

# Portfolio Choice and Pricing in Illiquid Markets

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## Abstract

This paper studies portfolio choice and pricing in markets in which immediate trading may be impossible, such as the market for private equity and certain over-the-counter markets. Optimal positions are found to depend significantly and naturally on liquidity: when future liquidity is expected to be higher, agents take more extreme positions, given that they do not have to hold them for long when no longer desirable. Consequently, in markets with more frequent trading larger trades should be observed. The price, on the other hand, is not affected significantly by liquidity, due to the mitigating effect of endogenous position choice. Extensions with transaction costs, multiple assets, and heterogenous agents are considered among others.

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In many markets completing a trade may require a significant amount time. For instance, some investments, such as private equity, are essentially non-tradeable for certain periods. Another example is the over-the-counter (OTC) markets, in which a large number of assets are traded, including corporate and municipal bonds. Here, the need to locate a suitable counterparty can induce delays, further lengthened when the possibility of asymmetric information makes complex due diligence processes necessary. Finally, even centralized markets do not operate continuously, in particular in less developed economies.

This paper studies the equilibrium price and optimal portfolio choice in such markets. Given an investor's inability to change his corporate-bond position quickly, what price should he pay for these bonds? How large of a stake in private equity should one take, given that it cannot be changed for a lengthy period of time?

Using an approximation, I derive closed-form expressions for optimal positions and price. In particular, I find that the positions depend on the liquidity level in a natural way: the less easily agents can trade, the less extreme positions they take, in order to avoid being forced into holding for a long time positions that become disadvantageous. For instance, when future trading is expected to be difficult, an institution with current high value for a particular corporate bond – say, due to low correlation with the rest of its portfolio – should buy a smaller amount of the bond than it would in a perfect market.

The price, on the other hand, does not depend on the liquidity level in the approximation studied, contrary to the common intuition that the inability to trade in a timely fashion lowers the price of an asset – equivalently, increases its discount rate. Exact numerical results confirm these findings: positions react strongly to the level of liquidity, while the price changes very little with liquidity.

So far, price formation in the presence of illiquidity in the sense of inability to trade immediately has been studied in two kinds of models. On one hand, several papers, such as Duffie, Gârleanu, and Pedersen (2003) (hereafter, DGP), Weill (2002), and Vayanos and Wang (2002), have developed search-based models, in which an agent can only trade at discrete, random times, usually given by a Poisson process. In these models, positions are held to exogenously imposed values, making the derived prices sensitive to these choices and rendering the study of portfolio choice impossible. On the other hand, Longstaff (2005) studies a situation in which one of two assets is traded once, then cannot be traded for a given period of time, after which it can be traded perfectly, while the other asset is perfectly liquid throughout. This is a stylized framework, but it allows for a meaningful portfolio-choice problem. Using numerical methods, Longstaff (2005) finds liquidity impacts on both asset prices, as well as a significant allocation impact. In Longstaff (2005), gains from trade stem from different patience levels, which make one agent keener to sell in order to finance consumption. In contrast, I analyze agents with hedging needs that change dramatically over time – for instance, an underwriter may take on a new issue, a bank may sell to a client insurance

for a particular portfolio, or an agent might change the exposure of her human capital to the risk in certain investments.

The model here works as follows. Time is continuous and infinite. Each of a continuum of agents is assumed to be able to trade only at random times corresponding to arrivals of a Poisson process, capturing the discontinuous nature of trading opportunities.<sup>1</sup> Unlike in the search-based asset-pricing literature, though, trade is not bilateral – rather, it is set competitively at the trading posts that the agents access from time to time. Furthermore, and important, agents are free to choose their holdings without restrictions. This is made possible by assuming that agents are risk averse, while the gains from trade stem from idiosyncratic shocks to the covariances between asset payoffs and agents' private endowments.

The model is of the CARA-normal class. The approximation I consider fixes the risk faced by agents in utility terms – that is, the product between the risk-aversion coefficient and the payoff covariances – while letting the risk-aversion coefficient go to zero. It follows that any agent's marginal benefit from holding the asset, per unit of time, is linear in the position and the correlation between endowment and dividends, which I refer to as agent type. (I further refer to low-correlation agents as high type, since they have higher marginal valuations for the asset.) Furthermore, even if not all agents can trade when they want, the ones that can trade are representative of all the agents. As a consequence, aggregating the optimality conditions of all trading agents yields the optimality conditional of a (fictional) representative agent holding the per-capita supply and having the average type in the economy, independent of the liquidity level. This observation yields the equilibrium price.

Using the price obtained this way, the optimal positions are in turn easily derived as solutions to linear equations. The positions are impacted naturally by the liquidity level. The more liquid the market is going to be in the future, the more any given agent can deviate currently from the autarchy allocation, since he can subsequently change his holding more easily if desired, rather than incur the utility cost of a disadvantageous position. Equivalently, the closer his position can be to the Pareto-optimal allocation.

The portfolio and price results are similar in spirit to those obtaining in exogenous transaction-cost settings, as studied by Constantinides (1986) or Vayanos (1998), in that positions are closer to the Pareto-efficient allocation if liquidity is high, while the price is affected little by liquidity. The mechanism is quite different, though. With exogenous transaction costs, agents choose to trade less frequently in order to avoid incurring transaction costs, thus allowing holdings to deviate from the perfect-market optimum and attenuating the impact of transaction costs on prices.

In this paper, on the other hand, worse liquidity means that an agent's current position has more weight in the determination of her marginal utility, as the position will be kept unchanged for a longer period. In particular, if the agent has a large holding

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<sup>1</sup>Trading blackout periods are also modeled, with the same results, in an extension.

relative to the per-capita supply, her marginal utility will depend more strongly on this holding and consequently decrease as liquidity worsens, since the marginal utility flow decreases in the position. At unchanged prices, therefore, the agent would prefer to choose a lower position to start with. At the same time, though, low-type agents would choose larger positions. The ultimate effect is that markets may clear with minimal adjustments in prices,<sup>2</sup> while the positions move further from the Pareto-optimal levels as liquidity becomes lower.

The position adjustment to liquidity has implications for the volume of trade. In particular, the exogenous liquidity level – the ability to trade frequently – has a larger effect on turnover than if the positions were fixed. A testable implication is that, in markets where completing trades is (expected to be) faster,<sup>3</sup> the average trade size is higher (larger blocks are traded).

The framework allows for several extensions of interest. For instance, studying multiple risky assets reveals an interesting impact of one asset’s liquidity on the holdings in another asset. Suppose, for example, that one asset is illiquid and a second asset perfectly liquid. It follows, naturally, that a decrease in the first asset’s liquidity results *(i)* in more extreme positions, and higher volume, in the second asset if the two assets are substitutes – say, both are used to hedge the same kind of risk, e.g., on-the-run and off-the-run treasuries are used to hedge interest-rate risk – and *(ii)* in less extreme positions if the two assets are complements – say, the second asset is used to hedge risk introduced by the first asset, e.g., corporate bonds are used to hedge endowment risk and treasuries are used to hedge the interest-rate risk in corporate bonds. Another extension concerns exogenous transaction costs: in this context, it is shown that the quantity that governs the ranking of the equilibrium price and the no-transaction-cost one is the difference between the number of buyers and the number of sellers. In particular, in the absence of forced exit, transaction costs decrease the price if and only if there are more buying than selling agents. Forced exit, on the other hand, has the usual price-reduction effect reflecting the amortized future transaction costs, as in Amihud and Mendelson (1986).

Despite the difference in price-setting mechanism, the results of this model are consistent with those of previous search-based studies, such as DGP. In particular, binding constraints on portfolio holdings do lead to a liquidity price effect. This is intuitive, since closing the endogenous adjustment channel leaves only the direct effect of liquidity on the marginal utility of an agent. Thus, with binding short-sale constraints,

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<sup>2</sup>In the approximation studied, there is, of course, no price adjustment, while numerical calculations with reasonable parameters yield small adjustments. Although these results depend on the CARA-normal assumptions, the intuition for the attenuating effect of the endogenous positions is general.

<sup>3</sup>Note that the positions chosen depend only on the future level of liquidity. Current and past liquidity levels, though, influence the number of agents wanting and able to trade. Trading volume, consequently, depends on all these liquidity levels, and not always in the same way. In particular, better *past* liquidity means that fewer agents need to trade today, in contrast to the effect on volume of better *future* liquidity.

for instance, the price increases with the liquidity level, as the marginal utilities of the high-type agents are weighted more and more when setting the price. This result is entirely compatible with the results of DGP, where the price also increases with liquidity if the price is set by the buyers. Furthermore, it complements such studies as Harrison and Kreps (1978) and Scheinkman and Xiong (2003), showing that over-pricing in the presence of shorting constraints increases with the level of liquidity, as liquidity enables agents to trade so as to take advantage of the changing relative valuations of the asset.

The paper includes a numerical calibration that illustrates, first of all, that the exact price effect is, indeed, very small<sup>4</sup> (of the order of basis points) for parameters deemed reasonable, while the impact on positions is considerable. In effect, in this model, the ability to trade is not important for price determination, but it is an important determinant of agents' derived utilities. The liquidity level, therefore, is not immaterial. The example also shows that the introduction of short-sale constraints, on the other hand, can lead to an important liquidity impact on the price.

In conclusion, the analysis here suggests that illiquid assets such as private equity should be priced similarly to their liquid counterparts, as long as investors are not constrained exogenously regarding the positions they can take. Furthermore, an investor in such an asset should consider the future desirability of the asset to herself and tilt her position choice towards the one that she would prefer in the future, given that subsequent adjustments may be difficult, even impossible.

Naturally, the paper is related to the large body of search-based literature, starting with Diamond (1982). Recently, this literature has been extended to address asset-pricing issues such as liquidity premia in various kinds of markets and marketmaking.<sup>5</sup> In particular, this paper complements the analysis in this body of literature by deriving optimal portfolios and the price impact of liquidity without position restrictions, pointing out the extent to which the position restrictions are important for a significant quantitative price impact.

