Bounds and Prices of Currency Cross-Rate Options

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Keywords: Option pricing, option bounds, exchange rates, cross-rate, correlation, copulas.
JEL Classification: F3, F4, G1
This version: February 15, 2006

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We would like to thank the seminar participants at National Chengchi University, National Chiao Tung University, National Taiwan University, National Tsing Hua University, and Yuan Ze University for helpful comments. The authors thank the National Science Council of Taiwan for financial support. Any comments please address them to Wang at ywang@cc.ncu.edu.tw.

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I. Introduction

In the option pricing literature, researchers are not only interested in pricing but also interested in bounding the option values. There are many useful techniques that can be employed to derive option pricing bounds. For example, Merton (1973), Garman (1976), Levy (1985), and Grundy (1991) use the arbitrage-free approach to derive option pricing bounds. The fundamental idea behind this approach is that it is not possible to formulate a dominant portfolio using the underlying stock, the risk-free bond, and the options if the market is absent of arbitrage opportunities. Ritchken (1985), Ritchken and Kuo (1989), Basso and Pianco (1997), Mathur and Ritchken (2000), and Ryan (2003) use the linear programming methods to derive option pricing bounds. These studies model option pricing bounds as a linear programming problem with a discrete state space, which involves complicated calculations. In addition to the above two types of techniques, some other approaches, such as the optimization methods[1] and the restrictions on the volatility of the pricing kernel,[2] have also been used in the literature.

Most if not all of the previous studies derive option pricing bounds by directly using the price information (such as the price distribution or price process) of the underlying asset. In contrast to the previous literature, this study uses the prices of the related dollar-rate options to derive the pricing bounds for the cross-rate option. In other words, we bind cross-rate option values using the prices of the dollar-rate options.[3] From this sense, the idea of this paper is close to that in the static hedge litera-

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[1] See Boyle and Lin (1997) and Bertsimus and Popescu (2002) for the applications of the optimization methods.
[2] Please see Cochrane and Saa-Requejo (2000) for details.
[3] The motivation for doing this is as follows. It is generally observed that options on dollar-denominated exchange rates are traded under satisfactory liquidity, while cross-rate option markets are

ture where the exotic options are priced (and hedged) in terms of the prices of standard options.[4] Our option pricing bounds are very general and do not rely on the distribution assumptions of the state variables. Moreover, our pricing bounds have economic meanings, because they are portfolios composed of the dollar-rate options (and sometimes also composed of spot dollar-rates).

Since a cross-rate option under the dollar measure is equivalent to an option that allows the buyer to exchange one asset (one dollar-rate) for the other asset (the other dollar-rate), this study derives the price bounds for cross-rate options by utilizing the exchange option price bounds implied in the copula theory.[5] Nonetheless, the bounds do not rely on the selection of the copula function. Using the prices of options on foreign exchange rates among the US dollar, euro, and pound sterling, we empirically test the relationship between the market prices and the estimated bounds of the cross-rate options. We estimate a time-series regression model where the market prices of one-month cross-rate (€/£) options are regressed on the estimated bounds. The average adjusted R^2 of our regressions across all cross-rate options is as high as 0.8341. Moreover, the regression coefficients of both the upper bound and the lower bound are also highly significant. In other words, our results suggest that there are strong and stable relationships between the market prices of cross-rate options (particularly for the deep-in-the-money and deep-out-of-the-money options) and the pricing bounds obtained from the market prices of the two dollar-rate options.

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much less liquid. Thus, the pricing bounds obtained from the liquid market prices of dollar-rate options are useful for pricing, hedging, and arbitraging.

[4] See Carr, Ellis, and Gupta (1998) for an example of static hedge.

[5] The details of the copula theory can be found in Joe (1997) and Nelsen (1999). Cherubini, Luciano, and Vechiato (2004) first applied the copula theory to derive the pricing bounds for the exchange options.

Because the correlation between two risky assets of an exchange option affects the pricing of this option, we expect that the correlation between two dollar rates, whose information is not included in our bounds, has the explanatory power for the market prices of the corresponding cross-rate options. Our empirical results show that the adjusted R^2 increases substantially when the historical correlation is added into the regression model.[6] In particular, the correlation between two dollar rates effectively improves the explanatory power for the at-the-money cross-rate option prices, which have the lowest adjusted R^2 in the previous regression model.

We also infer the cross-rate option prices from our pricing bounds and the historical correlation using the estimated regression model. The inferred prices are very close to the market prices of the cross-rate options across deltas. Particularly, the pricing errors for the deep-in-the-money and deep-out-of-the-money options are relatively small (smaller than 0.13%). Therefore, this study provides an accurate pricing model in comparison to the traditional option pricing methods which use the underlying asset price information only.

Since there is a triangular relationship between the foreign exchange rates among three currencies, Taylor and Wang (2005) show that it is plausible to estimate risk-neutral densities and option prices of a cross-rate under the correct numeraire using the market prices of two related dollar-rate options.[7] Instead of directly exploring the option pricing formula as in Taylor and Wang (2005), this paper provides pricing

[6] Our results are in line with the analysis of Driessen, Maenhout, and Vilkov (2005) who find that the risk of changes in equity correlations is priced, using data on S&P 100 options and options on all the stocks in the index.

[7] Taylor and Wang (2005) first establish the theoretical relationship between the risk-neutral density (RND) of the cross-rate and the bivariate RND of two related dollar rates. They then estimate the bivariate RND using the marginal RNDs for the two dollar rates and a copula function where the marginal RNDs are obtained from the market prices of the dollar-rate options. Taylor and Wang (2005) demonstrate that it is effective to use the price information of dollar-rates options to price cross-rate options when historical or implied correlations are available.

bounds for the cross-rate options. In compassion to Taylor and Wang (2005), our option pricing bounds (and the inferred option prices) are of economic meanings, because they are portfolios of the related dollar-rate options.[8] Moreover, our pricing bounds are also effective for inferring the prices of cross-rate options.

The remainder of this paper is organized as follows. Section II derives option pricing bounds for the cross-rate options, using the exchange option pricing bounds implied from the copula theory. Data and methodologies for generating the risk-neutral densities and option pricing bounds are presented in Section III. Section IV discusses the empirical results while Section V concludes the paper.

II. Pricing Bounds of the Cross-Rate Options

By applying the copula theory, Cherubini, Luciano, and Vechiato (2004) show that the super-replication bounds of the option to exchange one asset for the other asset are composed of the prices of the univariate options on the two individual exchanged assets. To utilize their results, we first show that the payoff of a cross-rate option under the dollar measure is equivalent to that of an exchange option where the two risky assets are the corresponding dollar rates. We then verify that our pricing bounds for the cross-rate option are portfolios composed of the dollar-rate options (and sometimes also composed of spot dollar-rates).

Consider options whose payoffs depend on the exchange rates among the following three currencies: US dollars ($, USD), British pounds (£, GBP), and euros (€, EUR). We denote the dollar price of one pound at time *t* by $S_t^{\$/£}$ and likewise the dollar price of one euro at the same time is denoted by $S_t^{\$/€}$. The cross-rate price of one pound in euros is then given by $S_t^{€/£} = S_t^{\$/£} \big/ S_t^{\$/€}$ under the no-arbitrage argument.

[8] The choice of copula functions in Taylor and Wang (2005) is arbitrary and lacks economic meaning.

Now consider a European call option where the holder has the right to buy £1 for €K at time T. Under the dollar measure (or from the viewpoint of U.S. residents), the above option is identical to an option to exchange $KS_T^{\$/€}$ dollars for $S_T^{\$/£}$ dollars at time T. Hence, a cross-rate call option under the dollar numeraire is equivalent to an option to exchange one asset for the other asset and its dollar payoff equals $\max(S_T^{\$/£} - KS_T^{\$/€}, 0)$. This payoff can be re-arranged as a function of the payoff of an option on the minimum of two risky asset prices ($S_T^{\$/£}$ and $KS_T^{\$/€}$), with strike price 0, as the following:

(1) $$S_T^{\$/£} - \max[\min(S_T^{\$/£}, KS_T^{\$/€}),0].$$

Hence, the current price of an exchange option is determined as follows:

(2) $$Call_{\$}^{€/£} = S_t^{\$/£} - Call_{\min}(S^{\$/£}, KS^{\$/€}, 0, t, T),$$

where $Call_{\min}(S_1, S_2, 0, t, T)$ represents the price at time t of an option on the minimum of S_1 and S_2 with strike price 0 and maturity time T.

