Portfolio Choice under Convex Transaction Costs

Sergei Isaenko*†

Concordia University

First version: September 2005

Abstract

We consider a portfolio optimization problem for an investor who faces convex transaction costs on trading a stock. The linear component in these costs defines a no–trading zone, while the nonlinear component does not allow trading the stock at an arbitrary rate. The latter component results from a stock illiquidity for a market maker and causes a stock illiquidity for a price–taking investor. We consider the effects that stock illiquidity has on an investor's optimal trading and his welfare when the stock path is continuous, when the stock may crash, and when there is a second stock that is perfectly liquid. We find that even though the illiquid stock allocation could be of any proportion in the portfolio, an investor has a highest expected utility when this allocation is the same as that in the liquid market. Thus, no flight to liquidity in the presence of liquid securities. This conclusion remains unchanged for any degree of convexity of a nonlinear term in the transaction costs. Moreover, this degree also does not affect the shape of the no–trading zone. We find that portfolio volatility can often be too high, making a risk–averse investor willing to hold a liquid stock with negative risk premium, as if he is a risk lover, just to avoid facing illiquidity. Thus, a substantial discount of the illiquid stock price. If the illiquid stock undergoes crashes, then in many states, an investor discounts the stock return for crashes at a much higher rate than he does the liquid stock for the same crashes. If an investor has an access to a second, perfectly liquid, stock, then allocations to the two stocks are considerably different from those in a liquid market. Finally, we confirm the limits to arbitrage by showing that an investor takes a risky position in the presence of arbitrage opportunities.

Keywords: Portfolio Choice; Liquidity; Transaction Costs

JEL Classification: G11

*The John Molson School of Business, Concordia University, 1455 de Maisonneuve Blvd. W. Montreal, Quebec, H3G 1M8. Tel: 1-514-848-2424 ext. 2797. E-mail: sisaenko@jmsb.concordia.ca.

†The author gratefully acknowledges financial support under a grant from Institut de Finance Mathematique de Montreal. I am thankful to Domenico Cuoco, Stylianos Perrakis and to participants in the seminars at Concorida University and SUNY at Buffalo for the helpful comments, and to Olga Babarina for computational support.

1 Introduction

Market illiquidity has recently become a topic of active theoretical research. One can distinguish between at least three directions in this research. The first direction considers illiquidity caused by the market power of some investors.[1] The second direction assumes that difficulties in trading result from the presence of proportional or fixed transaction costs[2], while the third direction assumes limits on the rate of trading securities.[3] In the former direction of research trading is costly, so an investor chooses to trade only at discrete random times at the infinitely large rate, while in the latter direction, trading is costless, but its rate is exogenously limited or zero.[4]

Treatment of market illiquidity is often related to a market microstructure. An investor trading in the market has to pay transaction fees, face a bid–ask spread and may also encounter with a price impact. The transaction fee is typically only a fraction of a percent of traded volume or a number of shares and contributes very little to the market illiquidity. On another hand, a bid–ask spread implicitly charged by a trader[5] could be more substantial. It depends on the search costs, inventory costs and also possible asymmetric information faced by a trader. Existing treatments of a portfolio choice of an investor in the presence of transaction costs traditionally assume that these costs are linear and include the bid–ask spread.[6] It follows that the first and the second views on market illiquidity are related to the market microstructure. However, the economic origin of the third view is not always well–established.

The goal of our paper is to understand the optimal behavior of an investor who trades continuously, but cannot trade as many shares as he would like to and is unable to reach his best allocations. We model a stock illiquidity for an investor by assuming that transaction costs on trading this stock are convex. We consider an investor who is a price–taker, thus neglecting the illiquidity of the market resulting from a price–impact. If the market becomes illiquid, a trader may have to hold shares which decrease his utility, spend considerable time searching for another investor to close a transaction or face substantial asymmetry of information. Naturally, if a stock becomes less liquid, a trader will increase the costs for

[1]This view on liquidity is developed by a number of authors including Frey (1998), Cetin, Jarrow, and Protter (2004), Cetin, Jarrow, Protter, and Warachka, (2004), He and Mamaysky (2005), Liu and Yong (2005), and Pereira and Zhang (2005).

[2]It has been explored by many authors including Constantinides (1986), Vayanos (1998, 2003), Lo, Mamaysky, and Wang (2004), and Jang, Koo, Liu, and Loewenstein (2004).

[3]This view is advocated by a number of authors including Longstaff (2001, 2005) and Kahl, Liu, and Longstaff (2003).

[4]See Longstaff (2001) or Kahl, Liu, and Longstaff (2003), and Longstaff (2005), respectively.

[5]We assume that a trader is a market maker or a dealer who trades from his own account.

[6]The literature on portfolio optimization in the presence of proportional transaction costs is vast. Its well–known examples include Constantinides (1986), Davis and Norman (1990), Duffie and Sun (1990), Dumas and Luciano (1991), Vayanos (1998, 2003), Liu and Loewestein (2002), Liu (2004), and many others.

an investor to trade this stock to compensate for possible losses resulting from illiquidity. Higher costs are defined by an increasing bid–ask spread included in the costs. Moreover, the faster an investor needs the stock shares to be traded, the higher the costs would be. As a result, a market becomes illiquid for an investor who endogenously chooses to trade at a finite rate and is not able to reach his best allocations. Therefore, we connect the third view on market illiquidity discussed above with the market microstructure.

Our approach stands between two extreme cases: One when a stock cannot be traded at all and the other when an investor can trade it at an infinite rate. Both cases are the limits of our model. In many economic settings, these two extreme approaches are more convenient due to their relative simplicity. Examples include an investor's trading in a perfectly liquid market and optimization by an insider who cannot sell his shares. On the other hand, many markets are captured much better by our model. Examples include extreme market episodes such as the 1987 stock market crash and the 1998 financial crisis.[7]

In our approach we do not model a trader. A trader provides market liquidity and conducts actual trading between investors. Modelling his behavior depends on numerous details which dramatically affect the resulting costs for an investor. These details include the preferences of a trader, the search procedure, the information available, and etc. Since many of these details are unknown, their explicit modelling will result in costs for an investor which are likely to be too specific. We take a phenomenological approach to market illiquidity by describing the presence of a trader through the convex costs faced by an investor. It turns out that our conclusions remain unchanged qualitatively and a number of them does not change even quantitatively if we vary the convexity of the transaction costs. Therefore, our results are robust with respect to many details of the behavior of a trader.

We assume that transaction costs are linear in the rate of trading when the trade is slow, and convex when the trade is fast. The linear term in the transaction costs generates a no–trading zone. Because the effects resulting from this term have been thoroughly studied, we concentrate on studying the impact from nonlinearity in the transaction costs. Hence, our further discussion in Introduction is related to the case where the linear component in the transaction costs is absent. In Sections 2–5, we will also present new results related to the linear component in the transaction costs.

We consider a portfolio choice of an investor who trades a liquid bond and an illiquid stock market in the economy with a constant investment opportunity set. The stock market can have a continuous path or it can be susceptible to crashes. The last situation is rather common since a market's illiquidity is often triggered by its crash (for example, the 1987 stock market crash and its resulting illiquidity). We also identify optimal strategies for an

[7]Our analysis can be easily extended to an illiquid bond market. For the empirical results, see Amihud and Mendelson (1991), Boudoukh, and Whitelaw (1991), and Kamara (1994).

investor who, besides having a liquid bond and an illiquid stock, has access to a perfectly liquid stock. Both stocks are significant for the stock market. This situation is typical as an investor often has to choose between liquid and illiquid stocks. Moreover, two stocks can be perfectly correlated, generally implying access to arbitrage opportunities. The given situation is related to markets where two securities providing identical cash flows, but having different liquidities, also have different price dynamics which may allow an arbitrage.

Our main conclusions are following:

1. We find that in the presence of stock illiquidity, an investor can be forced to be in states with any relative amount of stock holdings. However, he achieves a highest indirect utility when his allocations lie on the Merton line. This result is independent from the degree of convexity of the transactions costs. It implies that if an investor trades an illiquid stock, then no matter whether the liquid security is a bond or another stock, or whether the illiquid stock may undergo crashes, an investor does not try to reduce his holdings of the illiquid stock. Thus, flight to liquidity does not occur. Notice that the flight to liquidity cannot be caused by a stock illiquidity *per se*, but it could be triggered by additional features attributed to an illiquid stock. For example, it could be caused by an increasing volatility of an illiquid stock return.

No flight to liquidity does not mean, however, that the price of the illiquid stock does not change. Welfare analysis shows that an investor substantially discounts his holdings of the illiquid stock. This could lead to a noticeable fall of the illiquid stock price. We find that the presence of stock illiquidity can put an investor in states where his portfolio has excessive volatility. This volatility could be so strong, that a risk averse investor would be willing to replace his illiquid stock holdings with the same value holdings of a stock that has negative risk premium. Therefore, the expected returns of the illiquid stock could be very high.

2. If an investor trades two stocks one of which is illiquid, then holdings of both risky securities would be substantially different from those where both stocks are liquid. If there is a noticeable correlation between two stocks and the illiquid stock holdings are small, then an investor always reduces his allocations in the liquid stock to that in the liquid market with one stock. If volatility of the illiquid stock holdings is too high, he adjusts allocations in the liquid stock to reduce cumulative volatility. In particular, if a correlation between two stocks is positive, then an investor can take a short position in the liquid stock even if this position is significantly positive in a perfectly liquid market. Allocations to a liquid stock are similar but less pronounced, if two stocks are independent.

We find only minor quantitative differences in the last results between the cases where illiquidity is very strong, moderate or even small. It follows that the described impacts on liquid stock portfolio allocations can be significant even when stock illiquidity is minor, leading to incorrect conclusions about investors' rationality and market efficiency, if illiquidity is neglected.

3. In the special case when two traded stocks have perfect correlation, we recover the familiar story of limits to arbitrage: An investor considers arbitrage opportunities on an equal basis with other investment opportunities. He offsets the risk of his position in an illiquid stock with the one in a liquid stock and takes additional position in a liquid stock that is the same as in the absence of an illiquid stock.[8]

4. We consider optimal portfolio rules when, in addition to stock illiquidity, an investor faces the possibility that this stock may crash. We find that in the presence of stock crashes, an investor can easily find himself in states where he is willing to substitute his stock holdings with substantially positive risk premium by those of a stock with a continuous path and negative risk premium. Moreover, at many states he discounts the illiquid stock return for crashes at a much higher rate than he does the returns of a liquid stock experiencing the same crashes.

5. If the linear term is present in the transaction costs, then the shape of the no–trading zone is independent from the degree of convexity of the costs. This result indicates that our approach can be used to study the no–trading zone. A traditional analysis of an investor's trading in the presence of proportional transaction costs requires solving a challenging stochastic singular control problem (e.g., Shreve and Soner, 1992). Our approach can be used to study the no–trading zone for an investor with arbitrary preferences in the arbitrary market (see also conclusion 6) avoiding singularities of controls.

6. Finally, in Appendix C we verify that the above conclusions hold if the investment opportunity set is stochastic. In particular, we consider portfolio rule for an investor in the market with stochastic volatility of the stock return. We confirm that the expected utility function is maximized at allocations which are also optimal in the liquid market. Moreover, the no–trading zone is not affected by the convexity of the transaction costs.

An approach related to ours is taken by Longstaff (2001) who assumes that stock shares can be traded at a rate which is bounded from above and below by constants. As a result, an investor always chooses to buy or sell shares at the constant rate. This behavior implies that some compositions of the portfolio cannot be reached and that the expected utility function of an investor is maximized at portfolio weights which are substantially different from those in the perfectly liquid case. Longstaff maximizes expected utility function of an investor with respect to his initial allocations and studies its deviation relatively to his expected utility function in the perfect market. In our model, stock illiquidity is related to the market microstructure and an investor can choose any finite rate of trading and reach any proportion of stock holdings, by possibly paying very high fees. Contrary to Longstaff's

[8]Related results can be found in Basak and Croitoru (2000, 2003), Liu and Longstaff (2004), and Isaenko (2004).

model, our investor achieves the highest expected utility function at allocations which are optimal in the liquid market.[9] Moreover, we analyze optimal trading and welfare of an investor when his allocations are not utility–maximizing. A situation resulting from the market illiquidity that is not studied elsewhere. In addition, we analyze trading volume and optimal allocations by an investor in the presence of a second, perfectly liquid stock, as well as in the case where an illiquid stock may undergo crashes. Both markets are very common in practice.

The second paper which is related to ours is the one by Kahl, Liu, and Longstaff (2003). These authors consider a portfolio choice by an entrepreneur who cannot trade his firm's stock shares but is not restricted in trading other securities. Inability of investor to trade stock shares of his firm can be formally interpreted as if his firm's stock is illiquid. If so, our special case when the cost of trading of an illiquid stock is very high and a liquid stock is present resembles their model and some of their conclusions are similar to ours. However, illiquidity in our model is not extreme as investors can trade the security at a flexible and possibly high rate. Moreover, an investor in our model trades an illiquid stock that is essential for the stock market rather than a small stock with idiosyncratic risk as in Kahl, Liu, and Longstaff (2003). Overall, our analysis applies to trading by an investor in an illiquid market, rather than to an insider's transactions in a perfectly liquid market.

Finally, Cetin and Rogers (2005), and Rogers and Singh (2006) have independently from us used convex transaction costs for defining market illiquidity for an investor in discrete and continuous time settings, respectively. However, the goal of the first paper is a derivation of the equation for the value function, while the second paper analyzes option prices.

Our paper contributes to growing theoretical literature on the market's illiquidity. In addition to the above–cited references, important contributions include Lipman and McCall (1986), Amihud and Mendelson (1986), Grossman and Miller (1988), Grossman and Laroque (1990), Boudoukh and Whitelaw (1993), Holmstrom and Tirole (2001), Weill (2002), Acharya and Pedersen (2003), Huang (2003), Duffie, Garleanu, and Pedersen (2003a, 2003b), O'Hara (2003), Vayanos and Wang (2003), Eisfeldt (2004), Brunnermeir and Pederson (2005), Huang and Wang (2005), and many others.

The rest of the article is organized as follows. Section 2 describes the model. Section 3 presents results of the optimal control problem faced by an agent who trades a bond and one illiquid stock with a continuous path. Section 4 derives portfolio rules for an investor who, besides holding an illiquid stock, can also trade a perfectly liquid one. Both stocks have continuous paths. Section 5 presents optimal allocations of an investor who trades an illiquid stock that may undergo crashes. Section 6 summarizes. Appendix A describes

[9]Contrary to our assumption, Longstaff (2001) assumes that the volatility of a stock return is stochastic. However, this assumption does not change the nature of our conclusions, as verified in Appendix C.

the solutions of the optimization problems introduced in Sections 2, 4, and 5, Appendix B identifies the conditions for absence of arbitrage opportunities in the market with two stocks, while Appendix C solves a portfolio–choice problem in the market with stochastic volatility of the stock return.

2 The Basic Model

2.1 The Asset Market

We consider a markovian economy with a finite horizon T where there is a single perishable consumption good that we treat as the numeraire. We assume a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, Q)$. Uncertainty in the model is generated by a standard one–dimensional Brownian motion W. In Section 4, we assume that uncertainty is generated by a standard two–dimensional Brownian motion $W = (W_1, W_2)$. In Section 5, we assume that uncertainty in the model is generated by a standard one–dimensional Brownian motion W and by independent Poisson process P. In all cases W_t is adapted and so is P_t.

Investors can continuously trade two securities: a riskless bond and a stock. The riskless bond has the price dynamics

$$dB_t = B_t r dt, \tag{1}$$

where r denotes a constant interest rate. The price of the stock S_t follows the geometric Brownian process

$$dS_t \quad = \quad S_t(\mu dt + \sigma dW_t), \tag{2}$$

where σ is positive.

Because only one stock is available, we interpret it as a stock market. In Sections 4 and 5, we consider trading two stocks and trading one stock with crashes, respectively, which will be described therein.

