A Note on Skewness in The Stochastic Volatility Models

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Abstract

In this paper we propose a different point of view on the effect of the stochastic volatility on the skewness of the density of the underlying asset when the underlying price is not correlated with its volatility. Heston (1993) established that when the state variables are not correlated, stochastic volatility changes only the kurtosis of underlying asset density. We demonstrate that stochastic volatility always affects the higher moments of underlying asset distribution even if the price is not correlated with its volatility. Also, we show that when the state variables are not correlated, the risk-neutral probabilities are symmetrical.

Keywords: Skewness, Kurtosis, Option pricing, Stochastic volatility, Closed-form formula.

JEL Classification: G10, G12, G13

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1 Introduction

The most important models of options pricing with stochastic volatility are those proposed by Hull and White (1987), Stein and Stein (1991) and Heston (1993). Bates (1996 and 2000) and Pan (2002) extended the jump-diffusion model as to incorporate stochastic volatility in order to explain the structure of option prices while Bakshi *et al.* (1997 and 2000) developed option pricing models that simultaneously admit stochastic volatility, stochastic interest rate and random jump. Hull and White proved that the option price with stochastic volatility is the price of Black and Scholes integrated over the probability distribution of the average of future volatilities during the life of the option. Heston, Bates or Bakshi, Cao and Chen use the characteristic function of risk-neutral probabilities of final prices of the underlying asset.

We consider an option pricing evaluation model with two state variables. Thus, the price of the European call option depends on the price of the underlying asset and on its volatility. In the Heston model, in a risk-neutral world these two state variables verify the stochastic differential equations as follows:

$$dS_t = rS_t dt + \sigma_t S_t dw \tag{1}$$

$$d\sigma_t^2 = k\left(\theta - \sigma_t^2\right) dt + \sigma_v \sigma_t dz \tag{2}$$

where S represents the underlying nondividend-paying stock price, r represents the risk-free interest rate and σ represents the volatility. The Brownian motions w and z are correlated ($dw_t \cdot dz_t = \rho dt$) and the coefficient of correlation is ρ ($-1 < \rho < 1$). k represents the speed of adjustment of the volatility, θ represents the long-run mean of the volatility and σ_v represents the volatility of the volatility.

In a risk-neutral world, the option price formula with stochastic volatility is analogous to the Black-Scholes formula:

$$C = SP_1 - Ke^{-r\tau} P_2 \tag{3}$$

where S is the present value of the underlying asset and K is the strike price. P_1 and P_2 are the risk-neutral probabilities that the log-price of underlying asset is greater than $\ln K$.

Heston obtains the characteristic function of risk-neutral probabilities using the Fokker-Planck forward equation. When the underlying asset price is not correlated with the volatility ($\rho = 0$), these characteristic functions are defined by:

$$f_j(\alpha) = \exp\left(C + D\sigma^2 + i\alpha \ln S\right) \tag{4}$$

where

$$C = i\alpha r\tau + \frac{a}{\sigma_v^2}\left\{(b_j + \gamma)\tau - 2\ln\left[\frac{1 - \delta\exp(\gamma\tau)}{1 - \delta}\right]\right\}$$

$$D = \frac{b_j + \gamma}{\sigma_v^2}\left[\frac{1 - \exp(\gamma\tau)}{1 - \delta\exp(\gamma\tau)}\right]$$

$$\delta = \frac{b_j + \gamma}{b_j - \gamma}$$

$$\gamma = \sqrt{b_j^2 - \sigma_v^2\left(2u_j\alpha i - \alpha^2\right)}$$

where $j = 1, 2$, $u_1 = 1/2$, $u_2 = -1/2$, $a = k\theta$, $b_1 = b_2 = b = k + \lambda$ and λ is the market price of volatility risk.

Using this Heston definition of the characteristic function of the risk-neutral probabilities, we show that the probability densities are asymmetrical. The paper is organized as follows. In section 2, we use the characteristic functions to obtain a relation between the risk-neutral probabilities. In section 3, we present a new closed-form formula of the option price with stochastic volatility when the state variables are not correlated. Section 4 shows that the stochastic volatility always affects the skewness of the probability density of the underlying asset even if the stock price is not correlated with the volatility. Section 5 summarizes and concludes.

