

Asset Pricing in Dynamic Stochastic General Equilibrium Models with Indeterminacy*

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Abstract

We explore asset pricing in the context of the one-sector Benhabib-Farmer-Guo (BFG) model with increasing returns to scale in production and contrast our results with financial implications of the standard dynamic stochastic general equilibrium (DSGE) model. Our main goal is to determine the effects of local indeterminacy and the presence of sunspot shocks on asset pricing. We find that the BFG model does not adequately represent key stylized facts of U.S. capital markets and does not improve on the asset pricing results, obtained in the standard DSGE model.

1 Introduction

Over more than two decades financial economists have been striving to explain time-variation in the interest rates and cross-sectional variation of the returns on average stocks and bonds in the dynamic stochastic general equilibrium (DSGE) framework.¹ The DSGE models (also known as Real Business Cycle (RBC) models), in which macroeconomic factors affect both output and asset prices, seem to be a natural context for asset pricing explorations. The common feature

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¹Examples of such investigations include Brock [6], Donaldson and Mehra [13], Naik [30], Rauwenhorst [34], Boldrin, Christiano, and Fisher [5], Jermann [19], Tallarini [35], Danthine and Donaldson [10], Lettau [27], Gomes, Yaron, and Zhang [15] among others.

of neo-classical RBC models is that they rely on technology shocks as the main source of fluctuations. Although remarkably successful in matching real business cycle statistics, most neo-classical RBC models are unable to replicate one or several stylized financial facts: the high equity premium, the low risk free rate and the high volatility of equity returns with corresponding low volatility of bond returns².

More recently, starting with the pioneering work of Benhabib and Farmer [3] and Farmer and Guo [14], a large body of literature has developed in which DSGE models, modified to include increasing returns to scale in production, can result in a continuum of equilibria indexed by agents' expectations.³ In these models economic agents' self-fulfilling beliefs, also referred to as sunspots or animal spirits, alone can generate business cycle fluctuations, which are difficult to distinguish from the dynamics of the neo-classical RBC models driven by technology shocks. To our knowledge, the financial implications of models with animal spirits have not been investigated.

In this research we attempt to study financial properties of DSGE models, where economic agents' (investors') beliefs alone or in combination with technology shocks generate fluctuations. Since financial markets are theorized to be driven, at least in part, by agents' expectations, one might expect that models with indeterminacy would reflect well the behavior of such markets. Our main objective therefore is to explore whether inclusion of non-fundamental belief shocks and a different (endogenous) shock propagation mechanism, which arises in indeterminate models, enhances the asset pricing performance.

As a framework for examining the behavior of financial assets in an indeterminate one-sector RBC economy, we adopt the Benhabib-Farmer-Guo (BFG) model. It is well known, that the one-sector BFG model requires large increasing returns-to-scale to support sunspot equilibria. To the extent that one objects to the high returns-to-scale calibration, the quantitative experiments, which we present below, should be viewed more from a methodological perspective as illustrating the exact value added of indeterminacy and sunspot in accounting for the stylized financial facts⁴. We also realize that it is not possible to adequately represent stylized financial and macroeconomic facts in the context of the BFG model at the same time because the volatility of the pricing kernel in the BFG model depends solely on the volatility of the consumption growth, just as in the standard consumption-based asset pricing. In the data, consumption growth does not vary much and therefore the standard deviation of the pricing kernel is bound to be low if one wishes to respect this feature of the data. In contrast, the work of Hansen and Jagannathan [17] implies that accounting for the equity premium in the US data requires the volatility of the stochastic discount factor of at least 50% annually. Our goal in this research is not to resolve the financial puzzles but to understand if sunspot and indeterminacy help to alleviate them

²Excellent reviews of literature on asset pricing puzzles include Kocherlakota [25], Mehra and Prescott [29], and Campbell [8].

³Benhabib and Farmer [4] present a comprehensive survey of the indeterminacy literature.

⁴We discuss the issue of returns-to-scale calibration further in the Calibration Section of this paper.

to some degree.

We compare the asset pricing results obtained in the indeterminate BFG model with the results from the indivisible labor business cycle model of Hansen [16]. Two models are canonical in their respective classes, indeterminate versus neo-classical. Apart from the nature of shocks, they differ only in the level of returns to scale in production, facilitating a controlled comparison of financial implications of two competing paradigms.

In this comparison, we see the second contribution of our research. The empirical productivity analysis literature (e.g. Cabaliero and Lyons [7], Basu and Fernald [2], Laitner and Stolyarov [26] to name just a few examples) still did not achieve a broad consensus on the degree of the aggregate increasing returns and therefore on the possibility of the sunspots in many models with indeterminacy. This problem is well understood in light of Kamihigashi’s [21] observational equivalence argument: if the shock behind economic fluctuations is left unrestricted, the observed time series of consumption, investment, output, capital, and labor input can be generated by the economy with any value of returns to scale. Cole and Ohanian [9] showed that even with the restricted shock process the measurement of increasing returns is imprecise, making it difficult to discriminate between models. Two formulations, however, have different implications for economic policy and for that reason any additional information in favor of one setting over the other should be of value. Several studies (for instance Farmer and Guo [14] and Thomas [36]) showed both types of models to be comparably successful in replicating essential macroeconomic features of the business cycle. Success along asset pricing dimensions would certainly present a strong support for sunspot models.

Unfortunately, our results indicate that the BFG model does not improve upon the performance of the neo-classical RBC model of Hansen [16] in representing stylized financial facts of the data. The 4.04 percent return on equity obtained in the Benchmark formulation is significantly below the 7.87 percent average return in equity in the US data. The risk free rate is high – 4.039 percent versus 1.55 percent in the data. The reported numbers clearly illustrate the equity premium puzzle of Mehra and Prescott [28] and the risk free rate puzzle of Weil [37] in the BFG model. In addition, the volatility of the return on equity in the model economy is only 0.32 percent in contrast to the 15.77 percent in the US economy, indicating the volatility puzzle. The corresponding numbers from Hansen’s model are almost identical.

Our conclusion is that the indeterminate BFG model fails to resolve financial puzzles for the same reason as neo-classical Hansen’s model: in both settings agents can adjust consumption, labor supply and the rate of capital accumulation in response to shocks. They smooth consumption “too much” and as a result the stochastic discount factor, which is equal to consumption growth, is not very volatile. The presence of the sunspot shock and the endogenous shock propagation mechanism do not have any significant impact on the financial performance. The natural conclusion is that some technology must be in place that breaks the link between the pricing kernel and the consumption growth and at the same time prevents the easy consumption smoothing,

which characterizes DSGE models. This technology however would also prevent indeterminacy, since for swings in optimism and pessimism to translate into corresponding movements in economic activity there must be enough flexibility in the model economy to allow agents to act on their expectations. For example, Kim [22] showed that indeterminacy disappears if capital adjustment costs are incorporated into the BFG model.