We assume that trading stock shares takes place at a finite rate u, that is

$$dN_t = u_t dt, \tag{3}$$

where N is a number of shares hold by an investor.

This assumption is justified by the costs of trading stock shares. We require that trading ΔN shares of stock within time interval Δt will costs $\alpha|\Delta N| = \alpha|u|\Delta t$ dollars. We also assume that the cost percentage α increases with the rate of trading $|u|$ as $\alpha_1 + \alpha_2|u|^{\varepsilon}$, where α_1, α_2, and ε are positive constants. Hence, for trading ΔN shares an investor must pay a fee equal to $(\alpha_1|u| + \alpha_2|u|^{1+\varepsilon})\Delta t$ dollars. For simplicity of exposition, we require that coefficients α_1 and α_2 be the same for the purchasing and selling of shares. If the market is liquid, the dependence of α on $|u|$ is very weak and could be neglected. However, if the market becomes illiquid then coefficient α_2 becomes significant. The chosen representation of the transaction costs is intuitive. When the market is illiquid, it becomes more difficult for a trader to find stock shares. The faster shares have to be bought (sold) the more difficult it is to find their seller (buyer). To compensate for facing extra risk and effort, the trader imposes higher costs on an investor.

The term $\alpha_1|u|$ in the transaction costs represents the common proportional transaction fees paid for trading stock shares, while the term $\alpha_2|u|^{1+\varepsilon}$ captures the difficulty of trading shares at a high rate. The latter term does not allow the rate of trading u to be infinite, even though it can be large: The higher the u at fixed ΔN, the more costly it is to trade for an investor. Naturally, share holding N becomes absolutely continuous and can be given by expression (3). We emphasize the difference between our way of making u finite and that of Longstaff (2001). Longstaff simply requires that the rate of trading be bounded from above and below, implying that some portfolio compositions cannot be reached if S is a diffusion process and consumption is not allowed. In our model, any portfolio composition can be reached, but at the expense of losing money on costly portfolio adjustments.

Notice that an investor could have placed the buy (sell) order between many traders to buy (sell) ΔN shares within time Δt and considerably reduced the nonlinear component in transaction fees. Then stock illiquidity substantially decreases for this investor. We assume that this opportunity is limited either by a small number of traders, considerable time and effort required to place the orders between many traders, or fixed transaction fees charged by each trader. The last factor is not present in our model since it is not important for our goal, but it can become important in preventing diversification of the order.

2.2 The Investor's Problem

An investor is a price–taker who has CRRA–preferences that support only consumption at time T. The optimization problem faced by this investor is[10]

$$\max_{u \in R} \quad E_0\left(\frac{X_T^\gamma}{\gamma}\right) \tag{4}$$

$$dX_t = [rX_t + N_t S_t(\mu - r) - \alpha_1|u_t| - \alpha_2|u_t|^{1+\varepsilon}]dt + N_t S_t \sigma dW_t, \tag{5}$$

[10]Existing papers on portfolio optimization in the presence of proportional transaction costs often assumes that an investor sells all his stock shares at time T and consumes from his money market account. In our approach, an investor chooses not to sell the stock shares at time T. It is likely that an investor will continue to trade his portfolio after time T but he expects that the liquidity of the market will change and so a new optimization problem will be formulated. See also Longstaff (2001).

where $\gamma < 1$, $\gamma \neq 0$, and X stands for an investor's wealth.
Since dX depends explicitly on S and N, and u is a rate of trading N (not NS), the description of the problem is completed by adding equations 2 and 3.

Two remarks are in order. First, we assume that there is no explicit utility maximization over different initial conditions, that is, an investor cannot choose to be at a given point of the state variable space at time zero. If an investor can optimize throughout the initial conditions, then all the effects discussed in this paper would be present, but the expected magnitudes for some of them could be smaller than at other choices for the initial conditions. Moreover, there is no solid economic reason that would require utility maximization over different initial conditions.

Second, we assume that transaction charges are placed on the number of shares instead of the volume of the stock traded by an investor. Often, charges are placed on the volume, at least for the linear term in the costs. It is straightforward to modify our model by assuming that the linear term in the costs is proportional to the volume traded without any qualitative consequences for our conclusions. If we assume that all charges are proportional to the volume traded, then the stock allocation $\theta = NS$ will have finite first order variation and the dimension of the state variable space reduces to two. The shortcoming of this assumption is that the volatility of θ could be zero only if $\sigma_N = -\sigma$, where $N\sigma_N$ is the diffusion coefficient of N. It is hard to believe that the last equality is true. More realistically, one would expect that $\sigma > 0$ and $\sigma_N = 0$.[11]

2.3 Optimal Policies with no Transaction Costs

For the purpose of comparison, we describe the optimal strategy of an investor with CRRA–preferences in the absence of transaction costs (see Merton, 1969). In this case, we call a stock as (perfectly) liquid. Here, an investor chooses to invest a fixed fraction of his wealth in the stock and this fraction is independent of the investor's horizon:

$$\frac{N_t S_t}{X_t} = \hat{\pi} \equiv \frac{\mu - r}{(1-\gamma)\sigma^2}. \tag{6}$$

This result allows us to say that optimal allocations lie on the "Merton line". Loosely, we also say that optimal allocations to a liquid stock lie on the Merton line when this stock may crash, or when there is a second liquid stock.
The lifetime expected utility of this investor is

$$U = \frac{X^\gamma}{\gamma} \exp\left\{\gamma\left[r + \frac{(\mu - r)^2}{2\sigma^2(1-\gamma)}\right]T\right\}. \tag{7}$$

[11]Most of our results remain valid after appropriate modifications in state variables even if we assume that all fees are proportional to the volume of traded stock.

Notice that U is quadratic in $(\mu - r)$ implying that an investor will benefit from his stock positions even if $\mu - r < 0$. In this case, the benefits are achieved by taking a short position in the stock.

Sections 4 and 5 will present the portfolio rules and the expected utilities in an economy with two liquid stocks, and in an economy with one liquid stock susceptible to crashes, respectively.

2.4 Optimal Policies with Transaction Costs

We now consider the case when transaction costs are present. Moreover, for simplicity of calculations we assume from now on that $\varepsilon = 1$. Appendix A analyzes the problem for arbitrary $\varepsilon > 0$.

One can show that the investor's problem has a solution only if $X \geq NS \geq 0$, so we name a line $X = NS$ on the $NS - X$ plane as a solvency line and the area between NS–axis and a solvency line as a solvency region. The proof of the last result is very similar to that in Longstaff (2001) and is not presented. In essence, an investor cannot borrow, because a stock can quickly fall and he would not be able to sell enough stock to have positive terminal wealth. Furthermore, he cannot short–sell because the stock can rapidly rise and an investor would not be able to unwind his obligation before his wealth becomes negative. Thus, a dramatic effect of illiquidity on portfolio composition. Notice that an investor may borrow or take a short position in the stock if he has an additional source of earnings, for example from labor income which can hedge the variation in the stock.

The optimization problem (4) can be solved only by the dynamic programming approach. Furthermore, equations (2), (3), and (5) show that the markovian set of state variables in the given economy is (S, X, N).

As in Merton (1969), we define an investor's indirect utility to be

$$V(t, S, X, N) = \max_{u \in R} E_t \left(\frac{X_T^\gamma}{\gamma} \right). \tag{8}$$

The appendix shows the Hamilton–Jacobi–Bellman (HJB) equation for this value function. The first–order condition in the HJB equation gives the optimal rate of trading

$$u = \begin{cases} \frac{V_N - V_X \alpha_1}{2 V_X \alpha_2} & if \;\; V_N/V_X > \alpha_1 \\ 0 & if \;\; -\alpha_1 \leq V_N/V_X \leq \alpha_1 \\ \frac{V_N + V_X \alpha_1}{2 V_X \alpha_2} & if \;\; V_N/V_X < -\alpha_1. \end{cases} \tag{9}$$

The last result shows that an investor does not trade when $|V_N/V_X| < \alpha_1$, but a no–trading zone is absent if the linear term in the transaction costs is zero. To understand the nature of the existence of the no–trading zone, one has to consider change in the indirect utility resulting from a deviation from its maximizing allocations in a stock and that resulting from paying fees to adjust the allocations back. Suppose that the original allocation to the stock N is such that the indirect utility is maximal. (See the next section where we show that the indirect utility is maximal when $NS/X = \hat{\pi}$.) We now perturb this allocation by a small ΔN which will result in negative change in the indirect utility given by $V_N \Delta N$. To trade back to the original allocations within a small time Δt an investor has to pay $(\alpha_1 |\frac{\Delta N}{\Delta t}| + \alpha_2 (\frac{\Delta N}{\Delta t})^2)\Delta t$ dollars so the corresponding change in the value function is $-V_X(\alpha_1|u| + \alpha_2 u^2)\Delta t$, where $u = \frac{\Delta N}{\Delta t}$. To see whether an investor will or will not trade we have to compare $|V_N|$ and $V_X(\alpha_1 + \alpha_2|u|)$. Suppose that $\alpha_1 = 0$ and $\alpha_2 > 0$, then at an optimal level of trading the second term becomes $V_X \alpha_2 |u| = |V_N|/2$ which is always less than the first term. Hence, an investor will always prefer to trade to return to utility maximizing allocations. Now let us assume that $\alpha_1 > 0$ and $\alpha_2 > 0$. Then it is easy to show that after the substitution of optimal trading rate into $V_X(\alpha_1 + \alpha_2|u|)$ this term could be above $|V_N|$ (if $|V_N/V_X| < \alpha_1$) or never above it (if $|V_N/V_X| \geq \alpha_1$). Thus, an existence of the no–trading zone. We summarize that the existence of the no–trading zone when $\alpha_1 > 0$ and $\alpha_2 > 0$, and its absence when $\alpha_1 = 0$, and $\alpha_2 > 0$ follows from an investor's ability to recover quadratic costs from stock returns by slow trading and the inability to do so for linear costs (at least for a range of state variables), since they are independent from the rate of trading for a given ΔN. These conclusions are independent from whether trading is continuous or the stock path has an infinite first order variation. Even though the linear term $\alpha_1|\Delta N|$ alone cannot cause a market illiquidity for an investor in our approach, we will consider its impact on optimal strategies and document new findings.

Unfortunately, the equation for indirect utility function cannot be solved analytically and we reserve to the traditional finite–difference numerical approach. The following section presents the results of the numerical analysis. We are interested in states of the economy where stock liquidity is limited. Assuming that these states exist for a reasonably short period of time, we set the time horizon of an investor to be only one year. In addition, we choose the following parameters unless specified otherwise: $\gamma = -1$, $\sigma = 0.2$, $\mu = 0.07$, and $r = 0.01$.[12]

We show results when some of the state variables are fixed. Consideration of the results for other values of these state variables does not bring any new intuition and thus not presented. Moreover, some of the results are not new qualitatively, but we still document them for completeness.

[12] In the calculations, we choose state variable $\ln(S)$ instead of S and consider time and state variables $(t, \ln(S), X, N)$ on the set of values $[0,1] \times [-3,3] \times [0,10] \times [0,3]$, where the corresponding numbers of the grid–points are $10 \times 100 \times 100 \times 800$. The condition $0 \leq NS \leq X$ is maintained on each point of the grid.

3 Optimal portfolio with One Stock

In this section we consider an optimal portfolio held by an investor in the presence of nonlinear transaction costs. We use the solution of the Merton problem described in Section 2.3 as a benchmark.

First let us assume that only the quadratic term in the transaction costs is present. We notice a fundamental difference between optimal trading in our setting and that of the Merton problem. In the latter case, an investor will have a fixed proportion of his wealth invested to a stock at any state of the world, which is possible because of an investor's ability to trade an arbitrary large number of shares within a very short time interval. Because this is not true in our setting, an investor's allocations of stock could be way off from the Merton line and these allocations would be optimal in the sense that his indirect utility is maximal for given values of state variables. Depending on the dollar volume allocated to the stock, an investor prefers to buy or sell shares at an optimal rate. Interestingly, we find that the boundary between zones of selling and buying stock is a plane in the $N-S-X$ space defined by the equation $NS/X = \hat{\pi}$. That is, if we fix one of the two state variables N or S, then the value function $V(t,\cdot,\cdot,\cdot)$ reaches its maximum on the Merton line. The size of this maximum is probably affected by the value of the fixed state variable. However, this effect is very weak and well within the calculation error. Notice that this result is different in nature from that of Longstaff (2001): He finds that the border between buy and sell regions is dramatically affected by stock illiquidity. The difference between our conclusions is caused by the gradual nature of trading in our case: As NS/X gets further away from the utility–maximizing ratio $\hat{\pi}$ the rate of trading increases in magnitude, while in Longstaff (2001) this rate is always constant in magnitude. The rate of trading in our case is defined by the tradeoff between losses in utility due to not optimal allocations to the stock and the costs required to be paid in order to reach these allocations faster.

Remark 1 *We verify numerically that an investor reaches maximum of his expected utility function on the plane $NS/X = \hat{\pi}$ for any level of convexity ε.*

It follows that in our model, the flight to liquidity, that is the will to reduce illiquid stock holdings in favor of bond holdings, does not exist if the illiquid stock is the market stock. In Appendix C, we confirm that this conclusion also holds if an investment opportunity set is stochastic. We assume that volatility of a stock return is stochastic and show that the expected utility function is maximal at allocations that are optimal in the liquid market. This result does not imply that the price of illiquid stock is the same as the price of stock in a liquid market. Welfare analysis, to be discussed later, shows that an investor substantially discounts his holdings of the illiquid stock. This discount may cause a significant fall of the illiquid stock price. We also notice that the stock illiquidity is often caused by its sudden fall. As a result, investors lower their expectations about stock returns and try to decrease their illiquid stock holdings to reach the Merton line. This reduction has a behavioral nature but could be incorrectly interpreted as the flight to liquidity. In addition, the fall of the illiquid stock holdings can be caused by a sudden increase in this stock return volatility. An increase in the stock return volatility is often comes along with a decrease in this stock liquidity.[13] As a result, investors reduce their holdings of a less liquid stock because of its increased volatility and not decreased liquidity. The last three remarks also apply to the market with two stocks and the market with illiquid stock susceptible to crashes to be analyzed in Sections 4 and 5, respectively.

For welfare analysis, let us define the liquidity premium Δ_l to be the amount of the conditional stock return that an investor is willing to give up to avoid trading the illiquid stock:

$$V(t,S,X,N) = \frac{X^\gamma}{\gamma} \exp\left\{\gamma\left[r + \frac{(\mu - \Delta_l - r)^2}{2\sigma^2(1-\gamma)}\right](T-t)\right\}, \tag{10}$$

or

$$\Delta_l(t,S,X,N) = \mu - r - \sqrt{2\sigma^2(1-\gamma)\left(\frac{1}{\gamma(T-t)}\ln\frac{V(t,S,X,N)}{X^\gamma/\gamma} - r\right)}, \tag{11}$$

where we took into account that the expected utility function in the Merton problem is given by (7).

Table 1 shows the rate of trading, the expected utilities, and the liquidity premia for different values of state variable N at $t=0$ and fixed S and X when $\alpha_2 = 0.2$ and $\alpha_2 = 0.005$. This table confirms the intuition about the rate of trading discussed at the beginning of this section. We also notice that if $\alpha_2 = 0.2$, then the rate of trading is so small that an investor can never reach his highest expected utility function if the initial allocations are off the Merton line. As a result, this case is similar to the situation when a stock cannot be traded at all (for example, see Kahl, Liu, and Longstaff, 2003, and Longstaff, 2005).

