Small Caps in International Equity Portfolios: The Effects of Variance Risk*

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Abstract

This paper investigates how variance risk affects the portfolio choice of an investor faced with an international asset menu that includes European and North American small equity portfolios. Small capitalization stocks are known to display asymmetric risk across bull and bear markets. Therefore we model stock returns as generated by a multivariate regime switching process that is able to account for both non-normality and predictability of stock returns. Non-normality matters for portfolio choice because the investor has a power utility function, implying a preference for positively skewed returns and aversion to kurtosis. We find that small cap portfolios, that are shown to display negative co-skewness with other assets, command large optimal weights only when regime switching, and hence variance risk, is ignored. Otherwise a rational investor ought to hold a well-diversified portfolio. However, the availability of small caps substantially increases expected utility, in the order of riskless annualized gains of 3 percent and higher.

Keywords: international portfolio diversification; regime switching; co-skewness and co-kurtosis; variance risk.

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This paper investigates how variance risk affects the portfolio choice of an investor faced with an international asset menu that includes European and North American small equity portfolios. Small capitalization stocks are known to display asymmetric risk across bull and bear markets. Therefore we model stock returns as generated by a multivariate regime switching process that is able to account for both non-normality and predictability of stock returns. Non-normality matters for portfolio choice because the investor has a power utility function, implying a preference for positively skewed returns and aversion to kurtosis. We find that small cap portfolios, that are shown to display negative co-skewness with other assets, command large optimal weights only when regime switching, and hence variance risk, is ignored. Otherwise a rational investor ought to hold a well-diversified portfolio. However, the availability of small caps substantially increases expected utility, in the order of riskless annualized gains of 3 percent and higher.

1. **Introduction**

Small capitalization stocks have become important to international investors with the development of new technologies and venture capital. They are however known to be rather peculiar assets in that their returns display − along with higher average risk premium − asymmetric risk across bull and bear markets. Indeed, they generally imply higher risk in cyclical downturns due to tighter credit constraints associated to lower firm collateral (Ang and Chen, 2002; Perez-Quiros and Timmermann, 2000). Several papers focus on international portfolio choice under a variety of assumptions concerning the asset menu and the process generating asset returns, e.g. Ang and Bekaert (2002). However, no specific attention has been given to small capitalization firms. Our paper studies the contribution of small caps to the international diversification of stock portfolios under realistic specifications for the stochastic process driving asset returns, that allow for asymmetric risk.

Developing such a perspective on small capitalization firms appears to be warranted also in the light of recent asset pricing research showing that the cross-sectional distribution of the equity risk premium is related to variance risk, i.e. the correlation both between returns and aggregate volatility (Harvey and Siddique, 2000; Barone-Adesi, Gagliardini, and Urga, 2004) as well as between individual stock volatility and aggregate volatility (Dittmar, 2002).While these relationships between expected returns and asymmetric risk characterize multi-factor extensions of the standard CAPM, the literature on dynamic portfolio choice has mostly focused on the ability to forecast expected returns (Campbell and Viceira, 1999), even in multivariate asset menus (Campbell, Chan, and Viceira, 2003). However, it is clear that the partial equilibrium, portfolio choice counterparts of these findings in the asset pricing literature ought to show that optimal portfolio weights respond to predictable changes in the covariances between returns and volatility as well as among cross-sectional volatilities. So far, predictable volatility has been analyzed in the case of one risky asset only, and in such context it is unable to generate large hedging demands (Chacko and Viceira, 2005). Our paper investigates intertemporal portfolio choice taking into account not only predictable volatility but also predictable changes in covariance terms involving returns and volatilities in a multivariate asset menu.

We use international data to investigate how variance risk affects portfolio composition for a power utility investor under several alternative assumptions on risk aversion and investment horizon. We find that small caps returns imply above-average levels of such risk, which may substantially reduce their appeal in a portfolio. In fact, we provide three alternative measures of the relevance of variance risk. The first is directly based on a few selected features of the predictive joint distribution for stock portfolio returns, the co-skewness and co-kurtosis of each portfolio vs. both other portfolios as well as the aggregate market portfolio. The second measure is the welfare cost induced by restricting investors to stick to a myopic portfolio rule that ignores variance risks. Finally, we also compute the welfare cost that an investor would incur in case he were restricted to asset menus excluding vehicles − small capitalization stocks − that are more prone to such risks.

Our paper makes two choices in order to tease out the effects of variance risk on optimal asset allocation. First, we focus on an international equity diversification problem in which both US and European small cap portfolios figure prominently. While US small caps have already attracted attention from both asset pricers (e.g. Fama and French, 1993) and financial econometricians (Perez-Quiros and Timmermann, 2000), the case for European small caps is based on two observations. First, the European size effect has been almost neglected by the asset pricing literature (with the exception of Annaert et al., 2002) that instead has focused primarily on U.S. data.[1] Since such a focus poses data-snooping problems, it is important to prevent our estimates of the share of small caps in optimal portfolios to depend entirely on some well-known but possibly random features of North American data. Second, U.S. small caps have experienced an unprecedented performance in the first part of our sample period from January 1999 to June 2001. Since a concern has been expressed that the size premium may contain long and persistent swings (see e.g. Pástor, 1999 and Guidolin and Timmermann, 2004a), it seems useful to obtain broader evidence involving other major markets for small caps.

Second, in order to build a parametric model that allows accurate measurement of variance risk effects, we abandon the traditional approach that assumes joint normality of the distribution of asset returns (e.g. Elton, Gruber, Brown, and Goetzmann, 2003). It is now well known that stock portfolios exhibit non-normal features, such as asymmetric distributions with fat tails and the tendency for returns to be more highly correlated when below the mean (i.e. in bear markets) than when above the mean (in bull markets), see Longin and Solnik (2001) and Butler and Joaquin (2002). Asymmetries have been shown to be especially relevant for small caps. Furthermore, there has long been evidence of predictable returns (Keim and Stambaugh, 1986; Pesaran and Timmermann, 1995). This is why we represent stock returns through a Markov switching process, that is able to account for both non-normality, asymmetric correlations, and predictability.[2] Differently from previous papers, we characterize *endogenously* the number of regimes and the number of lags. As recently discussed by Ang and Bekaert (2001), Guidolin and Timmermann (2005c), and Jondeau and Rockinger (2004), possible departures of excess stock returns from joint multivariate normality may be of first-order importance for long-run optimal asset allocation when investors are characterized by power utility, implying a preference for a positively skewed final wealth process (besides a higher mean) and aversion to the kurtosis (besides variance) of final wealth.

Using a 1999-2003 weekly MSCI data set for four major portfolios, we find that the joint distribution of international excess stock returns is well captured by a three-state multivariate regime switching model. The states can be ordered by increasing risk premia. In the intermediate regime − that we label *normal* because of its high average duration − European small caps returns exhibit both an extremely low variance and a high Sharpe ratio. Thus a risk averse investor, who is assumed to believe to start from this regime, would invest close to 100% of her stock portfolio in European small caps for horizons up to two years. On the other hand, the change in regime-specific variance is the highest just for European small caps: in particular, variance almost doubles when the regime shifts from normal to bear. The high variance 'excursion' across regimes is compounded by the presence of high and negative co-skewness with other asset returns, which means that the European small variance is high when other excess returns are negative, and European small returns are small when the 'market' is highly volatile. Similarly, the co-kurtosis of

[1] The size of the U.S. small cap premium has been scrutinized for more than twenty years. Pástor (2000) reports that a portfolio comprising small firms paid 0.17% per month in excess of the risk-adjusted return on a portfolio composed of large firms from 1927 to 1996. Earlier, Fama and French (1993) estimated a premium of 0.74 per annum, and provided some international evidence (Fama and French, 1998). The size effect debate is summarized by Schwert (2003).

[2] Ang and Chen (2002) report that regime switching models may replicate the asymmetries in correlations observed in stock returns data better than GARCH-M and Poisson jump processes. There is now a large body of empirical evidence suggesting that returns on stocks and other financial assets can be captured by this class of models. While a single Gaussian distribution generally does not provide an accurate description of stock returns, the regime switching models that we consider have far better ability to approximate the return distribution and can capture outliers, fat tails, and skewness.

European small excess returns with other excess returns series is high – i.e. the variance of the European small class tends to correlate with the variance of other assets. Both these features suggest a tendency of European small caps to display a disproportionate variance risk. The striking implication is then that a rational investor ought to give European small caps a rather limited weight (as low as 10% only) when she is ignorant about the nature of the current regime, which is a realistic situation. Further experiments reveal that the dominant factor in inducing such shifts in optimal weights is represented by the co-skewness, the predictable, time-varying covariance between returns and volatilities. This shows that higher moments of the return distribution considerably reduce the desirability of small caps for portfolio diversification purposes. We quantify such an effect in about 300 basis points per year under the long-run, steady-state distribution for returns. These results provide a demand-side justification for the dependence of asset prices on co-skewness – as uncovered by Harvey and Siddique (2000).

Our results are qualitatively robust when *both* European and North American small caps are introduced in the analysis. In this case, even initializing the experiment to a state of ignorance on the regime, we obtain that small caps – both North American and European – enter optimal long-run portfolios with a weight exceeding 50% for all investment horizons. Moreover, the demand for small caps appears much more stable across regimes, which is easily explained by the finding that both North American small caps and Pacific stocks represent good hedges for European small caps that improve portfolio performance outside the normal regime. However, the fact remains that equity portfolios with excellent Sharpe ratio properties may command a limited optimum weight because of their variance risk properties.

One side implication of our paper is that the scarce interest for small capitalization firms of important classes of investors – those with long horizons that are unlikely to incur in high transaction costs due to the limited liquidity of small stocks, see Gompers and Metrick (2001) – may be a rational response to the statistical properties of the returns on small caps, in particular of high variance risk.[3] The claim that it may be rational to limit the holdings of small caps does not imply that small caps are irrelevant in international portfolio diversification terms. Even when their weight is moderate, we find that the welfare loss from excluding small caps from the asset menu may lead to first-order magnitude costs (e.g. 3 percent for long horizons).

Our work is closely related to Ang and Bekaert (2002), and Guidolin and Timmermann (2004a) who investigate the effects on portfolio diversification of time-varying correlations across markets when regime shifts are accounted for. Similarly to these papers, we overlook the analysis of inflation risk, informational differences, and currency hedging costs that – while generally important – may not radically affect rational choices of a large investor who can hedge currency risk. Ang and Bekaert work with US, German and UK excess stock returns. They fail to reject the hypothesis that correlations are constant across regimes, and test whether the US portfolio weight in each regime is different from 100%, conditional on assuming – as we do – that regimes are perfectly correlated across countries. Differently from Ang and Bekaert, we focus here on issues of international diversification across small and large capitalization firms. Guidolin and Timmermann (2004a) find strong evidence of time-variation in the joint distribution of US returns

[3]The size premium has been often interpreted as a reward for the lower liquidity of small caps. If this is the case, then investors with longer horizons (hence unlikely to actively trade stocks) ought to consider small caps an attractive diversification vehicle, since they would earn the small cap premium without incurring into large illiquidity costs (Amihud and Mendelsohn, 1986). However, the results in Gompers and Metrick (2001) imply that institutional investors such as pension funds and university endowments – which often have longer horizons than individuals – have low ownership shares in small caps.

on a stock market portfolio and portfolios reflecting size- and value effects. Mean returns, volatilities and correlations between these equity portfolios are found to be driven by regimes that introduce short-run market timing opportunities for investors. However, their asset allocation exercises are limited to menus including Fama and French's (1993) value- and size-tracking zero-investment portfolios, while in our paper we are interested in a standard portfolio exercise in which positive net investments in large and small cap equity portfolios are allowed.

Das and Uppal (2004) study the effects of infrequent price changes on international equity portfolios. Equity returns are generated by a multivariate jump diffusion process where jumps are simultaneous and perfectly correlated across assets. We also assume that regimes are perfectly correlated across stock portfolio returns, but allow for persistence of regimes. While this prevents us from obtaining their simple analytic results, it allows to compute portfolio allocations conditional on a given regime when the investor anticipates the probability of a regime shift next period. While the ex-ante cost of overlooking shifts is small both in Das and Uppal (2004) and in our paper, it is high when a normal state is prevailing. This observation can be especially important for shorter-term investors, who tailor their allocations to the state.

Last but not least, Harvey and Siddique (2000) show that conditional skewness contributes to the explanation of cross sectional US expected returns, commanding an average premium of 3.6% per year. They highlight that small cap portfolios have high expected returns together with negative co-skewness while low expected returns, large cap portfolios have positive co-skewness. Therefore they suggest analyzing portfolio choice in a richer conditional mean-variance-skewness framework, which is what we do. Dittmar (2002) allows for expected returns to be related also to co-kurtosis between returns and aggregate wealth. In our framework, all moments above the second may be responsible for departures of optimal portfolio shares from the mean variance ones. We are able to show, however, that a large fraction of such departures come from the (co-) skewness of the multivariate return distribution. Thus, we provide a portfolio choice motivation for the statistical significance of skewness in equity prices found by Harvey Siddique (2000) and Dittmar (2002).[4]

This paper is organized as follows. Section 2 presents the portfolio choice problem and gives details on the multivariate regime switching model used in this paper to represent the return process. Section 3 describes the data, while Section 4 reports our econometric estimates and provides an assessment of their economic implications for portfolio choice. This section presents the most interesting results of the paper and is organized around three sub-sections, each describing homogeneous sets of experiments for alternative asset menus. Section 5 performs a number of robustness checks. Section 6 concludes. We collect technical details in a few short Appendices.

[4]A related point is made by Ang, Chen, and Xing (2005) who use the notion of a downside beta − the loading on the market portfolio factor when returns on the latter are below their unconditional mean − to contribute to explaining the cross section of U.S. stock returns and show that this would contain a downside risk premium of approximately 6 percent per annum. While Ang et al. (2005) show that downside risk and negative co-skewness are different concepts and that downside risk is compensated in equilibrium in addition to co-skewness, our paper focuses on the international portfolio implications of co-higher order moments (not limited to co-skewness) in a standard constant relative risk aversion model.

2. **The Model**

2.1. *The General Portfolio Problem*

Consider an investor with power utility defined over terminal wealth, W_{t+T}, coefficient of relative risk aversion $\gamma > 0$, and horizon T:

$$u(W_{t+T}) = \frac{W_{t+T}^{1-\gamma}}{1-\gamma} \tag{1}$$

The investor is assumed to maximize expected utility by choosing a vector of portfolio shares at time t, that can be adjusted every $\varphi = \frac{T}{B}$ months at B equally spaced points. When $B = 1$ the investor simply implements a buy-and-hold strategy. Let $\boldsymbol{\omega}_b$ be the portfolio weights on $m \geq 1$ risky assets at these rebalancing times. Defining $W_B \equiv W_{t+T}$, and assuming for simplicity a unit initial wealth, the investor's optimization problem is:

$$\max_{\{\boldsymbol{\omega}_j\}_{j=0}^{B-1}} \quad E_t\left[\frac{W_B^{1-\gamma}}{1-\gamma}\right]$$
$$s.t.\ W_{b+1} = W_b \boldsymbol{\omega}_b' \exp\left(\mathbf{R}_{b+1}\right) \tag{2}$$

where $\exp\left(\mathbf{R}_{b+1}\right) \equiv [\exp\left(R_{1,b+1}\right)\ \exp\left(R_{2,b+1}\right)\ ...\ \exp\left(R_{m,b+1}\right)]'$ denotes an $m \times 1$ vector of cumulative, gross returns between two rebalancing points (under continuous compounding). The derived utility of wealth function can be simplified, for $\gamma \neq 1$, to:

$$J(W_b, \mathbf{r}_b, \boldsymbol{\theta}_b, \boldsymbol{\pi}_b, t_b) \equiv \max_{\{\boldsymbol{\omega}_j\}_{j=b}^{B-1}} E_b\left[\frac{W_B^{1-\gamma}}{1-\gamma}\right] = \frac{W_b^{1-\gamma}}{1-\gamma} Q(\mathbf{r}_b, \boldsymbol{\theta}_b, \boldsymbol{\pi}_b, t_b), \tag{3}$$

i.e. the optimal value function can be factored in such a way to be homogeneous in wealth. Here $\boldsymbol{\theta}_b$ and $\boldsymbol{\pi}_b$ are both vectors that collect the parameters of the return generating process, conditional on information at time b, the precise content of which will be specified later on.

2.2. *The Return Generating Process*

The popular press often acknowledges the existence of stock market states by referring to them as "bull" and "bear" markets. Here we consider that the distribution of each international equity index may depend on states characterizing international stock markets. Thus we write the joint distribution of a vector of m returns, conditional on an *unobservable* state variable S_t, as:[5]

$$\mathbf{r}_t = \boldsymbol{\mu}_{S_t} + \sum_{j=1}^{p} \mathbf{A}_{j,S_t} \mathbf{r}_{t-j} + \boldsymbol{\varepsilon}_t \quad \boldsymbol{\varepsilon}_t \sim N(\mathbf{0}, \boldsymbol{\Sigma}_{S_t}) \tag{4}$$

$\mathbf{r}_t$ is the $m \times 1$ vector collecting asset returns, $\boldsymbol{\mu}_{S_t}$ is a vector of intercepts (corresponding to $\mathbf{r}_{t-j} = \mathbf{0}$ for $j = 1, ..., p$) in state S_t, $\mathbf{A}_{j,S_t}$ is the matrix of autoregressive coefficients at lag j in state S_t, and $\boldsymbol{\varepsilon}_t \sim N(\mathbf{0}, \boldsymbol{\Sigma}_{S_t})$ is the vector of return innovations which are assumed to be jointly normally distributed

[5] While many papers have found evidence of regimes in univariate stock portfolio returns (e.g., Perez-Quiros and Timmermann (2000), Ramchand and Susmel (1998), Turner, Startz and Nelson (1989), Whitelaw (2001)), we model the joint conditional distribution of m returns.

with zero mean and state-specific covariance matrix $\mathbf{\Sigma}_{S_t}$. S_t is an indicator variable taking values $1, 2, ...k$, where k is the number of states. The presence of heteroskedasticity is allowed in the form of regime-specific covariance matrices.[6]

Crucially, S_t is never observed and the nature of the state at time t may at most be inferred (filtered) by the econometrician (i.e. our investor) using the entire history of asset returns. Similarly to most of the literature on regime switching models (see e.g. Ang and Bekaert, 2002), we assume that S_t follows a first-order Markov chain. Moves between states are assumed to be governed by the transition probability matrix, $\mathbf{P}$, with generic element p_{ij} defined as

$$\Pr(s_t = i | s_{t-1} = j) = p_{ij}, \quad i, j = 1, .., k, \tag{5}$$

i.e. the probability of switching to state i between t and $t+1$ given that at time t the market is in state j.

While we allow for the presence of regimes, we do not exogenously impose or characterized them, consistently with the true unobservable nature of the state of the markets in real life. On the contrary, in the sections that follow we will conduct a thorough specification search − based on both information criteria and standard misspecification tests − for each asset menu, letting the data endogenously determine the number of regimes k (as well as the VAR order, p).[7]

Notice that (4) nests several return processes as special cases. If there is a single market regime, we obtain the linear VAR model with predictable mean returns that is commonly used in the literature on strategic asset allocation, see e.g. Campbell and Viceira (1999), and Kandel and Stambaugh (1996).[8] However, when multiple regimes are allowed, (4) implies various types of predictability in the return distribution. When either $\boldsymbol{\mu}_{S_t}$ or $\mathbf{A}_{j,S_t}$ $(j = 1, ..., p)$ do depend on the underlying, latent regime, then expected returns vary over time. Similarly, when the covariance matrices differ across states there will be predictability in higher order moments such as volatilities, correlations, skews and tail thickness, see Timmermann (2000). Predictability is therefore not confined to mean returns but carries over to the entire return distribution.

Notice further that while current returns are normally distributed conditional on the state, the one-period ahead return distribution is *not* simply normal with regime dependent conditional mean and/or regime dependent conditional volatility, because it is instead a mixture of normal variates. Furthermore, the two-period ahead distribution is a mixture of a mixture, thus higher order moments become more relevant as T grows − given the number of regimes. Appendix C explicitly computes skewness and kurtosis of T-period ahead portfolio returns, when conditional mean and variance are regime dependent.

2.3. *The Dynamics of Beliefs about the Prevailing State*

Since we treat the state of the market as unobservable − which is consistent with the idea that investors cannot observe the true state but can use the time-series of returns to obtain information about it − we

[6]Unconditional returns thus follow a Gaussian mixture distribution, the weighted average of the conditional distributions, with weights - the regime probabilities - that are updated as new return data arrive. As pointed out by Marron and Wand (1992), mixtures of normal distributions provide a flexible family that approximates many other distributions.

[7]See also Appendix B. On the contrary, Butler and Joaquin (2002) exogenously define bear, normal, and bull regimes according to the level of US returns. Each regime is constrained to collect one-third of the sample.

