# Heterogeneous Basket Options Pricing Using Analytical Approximations

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**Abstract**

This paper proposes the use of analytical approximations to price an heterogeneous basket option combining commodity prices, foreign currencies and zero-coupon bonds. We examine the performance of three moment matching approximations: inverse gamma, Edgeworth expansion around the lognormal and Johnson family distributions. Since there is no closed-form formula for basket options, we carry out Monte Carlo simulations to generate the benchmark values. We perform a simulation experiment on a whole set of options based on a random choice of parameters. Our results show that the lognormal and Johnson distributions give the most accurate results.

**Keywords :** Basket Options, Options Pricing, Analytical Approximations, Monte Carlo Simulation.

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# 1 Introduction

The use of derivative securities in risk management activities emerged in the early 1990s and has evolved rapidly since. They now are the most important tool in financial risk management. According to the 1998 Wharton school survey of financial risk management by US non-financial firms,[1] over 50% of the responding firms use derivatives products to hedge their exposure, 83% of them use derivatives to hedge foreign-exchange risk, 76% to hedge interest-rate risk, and 56% to hedge commodity-price risk. In the current economic situation, many non-financial institutions such as gold-mining firms, energy companies or airlines companies may face different financial risks simultaneously and hence look for the most efficient way to hedge their portfolio.

For example, gold-mining firms may be exposed to different types of risk: commodity risk, which can include uncertainty about the price of their primary product, gold, as well as their by-products such as silver and copper; since these firms sell their products in other countries, they are exposed to currency risk; and their interest rate risk exposure is primarily related to their fixed-rate and variable-rate debt. Though all these markets are very active and very liquid so that a firm could hedge all its different risk exposures separately, it would be more interesting to adopt a portfolio approach because it allows the firm to account for the correlations between these different financial markets and hedge simultaneously a great variety of different financial risks. To attain this goal, basket options are an efficient instrument to use.

Basket options are a type of exotic option whose payoff depends on the value of a basket of assets. They offer the flexibility of being able to include virtually any kind and any number of assets, and hence can suitably respond to the specific needs of a firm in hedging its risk exposures. A risk manager may have other incentives in using basket options: depending on the design of each option, they are usually cheaper than a portfolio of standard options. In practice, basket options are traded over-the-counter and are designed specifically to meet the needs of the buyer. For these reasons, the liquidity premium required by the counterpart may annihilate some of the advantages coming from the correlation structure of the basket.

The pricing of basket options is more challenging than that of standard options because there is no explicit analytical solution for the density function of a weighted sum of correlated assets. Several approaches are proposed in the literature to price basket options. They can be categorized as follows:

1. Numerical methods: Monte Carlo simulations [Barraquand (1995), Pellizzari (2001)] and Quasi-Monte Carlo [Dahl (2000) and Dahl and Benth (2001)], and multinomial trees [Rubinstein (1994), Wan(2002)],

2. Upper and lower bounds: Curran (1994), Vanmaele, Deelstra and Liinev (2004), Laurence and Wang (2004) and Deelstra, Liinev and Vanmaele (2004),

3. Analytical approximations: Gentle (1993), Huynh (1994), Milevsky and Posner (1998a, 1998b, 1999), Posner and Milevsky (1999), Ju (2002), Datey, Gauthier and Simonato (2003) and Brigo, Mercurio, Rapisarda and Scotti (2004).

[1] Financial management, Vol. 27, No. 4, winter 1998.

However, all these papers are based on two main hypothesis. They assume constant interest rates and homogeneous basket options; that is, all the assets underlying the basket are of the same nature and thus hedge the same type of risk. Unfortunately, this homogeneity in basket assets does not always match a firm's needs, and with a growing diversification in investors' portfolios, heterogeneous basket options become a very efficient tool to hedge multiple risk exposures simultaneously.

This article proposes analytical approximations to price heterogeneous basket options, consisting of commodities, foreign currencies and zero-coupon bonds, with stochastic interest rates, and compares the accuracy and the performance of these approximations for different sets of parameters. Three distributions based on the moment matching technique will be used to approximate the basket density function: inverse gamma distribution, Edgeworth expansion around the lognormal distribution and Johnson distribution. It is found that the Edgeworth-lognormal and Johnson approximations perform better than the inverse gamma approximation. Our contributions are: (1) to design an arbitrage-free framework in which the domestic and the foreign interest rates, the exchange rate, the commodity price and the convenience yield are stochastic, (2) to compute the moments of the heterogeneous basket under the $T-$forward measure, (3) to show that some of the existing analytical approximations may be used in this very general setting and (4) quantify the approximation errors.

The next section presents the pricing model under the forward measure. Section 3 derives the inverse gamma, the Edgeworth-lognormal and the Johnson approximations. Section 4 compares the performance of the three approximations. Section 5 computes the hedging ratios and Section 6 concludes.

## 2 Pricing of a European heterogeneous basket option

Let us consider a non-financial firm looking for an alternative way to simultaneously hedge its different financial risk exposures: commodity price risk, exchange rate risk and interest rate risk. In the following, $S_t$ denotes the time $t$ value of the commodity, $\delta_t$ is its continuously-compounded convenience yield at time $t$, $C_t$ is the value at time $t$ of one unit of the foreign currency expressed in the domestic currency, $P(t, T_1)$ is the time $t$ value of a zero-coupon bond paying one unit of the domestic currency at time $T_1$, $P^*(t, T_2)$ is the time $t$ value of a foreign zero-coupon bond paying one unit of the foreign currency at time $T_2$. We assume that the firm's financial assets dynamics are given by the following stochastic differential equations[2] (SDE hereafter) under the historical

[2] In this setting, the convenience yield and the commodity price share the same source of risk to ensure the market completeness. For more details, see Dionne, Gauthier and Ouertani (2006).

probability measure $P$:

$$dS_t = S_t\left[(\alpha_s - \delta_t)\,dt + \sigma_s dW_t^{(1)}\right], \tag{1a}$$

$$d\delta_t = \kappa(\theta - \delta_t)dt + \sigma_\delta dW_t^{(1)}, \tag{1b}$$

$$dC_t = C_t\left[\alpha_c dt + \sigma_c dW_t^{(2)}\right], \tag{1c}$$

$$dP\left(t,T_1\right) = P\left(t,T_1\right)\left[\left(r_t - \beta_{t,T_1}\left(\frac{\gamma_{t,T_1}}{\eta_{t,T_1}} - \beta_{t,T_1}\right)\right)dt - \beta_{t,T_1}dW_t^{(3)}\right],\ 0 \le t \le T_1, \tag{1d}$$

$$dP^*\left(t,T_2\right) = P^*\left(t,T_2\right)\left[\left(r_t^* - \beta^*_{t,T_2}\left(\frac{\gamma^*_{t,T_2}}{\eta^*_{t,T_2}} - \beta^*_{t,T_2}\right)\right)dt - \beta^*_{t,T_2}dW_t^{(4)}\right],\ 0 \le t \le T_2 \tag{1e}$$

The expressions for the bond prices are obtained from the domestic and the foreign instantaneous forward rates:

$$df\left(t,T_1\right) = \gamma_{t,T_1}\ dt + \eta_{t,T_1}\ dW_t^{(3)},\ 0 \le t \le T_1, \tag{1f}$$

$$df^*\left(t,T_2\right) = \gamma^*_{t,T_2}\ dt + \eta^*_{t,T_2}\ dW_t^{(4)},\ 0 \le t \le T_2, \tag{1g}$$

in which the volatility parameters $\eta_{t,T_1}$ and $\eta^*_{t,T_2}$ are deterministic functions of time $t$ and maturity. Consequently, $\beta_{t,T_1} = \int_t^{T_1}\eta_{t,s}ds$, $\beta^*_{t,T_2} = \int_t^{T_2}\eta^*_{t,s}ds$ and $r_t = f\left(t,t\right)$, $r_t^* = f^*\left(t,t\right)$ are respectively the continuously-compounded domestic and foreign risk-free spot interest rates. We do not specify the drift terms $\gamma_{t,T_1}$ and $\gamma^*_{t,T_2}$ since they do not appear in the pricing formulae. The four-dimensional Brownian motion $\left\{\left(W_t^{(1)},\ W_t^{(2)},\ W_t^{(3)},\ W_t^{(4)}\right)' : t \ge 0\right\}$ is constructed on a filtered probability space $\left(\Omega, \mathcal{F}, \left\{\mathcal{F}_t : t \ge 0\right\}, P\right)$ with the correlation structure:

$$\mathrm{Corr}^P\left(W_t^{(i)}, W_t^{(j)}\right) = \rho_{i,j},\ \text{for each } i,j = \{1,2,3,4\} \text{ and } t > 0.$$

We consider an European call option that gives the firm the right to buy a basket consisting of the commodity, the domestic zero-coupon bond and the foreign zero-coupon bond expressed in domestic currency units, at a strike price $K_B$. The payoff at maturity $T$ is given by

$$X_B = \max\left[B_T - K_B, 0\right],$$

where the basket value at time $t$ is $B_t = w_1 S_t + w_2 P(t,T_1) + w_3 C_t P^*(t,T_2)$, $0 \le t \le T \le T_1, T_2$, $w_1$, $w_2$ and $w_3$ correspond to the numbers of shares invested respectively in the commodity $S_T$, the domestic zero-coupon bond $P(T,T_1)$ and the foreign zero-coupon bond expressed in domestic currency units $C_T P^*(T,T_2)$.