Now let us discuss the liquidity premium. Table 1 shows that this premium rises as α_2 goes up. It follows from the table that when α_2 is large and $NS/X \ll \hat{\pi}$, the expected utility function is close to the one when no stock is traded ($U = U_{min} \equiv \frac{X^\gamma}{\gamma}\exp(T\gamma r)$). Thus, the liquidity premium is close to $\mu - r$. It decreases as NS/X approaches $\hat{\pi}$ and increases again when $NS/X > \hat{\pi}$. The liquidity premium becomes even more extreme than in Table 1 if stock volatility increases: At high proportions of stock holdings, the expected utility of an investor becomes even smaller than the one he would have from holding only bonds. For example, Table 2 shows that if we augment σ from 0.2 to 0.3, the expected utility function

[13]According to CGFS, during the 1998 crisis, the implied volatility of the S&P index increased from 23% to 43%, that of the three–month US eurodollar rate from 8% to 33%, and that of the thirty–year US T–bonds rate from 7% to 14%. See Vayanos (2003).

of an investor falls below U_{min} in states with a high proportion of NS/X. This behavior results from the excessive volatility of an investor's portfolio.[14] We conclude that in some states of the economy an investor holds so much illiquid stock that he values these holdings not only below their market value, but also is willing to hold a liquid stock with negative risk premium, just to avoid facing illiquidity.

Longer than optimal position in the illiquid stock may result from a lowering in investor's expectations on stock returns so the stock becomes less attractive implying an excessive proportion NS/X. It may also follow from a sudden increase in the volatility of the illiquid stock returns (see footnote 13). For example, suppose that the stock market is liquid and $\hat{\pi} = 1.0$. Then right before time zero, this market becomes illiquid triggered possibly by its significant fall, and, as a result, the volatility of its return increases by $\sqrt{3}$. According to formula (6), an investor's expected utility is now maximized when $NS/X = 0.33$ (see also Table 2). But an investor cannot reach this allocation immediately because it is extremely expensive for him. Thus, his initial condition would be $NS/X = 1.0$ and the discount of the illiquid stock holdings is very high.[15]

The high values of the liquidity premium show that the existence of times when the market liquidity is low can explain why the historical stock returns are much higher than many equilibrium asset pricing models predict. Complete evaluation of the contribution of the liquidity premium to stock returns requires general equilibrium analysis of the economy.

The shown results are found when $T = 1$ and $\gamma = -1$. If we increase an investor's horizon, then the rate of trading grows in magnitude in both directions. This is due to an extra expected wealth that an investor obtains because of longer stock growth, so he can pay higher fees for trading shares to improve his allocations. In addition, the liquidity premium decreases in all states as an investor can reach the highest utility faster. If we increase an investor's risk aversion, then the Merton line will increase its slope in the $NS - X$ plane, so the rate of trading declines at small N and increases in magnitude at large N. Because higher risk aversion implies higher impassion for volatility, the utility in states with high N could easily be below U_{min}. For example, if we set $\gamma = -3$ while keeping all other parameters and state variables X, S as in Table 1, then $\Delta_l = 4.6\%$ and increases fast at $NS/X = 0.75$ versus $\Delta_l = 0.01\%$ in Table 1.

[14] As an illustration of this claim, let us consider two securities that have terminal values $\tilde{x}$ and a, where the former is random and the latter is constant. Assume that $E\tilde{x} > a$. Because the utility function is concave, $EU(\tilde{x}) < U(E\tilde{x})$. However, the last two inequalities do not imply that $EU(\tilde{x}) > U(a)$. In fact, the opposite inequality can be true if the variance of $\tilde{x}$ is large. For example, if $\tilde{x} = 2$ or 1 with the same probability 1/2 and $a = 1.4$, $\gamma = -1$, then $EU(\tilde{x}) = -0.75 < U(a) = -0.714$. If we decrease the variance of the portfolio by diluting $\tilde{x}$ with a to obtain $w\tilde{x} + (1 - w)a$, where $0 < w < 1$, then we can easily make $EU(w\tilde{x} + (1 - w)a) > U(a)$. This can be seen from the last example if we set $w = 0.5$, then $EU(w\tilde{x} + (1 - w)a) = -0.711 > U(a)$.

[15] In extreme cases with initial conditions $NS/X > 1.0$ or $N < 0$, an investor may have to default on his obligations.

Now let us consider the case when both α's are positive. In this case the solvency region splits into the buy, sell, and no–trading zones where an investor buys, sells, and does not trade stock shares, respectively. The no–trading zone lies between the first two and is separated from them by straight planes. The linearity of boundaries is defined by the linearity of the first term in the transaction costs and not by the convexity of the second one.

Remark 2 *We find that the shape of the no–trading zone remains unchanged for any choice of the convexity* ε*.*

The last remark follows from the ability of an investor to recover convex costs by slow trading and independence of the proportional transaction costs from the rate of trading at the given ΔN. This remark also applies to the case when there is a second, perfectly liquid stock and to the case when illiquid stock may crash, that are considered in Sections 4 and 5, respectively. Overall, our approach provides an alternative tool for studying the no–trading zone comparing to its traditional treatment which deals with a challenging singular stochastic control problem.

Figure 1 depicts the no–trading zones for an investor in the presence of the linear component in the transaction costs. Thick solid and dashed lines show the no–trading zone for $S = 1.0$ and $S = 0.5$, respectively, with all parameters being identical. As seen, the no–trading zone is very wide. This width results from a short time–horizon of an investor and the no–trading zone narrows as this horizon extends. The shorter the horizon, the smaller the expected stock return that can be used for paying transaction costs. Hence, the trade occurs further away from the Merton line (see also Liu and Loewenstein, 2002). Furthermore, the no–trading zone is wider for the smaller stock price. The reason follows from the fact that transaction fees are charged on the number of shares traded rather than the volume of the traded stock, so it is less costly to adjust the stock allocation for a more expensive stock at a given total stock investment of NS. Therefore, an investor can trade more volume with more expensive stock. When fees are charged on the volume of the stock traded, the no–trading zone is independent from the stock price.

The dependence of the no–trading zone on the stock price causes the buy and sell zones disappear at small S, so that at these values an investor never trades the stock. The stock is not traded because the loss in the expected utility function due to deviations of allocations to the stock from the Merton line are always smaller then transaction costs required to pay for staying inside the solvency region. This effect is illustrated in Figure 2, where we show the no–trading zone for different stock prices.

Table 3 shows the rate of trading when linear component α_1 is equal to 0.005. Not surprisingly, this rate decreases as the state of an investor approaches the no–trading zone. Moreover, the rate of trading is dramatically affected by the width of the no–trading zone

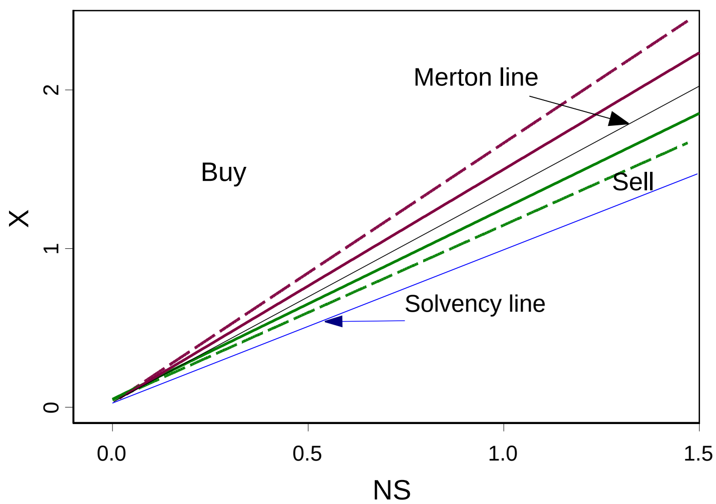

Figure 1: This Figure shows the no–trading zones of an investor when $S = 1.0$ (thick solid line) and when $S = 0.5$ (thick dashed line). The time is zero, and the parameters are fixed at $\alpha_1 = 0.005$, $\alpha_2 = 0.2$, $\gamma = -1$, $\mu = 0.07$, $\sigma = 0.2$, $r = 0.01$, and $T = 1$.

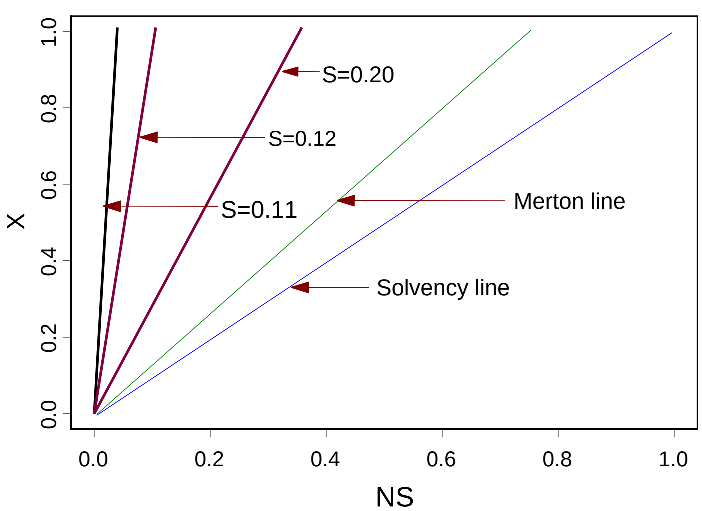

Figure 2: This Figure shows the boundaries between the buy and no–trading zones for different stock prices when $\alpha_1 = 0.005$ and $\alpha_2 = 0.2$. The boundaries between the sell and no–trading zones are absent since they lie below the solvency line. The time is zero, and the other parameters are fixed at $\gamma = -1$, $\sigma = 0.2$, $\mu = 0.07$, $r = 0.01$, and $T = 1$.

(parameter α_1): Even slight decrease in this width considerably increases the rate of trading in both directions.

Comparison of the expected utilities shows that the introduction of the linear term brings a noticeable change into the welfare of an investor when the nonlinear term is reasonably small (see Tables 1 and 3 for $\alpha_2 = 0.005$). It implies that if the horizon is short, then the impact of the no-trading zone on prices could be significant, contrary to the one in the long horizon (see Constantinides (1986)). Moreover, if the convex term is significant, the introduction of the linear term has no effect within the accuracy of the display (see Tables 1 and 3 for $\alpha_2 = 0.2$). The last result confirms that $\alpha_2 = 0.2$ is large enough to make the trading of an investor be essentially negligible.

In Appendix C we study investor's trading in the presence of linear term in the transaction costs when volatility of the market stock return is stochastic. Comparison of Tables 3 and 13 shows that the no–trading zone becomes wider if the volatility σ is stochastic and we condition the state of economy on its value in an economy with the constant investment opportunities set. If volatility of a stock return is stochastic, then an investor has to trade more often to stay near the Merton line. Thus, the transaction costs increase making him trade further away from the Merton line. In turn, widening of the no–trading zone causes the rate of trade to decrease in both directions.

4 Optimal Portfolio with Two Stocks

In this section we consider the optimal portfolio of an investor who can trade two stocks, one of which is illiquid (stock 1), while the other is perfectly liquid (stock 2). One can think of these stocks as those making the stock market. Our goal is to understand how the presence of the illiquid stock affects the allocation of an investor to the liquid stock. This goal is motivated by a common situation in markets when the presence of illiquid securities has a significant effect on the demand for liquid securities. In special instances, these two securities may be strongly or even perfectly correlated in their returns. In the last case, arbitrage opportunities may exist in the market.

We assume that the dynamics of the stocks are given by

$$dS_{1t} = S_{1t}(\mu_1 dt + \sigma_1 dW_{1t}), \tag{12}$$

$$dS_{2t} = S_{2t}(\mu_2 dt + \sigma_2 dW_{2t}), \tag{13}$$

where the standard Brownian motions W_1 and W_2 have the correlation coefficient ρ and μ_1, μ_2, σ_1, and σ_2 are constants, the last two of which are positive. The dynamics of an investor's wealth follows

$$
\begin{aligned}
dX_t &= [rX_t + N_{1t}S_{1t}(\mu_1 - r) + X_t\pi_{2t}(\mu_2 - r) - \alpha_1|u_{1t}| - \alpha_2 u_{1t}^2]dt \\
&+ N_{1t}S_{1t}\sigma_1 dW_{1t} + \pi_{2t}X_t\sigma_2 dW_{2t}, \qquad (14)
\end{aligned}
$$

where N_1 is the number of stock 1 shares and π_2 is the proportion of an investor's wealth allocated to stock 2.

First we identify the conditions for the absence of arbitrage. To do so, we introduce its definition:

Definition 1 *An arbitrage (a free lunch) is a self–financed portfolio process such that the associated wealth process $X(\cdot)$ satisfies*

$$X_0 \geq 0, \qquad X_T > X_0 e^{rT}. \qquad (15)$$

An arbitrage is a portfolio process whose wealth grows locally risklessly with a rate higher than the instantaneous interest rate. In Appendix B we show that, similar to perfect markets, arbitrage opportunities are possible only if $\rho = 1$ and $\Delta \equiv \frac{\mu_1 - r}{\sigma_1} - \frac{\mu_2 - r}{\sigma_2} \neq 0$ or if $\rho = -1$ and $\frac{\mu_1 - r}{\sigma_1} \neq -\frac{\mu_2 - r}{\sigma_2}$.

We will determine the decision rules of an investor and compare them to the benchmark case when both stocks are liquid:

$$
\begin{aligned}
\hat{\pi}_i &\equiv \frac{1}{\sigma_i(1-\gamma)(1-\rho^2)}\left[\frac{(\mu_i - r)}{\sigma_i} - \rho\frac{(\mu_j - r)}{\sigma_j}\right], \qquad i, j = 1, 2, \qquad (16) \\
V(t, X) &= \frac{X^\gamma}{\gamma}\exp\Bigg\{\gamma\Bigg[r + \frac{1}{2(1-\gamma)(1-\rho^2)} \qquad (17) \\
&\times \left(\frac{(\mu_1 - r)^2}{\sigma_1^2} + \frac{(\mu_2 - r)^2}{\sigma_2^2} - 2\frac{(\mu_1 - r)(\mu_2 - r)}{\sigma_1\sigma_2}\rho\right)\Bigg](T - t)\Bigg\}.
\end{aligned}
$$

Formula (16) implies that holdings of one stock are independent from the presence of the other when the two stocks are independent. As the number of independent stocks increases, the aggregate risk exposure of an investor increases proportionally. Nonetheless, the probability for the portfolio wealth to fall below a certain threshold diminishes because of diversification. If stocks are positively/negatively correlated, then an investor takes a shorter/longer positions in stocks to reduce his exposure to the correlated parts of risks. The correlated risks can also be reduced by hedging. If stocks are perfectly correlated and their conditional Sharpe ratios are different, an investor takes infinite opposite positions in the two stocks to take advantage of arbitrage. If $\rho = 1$ and the conditional Sharpe ratios of the two stocks are the same, then an investor is indifferent between the two stocks. Finally, we notice that an investor always chooses a market portfolio since π_i is independent from an investor's wealth.

If stock 1 is illiquid, an investor cannot short or take a very long position in stock 1 if $|\rho| \neq 1$, because unpredictable variations in this stock cannot be completely hedged away. Perfect hedging is possible, however, if $|\rho| = 1$, thus allowing positions with $N_1 < 0$ and $N_1 S_1 > X$.

We present the HJB equation for the value function $V(T, S_1, X, N_1)$ of the given problem in Appendix A. The first–order conditions in this equation provide the optimal policies for an investor:

$$u_1 = \begin{cases} \frac{V_{N_1} - V_X \alpha_1}{2 V_X \alpha_2} & if \;\; V_{N_1}/V_X > \alpha_1 \\ 0 & if \;\; -\alpha_1 \leq V_{N_1}/V_X \leq \alpha_1 \\ \frac{V_{N_1} + V_X \alpha_1}{2 V_X \alpha_2} & if \;\; V_{N_1}/V_X < -\alpha_1, \end{cases} \tag{18}$$

$$\pi_2 = -\frac{V_X(\mu_2 - r) + (V_{XX} N_1 S_1 + V_{X S_1} S_1)\rho\sigma_1\sigma_2}{X V_{XX} \sigma_2^2}. \tag{19}$$

Clearly, an investor can maintain any proportion π_2 at any time by trading bond and stock 2 at a possibly infinite rate.

