A NON - UTILITY MAXIMIZATION APPROACH TO MULTIPERIOD PORTFOLIO SELECTION

by Klaus Hellwig

Faculty of Mathematics and Economics, University of Ulm, 89069 Ulm, Germany.

Phone: ++49731-5023540, Fax: ++49731-5023584, Email: hellwig@mathematik.uni-ulm.de.

JEL: G11

1. INTRODUCTION

Maximizing expected utility is the standard approach for the solution of multiperiod portfolio selection problems. However, the applicability of the approach is limited:

- It requires a multi-period utility function that reflects the time and risk preferences of the investor. Such a utility function can hardly be found.
- It requires a probability distribution of the multi-period portfolio cash flows which is difficult, if not impossible, to determine.
- The solution can be inefficient in the sense that the optimal portfolio may enable arbitrage (e.g., Copeland et al., 2004, pp. 66).
- It is assumed that that the utility function does not depend upon the menu over which choice is being made. This, for example, has been criticized by Sen (1997).

As an alternative Hellwig (2004), Hellwig et al (2000), Korn (2000) and Selinka (2005) proposed a different approach where a portfolio is determined based on two conditions. First, the portfolio is required to be (intertemporal) efficient. Second, the valuation of the portfolio cash flows is required to support the growth preferences of the investor concerning the portfolio value. It is shown that under reasonable assumptions a portfolio exists where both conditions are satisfied. However, the approach poses a number of problems. First, the portfolio value is defined as discounted consumption after present consumption is realized. This excludes cases where present consumption is part of the decision problem. Second, the approach rests on the assumption of a given multiperiod probability distribution that hardly can be found. Finally, the solution does not exclude consumption to be negative. How to handle such situations remains open.

The aim of this paper is to solve these problems. In the next two sections the case is treated where the portfolio value is defined as discounted consumption before present consumption is realized (which will be denoted as ex ante valuation) while section four treats the case, where the portfolio value is defined as discounted consumption after present consumption has been realized (which will be denoted as ex post valuation). For both cases it is shown that a solution with a non negative consumption vector exists under reasonable assumptions. Finally, in section five it is shown that under relaxed growth conditions a solution exists under less restrictive assumptions.

2. THE PORTFOLIO MODEL

The following analysis is based on a finite-state, discrete-time approach. Uncertainty is modelled by an event-tree with a finite set of events (nodes). $S = \{0, \ldots, n\}$ denotes the set of nodes, S_t the set of nodes at time t where $t = 0, \ldots, T$ and $S_0 = \{0\}$, $N(s)$ the set of nodes succeeding s, $F(s)$ the set of nodes that immediately follow s and s^- the immediate predecessor of s where it is assumed that s^- is unique for every s.

It is assumed that the investor is endowed with a node dependent income vector $b = (b_0, \ldots, b_n)'$ and can choose in every node s from a set of investment and financing opportunities. This set can be different for every s. $x = (x_1, \ldots, x_m)'$ denotes the activity level of all opportunities where $x \in X = \{x \mid 0 \leq x_i \leq x_i^u,\ i = 1, \ldots, m\}$. x_i^u is assumed to be finite for every i.

Let $A \in I\!R^{(n+1)\times m}$ denote the payoff matrix. Then Ax is the cash flow if the activity level is x. Let $c = (c_0, \ldots, c_n)'$ be the vector of consumption (withdrawals). c is called feasible if $c \in C = \{c \mid c = Ax + b \text{ for some } x \in X\}$.

Assume $c \in C$ and denote by $p = (p_0, \ldots, p_n) > 0$ the price vector. Then the ex ante portfolio value in node s will be defined as

$$V_s = V_s(c, p) = c_s + \sum_{k \in N(s)} \frac{p_k}{p_s} c_k = c_s + \sum_{k \in F(s)} \frac{p_k}{p_s} V_k. \tag{1}$$

It will be assumed that the desired price vector $\bar{p}$ and the desired consumption vector $\bar{c}$ are endogenously determined such that the following conditions are met.

Given $\bar{p} > 0$, $\bar{c} \in C$ should maximize the present portfolio value:

(C1) (Efficiency) $\bar{c}$ is an optimal solution of $V_0(\bar{c}, \bar{p}) = \max\{V_0(c, \bar{p}) \mid c \in C\}$.