Since the proposed market model is complete,[3] Harrison and Pliska (1981) allows the pricing of any contingent claim as the expectation of the discounted payoff under the risk-neutral measure $Q$. Consequently, the price of the European call option at time $t$ is given by:

$$\begin{aligned} V_t^B &= \mathrm{E}^Q\left[\exp\left(-\int_t^T r_v dv\right)\max\left(B_T - K_B, 0\right)\middle|\, \mathcal{F}_t\right] \\ &= P(t,T)\mathrm{E}^{Q_T}\left[\max\left(B_T - K_B, 0\right)\middle|\, \mathcal{F}_t\right] \\ &= P\left(t,T\right)\int_{-\infty}^{+\infty}\max\left(x - K_B, 0\right)v\left(x\right)dx, \end{aligned} \tag{2}$$

[3]The derivation of the unique risk-neutral measure is available upon request.

where $v(x)$ is the true (unknown) density function of the basket value $B_T$ under the $T-$forward measure $Q_T$. This forward measure has a Radon-Nikodym derivative with respect to $Q$ denoted $\frac{dQ_T}{dQ}$. The associated $Q-$martingale is given by:

$$\zeta_t = \mathrm{E}^Q\left(\left.\frac{dQ_T}{dQ}\right| \mathcal{F}_t\right) = \frac{\exp\left(-\int_t^T r_v dv\right)}{P(t,T)}.$$

The SDEs satisfied by the basket underlying assets under the $T$-forward measure $Q_T$ is given in Appendix A and its strong solution is, for any $0 \le t \le T$,

$$S_T = S_t \exp\left(\int_t^T \left(r_u - \delta_u - \frac{\sigma_s^2}{2} - \sigma_s \rho_{1,3} \beta_{u,T}\right) du + \sigma_s \int_t^T d\widehat{W}_u^{(1)}\right) \tag{3a}$$

$$\begin{aligned} \delta_T = {} & \delta_t e^{-\kappa(T-t)} + \left(\kappa\theta - \frac{\sigma_\delta}{\sigma_s}\alpha_s\right)\frac{1-e^{-\kappa(T-t)}}{\kappa} - \sigma_\delta \rho_{1,3}\int_t^T e^{-\kappa(T-u)}\beta_{u,T}\, du \\ & + \frac{\sigma_\delta}{\sigma_s}\int_t^T r_u e^{-\kappa(T-u)}\, du + \sigma_\delta \int_t^T e^{-\kappa(T-u)}\, d\widehat{W}_u^{(1)} \end{aligned} \tag{3b}$$

$$C_T = C_t \exp\left(\int_t^T \left(r_u - r_u^* - \frac{\sigma_c^2}{2} - \sigma_c \rho_{2,3} \beta_{u,T}\right) du + \sigma_c \int_t^T d\widehat{W}_u^{(2)}\right) \tag{3c}$$

$$P(T,T_1) = P(t,T_1) \exp\left(\int_t^T \left(r_u - \frac{1}{2}\beta_{u,T_1}^2 + \beta_{u,T_1}\beta_{u,T}\right) du - \int_t^T \beta_{u,T_1} d\widehat{W}_u^{(3)}\right) \tag{3d}$$

$$P^*(T,T_2) = P^*(t,T_2)\exp\left(\begin{array}{c} \int_t^T \left(r_u^* - \frac{1}{2}\left(\beta_{u,T_2}^*\right)^2 + \sigma_c\rho_{2,4}\beta_{u,T_2}^* + \rho_{3,4}\beta_{u,T_2}^*\beta_{u,T}\right) du \\ - \int_t^T \beta_{u,T_2}^* d\widehat{W}_u^{(4)} \end{array}\right) \tag{3e}$$

where $\widehat{\mathbf{W}} = \left(\widehat{W}^{(1)},\ \widehat{W}^{(2)},\ \widehat{W}^{(3)},\ \widehat{W}^{(4)}\right)'$ is a $Q_T-$Brownian motions with $\mathrm{Corr}^{Q_T}\left(\widehat{W}_t^{(i)}, \widehat{W}_t^{(j)}\right) = \rho_{i,j}$.

The evaluation of the basket call option is complicated by the absence of a closed-form equation for the density function $v(x)$ in Equation (2). Among the different approaches proposed in the basket options literature, we find some numerical techniques such as Monte Carlo, Quasi-Monte Carlo and lattice-based methods, the upper and lower bound computations,[4] and some analytical approximations.

Lattice-based approaches are widely used for options on a single asset. They are, exponentially complicated and computationally expensive for options on multiple assets. For example, our three-asset basket option needs $(n+1)^3$ terminal nodes on an $n$-step trinomial tree. On the other hand, Monte Carlo and Quasi-Monte Carlo methods can be used for multi-assets options and are less time-consuming than lattice-based approaches. The estimates can be as accurate as needed at a computational cost however: to improve the accuracy of an $n$-path simulation by one half, one needs to simulate $4n$ paths and thus needs 4 times more computing time.

A practitioner might be interested in slightly less accurate but very fast methods such as analytical approximations. These methods approximate the unknown basket density function with an alternative and easy-to-compute distribution. In the next sections, we will extend three well-known

[4] The upper and lower bounds methods are not useful in our case since the market model proposed is complete and thus a unique option price can be computed.

analytical approximations to heterogeneous basket options: the inverse gamma distribution, the Edgeworth expansion around the lognormal distribution and Johnson distribution.

# 3 Analytical approximations

In order to apply these moment matching-based approximations, we need to calculate the first four moments of the weighted sum underlying the European option under the $T-$forward measure $Q_T$.

We adopt the following notations:

$$\mu_n'(h) = \int_{-\infty}^{+\infty} x^n h(x)\,dx \tag{4a}$$

$$\mu_n(h) = \int_{-\infty}^{+\infty} \left(x-\mu_1'(h)\right)^n h(x)\,dx, \tag{4b}$$

$\mu_n'(h)$ and $\mu_n(h)$ represent respectively the $n^{th}$ non-centered and centered moments of the density function $h \in \{v, a\}$, where $h = v$ corresponds to the exact density of the basket value under the forward measure while $h = a$ corresponds to the approximate density. The first four cumulants of distribution $h$, that is, the mean, the variance, the skewness and the kurtosis are defined as:

$$\kappa_1(h) = \mu_1'(h) \tag{5a}$$

$$\kappa_2(h) = \mu_2(h) \tag{5b}$$

$$\kappa_3(h) = \mu_3(h) \tag{5c}$$

$$\kappa_4(h) = \mu_4(h) - 3\mu_2(h). \tag{5d}$$

**Lemma 1** *For any positive integer $n$, the $n^{th}$ non-centered moment of the true distribution of the weighted sum $B_T$, under the $T$-forward measure $Q_T$, is*

$$\begin{aligned}
\mu_n'(v) &= \mathrm{E}^{Q_T}\left[B_T^n\right] = \mathrm{E}^{Q_T}\left[(w_1 S_T + w_2 P(T,T_1) + w_3 C_T P^*(T,T_2))^n\right] \\
&= \sum_{k=0}^{n}\sum_{j=0}^{k} \frac{n!}{j!\,(k-j)!\,(n-k)!} w_1^j w_2^{(k-j)} w_3^{(n-k)} \mathrm{E}^{Q_T}\left[S_T^j \left(P(T,T_1)\right)^{(k-j)} C_T^{n-k} \left(P^*(T,T_2)\right)^{n-k}\right].
\end{aligned}$$

Note that the log normal distribution of $S_T$, $P(T,T_1)$, $C_T$, and $P^*(T,T_2)$ under the forward measure allows the computation of the last expectation of Lemma 1 from following identity:

$$\mathrm{E}\left[\exp(\mu+\sigma Z)\right] = \exp\left(\mu + \frac{\sigma^2}{2}\right) \quad \text{where } Z \sim N(0,1).$$

Details are given in Appendix B.