2 The Characteristic Function

In the expression of the characteristic functions of the risk-neutral probability density, the value of γ equals $\sqrt{b^2 + \sigma_v^2\left(\alpha^2 - \alpha i\right)}$ if $j = 1$ and equals $\sqrt{b^2 + \sigma_v^2\left(\alpha^2 + \alpha i\right)}$ if $j = 2$. Using the hyperbolic sine and hyperbolic cosine definitions, the expression (4) of the characteristic function can be written in a very simple form:

$$f_1(\alpha) = e^{i\alpha(\ln S + r\tau)} e^{\frac{ab\tau}{\sigma_v^2}} \left(\frac{1}{\cosh\left(\frac{\gamma_1 \tau}{2}\right) + \frac{b}{\gamma_1} \sinh\left(\frac{\gamma_1 \tau}{2}\right)} \right)^{\frac{2a}{\sigma_v^2}} \exp\left[-\sigma^2 \frac{\alpha^2 - i\alpha}{b + \gamma_1 \coth\left(\frac{\gamma_1 \tau}{2}\right)} \right] \quad (5)$$

$$f_2(\alpha) = e^{i\alpha(\ln S + r\tau)} e^{\frac{ab\tau}{\sigma_v^2}} \left(\frac{1}{\cosh\left(\frac{\gamma_2 \tau}{2}\right) + \frac{b}{\gamma_2} \sinh\left(\frac{\gamma_2 \tau}{2}\right)} \right)^{\frac{2a}{\sigma_v^2}} \exp\left[-\sigma^2 \frac{\alpha^2 + i\alpha}{b + \gamma_2 \coth\left(\frac{\gamma_2 \tau}{2}\right)} \right] \quad (6)$$

where:

$$\gamma_1 = \sqrt{b^2 + \sigma_v^2 (\alpha^2 - i\alpha)} \text{ and } \gamma_2 = \sqrt{b^2 + \sigma_v^2 (\alpha^2 + i\alpha)} \quad (7)$$

From now on we can use the Gil-Pelaez inversion theorem[1] in order to obtain the risk-neutral probabilities, $P_1(x < d)$ and $P_2(x < d)$, that the return of underlying asset is lower than threshold d. Hence,

$$P_1(x < d) = \frac{1}{2} + \frac{1}{2\pi} \int_0^\infty \frac{e^{i\alpha d} \phi_1(-\alpha) - e^{-i\alpha d} \phi_1(\alpha)}{i\alpha} d\alpha \quad (8)$$

$$P_2(x < d) = \frac{1}{2} + \frac{1}{2\pi} \int_0^\infty \frac{e^{i\alpha d} \phi_2(-\alpha) - e^{-i\alpha d} \phi_2(\alpha)}{i\alpha} d\alpha \quad (9)$$

where the characteristic functions, ϕ_1 and ϕ_2, and d are defined by:

$$\phi_1(\alpha) = e^{\frac{ab\tau}{\sigma_v^2}} \left(\frac{1}{\cosh\left(\frac{\gamma_1 \tau}{2}\right) + \frac{b}{\gamma_1} \sinh\left(\frac{\gamma_1 \tau}{2}\right)} \right)^{\frac{2a}{\sigma_v^2}} \exp\left[-\sigma^2 \frac{\alpha^2 - i\alpha}{b + \gamma_1 \coth\left(\frac{\gamma_1 \tau}{2}\right)} \right] \quad (10)$$

$$\phi_2(\alpha) = e^{\frac{ab\tau}{\sigma_v^2}} \left(\frac{1}{\cosh\left(\frac{\gamma_2 \tau}{2}\right) + \frac{b}{\gamma_2} \sinh\left(\frac{\gamma_2 \tau}{2}\right)} \right)^{\frac{2a}{\sigma_v^2}} \exp\left[-\sigma^2 \frac{\alpha^2 + i\alpha}{b + \gamma_2 \coth\left(\frac{\gamma_2 \tau}{2}\right)} \right] \quad (11)$$

$$d = \ln \frac{K \exp(-r\tau)}{S} \quad (12)$$

One of the properties of a characteristic function stipulates that $\phi_1(\alpha)$ and $\phi_1(-\alpha)$ and, respectively $\phi_2(\alpha)$ and $\phi_2(-\alpha)$ are complex conjugate[2]. Because of characteristic functions symmetry we note that the quantities $\phi_1(\alpha)$ and $\phi_2(\alpha)$ are complex conjugate:

$$\phi_1(-\alpha) = \phi_2(\alpha) \text{ and } \phi_1(\alpha) = \overline{\phi_2(\alpha)} \quad (13)$$

[1] See Gil-Pelaez (1951), "Note on the inversion theorem", Biometrika, 38, page 481-2.