Given $\bar{c} \in C$, $\bar{p}$ should support the desired increase of the portfolio value:

(C2) (Compatibility) $V_s(\bar{c}, \bar{p}) = (1 + g_s)V_{s^-}(\bar{c}, \bar{p}), s = 1, \ldots, n$

where g_s is the required growth rate of the portfolio value between nodes s^- and s.

Definition: $\bar{c}$ is called ex ante growth-oriented (with respect to $g_1, \ldots, g_n$), if a price vector $\bar{p} > 0$ exists such that (C1) and (C2) are satisfied.

Contrary to the expected utility maximizing approach the concept of a growth-oriented consumption vector neither requires a utility function nor a probability distribution.

Furthermore, efficiency is guaranteed by (C1). Finally, the concept is not independent of the menu over which the choice is made. A growth-oriented consumption vector in principle may be found by expected utility maximization. However - contrary to the standard approach - if C is changed, then expected utility maximization with the same utility function may not lead to a growth-oriented consumption vector with respect to the same growth rates (Hellwig, 2002).

3. EX ANTE VALUATION

What are the consequences of the growth requirements for consumption? Assume $c \in C$ and $p > 0$. Combining (1) with (C2) yields

$$c_s = (1 - \sum_{k \in F(s)} \frac{p_k}{p_s}(1+g_k))V_s = (1 - \sum_{k \in F(s)} \frac{p_k}{p_s}(1+g_k))\Pi_{\tau \in T(0,s)}(1+g_\tau)V_0 \quad (2)$$

where $T(0,s)$ denotes the set of nodes between 0 and s (excluding 0 and including s) and V_0 is the optimal present portfolio value given p.

Let $\hat{p} > 0$ be an arbitrary price vector and $c^{su}(\hat{p})$ an optimal solution of $\max\{V_0(c,\hat{p}) \mid c \in C\}$. $c^{su}(\hat{p})$ can be understood as consumption vector supplied by $\hat{p}$. Similarly, $c^d(\hat{p})$ given by (2) can be understood as consumption vector that is demanded by $\hat{p}$. Clearly, if "excess demand" $z(\hat{p}) := c^d(\hat{p}) - c^{su}(\hat{p})$ is zero, $c^d(\hat{p}) = c^{su}(\hat{p})$ is an ex ante growth-oriented consumption vector.

Suppose that $z(\hat{p}) \neq 0$. Then a new price vector may be chosen, for example, as an optimal solution $p(z)$ of $\max\{\sum_{s=0}^n z_s(\hat{p})p_s \mid p \in P\}$ where P is a suitable set of price vectors. This means that prices should be increased if demand exceeds supply and decreased if supply exceeds demand. Performing $z(p) : P \rightarrow Z$ where Z denotes the image of z and thereafter $p(z) : Z \rightarrow P$ leads to a multivalued mapping $\varphi = p(z(p)) : P \rightarrow P$. In the appendix it is shown that φ has a fixed-point $\bar{p} > 0$ such that that $z(\bar{p}) = 0$ if the following assumptions hold:

(A1) In every node $s \notin S_T$ funds can be invested for one period with a return r_{1k} in node k for all $k \in F(s)$.

(A2) Between two arbitrary succeeding nodes s and $k \in F(s)$ funds can be borrowed at a rate r_{2k}.

(A3) There exists a consumption vector $c^* \in C$ such that $c^* \geq 0,\ c^* \neq 0$.

(A4) $-1 \leq g_k \leq r_{1k} \quad (k = 1, \ldots, n)$.

The opportunities in (A1) and (A2) have to be upper bounded. These bounds are chosen such that they never become active.

Finally, in Lemma 4 it is proved that $\bar{c} \geq 0$ for every ex-ante growth oriented consumption vector $\bar{c}$ if (A4) holds.

This establishes the following theorem.

Theorem 1: Given (A1) - (A4) an ex ante growth-oriented consumption vector $\bar{c} \geq 0$ exists.

4. EX POST VALUATION

In the last section, the portfolio value V_s was defined as discounted consumption before consumption in node s is realized. Alternatively, the portfolio value in node s can be understood as discounted consumption after consumption in node s has been realized, i.e.

$$W_s = \sum_{k \in N(s)} \frac{p_k}{p_s}\, c_k = \sum_{k \in F(s)} \frac{p_k}{p_s}\, (c_k + W_k). \tag{3}$$

Two problems have to be considered.

First, by definition, $W_s = 0$ for every $s \in S_T$ which may be inconsistent with the growth requirements. To solve this problem c_s will be substituted by $c_s + W_s$ (s$\in S_T$), where c$_s$ is the amount that is actually consumed and W_s the the terminal portfolio value that remains according to the growth requirements.