Let us also introduce the liquidity premium Δ_{l1} of stock 1 to be the amount of this stock conditional return that an investor is willing to give up to avoid trading illiquid stock. As follows from equation (17), indirect utility is not always a monotonically increasing function of μ_1 when $\rho \neq 0$. Thus, Δ_{l1} is not an accurate description of an investor's welfare when $\rho \neq 0$ so we use it only when $\rho = 0$:

$$V(t, S_1, X, N_1) \;\; = \;\; \frac{X^\gamma}{\gamma} \exp\left\{ \gamma \left[r + \frac{1}{2(1-\gamma)} \left(\frac{(\mu_1 - \Delta_{l1} - r)^2}{\sigma_1^2} + \frac{(\mu_2 - r)^2}{\sigma_2^2} \right) \right] (T - t) \right\},$$

or

$$\Delta_{l1}(t, S_1, X, N_1) = \mu_1 - r - \sigma_1 \sqrt{2(1-\gamma)\left(\frac{1}{\gamma(T-t)} \ln \frac{V(t, S_1, X, N_1)}{X^\gamma/\gamma} - r \right) - \left[\frac{(\mu_2 - r)}{\sigma_2} \right]^2}. \tag{20}$$

In the following subsections we consider three cases: When two stocks are independent ($\rho = 0$), when they are imperfectly correlated ($0 < |\rho| < 1$), and when trading stocks provide arbitrage ($\rho = 1$ and $\Delta \neq 0$). One can verify from the following results that the expected utility function of an investor is maximal when $\frac{N_1 S_1}{X} = \hat{\pi}_1$ if $0 \leq |\rho| < 1$. We conclude that an investor does not try to get rid of the illiquid stock in the presence of the liquid one. Thus, the flight to liquidity does not exist. Finally, it is straightforward to verify in all cases, that an investor chooses not to hold the market portfolio, since the weight of each stock holding changes differently with the variations in state variables.

4.1 Trading when $\rho = 0$

In the case of uncorrelated stocks, expression (19) becomes

$$\pi_2 = -\frac{V_X(\mu_2 - r)}{XV_{XX}\sigma_2^2}. \tag{21}$$

The last result is formally the same as the one in the absence of the illiquid stock. However, given that the indirect utility depends on the state variable N_1, we expect that optimal allocations to the liquid stock would depend on allocations to the illiquid stock.

The last expectation is confirmed in Table 4 which describes optimal trading when $\alpha_1 = 0$, $\alpha_2 > 0$, $\sigma_1 = \sigma_2 = 0.2$ and $\mu_1 = \mu_2 = 0.07$. Contrary to the case of two liquid stocks, the presence of another stock has an effect on the other's stock trading. In particular, the comparison of Table 1 with Table 4 shows that the rate of trading the illiquid stock increases in the presence of the liquid stock. Managing stock 2 provides additional positive returns and an investor is better off by forwarding some of them towards covering costs for trading the illiquid stock. As a result, an investor prefers paying higher fees to achieve best allocation in the illiquid stock faster. On the other hand, the presence of the illiquid stock has an effect on the holdings of the liquid stock: When holdings of the illiquid stock are very low, the proportion of wealth invested in the liquid stock is greater than that when both stocks are liquid, and when holdings of the illiquid stock are too high, the proportion of wealth invested in the liquid stock is lower than that when both stocks are liquid. To explain these allocations, we notice that an investor who maximizes his expected utility function from the terminal consumption tries to increase a probability of a high consumption and decrease a probability of a low consumption. In the perfect markets, this goal is achieved by a combination of substantial risk exposure and diversification. If one stock is illiquid, then diversification is very restricted. As a result, an investor manages the risk by controlling the aggregate risk exposure. In particular, when the aggregate risk exposure is low, an investor increases his allocations to the liquid stock to increase terminal consumption. If the aggregate risk exposure is too high, an investor decreases his allocations to the liquid stock to decrease the risk of low consumption. Contrary to the case with liquid stocks, an investor adjusts position in the liquid stock directly to change the aggregate risk exposure. Moreover, changing π_2 from $\hat{\pi}_2$ decreases diversification of stocks. Finally, an investor is much less willing to increase the return and volatility of his portfolio when they are low, than to give up extra volatility at the expense of the portfolio returns when the former is too high. This effect follows from the risk aversion of an investor and becomes stronger as this aversion rises. It strengthens even further if stock 1 becomes more illiquid (α_2 increases) or volatility of its return rises.

Interesting that at low values of N_1 the allocations to the liquid stock are not monotonic. Because in the absence of the illiquid stock $\hat{\pi} = 0.75$, they increase from $\pi_2 = \hat{\pi}$ at $N_1 = 0$

and then decrease as N_1S_1/X approaches $\hat{\pi}_1$. It turns out that an investor assumes that the illiquid stock is absent for trading when N_1 is very small and chooses π_2 as if only one liquid stock is being traded. If N_1 increases, then an investor treats the illiquid stock as an underinvested opportunity which he compensates by increasing allocations to the liquid stock.

Table 4 also shows the liquidity premium Δ_{1l}. Its behavior versus allocation to the illiquid stock is very similar to that in the economy with one illiquid stock: It is very small at $N_1S/X = \hat{\pi}_1$ and then increases as N_1S/X moves away from $\hat{\pi}_1$. Because Δ_{1l} is a discount for one illiquid stock, it is close to Δ_l which is a similar discount in the economy with one illiquid stock (see Tables 1 and 4). Still, $\Delta_{1l} < \Delta_l$. The last relation results from interdependence between optimal policies for trading two stocks. As an investor gets an access to a liquid stock, he improves the management of the risk in an illiquid stock which allows him to demand a smaller discount for holding illiquid stock.

Finally for the case with $\alpha_1 = 0$, Table 5 presents the optimal trading and the expected utility function for a longer time horizon when we set $T = 5$ while $\alpha_1 = 0$, $\alpha_2 = 0.2$, $\sigma_1 = \sigma_2 = 0.2$, and $\mu_1 = \mu_2 = 0.07$. First, we point out a dramatic rise in the trading volume of the illiquid stock for higher T. It is caused by additional returns from both stock holdings which are forwarded to paying transaction costs. Second, we observe a noticeable change in the allocation π_2 for a longer time horizon: It further increases for low N_1S_1/X and further decreases when this ratio is high. As the terminal consumption becomes more remote, an investor is better off by further adjusting the aggregate risk exposure with stock 2 and decreasing the diversification of his position. This is done even though the increased rate of trading the illiquid stock allows for better risk managing by reaching $\hat{\pi}_1$ before an investor horizon. Finally, as the time horizon T is extended, an investor can reach $\hat{\pi}_1$ within a smaller time–fraction of his horizon. Hence, the effect of stock illiquidity on the expected utility function (or the illiquidity premium) weakens. Table 5 also shows the impact on π_2 coming from a combination of the long–time horizon and high volatility of the illiquid stock. As seen, π_2 can fall as low as 63% of $\hat{\pi}_2$ at high ratios of N_1S_1/X.

Now let us assume that $\alpha_1 = 0.005$ and find the resulting effect on the holdings of the liquid stock. Table 6 shows optimal trading for the two different values of α_2 when illiquidity of stock 1 is strong and when it is moderate. Remarkably, even though the rate of trading u_1 is substantially different for the two values of α_2, the allocations to the liquid stock are almost equal to those when $\alpha_1 = 0$ and $\alpha_2 = 0.2$. This behavior results from the presence of the no–trading zone: In the short time horizon, this zone for the illiquid stock is so wide that the allocations to the liquid stock become almost identical to those when the trading rate of the illiquid stock is close to zero (case with $\alpha_2 = 0.2$).

Comparison of Tables 3 and 6 shows that introduction of the liquid stock makes the no–trading zone narrower. The narrowing results from additional returns from the liquid stock which are used to pay transaction costs for trading illiquid stock. Staying closer to the Merton line improves the diversification of stocks.

4.2 Trading when $0 < |\rho| < 1$

The situation when stocks are completely uncorrelated is rather unusual, even though instructive. In this subsection, we consider the case when $0 < |\rho| < 1$.

Table 7 shows optimal trading and the expected utility function when $\rho = 0.5$ and stock returns have identical conditional moments: $\mu_1 = \mu_2 = 0.07$ and $\sigma_1 = \sigma_2 = 0.2$. As follows from equation (16), $\hat{\pi}_1 = \hat{\pi}_2 = 0.50$ so that the expected utility function is highest when $N_1 S_1 / X = 0.5$. Given the results for $\rho = 0$, one would expect π_2 to be reasonably close to $\hat{\pi}_2$. However, we find that π_2 undergoes very significant variations around $\hat{\pi}_2$: If $N_1 S_1 / X$ is small, then π_2 is close to $\hat{\pi} = 0.75$, where $\hat{\pi}$ is an optimal proportion of wealth allocated to stock when only a liquid stock is traded, and if $N_1 S_1 / X$ exceeds $\hat{\pi}_1$, then π_2 falls considerably below $\hat{\pi}_2$. This behavior results from the correlation between the two stocks. If $N_1 S_1 / X$ is substantially below $\hat{\pi}_1$, a trader can rely only on stock 2 for a risk exposure and will trade as if only this stock is available, choosing $\pi_2 \approx 0.75$. If $N_1 S_1 / X$ increases above $\hat{\pi}_1$, then the volatility of the portfolio goes up and the only way it can be decreased is by selling stock 2. Reduction of the liquid stock holdings has very limited effect on the correlated risk since it remains with the illiquid stock. Thus, an investor sells more liquid stock than he does when stocks are uncorrelated.

Allocation to stock 2 may change its dynamics at $N_1 S_1 / X > \hat{\pi}_1$ when the correlation between the two stocks is negative. To compensate for the excessive uncorrelated risk of his position in stock 1, an investor decreases his holdings of stock 2. However, to reduce correlated stock, he increases his holdings of stock 2 since it has the dynamics that is opposite to that of the illiquid stock. Therefore, if the correlation is significant and negative, an investor will increase his holdings of stock 2 at an excessive volatility of stock 1 holdings.

Remarkably, the allocation to the liquid stock is essentially the same for both shown values of α_2 even though the trading of stock 1 is negligible if $\alpha_2 = 0.2$ and an investor can reach the best proportion $\hat{\pi}_2$ before the end of his time horizon if $\alpha_2 = 0.005$. This effect is present if correlation between the two stocks is significant. To understand its nature, let us decompose two correlated sources of uncertainty to two independent ones (e.g., see Shreve, 2004): $W_1 = \sqrt{1-\rho^2} B_1 + \rho B_2$, $W_2 = B_2$, where B_1 and B_2 are two independent standard Brownian motions. Then, the wealth of an investor's portfolio follows

$$
\begin{aligned}
dX_t &= \left[r X_t + N_{1t} S_{1t} \sigma_1 \left(\frac{\mu_1 - r}{\sigma_1} - \rho \frac{\mu_2 - r}{\sigma_2} \right) + \Phi_{\rho t} (\mu_2 - r) - \alpha_1 |u_{1t}| - \alpha_2 u_{1t}^2 \right] dt \\
&+ N_{1t} S_{1t} \sigma_1 \sqrt{1-\rho^2} dB_{1t} + \Phi_{\rho t} \sigma_2 dB_{2t}, \qquad (22)
\end{aligned}
$$

where $\Phi_{\rho t} = \rho N_{1t} S_{1t} \frac{\sigma_1}{\sigma_2} + X_t \pi_{2t}$.
If both stocks are liquid, then it is easy to show that $\Phi_\rho / X = \frac{\mu_2 - r}{(1-\gamma)\sigma_2^2}$, which is a position in stock 2 when two stocks are uncorrelated. Table 7 shows that the risk exposure Φ_ρ / X is almost the same for all values of N_1 and equal to $\frac{\mu_2 - r}{(1-\gamma)\sigma_2^2}$. We conclude that for any position in illiquid stock 1, an investor takes an additional risk exposure in liquid stock 2, such that an aggregate exposure to source of uncertainty B_2 is equal to that in the absence of illiquidity. The exposure to the other source of uncertainty B_1 is defined by the current position in the illiquid stock.

Table 7 shows that the described strategy persists for both values of α_2. Because Φ_ρ is independent from u_1, π_2 does not change with illiquidity of stock 2 (parameter α_2). Thus, π_2 will not change even if illiquidity becomes small. We conclude that deviations of allocations to stock 2 from those in the perfectly liquid market could be very significant even when stock illiquidity is small. Thus, a full consideration of illiquidity is important for finding the best strategies even when this illiquidity may seem to be small. In the latter case, however, an investor reaches the Merton line relatively fast. As long as the correlation between stocks is substantial, we expect that the given strategy is stable with respect to changes in parameters of the model, the underlying processes as well as the preferences of an investor.

The deviation of π_2 from $\hat{\pi}_2$ becomes even more dramatic if the correlation between the two stocks increases further. Similar results will be observed if the economy horizon or the volatility of the illiquid stock return is increased. Table 8 shows optimal policies and the expected utility function when $\rho = 0.8$, $\sigma_1 = 0.2$, $T = 1.0$, and $\rho = 0.5$, $\sigma_1 = 0.3$, $T = 5.0$ while the other parameters are the same: $\mu_1 = \mu_2 = 0.07$, $\sigma_2 = 0.2$, $\gamma = -1$, and $r = 0.01$. If both stocks are liquid, then for the first choice of parameters the allocations are $\hat{\pi}_1 = \hat{\pi}_2 = 0.417$, while for the second choice of parameters they are $\hat{\pi}_1 = 0.111$, $\hat{\pi}_2 = 0.667$. As shown, even at $N_1 S_1 / X = 0.5$ the difference between $\hat{\pi}_2$ and π_2 is substantial (especially for the second set of parameters). It becomes extreme at $N_1 S_1 / X \approx 1$ when an investor takes a short position in the liquid stock even though $\hat{\pi}_2 > 0$.

At this point we notice that an analysis similar to ours for $\lambda_2 = 0.2$ was conducted by Kahl, Liu, and Longstaff (2003) for the case when the liquid stock is the stock market and the illiquid stock cannot be traded at all. While some of their conclusions are similar to ours, there are major differences as well. Similar results include the behavior of optimal allocations π_2 versus illiquid stock return volatility and risk aversion. The differences include the behavior of the expected utility function : In our case, it is maximal when $N_1 S_1 / X = \hat{\pi}_1$, not when $N_1 = 0$ as in Kahl, Liu, and Longstaff (2003). This difference is a consequence of their illiquid stock being a small stock with an idiosyncratic risk. Moreover, contrary to our conclusions, these authors find that the deviation of the optimal portfolio weight π_2 from this weight in the unconstrained case decreases as T increases. The difference is caused by the presence of intertemporal consumption in the model by Kahl, Liu, and Longstaff (2003). As the stock stays illiquid for longer time, the consumption in the illiquid market becomes more important for an investor because of his time preferences. Meantime, longer time horizon in our model weakens the effect of stock illiquidity because an investor has more time to reach the utility–maximizing allocations by the time of the terminal consumption. Strengthening of effects from stock illiquidity in the model by Kahl, Liu, and Longstaff (2003) and their weakening in our model defines the difference in the behavior of π_2 versus T. Above all, the economic setups in Kahl, Liu, and Longstaff (2003) and our paper are very different: Kahl, Liu, and Longstaff find portfolio rules of an insider in a perfectly liquid market, while we find portfolio rules of an investor in illiquid market.