[2] $\phi_1(\alpha) = \overline{\phi_1(-\alpha)}$ and $\phi_2(\alpha) = \overline{\phi_2(-\alpha)}$.

$$\phi_1(\alpha) = \phi_2(-\alpha) \text{ and } \overline{\phi_1(\alpha)} = \phi_2(\alpha) \tag{14}$$

and that involves

$$\int_0^\infty \frac{e^{i\alpha d}\phi_2(\alpha) - e^{-i\alpha d}\phi_2(-\alpha)}{i\alpha} d\alpha = \int_0^\infty \frac{e^{i\alpha d}\phi_1(-\alpha) - e^{-i\alpha d}\phi_1(\alpha)}{i\alpha} d\alpha \tag{15}$$

and

$$P_1(x < d) = \frac{1}{2} + \frac{1}{2\pi}\int_0^\infty \frac{e^{i\alpha d}\phi_2(\alpha) - e^{-i\alpha d}\phi_2(-\alpha)}{i\alpha} d\alpha \tag{16}$$

2.1 The Relation between Risk-neutral Probabilities

The fact that the first risk-neutral probability can be written as depending on the second characteristic function $\phi_2(\alpha)$ allows us to determine a relation between the risk-neutral probabilities which appear in the theoretical option price formula.

The first step is to determine the probability $P_1(x < -d)$. Using the Gil-Pelaez formula and the fact that the characteristic functions are complex conjugate, we obtain:

$$\begin{aligned} P_1(x < -d) &= \frac{1}{2} + \frac{1}{2\pi}\int_0^\infty \frac{e^{-i\alpha d}\phi_1(-\alpha) - e^{i\alpha d}\phi_1(\alpha)}{i\alpha} d\alpha \\ &= \frac{1}{2} + \frac{1}{2\pi}\int_0^\infty \frac{e^{-i\alpha d}\phi_2(\alpha) - e^{i\alpha d}\phi_2(-\alpha)}{i\alpha} d\alpha \end{aligned} \tag{17}$$

therefore:

$$\begin{aligned} P_1(x < -d) &= \frac{1}{2} - \frac{1}{2\pi}\int_0^\infty \frac{e^{i\alpha d}\phi_2(-\alpha) - e^{-i\alpha d}\phi_2(\alpha)}{i\alpha} d\alpha \\ &= \frac{1}{2} - \left[P_2(x < d) - \frac{1}{2}\right] = 1 - P_2(x < d) \end{aligned}$$

In the same way, we obtain an analogous relation between $P_2(x < -d)$ and $P_1(x < d)$. Summarizing, we found the following relation between risk-neutral probabilities:

$$\begin{cases} P_1(x < -d) = 1 - P_2(x < d) = P_2(x > d) \\ P_2(x < -d) = 1 - P_1(x < d) = P_1(x > d) \end{cases} \tag{18}$$

The next step is to obtain a relation between the risk-neutral probabilities which appear in the option price formula. Knowing that,

$$\begin{cases} P_1(-d < x < d) = P_1(x < d) - P_1(x < -d) \\ P_2(-d < x < d) = P_2(x < d) - P_2(x < -d) \end{cases} \tag{19}$$

and using the Gil-Pelaez formula for each probability, we found:

$$P_1(-d < x < d) = \frac{1}{2\pi}\int_0^\infty \left[\frac{\left(e^{i\alpha d} - e^{-i\alpha d}\right)}{i\alpha}\phi_1(-\alpha) + \frac{\left(e^{i\alpha d} - e^{-i\alpha d}\right)}{i\alpha}\phi_1(\alpha)\right] d\alpha \tag{20}$$
$$= \frac{1}{2\pi}\int_0^\infty \frac{\left(e^{i\alpha d} - e^{-i\alpha d}\right)}{i\alpha}\left[\phi_1(-\alpha) + \phi_1(\alpha)\right] d\alpha$$