Let Pr denote the probability, $F_i(x)$ the cumulative distribution function, and r the dollar risk-free interest rate. With probability distribution techniques, the price of an option on the minimum of two risky assets can be expressed as:[9]

(3) $$\begin{aligned} Call_{\min}(S^{\$/£}, KS^{\$/€}, 0, t, T) &= e^{-r(T-t)} \int_0^{\infty} \Pr(\min(S^{\$/£}, KS^{\$/€}) > x) dx \\ &= e^{-r(T-t)} \int_0^{\infty} \Pr(S^{\$/£} > x, KS^{\$/€} > x) dx \\ &= e^{-r(T-t)} \int_0^{\infty} \overline{C}(\overline{F}_{S^{\$/£}}(x), \overline{F}_{KS^{\$/€}}(x)) dx, \end{aligned}$$

[9] According to Breeden and Litzenberger (1978), $f(x) = e^{r(T-t)} \frac{\partial^2 C(x)}{\partial x^2}$ where $f(.)$ is the risk-neutral density of the underlying asset price. Thus, we have $C = e^{-r(T-t)} \int_K^{\infty} (1 - F(x)) dx = e^{-r(T-t)} \int_K^{\infty} \Pr(S > x) dx$ for a univariate option. Similarly, one can derive that $C_{\min}(S_1, S_2, 0, t, T) = e^{-r(T-t)} \int_0^{\infty} \Pr(\min(S_1, S_2) > x) dx$.

where $\overline{C}$ is a survival copula[10] and $\overline{F_i}(x) = 1 - F_i(x)$.

According to the Fréchet bounds in the copula theory, it is true that $\max(u+v-1,0) \le \overline{C}(u,v) \le \min(u,v)$ since $\overline{C}(u,v)$ is a copula. Consequently, the upper and lower bounds of the minimum option are given as the following, respectively:

(4)
$$\begin{aligned} Call^{+}_{\min}(S^{\$/£}, KS^{\$/€}, 0, t, T) &= e^{-r(T-t)} \int_0^\infty \min(\overline{F}_{S^{\$/£}}(x), \overline{F}_{KS^{\$/€}}(x))dx, \\ Call^{-}_{\min}(S^{\$/£}, KS^{\$/€}, 0, t, T) &= e^{-r(T-t)} \int_0^\infty \max(\overline{F}_{S^{\$/£}}(x) + \overline{F}_{KS^{\$/€}}(x) - 1, 0)dx. \end{aligned}$$

In conjunction with equation (2) and the put-call parity, we obtain the upper bound of the cross-rate option price ($Call^{€/£+}_{\$}$) as follows (the details of derivation are shown in Appendix A):

(5)
$$Call^{€/£+}_{\$} = Call(S^{\$/£}, K^{**}, t, T) + KPut(S^{\$/€}, K'', t, T),$$

where K^{**} is a constant satisfying that $\overline{F}_{S^{\$/£}}(K^{**}) + \overline{F}_{KS^{\$/€}}(K^{**}) = 1$, and

$K'' = K^{**}/K$. Similarly, we define another constant K^{*} which solves

$\overline{F}_{S^{\$/£}}(K^{*}) = \overline{F}_{KS^{\$/€}}(K^{*})$. If $\overline{F}_{S^{\$/£}}(K^{*}) < \overline{F}_{KS^{\$/€}}(K^{*})$ for $u < K^{*}$, then it is shown in Appendix that the lower bound of the cross-rate option price ($Call^{€/£-}_{\$}$) is as follows:

(6)
$$Call^{€/£-}_{\$} = Call(S^{\$/£}, K^{*}, t, T) - K\,Call(S^{\$/€}, K', t, T),$$

where $K' = K^{*}/K$, otherwise if $\overline{F}_{S^{\$/£}}(K^{*}) > \overline{F}_{KS^{\$/€}}(K^{*})$ for $u < K^{*}$, then we derive that:

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[10] If two uniform variables U and V are jointed with a copula function *C*, then the joint probability that U and V are greater than u and v, respectively, is given by a survival function:
$\tilde{C} = \Pr(U > u, V > v) = 1 - u - v + C(u,v) = \overline{C}(1-u, 1-v)$.

$$(7) \qquad Call_{\$}^{€/£-} = S_t^{\$/£} - KS_t^{\$/€} + KCall(S^{\$/€}, K^{'}, t, T) - Call(S^{\$/£}, K^{*}, t, T).$$

Because equations (6) and (7) do not guarantee that the lower bound is positive, it is necessary to constrain that $Call_{\$}^{€/£-} \geq 0$.

From equations (5) to (7), we observe that our pricing bounds for cross-rate options are portfolios of the corresponding dollar-rate options (and may be also of the spot assets). Therefore, different from most option pricing bounds in the literature, the derived pricing bounds have economic meanings.

The derivation of our cross-rate option pricing bounds do not rely on the selection of an appropriate copula function, but on the Fréchet bounds which represents a necessary relationship between any two individual distributions and their joint distribution. Therefore, we can apply the technique utilized here to derive the price bounds for any European-style derivatives whose payoffs can be rearranged as the same type as that of an exchange option. Some examples have been given in Cherubini et al. (2004)[11]. As a demonstration, we provide another example, the price bounds of quanto options, in Appendix B.

III. Data and Empirical Methodologies

A. Data

The primary data used in this article are option prices quoted as Black-Scholes implied volatilities for three currency options (\$/£, \$/€, and €/£). Some settlement prices are available for cross-rate options traded in the Chicago Mercantile Exchange, but they correspond to almost no trading volume. Consequently, we rely on over-the-counter (OTC) option prices, with which we have the same time-to-maturity option

[11] These examples include the bivariate digital option, the minimum option and the exchange option.

data every day. Such prices are not in the public domain to the best of our knowledge. We make use of a confidential file of OTC option price mid-quotes, supplied by the trading desk of an investment bank. Our currency option data cover the period from 15 March 1999 to 11 January 2001. The OTC quotes are for all three foreign exchange options, recorded at the end of the day in London. The data include option prices for seven exercise prices, based upon "deltas" equal to 0.1, 0.25, 0.37, 0.5, 0.63, 0.75, and 0.9. The maturity of the options is one month, with which options in the OTC market are most frequently traded.

The summary statistics of the quoted implied volatilities are shown in Table 1 and the patterns of average implied volatilities across deltas are shown in Figure 1. All implied volatility functions exhibit a smile shape with the level for the $/€ options being the highest while the level for the $/£ options being the lowest. The low standard deviations imply that the levels of implied volatilities for these three exchange rate options do not change much as time goes.

We also use the spot exchange rates of $/£, $/€, and €/£ and the euro-currency interest rates (proxies of risk-free rates) of $, £, and € recorded by DataStream as the inputs of all relevant calculations.

B. Empirical Methodologies for Generating the Bounds

Because K^* and K^{**} are determined by the risk-neutral densities of two dollar rates, we use the observed market prices of European call options on $/£ and $/€ and a parametric distribution specification to estimate their risk-neutral densities. After the risk-neutral densities, K^*, and K^{**} are obtained, we are able to price dollar rate options with all strikes and to get pricing bounds of cross-rate options using equations (5) to (7). The details of the empirical procedures are described as the following.

At time t we assume there is a complete market for two dollar-rate ($/£ and $/€) European call options, priced in US dollars and expiring at time T. This implies the existence of a unique risk-neutral density (RND) for $S_T^{\$/£}$ under the US dollar measure. The RND for $S_T^{\$/£}$ is denoted as $f_\$(S_T^{\$/£})$, where the dollar subscript emphasizes that the numeraire of the option's payoffs is US dollars. Likewise, the RND for $S_T^{\$/€}$ is denoted as $f_\$(S_T^{\$/€})$.

In the first step we use the observed market prices of options on two dollar-rates, $/£ and $/€, to estimate $f_\$(S_T^{\$/£})$ and $f_\$(S_T^{\$/€})$. Many types of univariate RNDs have been proposed, including lognormal mixtures (Ritchey (1990) and Melick and Thomas (1997)), generalized beta densities (Bookstaber and MacDonald (1987)), multi-parameter discrete distributions (Jackwerth and Rubinstein (1996)), and densities derived from fitting spline functions to implied volatilities44 (Bliss and Panigirtzoglou (2002)). Providing that options are traded for a range of exercise prices that encompass most area of the risk-neutral distribution, it is documented that several flexible density families provide similar empirical estimates. In this paper we use the generalized beta density of the second kind (GB2) to estimate the RNDs of two dollar rates. The GB2 density has few parameters, but preserves many desirable properties: general levels of skewness and kurtosis are allowed, the shapes of the tails are fat relative to the lognormal density, and there are analytic formulae for the density, its moments, and the prices of options. Furthermore, the parameter estimation of the GB2 density is easy and does not involve any subjective choices occurred to non-parametric approaches, and the estimated densities are never negative. The details of the estimation of the GB2 density are shown in Appendix C.