The issue of infrequent adjustment has also received attention recently, with papers such as Gabaix and Laibson (2002), Reis (2004), and Chetty and Szeidl (2004) modeling agents that adjust their consumption discretely. These papers do not concern themselves, however, with the determination of asset prices, or the choice of positions in financial assets. Costly consumption adjustments and portfolio choice, but not asset prices, are studied by Grossman and Laroque (1990).

Beside the infrequent-trading literature, this paper relates to the exogenous transaction cost literature, which it complements by deriving natural counterparts to the results found in that kind of environment. (See, for instance, Amihud and Mendelson (1986), Constantinides (1986), Vayanos (1998), and Huang (2003).) It is important, however, to note that the friction studied here is conceptually different from exogenous transaction costs. In particular, exogenous trading delays generate imperfect-trading

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<sup>4</sup>The illiquid-market price may be either larger or smaller than the Walrasian one.

<sup>5</sup>See, for instance, Duffie, Gârleanu, and Pedersen (2005) for a list of references.

utility losses endogenously.

The paper is organized as follows. Section 1 introduces and solves the basic model. Section 2 contains several extensions, treating deterministic market closures, exogenous transaction costs, multiple assets, heterogeneous investors, position constraints, and agent-specific liquidity, possibly chosen endogenously. Finally, Section 3 concludes and discusses future research avenues, while the Appendix contains proofs and some technical details.

## 1 Basic Model

This paper uses as a starting point the risk-aversion version of the model of DGP, with the difference that agents' holdings are unrestricted and that trades occur at prices set competitively by a continuum of agents. As in DGP, each agent trades at random times, independent of each other, but trades can also be permitted to take place at deterministic times or correlated random times.

More specifically, there are two assets in the economy: a riskless asset paying interest at a constant rate  $r$  and a risky asset paying a cumulative dividend with Gaussian increments:

$$dD(t) = m_D dt + \sigma_D dB(t), \quad (1)$$

where  $m_D$  and  $\sigma_D$  are constants,<sup>6</sup> and  $B$  is a standard Brownian motion with respect to the given probability space and filtration  $(\mathcal{F}_t)$ . The interest rate is given exogenously, and the riskless asset has a perfectly elastic supply. The per-capita supply of the risky asset is  $\Theta$ , and its price is determined in equilibrium.

There is a continuum of agents, with total mass normalized to 1. Agent  $i$  has a cumulative endowment process  $\eta^i$ , with

$$d\eta^i(t) = m_\eta dt + \sigma_\eta dB^i(t), \quad (2)$$

where the standard Brownian motion  $B^i$  is defined by

$$dB^i(t) = \rho^i(t) dB(t) + \sqrt{1 - \rho^i(t)^2} dZ^i(t), \quad (3)$$

with  $Z^i$  a standard Brownian motion independent of  $B$  and  $\rho^i(t)$  the instantaneous correlation between the asset dividend and the endowment of agent  $i$ . I assume that  $\rho^i$  follows a Markov process on a finite state space with  $J > 1$  points  $1 \geq \rho_1 > \dots > \rho_J \geq -1$ . The transition intensity from state  $j$  to state  $l$  is denoted by  $\alpha_{jl}$ . The processes  $B$ ,  $Z^i$ , and  $\rho^i$  for all  $i$  are mutually independent.

Agents have a time-separable constant-absolute-risk-aversion (CARA) utility function with coefficient  $\gamma > 0$ , and changes in correlation between dividends and endowment induce them to want to trade. I assume, however, that they cannot trade

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<sup>6</sup>The drift  $m_D$  could be taken mean-reverting with no loss of tractability, as in many other papers using the CARA-normal framework.

immediately. The primary interpretation is that the market for the asset is illiquid, in that an agent has to search for a qualified counterparty, or an opportunity to trade. For instance, there are many assets, such as given corporate bonds or shares in companies emerging from Chapter 11, that are only traded by a relatively small number of market participants, who have the required expertise. Finding such a participant that is able to take on a larger position, or willing to sell her stake, takes time. Additional time might be required to convince the counterparty that the sale is not motivated by information, too. Periods of market inactivity, as experienced by investors in private equity, for instance, can also be modeled, as done in an extension below. Another interpretation is that agents have a certain inertia concerning their asset positions, and therefore do not change their portfolios frequently.

I model the infrequent-trading feature by assuming that agents only have the possibility to trade at discrete times, given by jumps of Poisson processes. More specifically, each agent comes across a trading post (or competitive marketmaker) with Poisson intensity  $\lambda$ . For simplicity, I take  $\lambda$  constant. A time-varying intensity  $\lambda$  can be modeled with the same conclusions. Furthermore, allowing for additional trading times, random or deterministic, when the entire market convenes, leads to the same results. I discuss such a possibility as an extension to the model, in Section 2.1.

An agent possessing  $\theta$  shares of the asset has a value function defined as

$$J(w, \rho, \theta) = \sup_{\bar{c}, \bar{\theta}} E_t \left[ - \int_t^\infty e^{-r(s-t)} e^{-\gamma \bar{c}_s} ds \mid \rho(t) = \rho, W_t = w, \bar{\theta}(t) = \theta \right], \quad (4)$$

s.t.

$$dW_t = (rW_t - \bar{c}_t) dt + \bar{\theta}(t) dD_t + d\eta_t - P_t d\theta_t, \quad (5)$$

where  $W$  is the agent's total cash holding at any point in time,  $\bar{c}$  the agent's consumption, and  $\bar{\theta}$  the number of shares he owns in the risky asset. The optimization problem is further constrained by the requirement that the asset holding be chosen only at the arrival times of the Poisson process. To avoid Ponzi schemes, I impose the transversality condition

$$\lim_{T \rightarrow \infty} e^{-rT} E_t [e^{-r\gamma W_T}] = 0. \quad (6)$$

Standard calculations (see DGP for details) imply that the value function has the form

$$J(w, \rho, \theta, t) = -e^{-r\gamma(w + \bar{a} + a(\theta, \rho, t))}, \quad (7)$$

where

$$\bar{a} = \frac{1}{r} \left( \frac{\log r}{\gamma} + m_\eta - \frac{1}{2} r \gamma \sigma_\eta^2 \right) \quad (8)$$

is a constant.

The rest of the analysis will concentrate on stationary equilibria, independent of time, so the time argument will be dropped from all functions.<sup>7</sup> Let  $a_j(\theta) = a(\theta, \rho_j)$  be the value-function coefficient for an agent with correlation  $\rho_j$ . These coefficients obey the following Hamilton-Jacobi-Bellman equations.

$$-ra_j(\theta) = \sum_l \alpha_{jl} \frac{e^{-r\gamma(a_l(\theta) - a_j(\theta))} - 1}{r\gamma} + \lambda \sup_{\bar{\theta}} \frac{e^{-r\gamma(-P(\bar{\theta} - \theta) + a_j(\bar{\theta}) - a_j(\theta))} - 1}{r\gamma} - \kappa(\theta, \rho_j), \quad (9)$$

where

$$\kappa(\theta, \rho) = \theta m_D - \frac{1}{2} r\gamma \left( \theta^2 \sigma_D^2 + 2\rho\theta\sigma_D\sigma_\eta \right) \quad (10)$$

is the (mean-variance) instantaneous benefit to the agent from holding position  $\theta$  when having intrinsic type  $\rho$ .

In a stationary equilibrium, all agents with a given correlation  $\rho_j$  choose the same position  $\theta_j$ . The positions are determined so that agents maximize their utilities, implying that

$$P = a'_j(\theta_j). \quad (11)$$

Differentiating (9) with respect to  $\theta$ , we get

$$ra'_j(\theta_k) = \sum_l \alpha_{jl} e^{-r\gamma(a_l(\theta_k) - a_j(\theta_k))} (a'_l(\theta_k) - a'_j(\theta_k)) + \lambda e^{-r\gamma(-P(\theta_j - \theta_k) + a_j(\theta_j) - a_j(\theta_k))} (P - a'_j(\theta_k)) + \kappa_1(\theta_k, \rho_j), \quad (12)$$

where  $\kappa_1$  is the partial derivative of  $\kappa$  with respect to its first argument.

These equations cannot be solved in closed form. Consequently, I resort to an approximation that ignores terms of order higher than 1 in  $(a_l(\theta) - a_j(\theta))$ . The accuracy of this approximation depends on the size of  $r\gamma(a_l(\theta) - a_j(\theta))^2$ , which can be shown to be small when  $r^3\gamma^2(\rho_1 - \rho_j)^2\sigma_D^2\sigma_\eta^2$  is small. As noted by Vayanos and Weill (2005), another way to derive this result is in the limit as  $\gamma \rightarrow 0$  while holding  $\gamma\sigma_D\sigma_\eta$  constant. A rigorous statement is made in Theorem 1 below.

That gives

$$ra_j(\theta) = \sum_l \alpha_{jl} (a_l(\theta) - a_j(\theta)) + \lambda(-P(\theta_j - \theta) + a_j(\theta_j) - a_j(\theta)) + \kappa(\theta, \rho_j) \quad (13)$$

and

$$ra'_j(\theta_k) = \sum_l \alpha_{jl} (a'_l(\theta_k) - a'_j(\theta_k)) + \lambda(P - a'_j(\theta_k)) + \kappa_1(\theta_k, \rho_j). \quad (14)$$

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<sup>7</sup>Time variation, via fluctuation in  $\lambda$ ,  $\alpha_u$ , or  $\alpha_d$  can be incorporated in the analysis.



Note that the approximate HJB equations (13) obtain exactly when agents are risk-neutral, but the benefit from holding the asset is quadratic. More precisely, they obtain when the value functions are given by

$$a_j(\theta) = \sup_{\bar{\theta}} \mathbb{E}_t \left[ \int_t^\infty e^{-r(s-t)} \kappa(\bar{\theta}(s), \rho(s)) ds - \sum_{s=t}^\infty e^{-r(s-t)} P_s \Delta \bar{\theta}(s) \mid \rho(t) = \rho(0), \bar{\theta}(t) = \theta \right],$$

where trading is only possible at the arrival times of the individual Poisson process.