Using the fact that the $\phi_1(\alpha)$ and $\phi_1(-\alpha)$ are complex conjugate[3] and the fact that $\frac{e^{i\alpha d} - e^{-i\alpha d}}{i\alpha} = 2\frac{\sin \alpha d}{\alpha}$,

$$P_1(-d < x < d) = \frac{2}{\pi}\int_0^\infty \frac{\sin \alpha d}{\alpha} \operatorname{Re}\left[\phi_1(\alpha)\right] d\alpha \tag{21}$$

In the same way, we obtain:

$$P_2(-d < x < d) = \frac{2}{\pi}\int_0^\infty \frac{\sin \alpha d}{\alpha} \operatorname{Re}\left[\phi_2(\alpha)\right] d\alpha \tag{22}$$

Once again we take advantage of the relation between the two characteristic functions. They are complex conjugate, hence $\operatorname{Re}[\phi_1(\alpha)] = \operatorname{Re}[\phi_2(\alpha)]$. Consequently, the probabilities (21) and (22) are identical. We make the notation P for this probability:

$$P = P_1(-d < x < d) = P_2(-d < x < d) \tag{23}$$

The final step is to use the relations (18) and (23). Hence, the relation between risk-neutral probabilities is given by:

$$\begin{cases} P_1(x > d) = P_2(x < d) - P \\ P_2(x > d) = P_1(x < d) - P \end{cases} \tag{24}$$

3 A Closed-form Formula of The Option Price

In this section, we establish the theoretical price of an European call option when the volatility is stochastic and when the underlying price is not correlated with the volatility.

[3] Knowing that $\phi_1(\alpha)$ and $\phi_1(-\alpha)$ are complex conjugate, then:

$$\phi_1(\alpha) + \phi_1(-\alpha) = 2\operatorname{Re}[\phi_1(\alpha)]$$

$$\operatorname{Re}[\phi_1(\alpha)] = \operatorname{Re}[\phi_1(-\alpha)] \quad \text{and} \quad \operatorname{Im}[\phi_1(\alpha)] = -\operatorname{Im}[\phi_1(-\alpha)]$$

We can express the probability $P_2(x > d)$ using the Gil-Pelaez definition of the probability $P_2(x < d)$:

$$
\begin{aligned}
P_2(x > d) &= 1 - \left[\frac{1}{2} + \frac{1}{2\pi}\int_0^\infty \frac{e^{i\alpha d}\phi_2(-\alpha) - e^{-i\alpha d}\phi_2(\alpha)}{i\alpha} d\alpha\right] \\
&= \frac{1}{2} - \frac{1}{2\pi}\int_0^\infty \frac{e^{i\alpha d}\phi_2(-\alpha) - e^{-i\alpha d}\phi_2(\alpha)}{i\alpha} d\alpha
\end{aligned} \tag{25}
$$

From Euler formulas,

$$
\begin{aligned}
P_2(x > d) &= \frac{1}{2} - \frac{1}{2\pi}\int_0^\infty \left[\left(\frac{\sin\alpha d}{\alpha} - i\frac{\cos\alpha d}{\alpha}\right)\phi_2(-\alpha) - \left(-\frac{\sin\alpha d}{\alpha} - i\frac{\cos\alpha d}{\alpha}\right)\phi_2(\alpha)\right] d\alpha \\
&= \frac{1}{2} - \frac{1}{2\pi}\int_0^\infty \left\{\frac{\sin\alpha d}{\alpha}\left[\phi_2(-\alpha) + \phi_2(\alpha)\right] - i\frac{\cos\alpha d}{\alpha}\left[\phi_2(-\alpha) - \phi_2(\alpha)\right]\right\} d\alpha
\end{aligned} \tag{26}
$$

But the quantities $\phi_2(-\alpha)$ and $\phi_2(\alpha)$ are complex conjugate. Hence,

$$
P_2(x > d) = \frac{1}{2} - \frac{1}{\pi}\int_0^\infty \frac{\sin\alpha d\,\mathrm{Re}\left[\phi_2(\alpha)\right] - \cos\alpha d\,\mathrm{Im}\left[\phi_2(\alpha)\right]}{\alpha} d\alpha \tag{27}
$$

and that involves[4]:

$$
P_2(x < d) = \frac{1}{2} + \frac{1}{\pi}\int_0^\infty \frac{\sin\alpha d\,\mathrm{Re}\left[\phi_2(\alpha)\right] - \cos\alpha d\,\mathrm{Im}\left[\phi_2(\alpha)\right]}{\alpha} d\alpha \tag{28}
$$