Once $f_{\$}(S_T^{\$/£})$ and $f_{\$}(S_T^{\$/€})$ are obtained, K^* and K^{**} can be easily calculated with a numerical method (such as the Newton-Raphson method) to solve $\overline{F}_{S^{\$/£}}(K^*) = \overline{F}_{KS^{\$/€}}(K^*)$ and $\overline{F}_{S^{\$/£}}(K^{**}) + \overline{F}_{KS^{\$/€}}(K^{**}) = 1$, respectively. We then utilize $f_{\$}(S_T^{\$/£})$ and $f_{\$}(S_T^{\$/€})$ to estimate the prices of dollar rate options with strikes K^* and K^{**} numerically, e.g. $C_{\$/£}(K^*) = e^{-r_{\$}T}\int_{S_T^{\$/£}=K^*}^{\infty}[x-K^*]f_{\$}(S_T^{\$/£})dS_T^{\$/£}$. Consequently, the pricing bounds of cross-rate ($S^{€/£}$) options are generated using equations (5) to (7).

IV. Empirical Results

The empirical tests in this article contain four parts. We first analyze the properties of our pricing bounds and their relationships with the market prices of the cross-rate options. Next, we investigate the explanatory powers of the pricing bounds and the correlation between two dollar rates for the market prices of the cross-rate options. Then, this study examines the accuracy of our empirical models for pricing cross-rate options. Finally, some robust checks for the accuracy of our results are provided.

A. Empirical Pricing Bounds of the Cross-Rate Options

In order to have a standardized comparison, all the market prices and pricing bounds are converted to the implied volatilities of the Black-Scholes model. The estimated pricing bounds and the market implied volatilities across deltas are shown in Figure 2 and their descriptive statistics are shown in Table 2.

As shown in Figure 2, the evolution of market implied volatility of the cross-rate (€/£) option exhibits a similar pattern to that of the estimated bounds across deltas. As the foreign exchange market became more volatile from 1999 to 2000, the bound range, defined as the difference between the upper bound and the lower bound, turned wider as time went by during the period. Furthermore, it is clearly seen from Figure 2 that the deeper the moneyness is, the closer the market implied volatility is to the lower bound. For options with deltas 10 and 90, the market implied volatilities and the lower bounds almost overlap.

The results of Table 2 suggest that the variation of the upper bounds is the highest while the variation of the lower bounds is the lowest. For example, for at-the-money (ATM, delta 50) options, the standard deviations for the upper bounds, the market implied volatilities, and the lower bounds are 0.0331, 0.0176, and 0.0117, respectively. Although for deep-in-the-money and deep-out-of-the-money options (deltas 90 and 10), the average market implied volatilities (0.1054 and 0.1050) are slightly lower than the lower bound (0.1108 and 0.1066), the differences are tiny and smaller than the bid-ask spread observed in the OTC market. Therefore, there is no evidence to support the violation of our super-replication bounds and thus the existence of arbitrage opportunities in the foreign exchange option market, especially when we take market frictions into account.

It is apparent from Figure 2 and Table 2 that the level, the mean, and the volatility of the upper bounds are almost the same across deltas with an extremely shallow smile. In contrast, the lower bound and the market implied volatilities exhibit clearer smile shapes across deltas with the lower bound smile being deeper than the market implied smile. To deeply explore the relationships between the option market prices and the estimated bounds, we further look at the behavior of the bound range, the difference between the upper bound and the market implied (upper range), and the difference between the lower bound and the market implied (lower range). Their descriptive statistics are illustrated in Table 3.

Consistent with what we have observed in Figure 2 and Table 2 whereby the lower bounds exhibit a smile shape while the upper bounds are flat across deltas, Table 3 indicates that the deeper the moneyness is, the smaller the bound ranges and the lower ranges are. In contrast, we cannot statistically reject that the upper range is consistent across deltas (*F* statistic: 1.21; *p* value: 0.2993). Moreover, this study finds that the volatilities of all of the ranges are very small (between 0.51% and 3.24%), which signals that the relationships between the market implied and the estimated bounds are persistent across time.

In summary, the lower bounds exhibit a deep smile shape while the upper bounds and the market implied volatilities are relatively flat across deltas. Both the upper and lower bounds exhibit tractable and persistent relationships with the market prices of cross-rate options. The divergences between the lower bounds and the market prices are significantly different across deltas, but stable across time.

B. Pricing Bounds, Correlation, and the Cross-Rate Option Prices

Since there are persistent relationships between the market prices of the cross-rate options and the pricing bounds estimated from two corresponding dollar-rate options, we further use a regression model to measure the extent where the cross-rate option prices can be explained by our pricing bounds. We regress the market implied volatilities either on the bound ranges or on the upper and lower bounds. The regression models are specified as follows:

(8) Model 1: $MIV_t = c + \beta\, BR_t + \varepsilon_t$

(9) Model 2: $MIV_t = c + \beta_1\, UB_t + \beta_2\, LB_t + \varepsilon_t$,

where MIV_t, BR_t, UB_t, and LB_t respectively denote the market implied volatility of the cross-rate option on €/£, the bound range, the upper bound, and the lower bound at time *t*, and ε_t is the residual term. Intuitively, when the market is more turbulent, the bound range should be wider and then β is expected to be positive. The estimates for these two models are shown in Panels 1 and 2 of Table 4.

From Panel 1 of Table 4, we find that the regression coefficient β in Model 1 is highly significant across deltas (all p values <0.005) and that the bound ranges can explain about 35% to 51% of the cross-rate option prices. The positive β also confirms the intuition that the larger the bound range is, the higher the market implied volatility will be. Similarly, from Panel 2 of Table 4, we also find highly significant regression coefficients (β_1 and β_2) in Model 2. In comparison to Model 1, the adjusted R^2 are much higher in Model 2 (between 0.76 and 0.91), particularly for deep-in-the-money and deep-out-of-the-money options. It is noticeable that β_1 is adhered to a small range (between 0.33 and 0.41) while β_2 ranges from 0.39 to 0.79 with β_2 the smallest for the ATM option. In other words, the upper bound contains almost same level of information content for the cross-rate options across delta, while the lower bound contains much less for the ATM option. This finding corresponds to our previous observation that the deeper the moneyness is, the closer the lower bound to the market implied and also explains why the adjusted R^2 is the highest for the deep-in-the-money and deep-out-of-the-money options. In conclusion, we confirm that there are strong and stable relationships between the market prices of cross-rate options and the pricing bounds estimated from the market prices of the two dollar-rate options.

By analyzing the relationship between the prices of stock index options and the prices of individual stock options included in the index, Driessen, Maenhout, and

Vilkov (2005) show the relevance of correlation risk and the associated premium for stock index options pricing.[12] Moreover, according to compositions of the bounds in equations (5) to (7), we find that no correlation information is used in the calculation of the pricing bounds of the cross-rate options, for which we only utilize the price information of two dollar rate options individually. As a result, this paper includes an extra explanatory variable, the historical correlation of two dollar rates, into Model 2 to see whether the correlation is able to provide any additional explanatory power. Thus, the regression model is modified as the following:

(10) Model 3: $MIV_t = c + \beta_1 UB_t + \beta_2 LB_t + \beta_3 Corr_t + \varepsilon_t$,

where $Corr_t$ is the correlation coefficients of two dollar rates at time t. When the correlation of two dollar rates increases, the variance of the cross rate decreases and thus the cross-rate option price also decreases. Therefore, the regression coefficient of the historical correlation (β_3) is expected to be negative.

The correlation coefficients are estimated using the dynamic conditional correlation (DCC) multivariate GARCH model proposed by Engle (2002) using the historical time series data of two dollar spot rates. The fact that correlations between financial assets are usually time-varying has crucial important implication in many ways such as portfolio hedging and multivariate asset pricing. This model overcomes the complexity of conventional multivariate GARCH models in computation by directly modeling the time-varying correlation as a conditional process. The procedure of using the DCC GARCH model to generate the time-varying correlation series is detailed in Appendix D and the regression results for Model 3 are shown in Panel 3 of Table 4.