[8]The i.i.d. Gaussian model − also often adopted as a benchmark in the portfolio choice literature (see e.g. Barberis, 2000 and Brennan and Xia, 2001) − obtains instead when both $k = 1$ and $p = 0$.

model the evolution of the investors' beliefs using the standard Bayesian updating algorithm. Investors optimally update their beliefs about the prevailing state at the next rebalancing point using (see Hamilton, 1994):

$$\boldsymbol{\pi}_{b+1}(\hat{\boldsymbol{\theta}}_b) = \frac{\left(\boldsymbol{\pi}_b'(\hat{\boldsymbol{\theta}}_b)\hat{\mathbf{P}}_b^{\varphi}\right)' \odot \mathbf{f}(\mathbf{r}_{b+1};\hat{\boldsymbol{\theta}}_b)}{[(\boldsymbol{\pi}_b'(\hat{\boldsymbol{\theta}}_b)\hat{\mathbf{P}}_b^{\varphi})' \odot \mathbf{f}(\mathbf{r}_{b+1};\hat{\boldsymbol{\theta}}_b))]'\boldsymbol{\iota}_k}. \tag{6}$$

Here $\boldsymbol{\pi}_b(\hat{\boldsymbol{\theta}}_b)$ collects the $k \times 1$ vector of state probabilities and $\hat{\boldsymbol{\theta}}_b$ all the estimated parameters characterizing (4); $\odot$ denotes the element-by-element product, $\hat{\mathbf{P}}_t^{\varphi} \equiv \prod_{i=1}^{\varphi} \hat{\mathbf{P}}_t$ is the Markov transition matrix relevant to periods of length $\varphi \geq 1$, and $f(\cdot)$ is the density of returns at the next rebalancing point conditional on the regime, on past returns and on estimated parameters:

$$\mathbf{f}(\mathbf{r}_{b+1};\hat{\boldsymbol{\theta}}_b) \equiv \begin{bmatrix} f(\mathbf{r}_{b+1}|s_{b+1}=1,\{\mathbf{r}_{b-j}\}_{j=0}^{p-1};\hat{\boldsymbol{\theta}}_b) \\ \vdots \\ f(\mathbf{r}_{b+1}|s_{b+1}=k,\{\mathbf{r}_{b-j}\}_{j=0}^{p-1};\hat{\boldsymbol{\theta}}_b) \end{bmatrix}$$

$$= \begin{bmatrix} (2\pi)^{-\frac{N}{2}}|\hat{\boldsymbol{\Sigma}}_1^{-1}|^{\frac{1}{2}} \exp\left[-\frac{1}{2}\left(\mathbf{r}_b - \hat{\boldsymbol{\mu}}_1 - \sum_{j=0}^{p-1}\hat{\mathbf{A}}_{1j}\mathbf{r}_{b-j}\right)' \hat{\boldsymbol{\Sigma}}_1^{-1}\left(\mathbf{r}_b - \hat{\boldsymbol{\mu}}_1 - \sum_{j=0}^{p-1}\hat{\mathbf{A}}_{1j}\mathbf{r}_{b-j}\right)\right] \\ \vdots \\ (2\pi)^{-\frac{N}{2}}|\hat{\boldsymbol{\Sigma}}_k^{-1}|^{\frac{1}{2}} \exp\left[-\frac{1}{2}\left(\mathbf{r}_b - \hat{\boldsymbol{\mu}}_k - \sum_{j=0}^{p-1}\hat{\mathbf{A}}_{kj}\mathbf{r}_{b-j}\right)' \hat{\boldsymbol{\Sigma}}_1^{-1}\left(\mathbf{r}_b - \hat{\boldsymbol{\mu}}_k - \sum_{j=0}^{p-1}\hat{\mathbf{A}}_{kj}\mathbf{r}_{b-j}\right)\right] \end{bmatrix},$$

which exploits the fact that on conditional on the state, asset returns have in fact a Gaussian distribution. In essence, (6) implies that the probability of the states at the rebalancing date $b+1$ is a weighted average of the φ-step ahead predicted probabilities $(\boldsymbol{\pi}_b'(\hat{\boldsymbol{\theta}}_b)\hat{\mathbf{P}}_b^{\varphi})$, with weights provided by the likelihood of observing the realized returns $\mathbf{r}_{b+1}$ conditional on each of the possible states, as represented by scaled versions of $\mathbf{f}(\mathbf{r}_{b+1};\hat{\boldsymbol{\theta}}_b)$.

Appendix A gives further details on the methods applied to solve (2) under multivariate regime switching returns. Here we only stress that, since Appendix A reminds us that the backward solution of (2) implies the relationship

$$Q(\mathbf{r}_b, \boldsymbol{\pi}_b, t_b) = \max_{\boldsymbol{\omega}_b} E_{t_b}\left[\left(\frac{W_{b+1}}{W_b}\right)^{1-\gamma} Q\left(\mathbf{r}_{b+1}, \boldsymbol{\pi}_{b+1}, t_{b+1}\right)\right],$$

it is clear that portfolio choices will reflect not only hedging demands for assets due to stochastic shifts in investment opportunities but also a hedging motive caused by changes in investors' beliefs concerning future state probabilities, $\boldsymbol{\pi}_{b+1}$.

2.4. *The Buy-and-Hold Problem*

One interesting special case is the buy-and-hold framework in which $\varphi = T$. Under this assumption the Appendix implies that, similarly to Barberis (2000), the integral defining the expected utility functional can be approximated as follows:

$$\max_{\boldsymbol{\omega}_t} N^{-1} \sum_{n=1}^{N} \left\{ \frac{\left[\boldsymbol{\omega}_t' \exp\left(\sum_{i=1}^{T} \mathbf{r}_{t+i,n}\right)\right]^{1-\gamma}}{1-\gamma} \right\},$$

where N is the number of simulations, and $\boldsymbol{\omega}_t' \exp\left(\sum_{i=1}^{T} \mathbf{r}_{t+i,n}\right)$ is the portfolio return in the n-th Monte Carlo simulation when the portfolio structure is given by $\boldsymbol{\omega}_t$. Each simulated path of portfolio returns is generated using draws from the model (4)-(5) that allow regimes to shift randomly as governed by the transition matrix, $\mathbf{P}$. We use $N = 30,000$ simulations.[9] The Appendix provides details on the numerical techniques employed in the solutions and extends these methods to the case of an investor who adjusts portfolio weights every $\varphi < T$ months.

2.5. *Welfare Cost Measures*

To quantify the utility costs of restricting the investor's asset allocation problem, we follow Ang and Bekaert (2002), Ang and Chen (2002), and Guidolin and Timmermann (2004a, 2005a). Call $\hat{\boldsymbol{\omega}}_t^R$ the vector of portfolio weights obtained by imposing restrictions on the portfolio problem, for instance, when the investor is forced to avoid small caps. We aim at comparing the investor's expected utility under the unrestricted model – leading to some optimal set of controls $\hat{\boldsymbol{\omega}}_t$ – to the utility derived assuming the investor is constrained. Since a restricted model is a special case of an unrestricted model, the following relationship between the value functions holds:

$$J(W_t, \mathbf{r}_t, \hat{\boldsymbol{\pi}}_t; \hat{\boldsymbol{\omega}}_t^R) \leq J(W_t, \mathbf{r}_t, \hat{\boldsymbol{\pi}}_t; \hat{\boldsymbol{\omega}}_t),$$

i.e. imposing restrictions reduces the derived utility from wealth. The compensatory premium, λ_t^R, is then computed as:

$$\lambda_t^R = \left[\frac{J(W_t, \mathbf{r}_t, \hat{\boldsymbol{\pi}}_t; \hat{\boldsymbol{\omega}}_t)}{J(W_t, \mathbf{r}_t, \hat{\boldsymbol{\pi}}_t; \hat{\boldsymbol{\omega}}_t^R)}\right]^{\frac{1}{1-\gamma}} - 1. \tag{7}$$

The interpretation is that an investor would be willing to pay λ_t^R in order to get rid of the restriction. Several types of restrictions are analyzed in what follows.

3. Data

We use weekly data from the MSCI total return indices data base for Pacific, North American, European Small Caps and European Large Caps (MSCI Europe Benchmark). Returns on North American Large Caps are computed as a weighted average of the MSCI US Large Cap 300 Index and the D.R.I. Toronto Stock Exchange 300, using as weights the relative capitalizations of US and Canada.[10] In practice, the US large caps index receives a weight of 94.4% vs. a 5.6% for the Canadian index.

We use total return data series, inclusive of dividends, adjusted for stock splits, etc. Returns are expressed in the local currencies as provided by MSCI. This implies a rather common assumption – see e.g. De Santis and Gerard (1997), Ang and Bekaert (2002), and Butler and Joaquin (2002) – that our investor is sophisticated enough to fully hedge her currency positions, so that her wealth is unrelated to the dynamics of exchange rates.

[9] Experiments with similar problems in Guidolin and Timmermann (2004a) indicated that for $m = 4$, a number of simulations N between 20,000 and 40,000 trials delivers satisfactory results in terms of accuracy and minimization of simulation errors vs. computational speed. To provide a rough sense of the latter dimension, with $N = 30,000$ and $m = 4$, the calculation of each long-run vector of optimal portfolio weights requires 51 minutes using a Pentium IV 3.60 GHz CPU).

[10] While the MSCI Europe Benchmark index targets mainly large capitalization firms, no equivalent for North America (i.e. US and Canada) is available from MSCI.

The sample period is January 1, 1999 - June 30, 2003. A Jan. 1, 1999 starting date for our study is justified by the evidence of substantial portfolio reallocations induced by the disappearing currency risk in the European Monetary Union (Galati and Tsatsaronis, 2001; European Central Bank, 2001). Given the relatively short sample period enforced by the 'Euro structural break' in an asset menu that includes European stock returns, we employ data at a weekly frequency, which anyway guarantee the availability of 234 observations for each of the series. Furthermore, notice that our sample does straddle one complete stock market cycle, capturing both the last months of the stock market rally of 1998-1999, its fall in March 2000, the crash of September 11, 2001, and the subsequent, timid recovery.

Tables 1 and 2 report basic summary statistics for stock returns. Since about half of our sample is characterized by bear markets, average mean returns are low for all portfolios under consideration. However − as discussed in the Introduction − small caps represent an exception. In particular, European small caps are characterized by a non-negligible annualized 14.4% positive median return, followed by North American small caps with 12.8% per year.[11] The resulting (median-based) Sharpe ratios for small capitalization firms make them highly appealing in a portfolio perspective: North American small caps display a 0.59 Sharpe ratio, while European small caps score a stunning 0.89.[12]

On the other hand, Table 1 questions the validity of an approach that relies only on the Sharpe ratio: the small caps skewness is negative, indicating that there are asymmetries in the marginal density that make negative returns more likely than positive ones; their kurtosis exceeds the Gaussian benchmark (three), indicating that extreme realizations are more likely than in a simple Gaussian i.i.d. framework. Second, opposite remarks apply to other stock indices, in particular the North American large caps and Asian Pacific ones: their skewness is positive, which may be seen as an expected utility-enhancing feature by many investors; their kurtosis is moderate, close to what a Gaussian i.i.d. framework implies. These remarks beg the question: When and how much do higher order moment properties matter for optimal asset allocation?

The last two columns reveal that while serial correlation in levels is limited to European and small caps portfolios, the evidence of volatility clustering − i.e. the tendency of squared returns to concentrate in time − is widespread, which points to the possible need of models that capture heteroskedastic patterns.

Finally, Table 2 reports the correlation coefficients of portfolio returns. Pacific stock returns have lower correlations (around 0.4 - 0.6 only) with other portfolios than all other pairs in the table. This feature makes Pacific stocks an excellent hedging tool. All other pairs display correlations in the order of 0.7 - 0.8, which is fairly high but also expected in the light of the evidence in the literature that all major international stock markets are becoming increasingly prone to synchronous co-movements (e.g. Longin and Solnik, 1995).

[11] We use the median of returns as estimators of location: for variables characterized by substantial asymmetries (negative skewness), the median is a more representative location parameter than the mean.

[12] Alternatively, we take the ratio between median returns and their interquartile range, a measure of risk that does not rely on the standard variance measure. We find ratios of 0.87 and 0.47 for European and North American small caps, respectively. The ratio is only 0.03 for Asian Pacific stocks and it is obviously negative for European and North American large caps.

4. **International Portfolio Diversification**

In this section, we present the main results of the paper. The section is organized around three subsections, each devoted to a distinct asset menu. In each case, we start by presenting econometric estimates of the return generating process and proceed to calculate optimal portfolio weights.[13] The sequence of asset menus is as follows: first, we set up a benchmark by studying a traditional portfolio problem in which the asset menu is restricted to Asian Pacific, North American large, and European large caps equity portfolios ($m = 3$). Next, we allow our investor to buy European small caps ($m = 4$). The choice to expand the asset menu leveraging on European small caps first is justified by their high ratio between median returns and their risk measures. However, European small caps are also the stock portfolio exhibiting the worst third and fourth moment properties. Hence they represent a natural starting point. Finally, we further expand the asset menu and add to our North American large stocks equity portfolio the MSCI North American small portfolio ($m = 5$). For the time being we impose no-short sale restrictions; this assumption is removed in Section 5. Similarly, we focus initially on the simpler buy-and-hold case and then analyze results under dynamic rebalancing in Section 5.

4.1. *Benchmark Results: Restricted Asset Menu*

4.1.1. **The estimated return generating process**

Appendix B reports the results of a model specification search concerning the case in which the asset menu consists of European large caps, North American large, and the Pacific equity portfolios. We estimate a variety of multivariate regime switching models, including the special cases of no regimes, and/or no VAR, and/or homoskedasticity.[14] Several, alternative statistical indicators allow us to conclude that the absence of regime switching in international stock returns data is rejected, similar to the findings in Ang and Bekaert (2002) and Ramchand and Susmel (1998). Information criteria (such as the Bayesian and Hannan-Quinn statistics) that trade-off in-sample fit against parsimony and hence out-of-sample forecasting accuracy agree in favoring to a relatively simple and parsimonious (20 parameters vs. a total of 702 observations) model with $k = 2$, $p = 0$, and regime-dependent covariance matrix.

Table 3 details the estimates of such a two-state model in panel B. Panel A refers to a benchmark i.i.d. case, with constant mean and variance. For this restricted asset menu, estimated means are never significant, which is not a new finding in the regime switching class, while second moments are precisely estimated. This suggests that the two regimes are accurately characterized by their second moments than by the first ones. The very persistent regime (average duration exceeds 6 months), which we label “normal”, implies moderate volatilities (roughly 17-18% on annualized basis) and high correlation across pairs of stock indices. The “bear” state is less persistent (its average duration is only 9 weeks) and implies much higher volatilities (as high as 40% a year in the case of European large caps) as well as lower mean returns (in the order of -0.2 to -0.5% per week).

[13] However, all details concerning the model specification search are collected in Appendix B.

[14] Estimation of the model is relatively straightforward and proceeds by optimizing the likelihood function associated with our model. Since the underlying state variable, S_t, is unobserved we treat it as latent and use the EM algorithm to update our parameter estimates, c.f. Hamilton (1989).

4.1.2. **Implied portfolio weights**

We discuss two sets of portfolio weights estimates. A first exercise computes optimal asset allocation at the end of June 2003 for an investor who, using all past data for estimation purposes, has obtained the parameter estimates in Table 3. This is a simulation exercise in which the unknown model parameters are calibrated to coincide with the full-sample estimates. In such a type of exercise the assessment of the role played by the different equity portfolios in international diversification may dramatically depend on the peculiar set of parameter estimates one obtains. As a result, we supplement this first exercise with calculations of real time optimal portfolio weights, each vector being based on a *recursively updated* set of parameter estimates.

Figure 1 shows optimal portfolio shares as a function of the investment horizon for a buy-and hold investor who employs parameter estimates at end of June 2003. Results for two alternative levels of relative risk aversion are reported, $\gamma = 5$ and 10. Each plot concerns one of the available equity portfolios and reports four alternative schedules: two of them condition on knowledge of the current, initial state of the markets (normal or bear); one further schedule implies the existence of uncertainty on the nature of the regime and assumes that the regime probabilities are set to match their long run, ergodic frequencies (in this case 0.73 and 0.27, for normal and bear states); one last schedule depicts the optimal choice by a myopic investor who incorrectly believes that international stock returns are drawn by a multivariate Gaussian model with time invariant means and covariance matrix.[15] Importantly, this last set of results corresponds to the case in which variance risk is disregarded altogether. The only demand for European large stocks is generated for $\gamma = 10$ and the normal state, when the variance of European large stocks is particularly small. Investors should otherwise demand North American large and Pacific stocks. North American large stocks are more attractive in the short-run and in the bear state (regime 2) when their mean returns are higher than all other stock portfolios. However, as the horizon T grows, the weight in North American large stocks generally declines.

In the normal state, the slopes are reversed: the North American schedule becomes upward sloping while the Pacific one is downward sloping. This occurs because Pacific stocks have the highest Sharpe ratio in the normal state, but the probability of a switch from the normal to the bear regime increases over time thus justifying increased caution towards these stocks.

Importantly, there are marked differences between the regime-switching portfolio weights and the IID benchmark that ignores predictability, especially for the case of the normal regime when $\gamma = 5$: while the IID weights are 38% in North American large stocks and 62% in Pacific stocks, the regime-dependent optimal choices assign much less weight to the former portfolio (the difference is almost 20% at long horizons when the comparison is performed with the steady-state schedule).[16]

We have calculated, but do not report for brevity, the welfare costs of ignoring regimes and adopting instead a IID return generating process. These are the utility losses from ignoring the existence of state-dependence in the moments of the joint distribution of asset returns, and hence variance risk itself. The

[15] These schedules are flat, implying that the investment horizon is irrelevant for asset allocation purposes.

[16] There is no reason to think that the IID schedule ought to be an average of the regime-specific ones: the unconditional (long-run) joint distribution implied by a Gaussian IID and a multivariate regime switching model need not be the same; on the opposite, our specification tests offer evidence that the null of a Gaussian IID model is rejected, an indication that the unconditional density of the data differs from the one implied by a switching model.

welfare costs strongly depend on the assumed initial state as well as on risk-aversion, being higher under moderate values for γ and in regime 1 (normal). However, an investor who ignores the initial regime and purely conditions on long-run ergodic probabilities would 'feel' a long-run (for $T = 2$ years) welfare loss of almost 20% of her initial wealth. This estimate is large and stresses that regimes should not be ignored when approaching international diversification problems.

In order to assess how sensitive portfolio choice is to the arrival of new information on the prevailing regime, we recursively estimate the parameters of the regime switching model with data covering the expanding samples Jan. 1999 - Dec. 2001, Jan. 1999 - first week of Jan. 2002, etc. up to the full sample Jan. 1999 - June 2003 previously employed. Unreported plots show that our previous remarks are not an artifact of the particular sample period we have selected: The demand for Pacific stocks is relatively stable, both over time and over investment horizons. Even though European large caps have become less attractive over time, as the incidence of the bear state increased, their demand has always been always limited.[17]

These results set up the background against which we proceed to measure the variance risk characterizing small caps. When the asset menu is restricted to European and North American large caps only − besides an overall Pacific portfolio − international diversification is substantial both in end-of-sample simulations and in real time experiments, although the highest proportions go to North American large and Pacific equities. This result echoes De Santis and Gerard's (1997) multivariate GARCH results for a larger set of national equity indices.

4.2. *Diversifying with European Small Caps*

4.2.1. **The estimated return generating process**

Appendix B repeats our specification search with reference to a model with four equity portfolios: European large and small stocks, North American large, and Pacific. Also in this case, the evidence against the null of a linear, IID Gaussian model is overwhelming in terms of likelihood ratio tests. The information criteria provide contrasting indications, but in the light of the pervasive evidence of volatility clustering in table 1, we select a three-state model that allows greater flexibility in capturing heteroskedastic patterns. Panel B of Table 4 reports parameter estimates for the selected 3-state process. In this case most of the estimated mean returns − beside covariance matrices − are highly significant. The second regime, that we label *normal*, is the dominant one in terms of long-run ergodic probability (72%). In this state, mean returns are essentially zero, volatilities are moderate (around 15% a year for all portfolios), correlations are high. This regime is highly persistent with an average duration in excess of 7 months.

When international equity markets are not in a normal state, there are two possibilities. With an ergodic probability of 13%, they are in the first, *bear* regime, when mean returns are negative across all portfolios.[18] The bear regime is also a high-volatility state: the variance of all portfolios drastically

[17] We also compute recursive estimates of the utility costs of ignoring regimes and observe that for long enough horizons the loss oscillates between 1 and 3 % in annualized terms over most of the sample. Peaks of 5 % (in annual terms) and higher are reached in correspondence to periods characterized as a bear state (e.g. the Summer of 2002).

[18] Readers may be concerned for the equilibrium justification of the existence of a state with negative stock returns. However − unless all investors have 1-week investment horizons − this does not imply a zero or negligible demand for stocks, as for longer horizons switching to better states with zero or positive mean returns is not only possible, but almost sure provided

increases when markets switch from normal to bear states, with peaks of volatility in excess of 21% per year (for European stocks). Interestingly, some of the implied correlations strongly decline when going from regime 2 to 1, with Pacific stocks being almost uncorrelated with both North American and European large caps. However, the persistence of regime 1 is low: starting from a bear state there is only a 22% probability of staying in such a state. As a result, the average duration of such a state is less than 2 weeks. This fits the common wisdom that sharp market declines happen suddenly and tend to span only a few consecutive trading days.

The rest of the time (15%), the markets are in a bull regime in which mean returns are positive, high, and significant. European large caps are characterized by the highest mean, 3.7% in a week. Once more, volatility is high in the bull regime: this is true for all markets, although the wedge vs. the normal volatilities are extreme for both large caps portfolios, for which bull volatility is almost twice the normal one (e.g. 27% in annualized terms for European large caps). Correlations decline when compared to the normal regime. Those involving Pacific stocks become systematically negative, which makes of Pacific equities an excellent hedge in this regime. The bull regime has low persistence, with a 'stayer' probability of 29% only and an average duration of less than 2 weeks.

An unreported plot for the smoothed state probabilities reveals that the bear state occurs relatively frequently in our sample (e.g. the week of September 11, 2001 is picked up by this state) but it rarely lasts more than 3 weeks. It also shows that bull states tend to cluster in the same periods in which bear states appear. The sum of the probability of the two regimes gives an estimate of the probability of being in a high volatility state, revealing that the 'high volatility' regime is persistent although its components are not. It captures periods which have been ex-post recognized as extremely volatile, e.g. early 2001 with the accounting scandals in the U.S. or the Fall of 2001, after the terror attacks to New York City. This is confirmed by the structure of the estimated transition matrix in Table 6: although the 'stayer' probabilities of bull and bear regimes are small, they both have rather high probabilities (0.78 and 0.54, respectively) of switching from bear to bull and from bull to bear. Thus several weeks may be characterized by highly volatile returns, although the signs of the means may be quickly switching back and forth.