In order to determine the probability $P_1(x > d)$, we use the relation (24) between the risk-neutral probabilities:

$$
\begin{aligned}
P_1(x > d) = P_2(x < d) - P &= \frac{1}{2} + \frac{1}{\pi}\int_0^\infty \frac{\sin\alpha d\,\mathrm{Re}\left[\phi_2(\alpha)\right] - \cos\alpha d\,\mathrm{Im}\left[\phi_2(\alpha)\right]}{\alpha} d\alpha \\
&\quad - \frac{2}{\pi}\int_0^\infty \frac{\sin\alpha d}{\alpha}\mathrm{Re}\left[\phi_2(\alpha)\right] d\alpha
\end{aligned}
$$

Therefore, the expression of the risk-neutral probability $P_1(x > d)$ is defined by:

$$
P_1(x > d) = \frac{1}{2} - \frac{1}{\pi}\int_0^\infty \frac{\sin\alpha d\,\mathrm{Re}\left[\phi_2(\alpha)\right] + \cos\alpha d\,\mathrm{Im}\left[\phi_2(\alpha)\right]}{\alpha} d\alpha \tag{29}
$$

Knowing the expressions (27) and (29) of the risk-neutral probabilities, the closed-form formula of the option price with stochastic volatility is given by:

$$
\begin{aligned}
C = S&\left\{\frac{1}{2} - \frac{1}{\pi}\int_0^\infty \frac{\sin\alpha d\,\mathrm{Re}\left[\phi_2(\alpha)\right] + \cos\alpha d\,\mathrm{Im}\left[\phi_2(\alpha)\right]}{\alpha} d\alpha\right\} \\
-Ke^{-r\tau}&\left\{\frac{1}{2} - \frac{1}{\pi}\int_0^\infty \frac{\sin\alpha d\,\mathrm{Re}\left[\phi_2(\alpha)\right] - \cos\alpha d\,\mathrm{Im}\left[\phi_2(\alpha)\right]}{\alpha} d\alpha\right\}
\end{aligned} \tag{30}
$$

[4] See the same definition in Kendall and Stuart, (1977), "The Advanced Theory of Statistics", Volume 1, page 96.

In this closed-form formula, the risk-neutral probabilities are explained only by one characteristic function, $\phi_2(\alpha)$. Moreover, we note that the risk-neutral probabilities are symmetrical[5].

3.1 A Heston-like Option Price Formula

Using the following relation[6]:

$$\frac{\sin \alpha d \,\mathrm{Re}\left[\phi_j(\alpha)\right] - \cos \alpha d \,\mathrm{Im}\left[\phi_j(\alpha)\right]}{\alpha} = -\mathrm{Re}\left[\frac{e^{-i\alpha d}}{i\alpha}\phi_j(\alpha)\right] = \mathrm{Im}\left[\frac{e^{-i\alpha d}}{\alpha}\phi_j(\alpha)\right] \tag{31}$$

we obtain:

$$P_2(x > d) = \frac{1}{2} + \frac{1}{\pi}\int_0^\infty \mathrm{Re}\left[\frac{e^{-i\alpha d}}{i\alpha}\phi_2(\alpha)\right] d\alpha \tag{32}$$

and

$$P_1(x > d) = \frac{1}{2} + \frac{1}{\pi}\int_0^\infty \mathrm{Re}\left[\frac{e^{-i\alpha d}}{i\alpha}\phi_1(\alpha)\right] d\alpha \tag{33}$$

and we found the Heston closed-form formula:

$$C = S\left\{\frac{1}{2} + \frac{1}{\pi}\int_0^\infty \mathrm{Re}\left[\frac{e^{-i\alpha d}}{i\alpha}\phi_1(\alpha)\right] d\alpha\right\} - Ke^{-r\tau}\left\{\frac{1}{2} + \frac{1}{\pi}\int_0^\infty \mathrm{Re}\left[\frac{e^{-i\alpha d}}{i\alpha}\phi_2(\alpha)\right] d\alpha\right\} \tag{34}$$

