$ and the signal sets S_1, S_2 are all binary: $\mathcal{X} = \{A, B\}$, $S_1 = \{a_1, b_1\}$ and $S_2 = \{a_2, b_2\}$. For this class of economies we are able to completely and succinctly characterize when trade is — and is not — possible, interpret our conditions, and derive economic implications.

Without loss, we assume that for agents $i = 1, 2$, $\Pr(a_i|\alpha) > \Pr(a_i|\beta)$, so that signal realization a_i is strictly more pro-α than signal realization b_i. Together with condition (1), this normalization implies that

$$a_1 a_2 \succeq a_1 \succeq a_1 b_2 \succeq b_1 a_2 \succeq b_1 \succeq b_1 b_2,$$

with either $a_1 \succeq a_2$ or $b_2 \succeq b_1$. Also without loss we assume that the highest asset payoff is achieved when action A is selected and $\omega = \alpha$:

$$v(A, \alpha) = \max\{v(A, \alpha), v(B, \alpha), v(A, \beta), v(B, \beta)\}. \tag{4}$$

To ease notation, throughout the section we write $V(s_1 s_2)$ in place of $V(\{s_1 s_2\})$, and $V(s_i)$ in place of $V(\{s_i\} \times S_j)$.

Necessary conditions for the binary economy

In the binary economy, Propositions 1 implies a succinct set of necessary conditions for trade:

Proposition 3. Ex post *individually rational trade is possible only if either*

$$\min\{V(b_1b_2), V(b_1a_2)\} > V(b_1) \tag{TC1}$$

or

$$\min\{V(a_1a_2), V(a_1b_2)\} > V(a_1) \tag{TC2}$$

holds. If the former is satisfied and trade occurs, it does so precisely in states $\{b_1a_2, b_1b_2\}$*. If the latter is satisfied and trade occurs, it does so precisely in* $\{a_1a_2, a_1b_2\}$*.*

Proof of Proposition 3: Suppose that (κ, τ) is an *ex post* individually rational trade. From Proposition 1, there must exist some signal realization $s_1^*s_2^* \in S_1 \times S_2$ at which the buyer acquires the asset and learns something about the seller's signal. Formally, $\mathcal{P}_2^{\kappa,\tau}(s_1^*s_2^*) \neq S_1 \times \{s_2^*\}$. Because S_2 is binary, the only possibility is $\mathcal{P}_2^{\kappa,\tau}(s_1^*s_2^*) = \{s_1^*s_2^*\}$: that is, the buyer learns the seller's signal at $\{s_1^*s_2^*\}$. Since the buyer learns the seller's signal at $s_1^*s_2^*$, the seller cannot learn the buyer's signal at $s_1^*s_2^*$, for otherwise there is no way both the buyer and seller *ex post* incentive compatibility conditions can hold. That is, $\mathcal{P}_1^{\kappa,\tau}(s_1^*s_2^*) = \{s_1^*\} \times S_2$.

The seller learns at least the information conveyed by the terms of trade, and so trade must occur in both $s_1^*a_2$ and $s_1^*b_2$, and must do so at the same terms. Let p be the common price (i.e., $p = \tau(s_1^*s_2^*)$). *Ex post* incentive compatibility implies $V(s_1^*s_2^*) > p \geq V(s_1^*)$.

Throughout the proof, we write s_i for the realization of agent i's signal that is *not* s_i^*. There are two cases to consider, depending on what the buyer learns when his own signal is $s_2 \neq s_2^*$:

Case: Buyer learns nothing at s_2, i.e., $\mathcal{P}_2^{\kappa,\tau}(s_1^*s_2) = S_1 \times \{s_2\}$.

We prove, by contradiction, that this case cannot arise. Suppose to the contrary that a trade with property exists, and $s_1^* = a_1$. (The proof when $s_1^* = b_1$ is symmetric.) Since the buyer learns nothing, trade must occur at price p in $s_1^* s_2$ and $s_1 s_2$, and so buyer *ex post* incentive compatibility implies $V(s_2) > p$.

First, suppose that the seller learns the buyer's signal when he observes $s_1 = b_1$, i.e., $\mathcal{P}_1^{\kappa,\tau}(s_1 s_2) = \{s_1 s_2\}$. In this case, seller *ex post* incentive compatibility implies that $p \geq V(b_1 s_2)$. As such, if $s_2^* = a_2$ we have $V(b_2) > \max\{V(a_1), V(b_1 b_2)\}$, while if $s_2^* = b_2$ we have $V(a_1 b_2) > \max\{V(a_1), V(b_1 a_2)\}$. In either instance, Lemma 1 gives the necessary contradiction.

Second, suppose that the seller does not learn the buyer's signal when he observes $s_1 = b_1$, i.e., $\mathcal{P}_1^{\kappa,\tau}(s_1 s_2) = \{s_1\} \times S_2$. Since the seller cannot distinguish signal realizations $s_1 s_2$ and $s_1 s_2^*$, trade must occur at a price p in both. Since $\{s_1^* s_2^*\}$ is in the buyer's information partition $\mathcal{P}_2^{\kappa,\tau}(s_1^* s_2^*)$, the set $\{s_1 s_2^*\}$ is also (the buyer directly observes s_2^*). *Ex post* incentive compatibility implies that $V(b_1 s_2^*) > p \geq V(b_1)$. As such, if $s_2^* = a_2$ we have $V(b_1 a_2) > \max\{V(a_1), V(b_1)\}$, while if $s_2^* = b_2$ we have $V(a_1 b_2) > \max\{V(a_1), V(b_1)\}$. In either instance, Lemma 1 gives the necessary contradiction.

Case: Buyer learns the seller's signal at s_2, i.e., $P_2^{\kappa,\tau}(s_1^* s_2) = \{s_1^* s_2\}$.
Buyer *ex post* incentive compatibility implies $V(s_1^* s_2) > p$. Substituting in for $s_1^* = b_1$ and a_1 respectively gives trade conditions (TC1) and (TC2).

To complete the proof, note that the two conditions (TC1) and (TC2) cannot hold simultaneously: for if they were to do so,

$$\max\{V(b_1 a_2), V(a_1 b_2)\} > \max\{V(a_1), V(b_1)\},$$

which is impossible by Lemma 1. ∎

Sufficient conditions for the binary economy

Proposition 3 establishes that conditions (TC1) and (TC2) are necessary for trade. Are they also sufficient? The answer clearly depends on the institutional setting in which trade occurs. However, for at least two simple trading mechanisms the answer is yes.

The first specific trading mechanism we consider is one in which: (1) a non-strategic third-party (the broker) sets a price p, (2) the seller can place his asset for sale at price p, in which case (3) the buyer decides whether or not to buy at this price. For this trading mechanism, we establish:

Proposition 4. *Suppose that one of trade conditions (TC1) and (TC2) holds. Then for sufficiently small trading costs δ there exists an equilibrium of the "third-party posts a price" mechanism described in which trade occurs. The equilibrium satisfies* ex post *individual rationality.*

The proof of Proposition 4 makes use of the following straightforward result, which we state separately for future reference:

Lemma 3. *If* $\min\{V(b_1b_2), V(b_1a_2)\} > V(b_1)$ *then* $V(a_1) > V(b_1)$*; if* $\min\{V(a_1a_2), V(a_1b_2)\} > V(a_1)$ *then* $V(b_1) > V(a_1)$.

Proof of Lemma 3: We prove the first statement. (The second statement is symmetric.) Suppose to the contrary that $\min\{V(b_1b_2), V(b_1a_2)\} > V(b_1)$ and $V(b_1) \geq V(a_1)$. Since $a_1 \succeq b_1a_2 \succeq b_1$, Lemma 1 gives an immediate contradiction. ∎

Proof of Proposition 4: We focus on the case in which trade condition (TC1) holds. (The case in which condition (TC2) holds is symmetric.)

We claim that for all posted prices $p \in [V(b_1), \min\{V(a_1), V(b_1a_2) - \delta, V(b_1b_2) - \delta\}]$, there exists an equilibrium in which the seller (agent 1) places the asset for sale if and only if he observes signal b_1; and whenever the seller offers to sell, the buyer agrees to buy. Note that by Lemma 3, such a choice of p exists whenever δ is sufficiently small.

In the equilibrium described the seller learns nothing. Since $p \in [V(b_1), V(a_1)]$, he is happy to sell after seeing signal b_1, and is happy to keep the asset after seeing signal a_1. Turning to the buyer, he knows that the seller only offers the asset for sale when he has observed signal b_1. So if the buyer himself observes signal a_2 he values the asset at $V(b_1 a_2) - \delta$, while if he observes signal b_2 he values the asset at $V(b_1 b_2) - \delta$. By construction both are higher than the price p, and so he will agree to buy whenever the seller places his asset on the market.

Finally, given that the seller learns nothing, and $\min\{V(b_1 b_2) - \delta, V(b_1 a_2) - \delta\} \geq p$, *ex post* individual rationality is clearly satisfied. ∎

The second simple trading mechanism we consider dispenses with the third-party, and instead entails the buyer suggesting the price. Specifically: (1) the buyer makes a take-it-or-leave-it offer to purchase the asset for a price p; (2) the seller either accepts the offer, in which case he delivers the asset and receives p; or he rejects the offer, in which case he keeps the asset.