We therefore conclude that the resolution of long-standing financial puzzles is a more challenging task in the context of models with self-fulfilling expectations than in the standard RBC framework because of two conflicting requirements. On the one hand, frictions to restrict the mobility of factors of production, especially of capital, are needed in order to generate a sufficient equity premium and volatility of the asset returns. On the other hand, the flexible factors of production adjustments in response to shocks are essential for the model economy to exhibit local indeterminacy with realistic increasing returns. Perhaps a model, in which indeterminacy is introduced through channels other than aggregate increasing returns in production will fare better in this regard.

The rest of this paper proceeds as follows: In Section 2 we describe the model and its equilibrium; in Section 3 we discuss the solution method and its application to asset pricing, in Section 4 we choose parameter values and present our results; in Section 5 we conclude.

2 The Model

The economy is populated by a continuum of identical households indexed by $[0,1]$ and by a representative "stand in" firm. There exists a legitimate financial market in which equity claims to the representative firm's net income stream and possibly other assets are traded.

2.1 Households

Households maximize their expected lifetime utility defined over consumption and leisure by deciding on the time they wish to work and by choosing their financial asset holdings:

$$Max_{\{Z_{t+1}N_t^h\}} E \left(\sum_{t=0}^{\infty} \beta^t \frac{(C_t)^{1-\xi} - 1}{1-\xi} - \Lambda N_t^h \right)$$

subject to:

$$C_t + Z'_{t+1}P_t^z \leq Z'_t(P_t^z + D_t^z) + W_tN_t^h, \quad (1)$$

where β ($0 < \beta < 1$) is the subjective time discount factor and parameter ξ ($0 < \xi < \infty$) is the coefficient of the relative risk aversion; C_t and N_t^h are per capita consumption and labor services respectively, each in period t . Each household is endowed with one unit of time and the parameter Λ in the utility function is chosen to match the fraction of that time devoted to work in the

data⁵. The wage rate W_t is determined competitively. Z_t is a vector of financial assets held at period t and chosen at $t-1$, P_t^z and D_t^z are vectors of asset prices and current period payouts (dividends).⁶ The period preference ordering of the representative household is assumed to be separable in consumption and leisure and has its origins in Hansen [16]. The representative household's marginal utility of consumption is given by $U_c(C_t, N_t^h) = (C_t)^{-\xi}$ and its intertemporal marginal rate of substitution in consumption, also known as the stochastic discount factor or pricing kernel, by:

$$M_{t+1} = \beta \frac{U_c(C_{t+1}, N_{t+1}^h)}{U_c(C_t, N_t^h)} = \beta \left(\frac{C_{t+1}}{C_t} \right)^{-\xi} \quad (2)$$

The first order conditions for optimization program (1) with respect to financial asset holdings produce asset pricing equations. For instance, the equation for the price of the equity (P_t^e) security, which is a claim to the infinite sequence of dividends, paid by the firm $\{D_{t+j}^e\}_{j=1}^{\infty}$, is given by:

$$P_t^e = E_t [M_{t+1} (P_{t+1}^e + D_{t+1}^e)] \quad (3)$$

Equation (3) means that in his intertemporal choice problem, a typical investor equates the loss in utility associated with buying an additional unit of financial asset (equity) at time t ($P_t^e U_c(C_t, N_t^h)$) to the discounted expected utility of the resulting additional consumption in the next period ($\beta E_t [(P_{t+1}^e + D_{t+1}^e) U_c(C_{t+1}, N_{t+1}^h)]$). Substituting forward for P_{t+j}^e ($j = 1, \dots, \infty$) and using the law of iterated expectations, we obtain a unique non-explosive solution for (3):

$$P_t^e = E_t \left[\sum_{j=1}^{\infty} M_{t+j} D_{t+j}^e \right] \quad (4)$$

The price of one-period risk free bond – an asset, which pays one unit of consumption in every state next period – is just the expectation of the stochastic discount factor:

$$P_t^b = E_t [M_{t+1}] \quad (5)$$

The first order condition for the household's labor decision equates the utility of extra consumption obtained by working longer to the disutility of the additional work effort:

$$(C_t)^{-\xi} W_t = \Lambda \quad (6)$$

The conditional expectations in (3) and (5) are taken over two exogenous sources of fluctuations: technology shock and a non-fundamental belief or sunspot shock. The latter is an extra return on the equity security (in terms of a utility

⁵We choose Λ so that the steady state value of N is $1/3$

⁶We do not consider leverage effects because they have a negligible effect on asset prices in our economy.

increment), which the household believes to materialize over the period. Under certain parameterization of the model, beliefs become self-fulfilling. The belief shock is known at time t . Therefore, in the absence of technology shocks, the expectations in equations (3) and (5) are point expectations. Dynamically, however, there is an uncertainty related to the sunspots because future realizations of the sunspot shock are unknown.

2.2 Firms

A representative firm begins period t with the stock of capital, K_t , carried over from the previous period. The evolution of the capital stock is given by:

$$K_{t+1} = (1 - \Omega) K_t + I_t \quad (7)$$

K_0 given,

where I_t is period t investment and Ω is the depreciation rate. The firm produces output via standard Cobb-Douglas function:

$$Y_t = A_t X_t (K_t)^\alpha (N_t^f)^{1-\alpha} \quad (8)$$

with two inputs – capital, K_t , and labor, N_t^f , and the current level of technology A_t , the log of which is assumed to follow an AR(1) process with the persistence coefficient $\rho \in (0, 1)$:

$$a_t = \ln A_t = (1 - \rho) \ln \bar{A} + \rho \ln A_{t-1} + \varepsilon_t \quad (9)$$

A_0 given.

There is an external effect, X_t , which depends on the economy-wide quantities of capital and labor, denoted by variables with bars:

$$X_t = \bar{K}_t^{\alpha\eta} \bar{N}_t^{(1-\alpha)\eta}$$

The parameter η ($\eta > 0$) captures the size of the aggregate production externality.

The firm takes external effect as given and views its production function as constant returns to scale. As a result, the firm behaves competitively. However, there are increasing returns to scale in production on the aggregate level because of the externality. If $\eta = 0$, private and social returns to scale are both constant.