An immediate consequence is that, in equilibrium, for all  $k = 1, \dots, J$  it holds approximately<sup>8</sup> that

$$P_t = \mathbb{E}_t \left[ \int_t^\infty e^{-r(s-t)} \kappa_1(\theta(s), \rho(s)) \mid \theta(t) = \theta_k, \rho(t) = \rho_k \right], \quad (15)$$

where I re-introduce the time argument to make the point that the results are not dependent on concentrating on the steady state. Equation (15) is intuitive, stating that the price equals the sum of the stream of discounted marginal utilities from the asset at all future times. (The equation is easily derived by considering permanent deviations in holdings from the optimal ones.)

The main pricing result of the paper follows now from two simple observations. First, by the nature of the choice of trading agents, the agents accessing the market at time  $t$  are representative of the population, in the following sense: (i) they hold the average supply of the asset at any time  $s \geq t$ , and (ii) the distribution of intrinsic types among them is the same as in the population at any time  $s \geq t$ . Second, the function  $\kappa_1$  is linear. Consequently, when aggregating Equation (15) over all types trading at  $t$ , one gets

$$\begin{aligned} P_t &= \mathbb{E}_t \left[ \int_t^\infty e^{-r(s-t)} \kappa_1(\Theta, \bar{\rho}(t)) \right] \\ &= P_t^W, \end{aligned} \quad (16)$$

where  $\Theta$  is the average intrinsic type in the economy, independent of the trading process. Here,  $P^W$  is the price that would obtain in the corresponding Walrasian market. Note that stationarity is not required for this result.

In order to be more explicit, I introduce the following notation. First, let  $\mu_{jk}$  be the mass of agents of intrinsic type  $j$  that had type  $k$  the last time they traded. Let  $\mu_{k\cdot} := \sum_i \mu_{ki}$  be the mass of agents of intrinsic type  $\rho_k$ . Note that  $\mu_{k\cdot}$  depends only on the transition intensities  $\alpha$  and not on the trading technology. Also, let  $\tilde{\mu}$  be defined by

$$\tilde{\mu}_{jk} = \left( \mathbb{E}_t \left[ \int_t^\tau e^{-r(s-t)} ds \right] \right)^{-1} \mathbb{E}_t \left[ \int_t^\tau e^{-r(s-t)} 1_{(\rho(s)=\rho_j)} ds \mid \rho(t) = \rho_k \right], \quad (17)$$

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<sup>8</sup>Throughout the paper this word will be used to mean up to a term in  $O([\gamma(\rho_1 - \rho_J)\sigma_D\sigma_\eta]^2)$ .

where  $\tau$  is the first arrival time of a trading opportunity after time  $t$ . Thus  $\tilde{\mu}_{jk}$ , for various  $j$ , give the relative payoff weights of the promises to receive a dollar for any future time  $s$  such that  $\rho(s) = \rho_j$ , as long as  $\tau$  has not occurred by  $s$ , given that  $\rho(t) = \rho_k$ . Finally, let  $\hat{\mu}_{jk} = \mu_k \tilde{\mu}_{jk}$ . The quantities  $\mu$  and  $\tilde{\mu}$  are easily computed using standard Markov-chain calculations.

Consider an agent of type  $\rho_k$  given the opportunity to trade. The optimality of the choice  $\theta_k$  means that

$$P = E_0 \left[ \int_0^\tau e^{-rs} \kappa_1(\theta_k, \rho(s)) ds \mid \rho(0) = \rho_k \right] + E_0 [e^{-r\tau}] P, \quad (18)$$

implying

$$P = \frac{1}{r} \kappa_1 \left( \theta_k, \sum_j \tilde{\mu}_{jk} \rho_j \right). \quad (19)$$

Note that multiplying this relation by  $\mu_k$  and summing over all  $k$  yields (16) again, in the stationary equilibrium:<sup>9</sup>

$$P = \frac{1}{r} \kappa_1 (\Theta, \bar{\rho}). \quad (20)$$

Using the expression for  $P$  in Equation (19) the optimal quantity choice  $\theta_k$  is calculated to be

$$\theta_k = \Theta + \frac{\sigma_\eta}{\sigma_D} \left( \bar{\rho} - \sum_j \tilde{\mu}_{jk} \rho_j \right). \quad (21)$$

The Walrasian holdings can be obtained in the limit as  $\lambda \rightarrow \infty$ , which gives  $\tilde{\mu}_{jk} \rightarrow 1_{(j=k)}$ , thus implying

$$\theta_k^W = \Theta + \frac{\sigma_\eta}{\sigma_D} (\bar{\rho} - \rho_k). \quad (22)$$

The expression for the equilibrium holdings is natural. The first term is the per-capita supply. The second reflects the difference in vulnerability to the asset-payoff risk between the average agent and the agent considered. Thus, if the correlation between the agent's endowment and the asset dividend is going to be relatively high, in expectation, until the next trading opportunity, then the agent will hold a lower position, and vice-versa. In particular, if the agent can trade continuously, then it is the difference between the average correlation and his current correlation that gives the holding, as can be seen in Equation (22).

The results derived above are collected in the following.

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<sup>9</sup>It is shown in the Appendix that, in a stationary equilibrium,  $\sum_j \hat{\mu}_{jk} = \sum_j \hat{\mu}_{kj} \equiv \mu_k$ .

**Theorem 1** *The economy studied has a stationary equilibrium, determined by equations (9), (11), and (12). The value function and consumption are given by*

$$\begin{aligned} J(w, \rho, \theta) &= -e^{-r\gamma(w+\bar{a}+a(\rho,\theta))} \\ c(w, \rho, \theta) &= -\frac{\log(r)}{\gamma} + r(w + \bar{a} + a(\rho, \theta)). \end{aligned}$$

Furthermore, fix parameters  $\bar{\gamma}$ ,  $\bar{\sigma}_D$ , and  $\bar{\sigma}_\eta$  and let  $\sigma_D = \bar{\sigma}_D\sqrt{\bar{\gamma}/\gamma}$  and  $\sigma_\eta = \bar{\sigma}_\eta\sqrt{\bar{\gamma}/\gamma}$ . Then, as  $\gamma$  goes to zero, the limit price is

$$P = \frac{\mu_D}{r} - \bar{\gamma} (\bar{\sigma}_D^2 \Theta + \bar{\rho} \bar{\sigma}_D \bar{\sigma}_\eta)$$

with  $\bar{\rho} = \sum_j \mu_j \rho_j$ , while the limit positions equal

$$\theta_k = \Theta + \frac{\bar{\sigma}_\eta}{\bar{\sigma}_D} \left( \bar{\rho} - \sum_j \tilde{\mu}_{jk} \rho_j \right).$$

The liquidity effect on positions is intuitive and easily understood: knowing that she may get stuck with an undesirable position for a period of time, an agent will tilt her choice towards the positions desired in the other states in which she is most likely to have to keep the position chosen now. In particular, this suggests that agents take less extreme positions in illiquid markets. Furthermore, it follows that the average trade size is smaller, which reduces volume beyond the direct effect of a worse ability to conduct a trade.

Without restrictions on the transition matrix of the correlation process, though, it does not follow that less liquidity always results in positions closer to the per-capita supply. In fact, part (i) of the proposition below states a non-trivial necessary condition for such a result, namely that the expected correlation conditional on a transition during the next instant be smaller than the current correlation if and only if the average correlation is.<sup>10</sup> This is true because, for liquidity levels high enough, an agent's quantity choice increases with liquidity if his expected type change is positive: a positive expected type change leads to a smaller position than in a Walrasian world, in which only the current type matters.

The following statements hold.

**Proposition 2** (i) *For any trading frequency  $\lambda < \infty$ ,  $\theta_1^W < \theta_k < \theta_J^W$  for all  $k$ . There exists  $\underline{\lambda} < \infty$  such that, for  $\lambda > \underline{\lambda}$ ,  $\theta_k$  is monotonic in  $\lambda$  for all  $k$ . Furthermore,  $\theta_k$  increases strictly in  $\lambda$  for  $\lambda > \underline{\lambda}$  if and only if*

$$\frac{\sum_{j \neq k} \alpha_{kj} \rho_j}{\sum_{j \neq k} \alpha_{kj}} > \rho_k, \quad (23)$$

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<sup>10</sup>I am indebted to Pierre-Olivier Weill for suggesting first-order approximations in  $1/\lambda$ .

$\alpha_{12}$	$\alpha_{21}$	$\gamma$	$\rho_l$	$\rho_h$	$r$	$m_D$	$\sigma_D$	$\sigma_\eta$	$\Theta$
10	2	0.8	0.8	0.2	0.1	50	20	20	0.25

Table 1: Parameters used to illustrate price and position impact of variable liquidity level.

and vice-versa. In particular,  $\theta_1$  is decreasing and  $\theta_J$  is increasing in  $\lambda$  for  $\lambda > \underline{\lambda}$ .  
(ii) If there are only two types ( $J = 2$ ), then

$$\begin{aligned}
\theta_1 &= \Theta - \alpha_{12} \left( \frac{1}{\alpha_{12} + \alpha_{21}} - \frac{1}{r + \lambda + \alpha_{12} + \alpha_{21}} \right) \frac{\sigma_\eta(\rho_l - \rho_h)}{\sigma_D} \\
&= \theta_1^W + \frac{\alpha_{12}}{r + \lambda + \alpha_{12} + \alpha_{21}} \frac{\sigma_\eta(\rho_l - \rho_h)}{\sigma_D} \\
\theta_2 &= \Theta + \alpha_{21} \left( \frac{1}{\alpha_{12} + \alpha_{21}} - \frac{1}{r + \lambda + \alpha_{12} + \alpha_{21}} \right) \frac{\sigma_\eta(\rho_l - \rho_h)}{\sigma_D} \\
&= \theta_2^W - \frac{\alpha_{21}}{r + \lambda + \alpha_{12} + \alpha_{21}} \frac{\sigma_\eta(\rho_l - \rho_h)}{\sigma_D},
\end{aligned}$$

and  $\theta_1$  and  $\theta_2$  are monotonically decreasing, respectively increasing, in  $\lambda$  for all  $\lambda$ . Furthermore, the rate with which agents trade,

$$\lambda(\mu_{12} + \mu_{21}) = 2\lambda \frac{\alpha_{12}\alpha_{21}}{(\alpha_{12} + \alpha_{21})(\lambda + \alpha_{12} + \alpha_{21})},$$

and the trading volume,

$$\lambda(\mu_{12} + \mu_{21})(\theta_2 - \theta_1),$$

increase with  $\lambda$ .