4.3 Limits to Arbitrage, or Trading when $|\rho| = 1$

Now we analyze the case when $\rho = 1$. The case when $\rho = -1$ is similar and will not be considered. If $\rho = 1$ and the conditional Sharpe ratios are the same, then an investor trades only the liquid stock and the problem becomes trivial. Hence, we assume that $\Delta \neq 0$ and arbitrage is present in the market. This analysis is motivated by a common situation in markets where financial securities, which are identical in provided cash flows but have different liquidities, may have substantially different prices. As a result, these securities may provide arbitrage opportunities. These opportunities will persist in the long run because an investor can take only a limited advantage of them. Thus, it is interesting to understand the optimal behavior of an investor in the presence of arbitrage opportunities.

As follows from our discussion in the prologue to this section, the market allows arbitrage in states where N_1 has the same sign as Δ [see equation (47)]. This arbitrage is limited by the rate of transactions for stock 1 and applicable fees. Arbitrage may also be available at states where N_1 and Δ have opposite signs.

Notice that the rate of portfolio growth in equation (47) cannot be made arbitrarily high since an investor can change N_1 only at a finite rate. Accordingly, the expected utility function of an investor is finite even in the presence of arbitrage opportunities. The finiteness of the expected utility function makes an investor consider arbitrage as one of many strategies available for trading, rather than a dominant one. This interpretation is consistent with the size of the optimal allocation π_2 when $\rho = 1$:

$$\pi_2 = -\frac{V_X(\mu_2 - r) + (V_{XX} N_1 S_1 + V_{XS_1} S_1)\sigma_1\sigma_2}{X V_{XX} \sigma_2^2}. \tag{23}$$

Contrary to the case where all stocks are liquid, this allocation is finite.

We confirm the above expectations by considering an investor's optimal portfolio which discounted value follows

$$d\left(\frac{X_t}{B_t}\right) = \frac{1}{B_t}\left\{\left[\Phi_t(\mu_2 - r) + N_{1t}S_{1t})\sigma_1\Delta - \alpha_1|u_{1t}| - \alpha_2 u_{1t}^2\right]dt + \sigma_2\Phi_t dW_{1t}\right\},$$

where $\Phi_t = N_{1t}S_{1t}\frac{\sigma_1}{\sigma_2} + \pi_{2t}X_t$ is the cumulative risk exposure of an investor.
Table 9 reports optimal policies at different values of state variable N_1 when $(\mu_1 - r)/\sigma_1 = 0.1$ and $(\mu_2 - r)/\sigma_2 = 0.3$, so the illiquid stock is overpriced with respect to the liquid one. In all shown states, an investor keeps Φ constant at the level corresponding to the market with only liquid stock 2. That is, for any position in the illiquid stock, an investor takes a risk–offsetting position in the liquid stock plus an additional positions in this stock and a bond which match those in the absence of the illiquid stock. First two positions define a locally riskless strategy which associated wealth follows $X_a(t) = \int_0^t(\Delta N_1 S_1\sigma_1 - \alpha_1|u_1| - \alpha_2 u_1^2)d\tau$. The last strategy is an arbitrage for states with a significantly short position in stock 1, since its drift is nonnegative $\forall t \in [0, T]$. It can also be arbitrage for other states N_1 which are not far from $N_1 = 0$. We expect that the described pattern of trading is stable with respect to changes in parameters of the model, the underlying processes, and the preferences of an investor. Thus, we recover a result similar to that in Basak and Croitoru (2000) who show that in the presence of market frictions an investor considers arbitrage as one of many strategies if taking advantage of arbitrage is limited.[16] Instead of allocating to only arbitrage strategy, an investor also trades a risky strategy. In our case, the frictions are caused by stock illiquidity, while Basak and Croitoru treat liquid markets in the presence of short–sale and long–position constraints. We also conclude that arbitrage opportunities may exist in a general equilibrium if markets are illiquid.

Notice that an investor offsets the risk of his position in the illiquid stock even when N_1 is significantly positive and an arbitrage strategy is not available. By doing this, he removes the risk related to his position in the illiquid stock which is very difficult to manage by trading the illiquid stock itself.

Table 10 presents optimal trading in the presence of arbitrage opportunities when $\alpha_1 = 0.005$. While in the case where $|\rho| \neq 1$ positive α_1 leads to the existence of the no–trading zone around $N_1S_1/X = \hat{\pi}_1$, this zone disappears if $|\rho| = 1$ and the difference between the conditional Sharpe ratios of the two stocks is substantial. In the latter case, an investor chooses allocations so that the volatility of his portfolio equals to the one in the absence of the illiquid stock, and trades the illiquid stock at an almost constant rate. In this way, losses due to the linear term in the transaction costs are not big enough to avoid trading the illiquid stock, and an investor can take advantage of arbitrage opportunities. If

[16] See also Basak and Croitoru (2003), Liu and Longstaff (2004), and Isaenko (2004).

the difference between the conditional Sharpe ratios of the two stocks decreases, and arbitrage opportunities provide smaller premiums, an investor stops trading the illiquid stock near the negative part of the NS–axis to avoid losses.

Tables 9 and 10 show a dramatic difference in the rate of trading u_1 between the cases when illiquidity is strong ($\alpha_2 = 0.2$) and moderate ($\alpha_2 = 0.005$). However, this rate has a very small effect on the allocations π_2 that are almost equal in the two cases. The reason is similar to the one in the presence of partial correlation between stocks. An investor keeps his cumulative risk exposure Φ constant at any level of stock 1 illiquidity. Because Φ is independent from u, π_2 is the same for any level of illiquidity at a given position in illiquid stock 1. Finally, comparison of Tables 9 and 10 shows that even though the no–trading zone is absent in Table 10, the presence of the linear term in the transaction costs has a very strong effect on the rate of trading u_1, but a negligible effect on the allocation π_2.

5 Optimal Portfolio with Stock Crashes

It is interesting to consider portfolio rules when a stock may undergo crashes. The motivation behind this consideration is that stock illiquidity is often triggered by its sudden fall in value or the possibility of a such fall. As a result, stock illiquidity and the probability of its crash often coexist. Examples include the 1987 stock market crash and the 1998 financial crisis. During the periods following these crashes, trading in many securities was available but limited, for some of them substantially. In this section, we consider trading a bond and an illiquid stock which can be thought of as a market index vulnerable to crashes. But we first identify portfolio rules when a stock susceptible to crashes is perfectly liquid.

We describe crashes by a Poisson process, so that the stock and an investor's wealth follow

$$dS_t = (\mu + \lambda\delta)S_t dt + \sigma S_t dW_t - \delta S_{t-} dP_t, \tag{24}$$

$$dX_t = [rX_t + \pi X_t(\mu + \lambda\delta - r)]dt + \pi X_t \sigma dW_t - \delta\pi X_{t-} dP_t, \tag{25}$$

where P is a poisson process with arrival intensity λ and δ is a jump percentage. We assume that λ and δ are constants. Making λ and δ random will not change the essence of our conclusions.

Notice that the term $\lambda\delta$ in the stock return drift stands to compensate an additional negative return introduced by crashes. However, following tradition, we do not compensate an additional variance resulting from crashes.

Appendix A presents the HJB equation for an investor in this economy. The solution of this equation reveals his portfolio rule and the value function:

$$\hat{\pi} = \frac{\mu + \lambda\delta - r}{(1-\gamma)\sigma^2} - \frac{\lambda\delta}{(1-\gamma)\sigma^2(1-\delta\hat{\pi})^{1-\gamma}}, \tag{26}$$

$$V(t, X) = \frac{X^\gamma}{\gamma} \exp\left\{\gamma\left[r + \hat{\pi}(\mu + \lambda\delta - r) - \frac{1}{2}\hat{\pi}^2\sigma^2(1-\gamma) - \frac{\lambda}{\gamma}(1 - (1-\hat{\pi}\delta)^\gamma)\right](T-t)\right\}. \tag{27}$$

It follows that $\hat{\pi}$ is independent from an investor's wealth and time horizon. Notice that even in the presence of stock crashes an investor maintains the fixed proportion of investments to the stock: Immediately after the stock crashes, and the allocation NS/X falls below $\hat{\pi}$, an investor buys as many shares as needed at an infinite rate to reach optimal proportion. This proportion is, however, always less than the one in the absence of stock crashes.[17] A more detailed discussion of the effect of crashes on an optimal portfolio with a liquid stock can be found in Liu, Longstaff, and Pan (2003).

To describe an investor's aversion to stock crashes, we introduce a crash premium Δ_c defined as the amount of the conditional return of a stock without crashes that an investor is willing to give up to avoid crashes:

$$V(t, X) = \frac{X^\gamma}{\gamma} \exp\left\{\gamma\left[r + \frac{(\mu - \Delta_c - r)^2}{2\sigma^2(1-\gamma)}\right](T-t)\right\}, \tag{28}$$

or, after taking into account result (27)

$$\Delta_c = \mu - r - \sqrt{2\sigma^2(1-\gamma)\left(\hat{\pi}(\mu + \lambda\delta - r) - \frac{1}{2}\hat{\pi}^2\sigma^2(1-\gamma) - \frac{\lambda}{\gamma}[1 - (1-\hat{\pi}\delta)^\gamma]\right)}. \tag{29}$$

Notice that Δ_c is constant and is always not bigger than $\mu - r$. The first remark follows from the nature of the CRRA preferences, while the second remark results from the fact that the expected utility function is always higher in the absence of crashes.

Now we turn to the case when a stock is illiquid. Similar to the economic settings considered in Sections (3) and (4), the position in the illiquid stock is limited by the constraints: $N \geq 0$ and $NS \leq X$. The formulation of the HJB equation for the value function $V(t, S, X, N)$ in this case is given in Appendix A. It follows from the first–order condition of this equation that optimal trading is affected by crashes only implicitly through marginal indirect utilities:

$$u = \begin{cases} \frac{V_N - V_X\alpha_1}{2V_X\alpha_2} & if \;\; V_N/V_X > \alpha_1 \\ 0 & if \;\; -\alpha_1 \leq V_N/V_X \leq \alpha_1 \\ \frac{V_N + V_X\alpha_1}{2V_X\alpha_2} & if \;\; V_N/V_X < -\alpha_1. \end{cases} \tag{30}$$

[17]When we add crashes to the stock, its expected return does not change but the variance increases. Thus, the return to risk ratio of the stock decreases making it less valuable to an investor.

Clearly, when stock crashes are present along with stock illiquidity, the crash premium cannot be sufficient. Here we introduce a crash–liquidity premium Δ_{c-l} as follows:

$$V(t,S,X,N) \;=\; \frac{X^\gamma}{\gamma}\exp\left\{(T-t)\gamma\left[r+\frac{(\mu-\Delta_{c-l}-r)^2}{2\sigma^2(1-\gamma)}\right]\right\}, \tag{31}$$

or:

$$\Delta_{c-l}(t,S,X,N)=\mu-r-\sqrt{2\sigma^2(1-\gamma)\left(\frac{1}{\gamma(T-t)}\ln\frac{V}{X^\gamma/\gamma}-r\right)}. \tag{32}$$

Apparently, Δ_{c-l} is the return of the liquid stock without crashes that an investor is willing to give up to avoid facing two imperfections. Table 11 shows the solution of an investor's problem when $\alpha_2=0.2$ and $\alpha_2=0.005$. The crash has a scale of 10% occurring each ten years on average, so in the absence of stock illiquidity, an investor will always hold proportion $\hat{\pi}=0.73$ of the stock.

Similar to the results obtained for an illiquid stock without crashes, an investor can be at any point in the solvency region. He will trade at a rate depending on how far he is from the plane $NS/X=\hat{\pi}$, where $\hat{\pi}$ is the solution of equation (26). The rate becomes higher as he gets further away from the plane. At the fixed value of one of the two state variables N or S, an investor's utility is maximal on the Merton line where he also switches the direction of trading. We conclude that even in the presence of illiquid stock crashes, the flight to liquidity, or the willingness to buy more bonds in favor of the stock, does not exist.

Now let us consider the premium. As seen from Table 11, the crash–liquidity premium decreases at small NS/X until $NS/X=\hat{\pi}$ and then starts increasing until $NS/X=1$. Moreover, at very small values of NS/X, Δ_{c-l} is very close to Δ_l defined in (11): If the proportion of stock holdings is small, then the possibility of a stock crash (and hence further decrease of this proportion) has a relatively small effect on an investor's ability to substantially increase stock holdings. When $NS/X=\hat{\pi}$, the crash–liquidity premium is defined by the crash premium, as the liquidity premium is very small. The impact of illiquidity shows off again for larger proportions NS/X: As NS/X increases from $\hat{\pi}$, the liquidity and crash–liquidity premiums go up. However, they do this at a different rate: $\Delta_{c-l}/(\Delta_l+\Delta_c)$ is one on the Merton line and increases with NS. For example, if $NS/X=\hat{\pi}=0.730$, then $\Delta_{c-l}/(\Delta_c+\Delta_l)\approx 1$, however if $NS/X=0.983$ and $\alpha_2=0.2$ then $\Delta_{c-l}/(\Delta_l+\Delta_c)=1.19$. Thus, the two types of imperfections are interdependent and their impact substantially increases when both are present at high proportion of NS/X.

An investor requires an additional to $\Delta_l+\Delta_c$ premium at high proportions of NS/X because if a stock crashes, he loses more of his wealth than when this proportion is small. For example, if $\delta=10\%$ and $NS=X$, then in the case of a stock crash an investor loses 10% of his wealth. If, on another hand, $NS/X\leq\hat{\pi}$, then the loss is less than or equal to 10% of

$\hat{\pi}X$ which can be much smaller than $0.1X$. As a result, an investor requires an additional premium for facing both imperfections at the same time.

For a better description of the last effect, we introduce the crash premium in the presence of illiquidity Δ_c^i which captures the impact of stock crashes when the stock is illiquid:

$$V(t,S,X,N) \;\; = \;\; \hat{V}(t,S,X,N;\mu-\Delta_c^i), \qquad (33)$$

where $\hat{V}(t,S,X,N;\mu-\Delta_c^i)$ is the indirect utility in the market having the illiquid stock with a continuous path and a return drift $\mu-\Delta_c^i$, while the other market characteristics are the same as in the market under consideration. Equation (33) is solved numerically with respect to Δ_c^i. Clearly, Δ_c^i is a function of time and the current state of the economy. We find that Δ_c^i is zero at low values of NS/X and increases above Δ_c at high values of NS/X. It follows that an investor often discounts the illiquid stock return for crashes at a higher rate than he does the liquid stock returns for the same crashes.

The last results are defined by the size and intensity of the stock crashes: As λ or δ goes up, so does the ratios $\Delta_{c-l}/(\Delta_l+\Delta_c)$ and Δ_c^i/Δ_c. As an illustration, Table 12 reports the optimal trading rate and premiums when $\lambda=3$ and $\delta=0.1$. As seen, $\Delta_{c-l}/(\Delta_l+\Delta_c)$ and Δ_c^i/Δ_c increase with NS/X much faster than they do when crashes occur rarely (see Table 11). As the expected effect from the crashes on the stock value increases, the more this value is discounted in the illiquid market since an investor will not be able to quickly adjust to changes in the stock making the possible losses increase.

Table 12 also shows that when $\delta=0.1$ and $\lambda=3$ the expected utility function of an investor can fall below U_{min}. This result has the same nature as the one in the absence of stock crashes when stock volatility is high, and follows from the excessive volatility in an investor's portfolio. Since the addition of Poisson crashes does not change the stock's expected return, but it increases its variance, an investor can easily find himself in a state where he would be happy to hold a liquid stock with a continuous path and a negative risk premium instead of an illiquid stock of the same volume which is susceptible to crashes. Clearly, this situation would occur at stock return volatility σ being much lower than in the case of the illiquid stock with no crashes (compare Δ_l in Table 1 with Δ_{c-l} in Table 12).

6 Conclusion

We analyze the optimal behavior of an investor who trades an illiquid stock which we suppose to be either the whole market index (Sections 3 and 5) or its substantial part (Section 4). We model stock illiquidity for an investor through the convex costs he has to pay for its trading. In the absence of the linear term in the transaction costs, the following conclusions are obtained. The new findings resulting from the presence of this term can be found in Sections 2–5.