## 3.1 Inverse gamma approximation

In this section, we use the inverse gamma distribution to approximate the sum of correlated lognormal variables. This approximation was first used by Milevsky and Posner (1998a, 1998b) to price Asian and basket options. In fact, a finite sum of correlated lognormal variables converges asymptotically to an inverse gamma variable. Under an inverse gamma distribution for the underlying basket, an European basket call option has a closed-form solution that looks like a Black and Scholes (1973) (B&S hereafter) formula:

$$V_{gamma}^{B} = P(t,T)\left(\mu_1'(v)\,\mathrm{G}\left(\frac{1}{K_B}\middle|\,\alpha-1,\beta\right) - K_B\mathrm{G}\left(\frac{1}{K_B}\middle|\,\alpha,\beta\right)\right), \tag{6}$$

where $G(\bullet \mid \alpha, \beta)$ is the cumulative function of a gamma distribution with parameters $(\alpha, \beta)$. These parameters are determined by matching the first two moments of the exact and the approximate distributions to obtain:

$$\alpha = \frac{\mu_1'^2(v) - 2\mu_2'(v)}{\mu_1'^2(v) - \mu_2'(v)} \text{ and } \beta = \frac{\mu_2'(v) - \mu_1'^2(v)}{\mu_1'(v)\,\mu_2'(v)}. \tag{7}$$

Mathematical details of the pricing formula are provided in Appendix C.

## 3.2 Edgeworth expansion around the lognormal distribution

We now present an analytical approximation based on a generalized Edgeworth expansion around the lognormal distribution. This approach, introduced by Jarrow and Rudd (1982) in option pricing, substitutes an unknown density function $v(\bullet)$ with a Taylor-like expansion around an easy-to-use density function denoted $a(\bullet)$. Notice, however, that Edgeworth expansions usually lead to a function which is not a true density function. Barton and Denis (1952) derive some conditions on the third and fourth moments of the unknown distribution to guarantee that the approximation obtained with a truncated Edgeworth expansion is positive and unimodal. Moreover, Ju (2002) points out that the Edgeworth expansion may diverge for some parameter values, which consequently can give incorrect prices for high volatility and long maturity options. However, in this paper we do not have this problem in the application of the Edgeworth expansion.

Following Huynh (1994) who uses this approach for basket options, we will use an Edgeworth expansion of order 4 and we will match the first and second moments of the exact and the lognormal distributions. Under this approximation, an European basket call option can be obtained as a sort of Black and Scholes price adjusted for the excess skewness and the excess kurtosis from the lognormal density:

$$V_{\log normal}^{B} = P(t,T)\left[V - \frac{\kappa_3(v) - \kappa_3(a)}{3!}\frac{da(K_B)}{dx} + \frac{\kappa_4(v) - \kappa_4(a)}{4!}\frac{d^2a(K_B)}{dx^2}\right], \tag{8a}$$

where

$$V = \mu_1'(v)\,N(d_1) - K_B N(d_2), \tag{8b}$$

$$d_1 = d_2 + \beta, \tag{8c}$$

$$d_2 = \frac{\alpha - \ln(K_B)}{\beta}, \tag{8d}$$

$$\alpha = \ln\left((\mu_1'(v))^2\right) - \frac{1}{2}\ln\left((\mu_1'(v))^2 + \mu_2(v)\right), \tag{8e}$$

$$\beta = \sqrt{\ln\left(1 + \mu_2(v)\,(\mu_1'(v))^{-2}\right)}, \tag{8f}$$

$a(\bullet)$ is the density function of a log-normal distribution (see Equation (19) in Appendix D), and $N(\bullet)$ represents the cumulative function of the standard normal distribution. The third and fourth moments of the lognormal distribution needed for the Edgeworth expansion depend only on the first and second moments of the exact distribution and are given by:

$$\mu_3'(a) = \left(\frac{\mu_2'(v)}{\mu_1'(v)}\right)^3 \text{ and } \mu_4'(a) = \frac{\mu_2'(v)^6}{\mu_1'(v)^8}. \tag{8g}$$

Details about the Edgeworth approximation formula are given in Appendix D.

### 3.3 Johnson approximation

Johnson (1949) proposes a family of density functions, obtained via a transformation of a standard normal variable, that can be used to approximate unknown distributions. Let $Z$ be a standard normal variable and $X$ be a random variable with an unknown density function, Johnson (1949) suggests the following transformations between $Z$ and $X$:

$$Z = \gamma + \delta\psi\left(\frac{X-\varepsilon}{\lambda}\right), \tag{9a}$$

$$X = \varepsilon + \lambda\psi^{-1}\left(\frac{Z-\gamma}{\delta}\right), \tag{9b}$$

where $\gamma$ and $\delta$ are the shape parameters of the Johnson distribution, $\lambda$ is the scale parameter, $\varepsilon$ is the threshold parameter, and $\psi(\bullet)$ is one of the following Johnson functions:

$$\begin{aligned} \psi_L(x) &= \ln(x), && \text{(Lognormal system)} \\ \psi_U(x) &= \ln\left(x + \sqrt{x^2+1}\right), && \text{(Unbounded system)} \\ \psi_B(x) &= \ln\left(\frac{x}{1-x}\right). && \text{(Bounded system)} \end{aligned}$$

The choice of the system and fitting parameters provides a great flexibility in adjusting the curve to match the first four moments of the unknown distribution. We use the lognormal ($\psi_L$) and the unbounded ($\psi_U$) systems that are common in the literature. We apply the Hill, Hill and Holder (1976) algorithm, based on the true skewness and kurtosis of the basket distribution, to determine which of the Johnson systems ($\psi_L$ or $\psi_U$) should be used in the approximation. Unlike approximations obtained with a truncated Edgeworth expansion, approximations based on Johnson (1949) systems correspond to true density functions with a perfect match of the first four moments.

Following Posner and Milevsky (1999), we substitute the unknown distribution of the underlying basket with the lognormal and the unbounded Johnson functions where the system parameters are calculated by matching the four moments. Under a Johnson density function, a European basket call option can be priced as:

$$\begin{aligned} V^B_{Johnson} &= P(t,T)\int_{K_B}^{+\infty}(x-K_B)\,\psi(x)\,dx \\ &= P(t,T)\left(\int_0^{+\infty}x\psi(x)\,dx - K_B\int_0^{+\infty}\psi(x)\,dx - \int_0^{K_B}(x-K_B)\,\psi(x)\,dx\right) \\ &= P(t,T)\left(\mu_1'(v) - K_B + \int_0^{K_B}\left(\int_0^x\psi(y)\,dy\right)dx\right) \\ &\cong P(t,T)\left(\mu_1'(v) - K_B + \int_{-\infty}^{K_B}\left(\int_0^x\psi(y)\,dy\right)dx\right). \end{aligned} \tag{11}$$

The third line in the equation is obtained using an integration by parts. Milevsky and Posner (1999) show that the last double integral, involving a Johnson density function, can be computed as follows:

1. Lognormal system $\psi_L : X = \varepsilon + \lambda \exp\left(\frac{Z-\gamma}{\delta}\right)$

$$\begin{aligned}\int_{-\infty}^{K_B}\left(\int_0^x \psi(y)\,dy\right)dx &= (K_B-\varepsilon)\,N\left(\gamma+\delta\ln\left(\frac{K_B-\varepsilon}{\lambda}\right)\right)\\ &\quad -\lambda\exp\left(\frac{1-2\gamma\delta}{2\delta^2}\right)N\left(\gamma+\delta\ln\left(\frac{K_B-\varepsilon}{\lambda}\right)-\frac{1}{\delta}\right). \qquad (12)\end{aligned}$$

2. Unbounded system $\psi_U : X = \varepsilon + \lambda \sinh\left(\frac{Z-\gamma}{\delta}\right)$

$$\begin{aligned}\int_{-\infty}^{K_B}\left[\int_0^x \psi(y)\,dy\right]dx &= (K_B-\varepsilon)\,N(q)+\frac{\lambda}{2}\exp\left(\frac{1}{2\delta^2}\right)\exp\left(\frac{\gamma}{\delta}\right)N\left(q+\frac{1}{\delta}\right)\\ &\quad -\frac{\lambda}{2}\exp\left(\frac{1}{2\delta^2}\right)\exp\left(-\frac{\gamma}{\delta}\right)N\left(q-\frac{1}{\delta}\right), \qquad (13)\end{aligned}$$

   where $q = \gamma + \delta \sinh^{-1}\left(\frac{K_B-\varepsilon}{\lambda}\right)$.[5]

# 4 Performance analysis of approximations

In this section, we analyze the performance of the three approximations presented previously. We will use prices obtained by Monte Carlo simulations as benchmarks since there is no closed-form solution for basket options. Following Barraquand (1995), we will apply a variance reduction technique based on the matching of the first and second moments. This will ensure that sample mean and variance are equal to the their theoretical counterparts. The Monte Carlo basket option price combined with the variance reduction technique is given by:

$$\begin{aligned}\overline{V}_0^B &= \frac{1}{N}\sum_{i=1}^{N} P(0,T)\max\left(B_{i,T}^* - K_B, 0\right)\\ \text{where } B_{i,T}^* &= \left(B_{i,T}-\overline{B}\right)\sqrt{\mu_2(v)}\widehat{S}^{-1}+\mu_1'(v),\end{aligned}$$

$B_{i,T}$ is the time $T$ basket value obtained with sample path $i$, and $\overline{B} = \frac{1}{m}\sum_{i=1}^{m} B_{i,T}$ and $\widehat{S} = \sqrt{\frac{1}{m}\sum_{i=1}^{m} B_{i,T}^2 - \overline{B}^2}$ are respectively the sample basket mean and standard deviation and $m$ is the number of simulated paths.