4.2.2. **Implied portfolio weights**

The role of European small caps (henceforth EUSC) in portfolio choice may strongly depend on the regime: indeed they have the best and second-best Sharpe ratios in the normal and bull states (a non-negligible 0.21 and a stellar 0.77, respectively), and display the worst possible combination (negative mean and high variance) in the bear state. However, it is not clear how this contrasting information may influence the choice of investors who cannot observe the state. Furthermore, speculating on the Sharpe ratio to trace back portfolio implication may be incorrect when portfolios have higher-moment properties featuring high variance risk.

Figure 2 shows the end-of-sample portfolio results. The demand for EUSC is roughly independent of the horizon and of γ when the state is normal. Approximate independence of the horizon is justified by the fact that the normal state is highly persistent. The schedule for the bull state provides first evidence that using the Sharpe ratio may be misleading: in regime 3, EUSC are never demanded as all the weight is given to North American large and Pacific stocks (plus European large caps for horizons between 1 and

the horizon is long enough.

3 weeks). Even though European large stocks have the best Sharpe ratio in the bull state, Even though European large stocks have the best Sharpe ratio in the bull state, the intuition behind the finding that their demand does not survive the test of longer horizons is that, while North American large caps still provide a respectable 0.62 Sharpe ratio, Pacific stocks provide their perfect hedge. Unsurprisingly, EUSC fail to enter the optimal portfolio in the bear state.

Even more interesting is the result concerning the 'steady-state' allocation to EUSC, when the investor assumes that all regimes are possible with a probability equal to their long-run measure. In this case − the most realistic situation since regimes are in fact not observable − EUSC play a limited role. Their weight is zero for short horizons ($T = 1, 2$ weeks) and grows to an unimpressive 10% for longer horizon. Once more, the steady-state portfolio puts almost identical weights on North American and Pacific equities. On the opposite, the IID myopic portfolio would be grossly incorrect, when compared to the steady-state regime switching weights, as it would place high weights on EUSC (87%) and Pacific stocks (13%). Finally, European large caps keep playing a modest role.

Figure 3 shows our estimates of the welfare costs of ignoring the existence of variance risks (regimes). Since Figure 2 stresses the existence of large differences between regime-switching and IID myopic weights, it is less than surprising to see that the utility loss from ignoring variance risk is of a first-order magnitude: for instance, a highly risk-averse ($\gamma = 10$), long-horizon ($T = 2$ years) investor who assigns ergodic probabilities to the states would be indifferent to account for regimes if compensated by a sum equal to roughly 4% of her initial wealth. These sums are of course much larger should we endow the investor with precise information on the nature of the current state (especially when the information is profitable, as it is in the bear and bull regimes), as the welfare loss climbs to 15-20% of wealth.

These results do not seem to entirely depend on the point in time in which they have been performed. We recursively estimate our three-state model and compute optimal portfolio weights similarly to Section 4.1.2. The average (over time) weight assigned to EUSC remains only approximately 39%, while also European large caps acquire substantial importance (26%), followed by North American large and Pacific stocks (23 and 12%).[19] Also in this case, ignoring variance risk would assign way too high a weight to EUSC, in excess of 80% on average (the rest goes to Pacific stocks). As a result, our recursive estimates of the welfare loss of ignoring regime switching (not reported) are extremely large over certain parts of the sample, exceeding annualized compensatory variation of 5-10% even under the most adverse parameters and investment horizons.

4.2.3. **Making sense of the results: variance risk**

Our simulations find that, under realistic assumptions concerning knowledge of the state, a rational investor should invest a limited proportion of her wealth in EUSC despite their high (median-based) Sharpe ratio. Tables 5 and 6 report several statistical findings that help us put this result into perspective. Many recent papers (Das and Uppal, 2004; Jondeau and Rockinger, 2003; Guidolin and Timmermann, 2005c) have stressed that investors with power utility functions are not only averse to variance and high correlations between pairs of asset returns − as normally recognized − but also averse to negative co-skewness and to high co-kurtosis, i.e. to properties of the higher order co-moments of the joint distribution of asset returns.

[19]These weights are also obtained by averaging across investment horizons, although slopes tend to be moderate, consistently with the shapes reported in Figure 2. These results are for the $\gamma = 5$ case. Under $\gamma = 10$, they are 36, 23, 26, and 15 percent.

For instance, investors dislike assets whose returns tend to become highly volatile at times in which the price of most of the other assets declines: in this situation, the expected utility of the investor is hurt by both the low expected mean portfolio returns as well as the high variance contributed by the asset.[20] Similarly, investors ought to be wary of assets the price of which declines when the volatility of most other assets increases. Investors will also dislike assets whose volatility increases when most other assets are also volatile. We say that an asset that suffers from this bad higher co-moment properties is subject to high *variance risk*.

Tables 5 and 6 pin down these undesirable properties of EUSC. In Table 5 we calculate the co-skewness coefficients,

$$S_{i,j,l} \equiv \frac{E[(r_i - E[r_i])(r_j - E[r_j])(r_l - E[r_l])]}{\{E[(r_i - E[r_i])^2]E[(r_j - E[r_j])^2]E[(r_l - E[r_l])^2]\}^{1/2}},$$

between all possible triplets of portfolio returns i, j, l. We do that both with reference to the data as well for the three-state model estimated in Section 4.2.1. In the latter case, since closed-form solutions for higher order moments are hard to come by in the multivariate regime switching case, we employ simulations to produce Monte Carlo estimates of the co-moments under regime switching. Calculations are performed both unconditionally (i.e. averaging across regimes) and conditioning on knowledge of the initial regime. In the latter case, the conditional co-moments refer to the one-step ahead predictive joint density of asset returns. Based on our definition, variance risk relates to the cases in which the triplet boils down to a pair, i.e. either $i = j$, or $i = l$, or $j = l$.[21] When $i = j = l$ we will be looking at the standard own skewness coefficient of some portfolio return. In Table 5, bold coefficients highlight point estimates' significance at standard levels (5 percent). There is an amazing correspondence between signs and magnitudes of co-skewness coefficients in the data and the unconditional estimates under our estimated regime switching model. Similarly to Das and Uppal (2004) we interpret this result as a sign of correct specification of the model.[22] Furthermore, notice that the co-skewness coefficients $S_{EUSC,EUSC,j}$ are all negative and large in absolute value for all possible js: the volatility of EUSC is indeed higher when each of the other portfolios performs poorly. On the opposite, similar co-skewness coefficients for most other indices (e.g. $S_{EU\_\text{large},EU\_\text{large},j}$ for varying js) are close to zero and sometimes even positive. Worse, a few of the $S_{EUSC,j,j}$ coefficients are also large and negative (when j = Pacific), an indication that EUSC may be losing ground exactly when some of the other assets become volatile. Therefore EUSC does display considerable variance risk. On the top of variance risk, from Tables 1 and 5 it emerges that EUSC also show high and negative own-skewness (i.e. left asymmetries in the marginal distribution which imply higher probability of below-mean returns), another feature a risk-averse investor ought to dislike.[23]

The results in the second column of Table 5 are relevant to interpret long-run portfolio choices, when the statistical properties of stock returns are well-approximated by their unconditional density. Table

[20]This is the case in the model of Vayanos (2004), where fund managers are subject to uncertain withdrawals in bear markets.

[21]Coefficient estimates for the cases in which $i \neq j \neq l$ are available but are hard to interpret. However our comments concerning the general agreements between sample and model-implied co-moment estimates also extend to the $i \neq j \neq l$ case.

[22]These findings confirm Ang and Chen's (2002) claim that markov switching models are fit to capture non-normalities in stock returns.

[23]This means that portfolios which already display an asymmetric marginal density of returns are also highly exposed to the possibility of their left tail 'thickening' when other portfolios become more volatile. This finding echoes a similar result in Acharya and Pedersen (2004) for idiosynchratic vs. systemic illiquidity.

5 also reports regime-specific, one-step ahead co-skewness coefficients, when the initial state is known (while the ending state is not). In the highly persistent normal regime 2, departure from multivariate normality are minimal and in fact none of the co-skewness coefficients is significantly different from zero. Therefore, at least for short investment horizons of a few weeks at most, using the Sharpe ratio for portfolio allocation purposes may be justified and $-$ consistently with the results in Figure 2 $-$ EUSC ought to receive considerable weight because of their excellent mean-risk trade-off properties. On the opposite, the bear and bull regimes 1 and 3 imply some important departures of the joint predictive density of stock returns even over short investment horizons. In particular, EUSC have a tendency to decline when the volatility of Pacific stocks is above average, while the volatility of EUSC tends to be high when each of the other markets is bearish. Because conditional on the current regime the properties of the predictive joint density are similar (although departures from normality are not as strong) to those found for the unconditional, long-run distribution, the optimal portfolio weights on regimes 1 and 3 in Figure 3 are relatively insensitive to the investment horizon and generally imply a modest role (or none at all) for EUSC.

Of course, it may be hard to balance off co-skewness coefficients involving EUSC with different magnitudes or signs. In these cases, it is sensible to calculate quantities similar to those appearing in Table 5 for portfolio returns vs. some *aggregate* portfolio benchmark. For our purposes we use an equally weighted portfolio (EW_ptf, 25% in each stock index), although results proved fairly robust to other notions (e.g. value-weighted) of benchmark portfolio. For instance, S_{i,EW_ptf,EW_ptf} for the generic portfolio i has expression

$$S_{i,EW\_ptf,EW\_ptf} \equiv \frac{E[(r_i - E[r_i])(r_{EW\_ptf} - E[r_{EW\_ptf}])^2]}{\sqrt{Var[r_i]}Var[r_{EW\_ptf}]},$$

the notion of co-skewness between a security i and the market portfolio employed in Harvey and Siddique (2000) and Ang, Chen, and Xing (2005). Once more the match between data- and model-implied coefficients is striking. In particular, in panel A of Table 6 we obtain model estimates $S_{EUSC,EUSC,EW\_ptf} = -0.60$ and $S_{EUSC,EW\_ptf,EW\_ptf} = -0.44$, i.e. the variance of EUSC is high when equally weighted returns are below average, and EUSC returns are below average when the variance of the equally weighted portfolio is high. This is another powerful indication of the presence of variance risk plaguing EUSC. For comparison purposes, in panel B of Table 6 we repeat calculations for European large stocks and obtain negligible (or even positive) coefficients.[24]

The co-skewness S_{EUSC,EW_ptf,EW_ptf} is the unconditional version of the moment used to amend several asset pricing models by Harvey and Siddique (2000). It is also akin to the covariance between EUSC return and market illiquidity , while $S_{EUSC,EUSC,EW\_ptf}$ is reminiscent of the covariance between EUSC illiquidity and market return that explains a large part of the small cap premium in the liquidity CAPM of Acharya and Pedersen (2004). In a sense, we are providing a portfolio choice rationale for their pricing formula, without resorting to exogenous illiquidity costs that are necessary in a mean-variance framework.

Table 7 performs an operation similar to Table 5, but with reference to the fourth co-moments of equity returns.[25] Once more $-$ although some discrepancies appear (as the order of moments grows their accurate

[24]Results are similar for North American large and Pacific portfolios and are available upon request.

[25]Also in this case, coefficient estimates for the cases in which $i \neq j \neq l \neq b$ are available on request.

estimation becomes more troublesome) − we find a striking correspondence between large co-kurtosis coefficients measured in the data and unconditional coefficients implied by our regime switching model (estimated by simulation). Generally speaking, EUSC have dreadful co-kurtosis properties: for instance $K_{EUSC,EUSC,j,j}$ exceeds 2.2 for all js and tends to be higher than all other similar coefficients involving other portfolios, which means that the volatility of EUSC is high exactly when the volatility of all other portfolios is high. As already revealed by Table 1, also the own-kurtosis of EUSC substantially exceeds a Gaussian reference point of 3. Table 6 confirms that also the model-implied $K_{EUSC,EUSC,EW\_ptf,EW\_pft}$ is 3.3, which is one of the highest among these types of coefficients. $K_{EUSC,EUSC,EW\_ptf,EW\_pft}$ is reminiscent of an indicator of covariance between EUSC illiquidity and market illiquidity. All in all, we have also some evidence that the extreme tails of the marginal density of EUSC tends to be fatter than for other portfolios and that their volatility might be dangerously co-moving with that of other assets. In conclusion, while the demand for European large caps is modest (with the exception of the bull state and $T = 1, 2$ weeks) because of their low Sharpe ratios and high correlation compared with Pacific stocks, the demand for EUSC is limited by their poor higher (co-) moment properties, in particular by their asymmetric marginal density variance risk.[26]

4.2.4. **Decomposing Variance Risk**

Tables 5 and 7 suggest a precise ranking of the contribution of higher order co-moments to EUSC's variance risk. First, the negative co-skewness of EUSC is particularly strong for all horizons, while the differences of co-kurtosis across stock portfolios are less striking. Second, for short investment horizons, the only important departure of our asset allocation framework under regimes from the single-regime IID Gaussian benchmark is provided by the co-skewness properties of EUSC. However, both these claims only rely on comparisons concerning the value taken by model-implied moments or at most their differential statistical significance. The actual relative importance of the factors underlying variance risk for optimal asset allocation can be directly assessed through the computation of portfolio weights when all but one of the sources of variance risk are left in operation.

We start by considering the case in which co-skewness effects are shut off so that only co-kurtosis effects − co-movements in the time-variation of the tail thickness of different portfolios − may affect asset allocation decisions. This can be obtained by calculating portfolio weights under a special regime switching model in which $\boldsymbol{\mu}$ is made independent of the state S_t, while $\boldsymbol{\Sigma}_{s_t}$ remains a function of the regime, i.e.

$$\mathbf{r}_t = \boldsymbol{\mu} + \boldsymbol{\varepsilon}_t \quad \boldsymbol{\varepsilon}_t \sim N(\mathbf{0}, \boldsymbol{\Sigma}_{s_t}), \tag{8}$$

This model aims at capturing pure heteroskedasticity effects since expected stock returns are constrained to be constant over time. It is possible to show (see Appendix C for a few details) that (8) implies that any departure from multivariate IID normality must come from *even* co-higher moments differing from their normal counterparts.

We therefore proceed to estimate (8) and to calculate optimal portfolio weights for an investor with $\gamma = 5$. Insofar as volatilities and correlations are concerned, the (unreported) parameter estimates are

[26] Table 9 also shows regime-specific, conditional (one-step ahead) co-kurtosis coefficients. They tend to be close to their multivariate Gaussian counterparts. This means that while long-run portfolio choices are also driven by the co-kurtosis properties of the stock portfolios under investigation, this is hardly the case for short horizons, when co-skewness is the only important factor.

generally rather close to those appearing in Table 4. Table 8 reports buy-and-hold optimal portfolio weights for a few alternative investment horizons and compares them with the results underlying Figure 2, for the unrestricted model. The table suggests that odd high-order co-moments (co-skewness) have first-order effects in reducing the portfolio role of EUSC. In fact, under model (8) the long-run, ergodic optimal portfolio weights (86% in EUSC and the reminder in Pacific stocks) are essentially the same as those obtainable under the false assumption of a multivariate Gaussian model with no regimes. Otherwise, EUSC remain very important when the investor is given information on the current, initial state being of a 'normal' type, but in the bear and bull regimes their weight differs dramatically from Figure 2.

A similar experiment involves the model in which $\mathbf{\Sigma}$ is made independent of the state S_t, while $\boldsymbol{\mu}_{S_t}$ is regime-specific, i.e.

$$\mathbf{r}_t = \boldsymbol{\mu}_{S_t} + \boldsymbol{\varepsilon}_t \quad \boldsymbol{\varepsilon}_t \sim N(\mathbf{0}, \mathbf{\Sigma}), \tag{9}$$

a model aimed at capturing pure regime switching predictability in expected returns, while homoskedasticity is (incorrectly) imposed. Appendix C argues that this restriction is unable to shut completely off the portfolio effects of even moments. However, under regime-independent conditional variance, departures of the third central moment from a Gaussian IID benchmark are of order 3, while fourth central moment deviations will be at most of order four. Therefore (9) represents a second-best device to investigate the portfolio weight effects of adding potential co-skewness driven variance risk.

Co-skewness turns out to have first-order effects. We estimate (9) and calculate optimal portfolio weights for an investor with $\gamma = 5$. The three regimes have characterizations and persistence very similar to those reported in Table 4, implying that − even though volatilities and correlations display rich cross-regime patterns in Table 4 − the definition of the nature of the three states is essentially dominated by the properties of expected stock returns. Additionally, the estimated regime-dependent expected returns are very similar to those appearing in Table 4. Interestingly, the demand for EUSC remains high and slowly changing with T when the investor is endowed with accurate knowledge that the current time t regime is the second, normal state. However, in the realistic case in which the regime is not known, the weight to EUSC is rather moderate, zero for short investment horizons and progressively increasing towards 30-35% for horizons exceeding one year. The reduction of the EUSC weight from 75-86% when only even co-moments are taken into account to 30-35% when odd co-moments effects are added, measures the first-order role played by co-skewness in portfolio diversification.

4.2.5. **Welfare Costs of Ignoring European Small Caps**

Gompers and Metrick (2002) observe that institutions do not usually invest in small caps, because they prefer liquid assets. This is surprising for long-horizon investors, such as pension funds and university endowments, that could profit from their higher Sharpe ratios and diversification potential without incurring too often large transaction costs (Amihud and Mendelsohn, 1986; Brennan and Subrahmanyam, 1996; Vayanos, 1998; Lo et al., 2004). Our evidence concerning the high variance risk of EUSC may in principle be able to explain their neglect as higher moments of their return distribution increase undesired skewness and kurtosis of wealth. However: Does this mean that there is no utility loss from restricting the available asset menu to exclude small caps?

We provide a preliminary answer by considering the case of EUSC. We consider this exercise informative because we found that EUSC have a limited role in optimal portfolios despite their promising full-sample unconditional Sharpe ratios; and display bad co-higher moment properties, i.e. *their variance risk is high*. Thus we may suspect that eliminating European small caps from the asset menu will only slightly reduce investors' welfare.

We compute compensatory variations similar to those in Sections 4.1.2 and 4.2.2. In this case we identify $J(W_t, \mathbf{r}_t; \hat{\boldsymbol{\omega}}_t^R)$ with the value function under a restricted asset menu that rules our EUSC; on the other hand, $J(W_t, \mathbf{r}_t; \hat{\boldsymbol{\omega}}_t)$ is the value function of the problem solved in this Section 4.2.[27] The conclusion drawn from Table 9 is that – in spite of their limited optimal weight – the loss from disregarding EUSC would be of a first-order magnitude. Therefore there is no direct mapping between Gompers and Metrick's result that small caps seem to be unimportant and the conclusion that their market is irrelevant. However, long horizon investors suffer a smaller loss than short horizon ones, which can exploit small caps for tactical purposes with lower probability of incurring into a regime shift. In particular, end-of-sample calculations (panel A, no short sales) show that the annualized utility loss of ignoring EUSC declines with the investment horizons, starts at exceptionally high levels (e.g. 60% a year in the ergodic probability case) for a weekly horizon to diminish to approximately 3 % when $T = 2$ years. Panel B documents real time results, distinguishing between three different samples (the last two break down Jan. 2002 - June 2003 into two shorter, 9-month periods to have a sense for the stability of the results over time). Interestingly, mean compensatory variations are now even higher, reaching levels in excess of 10 % per year even at long horizons and in the worst real time sub-samples.[28]

When faced with compensatory variation in excess of 3 % per year (up to 10 % per year) that can be considered as upper bounds for the transaction costs, it is difficult to think that small caps are not important for international diversification purposes. Although it is well-known that the effective costs paid when transacting on small caps strongly depend on the nature of the trader, on tax considerations, and on the frequency of trading, it is unlikely that any sensible estimate of the costs implied by long-run buy-and-hold positions (i.e. revised only every one or two years) may systematically exceed the spectrum of welfare loss estimates we have found. So, modest optimal weights and high doses of variance risk are still compatible with a claim that small caps are key to expected utility enhancing international portfolio diversification.

4.3. *The Role of Small Caps in an Extended Asset Menu*

Even though we have discussed our reasons to start the exercise by first augmenting the asset menu using EUSC, in this Section we proceed to further generalize the problem to also include North American small caps (NASC), besides the North American large portfolio, i.e. $m = 5$. We repeat the usual analysis of Sections 4.1 and 4.2 and therefore omit many details to save space.

[27] Notice that $J(W_t, \mathbf{r}_t; \hat{\boldsymbol{\omega}}_t^R) \leq J(W_t, \mathbf{r}_t; \hat{\boldsymbol{\omega}}_t)$ does not hold as the two value functions concern problems solved under different data, statistical models, and parameter estimates.

[28] Panel B of Table 10 also displays standard deviations for welfare loss estimations. In only one case the pseudo t-statistic is not significant at a standard 5% size. This means that our conclusion that omitting EUSC in real time implies high utility loss does not purely depend on some isolated peaks.

4.3.1. **The return generating process**

In Table 10, the characterization of the regimes is very similar to Section 4.2.1: the second regime is a normal, highly persistent state in which both mean returns (with the exception of NASC) and volatilities are small; correlations are all fairly high, including those involving Pacific stocks. The first regime is a bear state in which mean returns are significantly negative and large (e.g. -4% per week for European large caps), volatilities are high (between 25 and 50% higher than in the normal state), and correlations moderate. The third regime is a bull state implying high and significant means, high volatilities and modest correlations. Notice that once more all correlations involving Pacific stocks turn negative and some of them are now even significantly so. The bear and bull states are non-persistent; however, the structure of the estimated transition matrix is such that the world equity markets may easily enter a high volatility meta-state in which they cycle between regimes 1 and 3 with sustained fluctuations but relatively small chances to settle down to the normal state of affairs.