In the presence of stock illiquidity, portfolio decisions become partly irreversible and this substantially changes the decision making of an investor. We find that in the presence of illiquidity, an investor achieves the highest indirect utility when his allocations lie on the Merton line. This result is independent from the degree of convexity of the transaction costs. Thus, flight to liquidity does not occur. Furthermore, the presence of stock illiquidity can put an investor in states where his portfolio has excessive volatility. This volatility could be so high that an investor, as if he is a risk lover, would be willing to replace his illiquid stock holdings with a substantially positive risk premium by those of an liquid stock that has a negative risk premium.

If an investor trades two stocks, one of which is illiquid, then holdings of both risky securities will be considerably different from the holdings when both stocks are liquid, even if the two stocks are independent. In the special case where the two traded stocks have a perfect correlation, we recover the story of the limits to arbitrage: An investor considers arbitrage opportunities on an equal basis with other investment opportunities and takes a risky position in their presence.

We consider optimal portfolio rules when, in addition to stock illiquidity, an investor faces the possibility of a stock crash. We find that the premium for a stock crash could be much higher than it is when a stock is liquid.

The above findings remain unchanged if the linear term in the transaction costs is present and small. Furthermore, the shape of the no–trading zone is not affected by the convexity of the transaction costs. Finally, the listed conclusions hold even if the constant investment opportunity set is replaced with a stochastic one.

Appendix A

In this appendix we formulate the HJB equations to be solved for indirect utility functions when an investor trades a bond and one illiquid stock, one liquid and one illiquid stock, all with continuous paths, or an illiquid stock that may also undergo Poisson crashes. We also formulate and solve the HJB equation when the stock is liquid and undergoes Poisson crashes. The analysis is conducted for an arbitrary positive degree of convexity ε.

If an investor trades a bond and an illiquid stock, then his value function $V(t, S, X, N)$ solves the following PDE

$$\max_{u \in R} \Bigg\{ V_t + \frac{1}{2}\sigma^2 N^2 S^2 V_{XX} + \frac{1}{2}\sigma^2 S^2 V_{SS} + \sigma^2 N S^2 V_{XS}$$

$$+ \quad \Big[rX + NS(\mu - r) - \alpha_1 |u| - \alpha_2 |u|^{1+\varepsilon} \Big] V_X + \mu S V_S + u V_N \Bigg\} = 0, \tag{34}$$

$$V(T, S, X, N) = X^\gamma / \gamma. \tag{35}$$

The first–order condition implies that $u = 0$ or is a solution of

$$\alpha_1 + \alpha_2 (1 + \varepsilon) |u|^\varepsilon = \left| \frac{V_N}{V_X} \right|. \tag{36}$$

This equation has a solution for u only if the minimum of its left side is not above its right side. That is, if $\alpha_1 \leq |V_N / V_X|$. In the latter case we find $|u| = \left(\frac{|V_N/V_X| - \alpha_1}{\alpha_2 (1+\varepsilon)} \right)^{\frac{1}{\varepsilon}}$. Thus, the optimal rate of trading can be written as

$$u = \begin{cases} \left(\frac{V_N - \alpha_1 V_X}{\alpha_2 (1+\varepsilon) V_X} \right)^{\frac{1}{\varepsilon}} & if \;\; V_N / V_X > \alpha_1 \\ 0 & if \;\; |V_N / V_X| \leq \alpha_1 \\ -\left(\frac{-V_N - \alpha_1 V_X}{\alpha_2 (1+\varepsilon) V_X} \right)^{\frac{1}{\varepsilon}} & if \;\; -V_N / V_X > \alpha_1. \end{cases} \tag{37}$$

Result (37) provides expression (9) if we replace ε with 1. After substitution of result (37) into equation (34) we find the PDE for V:

$$V_t + \frac{1}{2}\sigma^2 N^2 S^2 V_{XX} + \frac{1}{2}\sigma^2 S^2 V_{SS} + \sigma^2 N S^2 V_{XS} + \Big[rX + NS(\mu - r) \Big] V_X + \mu S V_S$$

$$+ \quad \varepsilon \left(\frac{V_N - \alpha_1 V_X}{\alpha_2 (1+\varepsilon) V_X} \right)^{\frac{1}{\varepsilon}} (V_N - \alpha_1 V_X)^+ + \varepsilon \left(\frac{-V_N - \alpha_1 V_X}{\alpha_2 (1+\varepsilon) V_X} \right)^{\frac{1}{\varepsilon}} (-V_N - \alpha_1 V_X)^+ = 0, \tag{38}$$

where z^+ is a positive part of variable z.

If an investor trades a bond, an illiquid stock and a liquid stock, then his value function $V(t, S_1, X, N_1)$ solves the following PDE

$$
\begin{aligned}
\max_{u_1,\pi_2\in R^2} \quad & \Bigg\{V_t + \left[\frac{1}{2}(N_1 S_1 \sigma_1)^2 + \rho N_1 S_1 \pi_2 X \sigma_1 \sigma_2 + \frac{1}{2}(\pi_2 X \sigma_2)^2\right] V_{XX} + \frac{1}{2}\sigma_1^2 S_1^2 V_{S_1 S_1} \\
+ \quad & (N_1 S_1 \sigma_1 + \rho \pi_2 X \sigma_2)\sigma_1 S_1 V_{X S_1} + \mu_1 S_1 V_{S_1} + u_1 V_{N_1} \\
+ \quad & \left[rX + N_1 S_1(\mu_1 - r) + \pi_2 X(\mu_2 - r) - \alpha_1|u_1| - \alpha_2|u_1|^{1+\varepsilon}\right] V_X \Bigg\} = 0, \\
& V(T, S_1, X, N_1) = X^\gamma/\gamma.
\end{aligned} \tag{39}
$$

It is straightforward to verify that if $\varepsilon = 1$ then the first–order conditions in the last equation result in the trading rate u_1 and the optimal proportion π_2 given by expressions (18) and (19), respectively. When u_1 and π_2 are substituted into equation (39), we find:

$$
\begin{aligned}
& V_t + \frac{1}{2}(N_1 S_1 \sigma_1)^2 V_{XX} + \frac{1}{2}\sigma_1^2 S_1^2 V_{S_1 S_1} + N_1 S_1^2 \sigma_1^2 V_{X S_1} + \mu_1 S_1 V_{S_1} \\
+ \quad & [rX + N_1 S_1(\mu_1 - r)]V_X - \frac{[V_X(\mu_2 - r) + (V_{XX} N_1 S_1 + V_{X S_1} S_1)\rho\sigma_1\sigma_2]^2}{2 V_{XX}\sigma_2^2} \\
+ \quad & \varepsilon\left(\frac{V_{N_1} - \alpha_1 V_X}{\alpha_2(1+\varepsilon)V_X}\right)^{\frac{1}{\varepsilon}}(V_{N_1} - \alpha_1 V_X)^+ + \varepsilon\left(\frac{-V_{N_1} - \alpha_1 V_X}{\alpha_2(1+\varepsilon)V_X}\right)^{\frac{1}{\varepsilon}}(-V_{N_1} - \alpha_1 V_X)^+ = 0.
\end{aligned} \tag{40}
$$

Suppose now that an investor trades a bond and a liquid stock that may experience Poisson crashes. Then, as follows from the dynamics of portfolio wealth given by equation (25), the HJB equation for the value function $V(t, X)$ can be written as

$$
\begin{aligned}
& \max_{\pi\in R}\left\{V_t + \frac{1}{2}\sigma^2 X^2 \pi^2 V_{XX} + X\left[r + \pi(\mu + \lambda\delta - r)\right]V_X + \lambda\left[V(t, X(1-\delta\pi)) - V(t, X)\right]\right\} = 0, \\
& V(T, X) = X^\gamma/\gamma.
\end{aligned} \tag{41}
$$

The first–order condition reads

$$
\hat{\pi} = -\frac{(\mu + \lambda\delta - r)V_X - \lambda\delta V_X(t, X(1-\delta\hat{\pi}))}{\sigma^2 X V_{XX}}, \tag{42}
$$

which after substitution into equation (41), gives a PDE to be solved:

$$
\begin{aligned}
& V_t + XrV_X + \frac{1}{2}\sigma^2 X^2 \hat{\pi}^2 V_{XX} + X\left[r + \hat{\pi}(\mu + \lambda\delta - r)\right]V_X \\
+ \quad & \lambda\left[V(t, X(1-\delta\hat{\pi})) - V(t, X)\right] = 0, \qquad V(T, X) = X^\gamma/\gamma.
\end{aligned} \tag{43}
$$

Assume that $\hat{\pi}$ is a constant, then it is easy to verify that the last equation is solved by the indirect utility given by expression (27). This expression implies that the decision rule is in fact constant and given by expression (26).

Finally, let us suppose that the stock in the last problem becomes illiquid. Then the indirect utility function depends on variables S, X, N and the HJB equation for $V(t,S,X,N)$ becomes

$$\begin{aligned}
& \max_{u\in R}\Big\{V_t+\frac{1}{2}\sigma^2N^2S^2V_{XX}+\frac{1}{2}\sigma^2S^2V_{SS}+\sigma^2NS^2V_{XS} \\
+ \quad & \Big[rX+NS(\mu-r)-\alpha_1|u|-\alpha_2|u|^{1+\varepsilon}\Big]V_X+\mu SV_S+uV_N \qquad (44)\\
+ \quad & \lambda\Big[V(t,S(1-\delta),X-NS\delta,N)-V(t,S,X,N)\Big]\Big\}=0,\\
& V(T,S,X,N)=X^\gamma/\gamma. \qquad (45)
\end{aligned}$$

It is straightforward to verify that the first–order condition in the last equation results in the trading rate u given by formula (30). When u is substituted into equation (44), we find

$$\begin{aligned}
& V_t+\frac{1}{2}\sigma^2N^2S^2V_{XX}+\frac{1}{2}\sigma^2S^2V_{SS}+\sigma^2NS^2V_{XS}+\Big[rX+NS(\mu-r)\Big]V_X+\mu SV_S \\
+ \quad & \varepsilon\left(\frac{V_N-\alpha_1V_X}{\alpha_2(1+\varepsilon)V_X}\right)^{\frac{1}{\varepsilon}}(\varepsilon_2-1)(V_N-\alpha_1V_X)^+ +\varepsilon\left(\frac{-V_N-\alpha_1V_X}{\alpha_2(1+\varepsilon)V_X}\right)^{\frac{1}{\varepsilon}}(-V_N-\alpha_1V_X)^+ \\
+ \quad & \lambda\Big[V(t,S(1-\delta),X-NS\delta,N)-V(t,S,X,N)\Big]=0. \qquad (46)
\end{aligned}$$

Equations (38), (40), and (46) can be solved only numerically. Because all these equations are highly nonlinear, standard existence and uniqueness results for their solutions do not apply. Thus, we simply assume that the solutions exist and that they are unique. Moreover, we assume that our numerical solutions converge uniformly to the true solutions in the limit of infinitely small space (and so time) increment.

Appendix B

In this appendix we show that arbitrage opportunities in the setting of Section 4 are possible only if $\rho=1$ and $\Delta\equiv\frac{\mu_1-r}{\sigma_1}-\frac{\mu_2-r}{\sigma_2}\neq 0$ or if $\rho=-1$ and $\frac{\mu_1-r}{\sigma_1}\neq-\frac{\mu_2-r}{\sigma_2}$.

Indeed, if $\rho=1$ then W_1 and W_2 are identical and we can make the volatility of the portfolio be zero by setting $\pi_2=-\frac{N_1S_1}{X}\frac{\sigma_1}{\sigma_2}$. Consequently, equation (14) becomes

$$dX_t \;=\; \{rX_t+N_{1t}S_{1t}\sigma_1\Delta-\alpha_1|u_{1t}|-\alpha_2u_{1t}^2\}dt. \qquad (47)$$

If the allocation N_1 has the same sign as Δ, the portfolio return can be made locally riskless and higher than r, because an investor can choose u to be arbitrarily small. Assuming that an investor can be in a state where N_1 and Δ have the same sign, we find access to arbitrage opportunities. If $\Delta = 0$, then the locally riskless rate of portfolio growth is $r - \frac{\alpha_1|u|+\alpha_2 u^2}{X} \leq r$ and no arbitrage is available. The proof for $\rho = -1$ is similar. Finally, if $-1 < \rho < 1$, then the portfolio with stocks cannot be made locally riskless and arbitrage will not be available.

Appendix C

In this appendix we show that the main conclusions of this paper do not change if we assume that investment opportunity set is stochastic. In such a set, the interest rate, the drift and volatility of the stock return are stochastic. We consider the economy with liquid bond and illiquid stock market where only the volatility of the stock return is stochastic. Incorporation of stochastic spot interest rate and stock return drift is straightforward and will not change the essence of our conclusions.

We assume that stock is a positive process

$$dS_t = S_t(\mu dt + \sigma_t dW_t),$$

where μ is a constant, W is a standard Brownian motion and σ is a stochastic volatility following a geometrical Brownian motion

$$d\sigma_t = \sigma_t \upsilon dW_{\sigma t}, \tag{48}$$

where W_σ is a standard Brownian motion independent from W[18] and υ is a positive constant. First, we find optimal allocations when stock is liquid. Here, the economy is markovian and the HJB equation for $V(t, X, \sigma)$ is

$$\max_{\pi \in R}\left\{V_t + \frac{1}{2}\sigma^2\pi^2X^2V_{XX} + \frac{1}{2}\sigma^2\upsilon^2V_{\sigma\sigma} + [r + \pi(\mu - r)]XV_X\right\} = 0, \tag{49}$$

$$V(T, X, \sigma) = X^\gamma/\gamma. \tag{50}$$

The first order condition provides

$$\pi = -\frac{\mu - r}{\sigma^2}\frac{V_X}{V_{XX}X}.$$

Let us assume that $V(t, X, \sigma) = \frac{X^\gamma}{\gamma}F(t, \sigma)$, then the proportion of wealth allocated to the stock becomes

$$\hat{\pi} = \frac{\mu - r}{\sigma^2(1 - \gamma)} \tag{51}$$

[18] We assume a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, Q)$. Uncertainty in the model is generated by a standard two–dimensional Brownian motion (W, W_σ), which is also adapted.

and $F(t,\sigma)$ solves the following PDE

$$F_t + \frac{1}{2}\sigma^2 \upsilon^2 F_{\sigma\sigma} + \gamma\left[r + \frac{(\mu - r)^2}{2\sigma^2(1-\gamma)}\right]F = 0, \tag{52}$$

$$F(T,\sigma) = 1. \tag{53}$$

For the goal of our treatment the solution of the last PDE is not important, so we assume that it exists and unique but we do not find actual $F(t,\sigma)$. It follows that the proportion of investor's wealth allocated to the stock is not fixed and changes with stochastic volatility σ.

Now let us assume that the stock market becomes illiquid. Here, the set of state variables includes S, X, N, and σ. Thus, the HJB equation for $V(t,S,X,N,\sigma)$ is

$$\max_{u\in R}\left\{V_t + \frac{1}{2}\sigma^2 N^2 S^2 V_{XX} + \frac{1}{2}\sigma^2 S^2 V_{SS} + \sigma^2 N S^2 V_{XS} + \frac{1}{2}\sigma^2 \upsilon^2 V_{\sigma\sigma}\right.$$

$$+ \left.\left[rX + NS(\mu - r) - \alpha_1|u| - \alpha_2|u|^{1+\varepsilon}\right]V_X + \mu S V_S + u V_N\right\} = 0, \tag{54}$$

$$V(T,S,X,N,\sigma) = X^\gamma/\gamma. \tag{55}$$

Analysis of the first–order condition for the last equation provides the optimal rate of trading u given by (37), where V also depends on σ. Thus, the stochastic volatility affects the rate of trading only implicitly through the marginal indirect utilities. After substitution u into the last equation we find the PDE for $V(t,S,X,N,\sigma)$. We solve this PDE numerically and confirm that the indirect utility is maximal when $NS/X = \hat{\pi}$. Moreover, we find that, conditioned on the value of volatility of stock return, the no–trading zone has a shape of a positive cone and it is independent from the convexity of the transaction costs. As an illustration, Table 13 shows the rate of trading and the expected utility function at fixed values of state variables X, S and σ when $\varepsilon = 1$ and 2. We assume that σ takes values from 0.15 to 0.25 and is fixed at 0.20. Clearly, the expected utility function is maximal when $NS/X = \hat{\pi} = 0.75$ and the boundaries between trading zones are the same for both values of ε. This result can be easily extended to the two other economic settings considered in the paper without any changes. Moreover, our conclusions regarding an investor welfare, rate of trading and interdependence of allocations to liquid and illiquid stocks will be modified only quantitatively but not qualitatively if we condition them on the value of the stochastic volatility or average the results over distribution of σ.