After period t output is produced, the firm sells it and uses proceeds of the sale to pay the wage bill, $W_t N_t^f$ and to finance investments, I_t , under the knowledge of the equation of motion of the capital stock (7). The remaining output is distributed as dividends to the shareholders (households):

$$D_t = Y_t - W_t N_t^f - I_t \quad (10)$$

In this complete market setting the representative firm's objective is to maximize its pre-dividend stock market value, period by period, by choosing its

investment and labor input. The competitive firm realizes that shareholders' intertemporal marginal rates of substitution are crucial for asset pricing and uses investors' valuation for the price of equity⁷ provided by equation (4). Expression (4) simply equates the share price to the expected present discounted value of the infinite dividend stream, paid by the firm. The representative firm's dynamic optimization program is:

$$Max_{\{I_t, N_t^f\}} (D_t + P_t^e)$$

subject to:

$$\begin{aligned} P_t^e &= E \left(\sum_{j=1}^{\infty} M_{t+j} D_{t+j} \right) \\ K_{t+1} &= (1 - \Omega) K_t + I_t \\ Y_t &= A_t X_t (K_t)^\alpha (N_t^f)^{1-\alpha} \\ D_t &= Y_t - W_t N_t^f - I_t \end{aligned} \tag{11}$$

The program (11) is a decentralized version of the stochastic growth model proposed by Danthine and Donaldson [11]. This interpretation requires of shareholders to convey to the firm a complete listing of their future intertemporal marginal rates of substitution. Danthine and Donaldson [11] show that in the complete market setting with homogenous agents there is perfect unanimity about the provided information. Alternatively, shareholders appoint one of their own to manage the firm, realizing that her preferences over future consumption are identical to theirs.

The first order conditions for the firm's problem (11) with respect to its labor hiring and investment decisions are:

$$(1 - \alpha) \frac{Y_t}{N_t^f} = W_t, \tag{12}$$

and

$$-1 + E_t \left[M_{t+1} \left\{ \alpha \frac{Y_{t+1}}{K_{t+1}} + 1 - \Omega \right\} \right] = 0 \tag{13}$$

Since the private technology of a representative firm is convex, an interior solution to the model exists and the equilibrium is well-defined.

⁷The share price is equal to the ex-dividend value of the firm because the number of shares is normalized to one.

2.3 Equilibrium

Every period the state of the economy is completely determined by two endogenous state variables: the aggregate capital stock, K_t , and aggregate consumption, C_t , and by two exogenous state variables: the level of technology, A_t , and sunspot shock, ν_t . An equilibrium in this economy is a vector of price sequences: $\{W_t\}_{t=0}^{\infty}$, $\{P_t^z\}_{t=0}^{\infty}$ and the set of policy functions $\{Y_t\}_{t=0}^{\infty}$, $\{C_t\}_{t=0}^{\infty}$, $\{N_t\}_{t=0}^{\infty}$, $\{K_t\}_{t=0}^{\infty}$, $\{I_t\}_{t=0}^{\infty}$, and $\{D_t\}_{t=0}^{\infty}$ such that

1. The first order conditions of the representative household (3), (5), and (6), and of the representative firm (12) and (13) are satisfied together with the usual transversality condition $\lim_{t \rightarrow \infty} \beta^t U_c(C_t, N_t) K_{t+1} = 0$.
2. The labor, good, capital, and financial markets clear. Equilibrium in the financial market requires that investors hold all outstanding equity shares and all other assets are in zero net supply:

If the externality exists ($\eta > 0$), the decentralized equilibrium is not Pareto optimal because the representative firm fails to take the external effect into account while choosing optimal labor and investment.

The equilibrium representation of the decentralized model is given by the following system of equations:

$$(C_t)^{-\xi} = \frac{\Lambda}{(1-\alpha)} \frac{\bar{N}_t}{Y_t} \quad (14)$$

$$\bar{K}_{t+1} = (1-\Omega) \bar{K}_t + I_t \quad (15)$$

$$Y_t = A_t \bar{K}_t^{\alpha(1+\eta)} \bar{N}_t^{(1-\alpha)(1+\eta)} \quad (16)$$

$$Y_t = C_t + I_t \quad (17)$$

$$D_t = Y_t - W_t \bar{N}_t - I_t \quad (18)$$

$$1 = E_t \left[\beta \left(\frac{C_{t+1}}{C_t} \right)^{-\xi} \left\{ \frac{\alpha Y_{t+1}}{\bar{K}_{t+1}} + 1 - \Omega \right\} \right] \quad (19)$$

$$W_t = (1-\alpha) \frac{Y_t}{\bar{N}_t} \quad (20)$$

$$P_t^e = E_t \left[\sum_{j=1}^{\infty} \beta \left(\frac{C_{t+j}}{C_t} \right)^{-\xi} D_{t+j} \right] \quad (21)$$

$$P_t^b = E_t \left[\beta \left(\frac{C_{t+1}}{C_t} \right)^{-\xi} \right] \quad (22)$$

3 Model Solution and Asset Pricing

First, we solve the system of equations (14) – (20) for the approximate dynamics of the macroeconomic variables by log-linearizing the equations around the

unique steady state implied by the above equilibrium conditions as in King, Plosser and Rebelo [23]. The solution of the approximated model can be represented by a loglinear state space system with the vector of state variables, s_t , following a first order autoregressive process with multivariate normal i.i.d. impulses:

$$\begin{aligned} s_{t+1} &= P s_t + Q \epsilon_t \\ k_0, c_0 &\text{ given,} \end{aligned} \tag{23}$$

where the square matrix P governs the dynamics of the system. The small-case letters denote the log deviations of variables from their steady state values. For the economies considered in this research, s_t contains the capital stock, k_t , the consumption, c_t , the level of technology, a_t , and the sunspot, ν_t . Consistency with rational expectations requires for the sunspot to be i.i.d. with

$$E_t [\nu_t] = 0.$$

The vector of the exogenous shocks ϵ_t consists of two variables: the technology shock, ε_t , and the sunspot, ν_t . For asset pricing, we obtain the log of dividends d and the log of the stochastic discount factor, m , as a linear combinations of the state vector.

3.1 Rates of Return

The next step in our solution is to apply the lognormal pricing method developed by Jermann [19], which combines the linearization approach with non-linear asset pricing formulae. The main advantage of this technique over non-linear value function iteration, used for example in Danthine, Donaldson and Mehra [12] and Rouwenhorst [34], is its ability to handle a model with multiple endogenous state variables with ease. This would be a hurdle in purely non-linear methods. On the other hand, our return computations do not impose equal ex-ante returns across securities, as is generally true under pure linearization.