Proposition 5. *Suppose that one of trade conditions (TC1) and (TC2) holds. Then for sufficiently small trading costs δ there exists an equilibrium of the take-it-or-leave-it game described in which trade occurs. The equilibrium satisfies* ex post *individual rationality.*

Proof of Proposition 5: The proof is essentially an extension of that of Proposition 4. All that is required is to exhibit a set of out-of-equilibrium beliefs that deter the buyer from making an alternate offer. Details are in the Appendix. ∎

Finally, if one of trade conditions (TC1) and (TC2) holds, then under additional (but commonly satisfied) conditions, there is a trade equilibrium in the analogous "seller posts the price" trading mechanism. Details are available from the authors.

Discussion

Loosely speaking, gains to trade exist in our setting because information is valuable. Formally this can be seen by comparing the action taken by the buyer with the action the seller takes in the counterfactual event that he retains the asset and observes only his own signal.

Specifically, suppose that the trade condition (TC1) holds. By Proposition 3, when trade occurs it does so in states $\{b_1a_2, b_1b_2\}$. Our claim is that the optimal action after b_1 differs from the optimal action after either b_1a_2 or b_1b_2. To see this, simply suppose to the contrary that the optimal action is the same after all these signals, $b_1a_2 \succeq b_1 \succeq b_1b_2$. Since $V(A;p)$ and $V(B;p)$ are both monotone, this contradicts (TC1). Thus:

Corollary 2. *Suppose that trade occurs. Then there exists a state in which the buyer takes a different action than the seller would have taken had he retained the asset.*

In the previous sections, we gave necessary and sufficient conditions for trade in terms of the function V. These conditions can clearly be expressed in terms of the underlying asset payoffs $v(\cdot,\cdot)$ and signal qualities. In particular they imply:

Corollary 3. *Trade is possible only if (i) neither action dominates the other, i.e., $v(A,\alpha) > v(B,\alpha)$ and $v(A,\beta) < v(B,\beta)$; and moreover (ii) neither state dominates the other, i.e., $v(A,\alpha) > v(A,\beta)$ and $v(B,\alpha) < v(B,\beta)$.*

Proof of Corollary 3: Note that if V is monotone then neither trade condition (TC1) nor (TC2) can hold; and V is monotone if there exists either a dominating action or a dominating state. Recall that we normalized state and action names so that $v(A,\alpha)$ is the highest asset payoff (Assumption 4). It follows that $v(A,\alpha) >$

$v(B,\alpha)$, since if instead $v(A,\alpha) = v(B,\alpha)$, one action must weakly dominate the other; and $v(A,\alpha) > v(A,\beta)$, since if instead $v(A,\alpha) = v(A,\beta)$, one state must weakly dominate the other. These in turn imply $v(A,\beta) < v(B,\beta)$ (otherwise action A weakly dominates B) and $v(B,\alpha) < v(B,\beta)$ (otherwise state α weakly dominates β). ∎

For the special case in which the payoffs from the wrong actions in states α and β coincide, it is possible to express trade conditions (TC1) and (TC2) in terms of the economic fundamentals in a simple fashion:

Lemma 4. *Suppose $v(A,\beta) = v(B,\alpha)$. Trade is possible if and only if*

$$\frac{v(A,\alpha) - v(B,\alpha)}{v(B,\beta) - v(A,\beta)} \in \left(\frac{\Pr(\beta|a_1)}{\Pr(\alpha|a_1a_2)}, \frac{\Pr(\beta|a_1b_2)}{\Pr(\alpha|a_1)}\right) \cup \left(\frac{\Pr(\beta|b_1)}{\Pr(\alpha|b_1a_2)}, \frac{\Pr(\beta|b_1b_2)}{\Pr(\alpha|b_1)}\right).$$

Proof of Lemma 4: See Appendix. ∎

Lemma 4 makes clear that the conditions required for trade are in no way "knife-edge." Moreover, the bigger the difference between $\Pr(\beta|b_1b_2)$ and $\Pr(\beta|b_1)$, or between $\Pr(\alpha|b_1a_2)$ and $\Pr(\alpha|b_1)$, the easier it is to satisfy the trade conditions. Economically this makes sense: trade is possible under a wider range of asset payoffs when the buyer's (agent 2) signal contains additional valuable information. Similar statements apply with respect to $\Pr(\beta|a_1b_2) - \Pr(\beta|a_1)$, and $\Pr(\alpha|a_1a_2) - \Pr(\alpha|a_1)$.

From trade conditions (TC1) and (TC2) and Lemma 3, trade is possible only if either $V(a_1) > V(b_1)$ or $V(b_1) > V(a_1)$. That is, from the seller's perspective there exists a good and a bad signal. Economically this asymmetry can arise for the following three reasons: (i) the payoff from the right action is greater in one of the states α and β; (ii) the difference between the payoffs from the right and wrong actions is greater in one of α and β; or (iii) the signals are more informative in one of α and β:

Corollary 4. *Trade occurs only if there is a good and bad signal, and the asset owner observes the bad signal.*

Finally, consider what happens as information quality improves. First, observe that if the seller's information quality grows high enough, while the buyer's information quality remains unchanged, then trade becomes impossible. For in this case trade is possible only if $v(B,\alpha) < v(B,\beta)$ (see Corollary 3), and so action B is optimal after both b_1a_2 and b_1. But then $V(b_1a_2) > V(b_1)$ cannot hold, and so (TC1) is violated. By a parallel argument (TC2) cannot hold either.

On the other hand, the situation is very different if the buyer's and seller's information qualities grow high together. In particular, consider the leading special case in which $\Pr(a_i|\beta) = \Pr(b_i|\alpha) = \varepsilon$ for $i = 1, 2$, so that $\Pr(\alpha|a_1b_2) = \Pr(\alpha|b_1a_2) = \Pr(\alpha)$. Given Corollary 3, if trade is possible at all then as information quality grows high, i.e., $\varepsilon \to 0$, $V(b_1b_2) > V(b_1)$ must certainly hold. So for ε small trade condition (TC1) holds if and only if $V(b_1a_2) > V(b_1)$, which in turn holds for ε sufficiently small if and only if

$$\max\{\Pr(\alpha)v(A,\alpha) + \Pr(\beta)v(A,\beta), \Pr(\alpha)v(B,\alpha) + \Pr(\beta)v(B,\beta)\} > v(B,\beta). \tag{5}$$

This is clearly satisfied for some parameter values.

Information revelation, efficiency, and repeated trade

Propositions 4 and 5 established sufficient conditions for trade. The proofs explicitly construct equilibria in which the agents trade the asset in some states. A characteristic of both these equilibria is that when the seller keeps the asset he does not learn the buyer's signal. In contrast, in any state in which trade occurs the buyer learns the seller's signal.

Our first result in this section establishes that this property — that the seller does not learn the buyer's signal when he keeps the asset — must hold in any equilibrium satisfying *ex post* individual rationality, regardless of the trading mechanism used.

Proposition 6. *There does not exist a trading mechanism that implements an* ex post *individually rational trade as part of an equilibrium, and in which the seller learns the buyer's information in states in which the seller retains the asset.*

Proof of Proposition 6: See Appendix. ∎

Proposition 6 implies that even when trade is possible, information is not revealed all the time. However, no welfare loss is associated with this lack of revelation.

To see this, suppose that the trade condition (TC1) holds. (As usual, condition (TC2) is covered by symmetric arguments.) In this case, the lack of information revelation occurs in states $\{a_1a_2, a_1b_2\}$, where the seller keeps the asset. We use the following immediate consequence of Corollary 3: $V(p; A)$ is increasing and $V(p; B)$ is decreasing, and consequently if action A is optimal after s, it is optimal after $s' \succcurlyeq s$, while if action B is optimal after s, it is optimal after s' such that $s \succcurlyeq s'$.

What decision does the seller make in states a_1a_2 and a_1b_2, and what decision is socially optimal? Since $V(b_1a_2) > V(b_1)$ it must be the case that action A is optimal given signal b_1a_2: for if instead action B were optimal, it would be optimal given signals b_1 and b_1b_2 also, contradicting condition (TC1). Consequently, action A is optimal after signal $a_1b_2 \succcurlyeq b_1a_2$, and after signals a_1a_2 and a_1 also. In other words, the action chosen by the seller when he sees only his own signal a_1 matches the efficient action given full information.

Corollary 5. *Suppose that (TC1) or (TC2) holds. Then in any equilibrium in which trade occurs, for all $\theta \in \Theta$ the asset owner makes the same decision as he would if he had all information available to him.*

Unfortunately this strong efficiency result does not always hold if repeated trade opportunities exist, a possibility to which we now turn our attention. For specificity, we consider the following setting: agents $i \in \{1, 2, 3\}$ receive signals $s_i \in \{a_i, b_i\}$ about the state $\omega \in \{\alpha, \beta\}$. At date 0 agent 1 owns the asset. At date 1, agent 2 has an opportunity to buy the asset from agent 1. After observing whether or not trade took place at date 1 (but not the price paid), at date 2 agent 3 has an opportunity to buy the asset from its current owner. We assume that all trades take place according to the "buyer posts the price" mechanism of Proposition 5. When trade is possible, we focus on the equilibrium that is preferred by the buyer. We assume that the signals s_i are independently distributed conditional on the true state $\omega \in \{\alpha, \beta\}$; and that signals s_2 and s_3 are identically distributed, while the information quality of s_1 is weakly higher.