Second, c_0 has to be fixed a priori because it is not included in W_0. In what follows, $c_0 \equiv 0$.

With these changes a consumption sequence $\bar{c}$ will be called ex post growth-oriented if (C1) and (C2) are satisfied after substitution of V by W.

As an example let $C = \{c = (c_0, c_1) \mid x + c_0 = 110,\ 1.1x - c_1 = 0,\ x \geq 0\}$ be the feasible set underlying the ex ante valuation. For $g_1 = 0$ the ex ante growth-oriented consumption vector is $\bar{c} = (10, 110)$ where $V_0 = V_1 = 110$.

In case of an ex post valuation $C = \{c_1 \mid x = 110,\ 1.1x - c_1 = W_1,\ x \geq 0\}$. For $g_1 = 0$ the ex post growth-oriented consumption vector is $\bar{c} = (0, 11)$ where $W_0 = W_1 = 110$.

The example illustrates the difference between the ex ante and the ex post valuation. Using the ex ante valuation implies that the present value of the economic profit ($\frac{0.1 \cdot 110}{1.1} = 10$) is consumed at $t = 0$. Using the ex post valuation implies that the economic profit ($0.1 \cdot 110 = 11$) is consumed at $t = 1$. The ex post valuation complies with the usual procedure where profit is paid out only after it is realized. On the other hand, the ex ante valuation may be appropriate if, for example, an investor wants to determine the maximum amount that he presently can consume without being worse off in the future.

The existence of an ex post growth-oriented consumption vector can be proved similar to the case of an ex ante valuation. A solution is found by determining some $p \in P$ such that excess demand $z(p) = c^d(p) - c^{su}(p) = 0$ where $c^{su}(p)$ is an optimal solution of (C1) after V is substituted by W and $c^d(p)$ is determined as follows.

p_s/p_{s^-} $(p > 0)$ can be written as

$$\frac{p_s}{p_{s^-}} = \frac{\pi_s}{1 + r_{1s}} = \frac{\pi'_s}{1 + r'_{1s}} \tag{4}$$

where

$$\pi'_s := \frac{\pi_s}{\sum_{\tau \in F(s^-)} \pi_\tau} \text{ and } 1 + r'_{1s} := \frac{1 + r_s}{\sum_{\tau \in F(s^-)} \pi_\tau}.$$

π'_s are uniquely determined for every $s \in S_t,\ t = 1, \ldots, T$, and can be understood as (pseudo-) probability of node s after node s$^-$ has been realized.

Using (4), W_s can be written as

$$W_s = \sum_{\tau \in F(s)} \frac{(W_\tau + c_\tau)\pi_\tau}{1 + r_{1\tau}} = \sum_{\tau \in F(s)} \frac{(W_\tau + c_\tau)\pi'_\tau}{1 + r'_{1\tau}},\ s \in S_t,\ t = 1, \ldots, T - 1. \tag{5}$$

(5) is satisfied if

$$W_{s^-} = \frac{W_s + c_s}{1 + r'_{1s}}. \tag{6}$$

Combining (6) and the growth requirements $W_s = (1 + g_s)W_{s^-},\ s \in S_t,\ t = 1, \ldots, T,$ yields

$$c_s = (r'_{1s} - g_s)W_{s^-} = (r'_{1s} - g_s)\ \Pi_{\tau \in T(0,s^-)}\ (1 + g_\tau)\ W_0 \tag{7}$$

for $s = 1, \ldots, n$.

(7) is a condition for a consumption vector to be ex post growth-oriented. Therefore such a consumption vector can be understood as a consumption vector $c^d(p)$ that is demanded by p.

Similar to the proof of Theorem 1 in Lemma 5 the following theorem is proved in the appendix.

Theorem 2: Given (A1) - (A4) an ex post growth oriented consumption sequence $\bar{c} \geq 0$ exists.