The result on trade characteristics (trade size and volume) helps point out the complex impact of liquidity on trading volume: past liquidity determines the number of agents ( $\mu_{12} + \mu_{21}$ ) that would trade if given the opportunity (this decreases with the level of liquidity), current liquidity the rate ( $\lambda$ ) with which such agents actually get to trade, while future liquidity the positions to which they wish to trade, thus influencing the average trade size ( $\theta_2 - \theta_1$ ).<sup>11</sup>

<sup>11</sup>This point can be made even more saliently in a model that departs from steady-state analysis to allow for liquidity to change at some time  $T$  from  $\lambda$  to a different level,  $\lambda'$ . It follows then, under natural conditions, that  $\theta_2(t) - \theta_1(t)$  increases with  $\lambda'$  for  $t \leq T$ , while  $\mu_{12}(t) + \mu_{21}(t)$  decreases with  $\lambda$  for  $t \geq T$ . See Proposition 6 in the Appendix.

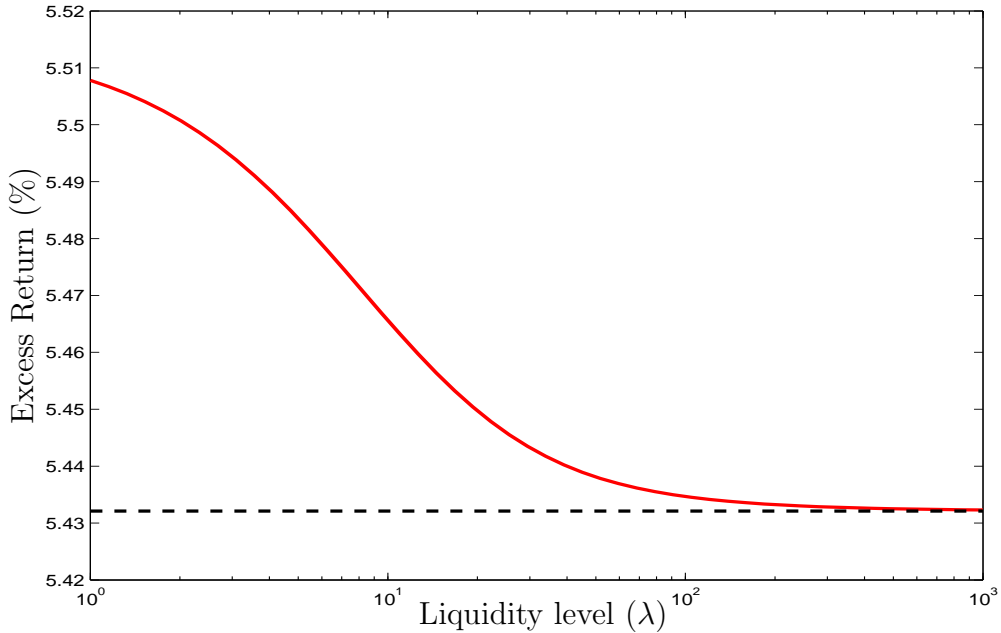


Figure 1: Excess-return impact of illiquidity (parameters given by Table 1). The continuous line plots the excess return computed numerically as a function of the meeting intensity  $\lambda$ , while the dashed line indicates the Walrasian excess return.

## 1.1 Numerical Example

To illustrate the theoretical results derived so far, I calibrated the model. I consider two types of agents and use the parameters in Table 1 to calculate the exact equilibrium price, as well as the linear approximation to the price for a range of liquidity values ( $\lambda$ ). Exact and approximated positions are also calculated.

The parameters are understood as follows: it takes, on average, one tenth of a year for an agent's endowment correlation with the asset to jump back to the low level ( $\rho_l = 0.2$ ) from the high level ( $\rho_h = 0.8$ ), and half a year for the opposite change. On average, there are  $\Theta = 0.25$  shares outstanding per agent. Together with the risk-aversion coefficient  $\gamma = 0.8$ , these parameters result in an equity premium around 5.4%.

Rather than reporting the price, I choose to report the more easily interpretable excess return on the asset, defined as  $m_D/P - r$ . Figure 1 shows that the excess return does, indeed, vary with liquidity, but that even for low levels of liquidity the impact is very small – for instance, when  $\lambda = 10$ , i.e., wait more than 1 month to trade, the return impact is smaller than 4bp. The excess return is noted to decrease with liquidity towards the Walrasian value, but it can also increase, for different parameters.

Positions, on the other hand, are much more sensitive to liquidity, as can be seen

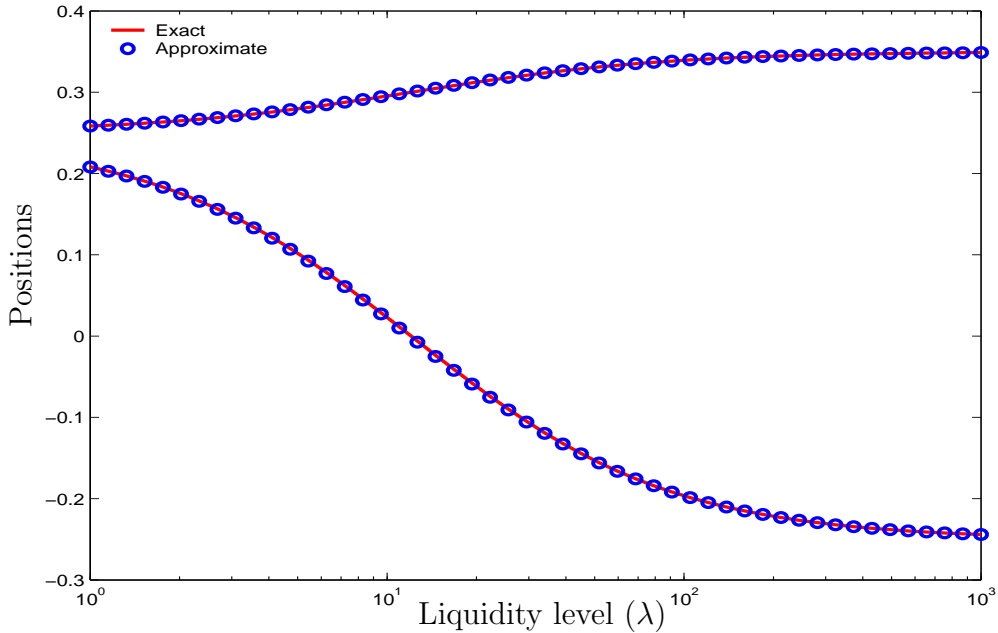


Figure 2: Portfolio sensitivity to illiquidity (parameters given by Table 1). The continuous line plots the exact positions, while the circles show the positions calculated using the approximation.

in Figure 2. For instance, if one can trade once a week on average ( $\lambda = 52$ ), the lower position is about -0.156, which makes it 20% closer to the per-capita supply  $\Theta = 0.25$  than the Walrasian position, -0.25. The same is true, of course, of the high position.<sup>12</sup> With less frequent trading, the deviation from the Walrasian positions is even stronger. For instance, with trades every month, on average, the low type is virtually abstaining from trading – more precisely, he has a long position of size 0.0005 in the asset.

## 2 Extensions

The analysis can be extended in multiple directions, to address features of interest of a given market. In particular, the set-up allows for the study of more than one assets, possibly of different liquidities. One can also consider position constraints and derive their pricing implications across different liquidity regimes. Furthermore, heterogeneous investor liquidity, be it exogenous or endogenous, can also be studied.

The result that the price is invariant to the liquidity level depends on two conditions:

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<sup>12</sup>The parameters were chosen to result in short selling in order to illustrate what happens when short sales are not allowed, in Section 2.5.

(i) marginal utility is linear, and (ii) the agents setting the price are representative of the entire population. The extensions below, then, can be divided into two classes: those that do not invalidate the conditions, and in which, consequently, there still is no price impact, and those that do. In particular, in the latter generalizations that I consider the second condition, on the representativeness of the price setters, no longer holds. This has an interesting impact on the price, which I analyze.

In the first kind of extensions, I consider multiple assets, non-constant liquidity, including market closures, and heterogeneous agents. While there still is no price effect in any of these situations, each of them has implications for portfolio choice. The second kind of extensions include position constraints (such as short-selling restrictions) and agent-type dependent, possibly endogenous, search ability.

These extensions illustrate the tractability of the approach and its applicability to a wide range of issues. Secondly, they shed further light on the robustness of the pricing and portfolio-choice results of the basic model.

## 2.1 Market Closures

The results of the basic model also hold when illiquidity takes the form of market closures. Whether the market operates at discrete, deterministic times, or whether it is open for certain time periods and closed for others, this feature can be easily accommodated by the model.

I illustrate here by changing the model as follows. First, let the trading intensity be time dependent – in particular, it can be 0 for given intervals, signifying a market closure. Second, let there be a sequence of pairs of times, possibly infinite,  $T_1^o \leq T_1^c < T_2^o \leq T_2^c < \dots$ , such that all agents have access to a centralized market between  $T_i^o$  and  $T_i^c$  regardless of the value of  $\lambda$ . (Informally speaking, one could think of  $\lambda$  as being infinite on these intervals.)