We claim that when the trade condition (TC1) holds, the following is an equilibrium. At date 1, agent 2 offers agent 1 a price $p_1 = V(b_1)$, and agent 1 accepts if and only if he observes $s_1 = b_1$. Conditional on agent 2 acquiring the asset in date 1, there are three possibilities at date 2 (depending on the underlying parameter values). (i) If

$$\min\{V(b_1b_2b_3), V(b_1b_2a_3)\} > V(b_1b_2)$$

then agent 3 offers agent 2 a price $p_2 = V(b_1b_2)$, and agent 2 accepts if and only if he observes $s_2 = b_2$. (ii) If

$$\min\{V(b_1a_2b_3), V(b_1a_2a_3)\} > V(b_1a_2)$$

then agent 3 offers agent 2 a price $p_2 = V(b_1a_2)$, and agent 2 accepts if and only if he observes $s_2 = a_2$. (iii) If neither inequality holds then no trade occurs. Finally, conditional on agent 1 keeping the asset in date 1, at date 2 agent 3 offers agent 2 a price $p_1 = V(b_1)$, and agent 1 accepts if and only if he observes $s_1 = b_1$.

In the Appendix we formally establish that this is indeed an equilibrium. The key observation we wish to make is that if the signal realizations of the three agents are a_1, b_2 and b_3 respectively, no trade occurs and agent 1 keeps the asset. His only information is that his own signal is a_1. As such, he takes action A (see above). However, it is quite possible that the optimal action given the signal combination

$a_1b_2b_3$ is action B. In this case, and in contrast to Corollary 5, the eventual asset owner fails to take the full information efficient action.

The correlation of actions with trade

The strongest empirical implication of our model is that the action taken is correlated with whether or not trade takes place. This implication is obtained even though the two agents have exactly the same ability to take the two actions under consideration, A and B.

To derive this implication, suppose first that condition (TC1) holds. If trade occurs, it does so in $\{b_1a_2, b_1b_2\}$. By prior arguments (see the text prior to Corollary 5) condition (TC1) implies that action A is optimal after signals a_1a_2, a_1, a_1b_2 and b_1a_2.

From the proof of Proposition 3, when agent 2 acquires the asset in b_1a_2 and b_1b_2 he learns agent 1's signal. In b_1b_2 he takes action B (for if instead he took action A, then action A would be optimal after all signals, contradicting condition (TC1)).

What happens when trade does not take place? In this case, agent 1 is the asset owner. From Proposition 6, agent 1 does not learn agent 2's signal. Thus he observes only his own signal, which is a_1. He takes action A.

Combined with a symmetric argument for the case in which (TC2) holds, we obtain:

Proposition 7. *Suppose that an equilibrium features trade. Then either:*
(I) Trade occurs when agent 1 *observes signal* b_1. *When trade does not occur, the asset owner takes action* A. *When trade does occur, the asset owner takes either action* A *or* B.
(II) Trade occurs when agent 1 *observes signal* a_1. *When trade does not occur, the asset owner takes action* B. *When trade does occur, the asset owner takes either action* A *or* B.

Proposition 7 says that the action taken by the asset owner is correlated with whether or not trade occurs. In particular, in any equilibrium in which trade occurs, there exists an action which is taken *only* in states in which trade takes place. Two possible applications include the role of vulture investors in debt restructuring, and corporate raiders. With regard to the former, it is widely perceived that vulture investors' behavior in restructuring negotiations differs from that of the original creditors (see, e.g., Morris 2002). With regard to the latter, there is evidence that large scale layoffs and divestitures follow takeovers (see, e.g., Bhagat *et al* 1990).

5 Continuous signals

In the previous section we established sufficient conditions for trade when agents' signal set are binary. For example, we showed that trade can occur when the seller observes signal b_1 if trade condition (TC1) holds. Under this condition both of the buyer's possible valuations, $V(b_1a_1)$ and $V(b_1b_2)$, are greater than the seller's valuation, $V(b_1)$. One possible concern with this observation is that it may appear overly dependent on the binary nature of the buyer's signal. Specifically, since $b_1a_2 \succcurlyeq b_1 \succcurlyeq b_1b_2$, *if the buyer's signal were drawn from a sufficiently fine* signal set, then there would exist some signal realization s_2 "between" b_2 and a_2 such that $\Pr(\alpha|b_1) \approx \Pr(\alpha|b_1s_2)$, and so $V(b_1) \approx V(b_1s_2)$ also. This argument establishes that if the buyer's signal is drawn from a fine-grained signal set it is very hard (if not impossible) to support an equilibrium in which he always acquires the asset after the seller observes b_1.

However, even when the buyer's signal set S_2 is of high cardinality, trade equilibria do exist. To establish this, it is convenient to examine the opposite extreme to binary signals and allow the signals of both[13] agents to be drawn from continuous

[13]A version of Proposition 8 below would hold if instead only one of the agents observes a contin-

distributions. This should be viewed as the limiting case of adding more and more signals to the signal sets S_i. We construct an equilibrium in which trade occurs when the seller observes a low signal ($s_1 \leq \hat{s}$, some $\hat{s}$), and when the buyer observes an extreme signal ($s_2 \leq \underline{s}$ or $s_2 \geq \bar{s}$, some $\underline{s}$, $\bar{s}$). In particular, because the buyer has two discrete "buying regions", the gains from trade that exist in our basic discrete model (i.e., that $V(b_1 a_1)$ and $V(b_1 b_2)$ exceed $V(b_1)$) exist in this equilibrium also. Observe that equilibria of this type are consistent with Proposition 2, which says that if trade occurs it must do so at extreme buyer signals.

In more detail, the extension of the model we consider is as follows. Each of agents $i = 1, 2$ observes the realization of a continuously distributed signal $s_i \in [0, 1]$.[14] Let $f_i(\cdot|\omega)$ and $F_i(\cdot|\omega)$ denote the density and distribution functions for agents $i = 1, 2$ and states $\omega = \alpha, \beta$. We assume that for both agents high realizations of the signal are more likely in state α, in the sense of the monotone like ratio property (MLRP): $\frac{f_i(s|\alpha)}{f_i(s|\beta)}$ is strictly increasing in s for $i = 1, 2$. Additionally, we assume that $\frac{f_1(s=1|\alpha)}{f_1(s=1|\beta)} \geq \frac{f_2(s=1|\alpha)}{f_2(s=1|\beta)}$ and $\frac{f_1(s=0|\alpha)}{f_1(s=0|\beta)} \leq \frac{f_2(s=0|\alpha)}{f_2(s=0|\beta)}$, i.e., extreme seller signals are weakly more informative than extreme buyer signals. This is akin to our previous assumption that the seller's information quality is (weakly) higher than the buyer's. Finally, we assume that the seller's signal satisfies the mild regularity condition that $\frac{f_1(s|\alpha)}{f_1(s|\beta)} / \frac{F_1(s|\alpha)}{F_1(s|\beta)}$ is increasing in s. In words, this condition says that the ratio of the information conveyed by knowing that the signal is s, to knowing only that the signal is less than s, is increasing in s. This condition is satisfied for many standard distributions.

In all other respects our model is identical to before. Our main objective in this section is to establish that trade is possible even when signals are continuously distributed. Accordingly, we content ourselves with exhibiting a set of sufficient conditions. We restrict attention to the special case in which the payoffs to the "wrong" actions are identical, $v(A, \beta) = v(B, \alpha)$. Under these conditions:

---

uous signal, while the other observes a discrete signal.

[14]Proposition 8 below would hold if instead the support of s_i were non-compact, e.g., $(0, 1)$.

Proposition 8. *There exists a pair of constants* $\underline{\phi}$ *and* $\bar{\phi} > \underline{\phi}$ *such that whenever* $\frac{v(A,\alpha)-v(B,\alpha)}{v(B,\beta)-v(A,\beta)} \in \left[\underline{\phi}, \bar{\phi}\right]$*, there exists a quadruple* $(\underline{s}, \bar{s}, \hat{s}, p)$ *such that the following is an equilibrium of the "third party posted price" trading mechanism:*[15] *(i) the buyer offers to buy at price* p *whenever* $s_2 \in [0, \underline{s}]$ *or* $s_2 \in [\bar{s}, 1]$*, (ii) the seller accepts whenever* $s_1 \in [0, \hat{s}]$*, and in this equilibrium (iii) the seller learns nothing from the buyer's behavior,*

$$\Pr\left(s_2 \in [0, \underline{s}] \cup [\bar{s}, 1] \,|\alpha\right) = \Pr\left(s_2 \in [0, \underline{s}] \cup [\bar{s}, 1] \,|\beta\right). \tag{6}$$

The equilibrium satisfies ex post *individual rationality.*

Proof of Proposition 8: See Appendix. ∎

6 Concluding remarks

In summary, we have shown that if asset payoffs are endogenously determined by the actions of agents, then trade based purely on informational differences is possible. This conclusion stands in sharp contrast to the existing literature, which takes asset values as exogenous. Even without the presence of noise traders, agents in our model would be prepared to spend resources to acquire information; and this information is subsequently impounded in the price at which trade occurs.