5. WEAKENING THE GROWTH REQUIREMENTS

The existence of an (ex ante or ex post) growth-oriented consumption sequence requires the possibility of borrowing between arbitrary succeeding nodes. Clearly this assumption is quite restrictive. Fortunately, it can be dropped if (C2) is weakened to

(C2') $V_s(\bar{c},\bar{p}) \geq (1+g_s)V_{s^-}(\bar{c},\bar{p}), s = 1,\ldots,n.$

To see how the case can be handled where (A2) does not hold assume $T = 1, b = (100, 10, 0)', g_1 = g_2 = 0.1$ and a riskless investment opportunity with a return of 10%. Clearly, an ex ante growth-oriented consumption sequence does not exist. However, changing g_1 to 0.2 leads to the growth-oriented consumption sequence $\bar{c} = (0, 120, 110)'$ where $\bar{c}$ is supported by the price vector $\bar{p} = (1, 0, 1.1^{-1})$ and $V_0(\bar{c},\bar{p}) = 100$. Note that $\bar{p}_1 = 0$. To understand this assume that there exists a lending opportunity between nodes 0 and 1 at a rate r_{21}. In this case an ex ante growth-oriented consumption sequence with respect to $g_1 = g_2 = 0.1$ exists and is given by $\hat{c}(r_{21}) = (0, 110 + \frac{11}{1+r_{21}}, 110 + \frac{11}{1+r_{21}})$ with the supporting price vector $\hat{p}(r) = (1, \frac{1}{1+r_{21}}, \frac{1}{1.1} - \frac{1}{1+r_{21}})$ and an initial value $V_0(\hat{c},\hat{p}) = 100 + \frac{10}{1+r_{21}}$. Letting $r_{21} \to \infty$ and dropping the borrowing opportunity results in $\bar{c}$ and $\bar{p}$.

The following theorem generalizes the result to the multi period case.

Theorem 3: Given (A1), (A3) and (A4) an (ex ante or ex post) growth-oriented consumption vector $\bar{c} \geq 0$ with respect to growth rates $g'_s \geq g_s,\ s = 1,\ldots,n$, exists.

Proof: Assume the ex post valuation. For $\bar{p} \in P$ let c^{su} be an optimal solution of $\max\{W_0(c,\bar{p}) = c_0 + \sum_{s\in N(0)} c_s\bar{p}_s \mid c \in C,\ c_s \equiv 0,\ s \notin S_T\}$ where the borrowing opportunities according to (A2) are dropped for the first T-1 periods and c^d satisfy (7) for $g'_s = r'_{1s},\ s = 1,\ldots,n$. Then analogue to the preceeding analysis a fixed point p^* of z(p) exists where $z_\tau(p^*) = 0$ for $\tau \in S_t,\ t = 0,...,T-1$. Because $-1+\sum_{\tau\in F(s)} (1+r_{1\tau})\frac{p_\tau}{p_s} = -1+\sum_{\tau\in F(s)} (1+r_{1\tau})\,\frac{\pi_\tau}{1+r_{1s}} \leq 0$ it follows that $\sum\limits_{\tau\in F(s)} \pi_\tau \leq 1$.

Thus $g'_s = r'_{1s} \geq r_{1s} \geq g_s$ for all $\tau \in F(s)$. The result follows after letting $r_{2s} \to \infty$ for all $s \in S_T$ and dropping all borrowing opportunities in period T. A simular argumentation applies to the ex ante valuation.

APPENDIX

Lemma 1: Let $P := \{p \mid p_0 = 1,\ p_s^l \leq \frac{p_s}{p_{s^-}} (s \notin S_0), 1 - \frac{1}{p_s} \sum_{k \in F(s)} p_k (1 + \overline{r}_{1k}) \geq 0\ (s \notin S_T)\}$ where $0 < \overline{r}_{1k} < r_{1k}$ and $p_s^l > 0$ are chosen such that $P \neq \emptyset$. Then φ has a fixed-point.

Proof: P is compact, non-void and convex where $p > 0$ for every $p \in P$. $c^{su}(\hat{p})$ is upper-semicontinuous and $V_0 = V_0(c(\hat{p}), \hat{p})$ (and consequently V_s for $s = 1, \ldots, n$) continuous (e.g., Luenberger, 1995, pp. 467). $c^d(\hat{p})$ is continuous. Therefore $z(\hat{p})$ is upper-semicontinuous. Because P is compact and non-void, $p(z)$ is upper-semicontinuous (Luenberger, 1995, p. 468). This implies that φ (as a combination of two upper-semicontinuous mappings) is upper-semicontinuous. φ is convex, because the set of optimal solutions of a convex optimization problem is convex. Therefore (applying Kakutani‘s fixed point theorem, Kakutani, 1948), a fixed point exists.