References

[1] Acharya, V., Pederson, L., 2003, "Asset Pricing with Liquidity Risk." Working paper, New York University.

[2] Amihud, Y., Mendelson, H., 1986, "Asset Pricing and the Bid–Ask Spread." *Journal of Financial Economics*, 17, 223–249.

[3] Amihud, Y., Mendelson, H., 1991, "Liquidity, Maturity, and the Yields on U.S. Treasury Securities." *Journal of Finance*, 46, 1411–1425.

[4] Basak, S., Croitoru, B., 2000, "Equilibrium Mispricing in a Capital Market with Portfolio Constraints." *The Review of Financial Studies*, 13, 715–748.

[5] Basak, S., Croitoru, B., 2003, "On the Role of Arbitrageurs in Rational Markets." *Journal of Financial Economics*, forthcoming.

[6] Boudoukh, J., Whitelaw, R. F, 1991, "The Benchmark Effect in the Japanese Government Bond Market." *Journal of Fixed Income*, September, 52–59.

[7] Boudoukh, J., Whitelaw, R. F, 1993, "Liquidity as a Chice Variable: A lesson from the Japanese Government Bond Market." *The Review of Financial Studies*, 6, 265–292.

[8] Brunnermeir, M., Pederson, L., 2005, "Market Liquidity and Funding Liquidity." Working paper, SSRN.

[9] Cetin, U., Jarrow, R., Protter, P., 2004, "Liquidity Risk and Arbitrage Pricing Theory." *Finance and Stochastics*, 8, 311–341.

[10] Cetin, U., Jarrow, R., Protter, P., Warachka, M., 2004, "Pricing Options in an Extended Black Scholes Economy with Illiquidity: Theory and Empirical Evidence." *The Review of Financial Studies*, forthcoming.

[11] Cetin, U., Rogers, L.C.G., 2005, "Modeling Liquidity Effects in Descrete Time." Working paper, University of Cambridge.

[12] Constantinides, G., 1986, "Capital Market Equilibrium with Transaction Costs." *Journal of Political Economy*, 94, 842–862.

[13] Davis, M. H. A., Norman, A.R., 1990, "Portfolio Selection with Transaction Costs." *Mathematics of Operations Research*, 15, 676–713.

[14] Duffie, D., Garleanu, N., Pederson, L., 2003a, "Marketmaking in Over–the–Counter Markets." Working Paper, Stanford University.

[15] Duffie, D., Garleanu, N., Pederson, L., 2003b, "Valuation in Over–the–Counter Markets." Working Paper, Stanford University.

[16] Duffie, D., Sun, T. 1990, "Transaction Costs and Por6tfolio Choice in a Discrete–Continuous Time Setting." *Journal of Economic Dynamics and Control*, 14, 35–51.

[17] Dumas, B., Luciano, E. 1991, "An Exact Solution to a Dynamic Portfolio Choice Problem under Transaction Costs." *Journal of Finance*, 48, 577–595.

[18] Eisfeldt, A., 2004, "Endogenous Liquidity in Asset Markets." *Journal of Finance*, 59, 1–30.

[19] Frey, R., 1998, "Perfect Option Hedging for a Large Trader." *Finance and Stochastics*, 2, 115–141.

[20] Grossman, S., Laroque, G., 1990, "Asset Pricing and Optimal Portfolio Choice in the Presence of Illiquid Durable Consumption Goods." *Econometrica*, 58, 25–52.

[21] Grossman, S., Miller, M., 1988, "Liquidity and Market Structure." *Journal of Finance*, 43, 617–633.

[22] He, H., Mamaysky, G., 2005, "Dynamic Trading Policy with Price Impact." *Journal of Economic Dynamics and Control*, 29, 891–930.

[23] Holmstrom, B., Tirole, J., 2001, "LAMP: A Liquidity–Based Asset Pricing Model." *Journal of Finance*, 56, 1837–1867.

[24] Huang, M., 2003, "Liquidity Shocks and Equilibrium Liquidity Premia." *Journal of Economic Theory*, 109, 104–129.

[25] Huang, J., Wang, J., 2005, "Market Liquidity and Asset Prices under Costly Participation." Working Paper, SSRN.

[26] Isaenko, S., 2004, "Dynamic Equilibrium with Overpriced Put Options." Working Paper, Concordia University.

[27] Jang, B., Koo, H., K., Liu, H., Loewenstein, M., 2005, "Transaction Cost Can Have a First–Order Effect on Liquidity Premium." Working Paper, SSRN.

[28] Kahl, M., Liu, J., Longstaff, F., 2003, "Paper Millionaires: How Valuable is Stock to a Stockholder who is Restricted from Selling it?" *Journal of Financial Economics*, 67, 385–410.

[29] Kamara, A., 1994, "Liquidity, Taxes, and Short–Term Treasury Yields." *Journal of Financial and Quantitative Analysis*, 29, 403–417.

[30] Lippman, S., McCall, J., 1986, "An Operational Measure of Liquidity." *American Economic Review*, 76, 43–55.

[31] Liu, H., 2004, "Optimal Consumption and Investment with Transaction Costs and Multiple Risky Assets." *Journal of Finance*, 59, 289–338.

[32] Liu, H., Loewestein 2002, "Optimal Portfolio Selection with Transaction Costs and Finite Horizons." *The Review of Financial Studies*, 15, 805–835.

[33] Liu, H., Yong, J., 2004, "Option Pricing with an Illiquid Underlying Asset Market." *Journal of Economic Dynamics and Control*, forthcoming.

[34] Liu, J., Longstaff, F., 2004, "Losing Money on Arbitrage: Optimal Dynamic Portfolio Choice in Markets with Arbitrage Opportunities." *The Review of Financial Studies*, 17, 611–641.

[35] Liu, J., Longstaff, F., Pan, J., 2003, "Dynamic Asset Allocation with Event Risk." *Journal of Finance*, 58, 231–259.

[36] Lo, A., X., Mamaysky, H., Wang, J. 2004, "Asset Prices and Trading Volume ander Fixed Transaction Costs." *Journal of Political Economy*, 112, 1054–1090.

[37] Longstaff, F., 1995, "How Much Can Marketability Affect Security Values." *Journal of Finance*, 50, 1767–1774.

[38] Longstaff, F., 2001, "Optimal Portfolio Choice and the Valuation of Illiquid Securities." *The Review of Financial Studies*, 14, 407–431.

[39] Longstaff, F., 2005, "Asset Pricing in Markets with Illiquid Assets." Working Paper, SSRN.

[40] Merton, R. C., 1969, "Lifetime Portfolio Selection under Uncertainty: The Continuous–Time Case." *The Review of Economics and Statistics*, 51, 247–257.

[41] O'Hara, M., 2003, "Presedential Address: Liquidity and Price Discovery." *Journal of Finance*, 58, 1335–1354.

[42] Pereira, P., Zhang, H., 2005, "The Liquidity Premium in a Dynamic Model with Price Impact." Working paper, SSRN.

[43] Rogers, L.C.G., Singh, S., 2006, "Modeling Liquidity and its Effects on Price." Working paper, University of Cambridge.

[44] Shleifer, A., Vishny, R., 1997. The Limits of Arbitrage. *Journal of Finance*, 52, 35–55.

[45] Shreve, S., 2004. *Stochastic Calculus for Finance II: Continuous–Time Models.* New York, Springer–Verlag.

[46] Shreve, S., Soner, H., 1992, "Op0timal Investment and Consumption with Transaction Costs" *Annals of Applied Probability*, 4, 609–692.

[47] Vayanos, D., 1998, "Transaction Costs and Asset Prices: A Dynamic Equilibrium Model." *The Review of Financial Studies*, 11, 1–58.

[48] Vayanos, D., 2003, "Flight to Quality, Flight to Liquidity, and the Pricing Risk." Working paper, MIT.

[49] Vayanos, D., Wang, T., 2003, "Search and Endogenous Concentration of Liquidity in Asset Markets." Working paper, MIT.

[50] Weill, O., 2002, "Liquidity Premia in Dynamic Bargaining Markets." Working paper, Stanford University.

Table 1: **Optimal trading of illiquid stock when $\alpha_1 = 0$**

		$\alpha_2 = 0.2$			$\alpha_2 = 0.005$		
N	NS/X	u	U	Δ_l	u	U	Δ_l
0.04	0.020	0.0660	-0.9787	0.0443	1.801	-0.9719	0.0230
0.50	0.246	0.0447	-0.9679	0.0150	1.235	-0.9648	0.0095
1.00	0.492	0.0227	-0.9609	0.0035	0.633	-0.9600	0.0022
1.52	0.749	0.00001	-0.9585	0.0001	0.005	-0.9584	0.0000
1.53	0.753	-0.0002	-0.9585	0.0001	-0.005	-0.9584	0.0000
1.70	0.836	-0.0074	-0.9587	0.0003	-0.207	-0.9585	0.0001
1.85	0.911	-0.0140	-0.9593	0.0012	-0.391	-0.9589	0.0007
2.00	0.983	-0.0205	-0.9604	0.0028	-0.571	-0.9597	0.0017

The table reports optimal trading rate u, expected utility function U, and liquidity premium Δ_l when $\alpha_1 = 0$, $\gamma = -1$, $\mu = 0.07$, $\sigma = 0.2$, $r = 0.01$, $T = 1$, and $t = 0$, $S = 0.497$, $X = 1.01$, so $\hat{\pi} = 0.75$, $U_{max} = -0.9584$, and $U_{min} = -0.9802$.

Table 2: **Optimal trading of illiquid stock when $\alpha_1 = 0$ and $\sigma = 0.3$**

		$\alpha_2 = 0.2$			$\alpha_2 = 0.005$		
N	NS/X	u	U	Δ_l	u	U	Δ_l
0.04	0.020	0.0637	-0.9800	0.0383	1.287	-0.9746	0.0142
0.50	0.246	0.0172	-0.9731	0.0182	0.358	-0.9708	0.0011
0.67	0.330	0.001	-0.9725	0.0005	0.005	-0.9706	0.0003
0.68	0.332	-0.0005	-0.9725	0.0005	-0.008	-0.9706	0.0003
1.00	0.492	-0.030	-0.9744	0.0076	-0.646	-0.9715	0.0032
1.30	0.587	-0.055	-0.9795	0.0336	-1.249	-0.9743	0.0132
1.85	0.911	-0.105	-0.9981	NA	-2.370	-0.9844	NA
2.00	0.983	-0.119	-1.0054	NA	-2.676	-0.9883	NA

The table reports optimal trading rate u, expected utility function U and the liquidity premium Δ_l when $\alpha_1 = 0$, $\gamma = -1$, $\mu = 0.07$, $\sigma = 0.3$, $r = 0.01$, $T = 1$, and $t = 0$, $S = 0.497$, $X = 1.01$, so $\hat{\pi} = 0.333$, $U_{max} = -0.9705$, and $U_{min} = -0.9802$.

Table 3: **Optimal trading of illiquid stock when** $\alpha_1 = 0.005$

		$\alpha_2 = 0.2$			$\alpha_2 = 0.005$		
N	NS/X	u	U	Δ_l	u	U	Δ_l
0.04	0.020	0.0537	-0.9788	0.0446	1.399	-0.9749	0.0304
0.50	0.246	0.0324	-0.9678	0.0148	0.972	-0.9664	0.0123
0.75	0.369	0.0213	-0.9636	0.0076	0.706	-0.9630	0.0067
1.0	0.492	0.0105	-0.9607	0.0032	0.354	-0.9605	0.0029
1.240	0.610	0.0001	-0.9589	0.0006	0.004	-0.9589	0.0006
1.244	0.612	0.0	-0.9589	0.0006	0.0	-0.9589	0.0006
1.525	0.750	0.0	-0.9585	0.0001	0.0	-0.9585	0.0001
1.820	0.895	0.0	-0.9589	0.0006	0.0	-0.9589	0.0006
1.824	0.897	-0.0001	-0.9589	0.0006	-0.006	-0.9589	0.0006
2.032	0.999	-0.0094	-0.9605	0.0029	-0.302	-0.9604	0.0028

The table reports optimal trading rate u, expected utility function U, and liquidity premium Δ_l when $\alpha_1 = 0.01$, $\gamma = -1$, $\mu = 0.07$, $\sigma = 0.2$, $r = 0.01$, $T = 1$, and $t = 0$, $S = 0.497$, $X = 1.01$, so $\hat{\pi} = 0.75$, $U_{max} = -0.9584$, and $U_{min} = -0.9802$.

Table 4: **Optimal trading of two stocks when** $\alpha_1 = 0$ **and** $\rho = 0$

		$\alpha_2 = 0.2$				$\alpha_2 = 0.005$			
N_1	N_1S_1/X	u_1	π_2	U	Δ_{l1}	u_1	π_2	U	Δ_{l1}
0.04	0.020	0.0703	0.751	-0.9569	0.0437	1.84	0.751	-0.9501	0.0225
0.50	0.246	0.0476	0.755	-0.9462	0.0147	1.26	0.753	-0.9431	0.0092
1.00	0.492	0.0239	0.750	-0.9393	0.0032	0.64	0.750	-0.9369	0.0020
1.52	0.749	0.0000	0.737	-0.9371	0.0000	0.01	0.740	-0.9371	0.0000
1.53	0.753	-0.0001	0.737	-0.9371	0.0000	-0.00	0.740	-0.9371	0.0000
1.70	0.836	-0.008	0.730	-0.9373	0.0003	-0.22	0.735	-0.9372	0.0002
1.85	0.911	-0.016	0.724	-0.9380	0.0012	-0.41	0.731	-0.9376	0.0007
2.00	0.983	-0.022	0.717	-0.9391	0.0029	-0.60	0.726	-0.9383	0.0017

The table reports optimal policies u_1, π_2, expected utility function U, and the liquidity premium Δ_{l1}, when $\alpha_1 = 0$, $\gamma = -1$, $\mu_1 = \mu_2 = 0.07$, $\sigma_1 = \sigma_2 = 0.2$, $\rho = 0$, $r = 0.01$, $T = 1$, and $t = 0$, $S_1 = 0.497$, $X = 1.01$, so $\hat{\pi}_1 = \hat{\pi}_2 = 0.75$, $U_{max} = -0.9371$, and $U_{min} = -0.9802$.