We conclude the paper with a discussion of two further implications of our model. First, we look at the price response to trade. Second, we consider the applicability of our analysis to circumstances in which asset payoffs are *exogenous* from the perspective of the trading agents, as is the case, for example, when small shareholders trade stocks.

[15] A parallel result would hold for the "buyer posts the price" trading mechanism.

The price response to trade

Suppose that trade condition (TC1) holds. Focusing on an equilibrium in which trade occurs in some states, if trade does not occur the expected payoff of the asset is $V(a_1)$; while if trade does occur, the expected payoff is either $V(b_1a_2)$ or $V(b_1b_2)$, depending on the signal observed by the buyer. Under condition (TC1), $V(a_1) > V(b_1a_2)$ from Lemma 3 and the convexity of V. As such, trade is often associated with a fall in asset value.

We can make this point more precise by focusing on the case in which signals are accurate and of equal quality: that is, $\Pr(a_1|\beta) = \Pr(b_1|\alpha) = \Pr(a_2|\beta) = \Pr(b_2|\alpha) \to 0$. Recall that by normalization $v(A,\alpha) \geq \max\{v(A,\beta), v(B,\alpha), v(B,\beta)\}$ (see Assumption 4). As such, (TC2) cannot possibly hold when signals are accurate. Moreover, trade condition (TC1) cannot hold if $v(A,\alpha) = v(B,\beta)$ (see inequality (5)). The only remaining possibility for trade is that $v(A,\alpha) > v(B,\beta)$ and (TC1) holds. In this case, $V(a_1) > \max\{V(b_1a_2), V(b_1b_2)\}$ when signals are accurate enough.

In summary, under many circumstances trade is associated with a reduction in the expected valuation of the asset. This prediction is consistent with at least some empirical claims. For example, in the context of mergers between firms this prediction implies that conditional on a merger occurring the combined valuation of the merging firms should fall. Moeller *et al* (2005) document just such value destruction in recent mergers.[16]

Likewise, many observers have expressed the view that Chapter 11 allows too many firms to reorganize.[17] One way to view negotiations in Chapter 11 is that a group of creditors "owns" the right to liquidate the firm, and management is allowed to make

[16]Studies of previous U.S. merger waves generally found that the combined valuation of merged firms rose. See, e.g., Bradley *et al* (1988).

[17]See, e.g., Baird (1986), along with other references cited by Hotchkiss (1995). Hotchkiss herself presents quantitative evidence that firms exiting Chapter 11 perform poorly.

an offer to purchase it from them. As such, our model predicts that conditional on exiting Chapter 11 (that is, on "trade") the value of the firm is low. Note that while this prediction matches what many observers have claimed, it does not stem from a bias towards too much reorganization.

Nonetheless, one needs to be very careful in taking our stylized model to the data. In particular, consider the following minor modification. In place of two underlying states, actions and signal realizations, suppose instead that there are three of each: $\omega \in \{\alpha, \beta, \gamma\}$, actions $\{A, B, C\}$, and signals $\{a, b, c\}$. We assume that states α and β are as before, while state γ has the following characteristics. First, action C is best in state γ. Second, the asset valuation is low: $v(C, \gamma) < \min\{v(A, \alpha), v(B, \beta)\}$. Third, agents know when the state is γ: $\Pr(c|\gamma) = 1$ and $\Pr(c|\alpha) = \Pr(c|\beta) = 0$.

Under these assumptions, over large segments of the state space both agents observe γ, there is no scope for trade, and the asset value is low. Trade is only potentially feasible when $\omega \neq \gamma$ and agents observe signals a or b. Consequently, although trade is associated with lower asset values when conditioned on agents receiving signals a or b, *unconditionally* trade is associated with an increase in asset valuations.

Trade when agents do not control assets

Thus far we have considered the possibility of trade in an asset when the eventual owner of the asset can directly affect the final cash flows produced. As we have noted, many applications fall within this framework. However, one important application that does not is the trade of a small number of shares. We conclude the paper by establishing that here, too, trade is possible.

Specifically, consider the following example. Agent 1 owns a single share of a large publicly traded firm, which will pay an *exogenously* determined terminal dividend at a future date. As before, there are two possible states of the world, α and β. The terminal dividend is perfectly correlated with the return on the market. However, the relation between the dividend and the market return differs across states α and β: the dividend in state $\omega \in \{\alpha, \beta\}$ is $D_\omega + \kappa_\omega r$, where $\kappa_\beta > \kappa_\alpha$ and r denotes the market return. That is, the dividend contains more market risk in state β than state α.

As in our main model, both agent 1 and a potential buyer, agent 2, observe noisy signals $s_i \in \{a_i, b_i\}$ about the true state of the world. In contrast to before, both agents are risk-averse, with identical preferences given by $u(\cdot)$. Both agents have initial wealth W_0; in addition, of course, agent 1 owns the share. After observing the signals, they have an opportunity to trade the share. Finally, following the trading opportunity both agents must choose how to allocate the remainder of their wealth between the market portfolio and a risk free asset.

A specific example of this framework, which we use below, is:

Example 1. *Preferences are constant absolute risk aversion, with a coefficient of absolute risk aversion of* 4*. The market risk premium is* 5% *(i.e.,* $r = 1.05$*). Market returns are distributed normally, with a standard deviation of* 20%*. Both agents start with wealth* $W_0 = 10$*. The share's terminal dividend in state* α *is* $1.63 - 0.6r$*, while the terminal dividend in state* β *is* $0.37 + 0.6r$*. Note that in both states the expected terminal dividend is* 1*. The two states are equally likely, and the signal qualities are given by* $\Pr(a_i|\beta) = \Pr(b_i|\alpha) = 1/4$ *for* $i = 1, 2$*.*

Although agents no longer have control over the terminal dividend paid by the share (as they do in our main model), they do have some ability to change the utility value of holding the share via the allocation of the rest of their portfolio. Intuitively, holding the share is more valuable in state α than in state β, because in state α the share is less exposed to market risk. But additionally, holding the share when one is certain that the state is β is likely to be more valuable than holding the share when one has some doubt. In the former case, the shareholder knows to allocate the rest of his wealth to risk free investments (or even to take a short position in the market). In contrast, in the latter case the shareholder would not take such an extreme position, and so is ultimately exposed to more market risk.

Notationally, let $U(W|s)$ be the expected utility of an agent with wealth W who owns the share and knows signal s. Normalizing the return on the risk free asset to unity, and assuming that the distribution of market returns is independent of the state ω,

$$U(W|s) \equiv \max_x \sum_{\omega=\alpha,\beta} \Pr(\omega|s) E_r \left[u \left(W - x + rx + D_\omega + \kappa_\omega r \right) \right].$$

Likewise, let $\bar{U}(W)$ be the expected utility of an agent with wealth W who does not own the share. Note that the state is irrelevant for an agent who does not hold the share, and so $\bar{U}$ is independent of any signals observed:

$$\bar{U}(W) \equiv \max_x E_r \left[u \left(W - x + rx \right) \right].$$

By previous arguments, an equilibrium with trade exists in the "third party posts the price" mechanism if there is a price p satisfying:

$$U(W_0|a_1) \geq \bar{U}(W_0 + p) \geq U(W_0|b_1) \tag{7}$$

$$\min \left\{ U(W_0 - p|b_1 a_2), U(W_0 - p|b_1 b_2) \right\} \geq \bar{U}(W_0) \tag{8}$$

Condition (7) says that the seller is happy to sell at price p when he sees signal b_1, but not when he sees signal a_1. Condition (8) says that, knowing the seller only sells when $s_1 = b_1$, the buyer is happy to buy at price p both when he observes a_2 and b_2.

It is readily verified that conditions (7) and (8) can be simultaneously satisfied: for instance, in Example 1 a trade equilibrium exists at a price $p = 0.965$. Intuitively, the seller is prepared to sell the share for less than the expected terminal dividend when he believes the state is β, since in that state the dividend is positively correlated with market risk. The buyer is then prepared to buy because his extra information allows him to hedge the market risk inherent in the terminal dividend more effectively than the seller is able to.

Whether or not a model of this type can account for a significant fraction of trading volume among non-controlling shareholder remains an open question. We leave this, together with a fuller analysis of when the trade conditions (7) and (8) are satisfied, for future research.

References

Douglas G. Baird. The uneasy case for corporate reorganizations. *Journal of Legal Studies*, 15(1):127–147, January 1986.

Sanjai Bhagat, Andrei Shleifer, and Robert W. Vishny. Hostile takeovers in the 1980s: The return to corporate specialization. *Brookings Papers on Economic Activity. Microeconomics*, pages 1–84, 1990.

Bruno Biais and Peter Bossaerts. Asset prices and trading volume in a beauty contest. *Review of Economic Studies*, 65(2):307–340, April 1998. 2.