Lemma 2: $z(\hat{p}) = 0$ for every fixed point $\hat{p}$ of φ if the following conditions hold:

(B1) $p \in P,\ 1 - \frac{1}{p_s} \sum_{k \in F(s)} p_k (1 + \overline{r}_{1k}) = 0 \Rightarrow z_s(p) \geq 0 \quad (s \notin S_T)$.

(B2) $p \in P,\ \frac{p_s}{p_{s^-}} = p_s^l \Rightarrow z_s(p) \geq 0 \quad (s \notin S_0)$.

Proof: Since $\hat{p}$ is a fixed point of φ, $\max\{\sum_s z_s(\hat{p})p_s \mid p \in P\} = \sum_s z_s(\hat{p})\hat{p}_s$. Furthermore $\sum_{s=0}^{n} c_s^d(\hat{p}_s)\hat{p}_s = \sum_{s=0}^{n} V_s\hat{p}_s - \sum_{s\notin S_T}\sum_{k\in F(s)}\hat{p}_k(1+g_k)V_s = \sum_{s=0}^{n} V_s\hat{p}_s - \sum_{s\notin S_T}\sum_{k\in F(s)} V_k\hat{p}_k = V_0$. Because $\sum_{s=0}^{n} c_s^{su}(\hat{p}_s)\hat{p}_s = V_0$ this implies $\sum_{s=0}^{n} z_s(\hat{p})\hat{p}_s = 0$.

The dual of $\max\{\sum_{s=0}^{n} z_s(\hat{p})p_s \mid p \in P\}$ is $\min y_0$ subject to

$$\sum_{k\in F(0)} p_k^l v_k - w_0 + y_0 \geq z_0(\hat{p}) \tag{8}$$

$$-v_s + \sum_{k\in F(s)} p_k^l v_k - w_s + (1+\overline{r}_{1s})w_{s^-} \geq z_s(\hat{p}) \quad (s \notin S_0, S_T) \tag{9}$$

$$-v_s + (1+\overline{r}_{1s})w_{s^-} \geq z_s(\hat{p}) \quad (s \in S_T) \tag{10}$$

$$v_s \geq 0 \quad (s \notin S_0), w_s \geq 0 \quad (s \notin S_T),\ y_0 \in I\!R.$$

Since $\hat{p} > 0$, (8), (9) and (10) hold as equalities for every optimal solution $\bar{v}_s$ $(s \notin S_0)$, $\bar{w}_s$ $(s \notin S_T)$, $\bar{y}_0$ where $\bar{y}_0 = 0$. Therefore $z_s(\hat{p}) < 0$ implies $\bar{v}_s > 0$ and/or $\bar{w}_s > 0$. By complementary slackness $\frac{\hat{p}_s}{\hat{p}_{s^-}} = p_s^l$ and/or $\sum_{k\in F(s)}\hat{p}_k(1+\overline{r}_{1k}) = \hat{p}_s$ which contradicts (B1) and (B2). Thus $z_s(\hat{p}) \geq 0$. Noting $\sum_s z_s(\hat{p}_s)\hat{p}_s = 0$ and $\hat{p} > 0$ completes the proof.

$\square$

Lemma 3: (B1) and (B2) follow from (A1) and (A2).

Proof: Assume, that (A1) holds. Choose $\hat{p} \in P$ such that $\sum_{k\in F(0)}\hat{p}_k(1+\overline{r}_{1k}) = 1$. Then the net present value of investing one unit according to (A1) in $t = 0$ is $-1 + \sum_{k\in F(0)}\hat{p}_k(1+r_{1k}) > -1 + \sum_{k\in F(0)}\hat{p}_k(1+\overline{r}_{1k}) = 0$. Value maximization therefore requires to invest as much as possible. As a result $c_0(\hat{p})$ strictly decreases with the amount invested. On the other hand $c^d(\hat{p}) = (1 - \sum_{k\in F(0)} \hat{p}_k(1+g_k))V_0 = 0$. Thus $z_0(\hat{p}) > 0$ if the investment is increased sufficiently. A similar argumentation applies to all nodes $s \in S_1$ and subsequently to all nodes $s \in S_t,\ t = 2, \dots, T-1$. This proves (B1).