Our performance study will be conducted with two analyses. In the first sensitivity analysis, we will compare the basket option price obtained with the analytical approximations to the Monte Carlo price obtained with 1,000,000 paths repeated for different maturities, different moneynesses and different levels for the basket volatility. The second sensitivity analysis is a more detailed analysis based on works done by Broadie and Detemple (1996). It computes the option prices over a wide range of parameters chosen randomly from a realistic set of values in order to generalise our

[5] Notice that $\sinh(x) = \frac{\exp(x)-\exp(-x)}{2}$ and thus $\sinh^{-1}(x) = \ln\left(x+\sqrt{x^2+1}\right) = \psi_U(x)$.

previous results independently of the model's parameters. For each combination, we compare the prices obtained with the approximations and Monte Carlo simulations.

Table 1 presents the set of parameters used in the first analysis. These values are based on estimations using real data on gold prices, CAD/USD exchange rate and Canadian and American 3-month zero-coupon bonds. Although we have positive and negative correlations in the set of parameters, the volatility of the basket will increase when individual assets volatilities increase.

Table 1: Parameters used in the first sensitivity analysis

| | |
|---|---|
| Domestic risk-free rate, $f(0,t) = f$ for any $t$ | 0.06 |
| Foreign risk-free rate, $f^*(0,t) = f^*$ for any $t$ | 0.05 |
| Commodity drift, $\alpha_s$ | 0.15 |
| Exchange rate drift, $\alpha_c$ | 0.04 |
| Commodity volatility, $\sigma_s$ | 0.15 |
| Exchange rate volatility, $\sigma_c$ | 0.06 |
| Domestic instantaneous forward rate volatility, $\eta_{t,T_1} = \eta$ for any $t$ and $T_1$ | 0.01 |
| Foreign instantaneous forward rate volatility, $\eta^*_{t,T_2} = \eta^*$ for any $t$ and $T_2$ | 0.01 |
| Convenience yield volatility, $\sigma_\delta$ | 0.3 |
| Convenience yield mean reversion parameter, $\kappa$ | 0.1 |
| Convenience yield long-run mean, $\theta$ | 0.15 |
| Commodity and exchange rate correlation, $\rho_{1,2}$ | 0.1 |
| Commodity and domestic instantaneous forward rate correlation, $\rho_{1,3}$ | -0.2 |
| Commodity and foreign instantaneous forward rate correlation, $\rho_{1,4}$ | -0.25 |
| Exchange rate and domestic instantaneous forward rate correlation, $\rho_{2,3}$ | 0.05 |
| Exchange rate and foreign instantaneous forward rate correlation, $\rho_{2,4}$ | 0.1 |
| Foreign and domestic instantaneous forward rates correlation, $\rho_{3,4}$ | 0.85 |
| Basket weights (commodity, domestic and foreign zero-coupon bonds) | 0.5; 0.25; 0.25 |
| Initial prices $S_0$, $C_0$ and $\delta_0$ are respectively | \$330; \$/CAD 0.65; 0.2 |

Table 2 presents sensitivity analysis of the basket option price with respect to moneyness, which corresponds to the ratio of the exercise price over the initial value of the basket $\frac{K_B}{B_0}$, and option maturity. The results show that Edgeworth-lognormal and Johnson approximations are much more accurate, with relative errors between $10^{-6}$ and $10^{-3}$, than the inverse gamma approximation with a relative error between $10^{-4}$ and $10^{-1}$.[6]

[6] This result is consistent with Milevsky and Posner (1998a) who showed that the convergence result for the inverse gamma works well when the risk-neutral drift of the underlying diffusion is negative or when the correlation matrix decays quickly.

Table 2: Sensitivity analysis w.r.t. the moneyness and option maturity

| **Moneyness** | **Monte Carlo** | **Standard Deviation** | **Inverse Gamma Approximation** | | **Lognormal Approximation** | | **Johnson Approximation** | |
|---|---|---|---|---|---|---|---|---|
| | | | Price | Relative error | Price | Relative error | Price | Relative error |
| 0.75-year maturity | | | | | | | | |
| **0.85** | 14.6677 | 9.16e-03 | 14.6496* | 1.23e-03 | 14.6678 | 4.60e-06 | 14.6676 | 5.06e-06 |
| **0.95** | 3.1876 | 5.33e-03 | 3.1890 | 4.50e-04 | 3.1879 | 7.94e-05 | 3.1879 | 8.57e-05 |
| **1** | 0.9173 | 2.84e-03 | 0.9400* | 2.47e-02 | 0.9175 | 2.87e-04 | 0.9177 | 4.75e-04 |
| **1.05** | 0.1842 | 1.21e-03 | 0.2008* | 9.02e-02 | 0.1843 | 7.06e-04 | 0.1844 | 1.36e-03 |
| **1.15** | 0.0026 | 1.28e-04 | 0.0039* | 4.69e-01 | 0.0026 | 4.91e-03 | 0.0027 | 7.76e-03 |
| 1-year maturity | | | | | | | | |
| **0.85** | 15.1267 | 1.12e-02 | 15.0876* | 2.59e-03 | 15.1266 | 1.02e-05 | 15.1263 | 3.07e-05 |
| **0.95** | 4.3557 | 7.08e-03 | 4.3517 | 9.09e-04 | 4.3557 | 1.80e-05 | 4.3557 | 1.65e-05 |
| **1** | 1.7612 | 4.54e-03 | 1.7915* | 1.72e-02 | 1.7616 | 2.26e-04 | 1.7619 | 3.67e-04 |
| **1.05** | 0.5828 | 2.54e-03 | 0.6180* | 6.06e-02 | 0.5830 | 3.82e-04 | 0.5833 | 8.51e-04 |
| **1.15** | 0.0359 | 5.79e-04 | 0.0460* | 2.81e-01 | 0.0360 | 1.97e-03 | 0.0361 | 3.89e-03 |
| 1.5-year maturity | | | | | | | | |
| **0.85** | 22.8033 | 2.19e-02 | 22.6061* | 8.65e-03 | 22.8037 | 1.70e-05 | 22.8022 | 4.84e-05 |
| **0.95** | 12.597 | 1.77e-02 | 12.465* | 1.05e-02 | 12.5976 | 4.47e-05 | 12.5967 | 2.19e-05 |
| **1** | 8.8786 | 1.52e-02 | 8.8241* | 6.14e-03 | 8.8792 | 6.63e-05 | 8.8790 | 4.15e-05 |
| **1.05** | 6.0473 | 1.28e-02 | 6.0704 | 3.81e-03 | 6.0470 | 5.47e-05 | 6.0474 | 6.31e-06 |
| **1.15** | 2.5478 | 8.33e-03 | 2.6693* | 4.77e-02 | 2.5475 | 1.11e-04 | 2.5485 | 2.91e-04 |
| 3-year maturity | | | | | | | | |
| **0.85** | 110.1708 | 1.35e-01 | 106.8891* | 2.98e-02 | 110.182 | 1.02e-04 | 110.1632 | 6.92e-05 |
| **0.95** | 99.7516 | 1.33e-01 | 95.7420* | 4.02e-02 | 99.7590 | 7.41e-05 | 99.7385 | 1.32e-04 |
| **1** | 94.8429 | 1.31e-01 | 90.5429* | 4.53e-02 | 94.8574 | 1.53e-04 | 94.8363 | 6.91e-05 |
| **1.05** | 90.1511 | 1.30e-01 | 85.5986* | 5.05e-02 | 90.1621 | 1.22e-04 | 90.1408 | 1.14e-04 |
| **1.15** | 81.3707 | 1.26e-01 | 76.4686* | 6.02e-02 | 81.3813 | 1.30e-04 | 81.3601 | 1.30e-04 |

The Monte Carlo benchmarks are based on $10^6$ trajectories using moment matching as a variance reduction technique. The estimated standard deviations of the Monte Carlo prices are reported. The parameters used for the simulation are provided in Table 1. The maturity dates of the domestic and foreign bonds are respectively $T_1 = T + 90/360$ and $T_2 = T + 120/360$. Relative errors are computed as $e = \left| V^B(a) - \overline{V}^B \right| / \overline{V}^B$, where $V^B(a)$ and $\overline{V}^B$ are the option prices obtained respectively with the analytical approximation and the Monte Carlo simulation. The prices marked with a star (*) are significantly different from their Monte Carlo benchmark at a level of 5%.