Up to the approximation made in the basic model, Equation (15) continues to hold in the modified model, and therefore the price is still approximately equal to the Walrasian price. The optimal portfolio choices can also be computed. To that end, consider an agent of intrinsic type  $k$  that trades at time  $t$ . Letting  $\tau$  be the first time larger than  $t$  when the agent can trade again, it holds that

$$\begin{aligned} P_t - \mathbb{E}_t \left[ e^{-r\tau} P_\tau \right] &= \mathbb{E}_t \left[ \int_t^\tau e^{-r(s-t)} \kappa_1(\theta_k(t), \rho(s)) ds \mid \rho(t) = \rho_k \right] \\ &= \mathbb{E}_t \left[ \int_t^\tau e^{-r(s-t)} \kappa_1(\Theta, \bar{\rho}) ds \right]. \end{aligned} \quad (24)$$

This is a linear equation in  $\theta_k(t)$ , the agent's choice of holding at time  $t$ . The equation only depends on the joint distribution of  $\tau$  and the intrinsic type  $\rho$ , and standard Markov-chain calculations yield the result immediately.

While the calculation details may be somewhat different from the case of random and independent trading times, the intuition for the determination of the position choice is very similar to that in the basic model: knowing that the positions may not be adjustable for a certain period of time, agents incorporate in their decision their future benefits from ownership. In particular, an agent with very high present need for the asset takes a smaller position than in a perfect market, given that he may be forced to keep the same position even if his need diminished.

## 2.2 Transaction Costs

In addition to the difficulty of accessing the market or finding a counterparty, trading often imposes exogenous transaction costs, such as brokerage fees or bid-ask spreads. This subsection extends the model to allow for such a case. In particular, let every agent trading have to pay transaction costs proportional to the dollar amount traded, so that the unit price paid by the buyer is  $P(1 + q)$  and the one received by the seller is  $P(1 - q)$ , for some  $q$  with  $0 \leq q < 1$ . It follows that, for any agent buying at time  $t$ ,

$$P_t(1 + q) = \mathbb{E}_t \left[ \int_t^\infty e^{-r(s-t)} \kappa_1(\theta(s), \rho(s)) ds \mid \theta(t) = \theta_k^b, \rho(t) = \rho_k \right], \quad (25)$$

while, for any seller,

$$P_t(1 - q) = \mathbb{E}_t \left[ \int_t^\infty e^{-r(s-t)} \kappa_1(\theta(s), \rho(s)) ds \mid \theta(t) = \theta_k^s, \rho(t) = \rho_k \right]. \quad (26)$$

Note that, for any intrinsic type  $\rho_i$  there are two positions an agent would trade to,  $\theta_i^b$  if he buys and  $\theta_i^s$  if he sells. Furthermore, an agent of type  $j$  holding the optimal position of an agent of type  $i \neq j$  may not trade when given the opportunity, if the transaction costs are large. If the transaction cost  $q$  is small enough, however, such an agent will always trade if he can. If, in addition, a stationary setting is considered, then agents whose type is the same as the last time they traded continue to be marginal. That is, Equation (25) holds for all agents of type  $(\theta_i^b, \rho_i)$ , while Equations (26) holds for all agents of type  $(\theta_i^s, \rho_i)$ . Consequently, one of Equations (25)–(26) holds for any agent accessing the market at time  $t$ .

Let  $\mu_{ij}^b$  represent the total mass of agents of intrinsic type  $\rho_i$  who holds  $\theta_j^b$ , and define  $\mu_{ij}^s$  similarly. Finally, let  $\mu^b = \sum_{i,j} \mu_{ij}^b$  and  $\mu^s = \sum_{i,j} \mu_{ij}^s$ . Aggregating Equations (25)–(26) over all agents accessing the market at time  $t$ , who are representative of the entire economy,

$$\begin{aligned} P(1 + q(\mu^b - \mu^s)) &= \mathbb{E}_t \left[ \int_t^\infty e^{-r(s-t)} \kappa_1(\theta(s), \rho(s)) ds \right] \\ &= \mathbb{E}_t \left[ \int_t^\infty e^{-r(s-t)} \kappa_1(\Theta, \bar{\rho}) ds \right] \\ &= P^W. \end{aligned} \quad (27)$$



Note that the mass of buyers need not equal that of sellers. (Market clearing means that the *number of shares* bought must equal that of shares sold.) Consequently, transaction costs have a price effect, which depends on the level of liquidity  $\lambda$ .<sup>13</sup> The price effect is negative if, in steady state, there are more buyers than sellers, and otherwise is positive. This feature is intuitive: Given the marginal benefit of holding (an additional unit of) the asset in perpetuity, buyers require price discounts, while sellers price premia, in order to trade. In equilibrium, the positions adjust so that all agents trade, but in order to attract more buyers than sellers, a price discount is required, and vice-versa.

Note that, in this expression for the price, there is no term capturing the frequency of trade as in Amihud and Mendelson (1986) and Vayanos (1998). Furthermore, transaction costs can make an asset price higher. Indeed, the intuition that the required return is increased by a measure of the amortized future transaction costs relies on the life-cycle of an agent, who initially buys the asset, then sells it and exits the economy. Over long periods of time, this is a reasonable description of market participants, but arguably less so over shorter periods: institutions, in particular, do not have severely limited life spans. In this case, the intuition in this paper may be the more relevant one: On one hand, an agent would value the asset less now because of transaction costs to be paid when selling in the future; on the other hand, buying the asset now means that the agent will *save* transaction costs when wanting to buy in the future, resulting in a higher current valuation.

The link with the literature can be made clearer by assuming finite agent life spans. Specifically, suppose that every agent may have to leave the economy with intensity  $\pi$ . In this case, the agent has immediate access to the market, where he liquidates his position. The bequeath function is defined as if the agent could only invest in the risk-free asset from then onwards:

$$V(W) = -e^{-r\gamma W}. \quad (28)$$

---

<sup>13</sup>It can be shown that, as  $\lambda$  tends to infinity, the price approaches

$$\lim_{\lambda \rightarrow \infty} P = P^W \left( \sum_i \frac{\sum_{j < i} \alpha_{ij} \mu_j - \sum_{j > i} \alpha_{ij} \mu_j}{\sum_{j \neq i} \alpha_{ij}} \right)^{-1},$$

where  $\mu_j$  is the steady-state mass of agents of type  $\rho_j$ .

Consequently, Equations (25)–(26) become

$$P(1 + q) = \mathbb{E}_t \left[ \int_t^{\tau_\pi} e^{-r(s-t)} \kappa_1(\theta(s), \rho(s)) ds \mid \text{buy at } t \right] + \quad (29)$$

$$\mathbb{E}_t \left[ e^{-r(\tau_\pi-t)} P(1 - q1_{(\theta_{\tau_\pi} > 0)} + q1_{(\theta_{\tau_\pi} < 0)}) \mid \text{buy at } t \right]$$

$$P(1 - q) = \mathbb{E}_t \left[ \int_t^{\tau_\pi} e^{-r(s-t)} \kappa_1(\theta(s), \rho(s)) ds \mid \text{sell at } t \right] + \quad (30)$$

$$\mathbb{E}_t \left[ e^{-r(\tau_\pi-t)} p(1 - q1_{(\theta_{\tau_\pi} > 0)} + q1_{(\theta_{\tau_\pi} < 0)}) \mid \text{sell at } t \right],$$

where  $\tau_\pi$  is the arrival time of exit.

To preserve stationarity, entry is assumed at the same rate as exit, (intrinsic) types being drawn from the stationary distribution. When aggregating the pricing equations above, only agents choosing their position freely are considered: agents already owning positions and trading to different positions, and agents trading for the first time. Equations (29)–(30) do not hold for agents exiting the economy.

Let  $\iota^b$  and  $\iota^s$  be the inflows of buyers and sellers — the ones who, in equilibrium, take a short position — with  $\iota^b + \iota^s = \pi$ ;  $\lambda\mu^b$  and  $\lambda\mu^s$  are the flows to the market of buyers and sellers from the pool of agents already in the economy. Aggregating Equations (29)–(30) yields

**Proposition 3** *With proportional transaction costs and entry and exit, the steady-state price satisfies*

$$P(\lambda + \iota + q(\lambda(\mu^b - \mu^s) + \iota^b - \iota^s)) \quad (31)$$

$$= (\lambda + \iota) \mathbb{E}_t \left[ \int_t^{\tau_\pi} e^{-r(s-t)} \kappa_1(\theta(s), \rho(s)) ds \right] + (\lambda + \iota) \mathbb{E}_t \left[ e^{-r(\tau_\pi-t)} P \right] +$$

$$q(\lambda + \iota) \mathbb{E}_t \left[ e^{-r(\tau_\pi-t)} P(-1_{(\theta_{\tau_\pi} > 0)} + 1_{(\theta_{\tau_\pi} < 0)}) \right].$$

*If there is no shorting, in equilibrium, the equation above simplifies to*

$$P(\lambda + \iota + q(\lambda(\mu^b - \mu^s) + \iota)) = (\lambda + \iota) \left( \frac{r}{r + \pi} P^W + (1 - q) \frac{\pi}{r + \pi} P \right). \quad (32)$$

*As  $\lambda \rightarrow \infty$ , the price approaches*

$$P = P^W \frac{r}{r + \pi q + q(r + \pi)(\mu^b - \mu^s)}. \quad (33)$$

Equation (33) captures both price effects that transaction costs have in this setting:  $\pi q$  represents the proportional loss  $q$  incurred with frequency  $q$ , and  $q(\mu^b - \mu^s)$  the imbalance between losses to buyers and losses to sellers. In particular, even in the case  $\mu^b = \mu^s$ ,  $P < P^W$ , a conclusion that does not depend on the level  $\lambda$  of market liquidity.

## 2.3 Multiple Assets

Suppose that the agents can invest in more than one illiquid risky assets, and that trading times for different assets are independent for any given agent.<sup>14</sup> Given the approximation used, the solution to the allocation problem, once again, is the same as that in a situation where utility flows are given by the instantaneous mean and variance of the flow of dividends and endowments. As an implication, however, if at least two assets are not fully liquid, then the steady-state distribution must have infinite support, since agents cannot update positions in these assets simultaneously. Explicit computation of positions, consequently, is quite difficult. One can say, nevertheless, that the price is not affected by the liquidity level to the first order in the gains from trade.