[12] In the same vein, a currency cross-rate can be regarded as an equal weight index of two dollar rates since its log return can be decomposed into the sum of the log returns of the two dollar rates.

It is clearly seen from Panel 3 of Table 4 that the correlations of two dollar rates provide incremental information in deciding cross-rate option prices as all adjusted R^2s increase. In particular, the correlation effectively improves the explanatory power for the ATM cross-rate option, which has the lowest R^2 in Model 2. The regression coefficients for the correlation across deltas are significantly negative and consistent with our expectation. Furthermore, our results are in line with the analyses and findings of Driessen, Maenhout, and Vilkov (2005).

The pricing bounds estimated from option prices of two dollar rates and the correlation of two dollar rates can provide almost perfect information for determining the cross-rate option prices across deltas. Specifically, the information contained in the dollar-rate option prices explains most proportion of the prices of the deep-in-the-money and deep-out-of-the-money cross-rate options, while the correlation of two dollar rates provides profound information for the prices of the ATM cross-rate options.

C. Pricing the Cross-Rate Options

Owing to the significant explanatory power of the estimated bounds and correlation to the market prices of cross-rate options, we are interested in the accuracy of our empirical models (Models 1 to 3) for pricing cross-rate options. Given the estimated parameters of the previous models, we infer the current implied volatility for the cross-rate (€/£) options from the current market prices of the dollar-rate options and the historical correlation. The actual and inferred implied volatilities of the cross-rate options across deltas are shown in Figure 3 and the descriptive statistics of the estimation errors are shown in Table 5. The estimation errors are defined as the absolute values of the actual values minus the inferred values.

From Table 5 we observe that the pricing errors of the inferred prices from Model 3 are the lowest while the errors of Model 1 are the highest. In Model 3 the precision of the estimations is very satisfactory across deltas, particularly for the deep-in-the-money and deep-out-of-the-money options. The average errors range from 0.12% to 0.33%, which are much lower than the bid-ask spread in the OTC market. The pricing errors for the options with deltas 90 and 10 are only 0.12% and 0.13%, respectively. In addition, the volatilities of the errors are very small as well (ranging from 0.11% to 0.29%), implying that the estimation performs consistently well across time.

Similar results are found for the inferred prices of Model 2. However, since Model 2 does not take the correlation into account, its accuracy is inferior to that of Model 3, especially for the ATM options. Nevertheless, the accuracy of the inferred prices from Model 2 is still remarkably small for deep-in-the-money and deep-out-of-the-money options. The results are valuable since most option pricing methods in the literature have difficulties in accurately pricing deep-in-the-money and deep-out-of-the-money options while our method performs particularly satisfactorily for these options. In other words, the inferred prices (or portfolios) are applicable to practical usage not only for pricing, but also for hedging.

D. Robustness Analysis

To investigate whether our results are robust, we analyze whether the accuracy of the inference of cross-rate option prices from Model 3 is sensitive to sample selection, and the implied volatility, the implied skewness, and the implied kurtosis levels estimated from the market prices of the cross-rate options.

To check whether sample selection affects our findings, we redo the option pricing inference for two evenly divided sub-samples. The average pricing errors across deltas are shown in Panel 1 of Table 6. Although the pricing errors are slightly higher in the second sub-periods, the patterns across deltas are the same. In other words, our finding, that the pricing errors are very small with the smallest for the deep-in-the-money and deep-out-of-the-money options, does not depend on sample selection.

As the volatility of exchange rates increases over our sample period, it is natural to check whether the increasing volatility changes the accuracy of information provided by our pricing bounds. The correlation coefficients between the pricing errors (in percentage) and market volatility levels[13] across deltas are shown in Panel 2 of Table 6. All correlations are very low and the sign is not all the same across deltas (-0.09 ~ 0.09).

To further check whether pricing errors depend on volatility, we run the following regression model.

(11) $$E_t = c + \alpha\, E_{t-1} + \beta\, Vol_t\,,$$

where E_t and Vol_t respectively denote the percentage pricing error and the volatility level at time *t*. The AR(1) specification is motivated by the high first-order autocorrelation of pricing errors. The estimates are reported in Panel 3 of Table 6. All β coefficients are insignificant under the 5% significance level. In summary, the accuracy of information provided by our pricing bounds is immune to the volatility change.

[13] The implied volatility of the ATM cross-rate option is chosen as the proxy of the market volatility level. Other proxies such as OTM (or ITM) cross-rate options or two ATM dollar-rate options are also used and the results (not reported here) are almost unchanged.

As discussed before, Figure 1 and Table 1 show that the average implied volatilities of all exchange rates exhibit a smile shape. However the slopes of the implied volatility curves vary from negatively sloped to positively sloped during our sample period. This implies that risk neutral skewness and kurtosis change substantially everyday. In order to investigate the impact of changes in implied volatility curves on our results, we first calculate the implied skewness and kurtosis using the Theorem 1 of Bakshi, Kapadia, and Madan (2003). We then run the following regression models to see whether the pricing errors are affected by changes in skewness and kurtosis:

(12) $$E_t = c + \alpha\, E_{t-1} + \beta\, Skew_t\,,$$

(13) $$E_t = c + \alpha\, E_{t-1} + \beta\, Kurt_t\,,$$

where $Skew_t$ and $Kurt_t$ are implied skewness and implied kurtosis, respectively.

Figure 4 indicates that the risk neutral distributions of the cross-rates are fat-tailed (average kurtosis equals 3.31) and slightly negatively skewed (average kurtosis equals -0.13).[14] Figure 4 also shows that the implied skewness changes noticeably over time. Nevertheless Panel 4 of Table 6 suggests that the pricing errors across deltas are not affected even though implied skewness changes much. All β coefficients are small and insignificant under the 5% confidence interval. Similarly Panel 5 of Table 6 shows that the implied kurtosis has little impact on the pricing errors across deltas.

In conclusion, our results seem robust across different levels and slopes of implied volatility curves.

[14] The risk neutral distributions of two dollar rates also exhibit the same pattern, i.e. fat-tailed and slightly negatively skewed.

V. Concluding Remarks

Instead of pricing cross-rate options directly, this study relates the option pricing bounds to the prices of the corresponding dollar-rate options. Our pricing bounds are derived from a general result of the copula theory and thus do not rely on the distribution assumptions of state variables. Different from most option pricing bounds in the literature, our cross-rate option bounds are functions of two dollar-rate option prices. In particular, our pricing bounds are portfolios of the dollar-rate options (and sometimes also of spot dollar-rates).

Using the prices of options on foreign exchange rates among US dollar, euro, and pound sterling for the empirical tests, we show the persistent relationships between the market prices of the cross-rate (€/£) options and the estimated bounds. This study also finds that the dollar-rate option prices and the correlation between two dollar rates provide almost perfect information in deciding the prices of the cross-rate options. Owing to the almost perfect explanatory power of the option bounds and the correlation to the market prices of the cross-rate options, we successfully price the cross-rate options under the circumstance where the current option prices of two dollar rates are available. In particular, the pricing errors are the smallest for deep-in-the-money and deep-out-of-the-money options, which is valuable, because most option pricing methods in the literature have difficulties in pricing these options accurately. Therefore, our results are useful for risk management and derivative pricing, particularly for those having cross-rate risk exposures.

The technique utilized to derive our cross-rate option bounds can be applied to any European derivative security whose payoff can be rearranged as the same type as that of an exchange option. As an example, we also derive the price bounds for quanto options using the same copula approach. Due to the lack of data, the empirical tests of the tightness and the information efficiency of our quanto option price bounds are left to interested readers for future research.