The basic pricing equation requires that the time t price of a claim to a single uncertain future payout (dividend), D_{t+j} , is equal to its expected present value discounted using the stochastic discount factor $M_{t+j} = \beta^j \left(\frac{C_{t+j}}{C_t} \right)^{-\xi}$:

$$P_t(D_{t+j}) = E_t [M_{t+j} D_{t+j}] = \beta^j D_t E_t [\exp \{ \xi (c_t - c_{t+j}) + d_{t+j} - d_t \}] \tag{24}$$

Since M and D are conditionally lognormal, their log-deviations with respect to the steady state values are conditionally normal with joint distribution given by (23). Using the well-known theorem about the expectation of lognormal variables, a closed-form solution for the Euler equation (24) can be written as:

$$P_t(D_{t+j}) = \beta^j D_t \exp \left\{ E_t [\xi (c_t - c_{t+j}) + d_{t+j}] + \frac{1}{2} Var_t [\xi (c_t - c_{t+j}) + d_{t+j}] \right\} \tag{25}$$

It is also possible to obtain closed-form solutions for first and second moments of returns on assets with a single period payout like in equation (24). For example, the return on a one-period bond, which pays one unit of consumption good in every state, i.e. per period risk free rate is given by:

$$R_{t,t+1}^b(1_{t+1}) = \frac{1}{P_t^b(1_{t+1})} = \beta^{-1} \exp\{E_t[\xi(c_t - c_{t+1})] + \frac{1}{2}Var_t[\xi(c_t - c_{t+1})]\} \quad (26)$$

The unconditional mean risk free rate and its variance can be shown to be equal to:

$$E[R_{t,t+1}^b(1_{t+1})] = \beta^{-1} \exp\{\frac{1}{2}Var[E_t[\xi(c_t - c_{t+1})]] - \frac{1}{2}Var[-\xi c_{t+1} + E_t[\xi c_{t+1}]]\} \quad (27)$$

$$Var(R_{t,t+1}^b(1_{t+1})) = \beta^{-2} \exp\{-\xi^2 E[Var_t(c_t - c_{t+1})] + Var(E_t(c_t - c_{t+1}))\} \\ \times (\exp\{Var(E_t(c_t - c_{t+1}))\} - 1) \quad (28)$$

An equity security is the claim to the infinite sequence of dividends, $\{D_{t+j}\}_{j=1}^{\infty}$ and can be regarded as an infinite composite of single strip securities, which are priced according to equation (25). The period gross return to the firm's equity is given by:

$$R_{t,t+1}^e(\{D_{t+j}\}_{j=1}^{\infty}) = \frac{P_{t+1}^e(\{D_{t+j}\}_{j=1}^{\infty}) + D_{t+1}}{P_t^e(\{D_{t+j}\}_{j=1}^{\infty})} \quad (29) \\ = \frac{D_{t+1}}{D_t} \frac{P_{t+1}^e(\{D_{t+j}\}_{j=1}^{\infty}) + 1}{P_t^e(\{D_{t+j}\}_{j=1}^{\infty})} \\ = \frac{\sum_{j=1}^{\infty} \beta^j E_{t+1}[\exp\{\xi(c_t - c_{t+j}) + d_{t+j+1}\}] + \exp\{d_{t+1}\}}{\sum_{j=1}^{\infty} \beta^j E_t[\exp\{\xi(c_t - c_{t+j}) + d_{t+j}\}]}$$

We can compute stock returns from the model linear solution, but to calculate the unconditional expectation and variance of the return on equity, it is necessary to use simulations. The detailed derivation of the financial asset returns is presented in the Appendix A.

3.2 Equity Premium and Volatility Puzzles in DSGE Models

The average excess stock return $E[R^e - R^b]$ in the U.S. data is almost 8 percent. The average excess stock return justified in the context of the standard DSGE model as a reward for bearing risk is close to zero. Hereby lies the equity

premium puzzle. To understand the determinants of the equity premium using DSGE framework, we re-write asset pricing Euler equations (22) and (21) in terms of returns:

$$1 = E_t [(1 + R_{t,t+1}^e) M_{t+1}] \quad (30)$$

$$1 + R_{t,t+1}^b = \frac{1}{E_t [M_{t+1}]} \quad (31)$$

The following discussion closely follows Campbell [8]. Denote the log gross returns on stocks and the risk free asset by $r_{t,t+1}^e = \ln(1 + R_{t,t+1}^e)$ and $r_{t,t+1}^b = \ln(1 + R_{t,t+1}^b)$ and the log of stochastic discount factor by $m_t = \ln(M_{t+1}) = \log \beta - \xi(c_{t+1} - c_t)$. Since the marginal rate of substitution and asset returns are jointly log-normally distributed and homoscedastic we can re-write the equations (30) and (31) in terms of logs⁸:

$$r_{t,t+1}^b = -E_t [m_{t+1}] - \frac{Var(m)}{2}$$

$$0 = E_t [r_{t,t+1}^e] + E_t [m_{t+1}] + \frac{1}{2} (Var(r^e) + Var(m) + Cov(r^e, m))$$

$$E_t [r_{t,t+1}^e - r_{t,t+1}^b] + \frac{Var(r^e)}{2} = -Cov(r^e, m) = -\xi Cov(r^e, \Delta c) \quad (32)$$

where Δc denotes log consumption growth. Equation (32) states that the log of the expected risk premium, adjusted for Jensen's inequality, is equal to the negative of the covariance between log return on equity and log consumption growth. Intuitively the asset, whose return positively co-varies with the consumption growth pays in "good" times when consumption level is high and the marginal utility of additional consumption is low. Such assets require high returns to induce investors to hold them. Alternatively, the assets, which pay in the "bad" states are very desirable and command low returns because they allow risk-averse agents to smooth their consumption patterns.

The equity premium results can be also examined in the framework provided by the work of Hansen and Jagannathan [17]. They show that the unconditional version of the first order condition for excess return on equity $E [M_{t+1} (R_{t,t+1}^e - R_{t,t+1}^b)] = 0$ implies the following restriction for the Sharpe ratio of any asset's excess return:

$$\frac{E [R_{t,t+1}^e - R_{t,t+1}^b]}{\sigma [R_{t,t+1}^e]} = -\rho (M_{t+1}, R_{t,t+1}^e - R_{t,t+1}^b) \frac{\sigma [M_{t+1}]}{E [M_{t+1}]} \leq \frac{\sigma [M_{t+1}]}{E [M_{t+1}]} \quad (33)$$

⁸If variable X is conditionally log-normal $\ln E_t [X] = E_t [\ln X] + \frac{1}{2} Var_t [\ln X]$ with $Var_t (\ln X) = Var (\ln X)$ if X is homoscedastic.

where ρ denotes the unconditional correlation between two variables, which cannot be higher than 1 in absolute value. The inequality in the expression (33) is the Hansen-Jagannathan lower bound (HJB) on the pricing kernel. In equation(33), $E[M_{t+1}]$ is the expected value of the price of a one-period risk-free discount bond and should be close to 1, which means that $\sigma[M_{t+1}]$ should be around 0.5 in annual terms.