Table 3 presents sensitivity analysis of the basket option price with respect to the basket volatility and option maturity, for in the money options, $\frac{K_B}{B_0} = 0.95$. The average level volatility corresponds to the values in Table 1, while high and low levels correspond respectively to an increase and a decrease of 50% in the volatility values given in Table 1.

Table 3: Sensitivity analysis with respect to the basket volatility and option maturity

| **Basket Volatility** | **Monte Carlo** | **Standard Deviation** | **Inverse Gamma Approximation** | | **Lognormal Approximation** | | **Johnson Approximation** | |
|---|---|---|---|---|---|---|---|---|
| | | | Price | Relative error | Price | Relative error | Price | Relative error |
| 0.75-year maturity | | | | | | | | |
| **Low** | 1.6122 | 2.63e-03 | 1.6132 | 5.76e-04 | 1.6124 | 7.90e-05 | 1.6124 | 8.36e-05 |
| **Average** | 3.1880 | 5.33e-03 | 3.1890 | 3.32e-04 | 3.1879 | 3.93e-05 | 3.1879 | 3.29e-05 |
| **High** | 4.5412 | 7.93e-03 | 4.5419 | 1.54e-04 | 4.5411 | 6.36e-06 | 4.5412 | 3.16e-06 |
| 1-year maturity | | | | | | | | |
| **Low** | 2.3602 | 3.59e-03 | 2.3594 | 3.62e-04 | 2.3602 | 1.59e-05 | 2.3602 | 1.72e-05 |
| **Average** | 4.3556 | 7.08e-03 | 4.3517 | 8.93e-04 | 4.3557 | 3.44e-05 | 4.3557 | 3.29e-05 |
| **High** | 6.0694 | 1.05e-02 | 6.0578 | 1.92e-03 | 6.0697 | 4.43e-05 | 6.0697 | 4.25e-05 |
| 1.5-year maturity | | | | | | | | |
| **Low** | 8.2211 | 9.43e-03 | 8.1845* | 4.45e-03 | 8.2214 | 3.72e-05 | 8.2211 | 4.92e-06 |
| **Average** | 12.5963 | 1.77e-02 | 12.4650* | 1.04e-02 | 12.5976 | 9.87e-05 | 12.5967 | 3.20e-05 |
| **High** | 16.9935 | 2.66e-02 | 16.6624* | 1.95e-02 | 16.9940 | 2.67e-05 | 16.9921 | 8.69e-05 |
| 3-year maturity | | | | | | | | |
| **Low** | 70.0721 | 5.75e-02 | 69.5164* | 7.93e-03 | 70.0736 | 2.17e-05 | 70.0701 | 2.73e-05 |
| **Average** | 99.7475 | 1.33e-01 | 95.7420* | 4.02e-02 | 99.7590 | 1.16e-04 | 99.7385 | 9.03e-05 |
| **High** | 149.5893 | 2.71e-01 | 138.7165* | 7.27e-02 | 141.3625* | 5.50e-02 | 149.5287 | 4.05e-04 |

The Monte Carlo benchmarks are based on $10^6$ trajectories using moment matching as a variance reduction technique. The estimated standard deviations of the Monte Carlo prices are reported. The parameters used for the simulation are provided in Table 1. The maturity dates of the domestic and foreign bonds are respectively $T_1 = T + 90/360$ and $T_2 = T + 120/360$ and the moneyness is $K_B/B_0 = 0.95$ Relative errors are computed as $e = \left| V^B(a) - \overline{V}^B \right| / \overline{V}^B$, where $V^B(a)$ and $\overline{V}^B$ are the option prices obtained respectively with the analytical approximation and the Monte Carlo simulation. The prices marked with a star (*) are significantly different from their Monte Carlo benchmark at a level of 5%.

The findings are similar to those in Table 2. The relative errors presented in Table 3 are very low with a magnitude between $10^{-6}$ and $10^{-2}$. For low volatility levels, Edgeworth-lognormal and Johnson approximate prices and Monte Carlo prices are very close. As for the inverse gamma approximation, our results show that it is much less accurate with relative errors between $10^{-4}$ and $10^{-2}$.

Notice that the accuracy of the three approximations decreases for longer maturities and higher volatilities, which supports the findings in Ju (2002).[7] Moreover, for all approximations, option prices increase with maturity and volatility.

The previous analysis is only local. Our findings depend on the set of parameters used and may change if we modify them. To confirm our conclusions, a more carefully designed numerical study where the parameters are randomly chosen is conducted. Following Broadie and Detemple (1996), the significant model parameters are chosen randomly from continuous or discrete uniform distributions. These uniform distributions are based on the estimation of the model parameters using real data. The correlation between the commodity and the exchange rate is expected to be positive, so we assume that $\rho_{12} \in [0.01, 0.55]$. Based on Schwartz (1997), the convenience yield long-run mean is positive for gold and the mean reversion parameter is small and less than 1; we assume thus that $\theta \in [0.05, 0.5]$ and $\kappa \in [0.05, 0.8]$. All paramater distributions used in the pricing are presented in Table 4.

Table 4: Parameters distributions

| | |
|---|---|
| Domestic risk free rate, $f(0,t) = f$ for any $t$ | U(0.02; 0.08) |
| Foreign risk free rate, $f^*(0,t) = f^*$ for any $t$ | U(0.02, 0.08) |
| Commodity drift, $\alpha_s$ | U(0.05, 0.35) |
| Exchange rate drift, $\alpha_c$ | 0.04 |
| Commodity volatility, $\sigma_s$ | U(0.1, 0.35) |
| Exchange rate volatility, $\sigma_c$ | U(0.02, 0.15) |
| Domestic instantaneous forward rate volatility, $\eta_{t,T_1} = \eta$ for any $t$ and $T_1$ | U(0.001, 0.06) |
| Foreign instantaneous forward rate volatility, $\eta^*_{t,T_2} = \eta^*$ for any $t$ and $T_2$ | U(0.001, 0.06) |
| Convenience yield volatility, $\sigma_\delta$ | U(0.1, 0.4) |
| Convenience yield mean reversion parameter, $\kappa$ | U(0.05, 0.8) |
| Convenience yield long-run mean, $\theta$ | U(0.05, 0.5) |
| Commodity and exchange rate correlation, $\rho_{1,2}$ | U(0.01, 0.55) |
| Commodity and domestic instantaneous forward rate correlation, $\rho_{1,3}$ | U(−0.5, 0.25) |
| Commodity and foreign instantaneous forward rate correlation, $\rho_{1,4}$ | U(−0.5, 0.25) |
| Exchange rate and domestic instantaneous forward rate correlation, $\rho_{2,3}$ | U(−0.35, 0.35) |
| Exchange rate and foreign instantaneous forward rate correlation, $\rho_{2,4}$ | U(−0.35, 0.35) |
| Foreign and domestic instantaneous forward rates correlation, $\rho_{3,4}$ | U(0.15, 0.9) |
| Option maturity (in years) | U{0.083; 0.25; 0.5; 0.75; 1; 1.5; 2; 3} |
| Moneyness | U{0.8; 0.9; 0.95; 1; 1.05; 1.1; 1.2} |
| Basket weights (commodity domestic and foreign zero-coupon bonds), | $\mathrm{U}\left\{\frac{1}{3}\ \frac{1}{3}\ \frac{1}{3}; \frac{1}{2}\ \frac{1}{4}\ \frac{1}{4}; \frac{1}{4}\ \frac{1}{2}\ \frac{1}{4}; \frac{1}{4}\ \frac{1}{4}\ \frac{1}{2}\right\}$ |
| Initial prices $S_0$, $C_0$ and $\delta_0$ are respectively | \$330; \$/CAD 0.65; 0.2 |

We compare the three previous analytical approximations and Monte Carlo simulation combined with the moment matching technique for 5000 random sets of parameters. Only 4740 sets of parameters provided positive definite correlation matrices. We also removed all basket options

[7] Ju (2002) proposes an analytical approximation to price Asian and basket options based on a Taylor expansion of the ratio of the characteristic function of the average of lognormal variables to that of the approximating lognormal random variable around zero volatility.

prices with a value lower than 5 cents. We obtained 4347 sets of parameters. First, we examine the accuracy of each approximation by calculating its root mean square error (RMSE). Second, we calculate the maximum relative error (MRE) for each approximation to examine the worst case scenario. More precisely, we define,

$$
\begin{aligned}
RMSE &= \sqrt{\frac{1}{n}\sum_{i=1}^{n}\left(\frac{V_i^B(a)-\overline{V}_i^B}{\overline{V}_i^B}\right)^2} \\
MRE &= \max_i \frac{\left|V_i^B(a)-\overline{V}_i^B\right|}{\overline{V}_i^B},
\end{aligned}
$$

where $n$ is the number of different sets of paramaters, that is, Monte Carlo prices are obtained with $1,000,000$ paths combined with the moment matching technique.