A comparison of Tables 10 and 4 shows that the characterization of the states is essentially unchanged when we add NASC to the asset menu: this is an important finding that corroborates the validity of our three-state regime switching model. The ergodic probabilities of the regimes are almost unchanged, 0.17, 0.65, and 0.18, respectively.

4.3.2. **Implied portfolio weights**

Although Section 4.2 has provided examples that both unconditional and regime-specific Sharpe ratios may be misleading, we start by stressing how in this metric NASC dominate EUSC and all other equity portfolios. Panel A of Table 10 shows that NASC have a Sharpe ratio of 0.06 vs. 0.01 for EUSC and negative ratios for all other portfolios. Figure 4 plots optimal portfolio schedules. As a reflection of the difference in Sharpe ratios, a myopic investor that ignores variance risk would invest most of her wealth (58%) in NASC, another important proportion in EUSC (29%), and the remainder (13%) in Pacific stocks, essentially for hedging reasons given the low correlations between Pacific and other portfolios. This means that a stunning 87% of the overall wealth ought to be invested in small caps, North American and European.

This portfolio recommendation would again be incorrect, both because it ignores the existence of predictability patterns induced by the structure of the transition matrix, and because it does not take into account variance risk. In fact, the regime switching portfolio schedules in Figure 5 contain dramatic departures from the solid, bold lines flattened by the IID myopic assumption: focussing on the case of $\gamma = 5$ and assuming the investor ignores the current regime, her commitment to NASC would remain large (and increasing in T) but would be in the 40-50% range; once more, EUSC imply large amounts of variance risk and poor third- and fourth-order moment properties, which brings their weights down to 15-20%. There is then the opportunity to invest between 30 and 45% in other portfolios, mainly the Pacific one.

Optimal allocations also turn out to be strongly regime-dependent: for instance, the bear state 1 is highly favorable to NASC investments as these stocks have the highest Sharpe ratio in this regime, while Pacific stocks provide a relatively good hedge; however as T grows the probability of leaving the bear state grows, so that investment schedules revert to their ergodic counterparts. Finally, North American large caps appear with moderate weights only in the extreme regimes 1 and 3, i.e. they should optimally be included in the portfolio only 35% of the time, which is quite a modest assessment of their overall importance.

Table 11 performs computations of co-skewness and co-kurtosis coefficients vs. an equally weighted portfolio, both under the available data and under the three-state regime switching model of Table10. In the latter case, simulations are employed. We find estimates $S_{NASC,EW\_ptf,EW\_ptf} = -0.29$ and $S_{NASC,NASC,EW\_ptf} = -0.25$ that approximately fit the sample moments; $K_{NASC,NASC,EW\_ptf,EW\_ptf} = 2.20$ is furthermore close to the sample estimate of 2.75.[29] This means that for both small cap portfolios we have evidence that their variance increases when the variance of the market is high, that their variance is high when the market is bear, and that their returns are below average when the market is unstable. These properties (along with own kurtosis and skewness) explain why our portfolio results do not completely reflect simple Sharpe ratio-based arguments and why both portfolios receive a much higher weight under the myopic IID calculations than in the plots in Figure 5. The estimates in Table 11 also make it clear that NASC imply less variance risk than EUSC − hence their higher weights in Figure 5.

Once more, real time results (for $\gamma = 5$) confirm that our conclusions are far from an artifact of the end-of-sample estimates: small caps play a substantial role in international diversification although their variance risk reduces somewhat their relevance, for instance from an average 90% myopic IID weight to less than 60% under regime switching. This wedge of roughly 30% in portfolio weight is a *prima facie* measure of the importance of variance risk in international diversification.

We conclude by performing the usual welfare cost calculations. While the utility loss of ignoring predictability remains large (especially when the investor is given knowledge of the current state), the most important result concerns the utility loss of ruling out diversification through small caps, similarly to Table 9. Assuming $\gamma = 5$, we find that the utility loss of excluding *both* NASC and EUSC from the asset menu is large (in annualized terms) over the short horizon (e.g. 39% for $T = 1$ week) and remains of the same order of magnitude as in Section 4.2.4 over long horizons (e.g. 4.7% for $T = 1$ year and 3.7% for $T = 2$ years). Results are only slightly smaller when risk aversion is set to higher levels (e.g. under $\gamma = 10$ we have 2.4% for $T = 1$ year and 1.5% for $T = 2$ years). Even a welfare loss of 'only' 150 basis points on an annualized, riskless basis appears enormous in light of the utility losses normally reported in the literature (e.g. Ang and Bekaert, 2001).

It may well be that total transaction costs associated with small caps exceeds 3-4%, the annualized welfare gain from including small caps into the portfolio of a 2-year investor. While the effective spread on the four most illiquid NYSE and AMEX stock deciles ranges from 0.98 to 4.16% (see Chalmers and Kadlec, 1998), the transaction costs associated with EUSC could be higher, for two reasons. First, some European markets are less liquid than the NYSE.[30] Second, total transaction costs include not only bid-ask costs but commissions as well. For instance, Lesmond (2004) estimates total round-trip costs to be equal, on average, to 8.5% in the Hungarian market. Hence, a 6% transaction cost over 2 years for the round-trip

[29] The evidence of variance risk remains strong for EUSC: the regime switching estimates are $S_{EUSC,EW\_ptf,EW\_ptf} = -0.31$, $S_{EUSC,EUSC,EW\_ptf} = -0.28$, and $K_{EUSC,EUSC,EW\_ptf,EW\_ptf} = 3.06$. Notice that these values are different from those in Table 8 as they are obtained for a different asset menu and statistical model.

[30] However Swan and Westerholm (2003) estimate the mean and standard deviation of effective spreads to be respectively equal to 1.28% and 1.95% on the NYSE, 0.3 and 0.7 on the London Stock Exchange, and 0.6 and 1.2 on the Milan Stock Exchange. In a global European definition, the latter market clearly lists many small capitalization firms.

transaction may be exceeded. However, a moderately risk averse investor with horizons shorter than 1 year and annualized welfare gains larger than 11.5%, should still have an incentive to invest in small caps in light of the above estimates. We therefore guess that − even after taking transaction costs into account − the availability of small caps significantly increases expected utility through better risk diversification opportunities.

5. **Robustness Checks**

5.1. *Dynamic Rebalancing*

Section 4 focusses on the buy-and-hold case, $\varphi = T$. We now repeat calculations of portfolio weights from Section 4.2, including EUSC, for $\gamma = 5$ and a few alternative assumptions on the rebalancing frequency, $\varphi = 1$ (weekly rebalancing) 4, 16, 26, and 52 (annual rebalancing). Given the short average durations of regimes 1 and 3, the cases $\varphi = 1$ and 4 seem the most plausible ones, although un-modeled transaction costs and other frictions may suggest in practice using higher values of φ.

Table 12 reports optimal weights.[31] Rebalancing hardly changes the main implications found under simpler, buy-and-hold strategies, although it makes portfolio weights much more reactive to the initial state, and much less sensitive to the investment horizon. Dynamic strategies imply positive and high weights on EUSC only when the investor knows the state is the normal one. In this case the optimal weight is extreme, 100%, because EUSC have excellent Sharpe ratio. Since this is also fairly high in the bull state, a positive demand exists also in this case, even though the proportions are small and limited to very high rebalancing frequencies. The demand for EUSC in the steady-state case is instead limited, zero for short horizons and up to 20% for $T = 2$ years. Rebalancing possibilities fail to overturn our previous finding that − because of their poor skewness properties − small caps may in practice result much less attractive than what their high Sharpe ratios may lead us to conjecture.

5.2. *Long Horizons*

Some institutional investor have horizons much longer than the 2-year ceiling we have used. Although some caution should be used when extending the horizon beyond the length of our data set (four and half years), we also calculate (unreported) optimal portfolio schedules when the investment horizon is extended up to $T = 5$ years. For simplicity, we comment on results only for buy-and-hold portfolio directly comparable to Section 4.2.2, i.e. when the asset menu includes EUSC. We notice a phenomenon already highlighted by Guidolin and Timmermann (2004a) in other applications: even though short- to medium-term horizon weights may strongly depend on the regime, as T grows all optimal investment schedules tend to converge towards their steady-state counterparts. Indeed, the best long-run forecast an agent may form about the future state is that all regimes are possible with probabilities identical to their ergodic frequencies. More importantly for our application, we obtain evidence that even for very long horizons compatible with the objectives of institutional investors, the optimal weight assigned to EUSC appears limited relative to the (local) mean-variance case as a result of their high variance risk. Furthermore, even assuming a strong initial belief in the normal regime 2, for $T = 5$ years we have that the EUSC weight will be at most 55%,

[31] Results are also available for the restricted asset menu case $m = 3$ but are not reported to save space.

since over long periods markets are bound to transition out of the normal state and spend a fair share of time in both bull and bear states where North American large stocks dominate.

5.3. *Short Sales*

Although selling short equity indices appears to be more problematic than shorting individual stocks, the strategic asset allocation literature has developed a tradition of allowing for both negative positions and positions exceeding 100% of the initial wealth. We therefore perform afresh portfolio calculations for the case in which weights are allowed to vary between -400% and +400%.[32]

Figure 5 shows a sample of the resulting optimal weights. Removing the no-short sale constraint hardly changes our conclusion concerning the desirability of EUSC in international diversification: while a myopic investor who operates under a (false) IID framework would in fact invest in excess of 130% of her initial wealth in EUSC to exploit their high Sharpe ratio, in a regime switching framework the demand for EUSC depends on the initial state. It is still very high under the second, normal regime (in excess of 250%!), but in the most plausible case of unknown regime, the weight is only 20%, not very different from the results of Section 4.2.2. Risk aversion increases this proportion to almost 40%, but it remains true that the highest regime switching weights still keep involving all other assets as well with the usual exception of European large caps.[33]

Table 9 contains compensatory variation estimates. In particular the ergodic panel of the table highlights that admitting short sales enhances our estimate of the welfare gains from using small caps in international portfolio diversification, as most estimates (for both $\gamma = 5$ and 10) do increase. The worst-case estimate remains a long-run annualized riskless 3%, obtained assuming $\gamma = 10$. Therefore also in this experiment, small caps command only moderate portfolio weight but also imply rather large welfare improvements.

5.4. *Longer Series*

One final check concerns the length of the series employed: one might doubt of the generality of results obtained for the period 1999-2003, characterized by one single complete stock market cycle and by relatively modest excess stock returns. We collect weekly European total return indices published by Standard & Poors in conjunction with Citigroup (SPCG) for the period July 1989 - December 2004, a total of 808 data points.[34] In particular, we focus on SPCG returns (expressed in local currencies) for two alternative portfolios: a large cap European portfolio that covers only companies with capitalization exceeding 1.5

[32]As discussed by Barberis (2000) and Kandel and Stambaugh (1996), allowing short-sales creates problems when returns come from an unbounded density, because bankruptcy becomes possible and expected utility is not defined for non positive terminal wealth. As stressed in Guidolin and Timmermann (2005a), when Monte Carlo methods are used, this forces the researcher to truncate the distribution from which returns are simulated to avoid instances of bankruptcy. Thus returns are not simulated from the econometric models estimated in Section 4, but from a suitably truncated distribution in which the probability mass is redistributed to sum to one. We accomplish the truncation by applying rejection methods.

[33]Since differences between IID and regime switching weights widen when short sales are admitted, we generally find that in this case the welfare costs of ignoring regimes are much higher than those reported in Sections 4.1.2 and 4.2.2.

[34]All companies in applicable markets are included provided they have available (float) market capitalisation greater than 100 million US dollars. Only issues that a non-domiciled investor may purchase are included. Each issue is weighted by the proportion of its available equity capital.

billion US dollars; a small cap portfolio that covers companies with market capitalization below 500 million US dollars. These SPCG series are completed by MSCI return series for Pacific stocks extended to cover the period 1989-2004, as well as a North American large cap portfolio constructed following the same criteria detailed in Section 3.

Table 1 provides further summary statistics for the new as well as the extended series over a common 1989-2004 sample. Spanning multiple market cycles delivers annualized mean returns in the range of 8-9 %, although pervasive negative skewness still makes mean much smaller than median stock returns (in the range 13-16 % per year). The unsurprising exception is represented by Asian Pacific markets, which failed to make any progress in this 15-year period. Pairwise correlations are similar to those reported in Table 2, again with Pacific stocks only weakly correlated with other indices (with correlations in the 0.39 - 0.47 range). Median Sharpe ratios for EUSC remain highly attractive, for instance 1.07 vs. 1.06 and 1.03 for European and North Americam large caps, respectively. Taking into account their imperfect correlation with other portfolios (e.g. 0.5 vs. North American large caps), there is little doubt that a naive portfolio strategy disregarding variance risk ought to assign considerable weight to EUSC.

However, we have shown that the differential variance risk of EUSC may drive an important wedge between simple portfolio strategies and more sophisticated ones. In fact, Table 1 already shows that EUSC have lower negative skewness and higher excess kurtosis than other portfolios. We therefore proceed to estimate the same multivariate three-state, heteroskedastic model of Section 4.2. Estimation outputs are reported in Table 13. Most of the specific parameter estimates fall within a two-standard deviation interval from the corresponding estimates obtained in Table 4.[35] The only relevant difference is that now the bear and bull states are also relatively persistent, with average durations of 9 and 6 weeks, respectively. As a result, the steady state probabilities of regimes 1 and 3 are now higher, 0.24 and 0.23, respectively.

Next, we proceed to calculate optimal portfolio weights for an investor with $\gamma = 5$. Figure 6 reports the corresponding optimal portfolio weights at alternative investment horizons. Although detailed results are different from those obtained employing a different data set in Section 4.2, a few broad implications still hold. First, while the myopic weight is approximately two-thirds in EUSC, the ergodic regime switching weight is below 10%.[36] Second, also in this case there is only one specific regime in which the EUSC demand should be of first-order magnitude (always in excess of 40%); however, in this case the regime is the bull state, when the EUSC portfolio gives an astonishing annualized 74% expected return and a Sharpe ratio in excess of 7. Third, the demand for European large caps remains modest and limited to short horizons and one specific regime (the normal state, when they display an annualized 1.3 Sharpe ratio), while the North American weight is now important and grossly in excess of the corresponding myopic weight, with upward sloping schedules.

In conclusion, although the optimal portfolio policies obviously remain a function of the quality and structure of the data used to estimate the econometric model in (4), this exercise shows that our results

[35] The only interesting exceptions concern volatilities in the bull regime (now very similar to the estimates characterizing the normal state, in line with the usual finding that volatilites are higher in bear states) and pairwise correlations between European large and small caps (now always higher than 0.82).

[36] Again, we find poor co-skewness and co-kurtosis properties of EUSC. For instance, when measured against an equally weighted portfolio, $S_{EUSC,EUSC,EW\_ptf} = -0.73$, $S_{EUSC,EW\_ptf,EW\_ptf} = -0.66$,and $K_{EUSC,EUSC,EW\_ptf,EW\_pft} = 5.30$. These values are closely matched by the three-state regime switching model under ergodic state probabilities (we obtain -0.58, -0.55, and 4.34, all coefficients differ significantly from Gaussian benchmark values).

are robust when it comes to highlighting the role of variance risk at reducing the demand for small caps.

6. **Conclusion**

A powerful display of the effects of variance risk on portfolio choice is our result that the optimal portfolio share of European small caps under state-dependent returns - when the state of the stock market is unobservable - is always less than 20%, while their optimal weight in a myopic portfolio ought to be close to 90%.

Our modelling of the return generating process allows to precisely measure three important components of the variance risk of an asset class that adversely affect the skewness and the kurtosis of wealth, in addition to the own- asset negative skewness and excess kurtosis. These are the negative covariance between its returns and the volatility of other assets, the negative covariance between its volatility and returns of other assets, and the covariance between volatilities, that remind of he priced factor in the cross section of returns reported by Harvey and Siddique (2000) and Dittmar (2002).In this metric, European small caps have large variance risk.

Their large variance risk does not make them irrelevant for portfolio diversification: for instance, our estimates of the annualized welfare loss associated with excluding them from the asset menu often exceed 5% of initial wealth. Even if our paper has ignored transaction costs, it is difficult to think that − when trading on illiquid small caps − an investor might systematically face costs of trading exceeding 500 basis points or more.

These results stand when the asset menu is extended to include a North American small capitalization portfolio. In spite of the exceptional average premia and Sharpe ratio that NASC have yielded, we find that under realistic assumptions the combined weights of European and North American small caps fails to exceed 50% and remains at least 30% below what we would have obtained assuming a simple IID framework that ignores variance risk and higher-moment properties.

There are several natural extensions of our paper. First, our results support an emerging view in the asset pricing literature that the so-called size premium (see Fama and French, 1993) may be not an anomaly but instead just a rational premium associated with the illiquidity and the variance risk of small caps. As a matter of fact, we have found that the demand for small caps might be severely limited by their variance risk, thus potentially explaining low equilibrium prices and high returns. However, our model is not yet an equilibrium model, while extensions in this direction would be interesting. Acharya and Pedersen (2004) is a first example, although in a mean-variance set up. Second, we have concluded that small capitalization stocks are helpful in international diversification programs, as revealed by welfare losses caused by excluding them. Needless to say, small caps are known to be traded on illiquid and expensive markets. It would be interesting to introduce transaction costs in our asset allocation exercise and explicitly check the robustness of our results. Balduzzi and Lynch (1999) and Lynch and Balduzzi (2000) show how this could be accomplished in discrete time frameworks akin to ours.

Finally, our results have rich implications for the general issue of the limits and benefits of international equity portfolio diversification. For instance, since Tesar and Werner (1995) it has been observed that investors in many countries and particularly in the U.S. tend to grossly under-diversify their equity portfolios. Our paper has shown that regime shifts, especially as they affect the covariance matrices of returns, deeply impact the composition of optimal stock portfolios. North American large caps are observed to be the least volatile asset in bear markets. Following Vayanos (2004), they can easily be construed as the quality asset to which investors should flee in market downturns. Indeed, their portfolio share grows from zero in the normal state to 30% in bear markets. However, flight to quality is not complete in our setting. Other equity portfolios remain in high demand: Pacific stocks allow to dampen portfolio volatility changes since they have low correlation with both North American large stocks in bear states and with both NASC and EUSC in bull states. Thus, the desire to hedge both potential losses and potential increases in portfolio variance preserves the diversification of international portfolios, contrary to results in Ang and Bekaert (2002) where the optimal portfolio may be entirely composed of US stocks.

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Appendix A – Solution Methods

A variety of solution methods have been applied in the literature on portfolio allocation under time-varying investment opportunities. Barberis (2000) employs simulation methods and studies a pure allocation problem without interim consumption. Ang and Bekaert (2001) solve for the optimal asset allocation using quadrature methods. Campbell and Viceira (1999, 2001) derive approximate analytical solutions for an infinitely lived investor when interim consumption is allowed and rebalancing is continuous. Campbell et al. (2003) extend this approach to a multivariate set-up and show that a mixture of approximations and numerical methods can deliver powerful results. Finally, some papers have derived closed-form solutions by working in continuous-time, e.g. Brennan et al. (1997) and Kim and Omberg (1996) for the case without interim consumption.

In our paper we make two choices that simplify the computational task with respect to competing approaches. First, solving (2) by standard backward induction techniques is, unfortunately, a formidable task (see e.g. the discussion in Barberis, 2000, pp. 256-260). Under standard discretization techniques the investor first needs to use a sufficiently dense grid of size $G, \{\boldsymbol{\theta}_b^j,\ \boldsymbol{\pi}_b^j\}_{j=1}^G$ to update both $\boldsymbol{\theta}_{b+1}$ and $\boldsymbol{\pi}_{b+1}$ from $\boldsymbol{\theta}_b$ and $\boldsymbol{\pi}_b$. In the presence of a high number of parameters implied by (4), standard numerical techniques are not feasible for this problem or would at best force us to use a very rough discretization grid, introducing large approximation errors. Therefore our approach simply assumes that investors condition on their current (as opposed to future ones, $\boldsymbol{\theta}_{b+1}$) parameter estimates, $\hat{\boldsymbol{\theta}}_t$. Under this assumption, since W_b is known at time t_b, $Q(.)$ simplifies to:

$$Q(\mathbf{r}_b, \boldsymbol{\pi}_b, t_b) = \max_{\boldsymbol{\omega}_b} E_b\left[\left(\frac{W_{b+1}}{W_b}\right)^{1-\gamma} Q\left(\mathbf{r}_{b+1}, \boldsymbol{\pi}_{b+1}, t_{b+1}\right)\right].$$

Second, we resort to simulation methods similarly to Barberis (2000) and Detemple, Garcia, and Rindisbacher (2003). Ang and Bekaert (2002) were the first to study this problem under regime switching. They consider pairs of international stock market portfolios under regime switching with observable states, so the state variable simplifies to a set of dummy indicators. This setup allows them to apply quadrature methods based on a discretization grid (see also Guidolin and Timmermann, 2004a). Our framework is quite different since we treat the state as unobservable and calculate asset allocations under optimal filtering (6).

To deal with the latent state we use Monte-Carlo methods for expected utility approximation. In the case in which dynamic rebalancing is admitted ($B \geq 2$), suppose that the optimization problem has been solved backwards at the rebalancing points t_{B-1}, ..., t_{b+1} so that $Q(\boldsymbol{\pi}_{b+1}^j, t_{b+1})$ is known for all values $j = 1, 2, ..., G$ on the discretization grid. For each $\boldsymbol{\pi}_b = \boldsymbol{\pi}_b^j$, it is then possible to find $Q(\boldsymbol{\pi}_b^j, t_b)$ at time t_b.For concreteness, consider the case of $p = 0$, i.e. the conditional mean function does not imply any autoregressive structure. Approximating the expectation in the objective function

$$E_{t_b}\left[\left\{\boldsymbol{\omega}_b' \exp\left(\mathbf{R}_{b+1}\right)\right\}^{1-\gamma} Q(\boldsymbol{\pi}_{b+1}^j, t_{b+1})\right]$$

by Monte Carlo methods requires drawing N samples of asset returns $\{\mathbf{R}_{b+1,n}(\boldsymbol{\pi}_b^j)\}_{n=1}^N$ from the $(b+1)\varphi$-step-ahead joint density of asset returns conditional on $\widehat{\boldsymbol{\theta}}_t$, assuming that $\boldsymbol{\pi}_b^j$ is optimally updated.