Michael Bradley, Anand Desai, and E. Han Kim. Synergistic gains from corporate acquisitions and their division between the stockholders of target and acquiring firms. *Journal of Financial Economics*, 21(1):3–40, May 1988.

Qi Chen, Itay Goldstein, and Wei Jiang. Price informativeness and investment sensitivity to stock price. *Working Paper*, 2005.

Douglas W Diamond. Rational expectations with asymmetric information about production technique. *PhD Dissertation, Yale University*, 1980.

James Dow, Itay Goldstein, and Alexander Guembel. Commitment to overinvest and price informativeness. *Working Paper*, 2006.

James Dow and Gary Gorton. Profitable informed trading in a simple general equilibrium model of asset pricing. *Journal of Economic Theory*, 67:327–369, 1995.

James Dow and Gary Gorton. Stock market efficiency and economic efficiency: Is there a connection? *Journal of Finance*, 52(3):1087–1129, July 1997.

James Dow and Rohit Rahi. Informed trading, investment, and economic welfare,. *Journal of Business*, 76:430–454, 2003.

Art Durnev, Randall Morck, and Bernard Yeung. Value-enhancing capital budgeting and firm-specific stock return variation. *Journal of Finance*, 59(1):65–105, February 2004.

Ronald A. Dye and Sri S. Sridhar. Resource allocation effects of prices reactions to disclosures. *Contemporary Accounting Research*, 19(3):385–410, Fall 2002.

Lawrence R Glosten and Paul R Milgrom. Bid, ask, and transaction prices in a specialist market with heterogenously informed traders. *Journal of Financial Economics*, 14:71–100, 1985.

Itay Goldstein and Alexander Guembel. Manipulation and the allocational role of prices. *Working Paper*, 2005.

Sanford J Grossman and Joseph E Stiglitz. On the impossibility of informationally efficient markets. *American Economic Review*, 70(3):393–408, June 1980.

Bruce D. Grundy and Maureen McNichols. Trade and the revelation of information through prices and direct disclosure. *Review of Financial Studies*, 2(4):495–526, 1998.

Yoram Halevy. The possibility of speculative trade between dynamically consistent agents. *Games and Economic Behavior*, 46:189–198, 2004.

Jack Hirshleifer. The private and social value of information and the reward to inventive activity. *American Economic Review*, 61(4):561–574, 1971.

Bengt Holmström and Roger Myerson. Efficient and durable decision rules with incomplete information. *Econometrica*, 51(6):1799–1819, 1983.

Edith Shwalb Hotchkiss. Postbankruptcy performance and management turnover. *Journal of Finance*, 50(1):3–21, March 1995.

Naveen Khanna, Steve L. Slezak, and Michael Bradley. Insider trading, outside search, and resource allocation: Why firms and society may disagree on insider trading restrictions. *Review of Financial Studies*, 7(3):575–608, Autumn 1994.

Albert S. Kyle. Continuous auctions and insider trading. *Econometrica*, 53(6):1315–1336, November 1985.

Paul Milgrom and Nancy Stokey. Information, trade and common knowledge. *Journal of Economic Theory*, 26(1):17–27, February 1982.

Sara B Moeller, Frederik P Schlingemann, and Rene M Stulz. Wealth destruction on a massive scale? a study of acquiring-firm returns in the recent merger wave. *Journal of Finance*, 60(2):757–782, April 2005.

John E Morris. Times are changing. *The Deal.com*, June 2002.

Stephen Morris. Trade with heterogeneous prior beliefs and asymmetric information. *Econometrica*, 62(6):1327–1347, November 1994.

Martin J Osborne and Ariel Rubinstein. *A Course in Game Theory.* The MIT Press, Cambridge, Massachusetts, 1994.

Avanidhar Subrahmanyam and Sheridan Titman. The going-public decision and the development of financial markets. *Journal of Finance*, 54(3):1045–1082, June 1999.

A Appendix

Proof of Lemma 2

Recall that s_i^α and s_i^β are, respectively, the most pro-α and most pro-β of agent i's signals. We start by establishing the following minor result:

Lemma 5. *Let signals $s_2, s_2' \in S_2$ be a pair of buyer signals (possibly the same), and $\hat{S}_1$ and $\hat{S}_2$ a signal subsets of S_1 and S_2 respectively. Then either $\{s_1^\alpha\} \times \hat{S}_2$ is more pro-α than $\hat{S}_1 \times \{s_2'\}$ or $\left\{s_1^\beta\right\} \times \hat{S}_2$ is more pro-β than $\hat{S}_1 \times \{s_2\}$.*

Proof of Lemma 5: From condition (1),

$$\frac{\Pr\left(s_1^\alpha s_2|\alpha\right)}{\Pr\left(s_1^\alpha s_2|\beta\right)} \geq \frac{\Pr\left(s_1^\alpha s_2^\beta|\alpha\right)}{\Pr\left(s_1^\alpha s_2^\beta|\beta\right)} \geq \frac{\Pr\left(s_1^\beta s_2^\alpha|\alpha\right)}{\Pr\left(s_1^\beta s_2^\alpha|\beta\right)} \geq \frac{\Pr\left(s_1^\beta s_2'|\alpha\right)}{\Pr\left(s_1^\beta s_2'|\beta\right)}.$$

Multiplying the first and last term by $\frac{\Pr\left(\hat{S}_1|\alpha\right)}{\Pr\left(\hat{S}_1|\beta\right)}\frac{\Pr\left(\hat{S}_2|\alpha\right)}{\Pr\left(\hat{S}_2|\beta\right)}$ gives

$$\frac{\Pr\left(s_1^\alpha|\alpha\right)}{\Pr\left(s_1^\alpha|\beta\right)}\frac{\Pr\left(\hat{S}_2|\alpha\right)}{\Pr\left(\hat{S}_2|\beta\right)}\frac{\Pr\left(s_2|\alpha\right)}{\Pr\left(s_2|\beta\right)}\frac{\Pr\left(\hat{S}_1|\alpha\right)}{\Pr\left(\hat{S}_1|\beta\right)} \geq \frac{\Pr\left(s_1^\beta|\alpha\right)}{\Pr\left(s_1^\beta|\beta\right)}\frac{\Pr\left(\hat{S}_2|\alpha\right)}{\Pr\left(\hat{S}_2|\beta\right)}\frac{\Pr\left(s_2'|\alpha\right)}{\Pr\left(s_2'|\beta\right)}\frac{\Pr\left(\hat{S}_1|\alpha\right)}{\Pr\left(\hat{S}_1|\beta\right)},$$

or equivalently,

$$\frac{\Pr\left(\{s_1^\alpha\} \times \hat{S}_2|\alpha\right)}{\Pr\left(\{s_1^\alpha\} \times \hat{S}_2|\beta\right)}\frac{\Pr\left(\hat{S}_1 \times \{s_2\}|\alpha\right)}{\Pr\left(\hat{S}_1 \times \{s_2\}|\beta\right)} \geq \frac{\Pr\left(\left\{s_1^\beta\right\} \times \hat{S}_2|\alpha\right)}{\Pr\left(\left\{s_1^\beta\right\} \times \hat{S}_2|\beta\right)}\frac{\Pr\left(\hat{S}_1 \times \{s_2'\}|\alpha\right)}{\Pr\left(\hat{S}_1 \times \{s_2'\}|\beta\right)}.$$

It follows that at least one of the following pair of inequalities hold:

$$\begin{aligned}
\frac{\Pr\left(\{s_1^\alpha\} \times \hat{S}_2|\alpha\right)}{\Pr\left(\{s_1^\alpha\} \times \hat{S}_2|\beta\right)} &\geq \frac{\Pr\left(\hat{S}_1 \times \{s_2'\}|\alpha\right)}{\Pr\left(\hat{S}_1 \times \{s_2'\}|\beta\right)} \\
\frac{\Pr\left(\hat{S}_1 \times \{s_2\}|\alpha\right)}{\Pr\left(\hat{S}_1 \times \{s_2\}|\beta\right)} &\geq \frac{\Pr\left(\left\{s_1^\beta\right\} \times \hat{S}_2|\alpha\right)}{\Pr\left(\left\{s_1^\beta\right\} \times \hat{S}_2|\beta\right)}.
\end{aligned}$$

This completes the proof of Lemma 5. ∎

We are now ready to establish Lemma 2. Let $s_2^{T\alpha}$ and $s_2^{T\beta}$ respectively be the most pro-α and pro-β signals in S_2^T. By Lemma 5, either (i) $\{s_1^\alpha\} \times S_2^T$ is more pro-α than $S_1 \times \{s_2^{T\alpha}\}$, or (ii) $\left\{s_1^\beta\right\} \times S_2^T$ is more pro-β than $S_1 \times \left\{s_2^{T\beta}\right\}$. We will establish the claim for case (i). (Case (ii) follows symmetrically.)