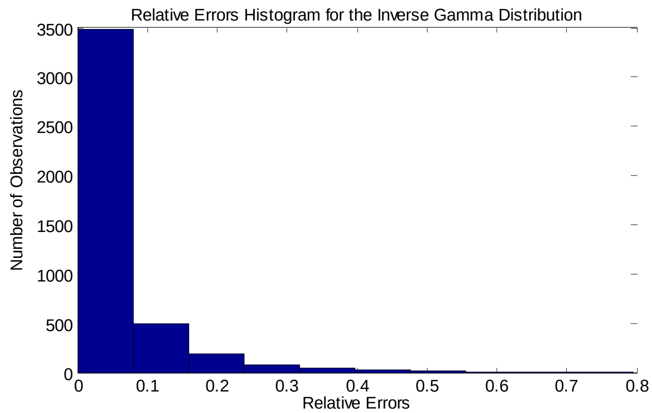
Table 5: RMSE and MRE for the three approximations

| | **Inverse Gamma Approximation** | **Lognormal Approximation** | **Johnson Approximation** |
|---|---|---|---|
| **RMSE** | 9.68% | 0.52% | 0.56% |
| **MRE** | 79.47% | 13.16% | 14.00% |

The results in Table 5 confirm those obtained with the first sensitivity analysis (Tables 2 and 3) and show that the Edgeworth-lognormal and Johnson approximations are much more accurate than the inverse gamma approximation. It is also found that, for the Edgeworth-lognormal and Johnson approximations, a very small proportion of options, 0.23% and 0.35% respectively, have relative errors above 5% while for the inverse gamma approximation, a larger proportion, 27.74%, of options have a relative error above 5%.

A more detailed look at the results shows that out-of-the-money and high volatility options have the largest relative errors, which confirms our findings in the first sensitivity analysis. However, since the out-of-the-money options have small prices, the augmentation of the relative errors in these cases may be attributed to the small denominators. Figures 1, 2 and 3 present respectively the histograms of the relative errors of the inverse gamma, the Edgeworth-lognormal and Johnson approximations. Table 6 shows a summary of descriptive statistics of the relative errors.

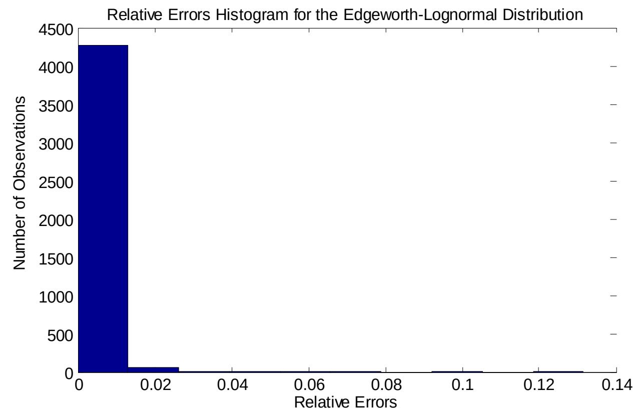
Figure 1: Histogram of relative errors of the inverse gamma approximation




X-axis represents relative errors

Y-axis represents the number of observations

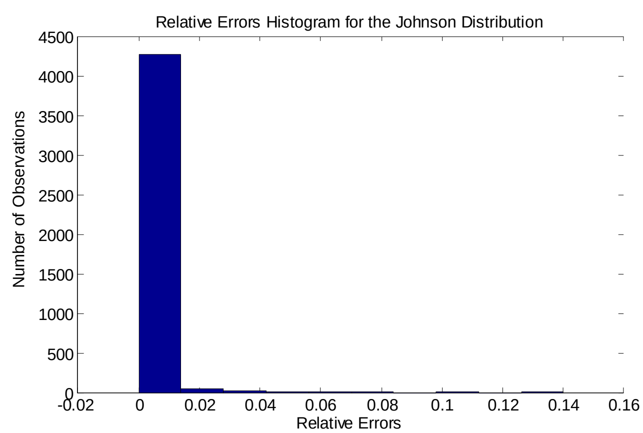
Figure 2: Histogram of relative errors of the Edgeworth approximation




X-axis represents relative errors

Y-axis represents the number of observations

Figure 3: Histogram of relative errors of Johnson approximation




X-axis represents relative errors

Y-axis represents the number of observations

Table 6: Descriptive statistics of relative errors

| | **Inverse Gamma Approximation** | **Lognormal Approximation** | **Johnson Approximation** |
|---|---|---|---|
| **Maximum** | 0.7947 | 0.1316 | 0.1400 |
| **Minimum** | 0 | 0 | 0 |
| **Mean** | 0.0493 | 0.0015 | 0.0014 |
| **Median** | 0.0136 | 0.0002 | 0.0001 |
| **Standard Deviation** | 0.0833 | 0.0050 | 0.0054 |
| **Nb of observations** | 4347 | 4347 | 4347 |

The histograms show that for the Edgeworth-lognormal and Johnson approximations, 99% of parameters sets (4300 out of 4347) have relative errors less than 2%. This demonstrates that these approximations are very accurate and that they give prices very close to Monte Carlo benchmarks. The inverse gamma approximation has 92% (4000 out of 4347) cases where the relative errors are less than 20%. We reach the same results by using larger and more general intervals for the parameters distributions.

To conclude this section regarding the pricing of heteregeneous basket options, we show that Edgeworth expansion around the lognormal distribution at order 4 and the Johnson distribution are equally accurate and very acceptable for practitionners. However, we suggest the use of the Edgeworth-lognormal distribution for two reasons: first, it is slightly more accurate, and second, the algorithm to calibrate Johnson distributions may not converge in a few cases, which can lead to mispriced options.

# 5 Hedging ratios

The analytical approximations allow for deriving the option price in a functional form which can also provide analytical expressions for the sensitivities with respect to the underlying parameters, such as deltas, vegas and theta, known as the hedging ratios or the *Greeks*. However, due to the complexity of the moments involved in our analytical approximations, deriving the hedging ratios analytically is beyond the scope of this article. Instead, we can compute them numerically as follows:

$$Ratio^B_{App} = \frac{V^{B(\varepsilon)}_{App} - V^B_{App}}{\varepsilon}, \tag{14}$$

where $\varepsilon > 0$ is a small number, $V^{B(\varepsilon)}_{App}$ and $V^B_{App}$ are respectively the approximate $\varepsilon$-disturbed and non-disturbed option prices. As an application, we compute numerically the delta with respect to the commodity price for different values of $\varepsilon$. We base our calculations on the parameters in Table 1. We consider an in-the-money call basket option $\left(\frac{K_B}{B_0} = 0.95\right)$ with a 9-month maturity. The deltas are presented in Table 7.

Table 7: Basket option Delta w.r.t. the commodity price

| **Commodity Price** | **Inverse Gamma Approximation** | | **Lognormal Approximation** | | **Johnson Approximation** | |
|---|---|---|---|---|---|---|
| | Price | Delta* | Price | Delta* | Price | Delta* |
| **$330** | 3.1890 | | 3.1879 | | 3.1879 | |
| **$330.33** | 3.1922 | 0.0097 | 3.1910 | 0.0094 | 3.1911 | 0.0097 |
| **$331.65** | 3.2049 | 0.0096 | 3.2037 | 0.0096 | 3.2038 | 0.0096 |
| **$333.30** | 3.2208 | 0.0096 | 3.2196 | 0.0096 | 3.2196 | 0.0096 |
| **$334.95** | 3.2367 | 0.0096 | 3.2355 | 0.0096 | 3.2355 | 0.0096 |

* Delta corresponds to the sensitivity with respect to the commodity price and is given by Equation (14). $V^B_{App}$, presented in the first row of the table, is the analytical basket option price obtained for a commodity price of $330.

Table 7 shows that, for the three approximations, the delta is very stable over different values of $\varepsilon$. Indeed, an increase of $1 in the commodity price leads to an increase of approximately $0.01 in the option price. The positiveness of delta is expected but its value depends on the set of parameters used. Following the same procedure, one can compute the other sensitivities with respect to other parameters of interest.

## 6 Conclusion

Firms can use basket options to hedge their exposure to different risks, such as commodity risk, interest rate risk and exchange rate risk. However, pricing this kind of options is not an easy task since no closed-form solution can be derived for the basket density function. Consequently, a standard pricing formula such as Black and Scholes cannot be derived. The main contribution of this article is the comparison of the performance of three analytical approximations to price a heteregeneous basket option, consisting of a commodity, a domestic and a foreign zero-coupon bonds, when interest rates are stochastic. The three approximations used are the inverse gamma proposed in Milevsky and Posner (1998a, 1998b), the Edgeworth expansion around the lognormal distribution as well as Johnson distribution developped in Posner and Milevsky (1999).

In order to assess and compare the accuracy of the approximations, we use two analyses. The first one is a local sensitivity analysis where the parameters of the model are fixed arbitrarily. Our findings show that both the Edgeworth-lognormal and Jonhson approximations are very accurate while the inverse gamma approximation is much less accurate.