The algorithm consists of the following steps:

1. For a given $\boldsymbol{\pi}_b^j$ calculate the $(b+1)\varphi$-step ahead probability of being in each of the possible future regime $s_{b+1} = j$ as $\boldsymbol{\pi}_{b+1|b} = (\boldsymbol{\pi}_b^j)'\hat{\mathbf{P}}_t^{\varphi}$, using that $\hat{\mathbf{P}}_t^{\varphi} \equiv \prod_{j=1}^{\varphi} \hat{\mathbf{P}}_t$ is the φ-step ahead transition matrix.

2. For each possible future regime, simulate N $\varphi-$period returns $\{\mathbf{R}_{b+1,s}(s_b)\}_{n=1}^N$ in calendar time from the regime switching model:
$$\mathbf{r}_{t_b+i,n}(s_b) = \hat{\boldsymbol{\mu}}_{s_{t_b+i}} + \boldsymbol{\varepsilon}_{t_b+i,n}.$$
At all rebalancing points this simulation allows for stochastic regime switching as governed by the transition matrix $\hat{\mathbf{P}}_t$. For example, if we start in regime 1, between t_b+1 and t_b+2 there is a probability $\hat{p}_{12} \equiv \mathbf{e}_1'\hat{\mathbf{P}}_t\mathbf{e}_2$ of switching to regime 2, and a probability $\hat{p}_{11} \equiv \mathbf{e}_1'\hat{\mathbf{P}}_t\mathbf{e}_1$ of staying in regime 1.

3. Combine the simulated $\varphi-$period asset returns $\{\mathbf{R}_{b+1,n}\}_{n=1}^N$ into a random sample of size N, using the probability weights contained in the vector $\boldsymbol{\pi}_b^j$:
$$\mathbf{R}_{b+1,n}(\boldsymbol{\pi}_b^j) = \sum_{i=1}^{k} (\boldsymbol{\pi}_b^j)'\mathbf{e}_i\mathbf{R}_{b+1,n}(s_b = i).$$

4. Update the future regime probabilities perceived by the investor using the formula:
$$\boldsymbol{\pi}_{b+1,n}(\boldsymbol{\pi}_b^j) = \frac{\left(\boldsymbol{\pi}_b'(\hat{\boldsymbol{\theta}}_b)\hat{\mathbf{P}}_b^{\varphi}\right)' \odot \boldsymbol{\eta}(\mathbf{r}_{b+1};\hat{\boldsymbol{\theta}}_b)}{[(\boldsymbol{\pi}_b'(\hat{\boldsymbol{\theta}}_b)\hat{\mathbf{P}}_b^{\varphi})' \odot \boldsymbol{\eta}(\mathbf{r}_{b+1};\hat{\boldsymbol{\theta}}_b))]'\boldsymbol{\iota}_k}$$
obtaining an $N \times 4$ matrix $\{\boldsymbol{\pi}_{b+1,n}(\boldsymbol{\pi}_b^j)\}_{n=1}^N$, each row of which corresponds to a simulated row vector of perceived regime probabilities at time t_{b+1}.

5. For all $n = 1, 2, ..., N$, calculate the value $\tilde{\boldsymbol{\pi}}_{b+1,n}^j$ on the discretization grid ($j = 1, 2, ..., G$) that is closest to $\boldsymbol{\pi}_{b+1,n}(\boldsymbol{\pi}_b^j)$ according to the metric $\sum_{i=1}^3 |(\boldsymbol{\pi}_{b+1}^j)'\mathbf{e}_i - \boldsymbol{\pi}_{b+1,n}'\mathbf{e}_i|$, i.e.
$$\tilde{\boldsymbol{\pi}}_{b+1,n}^j(\boldsymbol{\pi}_b^j) \equiv \arg\min_{\mathbf{x}\in\boldsymbol{\pi}_{b+1}^j} \sum_{i=1}^{3} |\mathbf{x}'\mathbf{e}_i - \boldsymbol{\pi}_{b+1,n}'\mathbf{e}_i|.$$
Knowledge of the vector $\{\tilde{\boldsymbol{\pi}}_{b+1,n}^j(\boldsymbol{\pi}_b^j)\}_{n=1}^N$ allows us to build $\{Q(\boldsymbol{\pi}_{b+1}^{(j,n)}, t_{b+1})\}_{n=1}^N$, where $\boldsymbol{\pi}_{b+1}^{(j,n)} \equiv \tilde{\boldsymbol{\pi}}_{b+1,n}^j(\boldsymbol{\pi}_b^j)$ is a function of the assumed vector of regime probabilities $\boldsymbol{\pi}_b^j$.

6. Solve the program
$$\max_{\boldsymbol{\omega}_b(\boldsymbol{\pi}_b^j)} N^{-1} \sum_{n=1}^{N} \left[\left\{ \boldsymbol{\omega}_b' \exp\left(\mathbf{R}_{b+1,n}(\boldsymbol{\pi}_b^j)\right) \right\}^{1-\gamma} Q(\boldsymbol{\pi}_{b+1}^{(j,n)}, t_{b+1}) \right],$$
which for large values of N provides an arbitrarily precise Monte-Carlo approximation of the expectation $E\left[\left\{ \boldsymbol{\omega}_b' \exp\left(\mathbf{R}_{b+1,n}(\boldsymbol{\pi}_b^j)\right) \right\}^{1-\gamma} Q(\boldsymbol{\pi}_{b+1}^j, t_{b+1}) \right]$.The value function corresponding to the optimal portfolio weights $\hat{\boldsymbol{\omega}}_b(\boldsymbol{\pi}_b^j)$ defines $Q(\boldsymbol{\pi}_b^j, t_b)$ for the jth point on the initial grid.

This algorithm is applied to all possible values $\boldsymbol{\pi}_b^j$ on the discretization grid until all values of $Q(\boldsymbol{\pi}_b^j, t_b)$ are obtained for $j = 1, 2, ..., G$. It is then iterated backwards until $t_{b+1} = t + \varphi$. At that stage the algorithm is applied one last time, taking $Q(\boldsymbol{\pi}_{t+\varphi}^j, t+\varphi)$ as given and using one row vector of perceived regime probabilities $\boldsymbol{\pi}_t$, the vector of smoothed probabilities estimated at time t. The resulting vector of optimal portfolio weights $\hat{\boldsymbol{\omega}}_t$ is the desired optimal portfolio allocation at time t, while $Q(\boldsymbol{\pi}_t, t)$ is the optimal value function.

Appendix B – Selection of the Return Generating Processes

We estimate a variety of multivariate regime switching models, including the special cases of no regimes, and/or no VAR, and/or homoskedasticity. Clearly, both $k = 1$ and $p = 0$ result in a multivariate Gaussian return distribution that implies the absence of predictability. Otherwise, our model search allows for $k = 1, 2, 3$, and 4, for $p = 0, 1, 2$, and entertains both homoskedastic and heteroskedastic models.

In Table B1, which refers to an asset menu without small caps, three different statistics are reported for specification purposes. The fourth column shows the likelihood ratio (LR) statistic for the test of $k = 1$, when the model reduces to a homoskedastic Gaussian VAR(p). Similarly to Guidolin and Timmermann (2005a,b) we report corrected, Davis (1977)-type upper bound for the associated p-values that correct for nuisance parameter problems. The high LR statistics (and the associated small p-values, generally equal to 0.000) show that most regime switching models ($k \geq 2$) perform better than simpler linear models in capturing the salient features of the joint density of the stock returns data. We conclude that the absence of regime switching in international stock returns data is rejected, similar to the findings in Ang and Bekaert (2002) and Ramchand and Susmel (1998).
The fifth and sixth columns of Table B1 present two information criteria, the Bayesian (BIC) and Hannan-Quinn (H-Q) statistics. Their purpose is allow the calculation of synthetic measures trading-off in-sample fit against parsimony and hence out-of-sample forecasting accuracy. By construction, the best performing model ought to minimize such criteria. Importantly, in this case we obtain that the same model minimizes both the BIC and the H-Q criteria. This is achieved by a relatively simple and parsimonious (20 parameters vs. a total of 702 observations) model with $k = 2$, $p = 0$, and regime-dependent covariance matrix.

Table B2 repeats our specification search with reference to a model with four equity portfolios, including European large and small stocks, North American large, and Pacific. Also in this case, the evidence against the null of a linear, IID Gaussian model is overwhelming in terms of likelihood ratio tests. The information criteria provide contrasting indications: while the H-Q sides for a rather 'expensive' (in terms of number of parameters, 52) two-regime model with a VAR(1) structure, the BIC is 'undecided' between a homoskedastic three-regime model and a heteroskedastic one (in both cases $p = 0$). Given the pervasive evidence of volatility clustering in Table 1 (see the Ljung-Box statistic for squared returns) − which is unsurprising in weekly data − we select the latter three-state model. The MSIH(3,0) model implies the estimation of 48 parameters, although with 936 observations this still amount to a reasonable saturation ratio of $936/48 = 19.5$, i.e. roughly 20 observations per parameter.

We perform once more our model selection search when we also allow for North American small caps. An unreported Table similar to Tables B1 and B2 shows that both the BIC and H-Q criteria keep selecting a three-state heteroskedastic regime switching model with $p = 0$ (MSIH(3,0)), i.e. in which regime switching is responsible of most of the autoregressive structure in levels noticed in Table 1. Such a model implies estimation of as many as 66 parameters, although with 1,170 observations this still gives an acceptable ratio of 18 observations per estimated parameter.

Appendix C – Moment Implications of Model Restrictions

Guidolin and Timmermann (2004b) show that when the VAR order (p) is zero, then the third and fourth central moments of portfolio returns are respectively equal to

$$E_t[(\boldsymbol{\omega}_t'\mathbf{r}_{t+T} - E_t[\boldsymbol{\omega}_t'\mathbf{r}_{t+T}])^3] = \sum_{j=1}^{k} \left(\boldsymbol{\pi}_t'\mathbf{P}^T\mathbf{e}_j\right) \left\{ \left[\boldsymbol{\mu}_j'\boldsymbol{\omega}_t - \boldsymbol{\pi}_t'\mathbf{P}^T\mathbf{M}\boldsymbol{\omega}_t\right]^3 + 3\left[\boldsymbol{\mu}_j'\boldsymbol{\omega}_t - \boldsymbol{\pi}_t'\mathbf{P}^T\mathbf{M}\boldsymbol{\omega}_t\right] \left(\boldsymbol{\omega}_t'\boldsymbol{\Sigma}_j\boldsymbol{\omega}_t\right)^2 \right\}, \tag{10}$$

and

$$E_t[(\boldsymbol{\omega}_t'\mathbf{r}_{t+T} - E_t[\boldsymbol{\omega}_t'\mathbf{r}_{t+T}])^4] = \sum_{j=1}^{k} \left(\boldsymbol{\pi}_t'\mathbf{P}^T\mathbf{e}_j\right) \Big\{ \left[\boldsymbol{\mu}_j'\boldsymbol{\omega}_t - \boldsymbol{\pi}_t'\mathbf{P}^T\mathbf{M}\boldsymbol{\omega}_t\right]^4 + 6\left[\boldsymbol{\mu}_j'\boldsymbol{\omega}_t - \boldsymbol{\pi}_t'\mathbf{P}^T\mathbf{M}\boldsymbol{\omega}_t\right]^2 \times \times \left(\boldsymbol{\omega}_t'\boldsymbol{\Sigma}_j\boldsymbol{\omega}_t\right)^2 + 3\left(\boldsymbol{\omega}_t'\boldsymbol{\Sigma}_j\boldsymbol{\omega}_t\right)^4 \Big\}, \quad (11)$$

where $\mathbf{M}$ is defined as a $k \times m$ matrix stacking in each of its rows the $1 \times m$ vectors $\boldsymbol{\mu}_1$, $\boldsymbol{\mu}_2$, ..., $\boldsymbol{\mu}_k$, ω_t is a weight vector, $\mathbf{e}_j$ is a $k \times 1$ vector with a 1 in its j-th position and zero everywhere else, so that $\boldsymbol{\pi}_t'\mathbf{P}^T\mathbf{e}_j$ is the time t predicted probability of regime $j = 1, ..., k$.

The third central moment of the T-step ahead portfolio returns,(10, is a predicted probability-weighted average of the sum of two type of terms over the k possible regimes: $\left[\boldsymbol{\mu}_j'\boldsymbol{\omega}_t - \boldsymbol{\pi}_t'\mathbf{P}^T\mathbf{M}\boldsymbol{\omega}_t\right]^3$, where $\boldsymbol{\pi}_t'\mathbf{P}^T\mathbf{M}\boldsymbol{\omega}_t$ is simply the T-step predicted portfolio return, i.e. cubic powers of the difference between the regime-specific expected return $\boldsymbol{\mu}_j'\boldsymbol{\omega}_t$ and the T-step forecast across regimes; $3\left[\boldsymbol{\mu}_j'\boldsymbol{\omega}_t - \boldsymbol{\pi}_t'\mathbf{P}^T\mathbf{M}\boldsymbol{\omega}_t\right]\left(\boldsymbol{\omega}_t'\boldsymbol{\Sigma}_j\boldsymbol{\omega}_t\right)^2$, an interaction term between the squared portfolio return regime-specific variance $(\boldsymbol{\omega}_t'\boldsymbol{\Sigma}_j\boldsymbol{\omega}_t)$ and once more the difference between the regime-specific expected return and the T-step forecast across regimes. Interestingly, when $\boldsymbol{\mu}_j' = \boldsymbol{\mu}$ for $j = 1, ..., k$ like in (8), $\boldsymbol{\mu}_j'\boldsymbol{\omega}_t = \boldsymbol{\pi}_t'\mathbf{P}^T\mathbf{M}\boldsymbol{\omega}_t$ by construction so that $E_t[(\boldsymbol{\omega}_t'\mathbf{r}_{t+T} - E_t[\boldsymbol{\omega}_t'\mathbf{r}_{t+T}])^3] = \mathbf{0}\ \forall t$. Since terms of the type $E_t[(\boldsymbol{\omega}_t'\mathbf{r}_{t+T} - E_t[\boldsymbol{\omega}_t'\mathbf{r}_{t+T}])^3]$ are at the numerator of any skewness coefficient, this shows that regime-independent conditional mean implies zero skewness and co-skewness for all the assets.

The fourth central moment, (11, involves terms with similar structure, but with different multiplicative coefficients and raised to even powers. Interestingly, a regime-independent mean, $\boldsymbol{\mu}_j'\boldsymbol{\omega}_t = \boldsymbol{\pi}_t'\mathbf{P}^T\mathbf{M}\boldsymbol{\omega}_t$, fails now to imply $E_t[(\boldsymbol{\omega}_t'\mathbf{r}_{t+T} - E_t[\boldsymbol{\omega}_t'\mathbf{r}_{t+T}])^4] = \mathbf{0}\ \forall t$. Since terms like $E_t[(\boldsymbol{\omega}_t'\mathbf{r}_{t+T} - E_t[\boldsymbol{\omega}_t'\mathbf{r}_{t+T}])^4]$ will appear at the numerator of any kurtosis coefficient, (8) may imply (time-varying) kurtosis and co-kurtosis in excess of a Gaussian benchmark.

Thus, both third and fourth central moments will differ from their Gaussian IID counterparts under regime-independent conditional variance (9). This means that the impact of odd-moment (skewness) variance risk may be measured only by taking the difference of the two portfolio weight vectors in Table 8. However, under (9) $E_t^{IID}[(\boldsymbol{\omega}_t'\mathbf{r}_{t+T} - E_t[\boldsymbol{\omega}_t'\mathbf{r}_{t+T}])^n] \equiv \sum_{j=1}^{k}\left(\boldsymbol{\pi}_t'\mathbf{P}^T\mathbf{e}_j\right)\left(\boldsymbol{\omega}_t'\boldsymbol{\Sigma}\boldsymbol{\omega}_t\right)^n = \left(\boldsymbol{\omega}_t'\boldsymbol{\Sigma}\boldsymbol{\omega}_t\right)^n$, so that departures in the third central moment from a Gaussian IID benchmark of an order equal to the third power of portfolio returns:

$$E_t[(\boldsymbol{\omega}_t'\mathbf{r}_{t+T}\text{-}E_t[\boldsymbol{\omega}_t'\mathbf{r}_{t+T}])^3] = \sum_{j=1}^{k}\left(\boldsymbol{\pi}_t'\mathbf{P}^T\mathbf{e}_j\right)\left[\boldsymbol{\mu}_j'\boldsymbol{\omega}_t\text{-}\boldsymbol{\pi}_t'\mathbf{P}^T\mathbf{M}\boldsymbol{\omega}_t\right]^3 + 3Var_t^{IID}\sum_{j=1}^{k}\left(\boldsymbol{\pi}_t'\mathbf{P}^T\mathbf{e}_j\right)\left[\boldsymbol{\mu}_j'\boldsymbol{\omega}_t\text{-}\boldsymbol{\pi}_t'\mathbf{P}^T\mathbf{M}\boldsymbol{\omega}_t\right],$$

On the contrary, deviations in the fourth central moment will be at most of order four:

$$E_t[(\boldsymbol{\omega}_t'\mathbf{r}_{t+T} - E_t[\boldsymbol{\omega}_t'\mathbf{r}_{t+T}])^4] = 3 + \sum_{j=1}^{k}\left(\boldsymbol{\pi}_t'\mathbf{P}^T\mathbf{e}_j\right)\left\{\left[\boldsymbol{\mu}_j'\omega_t\text{-}\boldsymbol{\pi}_t'\mathbf{P}^T\mathbf{M}\boldsymbol{\omega}_t\right]^4 + 6\left[\boldsymbol{\mu}_j'\omega_t\text{-}\boldsymbol{\pi}_t'\mathbf{P}^T\mathbf{M}\boldsymbol{\omega}_t\right]^2 Var_t^{IID}\right\},$$

(we define $Var_t^{IID} \equiv E_t^{IID}[(\boldsymbol{\omega}_t'\mathbf{r}_{t+T} - E_t[\boldsymbol{\omega}_t'\mathbf{r}_{t+T}])^2]$). Therefore (9) represents a good device to investigate the portfolio effects of adding co-skewness driven variance risk.

Table 1

Summary Statistics for International Stock Returns

The table reports basic moments for weekly equity total return series (including dividends, adjusted for stock splits, etc.) for a few international portfolios and two sample periods. All returns are expressed in local currencies. Means, medians, and standard deviations are annualized by multiplying weekly moments by 52 and $\sqrt{52}$, respectively. LB(j) denotes the j-th order Ljung-Box statistic.

Portfolio	Mean	Median	St. Dev.	Skewness	Kurtosis	LB(4)	LB(4)-squares
	January 1999 – June 2003						
MSCI Europe – Large Caps	-0.079	-0.081	0.267	0.186	4.975	20.031**	32.329**
MSCI Europe – Small Caps	0.012	0.144	0.161	-0.778	4.815	16.202**	29.975**
North America – Large Caps	-0.012	-0.114	0.206	0.277	3.673	6.981	12.396*
MSCI North America – Small Caps	0.101	0.128	0.218	-0.181	3.384	15.849**	11.374*
MSCI Asia Pacific	-0.035	0.006	0.187	-0.086	3.395	3.138	2.667
	July 1989 – December 2004						
SPCG Europe – Large Caps	0.092	0.160	0.151	-0.338	5.550	14.581**	175.8**
SPCG Europe – Small Caps	0.080	0.136	0.127	-0.824	6.740	144.6**	206.5**
North America – Large Caps	0.084	0.158	0.153	-0.460	5.921	13.821**	63.285**
MSCI Asia Pacific	-0.034	0.002	0.173	-0.119	3.950	2.114	108.5**

* denotes 5% significance, ** significance at 1%.

Table 2

Correlation Matrix of International Stock Returns

The table reports linear correlation coefficients for weekly equity total return series (including dividends, adjusted for stock splits, etc.) for a few international portfolios. The sample period is January 1999 – June 2003. All returns are expressed in local currencies.

	EU – Large	EU – Small	North America	North Am. – Large	North Am. – Small	Pacific
EU – Large Caps	1	0.782	0.747	0.754	0.695	0.509
EU – Small Caps		1	0.668	0.672	0.727	0.540
North America			1	0.997	0.795	0.484
North Am. – Large Caps				1	0.795	0.484
North Am. – Small Caps					1	0.427
Pacific						1

Table 3

Estimates of a Two-State Regime Switching Model for Large European, North American Large Caps, and Pacific Equity Portfolios

The table shows estimation results for the regime switching model:

$$r_t = \mu_{s_t} + \varepsilon_t$$

where $\mathbf{r}_t$ is a 3×1 vector collecting weekly total return series, $\boldsymbol{\mu}_{s_t}$ is the intercept vector in state s_t, and $\boldsymbol{\varepsilon}_t = [\varepsilon_{1t}\ \varepsilon_{2t}\ \varepsilon_{3t}]' \sim N(\mathbf{0}, \Sigma_{s_t})$. The sample period is January 1999 – June 2003. The unobservable state s_t is governed by a first-order Markov chain that can assume two values. The first panel refers to the single-state case *k* = 1. Asterisks attached to correlation coefficients refer to covariance estimates. For mean coefficients and transition probabilities, standard errors are reported in parenthesis.