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Appendix

A. Derivation of the Price Bounds for the Cross-Rate Option

Define K^{**} as the constant such that $\overline{F}_{S^{\$/£}}(K^{**})+\overline{F}_{KS^{\$/€}}(K^{**})=1$. Since $\overline{F}_i(u)$ is a decreasing function of *u*, it is true that $\overline{F}_{S^{\$/£}}(K^{**})+\overline{F}_{KS^{\$/€}}(K^{**})\geq 1$ for $u\leq K^{**}$. Therefore, the lower bound of the minimum option is

$$
\begin{aligned}
(\text{A.1})\qquad & Call^{-}_{\min}(S^{\$/£},KS^{\$/€},0,t,T)=e^{-r(T-t)}\int_0^{\infty}\max(\overline{F}_{S^{\$/£}}(u)+\overline{F}_{KS^{\$/€}}(u)-1,0)du \\
&=e^{-r(T-t)}\int_0^{K^{**}}\overline{F}_{S^{\$/£}}(u)du+e^{-r(T-t)}\int_0^{K^{**}}\overline{F}_{KS^{\$/€}}(u)du-e^{-r(T-t)}\int_0^{K^{**}}du \\
&=e^{-r(T-t)}\int_0^{\infty}\overline{F}_{S^{\$/£}}(u)du-e^{-r(T-t)}\int_{K^{**}}^{\infty}\overline{F}_{S^{\$/£}}(u)du+e^{-r(T-t)}\int_0^{\infty}\overline{F}_{KS^{\$/€}}(u)du \\
&\quad -e^{-r(T-t)}\int_{K^{**}}^{\infty}\overline{F}_{KS^{\$/€}}(u)du-e^{-r(T-t)}\int_0^{K^{**}}du \\
&=S_t^{\$/£}-Call(S^{\$/£},K^{**},t,T)+KS_t^{\$/€}-Call(KS^{\$/€},K^{**},t,T)-B(t,T)K^{**}.
\end{aligned}
$$

Substituting equation (A.1) into equation (1) and applying the put-call parity, we obtain the upper price bound of the cross-rate option as follows:

$$
\begin{aligned}
(\text{A.2})\qquad Call_{\$}^{€/£+}&=S_t^{\$/£}-Call^{-}_{\min}(S^{\$/£},KS^{\$/€},0,t,T) \\
&=Call(S^{\$/£},K^{**},t,T)+K\ Put(S^{\$/€},K'',t,T),
\end{aligned}
$$

where $K''=K^{**}/K$.

Assume that there exists a constant K^{*} such that $\overline{F}_{S^{\$/£}}(K^{*})=\overline{F}_{KS^{\$/€}}(K^{*})$. If $\overline{F}_{S^{\$/£}}(K^{*})<\overline{F}_{KS^{\$/€}}(K^{*})$ for $u<K^{*}$, then it is straightforward to show that the upper bound of the minimum option is:

$$
\begin{aligned}
(\text{A.3})\qquad & Call^{+}_{\min}(S^{\$/£},KS^{\$/€},0,t,T)=e^{-r(T-t)}\int_0^{\infty}\min(\overline{F}_{S^{\$/£}}(u),\overline{F}_{KS^{\$/€}}(u))du \\
&=e^{-r(T-t)}\int_0^{K^{*}}\overline{F}_{S^{\$/£}}(u)\,du+e^{-r(T-t)}\int_{K^{*}}^{\infty}\overline{F}_{KS^{\$/€}}(u)\,du \\
&=e^{-r(T-t)}\int_0^{\infty}\overline{F}_{S^{\$/£}}(u)\,du-e^{-r(T-t)}\int_{K^{*}}^{\infty}\overline{F}_{S^{\$/£}}(u)\,du+e^{-r(T-t)}K\int_{K^{*}/K}^{\infty}\overline{F}_{S^{\$/€}}(u)\,du \\
&=S_t^{\$/£}-Call(S^{\$/£},K^{*},t,T)+K\ Call(S^{\$/€},K',t,T),
\end{aligned}
$$

where $K^{'} = K^{*}/K$. Substituting equation (A.3) into equation (1) yields the lower price bound of the cross-rate option as follows:

$$\text{(A.4)} \qquad \begin{aligned} Call_{\$}^{€/£-} &= S_t^{\$/£} - Call_{\min}^{+}(S^{\$/£}, KS^{\$/€}, 0, t, T) \\ &= Call(S^{\$/£}, K^{*}, t, T) - K\,Call(S^{\$/€}, K^{'}, t, T). \end{aligned}$$

Similarly, if $\overline{F}_{S^{\$/£}}(K^{*}) > \overline{F}_{KS^{\$/€}}(K^{*})$ for $u < K^{*}$, then one can derive that:

$$\text{(A.5)} \qquad \begin{aligned} &Call_{\min}^{+}(S^{\$/£}, KS^{\$/€}, 0, t, T) = KS_t^{\$/€} - K\,Call(S^{\$/€}, K^{'}, t, T) + Call(S^{\$/£}, K^{*}, t, T), \\ &Call_{\$}^{€/£-} = S_t^{\$/£} - K\,S_t^{\$/€} + K\,Call(S^{\$/€}, K^{'}, t, T) - Call(S^{\$/£}, K^{*}, t, T). \end{aligned}$$

B. Price Bounds for the Quanto Options

Consider a European quanto call option where the holder has the right to buy 1 share of foreign asset by paying K units of domestic currency. In other words, the payoff in the domestic currency of a European quanto call option at maturity T equals

$$\max\left(X_T / S_T - K, 0\right),$$

where X_T and S_T are the prices in the foreign currency of the foreign asset price and one unit of domestic currency, respectively. Under the foreign-currency measure, the above quanto call option is identical to an option to exchange KS_T units of foreign currency for one share of foreign asset. Therefore, we can regard this option as an exchange option to obtain its super-replication price bounds under the foreign-currency measure and then apply the law of one price to convert the bounds to those in the domestic-currency measure.

Assume K^{**} is the constant such that $\overline{F}_X(K^{**}) + \overline{F}_{KS}(K^{**}) = 1$. The upper bound of the quanto option ($Quanto_d^{+}$) in the domestic-currency measure is

(A.6) $$Quanto_d^+ = \frac{1}{S_t}[C(X, K^{**}, t, T) + KP(S, K^{''}, t, T)],$$

where $K^{''} = K^{**}/K$.

Define a constant K^* such that $\overline{F}_X(K^*) = \overline{F}_{KS}(K^*)$. If $\overline{F}_X(K^*) < \overline{F}_{KS}(K^*)$ for $u < K^*$, the lower bound is

(A.7) $$Quanto_d^- = \frac{1}{S_t}[C(X, K^*, t, T) - KC(S, K^{'}, t, T)],$$

where $K^{'} = K^*/K$. If $\overline{F}_X(K^*) > \overline{F}_{KS}(K^*)$ for $u < K^*$, the lower bound becomes

(A.8) $$Quanto_d^- = \frac{1}{S_t}[X_t + KC(S, K^{'}, t, T) - C(X, K^*, t, T)] - K.$$

C. The GB2 Density and Its Estimation

Bookstaber and McDonald (1987) proposed the GB2 distribution for asset prices, with four positive parameters that define a parameter vector $\theta = (a, b, p, q)$. The risk-neutral density function for S_T under the GB2 distribution is defined as:

(A.9) $$f_{GB2}(x|a,b,p,q) = \frac{a}{b^{ap}B(p,q)} \times \frac{x^{ap-1}}{\left[1 + (x/b)^a\right]^{p+q}}, \quad x > 0,$$

where function B is defined in terms of the Gamma function by $B(p,q) = \Gamma(p)\Gamma(q)/\Gamma(p+q)$. European call option prices can then be derived as follows:

(A.10) $$C(X) = Fe^{-rT}[1 - F_\beta(u(X,a,b)|p + a^{-1}, q - a^{-1})] - Xe^{-rT}[1 - F_\beta(u(X,a,b)|p,q)],$$

where F_β is the incomplete beta function given by:

(A.11) $$F_\beta(u|p,q) = \frac{1}{B(p,q)}\int_0^u t^{p-1}(1-t)^{q-1}dt,$$

and $u(X,a,b) = \frac{(X/b)^a}{1+(X/b)^a}$. Risk-neutrality constrains the mean of the density to be:

(A.12) $$F = bB[p + a^{-1}, q - a^{-1}]\big/B(p,q)\,,$$

which also requires the constraint $aq > 1$.

Several loss functions can be considered when estimating the parameter vector θ of either density family. As is common in the RND literature, we minimize the sum of squared pricing errors for a set of market call prices denoted by $C_m(X_i)$. When there are market prices available for options with a common expiry time *T*, but with *N* distinct exercise prices, we estimate θ by minimizing:

(A.13) $$G(\theta) = \sum_{i=1}^{N} (C_m(X_i) - C(X_i|\theta))^2 \,.$$

D. The DCC GARCH Model

In the DCC GARCH model, returns[15] at time t from *k* assets (R_t) are assumed to be a conditional multivariate normal distribution with zero mean and covariance matrix H_t, i.e.