Now assume that (A2) holds. For $k \in F(0)$ choose p_k^l such that $p_k^l < (1+r_{2k})^{-1}$ and $\hat{p} \in P$ such that $\frac{\hat{p}_k}{\hat{p}_0} = p_k^l$. Then the time zero value of borrowing one unit between nodes $s = 0$ and k is $1 - \hat{p}_k(1+r_{2k}) = 1 - p_k^l(1+r_{2k}) > 0$. Therefore as much as possible should be borrowed and $c_k(\hat{p})$ strictly decreases with the amount borrowed. On the other hand $c_k^d(\hat{p}) = (1 - \frac{1}{\hat{p}_k}\sum_{\tau\in F(k)} \hat{p}_\tau(1+g_\tau))V_0 \geq 0$ since $\hat{p} \in P$ and $V_0 \geq 0$ by (A3). Thus $z_k(\hat{p}) > 0$ if the upper bound for borrowing is chosen sufficiently high. A similar argumentation subsequently applies to the succeeding nodes. This proves (B2).

□

Lemma 4: Given (A1) - (A4). Then $\bar{c} \geq 0$ for every ex ante growth-oriented consumption sequence $\bar{c}$.

Proof: Let $\bar{c}$ and $\overline{p}$ satisfy (C1) and (C2). (A3) and (A4) imply $V_s \geq 0$ for $s = 1, \dots, n$. Inserting (4) into (2) yields $c_s = (1 - \sum_{k\in F(s)} \frac{p_k}{p_s}(1+g_k))V_s = c_s = (1 - \sum_{k\in F(s)} \frac{\pi'_k}{1+r'_{1k}}(1+g_k))V_s$. Furthermore $-1 + \sum_{k\in F(s)} \frac{\pi_k(1+r_{1k})}{1+r_{1k}} \leq 0$ that is $\sum\limits_{k\in F(s)} \pi_k \leq 1$.Thus $r'_{1k} \geq r_{1k} \geq g_k$ for all k$\in F(s)$ which implies $c_s \geq 0$.

□

Proof of Theorem 2:

Under the ex post valuation $c^d(p)$ is given by (7). Lemma 1 remains valid. Lemma 2 remains valid because

$$
\begin{aligned}
& \sum_{s=0}^{n} c_s^d\ (\hat{p}_s)\ \hat{p}_s = 0 = \sum_{s=1}^{n}(r'_{1s} - g_s)W_s\ \hat{p}_s + \sum_{s\in S_T} W_{s^-}\ \hat{p}_s \\
= & \sum_{s=1}^{n}(1 + r'_{1s})W_{s^-})\ \hat{p}_s - \sum_{s=1}^{n}(1 + g_s)W_{s^-}\hat{p}_s + \sum_{s\in S_T} W_s\hat{p}_s \\
= & \sum_{s=1}^{n}(1 + r'_{1s})\frac{\hat{p}_{s^-}\ \pi'_s}{1 + r'_{1s}}W_{s^-} - \sum_{s=1}^{n} W_s\hat{p}_s + \sum_{s\in S_T} W_s\hat{p}_s \\
= & \sum_{s=1}^{n} \pi'_s\ \hat{p}_{s^-}\ W_{s^-} - \sum_{s\notin S_0,S_T} W_s\ \hat{p}_s = W_0
\end{aligned}
$$

Lemma 3 remains valid because, using $W_{s^-} \geq 0$ and $r'_{1s} \geq g_s$, $c_s^d = (r'_{1s} - g_s)\ W_{s^-} \geq 0$.

□

REFERENCES

Copeland, T.E; Weston, J.F. and Shastri, K. (2005). *Financial Theory and Corporate Policy (4th ed.), Addison-Wesley Series in Finance.*

Hellwig, K.; Speckbacher, G. and Wentges, P. *(2000). Utility maximization under capital growth constraints, Journal of Mathematical Economics, 33, pp. 1-22.*

Hellwig, K. *(2002). Growth and utility maximization, Economics Letters, 77, pp. 377-380.*

Hellwig, K. *(2004). Portfolio selection subject to growth objectives, Journal of Economic Dynamics and Control 28, pp. 2119-2128.*

Kakutani, S. *(1948). A generalization of Brower's fixed point theorem, Duke Mathematical Journal 8, pp. 457-459.*

Korn, R. *(2000). Value preserving strategies and a general framework for local approaches to optimal portfolios, Mathematical Finance, 10, pp. 227-241.*

Luenberger, D.C.(1995). Microeconomic Theory. Mc Graw-Hill.

Selinka, M. *(2005). Ein Ansatz zur wertorientierten Portfolioplanung mit nichtnegativen Konsumentnahmen, Logos Verlag.*

Sen, A, *(1997). Maximization and the act of choice*, Econometrica, 65, pp. 745-780.