	Panel A – Single State Model		
	Europe – Large caps	North America Large	Pacific
1. Mean return	-0.0015	-0.0008	-0.0007
2. Correlations/Volatilities			
Europe – Large caps	0.0370***		
North America - Large caps	0.7470***	0.0285***	
Pacific	0.5086***	0.4843***	0.0259***
	Panel B – Two State Model		
	Europe – Large caps	North America Large	Pacific
1. Mean return			
Normal State	-0.0002	-0.0003	0.0010
Bear State	-0.0046	-0.0020	-0.0048
2. Correlations/Volatilities			
Normal state:			
Europe – Large caps	0.0253***		
North America - Large caps	0.7318***	0.0231***	
Pacific	0.5845***	0.6077***	0.0227***
Bear state:			
Europe – Large caps	0.0559***		
North America - Large caps	0.7681***	0.0387***	
Pacific	0.4675**	0.3607*	0.0321***
3. Transition probabilities	Normal State	Bear State	
Normal State	0.9605***	0.0395	
Bear State	0.1084**	0.8916	

* denotes 10% significance, ** significance at 5%, *** significance at 1%.

Table 4

Estimates of a Three-State Regime Switching Model for European, North American, and Pacific Equity Portfolios – Effects of Adding European Small Caps

The table shows estimation results for the regime switching model:

$$r_t = \boldsymbol{\mu}_{s_t} + \boldsymbol{\varepsilon}_t$$

where $\mathbf{r}_t$ is a 4×1 vector collecting weekly total return series, $\boldsymbol{\mu}_{s_t}$ is the intercept vector in state s_t, and $\boldsymbol{\varepsilon}_t = [\varepsilon_{1t}\ \varepsilon_{2t}\ \varepsilon_{3t}\varepsilon_{4t}]' \sim N(\mathbf{0}, \Sigma_{s_t})$. The unobservable state s_t is governed by a first-order Markov chain that can assume three values. The first panel refers to the single-state case k = 1. Asterisks attached to correlation coefficients refer to covariance estimates. For mean coefficients and transition probabilities, standard errors are reported in parenthesis.

	Panel A – Single State Model			
	Europe – Large caps	North America Large	Pacific	Europe – Small caps
1. Mean return	-0.0015	-0.0008	-0.0007	0.0002
2. Correlations/Volatilities				
Europe – Large caps	0.0370***			
North America - Large caps	0.7470***	0.0285***		
Pacific	0.5086***	0.4843***	0.0259***	
Europe – Small caps	0.7816***	0.6680***	0.5403***	0.0222***
	Panel B – Three State Model			
	Europe – Large caps	North America Large	Pacific	Europe – Small caps
1. Mean return				
Bear State	-0.0501***	-0.0268***	-0.0256***	-0.0288***
Normal State	-0.0005	-0.0006	0.0007	0.0032**
Bull State	0.0374***	0.0214***	0.0157***	0.0136***
2. Correlations/Volatilities				
Bear state:				
Europe – Large caps	0.0300***			
North America - Large caps	0.6181***	0.0247***		
Pacific	0.1000	0.0544	0.0277***	
Europe – Small caps	0.7028***	0.5843***	0.5045**	0.0290***
Normal state:				
Europe – Large caps	0.0246***			
North America - Large caps	0.7182***	0.0226***		
Pacific	0.5694***	0.6022***	0.0219***	
Europe – Small caps	0.7062***	0.6369***	0.5759***	0.0153***
Bull state:				
Europe – Large caps	0.0370***			
North America - Large caps	0.5739***	0.0343***		
Pacific	-0.1242	-0.0515	0.0241***	
Europe – Small caps	0.7114***	0.5137***	-0.3581**	0.0177***

3. Transition probabilities	Bear State	Normal State	Bull State
Bear State	0.2190*	0.0012	0.7798
Normal State	0.0349	0.9650***	0.0001
Bull State	0.5416***	0.1698**	0.2886

* denotes 10% significance, ** significance at 5%, *** significance at 1%.

Table 5

Sample and Implied Co-Skewness Coefficients

The table reports the sample co-skewness coefficients,

$$S_{i,j,l} \equiv \frac{E[(r_i - E[r_i])(r_j - E[r_j])(r_l - E[r_l])]}{\{E[(r_i - E[r_i])^2]E[(r_j - E[r_j])^2]E[(r_l - E[r_l])^2]\}^{1/2}}$$

(*i, j, l* = Europe large, North America large, Pacific, Europe small) and compares them with the co-skewness coefficients implied by a three-state regime switching model:

$$\mathbf{r}_t = \boldsymbol{\mu}_{s_t} + \Sigma_{s_t}\boldsymbol{\varepsilon}_t .$$

$\boldsymbol{\varepsilon}_t \sim I.I.D.\ N(\mathbf{0}, \mathrm{I}_4)$ is an unpredictable return innovation. Coefficients under regime switching are calculated employing simulations (50,000 trials) and averaging across simulated samples of length equal to the available data (January 1999 – June 2003). In the table NA stands for 'North American small caps', and Pac for 'Pacific' portfolios. Bold coefficients are significantly different from zero.

Coeff.	Sample	MS – ergodic	Regime 1	Regime 2	Regime 3
$S_{EU\_large,EU\_large,NA}$	0.110	0.025	0.003	-0.006	0.051
$S_{EU\_large,EU\_large,Pac}$	-0.126	-0.131	**-0.155**	-0.016	-0.058
$S_{EU\_large,EU\_large,EU\_small}$	-0.167	**-0.228**	-0.101	-0.035	-0.047
$S_{NA,NA,Pac}$	0.005	-0.007	-0.021	0.006	0.027
S_{NA,NA,EU_small}	-0.111	-0.070	-0.014	-0.011	0.016
S_{NA,NA,EU_large}	0.149	0.095	0.079	0.004	0.105
S_{Pac,Pac,EU_small}	**-0.493**	**-0.341**	**-0.333**	-0.048	**-0.255**
S_{Pac,Pac,EU_large}	-0.203	-0.151	-0.174	-0.023	-0.103
$S_{Pac,Pac,NA}$	-0.140	-0.086	-0.128	-0.010	-0.071
$S_{EU\_small,EU\_small,EU\_large}$	**-0.467**	**-0.460**	**-0.240**	-0.063	**-0.187**
$S_{EU\_small,EU\_small,NA}$	**-0.367**	**-0.323**	**-0.195**	-0.046	**-0.152**
$S_{EU\_small,EU\_small,Pac}$	**-0.525**	**-0.487**	**-0.431**	-0.067	**-0.342**
$S_{EU\_large, EU\_large, EU\_large}$	0.186	0.110	0.023	-0.012	0.075
$S_{NA,NA,NA}$	**0.237**	0.170	0.140	0.012	0.147
$S_{Pac,Pac,Pac}$	-0.086	-0.169	-0.109	-0.022	-0.079
$S_{EU\_small, EU\_small, EU\_small}$	**-0.711**	**-0.722**	**-0.332**	-0.081	**-0.290**

Table 6

Sample and Implied Co-Skewness and C-Kurtosis Coefficients of European Small Caps vs. an Equally Weighted International Equity Portfolio

The table reports average sample co-skewness coefficients,

$$S_{i,j,l} \equiv \frac{E[(r_i - E[r_i])(r_j - E[r_j])(r_l - E[r_l])]}{\{E[(r_i - E[r_i])^2]E[(r_j - E[r_j])^2]E[(r_l - E[r_l])^2]\}^{1/2}}$$

$$K_{i,j,l,b} \equiv \frac{E[(r_i - E[r_i])(r_j - E[r_j])(r_l - E[r_l])(r_b - E[r_b])]}{\{E[(r_i - E[r_i])^2]E[(r_j - E[r_j])^2]E[(r_l - E[r_l])^2]E[(r_b - E[r_b])^2]\}^{1/2}}$$

(*i, j, l* = Europe large, North America large, Pacific, Europe small, Equally weighted portfolio) and compares them with the co-kurtosis coefficients implied by a three-state regime switching model. Coefficients under multivariate regime switching are calculated employing simulations. Bold co-skewness coefficients are significantly different from zero; bold co-kurtosis coefficients are significantly different from their Gaussian counterparts.

	Co-Skewness		Co-Kurtosis	
	Sample	**MS - ergodic**	**Sample**	**MS - ergodic**
	European Small Caps			
$S_{EU\_small,EU\_small,EW\_ptf}$	**-0.604**	**-0.566**	–	–
$S_{EU\_small,EW\_ptf,EW\_ptf}$	**-0.440**	**-0.412**	–	–
$S_{EU\_small,EU\_small,Pac,EW\_ptf}$	–	–	**2.094**	**2.133**
$S_{EU\_small,EU\_small,NA,EW\_ptf}$	–	–	**2.623**	**2.460**
$S_{EU\_small,EU\_small,EU\_large,EW\_ptf}$	–	–	**3.220**	**2.927**
$S_{EW\_ptf,EW\_ptf,EU\_small,Pac}$	–	–	1.945	**2.133**
$S_{EW\_ptf,EW\_ptf,EU\_small,NA}$	–	–	**2.680**	**2.428**
$S_{EW\_ptf,EW\_ptf,EU\_small,EU\_large}$	–	–	**3.168**	**2.790**
$S_{EW\_ptf,EW\_ptf,EU\_small,EU\_small}$	–	–	**3.460**	**3.262**
$S_{EW\_ptf,EW\_ptf,EU\_ptf,EU\_small}$	–	–	**3.903**	**3.713**
$S_{EU\_small,EU\_small,EU\_small,EU\_ptf}$	–	–	**3.315**	**3.071**
	European Large Caps			
$S_{EU\_large,EU\_large,EW\_ptf}$	0.031	-0.074	–	–
$S_{EU\_large,EW\_ptf,EW\_ptf}$	-0.097	-0.154	–	–
$S_{EU\_large,EU\_large,NA,EW\_ptf}$	–	–	**3.128**	**2.483**
$S_{EU\_large,EU\_large,Pac,EW\_ptf}$	–	–	1.465	1.616
$S_{EU\_large,EU\_large,EU\_small,EW\_ptf}$	–	–	**3.320**	**2.730**
$S_{EW\_ptf,EW\_ptf,EU\_large,Pac}$	–	–	1.691	1.841
$S_{EW\_ptf,EW\_ptf,EU\_large,NA}$	–	–	**2.997**	**2.521**
$S_{EW\_ptf,EW\_ptf,EU\_large,EU\_small}$	–	–	**3.168**	**2.790**
$S_{EW\_ptf,EW\_ptf,EU\_large,EU\_large}$	–	–	**3.650**	**3.005**
$S_{EW\_ptf,EW\_ptf,EU\_ptf,EU\_large}$	–	–	**3.458**	**3.021**
$S_{EU\_large,EU\_large,EU\_large,EU\_ptf}$	–	–	**4.119**	**3.190**

Table 7

Sample and Implied Co-Kurtosis Coefficients

The table reports the sample co-kurtosis coefficients,

$$K_{i,j,l,b} \equiv \frac{E[(r_i - E[r_i])(r_j - E[r_j])(r_l - E[r_l])(r_b - E[r_b])]}{\{E[(r_i - E[r_i])^2]E[(r_j - E[r_j])^2]E[(r_l - E[r_l])^2]E[(r_b - E[r_b])^2\}^{1/2}}$$

(*i, j, l, b* = Europe large, North America large, Pacific, Europe small) and compares them with the co-kurtosis coefficients implied by a three-state regime switching model:

$$\mathbf{r}_t = \boldsymbol{\mu}_{s_t} + \Sigma_{s_t} \boldsymbol{\varepsilon}_t,$$

where $\boldsymbol{\varepsilon}_t \sim I.I.D.\ N(\mathbf{0}, I_4)$ is an unpredictable return innovation. Coefficients under multivariate regime switching are calculated employing simulations (50,000 trials) and averaging across simulated samples. In the table NA stands for 'North American small caps', and Pac for 'Pacific' equity portfolios. Bold co-skewness coefficients are significantly different from zero; bold co-kurtosis coefficients are significantly different from their Gaussian counterparts.

Coeff.	Sample	MS – erg.	Regime 1	Regime 2	Regime 3
$K_{EU\_large, EU\_large, NA, EU\_small}$	**2.725**	**2.125**	1.884	1.667	1.877
$K_{EU\_large, EU\_large, NA, Pac}$	1.137	1.123	1.071	1.379	1.156
$K_{EU\_large, EU\_large, Pac, EU\_small}$	1.234	1.377	1.194	1.370	1.284
$K_{NA,NA,EU\_large,Pac}$	1.215	1.131	1.100	1.377	1.192
$K_{NA,NA,EU\_large,EU\_small}$	**2.395**	**2.002**	**1.908**	1.682	**1.906**
K_{NA,NA,Pac,EU_small}	1.086	1.129	1.023	1.317	1.141
$K_{Pac,Pac,EU\_large,EU\_small}$	1.330	1.496	1.495	1.466	1.505
$K_{Pac,Pac,EU\_large,NA}$	1.243	1.273	1.268	1.364	1.301
$K_{Pac,Pac,EU\_large,NA}$	1.117	1.221	1.322	1.467	1.356
$K_{EU\_small,EU\_small,EU\_large,NA}$	**2.505**	**2.191**	**1.918**	1.689	**1.900**
$K_{EU\_small,EU\_small,EU\_large,Pac}$	1.517	1.655	1.276	1.378	1.346
$K_{EU\_small,EU\_small,,NA,Pac}$	1.246	1.376	1.089	1.331	1.176
$K_{EU\_large,EU\_large,NA,NA}$	**2.985**	**2.412**	2.259	2.221	2.273
$K_{EU\_large,EU\_large,Pac,Pac}$	**1.229**	1.562	1.751	1.929	1.773
$K_{EU\_large,EU\_large,EU\_small,EU\_small}$	**3.324**	**2.856**	2.416	2.226	2.380
$K_{NA,NA,Pac,Pac}$	**1.510**	**1.495**	1.697	1.953	1.735
$K_{NA,NA,EU\_small,EU\_small}$	2.369	2.198	2.142	2.073	2.144
$K_{Pac,Pac,EU\_small,EU\_small}$	2.193	2.080	1.885	1.958	1.898
$K_{EU\_large,EU\_large,EU\_large,NA}$	**3.450**	**2.586**	2.199	2.131	2.212
$K_{EU\_large,EU\_large,EU\_large,Pac}$	1.354	1.457	1.365	1.619	1.452
$K_{EU\_large,EU\_large,EU\_large,EU\_small}$	**3.727**	**2.847**	2.376	2.122	2.352
$K_{NA,NA,NA,Pac}$	1.549	1.381	**1.200**	1.674	1.325
K_{NA,NA,NA,EU_small}	**2.463**	**2.212**	2.054	1.885	2.070
$K_{Pac,EU\_small,EU\_small,EU\_small}$	1.922	1.852	1.406	1.653	1.554
K_{NA,NA,NA,EU_large}	**2.955**	**2.536**	2.253	2.136	2.271
K_{Pac,Pac,Pac,EU_large}	1.469	1.606	1.471	1.628	1.541
$K_{EU\_small,EU\_small,EU\_small,EU\_large}$	**3.508**	**3.290**	2.469	2.132	2.419
$K_{Pac,Pac,Pac,NA}$	1.394	1.455	1.232	1.679	1.336
$K_{EU\_small,EU\_small,EU\_small,NA}$	**2.760**	**2.665**	**2.100**	1.891	2.090
$K_{EU\_small,EU\_small,EU\_small,Pac}$	**2.437**	**2.363**	1.454	1.666	1.585
$K_{EU\_large,EU\_large,EU\_large,EU\_large}$	**4.975**	3.646	2.994	3.130	3.008
$K_{NA,NA,NA,NA}$	3.689	3.434	3.094	3.136	3.110
$K_{Pac,Pac,Pac,Pac}$	3.395	3.258	3.105	3.124	3.094
$K_{EU\_small,EU\_small,EU\_small,EU\_small}$	**4.815**	**4.758**	3.164	3.152	3.140

Table 8

Comparing Optimal Buy-and-Hold Weights with and without Co-Skewness Risks

The table reports optimal portfolio weights obtained from two alternative regime switching models. The first one corresponds to the parameter estimates in Table 6 and implies regime-specific expected stock returns. Therefore it implies that all high co-moments (including co-skewness) may affect portfolio behavior ('Co-Skew' column). The second model constrains expected returns to be independent of regimes and therefore implies that only even co-higher moments (e.g. co-kurtosis) affect portfolio choices ('No Co-Skew' column). For comparison, the 'Gaussian IID' row reports results under a multivariate Gaussian benchmark with no regimes. The investor is assumed to have power utility and a constant relative risk aversion coefficient of 5.

Investment Horizon (T)	European small caps		European large caps		North American large		Pacific large	
	Co-Skew	No Co-skew	Co-Skew	No Co-skew	Co-Skew	No Co-skew	Co-Skew	No Co-skew
	Gaussian IID (Single-regime)							
	0.87		0.00		0.00		0.13	
	Regime 1 (Bear)							
1 week	0.00	0.21	0.00	0.00	0.44	0.42	0.66	0.27
1 month	0.00	0.59	0.00	0.00	0.59	0.09	0.41	0.32
2 months	0.00	0.68	0.00	0.00	0.61	0.03	0.39	0.29
4 months	0.00	0.73	0.00	0.00	0.60	0.00	0.40	0.27
1 year	0.00	0.82	0.04	0.00	0.57	0.00	0.39	0.18
2 years	0.00	0.83	0.05	0.00	0.57	0.00	0.38	0.17
	Regime 2 (Normal)							
1 week	1.00	1.00	0.00	0.00	0.00	0.00	0.00	0.00
1 month	1.00	1.00	0.00	0.00	0.00	0.00	0.00	0.00
2 months	1.00	1.00	0.00	0.00	0.00	0.00	0.00	0.00
4 months	1.00	0.99	0.00	0.00	0.00	0.00	0.00	0.01
1 year	0.99	0.93	0.00	0.00	0.01	0.00	0.00	0.06
2 years	0.97	0.91	0.00	0.00	0.03	0.00	0.00	0.06
	Regime 3 (Bull)							
1 week	0.00	0.68	1.00	0.00	0.00	0.02	0.00	0.30
1 month	0.00	0.70	0.37	0.00	0.30	0.01	0.23	0.29
2 months	0.00	0.72	0.03	0.00	0.56	0.01	0.41	0.27
4 months	0.00	0.75	0.00	0.00	0.57	0.00	0.43	0.25
1 year	0.01	0.82	0.00	0.00	0.57	0.00	0.42	0.18
2 years	0.06	0.84	0.00	0.00	0.54	0.00	0.40	0.16
	Steady-state (Ergodic)							
1 week	0.00	0.75	0.00	0.00	0.55	0.00	0.45	0.25
1 month	0.00	0.79	0.00	0.00	0.53	0.00	0.47	0.21
2 months	0.05	0.78	0.00	0.00	0.51	0.00	0.44	0.22
4 months	0.10	0.81	0.00	0.00	0.47	0.00	0.43	0.19
1 year	0.11	0.85	0.00	0.00	0.46	0.00	0.43	0.15
2 years	0.13	0.86	0.00	0.00	0.46	0.00	0.41	0.14

Table 9

Annualized Percentage Welfare Costs from Ignoring European Small Caps

The table reports the (annualized, percentage) compensatory variation from restricting the asset menu to exclude European small caps. The table shows welfare costs as a function of the investment horizon; calculations were performed under a variety of assumptions concerning the coefficient of relative risk aversion and the possibility to short-sell. The investor is assumed to have a simple buy-and-hold objective. Panel A and B present results for end-of-sample simulations (when assumptions are imposed on the regime probabilities) and for real-time portfolios, respectively.

	Investment Horizon T (in weeks)					
	T=1	**T=4**	**T=12**	**T=24**	**T=52**	**T=104**
Panel A – Simulations (based on end-of-sample parameter estimates)						
	Equal probabilities					
γ =5	34.94	11.87	5.92	4.38	4.33	2.96
γ =10	3.57	1.86	1.24	1.06	1.03	0.74
γ =5, short sales allowed	42.42	19.42	12.55	11.77	11.97	7.77
γ =10, short sales allowed	3.53	1.43	0.79	0.61	0.53	0.41
	Ergodic Probabilities					
γ =5	60.11	10.55	5.79	4.63	4.62	3.17
γ =10	8.40	2.19	1.18	0.97	0.88	0.69
γ =5, short sales allowed	77.90	9.95	5.68	4.95	5.02	3.51
γ =10, short sales allowed	41.81	9.86	5.21	4.26	3.89	3.00
Panel B – Real time recursive results						
	Full sample (Jan. 2002 – June 2003)					
Mean	40.31	21.21	22.11	22.86	23.79	16.26
Median	39.98	26.43	24.39	22.71	22.82	15.41
Standard deviation	23.16	8.44	6.23	8.49	14.58	15.76
t-stat	1.80	5.62	13.92	15.27	14.41	13.94
	First sub-sample (Jan. 2002 – Sept. 2003)					
Mean	21.27	24.63	27.71	29.12	30.36	20.47
Median	59.35	37.47	32.66	32.92	33.17	21.69
Standard deviation	22.14	8.91	6.42	8.34	14.47	15.92
t-stat	0.76	4.32	11.75	13.79	13.10	12.52
	Second sub-sample (Oct. 2002 – June 2003)					
Mean	62.28	17.88	16.70	16.76	17.22	11.88
Median	32.16	23.72	21.11	20.35	20.00	13.63
Standard deviation	24.26	7.99	5.18	6.88	11.52	12.14
t-stat	1.74	3.60	9.10	9.91	9.34	9.16

Table 10

Estimates of a Three-State Regime Switching Model – Effects of Adding European and North American Small Caps

The table shows estimation results for the regime switching model:

$$r_t = \boldsymbol{\mu}_{s_t} + \boldsymbol{\varepsilon}_t$$

where $\mathbf{r}_t$ is a 4×1 vector collecting weekly total return series, $\boldsymbol{\mu}_{s_t}$ is the intercept vector in state s_t, and $\boldsymbol{\varepsilon}_t = [\varepsilon_{1t}\ \varepsilon_{2t}\ \varepsilon_{3t}\varepsilon_{4t}\varepsilon_{5t}]' \sim N(\mathbf{0}, \Sigma_{s_t})$. The unobservable state s_t is governed by a first-order Markov chain that can assume three values. The first panel refers to the single-state case k = 1. Asterisks attached to correlation coefficients refer to covariance estimates.