Consider an element Q' of the seller's information partition $\mathcal{P}_1$ of the form $Q' = \left\{s_1^\beta\right\} \times \hat{S}_2$, where $\hat{S}_2 \subset S_2^T$. (The fact that $S_1 \times S_2^T$ is measurable with respect to $\mathcal{P}_1$ ensures that such an element exists.) Expanding,

$$\begin{aligned}
\Pr(\alpha|Q') &= \frac{\Pr(\alpha)\Pr(Q'|\alpha)}{\Pr(Q')} = \frac{\Pr(\alpha)\Pr\left(s_1^\beta|\alpha\right)\Pr\left(\hat{S}_2|\alpha\right)}{\Pr(Q')} \\
&= \Pr\left(s_1^\beta|\alpha\right)\sum_{s_2\in\hat{S}_2}\frac{\Pr(\alpha)\Pr(s_2|\alpha)}{\Pr(Q')} = \Pr\left(s_1^\beta|\alpha\right)\sum_{s_2\in\hat{S}_2}\frac{\Pr(s_2)\Pr(\alpha|s_2)}{\Pr(Q')}.
\end{aligned}$$

Observe that

$$\begin{aligned}
\frac{\Pr(Q')}{\Pr\left(s_1^\beta|\alpha\right)} &= \sum_{s_2\in\hat{S}_2}\left(\Pr(\alpha)\frac{\Pr\left(s_1^\beta|\alpha\right)}{\Pr\left(s_1^\beta|\alpha\right)}\Pr(s_2|\alpha) + \Pr(\beta)\frac{\Pr\left(s_1^\beta|\beta\right)}{\Pr\left(s_1^\beta|\alpha\right)}\Pr(s_2|\beta)\right) \\
&> \sum_{s_2\in\hat{S}_2}\left(\Pr(\alpha)\Pr(s_2|\alpha) + \Pr(\beta)\Pr(s_2|\beta)\right) = \sum_{s_2\in\hat{S}_2}\Pr(s_2),
\end{aligned}$$

where the inequality follows since the seller's signal is at least somewhat informative and so $\Pr\left(s_1^\beta|\alpha\right) < \Pr\left(s_1^\beta|\beta\right)$. Thus we can write $\Pr(\alpha|Q')$ in form

$$\Pr(\alpha|Q') = \sum_{s_2\in\hat{S}_2} w(s_2)\Pr(\alpha|s_2),$$

where $\left\{w(s_2) : s_2 \in \hat{S}_2\right\}$ is a set of weights summing to strictly less than unity. Consequently, there exists $\hat{s}_2 \in \hat{S}_2 \subset S_2^T$ such that

$$\Pr(\alpha|S_1 \times \{\hat{s}_2\}) = \Pr(\alpha|\hat{s}_2) > \Pr(\alpha|Q').$$

Clearly $S_1 \times \left\{s_2^{T\alpha}\right\}$ is more pro-α than $S_1 \times \{\hat{s}_2\}$. Recall, moreover, that $\{s_1^\alpha\} \times S_2^T$ is more pro-α than $S_1 \times \left\{s_2^{T\alpha}\right\}$ (we are in case (i)). So

$$\Pr\left(\alpha|\{s_1^\alpha\} \times S_2^T\right) \geq \Pr(\alpha|S_1 \times \{\hat{s}_2\}) > \Pr(\alpha|Q').$$

To complete the proof, simply observe that that the most pro-α element of the seller's information partition lying in $S_1 \times S_2^T$ is *at least* as pro-α as $\{s_1^\alpha\} \times S_2^T$.

Proof of Proposition 2:

From a general mechanism design perspective, a trading mechanism entails agents $1, 2$ submitting reports, $m_i \in M_i$ for $i = 1, 2$, to a central planner after observing their signals; and the planner then announcing an allocation $g(m_1, m_2)$, and possibly some additional information. By the revelation principle, we can focus on truth-telling mechanisms: $M_i = S_i$ for $i = 1, 2$.

Suppose to the contrary that a trading mechanism exists in which trade occurs in $S_1^T \times \{s_2'\}$ for some $S_1^T \subset S_1$, but in which for any $\hat{S}_1 \subset S_1$, there is no trade in $\hat{S}_1 \times \{s_2\}$ and $\hat{S}_1 \times \{s_2''\}$.

By hypothesis, then, there is an equilibrium of the direct-revelation mechanism in which both agents report truthfully, and the buyer receives the asset in states in $S_1^T \times \{s_2'\}$. Let p be the associated price. Also by hypothesis the seller keeps the assets in states $S_1 \times \{s_2, s_2''\}$. By *ex post* individual rationality no monetary transfer takes place in these states. Hence the buyer's payoff in these states is simply zero.

Consider any element Q of the buyer's information partition $\mathcal{P}_2^{\kappa,\tau}$ of the form $Q' = \tilde{S}_1' \times \{s_2'\}$ where $\tilde{S}_1' \subset S_1^T$. The buyer's individual rationality constraint is satisfied only if

$$p < V(Q').$$

Additionally, observe that by reporting s_2' to the planner after seeing signal s_2, the buyer could acquire the information $Q = \tilde{S}_1' \times \{s_2\}$. Similarly, by reporting s_2' to the planner after seeing signal s_2, the buyer could acquire the information $Q'' = \tilde{S}_1' \times \{s_2''\}$. After either report he receives the asset for a price p. In contrast, if he reports truthfully his expected payoff is zero, as argued above. Since truth-telling is an equilibrium,

$$p \geq \max \{V(Q), V(Q'')\}.$$

But by construction $Q \succeq Q' \succeq Q''$. Thus Lemma 1 delivers a contradiction.

Proof of Proposition 5

We focus on the case in which (TC1) holds. (The case in which condition (TC2) holds is symmetric.) For use below, observe that condition (TC1) implies both $V(a_1b_2) \geq V(b_1a_2)$ and $V(a_1a_2) \geq V(a_1b_2)$. To see this, suppose first that $V(b_1a_2) > V(a_1b_2)$. Since $V(b_1a_2) > V(b_1)$ also, and $a_1b_2 \succeq b_1a_2 \succeq b_1$, Lemma 1 gives a contradiction. Second, suppose that $V(a_1b_2) > V(a_1a_2)$. Since $V(a_1b_2) \geq V(b_1a_2) > V(b_1)$ also, and $a_1a_2 \succeq a_1b_2 \succeq b_1$, Lemma 1 gives a contradiction.

We claim that for all $p \in [V(b_1), \min\{V(a_1), V(b_1a_2) - \delta, V(b_1b_2) - \delta\}]$, there exists an equilibrium in which the buyer (agent 2) offers p independent of his signal, and the seller (agent 1) accepts the offer p if and only if he observes signal b_1. Agent 1's off-equilibrium beliefs are that an offer $\tilde{p} < p$ indicates $s_2 = b_2$, while an offer $\tilde{p} > p$ indicates $s_2 = a_2$.

In light of the proof of Proposition 4, it suffices to show that agent 2 prefers the offer p to all alternative offers $\tilde{p} \neq p$. Under the beliefs stated above, no downwards deviation is strictly more profitable than the equilibrium offer p, since agent 1 will never accept an offer $\tilde{p} < p$. This follows since

$$\tilde{p} < p \leq \min\{V(b_1a_2), V(b_1b_2)\} \leq \min\{V(a_1b_2), V(b_1b_2)\}.$$

Moreover, no upwards deviation $\tilde{p} > p$ is strictly more profitable than the equilibrium offer p. To see this, note first that regardless of whether the seller accepts or rejects the deviation $\tilde{p}$ when $s_1 = b_1$, the buyer makes less money conditional on $s_1 = b_1$ than he does using the offer p. As a consequence, a necessary condition for the deviation $\tilde{p}$ to be more profitable than p is that agent 1 must accept it when he sees $s_1 = a_1$. Given the off-equilibrium beliefs, this requires $\tilde{p} \geq V(a_1a_2)$. Since $V(a_1a_2) \geq V(a_1b_2)$, the buyer certainly loses money when the seller accepts when $s_1 = a_1$. But then $\tilde{p}$ is (weakly) less profitable for the buyer than the original offer p.

Proof of Lemma 4

Without loss, normalize $v(A,\beta) = v(B,\alpha) = 0$.

Case: Trade condition (TC1) holds.