(A.14) $$R_t \mid \Phi_{t-1} \sim N(0, H_t),$$

(A.15) $$H_t = D_t \Psi_t D_t,$$

where D_t is the $k \times k$ diagonal matrix of time varying standard deviation from univariate GARCH models with $\sqrt{h_{i,t}}$ on the i^{th} diagonal and Ψ_t is the time varying correlation matrix. Therefore, the first step is to estimate *k* univariate GARCH models to generate the conditional variance series.

[15] Note that the returns can be de-meaned assets returns or the residuals filtered from an econometric model.

The dynamic correlation structure is proposed as

(A.16) $$\Psi_t = Q_t^{*-1} Q_t Q_t^{*-1},$$

(A.17) $$Q_t = (1 - \sum_{m=1}^{M} \alpha_m - \sum_{n=1}^{N} \beta_n)\overline{Q} + \sum_{m=1}^{M} \alpha_m (z_{t-m} z'_{t-m}) + \sum_{n=1}^{N} \beta_n Q_{t-n},$$

where $\overline{Q}$ is the unconditional covariance of the standardized residuals (z_t) from the first step and

(A.18) $$Q_t^* = \begin{bmatrix} \sqrt{q_{11}} & 0 & \cdots & 0 \\ 0 & \sqrt{q_{22}} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sqrt{q_{kk}} \end{bmatrix}.$$

In other words, Q_t^* is a diagonal matrix composed of the square root of the diagonal elements ($\sqrt{q_{ii}}$) of Q_t.

The second step is to estimate the parameters in the dynamic correlation structure and generate the time-varying correlation series by maximizing the following log-likelihood function:

(A.19) $$L = -\frac{1}{2} \sum_{t=1}^{T} (k \log(2\pi) + 2\log(|D_t|) + \log(|\Psi_t|) + z_t' \Psi_t^{-1} z_t).$$

In this study, k = 2 and both M and N are set to be 1. This DCC(1,1) model is the most broadly accepted and reliable one in the literature.

Figure 1: The Average Implied Volatilities

This figure consists of the average implied volatilities for options on the three foreign exchange rates, \$/£, \$/€, and €/£. All implied volatilities are quoted in the OTC market.

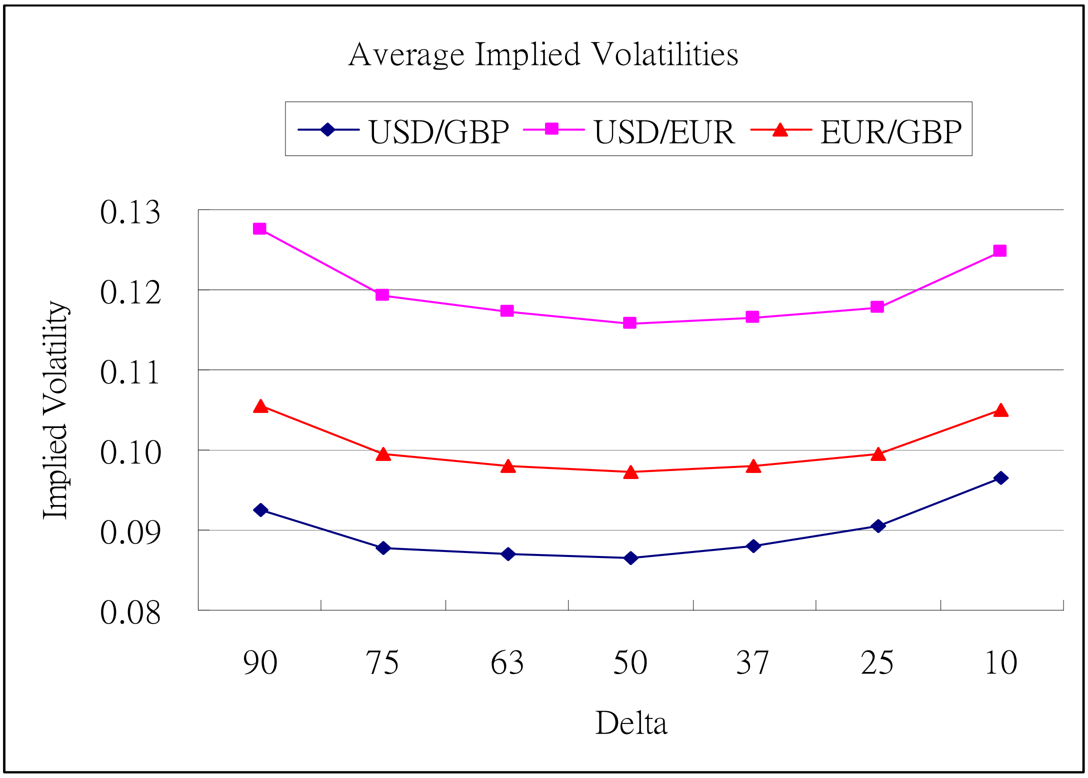


Figure 2: The Implied Volatilities from the Market Prices and the Estimated Bounds for the Cross-Rate Option

This figure shows the evolutions of the Black-Scholes implied volatilities from the market prices and the estimated bounds of the cross-rate (€/£) options across deltas. The option bounds are estimated by calibrating equations (5) to (7) using the option prices of two dollar rates, $/£ and $/€.

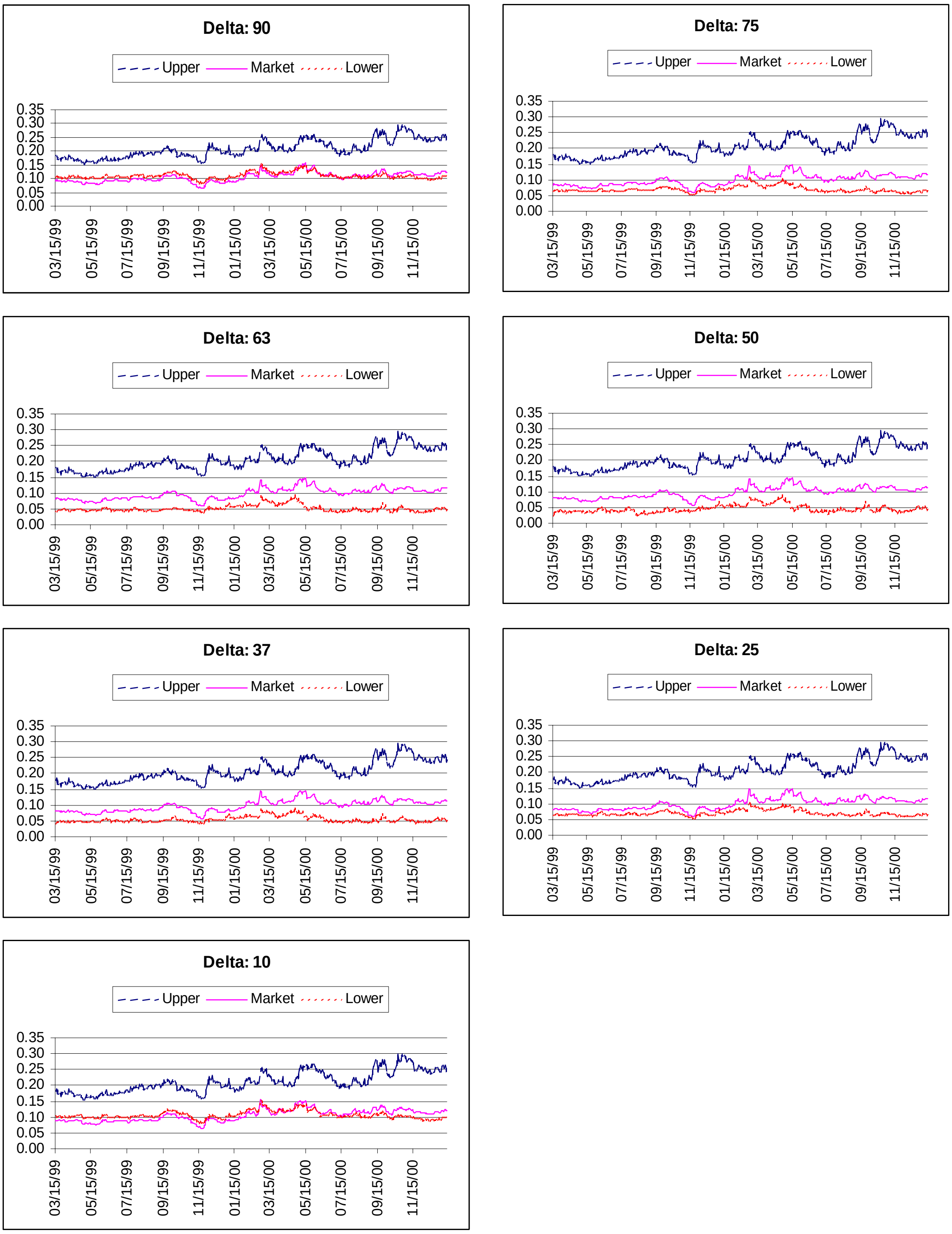
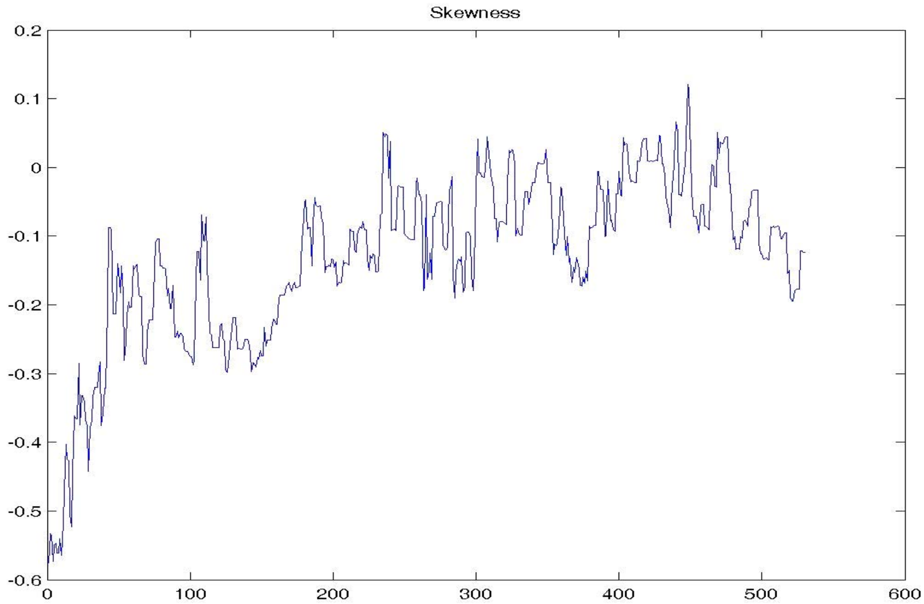