These results are confirmed with the second analysis where 5000 sets of parameters are chosen randomly from different uniform distributions. It is also found that the pricing relative errors, computed with Monte Carlo benchmark prices, are small and of an acceptable magnitude for practitionners. The accuracy of all approximations deteriorates for out-of-the-money as well as for high volatility options.

This article considers only plain vanilla basket options. However, extending the approach followed here to more exotic basket options is a promising area for future research.

# Appendices

## A Derivation of the forward measure

In this appendix, we will derive the model under the $T-$forward measure. Let $\mathbf{A}_i$ be the $i^{th}$ row of the matrix $\mathbf{A}$, where $\mathbf{A} = [a_{ij}]_{i,j=\{1,2,3,4\}}$ is the Cholesky decomposition of the correlation matrix of the four-dimensional $P-$Brownian motions $\mathbf{W} = \left(W^{(1)}, W^{(2)}, W^{(3)}, W^{(4)}\right)'$, and $\widetilde{\mathbf{B}}$ corresponds to the vector of independent Brownian motions under $Q$. The SDEs satisfied by all underlying assets under $Q$ can be written as:[8]

$$\begin{aligned} dS_t &= S_t\left[(r_t - \delta_t)\, dt + \sigma_s \mathbf{A}_1 d\widetilde{\mathbf{B}}_t\right] && (15a)\\ d\delta_t &= (\kappa\theta - \frac{\sigma_\delta}{\sigma_s}(\alpha_s - r_t) - \kappa\delta_t)dt + \sigma_\delta \mathbf{A}_1 d\widetilde{\mathbf{B}}_t && (15b)\\ dC_t &= C_t\left[(r_t - r_t^*)\, dt + \sigma_c \mathbf{A}_2 d\widetilde{\mathbf{B}}_t\right] && (15c)\\ dP(t,T_1) &= P(t,T_1)\left[r_t dt - \beta_{t,T_1}\mathbf{A}_3 d\widetilde{\mathbf{B}}_t\right] && (15d)\\ dP^*(t,T_2) &= P^*(t,T_2)\left[\left(r_t^* + \beta^*_{t,T_2}\sigma_c\rho_{2,4}\right)\, dt - \beta^*_{t,T_2}\, \mathbf{A}_4 d\widetilde{\mathbf{B}}_t\right] && (15e) \end{aligned}$$

The risk-neutral measure $Q$ corresponds to a numeraire equal to the domestic bank account $\exp\left(\int_t^T r_u du\right)$, while the $T-$forward measure $Q_T$ corresponds to a numeraire equal to the domestic zero-coupon bond with maturity date $T$, $P(t,T)$. Using Girsanov theorem, we have that:

$$d\widehat{\mathbf{B}}_t = d\widetilde{\mathbf{B}}_t + \beta_{t,T}\mathbf{A}_3' dt,$$

where $\widehat{\mathbf{B}}$ are independent Brownian motions under $Q_T$. The correlation structure is the same under $Q$ and $Q_T$, and the previous SDEs can be rewritten as:

$$\begin{aligned} dS_t &= S_t\left[\left(r_t - \delta_t - \rho_{13}\sigma_s\beta_{t,T}\right) dt + \sigma_s d\widehat{W}_t^{(1)}\right] && (16a)\\ d\delta_t &= (\kappa\theta - \frac{\sigma_\delta}{\sigma_s}(\alpha_s - r_t) - \kappa\delta_t - \rho_{13}\sigma_\delta\beta_{t,T})dt + \sigma_\delta d\widehat{W}_t^{(1)} && (16b)\\ dC_t &= C_t\left[\left(r_t - r_t^* - \rho_{23}\sigma_c\beta_{t,T}\right) dt + \sigma_c d\widehat{W}_t^{(2)}\right] && (16c)\\ dP(t,T_1) &= P(t,T_1)\left[\left(r_t + \beta_{t,T_1}\beta_{t,T}\right) dt - \beta_{t,T_1} d\widehat{W}_t^{(3)}\right] && (16d)\\ dP^*(t,T_2) &= P^*(t,T_2)\left[\left(r_t^* + \beta^*_{t,T_2}\sigma_c\rho_{2,4} + \rho_{3,4}\beta_{t,T}\beta^*_{t,T_2}\ \right)\ dt - \beta^*_{t,T_2}\ d\widehat{W}_t^{(4)}\right] && (16e) \end{aligned}$$

where $\widehat{\mathbf{W}} = \left(\widehat{W}^{(1)},\ \widehat{W}^{(2)},\ \widehat{W}^{(3)},\ \widehat{W}^{(4)}\right)' = \mathbf{A}\widehat{\mathbf{B}}$. The strong solution exists and is given by the system of equations (3).

## B Derivation of moments

This appendix gives the detailed calculation of the first four moments of the basket value at maturity $T$ under the $T-$forward measure $Q_T$. Using the strong solution of the model under the forward-measure $Q_T$,

[8] The details of the computation are available from the authors upon request.

one can write

$$
\begin{aligned}
S_T &= S_0 \exp\left(\phi_S(T) + \sum_{i=1}^{4} \int_0^T \psi_{S,i}(t)\ d\widehat{W}_t^{(i)}\right) \\
P(T,T_1) &= \exp\left(\phi_P(T) + \sum_{i=1}^{4} \int_0^T \psi_{P,i}(t)\ d\widehat{W}_t^{(i)}\right) \\
C_T P^*(T,T_2) &= C_0 \exp\left(\phi_{CP^*}(T) + \sum_{i=1}^{4} \int_0^T \psi_{CP^*,i}(t)\ d\widehat{W}_t^{(i)}\right),
\end{aligned}
$$

where

$$
\begin{aligned}
\phi_S(T) &= \left[\begin{array}{l}
\int_0^T \left(1 - \frac{\sigma_\delta}{\sigma_s \kappa}\left(1 - e^{-\kappa(T-u)}\right)\right) f(0,u)\ du - \delta_0 \frac{1}{\kappa}\left(1 - e^{-\kappa T}\right) \\
-\left(\theta - \frac{\sigma_\delta}{\sigma_s}\frac{\alpha_s}{\kappa} + \frac{\sigma_s^2}{2}\right) T + \left(\theta - \frac{\sigma_\delta}{\sigma_s}\frac{\alpha_s}{\kappa}\right)\frac{1}{\kappa}\left(1 - e^{-\kappa T}\right) \\
-\frac{1}{2}\int_0^T \beta_{u,T}^2 du + \frac{\sigma_\delta}{\sigma_s \kappa}\int_0^T \left(1 - e^{-\kappa(T-u)}\right)\left(\int_0^u \eta_{v,u}\left(\beta_{v,T} - \beta_{v,u}\right)\ dv\right) du \\
+\rho_{1,3}\int_0^T \beta_{u,T}\left(\frac{\sigma_\delta}{\kappa}\left(1 - e^{-\kappa(T-u)}\right) - \sigma_s\right)\ du
\end{array}\right], \\
\phi_P(T) &= -\int_T^{T_1} f(0,u)\,du - \frac{1}{2}\int_0^T \left(\beta_{u,T_1} - \beta_{u,T}\right)^2 du, \\
\phi_{CP^*}(T) &= \int_0^T f(0,u)\ du - \int_0^{T_2} f^*(0,u)\ du - \frac{1}{2}\int_0^T \left(\beta_{u,T}^2 + \left(\beta_{u,T_2}^*\right)^2\right)\ du - \frac{\sigma_c^2}{2} T \\
&\quad -\sigma_c \rho_{2,3}\int_0^T \beta_{u,T}\ du + \sigma_c \rho_{2,4}\int_0^T \beta_{u,T_2}^* du + \rho_{3,4}\int_0^T \beta_{u,T_2}^* \beta_{u,T}\ du, \\
\psi_{S,1}(t) &= \sigma_s - \frac{\sigma_\delta}{\kappa}\left(1 - e^{-\kappa(T-t)}\right), \\
\psi_{S,3}(t) &= \beta_{t,T} - \frac{\sigma_\delta}{\sigma_s \kappa}\int_t^T \left(1 - e^{-\kappa(T-v)}\right)\eta_{t,v}\ dv, \\
\psi_{P,3}(t) &= \beta_{t,T} - \beta_{t,T_1}, \\
\psi_{CP^*,2}(t) &= \sigma_c, \\
\psi_{CP^*,3}(t) &= \beta_{t,T}, \\
\psi_{CP^*,4}(t) &= -\beta_{t,T_2}^*, \\
\psi_{S,2}(t) &= \psi_{S,4}(t) = \psi_{P,1}(t) = \psi_{P,2}(t) = \psi_{P,4}(t) = \psi_{CP^*,1}(t) = 0,
\end{aligned}
$$