Subcase: B is optimal after b_1. In this case, A must be optimal after b_1a_2, for otherwise $V(b_1a_2) > V(b_1)$ cannot hold. To deliver these action choices, we need

$$\begin{aligned} v(A,\alpha)\Pr(\alpha|b_1a_2) &\geq v(B,\beta)\Pr(\beta|b_1a_2) \\ v(A,\alpha)\Pr(\alpha|b_1) &\leq v(B,\beta)\Pr(\beta|b_1). \end{aligned}$$

For $V(b_1a_2) > V(b_1)$ we need

$$v(A,\alpha)\Pr(\alpha|b_1a_2) > v(B,\beta)\Pr(\beta|b_1).$$

Under these conditions, $V(b_1b_2) > V(b_1)$. Since $\Pr(\beta|b_1) > \Pr(\beta|b_1a_2)$, trade condition (TC1) is satisfied in this case if and only if

$$\frac{v(A,\alpha)}{v(B,\beta)} \in \left(\frac{\Pr(\beta|b_1)}{\Pr(\alpha|b_1a_2)}, \frac{\Pr(\beta|b_1)}{\Pr(\alpha|b_1)}\right].$$

Subcase: A is optimal after b_1. In this case, B must be optimal after b_1b_2, for otherwise $V(b_1b_2) > V(b_1)$ cannot hold. To deliver these action choices, we need

$$\begin{aligned} v(A,\alpha)\Pr(\alpha|b_1b_2) &\leq v(B,\beta)\Pr(\beta|b_1b_2) \\ v(A,\alpha)\Pr(\alpha|b_1) &\geq v(B,\beta)\Pr(\beta|b_1). \end{aligned}$$

For $V(b_1b_2) > V(b_1)$ we need

$$v(A,\alpha)\Pr(\alpha|b_1) < v(B,\beta)\Pr(\beta|b_1b_2).$$

Under these conditions, $V(b_1a_2) > V(b_1)$. Since $\Pr(\alpha|b_1b_2) < \Pr(\alpha|b_1)$, trade condition (TC1) is satisfied in this case if and only if

$$\frac{v(A,\alpha)}{v(B,\beta)} \in \left[\frac{\Pr(\beta|b_1)}{\Pr(\alpha|b_1)}, \frac{\Pr(\beta|b_1b_2)}{\Pr(\alpha|b_1)}\right).$$

Combining the two cases, trade condition (TC1) is satisfied if and only if

$$\frac{v(A,\alpha)}{v(B,\beta)} \in \left(\frac{\Pr(\beta|b_1)}{\Pr(\alpha|b_1a_2)}, \frac{\Pr(\beta|b_1b_2)}{\Pr(\alpha|b_1)}\right).$$

Case: Trade condition (TC2) holds.

By a symmetric argument to above, (TC2) holds if and only if

$$\frac{v(A,\alpha)}{v(B,\beta)} \in \left(\frac{\Pr(\beta|a_1)}{\Pr(\alpha|a_1a_2)}, \frac{\Pr(\beta|a_1b_2)}{\Pr(\alpha|a_1)}\right).$$

Proof of Proposition 6

From Proposition 3, trade can occur only if either (TC1) or (TC2) holds. We focus on the case in which (TC1) holds; the proof of the other case proceeds symmetrically.

From a general mechanism design perspective, a trading mechanism entails agents $1, 2$ submitting reports, $m_i \in M_i$ for $i = 1, 2$, to a central planner after observing their signals; and the planner then announcing an allocation $g(m_1, m_2)$, and possibly some additional information. By the revelation principle, we can focus on truth-telling mechanisms: $M_i = \{a_i, b_i\}$ for $i = 1, 2$.

Suppose to the contrary that a trading mechanism exists in which there is an equilibrium with trade; and agent 1 learns agent 2's information in a state where trade does not take place; and the equilibrium is *ex post* individually rational. From Proposition 3 and its proof, we know that the outcomes $g(b_1a_2)$ and $g(b_1b_2)$ are: agent 2 acquires the asset and learns agent 1's signal, agent 1 does not learn agent 2's signal, agent 1 receives a transfer p. Again from Proposition 3, we also know that the the outcomes $g(a_1a_2)$ and $g(a_1b_2)$ entail agent 1 keeping the asset. To satisfy *ex post* individual rationality outcomes $g(a_1a_2)$ and $g(a_1b_2)$ cannot involve any monetary transfer. Moreover, since agent 2 learns agent 1's signal in equilibrium, *ex post* individual rationality is satisfied only if $p < \min\{V(b_1b_2), V(b_1a_2)\}$.

By supposition, agent 1 learns agent 2's signal in one of states a_1a_2 and a_1b_2. As such, he must learn agent 2's signal in both. Consider agent 1's incentive to report truthfully after observing signal b_1. If he reports b_1, under the mechanism agent 2 acquires the asset and agent 1 receives p. On the other hand, if he reports a_1 he keeps the asset. By supposition he learns agent 2's signal. As such, the asset is worth either $V(b_1 a_2)$ or $V(b_1 b_2)$. Since both exceed p, he prefers deviating and reporting a_1 to truthfully reporting b_1. This gives a contradiction, and completes the proof.

Formal analysis of the repeated trade equilibrium of Section 4

We claim that when the trade condition (TC1) holds, the following is an equilibrium:

At date 1, agent 2 offers agent 1 a price $p_1 = V(b_1)$. Agent 1's off-equilibrium beliefs are that an offer $p' < p_1$ indicates $s_2 = b_2$, while an offer $p' > p_1$ indicates $s_2 = a_2$. Agent 1 accepts this offer if and only if $s_1 = b_1$.

Conditional on agent 2 acquiring the asset in date 1, there are three possibilities at date 2 (depending on the underlying parameter values). If

$$\min\{V(b_1 b_2 b_3), V(b_1 b_2 a_3)\} > V(b_1 b_2) \tag{9}$$

then agent 3 offers agent 2 a price $p_2 = V(b_1 b_2)$. Agent 2 accepts if and only if $s_2 = b_2$. Agent 2's off-equilibrium beliefs are that an offer $p' < p_2$ indicates $s_3 = b_3$, while an offer $p' > p_2$ indicates $s_2 = a_3$. If instead

$$\min\{V(b_1 a_2 a_3), V(b_1 a_2 b_3)\} > V(b_1 a_2) \tag{10}$$

then agent 3 offers agent 2 a price $p_2 = V(b_1 a_2)$. Agent 2 accepts if and only if $s_2 = a_2$. Agent 2's off-equilibrium beliefs are that an offer $p' < p_2$ indicates $s_3 = a_3$, while an offer $p' > p_2$ indicates $s_2 = b_3$. If neither (9) nor (10) holds, then no trade occurs between agents 2 and 3.

Finally, conditional on agent 1 keeping the asset in date 1, at date 2 agent 3 offers agent 2 a price $p_1 = V(b_1)$. Agent 1's off-equilibrium beliefs are that an offer $p' < p_1$ indicates $s_3 = b_3$, while an offer $p' > p_1$ indicates $s_2 = a_3$. Agent 1 accepts this offer if and only if $s_1 = b_1$. (In equilibrium, he rejects the offer conditional on reaching this path of the game.)

The proof that this is indeed an equilibrium is as follows. Given that in the equilibrium described agent 2 acquires the asset only when $s_1 = b_1$, the trading round between agents 2 and 3 is exactly parallel to our basic model. Conditions (9) and (10) are simply straightforward adaptations of conditions (TC1) and (TC2).

Given the equilibrium strategies in the trading round between agents 2 and 3, agent 2 finds it weakly profitable to offer agent 1 an amount $p_1 = V(b_1)$ for the asset: agent 1 only accepts if $s_1 = b_1$, and so the value of the asset to agent 2 is either $V(b_1 a_2)$ or $V(b_1 b_2)$, both of which exceed $V(b_1)$ by assumption. As in the proof of Proposition 5, agent 2 has no profitable deviation available.

The trading round between agent 1 and agent 3 is again exactly analogous to our basic model.

Finally, given the payoffs available in date 2 if he keeps the asset, agent 1's strategy in date 1 is optimal — once again, for the same reasons as in the proof of Proposition 5.

Proof of Proposition 8

Since by assumption $v(A,\beta) = v(B,\alpha)$, without loss we normalize both to 0, i.e., $v(A,\beta) = v(B,\alpha) = 0$. Throughout the proof, we write $\phi = v(A,\alpha)/v(B,\beta)$, $L_i(s) = f_i(s|\alpha)/f_i(s|\beta)$, $M_i(s) = F_i(s|\alpha)/F_i(s|\beta)$, and $Q = \Pr(\alpha)/\Pr(\beta)$. Note that $M_i(0) = L_i(0)$, $M_i(1) = 1$, and $M_i(s) < L_i(s)$ for $s > 0$ (this is easily established using MLRP). By assumption, $L_2(1) \leq L_1(1)$, $L_2(0) \geq L_1(0)$, and $L_1(s)/M_1(s)$ is increasing in s.

Define $\bar{\phi} = (QL_1(0))^{-1}$ and $\underline{\phi}_0 = \frac{(QL_1(1))^{-1}+1}{QL_2(0)+1}$. Note that $\bar{\phi} > \underline{\phi}_0$ since $L_2(0) \geq L_1(0)$. Take $\frac{v(A,\alpha)}{v(B,\beta)} \in \left[\underline{\phi}_0, \bar{\phi}\right]$.

We establish the result for the case of $\delta = 0$. By continuity, our result holds for all δ sufficiently small.

Part I: There exists a quadruple $(\underline{s}, \bar{s}, \hat{s}, p)$ such that condition (6) holds (the buyer's behavior reveals nothing), along with

$$p = v(A,\alpha)\Pr(\alpha|s_1 = \hat{s}) \tag{11}$$

$$p = v(A,\alpha)\Pr(\alpha|s_1 \leq \hat{s}, s_2 = \bar{s}) \tag{12}$$

$$p = v(B,\beta)\Pr(\beta|s_1 \leq \hat{s}, s_2 = \underline{s}). \tag{13}$$

These three conditions say that the seller is indifferent between selling and not selling when he observes $\hat{s}$, and that knowing $s_1 \leq \hat{s}$ the buyer is indifferent between buying and not buying when he observes $s_2 = \underline{s}, \bar{s}$. (In making these statements we have assumed that action A is optimal given $s_1 = \hat{s}$, and given $s_1 \leq \hat{s}$ together with $s_2 = \bar{s}$; and that action B is optimal given $s_1 \leq \hat{s}$ together with $s_2 = \underline{s}$. We verify these in Part II below.)