The equity premium puzzle is related to the volatility puzzle because the standard deviation of stock returns also depends on the volatility of consumption growth in addition to the volatility of dividend growth. Campbell [8] shows that the standard deviation of dividends needs to be counterfactually high to achieve 16 percent standard deviation of stock returns found in U.S. data.

4 Quantitative Results

The main purpose of the quantitative evaluation presented in this section is to examine the potential of the indeterminate BFG model with respect to its ability to explain the historic equity premium, the average risk free rate and volatility of financial asset returns, while maintaining its adequate representation of stylized real business cycle facts and contrast its performance with the neoclassical RBC model of Hansen [16]. As a Benchmark case, we explore the BFG model with two sources of uncertainty: the non-fundamental belief shock and the productivity (technology) shock. As we have noted earlier, the BFG model and Hansen’s model differ by the degree of aggregate returns in production and by the nature of driving processes. We would like to disentangle the impact of increasing returns on asset pricing from the impact of the sunspot shock. To achieve this goal we consider several differently parameterized versions of BFG model, driven by belief shocks alone. Then we turn our attention to models with increasing returns, but driven solely by the technology shocks.

4.1 Calibration

With regards to calibration, parameters of the model can be divided into two groups. Parameters in the first group describe the long-run behavior of the model economy and are assigned values corresponding to their point estimates, obtained from the National Income and Product Accounts (NIPA) data, a standard in the RBC and indeterminacy literature. For the second parameter group, which relates to shock processes, precise a priori knowledge of their values is unavailable. We calibrate these parameters to maximize the model’s ability to replicate certain business cycle and financial moments, such as the volatility of output and consumption growth. In addition, we present a detailed sensitivity analysis, which provides some insights about the model’s mechanism at work.

Within the first group of parameters the capital’s share of output α is 0.3 and the quarterly depreciation rate Ω is 0.025. The subjective discount factor, β , is 0.99, corresponding to a steady state risk free rate of return of 4% per year. We choose parameter Λ to yield steady state work time of the representative

household equal to $1/3$ of its time endowment. In the Benchmark parameterization Λ is set to 2.867^9 . In the Benchmark case, the relative risk aversion (RRA) coefficient ξ is 1. Since RRA coefficient is a critical parameter for asset pricing, we also present cases, where the RRA coefficient is equal to 5. All of these values are line with the empirical estimates and the values commonly used by the literature in this field: see for instance Hansen [16], Mehra and Prescott [28], Juster and Stafford [20], Poterba [32], Jermann [19], Boldrin, Christiano and Fisher [5], King and Rebelo [24].

The degree of increasing returns to scale (IRS) in the model economy is given by $1 + \eta$. Despite the numerous attempts to estimate the level of returns-to-scale in the data, there is no broad agreement in the literature on its value. Cole and Ohanian [9] note that the estimates of IRS are imprecise even with the restrictions put on shock processes. Basu and Fernald [2] and Laitner and Stolyarov [26] show that returns-to-scale estimates reported vary dramatically depending on the type of data used, the level of aggregation, and the estimation method. In attempting to account for the wide range of estimates, Basu and Fernald [2] demonstrate that while the average U.S. industry exhibits approximately constant returns-to-scale, the aggregate private business economy appears to exhibit large increasing returns. The largest aggregate estimate they obtain is 1.72 (standard error equal to 0.36). However, when the aggregate returns-to-scale estimation procedure is corrected to account for reallocation of inputs across industries, the difference between the aggregate and industry returns-to-scale estimates shrinks. The largest corrected aggregate estimate is 1.03 (standard error 0.18). Despite these findings, in Section V, Basu and Fernald [2] suggest that uncorrected aggregate estimates might be more appropriate for calibration of one-sector models, which abstract from heterogeneity in production. This argument is very helpful for the proposed research because the one-sector BFG model requires the minimum externality of about 0.55 to support sunspot equilibria. For the Benchmark parameterization we choose η equal to 0.6. To gain a more detailed understanding of the model's responses to the change in the magnitude of the externality parameter, we present results for several different values of η .

Estimations of the Solow residuals, typically yield a highly persistent AR(1) process (see Prescott [33] for details). For the AR(1) process describing A_t , we choose the value of the persistence parameter, ρ , equal to 0.95. When the model economy is driven by the sunspot shocks alone or technology shocks, we set the variance of the shocks to match the standard deviation of output in the US data (1.82%).

In the Benchmark case we consider the model economy with sunspot and technology shocks simultaneously. Following Perli [31], we make the two shocks highly correlated with the correlation coefficient of 0.9^{10} . Again, we choose the total variance of both shocks to match the volatility of the US output. Since

⁹In other simulated cases we choose Λ in the same fashion.

¹⁰We considered a wide range of correlation coefficients between two shocks and found out that this parameter does not have any significant effect on the asset pricing implications of the studied economies.

there is no obvious way to estimate the variance of these shocks individually, we choose these parameters to maximize the return on equity and minimize the return on the risk free asset.

The results of the calibration exercise for the Benchmark case is presented in Table 1:

Table 1: Parameter Choices for the Benchmark Model

Parameter		Value
Subjective discount factor	β	0.99
Capital's share of output	α	0.3
Quarterly capital depreciation rate	Ω	0.025
Coefficient of relative risk aversion	ξ	1
Leisure parameter in the utility function	Λ	2.867
Level of increasing returns	η	0.6
Persistence of technology process	ρ	0.95
Correlation between technology and sunspot shocks	$\rho_{\varepsilon\nu}$	0.9
Standard deviation of the technology shock	σ_{ε}	0.00385
Standard deviation of the sunspot shock	σ_{ν}	0.0025

4.2 Financial Implications of the Benchmark Model

Table 2 presents main results. The information is provided in two panels. In Panel A the standard deviations of log output and consumption are in annual terms, while all other quantities are annualized. The first column displays the point estimates of moments of the U.S. data accompanied by their standard errors, the second column presents the corresponding statistics obtained from Hansen's [16] indivisible labor model driven by the technology shocks and the third column shows results from the BFG model with two exogenous shocks: the innovation to the technology level and sunspot. As discussed in the Calibration Section, we match the standard deviation of output to U.S. data. In Panel A we also present macroeconomic moments, which are essential for asset pricing. In both models consumption is less variable than in the data. In fact the standard deviation of consumption is less in the BFG model than in the standard determinate setting. The same holds for the volatility of the consumption growth: it is the least volatile in the BFG model. Both models display high standard deviation of the dividend growth rate found in the data.