Therefore,

$$
\mu_n' = \sum_{k=0}^{n}\sum_{j=0}^{k} \frac{n!}{j!\,(k-j)!\,(n-k)!} w_1^j w_2^{(k-j)} w_3^{(n-k)} \mathrm{E}^{Q_T}\left[S_T^j \left(P(T,T_1)\right)^{k-j} C_T^{n-k}\left(P^*(T,T_2)\right)^{n-k}\right]
$$

with

$$
\begin{aligned}
&\mathrm{E}^{Q_T}\left[S_T^j \left(P(T,T_1)\right)^{k-j} C_T^{n-k}\left(P^*(T,T_2)\right)^{n-k}\right] \\
&= S_0^j C_0^{n-k}\left(P^*(0,T_2)\right)^{n-k} \mathrm{E}^{Q_T}\left[\exp\left(\phi(T) + \sum_{i=1}^{4}\int_0^T \psi_i(t)\ d\widehat{W}_t^{(i)}\right)\right] \\
&= S_0^j C_0^{n-k}\left(P^*(0,T_2)\right)^{n-k} \exp\left(\phi(T) + \frac{1}{2}\sum_{i=1}^{4}\sum_{\ell=1}^{4}\rho_{i,\ell}\int_0^T \psi_i(t)\,\psi_\ell(t)\ dt\right)
\end{aligned}
$$

where

$$
\begin{aligned}
\phi(T) &= j\phi_S(T) + (k-j)\,\phi_P(T) + (n-k)\,\phi_{P^*}(T) \\
\psi_i(t) &= j\psi_{S.i}(t) + (k-j)\,\psi_{P.i}(t) + (n-k)\,\psi_{P^*.i}(t),\ i \in \{1,2,3,4\}.
\end{aligned}
$$

Table 8 presents the theoretical first four moments, calculated as explained previously, and those obtained by Monte Carlo simulation. This ensures that our theoretical formulas give the exact moments.

Table 8: Comparison of theoretical and simulated moments

| **Order** | **Theoretical Moments** | **Simulated Moments** |
|---|---|---|
| **1** | 157.21 | 157.20 |
| **2** | 24807 | 24807 |
| **3** | 3.9293e+6 | 3.9292e+6 |
| **4** | 6.2472e+8 | 6.2470e+8 |
| **Mean** | 157.21 | 157.20 |
| **Variance** | 93.13 | 93.13 |

# C Inverse gamma approximation

This appendix shows how we obtain the pricing formula using the inverse gamma approximation. The density function of a gamma random variable $X$ of parameters $(\alpha, \beta)$, $X \sim G(\alpha, \beta)$, is given by:

$$g(x\,|\alpha, \beta) = \frac{x^{\alpha-1}\exp\left(-\frac{x}{\beta}\right)}{\beta^{\alpha}\Gamma(\alpha)}, \quad x \geq 0$$

where $\alpha > 0$, $\beta > 0$ and $\Gamma(\alpha)$ is the Gamma function.

**Proposition 1** *Let $X$ be a gamma random variable with parameters $(\alpha, \beta)$. Then, the random variable $Y = \frac{1}{X}$, follows an inverse gamma distribution, $Y \sim G_R(\alpha, \beta)$, and its density function is given by:*

$$g_R(y\,|\alpha, \beta) = \frac{1}{y^2} g\left(\frac{1}{y}\,|\alpha, \beta\right) = \frac{\exp\left(-\frac{1}{y\beta}\right)}{y^{\alpha+1}\beta^{\alpha}\Gamma(\alpha)}, \quad y > 0, \quad \alpha, \beta > 0. \tag{17}$$

**Proposition 2** *The non-centered moments of the random variable $Y \sim G_R(\alpha, \beta)$ are given by:*

$$\mu_n'(g_R) = \frac{1}{\beta^n(\alpha-1)(\alpha-2)\ldots(\alpha-n)}, \quad n = 1, 2, 3, \ldots \tag{18}$$

We price the basket option by approximating the sum of lognormal variables by an inverse gamma distribution. We match the two first moments

$$\mu_1'(v) = \mu_1'(g_R) = \frac{1}{\beta(\alpha-1)} \text{ and } \mu_2'(v) = \mu_2'(g_R) = \frac{1}{\beta^2(\alpha-1)(\alpha-2)},$$

to get the two parameters of the inverse gamma density given at line (7). Using the inverse gamma density, the option price $P(t,T)\int_{-\infty}^{+\infty}\max(x-K_B,0)\,v(x)\,dx$ is approximated by:

$$\begin{aligned}
V_{gamma}^{B} &= P(t,T)\int_{K_B}^{+\infty}(x-K_B)\,g_R(x|\,\alpha,\beta)\,dx \\
&= P(t,T)\int_{K_B}^{+\infty}(x-K_B)\,x^{-2}g\left(\left.\frac{1}{x}\right|\alpha,\beta\right)dx \\
&= P(t,T)\int_{0}^{\frac{1}{K_B}}\left(\frac{1}{y}-K_B\right)g(y|\,\alpha,\beta)\,dy \;\left(\text{from the change of variable } y=\frac{1}{x}\right) \\
&= P(t,T)\left(\frac{1}{\beta(\alpha-1)}\int_{0}^{\frac{1}{K_B}}g(y|\,\alpha-1,\beta)\,dy-K_B\int_{0}^{\frac{1}{K_B}}g(y|\,\alpha,\beta)\,dy\right) \\
&\quad\text{since } \frac{1}{y}g(y|\,\alpha,\beta)=\frac{1}{\beta(\alpha-1)}g(y|\,\alpha-1,\beta) \\
&= P(t,T)\left(\frac{1}{\beta(\alpha-1)}\left(\mathrm{G}\left(\left.\frac{1}{K_B}\right|\alpha-1,\beta\right)\right)-K_B\left(\mathrm{G}\left(\left.\frac{1}{K_B}\right|\alpha,\beta\right)\right)\right)
\end{aligned}$$

leading to Equation (6). □

# D Edgeworth-lognormal expansion

This appendix shows how we obtain the pricing formula using an Edgeworth expansion around the lognormal distribution. Matching the first two moments, the lognormal density used is given by

$$a(x)=\frac{1}{\sqrt{2\pi}}\frac{1}{\beta}\frac{1}{x}\exp\left(-\frac{1}{2}\left(\frac{\ln x-\alpha}{\beta}\right)^2\right) \tag{19}$$

where $\alpha$ and $\beta$ are defined at lines (8e) and (8f) respectively. Following Jarrow and Rudd (1982), the unkown basket density function can be approximated by:

$$\begin{aligned}
v(x) &= a(x)+\frac{\kappa_2(v)-\kappa_2(a)}{2!}\frac{d^2a(x)}{dx^2}-\frac{\kappa_3(v)-\kappa_3(a)}{3!}\frac{d^3a(K)}{dx^3} \\
&\quad+\frac{(\kappa_4(v)-\kappa_4(a))+3(\kappa_2(v)-\kappa_2(a))^2}{4!}\frac{d^4a(x)}{dx^4}+\xi(x),
\end{aligned} \tag{20}$$

where $\xi(x)$ is an error term and $\kappa_i(h)$, $i=\{1,2,3,4\}$ are the first four cumulants of the density function $h=\{v,a\}$ defined by the system of Equations (5). Jarrow and Rudd (1982) state that, in general, there is no bound on the error term resulting from an Edgeworth expansion. Consequently, the error does not necessarily decrease with the expansion's order. Given that the two first moments are the same for the true density and for the approximated density, Equation (20) becomes

$$v(x)=a(x)-\frac{\kappa_3(v)-\kappa_3(a)}{3!}\frac{d^3a(K)}{dx^3}+\frac{\kappa_4(v)-\kappa_4(a)}{4!}\frac{d^4a(x)}{dx^4}+\xi(x). \tag{21}$$

The basket option price $P(t,T)\int_{-\infty}^{+\infty}\max(x-K_B,0)\,v(x)\,dx$ can thus be approximated by:

$$V_{\log normal}^{B}=P(t,T)\int_{K_B}^{+\infty}(x-K_B)\left(a(x)-\frac{\kappa_3(v)-\kappa_3(a)}{3!}\frac{d^3a(x)}{dx^3}+\frac{\kappa_4(v)-\kappa_4(a)}{4!}\frac{d^4a(x)}{dx^4}\right)dx.$$

Using the fact that

$$\int_{K_B}^{+\infty}(x-K_B)\frac{d^ja}{dx^j}(x)\,dx=\frac{d^{j-2}a}{dx_B^{j-2}}(K_B)\quad\text{for}\quad j\geq 2,$$

we obtain

$$V^B_{\log normal} = P(t,T)\left[\int_{K_B}^{+\infty}(x-K_B)\,a(x)\,dx - \frac{\kappa_3(v)-\kappa_3(a)}{3!}\frac{da}{dx}(K_B) + \frac{\kappa_4(v)-\kappa_4(a)}{4!}\frac{d^2a}{dx^2}(K_B)\right]. \quad (22)$$

Notice that the first integral in Equation (22) is very similar to a Black and Scholes price.

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