Table 5: **Optimal trading of two stocks when $\rho = 0$, $\alpha_1 = 0$, and $T = 5$**

		$\sigma_2 = 0.2$				$\sigma_2 = 0.3$			
N_1	N_1S_1/X	u_1	π_2	U	Δ_{l1}	u_1	π_2	U	Δ_{l1}
0.04	0.020	0.258	0.763	-0.8068	0.0232	0.166	0.756	-0.8179	0.0147
0.50	0.246	0.174	0.768	-0.7755	0.0088	0.046	0.746	-0.8012	0.0005
0.67	0.330	0.143	0.764	-0.7672	0.0056	0.0004	0.734	-0.8006	0.0000
0.68	0.332	0.142	0.764	-0.7670	0.0055	-0.003	0.734	-0.8006	0.0000
1.00	0.492	0.085	0.748	-0.7559	0.0014	-0.088	0.698	-0.8052	0.0036
1.52	0.749	0.0002	0.701	-0.7521	0.0000	-0.224	0.605	-0.8363	0.0387
1.53	0.753	-0.0005	0.701	-0.7521	0.0000	-0.225	0.603	-0.8367	0.0395
1.70	0.836	-0.037	0.673	-0.7523	0.0001	-0.269	0.564	-0.8542	NA
1.85	0.911	-0.065	0.648	-0.7534	0.0004	-0.309	0.521	-0.8741	NA
2.00	0.983	-0.093	0.621	-0.7574	0.0019	-0.349	0.470	-0.8982	NA

The table reports optimal policies u_1, π_2, expected utility function U, and the liquidity premium Δ_{l1}, when $\alpha_1 = 0$, $\alpha_2 = 0.2$, $\gamma = -1$, $\mu_1 = \mu_2 = 0.07$, $\sigma_1 = 0.2$, $\rho = 0$, $r = 0.01$, $T = 5$, and $t = 0$, $S_1 = 0.497$, $X = 1.01$, so $\hat{\pi}_1 = \hat{\pi}_2 = 0.75$, $U_{min} = -0.9418$, while $U_{max} = -0.7521$ for $\sigma_2 = 0.2$, and $U_{max} = -0.8006$ for $\sigma_2 = 0.3$.

Table 6: **Optimal trading of two stocks when $\alpha_1 = 0.005$ and $\rho = 0$**

		$\alpha_2 = 0.2$				$\alpha_2 = 0.005$			
N_1	N_1S_1/X	u_1	π_2	U	Δ_{l1}	u_1	π_2	U	Δ_{l1}
0.04	0.020	0.055	0.751	-0.9567	0.0431	1.506	0.752	-0.9529	0.0294
0.50	0.246	0.033	0.755	-0.9460	0.0142	0.944	0.755	-0.9447	0.0119
0.75	0.369	0.022	0.754	-0.9418	0.0071	0.643	0.754	-0.9414	0.0063
1.00	0.492	0.011	0.751	-0.9390	0.0027	0.334	0.752	-0.9389	0.0025
1.248	0.614	0.0001	0.746	-0.9373	0.0002	0.004	0.746	-0.9373	0.0002
1.252	0.616	0.00	0.746	-0.9373	0.0002	0.00	0.746	-0.9373	0.0002
1.525	0.750	0.00	0.738	-0.9371	0.0000	0.00	0.738	-0.9371	0.0000
1.796	0.811	0.00	0.726	-0.9373	0.0002	0.00	0.726	-0.9373	0.0002
1.800	0.813	-0.0001	0.726	-0.9373	0.0002	-0.004	0.726	-0.9373	0.0002
2.032	0.999	-0.010	0.715	-0.9390	0.0027	-0.301	0.720	-0.9389	0.0026

The table reports optimal policies u_1, π_2, expected utility function U, and the liquidity premium Δ_{l1}, when $\alpha_1 = 0.005$, $\gamma = -1$, $\rho = 0$, $\mu_1 = \mu_2 = 0.07$, $\sigma_1 = \sigma_2 = 0.2$, $\rho = 0$, $r = 0.01$, $T = 1$, and $t = 0$, $S_1 = 0.497$, $X = 1.01$, so $\hat{\pi}_1 = \hat{\pi}_2 = 0.75$, $U_{max} = -0.9371$, and $U_{min} = -0.9802$.

Table 7: **Optimal trading of two stocks when $\alpha_1 = 0$, $\rho = 0.5$, $T = 1$ and $\sigma_1 = 0.2$**

		$\alpha_2 = 0.2$				$\alpha_2 = 0.005$			
N_1	N_1S_1/X	u_1	π_2	U	Φ_ρ/X	u_1	π_2	U	Φ_ρ/X
0.04	0.020	0.034	0.739	-0.9576	0.749	0.997	0.740	-0.9558	0.750
0.50	0.246	0.018	0.628	-0.9529	0.751	0.525	0.628	-0.9524	0.751
1.01	0.499	0.0000	0.499	-0.9513	0.749	0.003	0.499	-0.9513	0.749
1.02	0.502	-0.0001	0.498	-0.9513	0.749	-0.001	0.498	-0.9513	0.749
1.53	0.749	-0.017	0.368	-0.9529	0.743	-0.513	0.370	-0.9524	0.745
1.70	0.836	-0.023	0.324	-0.9543	0.742	-0.687	0.326	-0.9534	0.744
1.85	0.911	-0.028	0.285	-0.9559	0.741	-0.842	0.288	-0.9546	0.744
2.00	0.983	-0.033	0.246	-0.9580	0.741	-0.993	0.249	-0.9560	0.744

The table reports optimal policies u_1, π_2, and expected utility function U when $\alpha_1 = 0$, $\gamma = -1$, $\mu_1 = \mu_2 = 0.07$, $\sigma_1 = \sigma_2 = 0.2$, $\rho = 0.5$, $r = 0.01$, $T = 1$, and $t = 0$, $S_1 = 0.497$, $X = 1.01$, so $\hat{\pi}_1 = \hat{\pi}_2 = 0.5$, $U_{max} = -0.9513$, and $U_{min} = -0.9802$.

Table 8: **Optimal trading of two stocks when $\alpha_1 = 0$ and $\rho = 0.5$.**

		$\rho = 0.5$, $\sigma_1 = 0.3$, $T = 5$			$\rho = 0.8$, $\sigma_1 = 0.2$, $T = 1$		
N_1	N_1S_1/X	u_1	π_2	U	u_1	π_2	U
0.04	0.020	0.044	0.736	-0.8394	0.014	0.734	-0.9580
0.22	0.110	0.0006	0.671	-0.8381	0.011	0.667	-0.9572
0.23	0.111	-0.0004	0.670	-0.8381	0.011	0.665	-0.9572
0.50	0.226	-0.066	0.564	-0.8414	0.006	0.553	-0.9563
0.84	0.417	-0.142	0.431	-0.8522	0.0000	0.417	-0.9560
0.85	0.419	-0.142	0.430	-0.8524	-0.0001	0.415	-0.9560
1.00	0.497	-0.173	0.372	-0.8613	-0.003	0.357	-0.9561
1.53	0.749	-0.280	0.167	-0.9115	-0.011	0.150	-0.9574
1.70	0.836	-0.314	0.100	-0.9358	-0.014	0.082	-0.9583
1.85	0.911	-0.345	0.042	-0.9608	-0.017	0.023	-0.9592
2.00	0.983	-0.377	-0.017	-0.9915	-0.019	-0.035	-0.9603

The table reports optimal policies u_1, π_2, and expected utility function U when $\alpha_1 = 0$, $\alpha_2 = 0.2$, $\gamma = -1$, $\mu_1 = \mu_2 = 0.07$, $\sigma_2 = 0.2$, $r = 0.01$ and $t = 0$, $S_1 = 0.497$, $X = 1.01$. The other parameters are shown at the top of columns 2 and 3. $U_{min} = -0.9418$, while $U_{max} = -0.8381$, $\hat{\pi}_1 = 0.111$, $\hat{\pi}_2 = 0.667$, and $U_{max} = -0.9560$, $\hat{\pi}_1 = \hat{\pi}_2 = 0.417$ in the setting of the second and the third columns, respectively.

Table 9: **Optimal trading of two stocks when $\alpha_1 = 0$ and $\rho = 1$**

	$\alpha_2 = 0.2$				$\alpha_2 = 0.005$			
N_1	u_1	π_2	μ_X	σ_X	u_1	π_2	μ_X	σ_X
-2.5	-0.0471	1.970	0.104	0.149	-1.925	1.968	0.091	0.149
-2.0	-0.0470	1.724	0.094	0.150	-1.917	1.722	0.082	0.149
-1.5	-0.0469	1.483	0.084	0.150	-1.909	1.481	0.072	0.150
-1.0	-0.0468	1.237	0.075	0.151	-1.901	1.235	0.062	0.150
-0.5	-0.0467	0.996	0.065	0.151	-1.893	0.993	0.053	0.151
0.0	-0.0466	0.750	0.055	0.152	-1.885	0.747	0.043	0.151
0.5	-0.0465	0.504	0.045	0.152	-1.877	0.501	0.033	0.151
1.0	-0.0464	0.263	0.036	0.152	-1.869	0.260	0.024	0.152
1.5	-0.0463	0.017	0.026	0.153	-1.861	0.014	0.014	0.152
2.0	-0.0462	-0.225	0.016	0.153	-1.852	-0.228	0.004	0.153
2.5	-0.0460	-0.471	0.006	0.154	-1.844	-0.474	-0.006	0.153

The table reports optimal policies u_1, π_2, as well as portfolio drift μ_X, and volatility σ_X when $\alpha_1 = 0$, $\gamma = -1$, $\mu_1 = 0.03$, $\mu_2 = 0.07$, $\sigma_1 = \sigma_2 = 0.2$, $\rho = 1$, $r = 0.01$, $T = 1$, and $t = 0$, $S_1 = 0.497$, $X = 1.01$.

Table 10: **Optimal trading of two stocks when $\alpha_1 = 0.005$ and $\rho = 1$**

	$\alpha_2 = 0.2$				$\alpha_2 = 0.005$			
N_1	u_1	π_2	μ_X	σ_X	u_1	π_2	μ_X	σ_X
-2.5	-0.0353	1.971	0.104	0.150	-1.419	1.969	0.093	0.149
-2.0	-0.0351	1.724	0.094	0.150	-1.412	1.722	0.083	0.150
-1.5	-0.0350	1.482	0.085	0.150	-1.404	1.480	0.073	0.150
-1.0	-0.0348	1.239	0.075	0.151	-1.396	1.238	0.064	0.150
-0.5	-0.0346	0.993	0.065	0.151	-1.388	0.991	0.054	0.151
0.0	-0.0344	0.750	0.055	0.152	-1.380	0.748	0.044	0.151
0.5	-0.0342	0.505	0.046	0.152	-1.372	0.506	0.035	0.152
1.0	-0.0340	0.261	0.036	0.152	-1.364	0.259	0.025	0.152
1.5	-0.0337	0.018	0.026	0.153	-1.355	0.017	0.015	0.152
2.0	-0.0336	-0.224	0.016	0.153	-1.347	-0.226	0.005	0.153
2.5	-0.0333	-0.471	0.006	0.153	-1.339	-0.473	-0.005	0.153

The table reports optimal policies u_1, π_2, as well as portfolio drift μ_X, and volatility σ_X when $\alpha_1 = 0.005$, $\gamma = -1$, $\mu_1 = 0.03$, $\mu_2 = 0.07$, $\sigma_1 = \sigma_2 = 0.2$, $\rho = 1$, $r = 0.01$, $T = 1$, and $t = 0$, $S_1 = 0.497$, $X = 1.01$.

Table 11: **Optimal trading with illiquid stock crashes when $\alpha_1 = 0$ and $\lambda = 0.1$.**

		$\alpha_2 = 0.2$				$\alpha_2 = 0.005$			
N	NS/X	u	U	Δ_{c-l}	Δ_c^i	u	U	Δ_{c-l}	Δ_c^i
0.04	0.020	0.066	-0.9787	0.0443	0.0000	1.75	-0.9717	0.0225	0.0001
0.5	0.2486	0.044	-0.9680	0.0151	0.0001	1.21	-0.9648	0.0096	0.0002
1.0	0.492	0.022	-0.9611	0.0038	0.0003	0.60	-0.9604	0.0027	0.0007
1.48	0.729	0.0001	-0.9589	0.0008	0.0008	0.0047	-0.9589	0.0008	0.0008
1.49	0.732	-0.0001	-0.9589	0.0008	0.0008	-0.0001	-0.9589	0.0008	0.0008
1.70	0.836	-0.010	-0.9593	0.0013	0.0011	-0.26	-0.9592	0.0010	0.0009
1.85	0.911	-0.016	-0.9602	0.0025	0.0013	-0.45	-0.9597	0.0018	0.0010
2.00	0.983	-0.023	-0.9614	0.0043	0.0015	-0.64	-0.9605	0.0030	0.0011

The table reports optimal trading rate u, expected utility function U, crash–liquidity premium Δ_{c-l}, and the crash premium Δ_c^i when $\gamma = -1$, $\mu = 0.07$, $\sigma = 0.2$, $r = 0.01$, $T = 1$, $\lambda = 0.1$, $\delta = 0.1$, and $t = 0$, $S = 0.497$, $X = 1.01$, so $\hat{\pi} = 0.730$, $\Delta_c = 0.0008$, $U_{max} = -0.9584$, and $U_{min} = -0.9802$. U_{max} is found from equation (7).

Table 12: **Optimal trading with illiquid stock crashes when $\alpha_1 = 0$ and $\lambda = 3$.**

		$\alpha_2 = 0.2$				$\alpha_2 = 0.005$			
N	NS/X	u	U	Δ_{c-l}	Δ_c^i	u	U	Δ_{c-l}	Δ_c^i
0.04	0.020	0.064	-0.9788	0.0444	0.000	1.45	-0.9737	0.0272	0.0070
0.5	0.249	0.026	-0.9699	0.0187	0.0081	0.62	-0.9690	0.0169	0.0111
0.84	0.415	0.0007	-0.9680	0.0152	0.0151	0.004	-0.9680	0.0151	0.0151
0.85	0.417	-0.0001	-0.9680	0.0151	0.0151	-0.003	-0.9680	0.0151	0.0151
1.00	0.492	-0.011	-0.9684	0.0159	0.0158	-0.28	-0.9681	0.0154	0.0154
1.50	0.738	-0.053	-0.9753	0.0316	0.0266	-1.19	-0.9717	0.0227	0.0197
1.70	0.836	-0.068	-0.9805	NA	0.0269	-1.57	-0.9745	0.0294	0.0217
1.85	0.911	-0.086	-0.9858	NA	0.0300	-1.86	-0.9771	0.0376	0.0230
2.00	0.983	-0.099	-0.9905	NA	0.0315	-2.14	-0.9802	0.0590	0.0247

The table reports optimal trading rate u, expected utility function U, crash–liquidity premium Δ_{c-l}, and the crash premium Δ_c^i when $\gamma = -1$, $\mu = 0.07$, $\sigma = 0.2$, $r = 0.01$, $T = 1$, $\lambda = 3$, $\delta = 0.1$, and $t = 0$, $S = 0.497$, $X = 1.01$, so $\hat{\pi} = 0.417$, $\Delta_c = 0.0151$, $U_{max} = -0.9584$, and $U_{min} = -0.9802$. U_{max} is found from equation (7).

Table 13: **Optimal trading of illiquid stock with stochastic volatility of return when** $\alpha_1 = 0.005$

		$\varepsilon = 1$		$\varepsilon = 2$	
N	NS/X	u	U	u	U
0.04	0.020	0.0512	-0.9793	0.181	-0.9786
0.50	0.246	0.0307	-0.9687	0.140	-0.9685
0.75	0.369	0.0200	-0.9647	0.113	-0.9646
1.00	0.492	0.0096	-0.9619	0.078	-0.9618
1.228	0.604	0.0001	-0.9603	0.008	-0.9603
1.232	0.606	0.0	-0.9603	0.0	-0.9603
1.525	0.750	0.0	-0.9595	0.0	-0.9595
1.832	0.901	0.0	-0.9603	0.0	-0.9603
1.836	0.903	-0.0001	-0.9603	-0.005	-0.9603
2.032	0.999	-0.0086	-0.9617	-0.067	-0.9616

The table reports optimal trading rate u, expected utility function U, and liquidity premium Δ_l when $\alpha_1 = 0.005$, $\alpha_2 = 0.2$, $\gamma = -1$, $\mu = 0.07$, $r = 0.01$, $T = 1$, and $t = 0$, $S = 0.497$, $X = 1.01$, $\sigma = 0.2$, so $\hat{\pi} = 0.75$.