Panel B presents financial results. The return on equity in the BFG model is 4.04 percent, which is much lower than the 7.87 percent in U.S. data and almost identical to the 4.039 percent return on the stock produced by Hansen's model. The risk free rate is 4.039 percent in comparison to the 1.55 percent risk free rate in the data. The risk free rate in Hansen's model is 4.038 percent. In

both models the equity premium is close to zero.

The reason for this failure is evident from equation (32), which states that the log risk premium is determined by the product of the RRA coefficient (ξ) and the covariance of the log consumption growth and log return on equity ($Cov(r^e, \Delta c) = Corr(r^e, \Delta c) \sigma_{\Delta c} \sigma_{R^e}$). In the data, covariance between two variables is 3.41 percent and even with this value, very high RRA coefficients are necessary to match the equity premium. In the model with indeterminacy, covariance between logs of consumption growth and stock returns is 0.057 percent, which is an order of magnitude lower than in the data and lower than in Hansen's model. The volatility of equity returns is slightly higher in the BFG model but it is a very minor improvement over the determinate model and clearly insufficient to increase the value of $Cov(r^e, \Delta c)$.

The inspection of the market price of risk – the ratio $\frac{\sigma_M}{E[M]}$ – confirms the severity of the equity premium puzzle. The market price of risk in the data, implied by the HJ bound, is at least 0.53, meaning that the standard deviation of the stochastic discount factor (σ_M) should be at least 50 percent annually. The corresponding quantity in the BFG model is 0.0033, which is an order of magnitude lower than in the data and almost identical to the ratio in Hansen's model. Low values of the market price of risk ratios result from the smooth pricing kernel

Our results clearly show that one-sector DSGE model, modified to include IRS in production, sufficient for indeterminacy, and the sunspot shock, does not explain stylized financial facts of the U.S. data any better than the standard RBC model of Hansen. In the next section we would like to inspect the robustness of the asset pricing implications of the BFG model with respect to changes in several key features and parameters. In particular, we examine asset pricing in the BFG framework when the sunspot shock is the only source of economic fluctuations. Next, we preserve increasing returns but remove the sunspot, in which case the model economy is driven only by technology shocks. Finally, we examine the changes in the financial performance of the model when the level of increasing returns is gradually reduced.

4.3 Robustness of Asset Pricing Results

Table 3 displays results from the several parameterization of the BFG model with only one extrinsic shock: the sunspot. For easy comparison the first column lists corresponding statistics from the Benchmark economy already discussed in Section 4.2. In Panel A, we note that in all considered "sunspot-only" cases, the standard deviation of consumption growth is lower than in the Benchmark. Among the sunspot driven economies, the economy with high relative risk aversion ($\xi = 5$) has the least volatile consumption growth (and consequently the least volatile pricing kernel). It is not surprising because more risk averse agents strive to achieve smoother consumption patterns. The increase in the level of returns to scale (η) from 0.6 to 0.72 (the value used in Farmer and Guo [14]) increases the volatility of consumption growth from 0.34 to 0.51. These effects are quite minor.

All sunspot-only models produce very similar asset pricing results: near-zero equity premium and very smooth asset returns. The economy with high relative risk aversion actually has a negative excess return on stocks because the negative effect of the reduction in the standard deviation of stochastic discount factor and the volatility of the equity return on premium outweighs the positive effect of the increase in RRA coefficient (see equation (32)). Therefore increasing the RRA coefficient within acceptable range of values does not help to solve the puzzle in the BFG model. Comparison of statistics from sunspot-only economies to the Benchmark shows slight deterioration in asset pricing results which is due to the reduction in the variation of the pricing kernel and the standard deviation of the return on stocks. Without a persistent technology shock, successive i.i.d. sunspot shocks cancel each other resulting in the lower covariance between the stochastic discount factor and stock returns.

In Table 4 we collect statistics from the economies driven by technology shocks only. Again, for the purposes of comparison we reproduce the corresponding quantities from the Benchmark economy. We gradually reduce the level of increasing returns from 0.72 to 0. The comparison of numbers in columns one and two indicates that the removal of the sunspot shock from the Benchmark economy has negligible effect on asset pricing. The slightly higher volatility of consumptions growth is offset by the less volatile dividend growth. All financial moments are almost identical in both cases. In columns three and four we reduce the size of externality to zero (Hansen’s model). As presented, these statistics indicate the level of increasing returns has almost no effect on the financial asset returns and their volatilities in the technology shock driven models.

5 Conclusion

We investigate the pricing of financial assets in the context of the one-sector indeterminate Benhabib-Farmer-Guo model with increasing returns to scale in productions and sunspot shocks and compare the asset pricing results from the models with indeterminacy with results obtained in Hansen’s [16] model, a standard in the RBC literature. Two formulations differ in the degree of increasing returns, the nature of shocks and the shock propagation mechanism. Our goal is to understand the impact of the sunspot shock and indeterminacy on asset pricing implications of the otherwise standard DSGE consumption-based asset pricing model. Our principal conclusion is that indeterminacy and the sunspot shock have almost no effect on the financial performance of the one-sector production model. We also found that the level of increasing returns does not influence financial statistics in any significant way. We show that neither the introduction of the sunspot shock nor the higher level of returns-to-scale in production increase the volatility of the stochastic discount factor and the standard deviation of the return on equity, which is necessary to account for the equity premium.

Moreover, it is not clear which technology would allow to increase the standard deviation of the pricing kernel and the return on equity and preserve inde-

terminacy in the model at the same time. In the previous asset pricing studies in the standard DSGE framework without indeterminacy (for example Boldrin, Christiano and Fisher [5], Jermann [19] and Avalos [1]) improvement in the asset pricing performance followed from the introduction of habit formation into the agent's preferences in combination with costs of adjustments in capital stock or other similar mechanisms, which prevented easy factor adjustments in response to shock. The combination of habit persistence and capital adjustment costs resolves the asset pricing puzzles in production economies because the agents whose preferences display habit persistence are very risk-averse locally and are eager to avoid fluctuations in their consumption. With frictions such as adjustment costs, the equity security becomes an unattractive instrument for consumption smoothing relative to the risk-free asset. As a result, agents require a higher return for holding equity and accept a lower return on bonds. On the other hand, adjustment costs prevent the instantaneous response of the capital stock to exogenous shocks and therefore increase the volatility of the return on equity. Similar mechanisms are not consistent with indeterminacy because full mobility of factors of production is needed for agents to act on their beliefs.