Heston doesn't take into consideration the relation between the risk-neutral probabilities and the fact that the characteristic functions are complex conjugate. Therefore, we can simplify the formula proposed by Heston using this relation between the probabilities. We found the following closed-form formula:

$$\begin{aligned} C = {} & S\left\{\frac{1}{2} - \frac{1}{\pi}\int_0^\infty \mathrm{Re}\left[\frac{e^{-i\alpha d}}{i\alpha}\phi_2(\alpha)\right] d\alpha - \frac{2}{\pi}\int_0^\infty \frac{\sin \alpha d}{\alpha}\,\mathrm{Re}\left[\phi_2(\alpha)\right] d\alpha\right\} \\ & -Ke^{-r\tau}\left\{\frac{1}{2} + \frac{1}{\pi}\int_0^\infty \mathrm{Re}\left[\frac{e^{-i\alpha d}}{i\alpha}\phi_2(\alpha)\right] d\alpha\right\} \end{aligned} \tag{35}$$

Obviously, it is better to use our formula when we pass on the numerical integration. The relation between the risk-neutral probabilities operates like a restriction. The first risk-neutral probability is forced to respect this restriction. Consequently, the degree of accuracy in the numerical integration must be higher.

[5] As concerning the accuracy of the computing, this symmetry will allow a better numerical approximation of the option price.

[6] See, for instance, Davis (1973) - "Numerical inversion of characteristic function", Biometrika 60, and Shephard (1991) - "From characteristic function to distribution function: A simple framework for the theory", Econometric Theory 7.

4 The Effect of The Stochastic Volatility on The Skewness of The Underlying Return Density

Reasoning *ab absurdo* we demonstrate that the density laws of P_1 and, respectively of P_2 are not symmetrical when the underlying price is not correlated with the volatility. Supposing *ab absurdo* that the density law of P_2 is symmetrical, we have[7]:

$$P_2^* \left(-d < x < d\right) = 2P_2 \left(x < d\right) - 1 \tag{36}$$

and, using the Gil-Pelaez inversion formula, we obtain[8]:

$$\begin{aligned} P_2^* \left(-d < x < d\right) &= 2 \left[\frac{1}{2} + \frac{1}{2\pi} \int_0^\infty \frac{e^{i\alpha d}\phi_2 \left(-\alpha\right) - e^{-i\alpha d}\phi_2 \left(\alpha\right)}{i\alpha} d\alpha\right] - 1 \\ &= \frac{1}{\pi} \int_0^\infty \frac{e^{i\alpha d}\phi_2 \left(-\alpha\right) - e^{-i\alpha d}\phi_2 \left(\alpha\right)}{i\alpha} d\alpha \end{aligned} \tag{37}$$

Moreover, using the Euler formulas, we have:

$$\begin{aligned} P_2^* \left(-d < x < d\right) &= \frac{1}{\pi} \int_0^\infty \left[\left(\frac{\sin \alpha d}{\alpha} - i\frac{\cos \alpha d}{\alpha}\right) \phi_2 \left(-\alpha\right) - \left(-\frac{\sin \alpha d}{\alpha} - i\frac{\cos \alpha d}{\alpha}\right) \phi_2 \left(\alpha\right)\right] d\alpha \\ &= \frac{1}{\pi} \int_0^\infty \left\{\frac{\sin \alpha d}{\alpha} \left[\phi_2 \left(-\alpha\right) + \phi_2 \left(\alpha\right)\right] - i\frac{\cos \alpha d}{\alpha} \left[\phi_2 \left(-\alpha\right) - \phi_2 \left(\alpha\right)\right]\right\} d\alpha \end{aligned} \tag{38}$$

therefore:

$$P_2^* \left(-d < x < d\right) = \frac{2}{\pi} \int_0^\infty \frac{\sin \alpha d \, \mathrm{Re} \left[\phi_2 \left(\alpha\right)\right] - \cos \alpha d \, \mathrm{Im} \left[\phi_2 \left(\alpha\right)\right]}{\alpha} d\alpha \tag{39}$$