	Europe – Large caps	North America – Large caps	Pacific	Europe – Small caps	North America – Small caps
Panel A – Single State Model					
1. Mean return	-0.0015	-0.0010	-0.0007	0.0002	0.0019
2. Correlations/Volatilities					
Europe – Large caps	0.0370***				
North America – Large caps	0.7537***	0.0285***			
Pacific	0.5086**	0.4822**	0.0259***		
Europe – Small caps	0.7816***	0.6718***	0.5403**	0.0222***	
North America – Small caps	0.6948***	0.7992***	0.4267**	0.7275***	0.0301
Panel B – Three State Model					
1. Mean return					
Bear State	-0.0403***	-0.0248***	-0.0218***	-0.0214***	-0.0216**
Normal State	-0.0015	-0.0009	0.0004	0.0024*	0.0046**
Bull State	0.0337***	0.0204***	0.0153***	0.0131***	0.0134**
2. Correlations/Volatilities					
Bear state:					
Europe – Large caps	0.0365***				
North America – Large caps	0.6850***	0.0256***			
Pacific	0.3579**	0.2229*	0.0285***		
Europe – Small caps	0.8049***	0.6547***	0.6004***	0.0324***	
North America – Small caps	0.7759***	0.6757***	0.3714**	0.7092***	0.0378***
Normal state:					
Europe – Large caps	0.0242***				
North America – Large caps	0.7443***	0.0216***			
Pacific	0.5445**	0.6008***	0.0212***		
Europe – Small caps	0.7096***	0.6616***	0.6046***	0.0146***	
North America – Small caps	0.6869***	0.8410***	0.5779**	0.7370***	0.0234***
Bull state:					
Europe – Large caps	0.0359***				
North America – Large caps	0.5386***	0.0330***			
Pacific	-0.0551	-0.0067	0.0245***		
Europe – Small caps	0.6581***	0.4863**	-0.3451*	0.0167***	
North America – Small caps	0.4895*	0.7983***	-0.2535*	0.5554***	0.0314***

3. Transition probabilities	Bear State	Normal State	Bull State
Bear State	0.2450**	0.0005	0.7545
Normal State	0.0457*	0.9542***	0.0001
Bull State	0.5351**	0.1656*	0.2993*

* denotes 10% significance, ** significance at 5%, *** significance at 1%.

Table 11

Co-Skewness and Co-Kurtosis Coefficients for Small Caps vs. an Equally Weighted Portfolio

Coefficients under multivariate regime switching are calculated employing simulations (50,000 trials) and averaging across simulated samples of length equal to the available data (January 1999 – June 2003).

	Co-Skewness		Co-Kurtosis	
	Sample	**MS - ergodic**	**Sample**	**MS - ergodic**
	European Small Caps			
$S_{EU\_small,EW\_ptf,EW\_ptf}$	**-0.422**	**-0.314**		
$S_{EU\_small,EU\_small,EW\_ptf}$	**-0.591**	-0.275		
$S_{EU\_small,EU\_small,NA\_large,EW\_ptf}$			**2.627**	**2.619**
$S_{EU\_small,EU\_small,NA\_small,EW\_ptf}$			**2.709**	1.700
$S_{EU\_small,EU\_small,Pac,EW\_ptf}$			**2.007**	**2.782**
$S_{EU\_small,EU\_small,EU\_large,EW\_ptf}$			**3.178**	**2.629**
$S_{EW\_ptf,EW\_ptf,EU\_small,NA\_large}$			**2.670**	**2.663**
$S_{EW\_ptf,EW\_ptf,EU\_small,NA\_small}$			**2.646**	1.872
$S_{EW\_ptf,EW\_ptf,EU\_small,Pac}$			1.827	**2.907**
$S_{EW\_ptf,EW\_ptf,EU\_small,EU\_large}$			**3.094**	**2.751**
$S_{EW\_ptf,EW\_ptf,EU\_small,EU\_small}$			**3.377**	**3.058**
$S_{EW\_ptf,EW\_ptf,EW\_ptf,EU\_small}$			**3.222**	**3.136**
$S_{EU\_small,EU\_small,EU\_small,EW\_ptf}$			**3.845**	**3.173**
	North American Small Caps			
$S_{NA\_small,EW\_ptf,EW\_ptf}$	-0.200	-0.286		
$S_{NA\_small,NA\_small,EW\_ptf}$	-0.174	-0.252		
$S_{NA\_small,NA\_small,NA\_large,EW\_ptf}$			**2.422**	1.655
$S_{NA\_small,NA\_small,EU\_small,EW\_ptf}$			1.869	1.827
$S_{NA\_small,NA\_small,Pac,EW\_ptf}$			1.431	1.991
$S_{NA\_small,NA\_small,EU\_large,EW\_ptf}$			**2.442**	1.793
$S_{EW\_ptf,EW\_ptf,NA\_small,NA\_large}$			**2.617**	1.767
$S_{EW\_ptf,EW\_ptf,NA\_small,EU\_small}$			**2.646**	1.872
$S_{EW\_ptf,EW\_ptf,NA\_small,Pac}$			1.576	**2.162**
$S_{EW\_ptf,EW\_ptf,NA\_small,EU\_large}$			**2.725**	1.930
$S_{EW\_ptf,EW\_ptf,NA\_small,NA\_small}$			**2.747**	**2.199**
$S_{EW\_ptf,EW\_ptf,EW\_ptf,NA\_small}$			**2.936**	**2.318**
$S_{NA\_small,NA\_small,NA\_small,EW\_ptf}$			**2.825**	**2.263**

Table 12

Effects of the Rebalancing Frequency

This table reports the optimal weight to be invested in the various equity portfolios as a function of the rebalancing frequency for an investor with power utility and a constant relative risk aversion coefficient of 5. Nominal returns are assumed to be generated by the three-state regime switching model:

$$r_t = \mu_{s_t} + \varepsilon_t$$

Rebalancing Frequency	Investment Horizon T (in months)					
	T=1	T=4	T=12	T=24	T=52	T=104
Panel A - Optimal Allocation to European Small Cap Stocks						
IID (no predictability)	0.87	0.87	0.87	0.87	0.87	0.87
Bear state 1						
Buy-and-hold	0.00	0.00	0.00	0.00	0.00	0.00
Bi-annually	0.00	0.00	0.00	0.00	0.00	0.00
Quarterly	0.00	0.00	0.00	0.00	0.00	0.00
Monthly	0.00	0.00	0.00	0.00	0.01	0.03
Weekly	0.00	0.05	0.01	0.02	0.03	0.04
Normal state 2						
Buy-and-hold	1.00	1.00	1.00	1.00	1.00	1.00
Bi-annually	1.00	1.00	1.00	1.00	1.00	1.00
Quarterly	1.00	1.00	1.00	1.00	1.00	1.00
Monthly	1.00	1.00	1.00	1.00	1.00	1.00
Weekly	1.00	1.00	1.00	1.00	1.00	1.00
Bull state 3						
Buy-and-hold	0.00	0.00	0.00	0.00	0.00	0.00
Bi-annually	0.00	0.00	0.00	0.00	0.03	0.04
Quarterly	0.00	0.00	0.00	0.00	0.04	0.05
Monthly	0.00	0.00	0.00	0.01	0.01	0.02
Weekly	0.00	0.00	0.00	0.01	0.01	0.01
Steady-state probabilities						
Buy-and-hold	0.00	0.00	0.05	0.11	0.10	0.10
Bi-annually	0.00	0.00	0.05	0.11	0.18	0.18
Quarterly	0.00	0.00	0.05	0.11	0.18	0.19
Monthly	0.00	0.00	0.08	0.13	0.20	0.20
Weekly	0.00	0.00	0.00	0.02	0.06	0.07
Panel B - Optimal Allocation to European Large Cap Stocks						
IID (no predictability)	0.00	0.00	0.00	0.00	0.00	0.00
Bear state 1						
Buy-and-hold	0.00	0.00	0.00	0.00	0.04	0.05
Bi-annually	0.00	0.00	0.00	0.00	0.08	0.09
Quarterly	0.00	0.00	0.00	0.00	0.09	0.10
Monthly	0.00	0.00	0.09	0.08	0.07	0.06
Weekly	0.00	0.00	0.04	0.02	0.01	0.00
Normal state 2						
Buy-and-hold	0.00	0.00	0.00	0.00	0.00	0.00
Bi-annually	0.00	0.00	0.00	0.00	0.00	0.00
Quarterly	0.00	0.00	0.00	0.00	0.00	0.00
Monthly	0.00	0.00	0.00	0.00	0.00	0.00
Weekly	0.00	0.00	0.00	0.00	0.00	0.00
Bull state 3						
Buy-and-hold	1.00	0.37	0.03	0.00	0.00	0.00
Bi-annually	1.00	0.37	0.03	0.00	0.00	0.00
Quarterly	1.00	0.37	0.03	0.00	0.00	0.00
Monthly	1.00	0.37	0.18	0.09	0.08	0.08
Weekly	1.00	1.00	0.97	0.90	0.88	0.87
Steady-state probabilities						
Buy-and-hold	0.00	0.00	0.00	0.00	0.00	0.00
Bi-annually	0.00	0.00	0.00	0.00	0.00	0.00
Quarterly	0.00	0.00	0.00	0.00	0.00	0.00
Monthly	0.00	0.00	0.00	0.00	0.00	0.00
Weekly	0.00	0.00	0.00	0.00	0.00	0.00

Table 12 (continued)

Effects of the Rebalancing Frequency

Rebalancing Frequency	Investment Horizon T (in months)					
Panel C - Optimal Allocation to North American Large Cap Stocks						
	T=1	**T=4**	**T=12**	**T=24**	**T=52**	**T=104**
IID (no predictability)	0.00	0.00	0.00	0.00	0.00	0.00
Bear state 1						
Buy-and-hold	0.44	0.59	0.60	0.60	0.57	0.57
Bi-annually	0.44	0.59	0.60	0.60	0.51	0.50
Quarterly	0.44	0.59	0.60	0.60	0.50	0.49
Monthly	0.44	0.59	0.49	0.50	0.50	0.49
Weekly	0.44	0.46	0.48	0.49	0.50	0.50
Normal state 2						
Buy-and-hold	0.00	0.00	0.00	0.00	0.00	0.00
Bi-annually	0.00	0.00	0.00	0.00	0.00	0.00
Quarterly	0.00	0.00	0.00	0.00	0.00	0.00
Monthly	0.00	0.00	0.00	0.00	0.00	0.00
Weekly	0.00	0.00	0.00	0.00	0.00	0.00
Bull state 3						
Buy-and-hold	0.00	0.30	0.56	0.57	0.57	0.56
Bi-annually	0.00	0.30	0.56	0.57	0.59	0.59
Quarterly	0.00	0.30	0.56	0.57	0.58	0.58
Monthly	0.00	0.30	0.42	0.50	0.54	0.53
Weekly	0.00	0.00	0.00	0.00	0.02	0.02
Steady-state probabilities						
Buy-and-hold	0.55	0.53	0.51	0.46	0.46	0.46
Bi-annually	0.55	0.53	0.51	0.46	0.40	0.40
Quarterly	0.55	0.53	0.51	0.46	0.40	0.39
Monthly	0.55	0.53	0.47	0.45	0.39	0.38
Weekly	0.55	0.51	0.46	0.43	0.38	0.36
Panel D - Optimal Allocation to Pacific Stocks						
	T=1	**T=4**	**T=12**	**T=24**	**T=52**	**T=104**
IID (no predictability)	0.13	0.13	0.13	0.13	0.13	0.13
Bear state 1						
Buy-and-hold	0.56	0.41	0.40	0.40	0.39	0.38
Bi-annually	0.56	0.41	0.40	0.40	0.41	0.41
Quarterly	0.56	0.41	0.40	0.40	0.41	0.41
Monthly	0.56	0.41	0.42	0.42	0.42	0.42
Weekly	0.56	0.49	0.47	0.47	0.46	0.46
Normal state 2						
Buy-and-hold	0.00	0.00	0.00	0.00	0.00	0.00
Bi-annually	0.00	0.00	0.00	0.00	0.00	0.00
Quarterly	0.00	0.00	0.00	0.00	0.00	0.00
Monthly	0.00	0.00	0.00	0.00	0.00	0.00
Weekly	0.00	0.00	0.00	0.00	0.00	0.00
Bull state 3						
Buy-and-hold	0.00	0.33	0.41	0.43	0.43	0.44
Bi-annually	0.00	0.33	0.41	0.43	0.38	0.37
Quarterly	0.00	0.33	0.41	0.43	0.38	0.37
Monthly	0.00	0.33	0.40	0.40	0.37	0.37
Weekly	0.00	0.00	0.03	0.09	0.09	0.10
Steady-state probabilities						
Buy-and-hold	0.45	0.47	0.44	0.43	0.44	0.44
Bi-annually	0.45	0.47	0.44	0.43	0.42	0.42
Quarterly	0.45	0.47	0.44	0.43	0.42	0.42
Monthly	0.45	0.47	0.45	0.42	0.41	0.42
Weekly	0.45	0.49	0.54	0.55	0.56	0.57

Table 13

Estimates of a Three-State Regime Switching Model for European, North American, and Pacific Equity Portfolios: Longer Time Series (1989-2004)

The table shows estimation results for the regime switching model:

$$r_t = \boldsymbol{\mu}_{s_t} + \boldsymbol{\varepsilon}_t$$

where $\mathbf{r}_t$ is a 4×1 vector collecting weekly total return series, $\boldsymbol{\mu}_{s_t}$ is the intercept vector in state s_t, and $\boldsymbol{\varepsilon}_t = [\varepsilon_{1t}\ \varepsilon_{2t}\ \varepsilon_{3t}\varepsilon_{4t}]' \sim N(\mathbf{0}, \Sigma_{s_t})$. The unobservable state s_t is governed by a first-order Markov chain that can assume three values. The first panel refers to the single-state case k = 1. Asterisks attached to correlation coefficients refer to covariance estimates. For mean coefficients and transition probabilities, standard errors are reported in parenthesis.

	Panel A – Single State Model			
	Europe – Large caps	North America Large	Pacific	Europe – Small caps
1. Mean return	0.0018	0.0016	-0.0007	0.0015
2. Correlations/Volatilities				
Europe – Large caps	0.0209***			
North America - Large caps	0.6156***	0.0212***		
Pacific	0.4669***	0.3844**	0.0241***	
Europe – Small caps	0.8951***	0.5000***	0.4466***	0.0176***

	Panel B – Three State Model			
	Europe – Large caps	North America Large	Pacific	Europe – Small caps
1. Mean return				
Bear State	-0.0094***	-0.0034*	-0.0070**	-0.0121***
Normal State	0.0028**	0.0021**	-0.0007	0.0023***
Bull State	0.0112***	0.0058***	0.0061**	0.0142***
2. Correlations/Volatilities				
Bear state:				
Europe – Large caps	0.0289***			
North America - Large caps	0.6620***	0.0309***		
Pacific	0.5139**	0.3782*	0.0320***	
Europe – Small caps	0.9253***	0.5947***	0.5151**	0.0227***
Normal state:				
Europe – Large caps	0.0153***			
North America - Large caps	0.5411***	0.0157***		
Pacific	0.5027***	0.4420***	0.0186***	
Europe – Small caps	0.8811***	0.4271**	0.4511***	0.0110***
Bull state:				
Europe – Large caps	0.0163***			
North America - Large caps	0.5587***	0.0185***		
Pacific	0.1426	0.2309*	0.0236***	
Europe – Small caps	0.8159***	0.3221**	0.1317	0.0131***

3. Transition probabilities	Bear State	Normal State	Bull State
Bear State	0.8820***	0.0463*	0.0717
Normal State	0.0440	0.9178***	0.0382
Bull State	0.0218	0.1440**	0.8342***

* denotes 10% significance, ** significance at 5%, *** significance at 1%.

Figure 1

Buy-and-Hold Optimal Allocation – Restricted Asset Menu

The graphs plot the optimal international equity portfolio weights when returns follow a two-state Markov Switching model as a function of: (i) the coefficient of relative risk aversion; (ii) the investment horizon. As a benchmark (bold horizontal lines) we also report the IID/Myopic allocation that obtains when returns have an IID multivariate Gaussian distribution.

$\gamma = 5$ | $\gamma = 10$

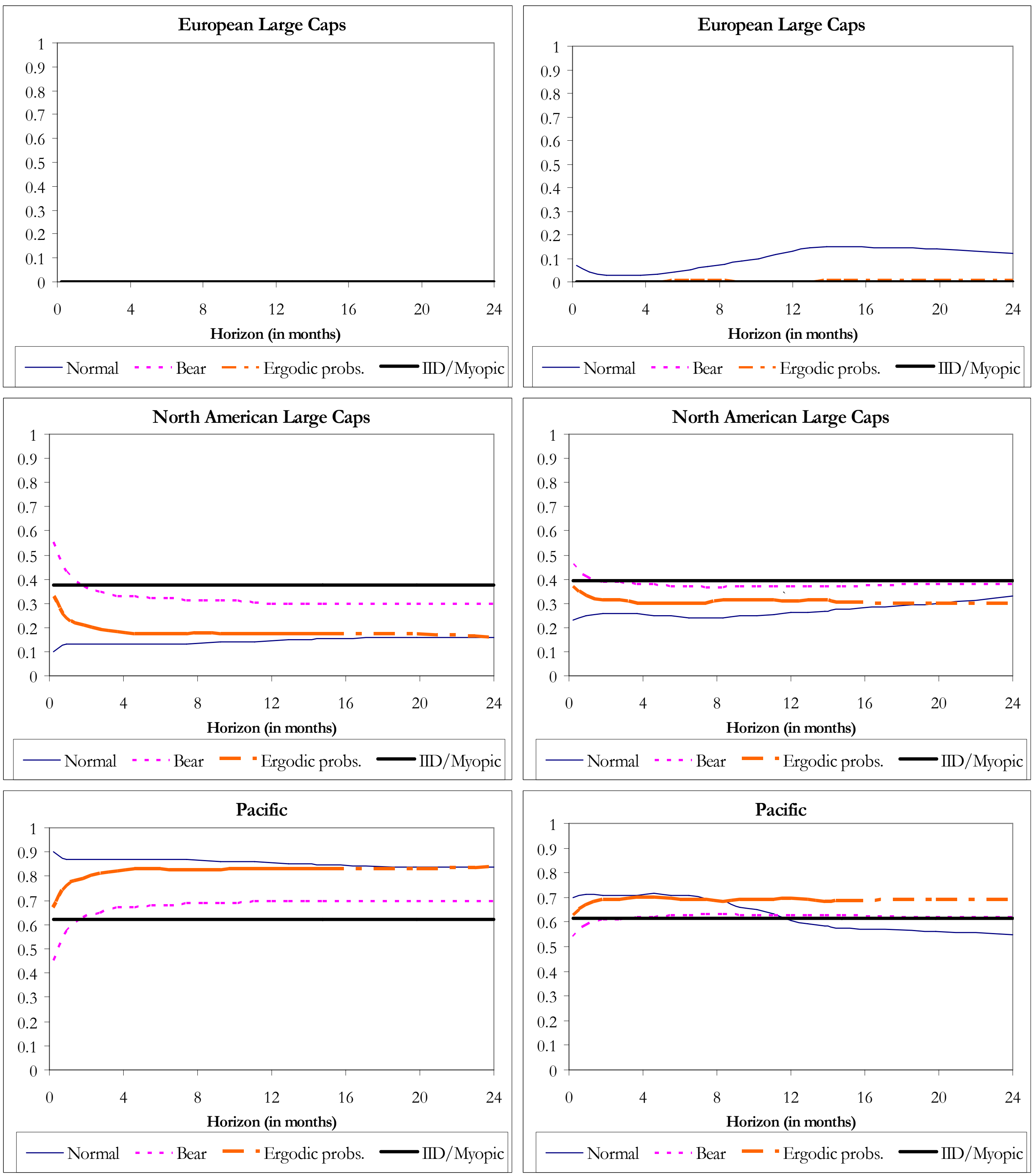


Figure 2

Buy-and-Hold Optimal Allocation

The graphs plot the optimal international equity portfolio weights when returns follow a three-state Markov Switching model as a function of: (i) the coefficient of relative risk aversion; (ii) the investment horizon. As a benchmark (bold horizontal lines) we also report the IID/Myopic allocation. The asset menu includes European small caps.