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A Derivation of Unconditional Moments of Returns on Financial Securities in Log-linear Log-normal Environment

The dynamics of the endogenous state variables is given by equation (23). The vector of endogenous variables e_t as a linear combination of the state vector s_t and the vector of innovations ϵ_t :

$$e_t = T s_t + S \epsilon_t$$

The risk-free rate is a return on a one-period bond with a sure payout of one unit of the consumption good. The price of the riskless bond is given by:

$$\begin{aligned} P_t^b(1_{t+1}) &= E_t [M_{t+1} 1_{t+1}] \\ &= \beta e^{\{E_t[\xi(c_t - c_{t+1})] + \frac{1}{2} \text{Var}_t[\xi(c_t - c_{t+1})]\}} \end{aligned}$$

The above expression is derived using a formula for the expectation of a log-normal variable.

The risk-free rate of return is:

$$\begin{aligned} R_{t,t+1}^b &= \frac{1}{P_t^b(1_{t+1})} \\ &= \beta^{-1} e^{\{-E_t[\xi(c_t - c_{t+1})] - \frac{1}{2} \text{Var}_t[\xi(c_t - c_{t+1})]\}} \end{aligned}$$

Utilizing the fact that conditional variances are constant and using the law of iterated expectations, we can derive an unconditional expectation of the risk-free rate:

$$\begin{aligned} E[R_{t,t+1}^b] &= \beta^{-1} e^{\{-E[E_t[\xi(c_t - c_{t+1})] + \frac{1}{2} \text{Var}_t[\xi(c_t - c_{t+1})]] + \frac{1}{2} \text{Var}[E_t[\xi(c_t - c_{t+1})]]\}} \\ &= \beta^{-1} e^{\{\frac{1}{2} \text{Var}[E_t[\xi(c_t - c_{t+1})]] - \frac{1}{2} \text{Var}[\xi(c_t - c_{t+1}) - E_t[\xi(c_t - c_{t+1})]]\}} \end{aligned}$$

Both variances in the above expression have closed form solutions as they are functions of consumption c_t .

In the indeterminacy case consumption is the element of the state vector s_t . Let P_c and Q_c be row vectors of the matrices P and Q respectively, which correspond to consumption. The variance terms are:

$$\text{Var}[E_t[\xi(c_t - c_{t+1})]] = \xi^2 P_c (P - I) \text{Var}(s_t) (P - I)' P_c' + \xi^2 (P_c Q - Q_c) \Sigma (P_c Q - Q_c)'$$

and

$$\text{Var}[\xi(c_t - c_{t+1}) - E_t[\xi(c_t - c_{t+1})]] = \xi^2 Q_c \Sigma Q_c'$$

In the determinacy case, c_t is a element of the vector e_t . Let T_c and S_c be the row vectors of matrices T and S , corresponding to consumption. Then the variance terms in equation (??) are given by:

$$Var[E_t[\xi(c_t - c_{t+1})]] = \xi^2 T_c (P - I) Var(s_t) (P - I)' T_c' + \xi^2 (T_c W - S_c) \Sigma (T_c W - S_c)'$$

and

$$Var[\xi(c_t - c_{t+1}) - E_t[\xi(c_t - c_{t+1})]] = \xi^2 S_c \Sigma S_c'$$

To calculate the unconditional variance of the risk-free rate, we use the formula for the variance of log-normal distribution. Suppose

$$\log(Z) \sim N(\mu, \sigma^2)$$

then the variance of log-normal variable Z is given by:

$$Var(Z) = e^{2\mu + \sigma^2} (e^{\sigma^2} - 1)$$

Using the equation (26) we compute the unconditional expectation of $\log(R_{t,t+1}^b)$

$$E[\log(R_{t,t+1}^b)] = -\log \beta - \frac{1}{2} \xi^2 E[Var_t(c_t - c_{t+1})]$$

The unconditional variance of $\log(R_{t,t+1}^b)$ is equal to:

$$Var(\log(R_{t,t+1}^b)) = Var(E_t \xi(c_t - c_{t+1}))$$

$$Var(R_{t,t+1}^b) = \beta^{-2} e^{-\xi^2 E[Var_t(\xi c_t - \xi c_{t+1})] + Var(E_t(\xi c_t - \xi c_{t+1}))} (e^{Var(E_t \xi(c_t - c_{t+1}))} - 1)$$

A.1 Unconditional Expectation and Variance of the Rate of Return on a Stock

Stocks constitute a claim to an infinite sequence of uncertain dividends paid off by a firm and can be priced as a collection of single future payoffs:

$$\begin{aligned} P_t^e \left(\{D_{t+j}\}_{j=1}^{\infty} \right) &= \sum_{j=1}^{\infty} P_t^e(D_{t+j}) = \sum_{j=1}^{\infty} E_t[M_{t+j} D_{t+j}] \\ &= \sum_{j=1}^{\infty} \beta^j E_t \left[e^{\xi(c_t - c_{t+j}) + d_{t+j}} \right] \end{aligned}$$

The one period return on the stock is:

$$\begin{aligned}
R_{t,t+1}^e \left(\{D_{t+j}\}_{j=1}^\infty \right) &= \frac{P_{t+1}^e \left(\{D_{t+j}\}_{j=2}^\infty \right) + D_{t+1}}{P_t^e \left(\{D_{t+j}\}_{j=1}^\infty \right)} \\
&= \frac{\sum_{j=1}^\infty \beta^j E_{t+1} \left[e^{\xi(c_t - c_{t+j}) + d_{t+j+1}} \right] + e^{d_{t+1}}}{\sum_{j=1}^\infty \beta^j E_t \left[e^{\xi(c_t - c_{t+j}) + d_{t+j}} \right]}
\end{aligned} \tag{34}$$

Stock returns can be computed from the model linear solution, but we have to simulate the model to find their unconditional moments. We will use an expectation property of log-normal distribution for simulations:

$$E_t \left[e^{\xi(c_t - c_{t+j}) + d_{t+j}} \right] = e^{E_t(\xi(c_t - c_{t+j}) + d_{t+j}) + \frac{1}{2} \text{Var}_t(\xi(c_t - c_{t+j}) + d_{t+j})} \tag{35}$$