But, without supposing the probability density symmetry, we know the expression of $P_2 \left(-d < x < d\right)$ given by (22). Consequently, the difference between the two expressions must be zero, $P_2 \left(-d < x < d\right) - P_2^* \left(-d < x < d\right) = 0$ or

$$\frac{2}{\pi} \int_0^\infty \frac{\sin \alpha d}{\alpha} \mathrm{Re} \left[\phi_2 \left(\alpha\right)\right] d\alpha - \frac{2}{\pi} \int_0^\infty \frac{\sin \alpha d \, \mathrm{Re} \left[\phi_2 \left(\alpha\right)\right] - \cos \alpha d \, \mathrm{Im} \left[\phi_2 \left(\alpha\right)\right]}{\alpha} d\alpha = 0$$

We conclude that the probability density of the underlying return is symmetrical if and only if:

$$\frac{2}{\pi} \int_0^\infty \frac{\cos \alpha d}{\alpha} \mathrm{Im} \left[\phi_2 \left(\alpha\right)\right] d\alpha = 0$$

[7] The sign * denotes the fact that the probability is determined under *ab absurdo* assumption.

[8] The same result is obtained by Gil-Pelaez (1951).

In other words, the underlying return density is symmetrical if and only if $\mathrm{Im}\left[\phi_2(\alpha)\right] d\alpha = 0$. Obviously, the skewness coefficient of underlying return density is non-zero because

$$\frac{2}{\pi}\int_0^\infty \frac{\cos\alpha d}{\alpha}\mathrm{Im}\left[\phi_2(\alpha)\right] d\alpha \neq 0 \quad \text{and} \quad \mathrm{Im}\left[\phi_2(\alpha)\right] d\alpha \neq 0 \tag{40}$$

We offer a second proof of an *ab absurdo* reasoning. If we suppose *ab absurdo* that the underlying asset density is symmetrical, the risk-neutral probability $P_1^*(x > d)$ can be determined knowing the expression of $P_2^*(-d < x < d)$.

$$\begin{aligned} P_1^*(x > d) &= P_2(x < d) - P_2^*(-d < x < d) \\ &= \frac{1}{2} + \frac{1}{\pi}\int_0^\infty \frac{\sin\alpha d\,\mathrm{Re}\left[\phi_2(\alpha)\right] - \cos\alpha d\,\mathrm{Im}\left[\phi_2(\alpha)\right]}{\alpha} d\alpha \\ &\quad - \frac{2}{\pi}\int_0^\infty \frac{\sin\alpha d\,\mathrm{Re}\left[\phi_2(\alpha)\right] - \cos\alpha d\,\mathrm{Im}\left[\phi_2(\alpha)\right]}{\alpha} d\alpha \end{aligned}$$

which involves:

$$P_1^*(x > d) = \frac{1}{2} - \frac{1}{\pi}\int_0^\infty \frac{\sin\alpha d\,\mathrm{Re}\left[\phi_2(\alpha)\right] - \cos\alpha d\,\mathrm{Im}\left[\phi_2(\alpha)\right]}{\alpha} d\alpha \tag{41}$$

or

$$P_1^*(x > d) = P_2(x > d) \tag{42}$$

This is not true because $P_1^*(x > d)$ must be identical with $P_1(x > d)$ and $P_1(x > d) \neq P_2(x > d)$. Therefore, it is not possible to have the equality (42) for all values taken by d. Consequently, we conclude that the underlying return density is not symmetrical and the skewness is changed by the stochastic volatility even if the underlying price is not correlated with its volatility.

The Figure 1 shows the densities which correspond to the probabilities P_1 and P_2. The Figure shows that the underlying return density is not symmetrical. Moreover, a certain "symmetry" between the densities can be observed. The two densities have identical skewness coefficients, but with opposite sign. We can note that one of the density is the "image in the mirror" of the other density. That is true because of the relations between the risk-neutral probabilities.

We conclude that the stochastic volatility also changes the skewness, not only the kurtosis, of the underlying return density when the price of underlying asset is correlated with the volatility. That is contrary to Heston who stipulates that, without correlation between the state variables, the stochastic volatility changes only the kurtosis of the underlying return density.