$\gamma = 5$ $\gamma = 10$

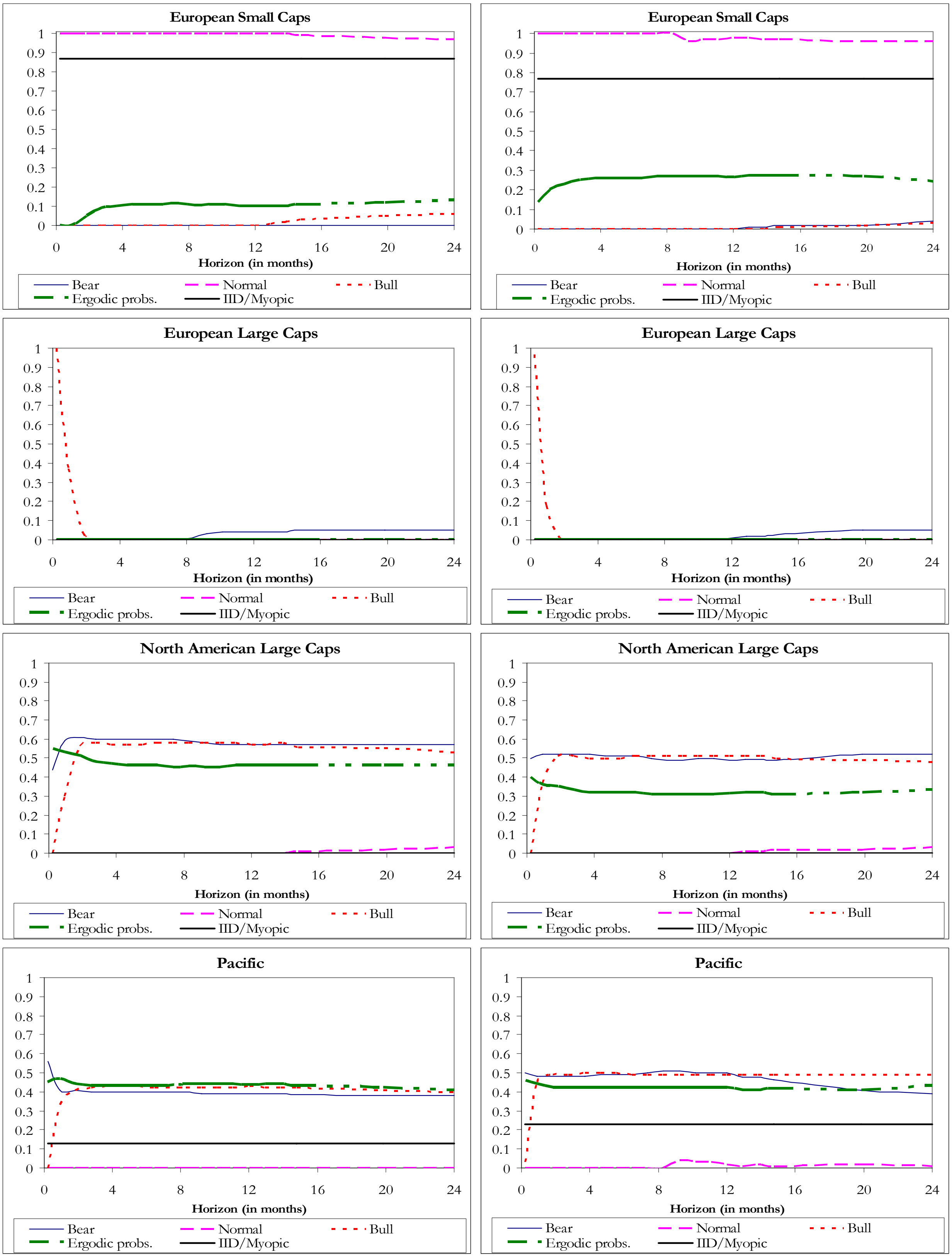
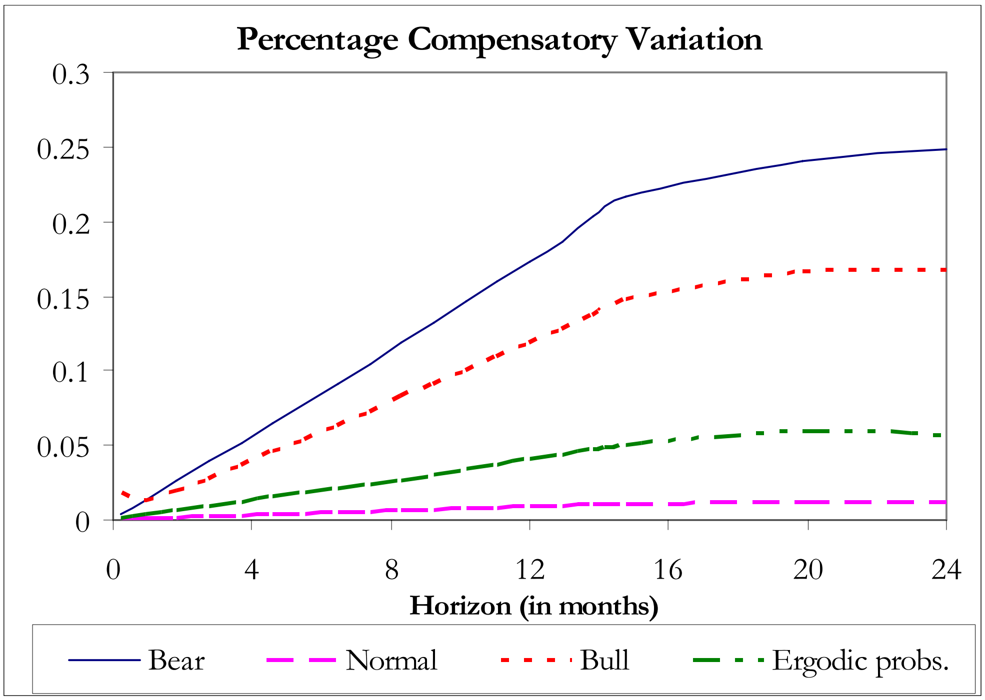


Figure 3

Welfare Costs of Ignoring Regime Switching

The graphs plot the percentage compensatory variation from ignoring the presence of regime switches in the multivariate process of asset returns. The graphs plot the welfare costs as a function of the investment horizon; calculations were performed for two alternative levels of the coefficient of relative risk aversion. The investor is assumed to have a simple buy-and-hold objective.

$\gamma = 5$



9

$\gamma = 10$

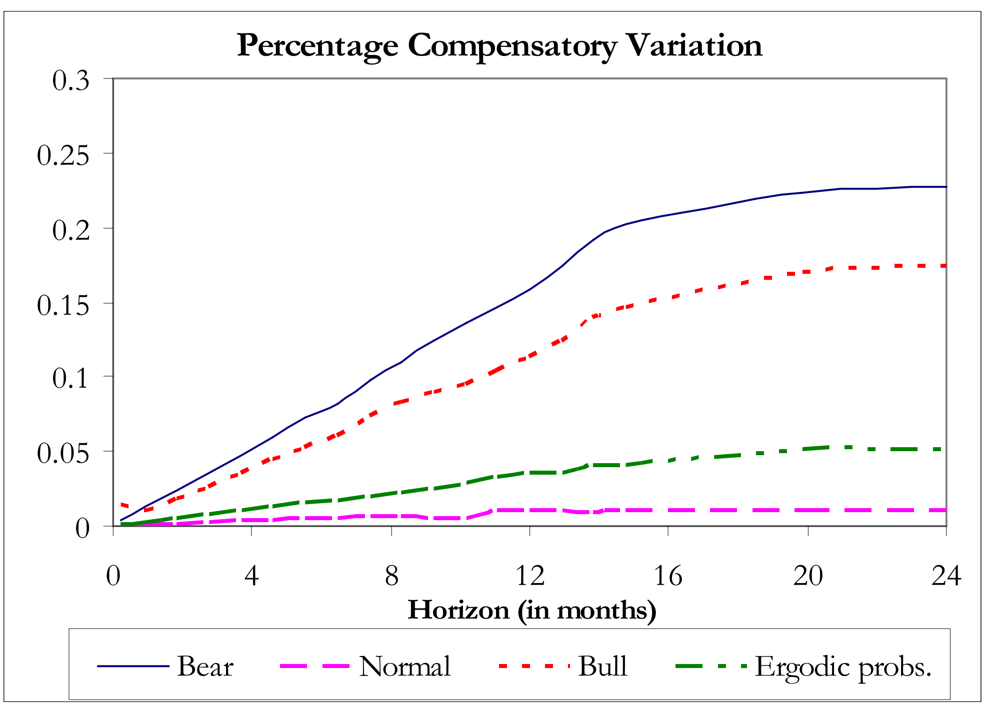


Figure 4

Buy-and-Hold Optimal Allocation – Asset Menu Expanded to North American Small Caps

The graphs plot the optimal international equity portfolio weights when returns follow a three-state Markov Switching model as a function of: (i) the coefficient of relative risk aversion; (ii) the investment horizon.

Figure 5

Buy-and-Hold Optimal Allocation – Short Sales Allowed

The graphs plot the optimal international equity portfolio weights when returns follow a three-state Markov Switching model as a function of: (i) the coefficient of relative risk aversion; (ii) the investment horizon. As a benchmark (bold horizontal lines) we also report the IID/Myopic allocation. The asset menu includes European small caps.

$\gamma = 5$ $\gamma = 10$

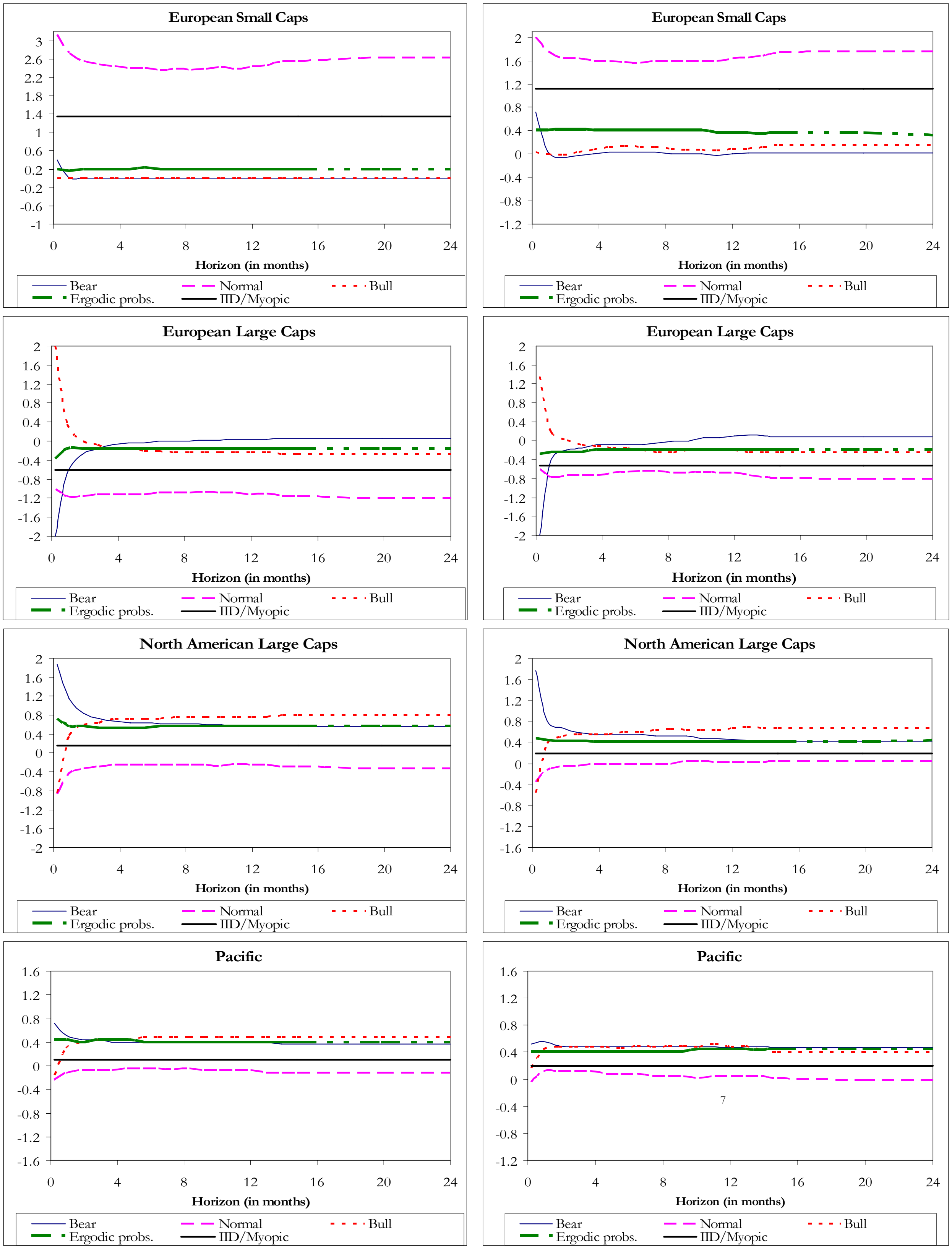


Figure 6

Buy-and-Hold Optimal Allocation – Longer Data Set

The graphs plot the optimal international equity portfolio weights when returns follow a three-state Markov Switching model estimated on July 1989 – December 2004 weekly returns data. As a benchmark (bold horizontal lines) we report the IID/Myopic allocation. The investor is assumed to have a simple buy-and-hold objective and constant relative risk aversion equal to 5.

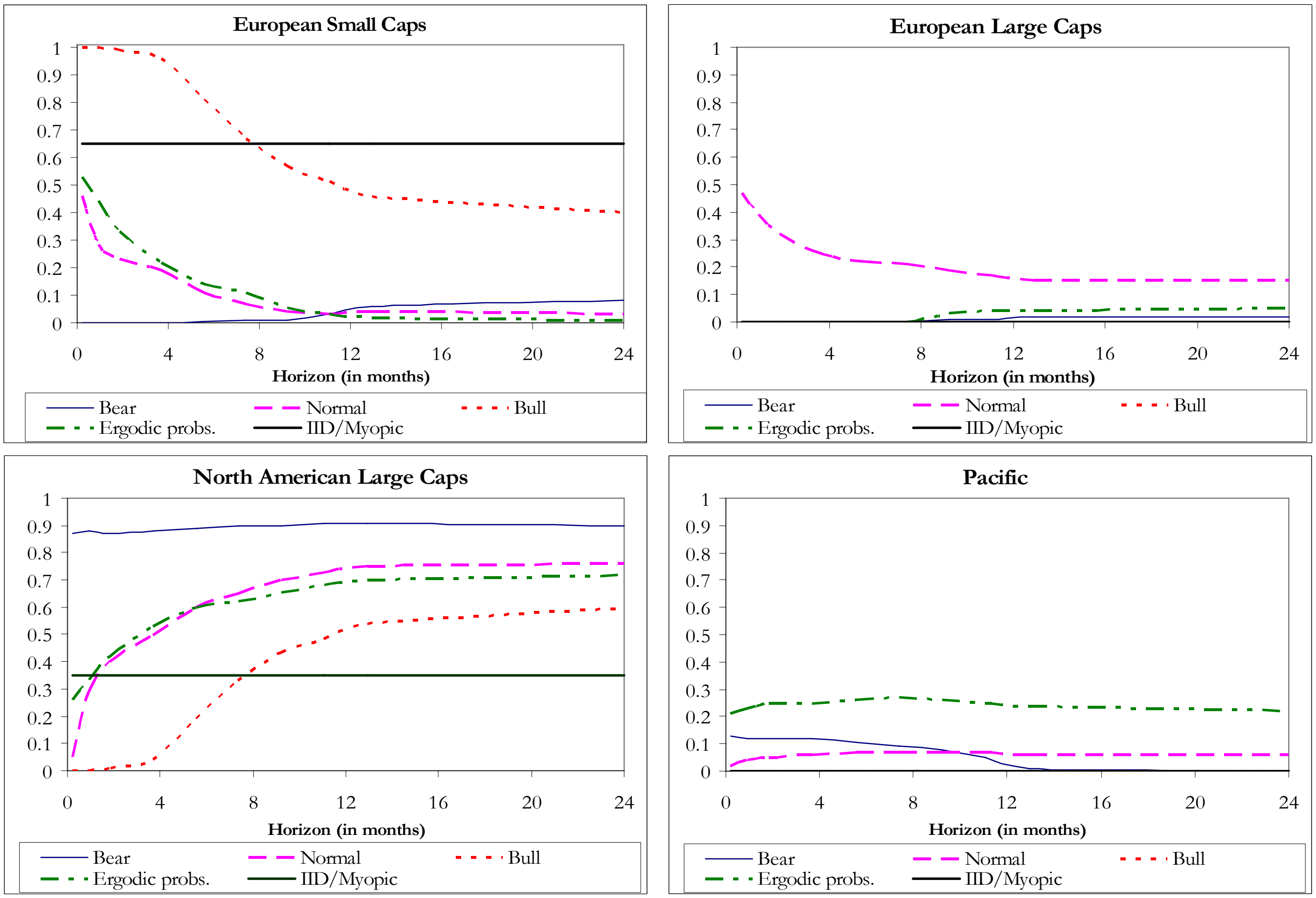


Table B1

Model Selection for Returns on European Large Caps, North American Large Caps, and Pacific Equity Portfolios

The table reports estimates for the multivariate Markov switching conditionally heteroskedastic VAR model:

$$\mathrm{r}_t = \mu_{s_t} + \sum_{j=1}^{p} A_{js_t}\mathrm{r}_{t-j} + \varepsilon_t$$

where $\boldsymbol{\mu}_{s_t}$ is the intercept vector in state s_t, A_{js_t} is the matrix of autoregressive coefficients associated with lag $j \geq 1$ in state s_t and $\boldsymbol{\varepsilon}_t = [\varepsilon_{1t}\ \varepsilon_{2t}\ \varepsilon_{3t}]' \sim N(\mathbf{0}, \Sigma_{s_t})$. The unobserved state variable s_t is governed by a first-order Markov chain that can assume *k* distinct values. *p* autoregressive terms are considered. The sample period is January 1999 – June 2003. MSIAH(k,p) stands for Markov Switching Intercept Autoregressive Heteroskedasticity model with k states and p autoregressive lags.

Model (k,p)	Number of parameters	Log-likelihood	LR test for linearity	BIC	Hannan-Quinn
Base model: MSIA(1,0)					
MSIA(1,0)	9	1597.00	NA	-13.4398	-13.5191
MSIA(1,1)	18	1607.08	NA	-13.3736	-13.5327
MSIA(1,2)	27	1610.42	NA	-13.2490	-13.4884
Base model: MSIA(2,0)					
MSIA(2,0)	14	1599.35	4.6972 (0.971)	-13.3433	-13.4666
MSIH(2,0)	20	1639.69	85.3730 (0.000)	**-13.5482**	**-13.7244**
MSIA(2,1)	32	1639.42	64.6713 (0.000)	-13.3236	-13.6064
MSIAH(2,1)	38	1642.85	71.5345 (0.000)	-13.2127	-13.5486
MSIA(2,2)	50	1663.94	107.0428 (0.000)	-13.1705	-13.6137
Base model: MSIA(3,0)					
MSIA(3,0)	21	1628.50	63.0003 (0.000)	-13.4292	-13.6143
MSIH(3,0)	33	1656.26	118.5173 (0.000)	-13.3867	-13.6775
MSIA(3,1)	48	1659.77	105.3812 (0.000)	-13.1240	-13.5483
MSIAH(3,1)	60	1681.08	147.9954 (0.000)	-13.0261	-13.5565
Base model: MSIA(4,0)					
MSIA(4,0)	30	1633.58	73.1593 (0.000)	-13.2628	-13.5272
MSIA(4,1)	66	1684.87	155.5868 (0.000)	-12.9184	-13.5017
MSIH(4,0)	48	1667.89	141.7696 (0.000)	-13.1364	-13.5594
MSIAH(4,1)	84	1703.65	193.1344 (0.000)	-12.6584	-13.4009

Table B2

Selection of Regime Switching Model for Returns on European, North American, and Pacific Equity Portfolios – Effects of Adding European Small Caps

The table reports estimates for the multivariate Markov switching conditionally heteroskedastic VAR model:

$$r_t = \mu_{s_t} + \sum_{j=1}^{p} A_{js_t} r_{t-j} + \varepsilon_t$$

where $\boldsymbol{\mu}_{s_t}$ is the intercept vector in state s_t, A_{js_t} is the matrix of autoregressive coefficients associated with lag $j \geq 1$ in state s_t and $\varepsilon_t = [\varepsilon_{1t}\ \varepsilon_{2t}\ \varepsilon_{3t}\varepsilon_{4t}]' \sim N(0,\Omega_{s_t})$. The unobserved state variable s_t is governed by a first-order Markov chain that can assume *k* distinct values. *p* autoregressive terms are considered. The sample period is January 1999 – June 2003. MSIAH(k,p) stands for Markov Switching Intercept Autoregressive Heteroskedasticity model with k states and p autoregressive lags.

Model (k,p)	Number of parameters	Log-likelihood	LR test for linearity	BIC	Hannan-Quinn
Base model: MSIA(1,0)					
MSIA(1,0)	14	2277.84	NA	-19.1423	-19.2657
MSIA(1,1)	30	2321.25	NA	-19.2230	-19.4882
MSIA(1,2)	46	2325.78	NA	-18.9699	-19.3777
Base model: MSIA(2,0)					
MSIA(2,0)	20	2293.17	30.6600 (0.000)	-19.1335	-19.3097
MSIH(2,0)	30	2309.30	62.9205 (0.000)	-19.0382	-19.3026
MSIA(2,1)	52	2377.18	111.8710 (0.000)	-19.1885	**-19.6281**
MSIAH(2,1)	62	2377.86	99.2137 (0.000)	-18.9002	-19.4482
MSIA(2,2)	84	2379.88	94.2066 (0.000)	-18.4838	-19.2285
Base model: MSIA(3,0)					
MSIA(3,0)	28	2328.06	100.4450 (0.000)	**-19.2452**	-19.4919
MSIH(3,0)	48	2373.25	190.8288 (0.000)	**-19.2252**	-19.5882
MSIA(3,1)	76	2384.26	126.0169 (0.000)	-18.6877	-19.3594
MSIAH(3,1)	96	2432.60	222.6945 (0.000)	-18.6347	-19.4832
Base model: MSIA(4,0)					
MSIA(4,0)	38	2330.84	106.0120 (0.000)	-19.0358	-19.3707
MSIA(4,1)	102	2429.12	215.7464 (0.000)	-18.4645	-19.3661
MSIH(4,0)	68	2393.42	231.1690 (0.000)	-18.8713	-19.4706