In the indeterminacy case the expectation term in (35) is given by:

$$E_t (\xi(c_t - c_{t+j}) + d_{t+j}) = [\xi P_c + (T_d - \xi P_c) P^j] s_t + [\xi Q_c + (T_d - \xi P_c) P^{j-1} Q] \epsilon_t$$

and the variance term is:

$$\text{Var}_t (\xi(c_t - c_{t+j}) + d_{t+j}) = \sum_{s=1}^{j-2} (T_d - \xi P_c) P^s Q \Sigma Q' P^{s'} (T_d - \xi P_c)' + (S_d - \xi Q_c) \Sigma (S_d - \xi Q_c)'$$

In the determinacy case the expectation and variance terms in (35) are as follows:

$$E_t (\xi(c_t - c_{t+j}) + d_{t+j}) = [\xi T_c + (T_d - \xi T_c) P^j] s_t + [\xi S_c + (T_d - \xi T_c) P^{j-1} Q] \epsilon_t$$

and

$$\text{Var}_t (\xi(c_t - c_{t+j}) + d_{t+j}) = \sum_{s=1}^{j-2} (T_d - \xi T_c) P^s Q \Sigma Q' P^{s'} (T_d - \xi T_c)' + (S_d - \xi S_c) \Sigma (S_d - \xi S_c)'$$

Table 2: Quantitative Results for the Benchmark Model

	U.S. data	Hansen's Model	Benchmark Model
A. Select Business Cycle Moments			
σ_y	1.82	1.82	1.82
σ_c	0.87	0.57	0.447
$\sigma_{\Delta c}$	1.08	0.76	0.66
$\sigma_{\Delta d}$	28	16.87	20.16
B. Financial Moments			
$E[R^e]$	7.87 (2.33)	4.039	4.04
σ_{R^e}	15.77 (0.13)	.2527	0.32
$E[R^b]$	1.55 (0.19)	4.038	4.039
σ_{R^b}	2.56 (0.002)	0.2017	0.2766
$E[R^e - R^b]$	6.34 (2.28)	0.001	0.001
$\sigma_M/E[M]$	≥ 0.53	0.0038	.0033
$Corr(r^e, \Delta c)$	0.2	0.59	0.27
$Cov(r^e, \Delta c)$	3.41	0.11	0.057

Notation: variables y , c and d denote log-deviations of the Hodrick-Prescott filtered series of output, consumption and dividends respectively; Δc and Δd are the logs of consumption growth and dividend growth and $\sigma_{\Delta c}$ and $\sigma_{\Delta d}$ are their annualized quarterly standard deviations. R^e and R^b denote the return on equity and on the one-quarter risk free bond; r^e denotes the log return on the stock. All financial statistics are reported in the annualized percentage points.

Panel A column U.S. data exhibits empirical moments computed using NIPA's quarterly information provided by DRI Database from 1947/Q1 to 2002/Q4. Panel B column U.S. data presents annualized quarterly return moments from CRSP for the same period. Stock returns correspond to returns on the NYSE value-weighted index and the risk free rate corresponds to zero-coupon yields data for 3-month T-bills. Standard errors are reported in parentheses.

Table 3: Quantitative Results for the BFG Model with Sunspot Shocks Only

	Benchmark Model	BFG with Sunspot Shocks Only		
		$\xi = 1, \eta = 0.6$	$\xi = 1, \eta = 0.72$	$\xi = 5, \eta = 0.6$
σ_ν	0.0025	0.004	0.0024	0.0003
A. Select Business Cycle Moments				
σ_y	1.82	1.82	1.82	1.82
σ_c	0.447	0.31	0.42	.06
$\sigma_{\Delta c}$	0.66	0.34	0.51	0.07
$\sigma_{\Delta d}$	20.16	23.04	19.32	24.58
B. Financial Moments				
$E[R^e]$	4.04	4.0413	4.0423	4.0392
σ_{R^e}	0.32	.1794	0.2342	0.182
$E[R^b]$	4.039	4.04	4.0394	4.04
σ_{R^b}	0.2766	0.1306	0.161	0.134
$E[R^e - R^b]$	0.001	0.0013	0.0029	-0.0008
$\sigma_M/E[M]$	0.0033	0.0017	0.0026	.0016
$Corr(r^e, \Delta c)$	0.27	0.81	0.66	0.79
$Cov(r^e, \Delta c)$	0.057	0.045	.079	0.01

Notation: variables y , c and d denote log-deviations of the Hodrick-Prescott filtered series of output, consumption and dividends respectively; Δc and Δd are the logs of consumption growth and dividend growth and $\sigma_{\Delta c}$ and $\sigma_{\Delta d}$ are their annualized quarterly standard deviations. R^e and R^b denote the return on equity and on the one-quarter risk free bond; r^e denotes the log return on the stock. All financial statistics are reported in the annualized percentage points.

Table 4: Effect of Increasing Returns. Models with Technology Shocks Only

	Benchmark Model	Models with Technology Shocks Only		
		$\eta = 0.6$	$\eta = 0.15$	$\eta = 0$ (Hansen)
σ_ε	0.0038	0.004	0.0057	0.007
A. Select Business Cycle Moments				
σ_y	1.82	1.82	1.82	1.82
σ_c	0.447	0.66	0.56	0.57
$\sigma_{\Delta c}$	0.66	0.9	0.75	0.76
$\sigma_{\Delta d}$	20.16	7.78	17.09	16.87
B. Financial Moments				
$E[R^e]$	4.04	4.036	4.036	4.039
σ_{R^e}	0.32	0.36	0.2509	0.2527
$E[R^b]$	4.039	4.034	4.035	4.038
σ_{R^b}	0.2766	0.3	0.2	0.2017
$E[R^e - R^b]$	0.001	0.002	0.001	0.001
$\sigma_M/E[M]$	0.0033	0.004	0.0038	0.0038
$Corr(r^e, \Delta c)$	0.27	0.39	0.59	0.59
$Cov(r^e, \Delta c)$	0.057	0.13	0.11	0.11

Notation: variables y , c and d denote log-deviations of the Hodrick-Prescott filtered series of output, consumption and dividends respectively; Δc and Δd are the logs of consumption growth and dividend growth and $\sigma_{\Delta c}$ and $\sigma_{\Delta d}$ are their annualized quarterly standard deviations. R^e and R^b denote the return on equity and on the one-quarter risk free bond; r^e denotes the log return on the stock. All financial statistics are reported in the annualized percentage points.