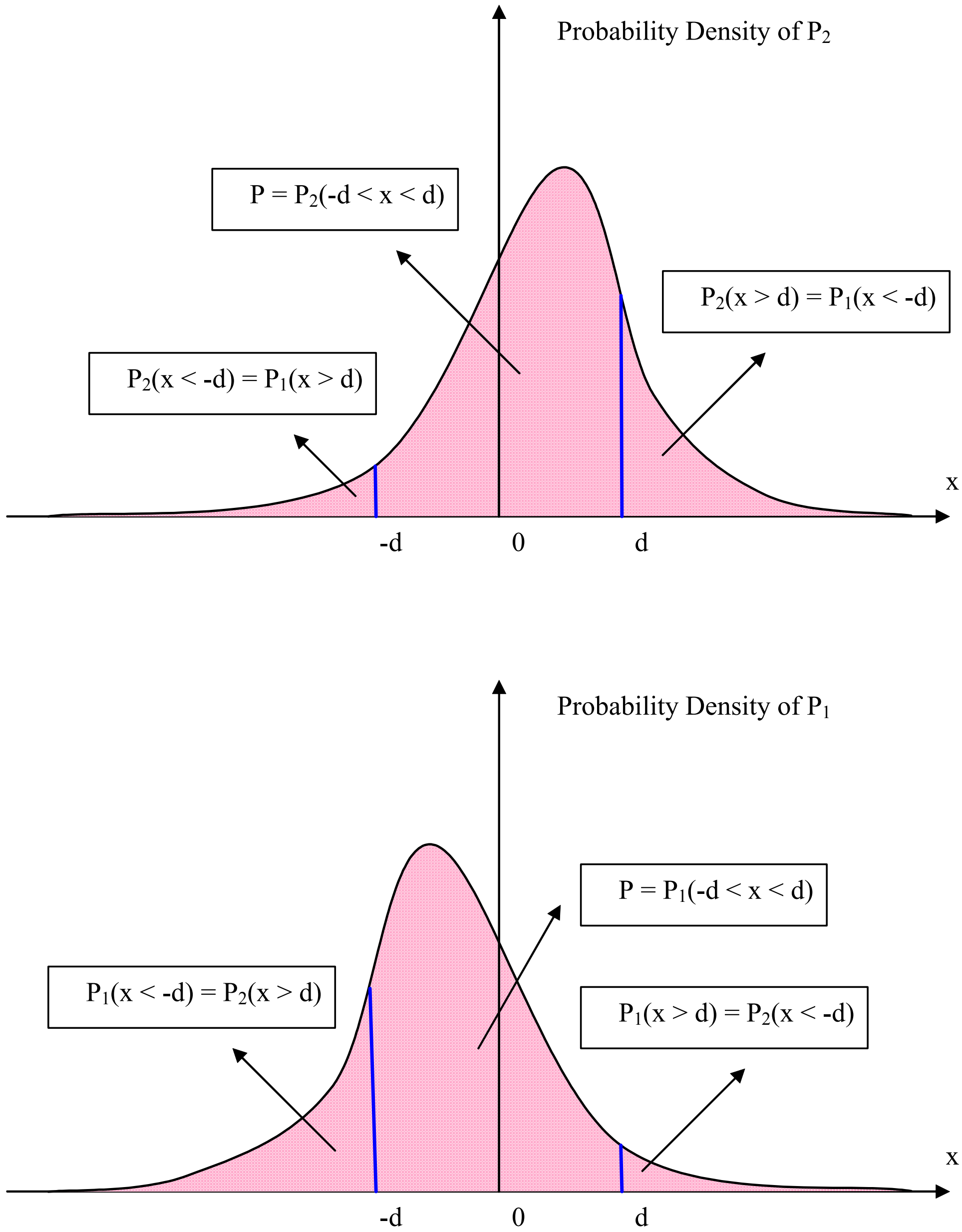

Figure 1
The asymmetrical densities

5 Conclusions

Heston (1993) established that stochastic volatility changes only the kurtosis of underlying asset density when the state variables are not correlated. This paper shows that stochastic volatility always affects the skewness of underlying asset distribution even if the price is not correlated with its volatility. Moreover, the risk-neutral probabilities are symmetrical when the state variables are not correlated. This symmetry is done by a relation which exists between the risk-neutral probabilities. The two densities which give the risk-neutral probabilities have identical skewness coefficients, but with opposite sign. In fact, one of the density is the "image in the mirror" of the other density.

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