Figure 3: Actual and Inferred Implied Volatilities for the Cross-Rate Options

This figure consists of the evolutions of the actual and inferred Black-Scholes implied volatilities of the cross-rate (€/£) options across deltas. The actual implied volatilities are backed out from the market prices of options. The inferred implied volatilities are obtained from Model 3 in Section IV using the current option prices and historical DCC correlation of two dollar rates, $/£ and $/€.

Figure 4: Implied Skewness and Kurtosis for the Cross-Rate Options

This figure consists of the evolutions of the implied skewness and kurtosis of the cross-rate (€/£) options. The implied skewness and kurtosis are calculated using the Theorem 1 of Bakshi, Kapadia, and Madan (2003). The results indicate that the risk neutral distributions of the cross-rates are fat-tailed (average kurtosis equals 3.31) and slightly negatively skewed (average kurtosis equals -0.13).



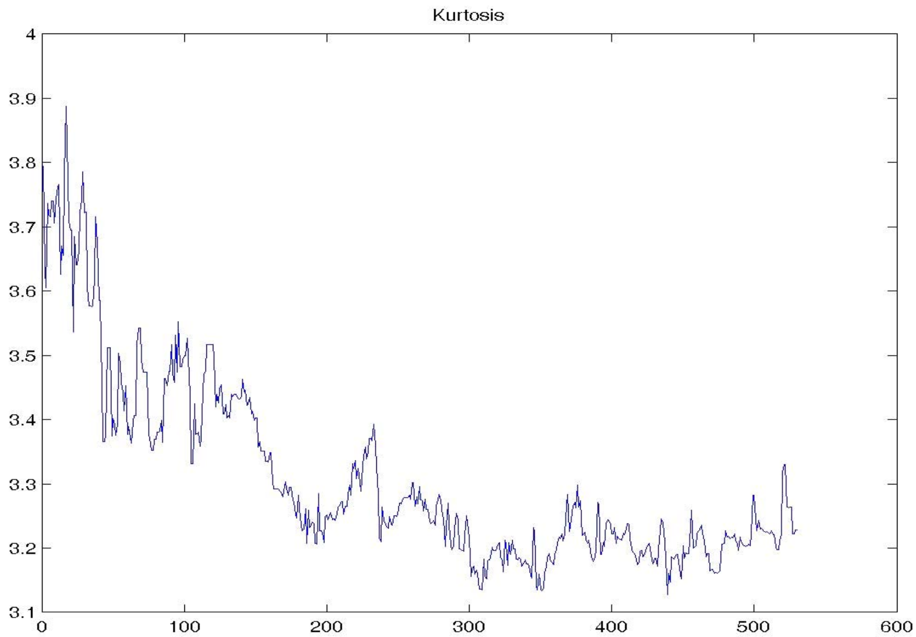


Table 1: Summary Statistics of Market Implied Volatilities for the Foreign Exchange Options

Panel 1: USD/GBP							
Delta	90	75	63	50	37	25	10
Mean	0.0926	0.0878	0.0870	0.0866	0.0881	0.0904	0.0964
Std. Dev.	0.0130	0.0130	0.0134	0.0136	0.0139	0.0142	0.0146
Skewness	0.5772	0.6170	0.6473	0.6703	0.6869	0.7037	0.7065
Kurtosis	2.6181	2.6621	2.6942	2.7323	2.7607	2.8066	2.8542
Panel 2: USD/EUR							
Delta	90	75	63	50	37	25	10
Mean	0.1276	0.1193	0.1172	0.1158	0.1166	0.1179	0.1248
Std. Dev.	0.0208	0.0213	0.0212	0.0212	0.0213	0.0215	0.0214
Skewness	0.3428	0.3291	0.3577	0.3483	0.4046	0.4401	0.4890
Kurtosis	2.2055	2.1561	2.1455	2.1479	2.1585	2.1915	2.2623
Panel 3: EUR/GBP							
Delta	90	75	63	50	37	25	10
Mean	0.1054	0.0996	0.0980	0.0972	0.0979	0.0995	0.1050
Std. Dev.	0.0169	0.0173	0.0176	0.0176	0.0180	0.0180	0.0183
Skewness	0.2457	0.2441	0.2288	0.2257	0.2222	0.2321	0.2484
Kurtosis	2.9339	2.7199	2.5933	2.5686	2.4881	2.4722	2.4511

This table consists of the summary statistics of the market implied volatilities of the two dollar-rate ($/£ and $/€) options. All implied volatilities are quoted in the OTC market.

Table 2: Summary Statistics of the Implied Volatilities from the Market Prices and the Estimated Bounds

Panel 1: Upper Bounds							
Delta	90	75	63	50	37	25	10
Mean	0.2087	0.2061	0.2056	0.2058	0.2064	0.2077	0.2118
Std. Dev.	0.0325	0.0329	0.0330	0.0331	0.0332	0.0333	0.0334
Skewness	0.4752	0.4798	0.4806	0.4807	0.4785	0.4753	0.4675
Kurtosis	2.3988	2.4038	2.4031	2.3989	2.3914	2.3809	2.3575
Panel 2: Market Implieds							
Delta	90	75	63	50	37	25	10
Mean	0.1054	0.0996	0.0980	0.0972	0.0979	0.0995	0.1050
Std. Dev.	0.0169	0.0173	0.0176	0.0176	0.0180	0.0180	0.0183
Skewness	0.2457	0.2441	0.2288	0.2257	0.2222	0.2321	0.2484
Kurtosis	2.9339	2.7199	2.5933	2.5686	2.4881	2.4722	2.4511
Panel 3: Lower Bounds							
Delta	90	75	63	50	37	25	10
Mean	0.1108	0.0694	0.0514	0.0462	0.0547	0.0705	0.1066
Std. Dev.	0.0119	0.0089	0.0102	0.0117	0.0096	0.0088	0.0119
Skewness	0.8569	1.2038	1.5257	1.0723	1.2282	1.0595	0.9384
Kurtosis	4.0491	4.4558	5.0837	3.7786	3.8409	3.5790	3.9496

This table consists of the summary statistics of the market implied volatilities and estimated upper and lower bounds of the cross-rate €/£ across deltas. The option bounds are estimated by calibrating equations (5) and (6) or (7) with the option prices of two dollar-rates, $/£ and $/€.