An interesting and simpler case is that in which one asset only is illiquid, while the other ones are perfectly liquid. For simplicity, let there be only two risky assets. Specifically, assume that asset  $i \in \{1, 2\}$  is characterized by trade intensity  $\lambda^i$  with  $\lambda^2 = \infty$ . Here, again, because the holding of asset 2 is adjusted simultaneously with the holding of asset 1 or with any change in intrinsic type, there are a finite number of types in a steady-state equilibrium ( $J^2$ , in fact). Clearly, the degree of liquidity  $\lambda^1$  of asset 1 impacts the positions in asset 1 that agents choose to take. Furthermore, if the two assets have correlated dividends, then the positions in asset 2 are also impacted. While less liquidity makes asset 1 be traded in smaller amounts, asset 2 may be traded either in larger or in smaller amounts, depending on whether the two assets are complements or substitutes in the agents' optimization problems.

I first describe the modified model formally, and then present two concrete examples. Let there be two assets, paying Gaussian dividends with volatilities  $\sigma_{D_1}$  and  $\sigma_{D_2}$ . The correlation between the dividend innovations is a constant  $\nu$ , and the variance-covariance matrix is denoted by  $\Sigma_D$ . The dividend innovations are also correlated with the endowment innovations, the correlation vector  $\rho(t) = (\rho^{(1)}(t), \rho^{(2)}(t))^\top$  being a Markov process for each agent.

Following an analogous argument to the case of one asset, the following obtains.

**Proposition 4** *In the setting with one illiquid and one liquid assets, the price satisfies, approximately,*

$$\begin{aligned} Pr &= \mu_D - \gamma r \Sigma_D \theta_k - \gamma r \sigma_\eta \sum_j \tilde{\mu}_{jk} \left( \sigma_{D_1} \rho_j^{(1)}, \sigma_{D_2} \rho_j^{(2)} \right)^\top \\ &= \mu_D - \gamma r \Sigma_D \Theta - \gamma r \sigma_\eta \sum_j \tilde{\mu}_{jk} \left( \sigma_{D_1} \bar{\rho}^{(1)}, \sigma_{D_2} \bar{\rho}^{(2)} \right)^\top, \end{aligned} \quad (34)$$

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<sup>14</sup>If, instead, every agent can trade all assets simultaneously, albeit not continuously – because of inertia or inattention, or because all assets are traded in the same market – then the solution is essentially the same as that for one asset, subject to the obvious generalization. Details are omitted.

while the optimal holdings are related to the Walrasian holdings by

$$\theta_k = \theta_k^W + \sigma_\eta \Sigma_D^{-1} \sum_j \tilde{\mu}_{jk} \left( \sigma_{D_1} (\rho_k^{(1)} - \rho_j^{(1)}), \sigma_{D_2} (\rho_k^{(2)} - \rho_j^{(2)}) \right)^\top. \quad (35)$$

For some concrete examples, consider first an illiquid asset used to hedge a certain kind of risk – say, corporate bonds used to hedge exposure to credit risk – and a liquid asset used to hedge another kind of risk carried by the first asset – say, treasury issues used to hedge the interest-rate exposure of the corporate bonds. To model this situation, I take  $\rho^{(2)} = 0$  and  $0 < \nu < 1$ .

Letting  $J = 2$  to gain simplicity, it follows that  $\theta_k^{(1)}$  and  $\theta_k^{(2)}$  are monotonic, with

$$\begin{aligned} \theta_1^{W(1)} &< \theta_1^{(1)} < \theta_2^{(1)} < \theta_2^{W(1)}, \\ \theta_1^{W(2)} &> \theta_1^{(2)} > \theta_2^{(2)} > \theta_2^{W(2)}. \end{aligned}$$

A high correlation  $\rho^{(1)}$  between the endowment and the first asset<sup>15</sup> induces the agent to take a low position in this asset, and consequently a large position in asset 2 (the dividends of the two assets are negatively correlated).

Furthermore, a deterioration in asset-1 (“corporate-bond”) liquidity results in positions closer to the per-capita supply, therefore smaller variation in the exposure that can be hedged with asset 2 (“interest-rate exposure”), whence smaller trades in asset 2 (“treasuries”).

As a second example, suppose that the liquid and illiquid asset are both used to hedge the same kind of risk — for instance, on-the-run and off-the-run treasuries used to hedge interest-rate risk. To capture this notion, let  $\rho^{(1)} = \rho^{(2)}$  and  $0 < \nu < 1$ .

Setting  $J = 2$  once again, it follows that  $\theta_k^{(1)}$  and  $\theta_k^{(2)}$  are monotonic, but this time

$$\begin{aligned} \theta_1^{W(1)} &< \theta_1^{(1)} < \theta_2^{(1)} < \theta_2^{W(1)}, \\ \theta_1^{(2)} &< \theta_1^{W(2)} < \theta_2^{W(2)} < \theta_2^{(2)}. \end{aligned}$$

This time, a high correlation  $\rho^{(1)} = \rho^{(2)}$  induces positions in both assets to be low. The interesting feature is that, since a deterioration in the liquidity of asset 1 (the “off-the-run treasuries”) results in less extreme positions in this asset, it also induces more variable positions in the second asset (the “on-the-run treasuries”), which is a substitute.

## 2.4 Heterogeneous Investors

One can easily extend the model to allow for agents to have, intrinsically, different abilities to search, or different access to markets. Provided that the intrinsic-type distribution is independent of trading ability, though, such heterogeneity makes no

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<sup>15</sup>Remember that  $\rho_1 > \rho_2$ .

difference to pricing. As argued before, the price-invariance result depends on the linearity of marginal utility and the fact that the agents trading (thus, setting the price) at a given time are representative of the entire market. While the agents with a higher trading ability are over-represented, they set approximately the same price as the ones with lower ability, namely the Walrasian one.

The positions of the different types of agents, of course, will not be the same. This setting allows, in fact, to derive within the same economy the comparative-static results across economies from Section 1. Going further, one could also study endogenous intermediation in this market,<sup>16</sup> provided that liquid agents are offered compensation for their services, but that is not done here.

## 2.5 Position Constraints

One way to summarize the price implication of the basic model is to say that agents adjust their investment strategy to the liquidity level, as a result diminishing considerably the sensitivity of the price to liquidity. If the position-adjustment channel is shut, on the other hand, intuition suggests that the price should suffer directly from changes in liquidity.

Indeed, assume that, at all times, every position  $\theta_k$  must satisfy  $\theta_k \geq \underline{\theta}$ .<sup>17</sup> For any position  $\theta_k > \underline{\theta}$  chosen optimally, the pricing equation (19) holds:

$$\mu_k \cdot Pr = \sum_j \hat{\mu}_{jk} \kappa_1(\theta_k, \rho_j). \quad (36)$$

Let us assume that  $\theta_k > \underline{\theta}$  if and only if  $k > k_0$ . Aggregating (36) over the values of  $k$  for which it holds, i.e.,  $k > k_0$ , yields

$$\left( \sum_{k > k_0} \mu_k \cdot \right) Pr = \sum_{j, k > k_0} \hat{\mu}_{jk} \kappa_1(\theta_k, \rho_j),$$

or

$$Pr = \kappa_1 \left( \left( \sum_{k > k_0} \mu_k \cdot \right)^{-1} \left( \sum_{k > k_0} \mu_k \cdot \theta_k \right), \left( \sum_{k > k_0} \mu_k \cdot \right)^{-1} \left( \sum_{j, k > k_0} \hat{\mu}_{jk} \rho_j \right) \right).$$

While lengthy, the expression above is as natural as Equation (20): the price is given by the per-capita supply of assets held away from the constraints, and the average discounted intrinsic type among unconstrained agents.

<sup>16</sup>Duffie, Gârleanu, and Pedersen (2005) study intermediation, including the endogenous choice of the level of intermediation, in a bilateral trade model.

<sup>17</sup>It is more meaningful to restrict  $\underline{\theta}$  to  $\underline{\theta} \leq 0$ .

Since the amount of the asset held by the constrained agents is independent of the liquidity in the market (as long as the same types are constrained), the average holding of unconstrained agents is constant. Consequently, the price dependence on the liquidity level is determined by the term

$$\sum_{j,k>k_0} \hat{\mu}_{jk} \rho_j.$$

Intuition suggests that, as liquidity improves, fewer agents, on average, will be forced to hold  $\theta_k$  when they prefer  $\theta_j$ . In other words,  $\hat{\mu}_{jj}$  goes up, while  $\hat{\mu}_{jk}$  for  $j \neq k$  goes down with  $\lambda$ . Since  $\rho$  decreases in  $j$ , this would imply that the average correlation affecting the price decreases, and thus the price increases.

While intuitive, the argument above is not correct, because the masses  $\hat{\mu}_{jk}$  need not be monotonic in  $\lambda$ . In general, though, the following holds.

**Proposition 5** *Assume that positions are constrained to satisfy  $\theta_k \geq \underline{\theta}$  and consider parameters for which the constraint binds. Then the following holds, up to second-order terms in  $\gamma(\rho_1 - \rho_2)\sigma_D\sigma_\eta$ .*

- (i) *There exists a value  $\underline{\lambda} > 0$  such that the price  $P$  increases in  $\lambda$  for  $\lambda > \underline{\lambda}$ .*
- (ii) *If there are only two types ( $J = 2$ ), then the price is given by*

$$Pr = \kappa_1(\theta_2, \rho_2) - \frac{\alpha_{21}}{r + \lambda + \alpha_{12} + \alpha_{21}} r \gamma (\rho_1 - \rho_2) \sigma_D \sigma_\eta \quad (37)$$

*and it increases in  $\lambda$  for all  $\lambda > 0$ .*

Naturally, the converse of Proposition 5, concerning binding upper limits on positions, is also true. It is debatable, though, whether binding upper limits, due perhaps to agency issues, arise as naturally as lower such as short-sale constraints. Interestingly, with more than 2 types, both lower and upper bounds can bind for open sets of parameters. In that case, the dependence of price on liquidity is parameter specific.