Proof of Part I:

Combining the seller indifference condition (11) with the buyer's first indifference condition (12) yields

$$\frac{QL_1(\hat{s})}{QL_1(\hat{s})+1} = \frac{QL_2(\bar{s})M_1(\hat{s})}{QL_2(\bar{s})M_1(\hat{s})+1} \tag{14}$$

while combining condition (11) with the buyer's second indifference condition (13) yields

$$\phi\frac{QL_1(\hat{s})}{QL_1(\hat{s})+1} = 1 - \frac{QL_2(\underline{s})M_1(\hat{s})}{QL_2(\underline{s})M_1(\hat{s})+1}. \tag{15}$$

Let s^* be the signal for which $L_2(s^*) = 1$. The lefthand side (LHS) of (15) is increasing in $\hat{s}$, while holding $\underline{s} \leq s^*$ fixed the righthand side (RHS) is decreasing. Moreover, at $\hat{s} = 0$ the LHS is strictly less than the RHS since

$$\begin{aligned}
\phi\frac{QL_1(0)}{QL_1(0)+1} &< \frac{1}{QM_1(0)+1} \\
&= 1 - \frac{QL_2(s^*)M_1(0)}{QL_2(s^*)M_1(0)+1} \leq 1 - \frac{QL_2(\underline{s})M_1(0)}{QL_2(\underline{s})M_1(0)+1},
\end{aligned}$$

where the first inequality follows from $\phi \leq \bar{\phi}$ and the second inequality from MLRP. Likewise, at $\hat{s} = 1$ the LHS of (15) strictly exceeds the RHS since

$$\phi \frac{QL_1(1)}{QL_1(1)+1} > 1 - \frac{QL_2(0)\, M_1(1)}{QL_2(0)\, M_1(1)+1} \geq 1 - \frac{QL_2(\underline{s})\, M_1(1)}{QL_2(\underline{s})\, M_1(1)+1},$$

where the first inequality follows from $\phi \geq \underline{\phi}_0$ and the second inequality from MLRP. As such, for each $\underline{s} \leq s^*$ there exists a unique $\hat{s}$ such that condition (15) holds. Let $g(\underline{s})$ denote the corresponding function from $[0, s^*]$ into $[0, 1]$. The function g is decreasing and continuous, and is easily seen to have range $[s_L, s_H]$ for some $0 < s_L < s_H < 1$.

We next show that conditions (6) and (14) together define a continuous function h mapping values $\underline{s} \in [0, s^*]$ into $[0, 1]$.

First, (6) rewrites as

$$F_2(\underline{s}|\beta) - F_2(\underline{s}|\alpha) = F_2(\bar{s}|\beta) - F_2(\bar{s}|\alpha).$$

The function $F_2(s|\beta) - F_2(s|\alpha)$ is continuous in s, equals 0 at $s = 0, 1$, and obtains its unique maximum at s^* (recall $f_2(s^*|\alpha) = f_2(s^*|\beta)$). As such, for any $\underline{s} \leq s^*$ there is a unique $\bar{s} \geq s^*$ such that (6) holds.

Second, condition (14) clearly holds if and only if

$$\frac{L_1(\hat{s})}{M_1(\hat{s})} = L_2(\bar{s}).$$

Fix any $\bar{s} \geq s^*$. Certainly $L_1(0)/M_1(0) = 1 \leq L_2(\bar{s}) \leq L_2(1) \leq L(1)/M(1)$. So by continuity (14) is satisfied for some $\hat{s}$. Moreover, by our regularity condition, $L_1(\cdot)/M_1(\cdot)$ is an increasing function, and so $\hat{s}$ is unique. Finally, note that if $\bar{s} = s^*$ then (14) is satisfied only if $\hat{s} = 0$, while if $\bar{s} = 1$ then (14) is satisfied only if $\hat{s} = 1$.

Together, the above observations imply that conditions (6) and (14) together define a continuous function h mapping values $\underline{s} \in [0, s^*]$ into $[0, 1]$, with $h(0) = 1$ and $h(s^*) = 0$.

By standard arguments, there exists $\underline{s} \in [0, s^*]$ such that $g(\underline{s}) = h(\underline{s})$. Define $(\bar{s}, \hat{s}, p)$ by $\hat{s} = g(\underline{s})$, $\bar{s}$ such that (6) holds, and $p = v(A, \alpha) \Pr(\alpha | s_1 = \hat{s})$. This completes the proof of Part I. ∎

Part II: Given $(\underline{s}, \bar{s}, \hat{s}, p)$ satisfying (6) and (11) - (13), action A is optimal given $s_1 = \hat{s}$, and given $s_1 \leq \hat{s}$ together with $s_2 = \bar{s}$; and action B is optimal given $s_1 \leq \hat{s}$ together with $s_2 = \underline{s}$.

Proof of Part II: If the seller knows only $s_1 = \hat{s}$, we must show that he chooses action A, that is,

$$\phi \frac{QL_1(\hat{s})}{QL_1(\hat{s}) + 1} \geq 1 - \frac{QL_1(\hat{s})}{QL_1(\hat{s}) + 1}.$$

From condition (15), this holds if and only if $L_2(\underline{s}) M_1(\hat{s}) \leq L_1(\hat{s})$. This satisfied, since from condition (14) $L_1(\hat{s}) = L_2(\bar{s}) M_1(\hat{s})$.

If the buyer knows $s_1 \leq \bar{s}$, and also $s_2 = \bar{s}$, we must again show that he chooses action A, that is,

$$\phi \frac{QL_2(\bar{s}) M_1(\hat{s})}{QL_2(\bar{s}) M_1(\hat{s}) + 1} \geq 1 - \frac{QL_2(\bar{s}) M_1(\hat{s})}{QL_2(\bar{s}) M_1(\hat{s}) + 1}.$$

From condition (14) $L_2(\bar{s}) M_1(\hat{s}) = L_1(\hat{s})$, and so the buyer chooses action A under these circumstances if and only if the seller chooses action A having observed just $s_1 = \hat{s}$ — which we have shown to be the case.

Finally, the buyer will choose action B if he knows $s_1 \leq \bar{s}$, and also $s_2 = \underline{s}$: for by construction his valuation under these conditions is the same as if he knows $s_1 \leq \bar{s}$ and $s_2 = \bar{s}$, and this can only be the case if he selects a different action in the two cases. ∎

Take $(\underline{s}, \bar{s}, \hat{s}, p)$ satisfying (6) and (11) - (13). To complete the proof of Proposition 8, it remains to show that the buyer prefers to buy whenever $s_2 \in [0, \underline{s}] \cup [\bar{s}, 1]$ and prefers not to buy whenever $s_2 \in (\underline{s}, \bar{s})$; and the seller prefers to sell whenever $s_2 \in [0, \hat{s}]$, and prefers not to sell whenever $s_2 \in (\hat{s}, 1]$.

With some abuse of notation, we write $V(s_1 \leq \hat{s}, s_2 = \bar{s})$ for the value of asset given the information that $s_1 \leq \hat{s}$ and $s_2 = \bar{s}$, etc. By construction,

$$p = V(s_1 \leq \hat{s}, s_2 = \underline{s}) = V(s_1 \leq \hat{s}, s_2 = \bar{s}).$$

Since $\Pr(\alpha | s_1 \leq \hat{s}, s_2)$ is increasing in s_2, and V is convex, it follows that $V(s_1 \leq \hat{s}, s_2) \geq p$ for $s_2 \leq \underline{s}$ and $s_2 \geq \bar{s}$, and that $V(s_1 \leq \hat{s}, s_2) \leq p$ for $s_2 \in (\underline{s}, \bar{s})$. Thus the buyer's equilibrium behavior is as described.

For the seller, note first that since he takes action A at signal $s_1 = \hat{s}$, he will certainly take action A for all higher signals. As such, $V(s_1) \geq V(\hat{s}) = p$ if $s_1 \geq \hat{s}$. Finally, we must show that $V(s_1) \leq V(\hat{s}) = p$ for $s_1 \leq \hat{s}$. If action A is optimal even at the lowest signal $s_1 = 0$, this is immediate. If instead action B is optimal at $s_1 = 0$, it suffices (given convexity) to show that

$$V(s_1 = 0) = \Pr(\beta | s_1 = 0) \, v(B, \beta) \leq \Pr(\alpha | s_1 = \hat{s}) \, v(A, \alpha) = V(\hat{s}).$$

Rewriting this condition gives

$$\phi \frac{QL_1(\hat{s})}{QL_1(\hat{s}) + 1} \geq 1 - \frac{QL_1(0)}{QL_1(0) + 1}.$$

Define

$$\underline{\phi} = \max\left\{\underline{\phi}_0, \frac{1}{QL_1(\hat{s})} \frac{QL_1(\hat{s}) + 1}{QL_1(0) + 1}\right\}.$$

Finally, note that since $\hat{s} > 0$, $L_1(\hat{s}) > L_1(0)$, and so $\underline{\phi} < \bar{\phi}$. As such, the interval $[\underline{\phi}, \bar{\phi}]$ is non-empty.