Table 3: Summary Statistics of the Estimated Bound Ranges, Upper Ranges, and Lower Ranges

Panel 1: Bound Ranges							
Delta	90	75	63	50	37	25	10
Mean	0.0979	0.1367	0.1543	0.1596	0.1518	0.1372	0.1053
Std. Dev.	0.0303	0.0322	0.0324	0.0313	0.0314	0.0318	0.0320
Skewness	0.8483	0.7479	0.6806	0.6425	0.6957	0.7450	0.8509
Kurtosis	2.8798	2.6479	2.5045	2.5029	2.5696	2.6288	2.7514
Panel 2: Upper Ranges							
Delta	90	75	63	50	37	25	10
Mean	0.1033	0.1018	0.1031	0.1042	0.1042	0.1041	0.1027
Std. Dev.	0.0208	0.0183	0.0178	0.0177	0.0175	0.0175	0.0174
Skewness	0.9414	1.4807	1.5144	1.4947	1.4593	1.3644	1.2319
Kurtosis	3.4138	5.6058	5.7064	5.6440	5.5473	5.2522	4.8148
Panel 3: Lower Ranges							
Delta	90	75	63	50	37	25	10
Mean	0.0115	0.0303	0.0466	0.0510	0.0433	0.0290	0.0129
Std. Dev.	0.0051	0.0143	0.0159	0.0154	0.0148	0.0145	0.0055
Skewness	-0.5655	0.3320	0.2528	0.0476	0.1595	0.2596	-0.0298
Kurtosis	2.4787	1.7984	2.1688	2.2946	1.9362	1.6386	2.5738

This table consists of the summary statistics of the estimated bound ranges, upper ranges, and lower ranges of the cross rate (€/£) options across deltas. The bound ranges, upper ranges, and lower ranges are the distances between the upper and lower bounds, between the upper bounds and market implieds, and between the lower bounds and market implieds, respectively.

Table 4: Explanatory Power of Estimated Bounds and Correlation to Market Implied Volatility

Panel 1: Model 1							
Delta	90	75	63	50	37	25	10
β	0.3322	0.3740	0.3884	0.3919	0.4101	0.4015	0.3606
	(16.25)	(21.05)	(22.20)	(21.13)	(22.47)	(21.90)	(17.86)
Adjusted R^2	0.3550	0.4804	0.5071	0.4823	0.5131	0.5004	0.3994
Panel 2: Model 2							
Delta	90	75	63	50	37	25	10
β_1	0.3336	0.3925	0.4144	0.4009	0.3973	0.3919	0.3899
	(43.58)	(41.46)	(35.18)	(31.99)	(33.88)	(38.79)	(44.21)
β_2	0.7231	0.7987	0.4930	0.3967	0.6078	0.8133	0.6786
	(34.50)	(22.81)	(12.88)	(11.23)	(14.94)	(21.36)	(27.54)
Adjusted R^2	0.9096	0.8529	0.7785	0.7639	0.7996	0.8483	0.8861
Panel 3: Model 3							
Delta	90	75	63	50	37	25	10
β_1	0.3554	0.4116	0.4312	0.4186	0.4191	0.4131	0.4055
	(50.12)	(52.82)	(48.90)	(44.72)	(45.37)	(46.99)	(48.91)
β_2	0.5916	0.6022	0.3958	0.3335	0.4705	0.6202	0.5578
	(26.81)	(19.35)	(13.67)	(12.60)	(14.39)	(17.38)	(21.34)
β_3	-0.0545	-0.0844	-0.1119	-0.1151	-0.1032	-0.0795	-0.0554
	(-11.18)	(-15.66)	(-19.55)	(-19.65)	(-17.50)	(-13.24)	(-9.30)
Adjusted R^2	0.9283	0.9028	0.8770	0.8695	0.8779	0.8890	0.9034

This table consists of the regression results of the following three models:

Model 1: $MIV_t = c + \beta BR_t + \varepsilon_t$

Model 2: $MIV_t = c + \beta_1 UB_t + \beta_2 LB_t + \varepsilon_t$

Model 3: $MIV_t = c + \beta_1 UB_t + \beta_2 LB_t + \beta_3 Corr_t + \varepsilon_t$.

Here, MIV_t, BR_t, UB_t, LB_t, and $Corr_t$ denote respectively the market implied volatility of an option on €/£, the bound range, the upper bound, the lower bound, and the historical DCC correlation between S/€ and $/£ at time *t*, and ε_t is the residual term. The numbers in the parentheses are t-statistics.

Table 5: Summary Statistics of Option Pricing Errors

Panel 1: Inferred from Model 1							
Delta	90	75	63	50	37	25	10
Mean	0.0062	0.0054	0.0048	0.0048	0.0048	0.0052	0.0059
Std. Dev.	0.0060	0.0051	0.0043	0.0043	0.0043	0.0050	0.0059
Skewness	2.2006	2.0448	1.6691	1.6355	1.6100	1.9590	1.9502
Kurtosis	10.3309	9.3174	6.5211	6.5356	6.0075	8.4346	8.6921
Panel 2: Inferred from Model 2							
Delta	90	75	63	50	37	25	10
Mean	0.0013	0.0024	0.0034	0.0037	0.0036	0.0027	0.0013
Std. Dev.	0.0012	0.0017	0.0027	0.0030	0.0028	0.0019	0.0009
Skewness	1.5945	1.9094	2.8443	2.3065	2.0411	1.6276	1.8203
Kurtosis	5.6991	8.5490	21.2433	13.5038	10.4800	5.7794	7.2845
Panel 3: Inferred from Model 3							
Delta	90	75	63	50	37	25	10
Mean	0.0012	0.0021	0.0031	0.0033	0.0031	0.0024	0.0013
Std. Dev.	0.0011	0.0021	0.0028	0.0029	0.0028	0.0024	0.0013
Skewness	1.7674	2.2141	3.5950	2.8487	2.9133	3.4914	2.0573
Kurtosis	7.0328	11.2172	34.3467	22.9182	21.9591	27.6906	8.8516

This table consists of the summary statistics of the pricing errors for the cross-rate (€/£) options. The pricing errors are defined as the absolute values of the actual values minus the estimated values of implied volatilities. The actual implied volatilities are backed out from the market prices of options. The estimated implied volatilities are inferred from the three regression models in Section IV using the current market prices of options on two dollar rates ($/£ and $/€) and/or their DCC historical correlations.

Table 6: Robustness Analysis for Option Pricing Errors

Panel 1: Mean Errors for Sub-samples							
Delta	90	75	63	50	37	25	10
First half sample	0.0010	0.0017	0.0028	0.0033	0.0030	0.0020	0.0012
Second half sample	0.0014	0.0025	0.0033	0.0033	0.0033	0.0028	0.0015
Panel 2: Correlation between Errors and Volatility Levels							
Delta	90	75	63	50	37	25	10
ρ	0.0030	0.0941	-0.0459	-0.0935	-0.0177	0.0852	-0.0105
Panel 3: Regression of Errors on Volatility Levels							
Delta	90	75	63	50	37	25	10
β	0.0057	0.1040	-0.0197	-0.0983	0.0269	0.1624	0.0198
	(0.13)	(1.37)	(-0.18)	(-0.85)	(0.26)	(1.94)	(0.43)
Panel 4: Regression of Errors on Implied Skewness							
Delta	90	75	63	50	37	25	10
β	0.0048	0.0138	0.0122	0.0111	0.0228	0.0175	0.0044
	(0.61)	(0.96)	(0.59)	(0.50)	(1.13)	(1.08)	(0.48)
Panel 5: Regression of Errors on Implied Kurtosis							
Delta	90	75	63	50	37	25	10
β	0.0012	-0.0035	0.0090	0.0193	-0.0032	-0.0148	-0.0045
	(0.15)	(-0.25)	(0.44)	(0.88)	(-0.16)	(-0.92)	(-0.50)

This table consists of the summary statistics used to check the sensitivity of the accuracy of option pricing for Model 3. The means of option pricing errors for two evenly divided sub-samples and the correlation coefficients between the errors and volatility levels across deltas are shown in Panel 1 and Panel 2, respectively. In addition, we run the following regression model to analyze whether pricing errors depend on volatility, skewness or kurtosis:

$$E_t = c + \alpha E_{t-1} + \beta X_t,$$

where E_t denotes the pricing error in percentage and X_t is the implied volatility, implied skewness and implied kurtosis estimated from the market prices of cross-rate options at time t in Panels 3, 4 and 5, respectively. The numbers in the parentheses are t-statistics.