The result of Proposition 5 relates to the findings of Harrison and Kreps (1978) and Scheinkman and Xiong (2003). In a setting with shorting constraints and differences of opinions, they find that the price is increased by the re-sale option of the asset in the future. Proposition 5 adds the natural refinement that the price increase is higher when trading is easier.

It is instructive to extend the numerical example to the case of shorting constraints. To that end, assume that short sales are not allowed, and compute the price again for a variety of levels of liquidity, given the parameters in Table 1. As can be seen in Figure 2, and is reflected in Figure 3, for low levels of liquidity the optimal holdings are both positive, so the constraints do not bind and the price is very close to the Walrasian price. Beyond a certain threshold, though, the high-correlation agents does not hold any amount of the asset, which fixes the holding of the other type of agent,

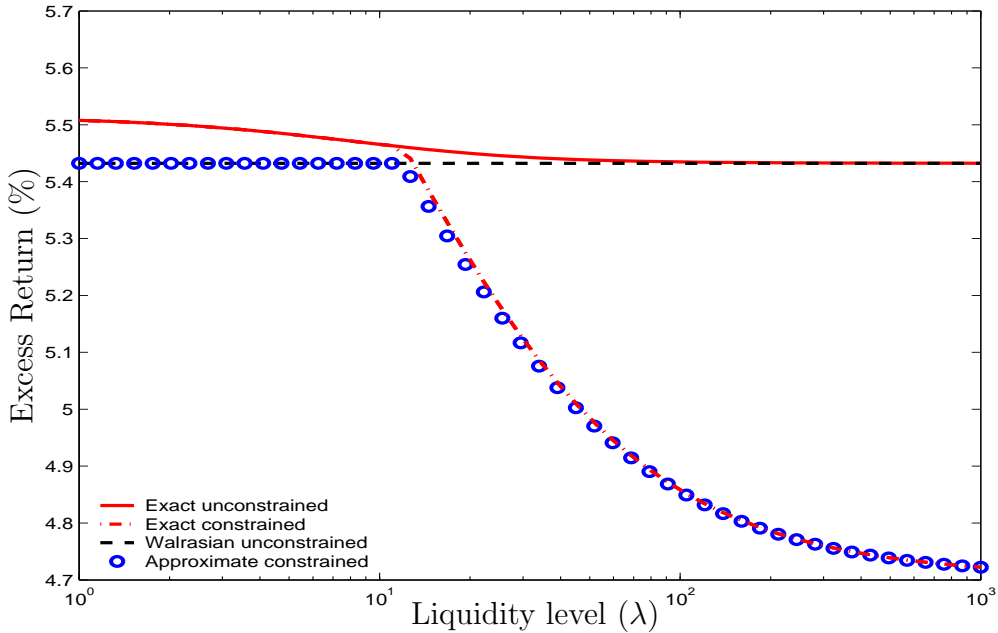


Figure 3: Excess-return impact of illiquidity in the presence of short-sale constraints (parameters given by Table 1). The continuous line plots the exact unconstrained-holding excess return, the dash-dot line plots the exact constrained-holding excess-return, while the circles graph the approximate constrained-holding excess-return. The dashed line shows the Walrasian level.

too. The excess return decreases as a consequence, to the effect that it becomes about 80bp lower than in the illiquid market, as  $\lambda$  goes to infinity. The reason, once again, is the agents' inability to adjust positions to the level of liquidity.

The numerical example shows that the direct effect of liquidity on the excess return when the positions are constrained can be significant (80bp), and at the same time virtually canceled by the effect of the endogenous position adjustment.

## 2.6 Endogenous Meeting Intensities

Trading delays can be the result of exogenous factors such as no-trading requirements or market closures. They can also come about because finding a trading partner consumes a significant amount of resources. In this case, it would be natural that agents choose the intensity with which they access the market as a function of its cost and the benefits from trading.

Taking the equilibrium price as given, the intensity choice can be written as  $\lambda(\theta, \rho)$ , or  $\lambda_k(\rho)$  if  $\theta = \theta_k$ . Let  $c(\lambda)$  be the per-unit-of-time cost for a choice  $\lambda$ . Then, in steady

state, the linearized HJB equation (13) becomes

$$\begin{aligned}
ra_j(\theta) &= \sum_l \alpha_{jl}(a_l(\theta) - a_j(\theta)) + \kappa(\theta, \rho_j) + \\
&\quad \max_{\lambda} (\lambda(a_j(\theta_j) - a_j(\theta) - P(\theta_j - \theta)) - c(\lambda)) \\
&= \sum_l \alpha_{jl}(a_l(\theta) - a_j(\theta)) + \kappa(\theta, \rho_j) + \lambda(\theta, \rho_j)(a_j(\theta_j) - a_j(\theta) - P(\theta_j - \theta)) - c(\lambda).
\end{aligned} \tag{38}$$

By virtue of the wealth-independence of the value function and the envelope theorem, Equation (18) continues to obtain, namely,

$$P = \mathbb{E}_0 \left[ \int_0^{\tau_k} e^{-rs} \kappa_1(\theta_k, \rho(s)) ds \mid \rho(0) = \rho_k \right] + \mathbb{E}_0 [e^{-r\tau_k}] P, \tag{39}$$

where  $\tau_k \geq 0$  is the arrival time of the first contact with the market for an agent starting at  $(\theta(0), \rho(0)) = (\theta_k, \rho_k)$ . Rewrite (39) as

$$\begin{aligned}
P(1 - \mathbb{E}_0 [e^{-r\tau_k}]) &= (\mu - 2\sigma_D^2 \theta_k) \mathbb{E}_0 \left[ \int_0^{\tau_k} e^{-rt} dt \right] - \\
&\quad 2\sigma_D \sigma_\eta \mathbb{E}_0 \left[ \int_0^{\tau_k} e^{-rt} \rho(t) ds \mid \rho(0) = \rho_k \right] \\
&= \frac{1}{r} (\mu - 2\sigma_D^2 \theta_k) \mathbb{E}_0 [1 - e^{-r\tau_k}] - \\
&\quad 2\sigma_D \sigma_\eta \mathbb{E}_0 \left[ \int_0^\infty \lambda_k(t) e^{-\int_0^t (\lambda_k(s) + r) ds} \rho(t) dt \mid \rho(0) = \rho_k \right].
\end{aligned} \tag{40}$$

Thus, if the price  $P$  is known, this equation gives the optimal quantity  $\theta_k$ . The only difference with the analogous result in the main model is that the average discounted future expected correlation  $\rho(t)$  until the next trading opportunity is affected by the endogenous trading-intensity choices.

To calculate the price, isolate  $P$  on the left-hand side, multiply each equation with the mass  $\mu_{\cdot k}$  of all agents holding a quantity  $\theta_k$  in equilibrium and adding all resulting equations, to obtain

$$\begin{aligned}
P &= \frac{1}{r} (\mu - 2\sigma_D^2 \Theta) - 2\sigma_D \sigma_\eta \sum_k \mu_{\cdot k} \frac{\mathbb{E}_0 \left[ \int_0^\infty \lambda_k(t) e^{-\int_0^t (\lambda_k(s) + r) ds} \rho(t) dt \mid \rho(0) = \rho_k \right]}{(1 - \mathbb{E}_0 [e^{-r\tau_k}])} \\
&= P^W + 2\sigma_D \sigma_\eta \left( \bar{\rho} - \sum_k \mu_{\cdot k} \frac{\mathbb{E}_0 \left[ \int_0^\infty \lambda_k(t) e^{-\int_0^t (\lambda_k(s) + r) ds} \rho(t) dt \mid \rho(0) = \rho_k \right]}{(1 - \mathbb{E}_0 [e^{-r\tau_k}])} \right).
\end{aligned}$$

A first observation to make is that, in general, the price no longer equals the Walrasian price, as some intrinsic types  $\rho_i$  are weighted more heavily than their preponderance in the population. This result is not surprising, given that agents with stronger



trading incentives have a larger market presence, and therefore greater impact on the price than in the case when market access is exogenously set to be the same for all agents. This point is also made and analyzed further in Huang and Wang (2005).

In special cases, however, the price is not affected by the endogeneity of the trading intensities. In particular, with only two types of agents, the utility gains from trading from the sub-optimal to the optimal position are the same, and so are the intensity choices.

### 3 Conclusion

This paper studies portfolio choice and pricing in markets in which trading may take place with considerable delay. Examples of assets in such a situation include private equity, which may have to be held without trading for several years, and small corporate and municipal bond issues and shares in firms recently emerged from Chapter 11 proceedings, for all of which finding an appropriate buyer or seller may require lengthy search.

The paper uses a first-order approximation, supported by numerical results, to derive closed-form price and holding expressions. Regarding the price, it is found that the endogenous adjustment in positions acts so as to keep prices (almost) the same as in a perfectly liquid world. Thus, as long as market participants are not restricted exogenously in their holdings, illiquid corporate bonds, for instance, should not be required to yield (significantly) more than otherwise identical liquid ones. Binding short-sale constraints, on the other hand, give rise to a liquidity premium.

The optimal portfolio choice, on the other hand, is impacted considerably by the liquidity level. For instance, when future trading is expected to be difficult, an institution with current high value for a particular corporate bond – say, due to low correlation with the rest of its portfolio – should buy a smaller amount of the bond than it would in a perfect market. The reason is that, as its value for the bond diminishes, the institution may have to continue maintaining its position for a while. Similarly, if its value from the asset could increase in the future, the institution should hold a larger amount if the market is illiquid. As a result, smaller blocks should be traded in illiquid markets – that is, in markets in which completing a trade takes time.

The irrelevance of the liquidity level for pricing found in this paper depends on specific assumptions. The dampening effect of the endogenous portfolio choice on the liquidity impact, however, is quite general: the price is generally affected less by liquidity than the marginal utility of any agent who is held exogenously to certain positions. The direction of the portfolio adjustment to illiquidity is also natural and general.

An interesting question not addressed in this paper, and which constitutes the focus of future work, concerns random liquidity correlated with asset fundamentals, perhaps

in conjunction with correlated personal liquidity events to individual agents. This would capture the notion of liquidity crunches, and would have an impact on the price.

# A Appendix

## Mass Distribution in Steady State

First of all, note that, under the standing assumptions, in equilibrium there are a total of  $J^2$  types of agents, where a type is defined by the tuple  $\sigma = (\rho, \theta)$ . Note that, in steady state, the mass of agents of type  $(\rho_j, \theta_k)$  is  $\mu_{jk}$ . Since the masses are constant, the net outflow from type  $(\rho_j, \theta_k)$  – in short, type  $jk$  – is 0:

$$\begin{aligned} 0 &= -\mu_{jk} \sum_{l \neq j} \alpha_{jl} + \sum_{l \neq j} \alpha_{lj} \mu_{lk} - 1_{(j \neq k)} \lambda \mu_{jk} + 1_{(j=k)} \lambda \sum_{l \neq j} \mu_{jl} \\ &= -\mu_{jk} \sum_{l \neq j} \alpha_{jl} + \sum_{l \neq j} \alpha_{lj} \mu_{lk} - \lambda \mu_{jk} + 1_{(j=k)} \lambda \sum_l \mu_{jl}. \end{aligned} \quad (\text{A.1})$$

The expressions in the first row above have a straightforward justification. (The second row is just a manipulation of the first.) The first term represents the rate at which agents of type  $jk$  change their type because the correlation coefficient changes. The second term represents the rate at which agents of other types holding  $\theta_k$  become of type  $jk$  as a result of an intrinsic-type change. The third and fourth terms record the results of changes due to trading: if an agent of type  $jk$  does not have the optimal position, that is,  $j \neq k$ , then upon trading he changes type, leaving the pool  $jk$ . If  $j = k$ , then all other types  $lk$  trade to become of type  $jk$ , joining the pool.

Note that summing over Equation (A.1) above over  $j$  gives

$$\sum_j \mu_{jk} = \sum_j \mu_{kj}, \quad (\text{A.2})$$

meaning that the total mass of agents holding  $\theta_k$  shares is the same as the total mass of intrinsic type  $k$ .

Finally, it holds that<sup>18</sup>

$$\sum_j \hat{\mu}_{kj} = \sum_j \hat{\mu}_{jk} = \mu_{k..}$$

### Proof of Theorem 1:

The theorem is proved almost entirely in the main body of the text. Existence of the equilibrium and the form of the value functions and optimal consumption follow along the same lines of reasoning as in DGP, and are consequently omitted. The approximations follow immediately from the fact that the equilibrium value-function coefficients  $a_j(\theta_k)$  and price are bounded as  $\gamma \rightarrow 0$  keeping  $\gamma \sigma_D \sigma_\eta$  fixed.  $\square$

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<sup>18</sup>In fact,  $\hat{\mu}_{jk}$  are the steady-state masses corresponding to a trading intensity  $\hat{\lambda} = \lambda + r$ .

**Proof of Proposition 2:**

For part (i), note from Equations (21) and (22) that

$$\theta_k = \theta_k^W - \frac{\sigma_\eta}{\sigma_D} \left( \sum_j \tilde{\mu}_{jk} \rho_j - \rho_k \right). \quad (\text{A.3})$$

Since  $\rho_1$  and  $\rho_J$  are the minimum, respectively maximum, values that  $\rho$  can take,  $\theta_1^W < \theta_k < \theta_J^W$ .

Furthermore, the quantities  $\tilde{\mu}_{jk}$  are rational functions of  $\lambda$ , and consequently so are  $\theta_k$ . Therefore, the quantities  $\theta_k$  have only a finite number of local maxima or minima. Consider  $\underline{\lambda}$  higher than all such local extremes.

Up to terms in  $O(\lambda^{-2})$  for large  $\lambda$ , it is easily seen that, with  $j \neq k$ ,

$$\tilde{\mu}_{kk} \simeq 1 - \frac{\sum_{i \neq k} \alpha_{ki}}{\lambda} \quad (\text{A.4})$$

$$\tilde{\mu}_{jk} \simeq \frac{\alpha_{kj}}{\lambda}, \quad (\text{A.5})$$

and, consequently,

$$\theta_k \simeq \theta_k^W - \frac{1}{\lambda} \frac{\sigma_\eta}{\sigma_D} \sum_{j \neq k} \alpha_{kj} (\rho_j - \rho_k). \quad (\text{A.6})$$

Since  $\theta_k$  is monotonic in  $\lambda$ , the sign of its dependence is given by that of  $\sum_{j \neq k} \alpha_{kj} (\rho_j - \rho_k)$ . It is clear that  $\theta_1$  decreases, while  $\theta_J$  increases with  $\lambda$  for  $\lambda > \underline{\lambda}$ .

For part (ii) one calculates explicitly the quantities

$$\tilde{\mu}_{kk} = \frac{r + \lambda + \alpha_k}{r + \lambda + \alpha_1 + \alpha_2} \quad (\text{A.7})$$

$$\mu_{1\cdot} = \frac{\alpha_{21}}{\alpha_{12} + \alpha_{21}} \quad (\text{A.8})$$

$$\mu_{12} = \frac{\alpha_{12} \alpha_{21}}{(\alpha_{12} + \alpha_{21})(\lambda + \alpha_{12} + \alpha_{21})} \quad (\text{A.9})$$

$$\mu_{21} = \mu_{12}. \quad (\text{A.10})$$

□

**Proposition 6** Consider a market with two types ( $J = 2$ ), and fix a certain distribution of holdings at time 0, with the intrinsic types distributed as in steady state. Let the meeting intensity be given by  $\lambda(t) = \lambda$  for  $t < T$  and  $\lambda(t) = \lambda'$  for  $t \geq T$ . Then

(i)  $\theta_1(t)$  decreases and  $\theta_2(t)$  increases with  $\lambda'$  for all  $t \leq T$ ;

(ii)  $\mu_{12}(t) + \mu_{21}(t)$  decreases with  $\lambda$  for  $t > T$ , provided that  $T > \underline{T}$  for some  $\underline{T}$  depending on the initial distribution  $\mu$ ;

(iii)  $\mu_{12}(t) + \mu_{21}(t)$  decreases with  $\lambda$  for all  $t$ , provided that  $\mu_{12} \geq \mu_{12}^*$  and  $\mu_{21} \geq \mu_{21}^*$ , where  $\mu_{jk}^*$  is the steady-state value of  $\mu_{jk}$  corresponding to  $\lambda$ .

**Proof of Proposition 6:**

For part (i), use Equation (24)

$$\mathbb{E}_t \left[ \int_t^\tau e^{-r(s-t)} \kappa_1(\theta_k(t), \rho(s)) ds \mid \rho(t) = \rho_k \right] = \mathbb{E}_t \left[ \int_t^\tau e^{-r(s-t)} \kappa_1(\Theta, \bar{\rho}) ds \right].$$

Using the linearity of  $\kappa_1$ , it follows that

$$\begin{aligned} \theta_k(t) &= \Theta + \left( \bar{\rho}_k - \frac{\mathbb{E}_t \left[ \int_t^\tau e^{-r(s-t)} \rho(s) \mid \rho(t) = \rho_k \right]}{\mathbb{E}_t \left[ \int_t^\tau e^{-r(s-t)} \right]} \right) \frac{\sigma_\eta}{\sigma_D} \\ &= \Theta + \left( \bar{\rho}_k - \int_t^\infty \Psi(s; \lambda, \lambda') \mathbb{E}_t [\rho(s) \mid \rho(t) = \rho_k] ds \right) \frac{\sigma_\eta}{\sigma_D}, \end{aligned}$$

where  $\Psi$  is defined by the last equation, and is the only quantity that depends on  $\lambda'$ . To finish the proof, note that

$$\int_t^\infty \Psi(s; \lambda, \lambda') ds = 1,$$

that  $\frac{\partial \Psi}{\partial \lambda'} > 0$  if and only if  $s > \bar{s}$  for some  $\bar{s} > T$ , and that  $\mathbb{E}_t [\rho(s) \mid \rho(t) = \rho_k]$  is monotonic in  $s$ .

For parts (ii) and (iii), let  $\bar{\mu} = (\mu_{12}, \mu_{21})^\top$ . For  $t \leq T$ ,

$$\bar{\mu}(t) = \bar{\mu}^* + e^{-At} (\bar{\mu}(0) - \bar{\mu}^*), \tag{A.11}$$

where

$$A = \begin{bmatrix} \lambda + \alpha_{12} & \alpha_{21} \\ \alpha_{12} & \lambda + \alpha_{21} \end{bmatrix}$$

and  $\bar{\mu}^*$  is the steady-state value of  $\bar{\mu}$  corresponding to  $\lambda$ . □

**Proof of Proposition 5:**

As noted in the text, the dependence of the price on liquidity is given by the term

$$\begin{aligned}
\sum_{j,k>k_0} \hat{\mu}_{jk} \rho_j &\simeq \sum_{j,k>k_0} \mu_{\cdot k} \tilde{\mu}_{jk} \rho_j \\
&\simeq \sum_{k>k_0} \mu_{\cdot k} \left( \sum_{j \neq k} \frac{\alpha_{kj}}{\lambda} (\rho_j - \rho_k) + \rho_k \right) \\
&\simeq \sum_{k>k_0} \mu_{\cdot k} \rho_k + \frac{1}{\lambda} \sum_{k>k_0, j \neq k} \mu_{\cdot k} \alpha_{kj} (\rho_j - \rho_k).
\end{aligned}$$

Now use the fact that  $\sum_{k,j \neq k} \mu_{\cdot k} \alpha_{kj} (\rho_j - \rho_k) = 0$  together with the assumption that  $\rho_1 > \dots > \rho_J$  to infer that  $\sum_{k>k_0, j \neq k} \mu_{\cdot k} \alpha_{kj} (\rho_j - \rho_k) > 0$ .

Part (ii) follows immediately from direct computation.

□

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