Parametric properties of semi-nonparametric distributions, with applications to option valuation*

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Abstract

We derive the statistical properties of the SNP densities of Gallant and Nychka (1987). We show that these densities, which are always positive, are more general than the truncated Gram-Charlier expansions of Jondeau and Rockinger (2001), who impose parameter restrictions to ensure positivity. We also use the SNP densities for option valuation. We relate real and risk-neutral measures, obtain closed-form prices for European options, and study the "Greeks". We show that SNP densities generate wider option price ranges than the truncated expansions. In an empirical application to S&P 500 index options, we find that the SNP model beats the standard and Practitioner's Black-Scholes formulas, and the truncated expansions.

Keywords: Kurtosis, Density Expansions, Gram-Charlier, Skewness, S&P index options

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1 Introduction

In recent years, many studies have attempted to overcome the limitations of the popular normality assumption on the returns of stocks and other financial assets, which is systematically rejected in the empirical finance literature. Although this assumption may still be reasonable if the interest focuses on the first two moments (see Bollerslev and Wooldridge, 1992), in many financial applications the features under study are higher order moments such as skewness and kurtosis. An important example is option pricing theory, in which the Black and Scholes (1973) pricing formula, which remains the benchmark model because of its analytical tractability, also relies on the normality of returns. Unfortunately, normality is too restrictive to approximate the complex shape of the distribution of most underlying asset returns, and more flexible distributions may help to explain the puzzles unresolved by the Black and Scholes (1973) framework, including smiles and smirks.

However, any successful generalisation of the Gaussian assumption must satisfy two crucial requirements: modelling flexibility and analytical tractability. Both needs are satisfied by Gram-Charlier expansions, which were introduced in option pricing theory by Jarrow and Rudd (1982), and have been used more recently by Corrado and Su (1996, 1997), Capelle-Blanchard, Jurczenko, and Maillet (2001), Jurczenko, Maillet, and Negrea (2002a), and Lim, Martin, and Martin (2005). As is well known, most density functions can be expressed as a possibly infinite expansion of the Gaussian density. In practice, however, the expansion is usually truncated after the fourth power. Unfortunately, such truncated expansions often imply negative densities over some interval of their domain of variation, as Jondeau and Rockinger (2001) emphasize. This feature is particularly worrying in option pricing applications because it allows some arbitrage opportunities. For instance, the price of a butterfly spread with positive payoff over an interval of negative density would necessarily be negative in those circumstances. As a solution to this problem, Jondeau and Rockinger (2001) propose to restrict the parameters of the expansion so that the density remains always positive. Unfortunately, their approach can be very difficult to implement even when the truncation order is low. Furthermore, the flexibility of the positivity restricted distributions to model skewness and kurtosis is rather limited. As we shall see, this lack of flexibility turns out to be empirically restrictive in option pricing applications.

In this context, we propose the use of the semi-nonparametric distribution (SNP), which was introduced by Gallant and Nychka (1987) for nonparametric estimation purposes, as an equally flexible and analytically tractable solution. The SNP density can be regarded as an alternative expansion of the Gaussian density function, which is always positive by definition.

The properties of the SNP density from the nonparametric estimation point of view have been studied in depth by Fenton and Gallant (1996) and Gallant and Tauchen (1999). However, this density has not been treated from a purely parametric point of view, that is, taking the SNP distribution as if it reflected the actual data generating process instead of an approximating kernel. In this sense, our starting point will be the assumption that asset returns follow a SNP distribution under the real measure. In this framework, we will study first the statistical properties of this distribution, including moments, distribution of linear combinations, standardised versions, as well as its relationship to the Gram-Charlier densities. Then, we will combine it with an exponentially affine assumption on the stochastic discount factor, which will enable us to transform the real measure into the risk neutral measure required for the valuation of derivative assets. We will obtain closed-form expressions for plain vanilla options by exploiting the analytical tractability of the SNP distribution. We will also obtain the different option sensitivities, commonly known as the "Greeks". Finally, we will carry out an empirical application to the S&P 500 options data of Dumas, Fleming, and Whaley (1998) in which we will evaluate the performance of our pricing formulas.

The paper is structured as follows. In the next section, we study the statistical properties of SNP densities, and compare them with those of Gram-Charlier expansions. In section 3, we first relate the real and risk neutral measures, and then focus on pricing European options. Finally, section 4 presents the empirical application, followed by our conclusions in section 5. Proofs and auxiliary results can be found in appendices.

2 Density definition

We want to analyse the statistical properties of a random variable z which can be expressed as a linear transformation of another random variable x, i.e. $z = a + bx$, where the density of x belongs to the semi-nonparametric class introduced by Gallant and Nychka (1987). Specifically,

$$f(x) = \frac{\phi(x)}{\boldsymbol{\nu}'\boldsymbol{\nu}} \left(\sum_{i=0}^{m} \nu_i H_i(x) \right)^2, \tag{1}$$

where $\boldsymbol{\nu} = (\nu_0, \nu_1, \cdots, \nu_m)' \in \mathbb{R}^{m+1}$, $\phi(\cdot)$ denotes the probability density function (pdf) of a standard normal random variable, and $H_i(x)$ is the normalised Hermite polynomial of order i. These polynomials can be defined recursively for $i \geq 2$ as

$$H_i(x) = \frac{x H_{i-1}(x) - \sqrt{i-1} H_{i-2}(x)}{\sqrt{i}}, \tag{2}$$

with initial conditions $H_0(x) = 1$ and $H_1(x) = x$. Importantly, $\{H_i(x)\}_{i \in \mathbb{N}}$ constitutes an orthonormal basis with respect to the weighting function $\phi(x)$, as illustrated by the following two conditions:

$$\begin{array}{ll} (a) & \int_{-\infty}^{+\infty} H_i^2(x)\,\phi(x)\,dx = 1, \quad \forall i \\ (b) & \int_{-\infty}^{+\infty} H_i(x)\,H_j(x)\,\phi(x)\,dx = 0, \quad \forall i \neq j. \end{array}$$

The change of variable formula implies that the density function of z will be

$$g(z) = \frac{1}{b} \frac{1}{\boldsymbol{\nu}'\boldsymbol{\nu}} \phi\left(\frac{z-a}{b}\right) \left[\sum_{i=0}^{m} \nu_i H_i\left(\frac{z-a}{b}\right) \right]^2, \tag{3}$$

where we could interpret a as a location parameter and b as a scale parameter. Note that both (1) and (3) are homogeneous of degree zero in $\boldsymbol{\nu}$, which implies that there is a scale indeterminacy that we will solve by imposing a single normalising restriction on these parameters, such as $\nu_0 = 1$ or preferably $\boldsymbol{\nu}'\boldsymbol{\nu} = 1$.

If we expand the squared expression in (1), we can express the density as

$$f(x) = \phi(x) \sum_{k=0}^{2m} \gamma_k(\boldsymbol{\nu})\, H_k(x), \tag{4}$$

where $\gamma_k(\boldsymbol{\nu})$ are functions of $\boldsymbol{\nu}$ that satisfy the following general expression:

Proposition 1

$$\gamma_k(\boldsymbol{\nu}) = \frac{\boldsymbol{\nu}' A_k \boldsymbol{\nu}}{\boldsymbol{\nu}'\boldsymbol{\nu}}, \tag{5}$$

where A_k is a $(m+1) \times (m+1)$ symmetric matrix whose typical element is

$$a_{ij,k} = \frac{(i!j!k!)^{1/2}}{\left(\frac{i+j-k}{2}\right)!\left(\frac{i+k-j}{2}\right)!\left(\frac{k+j-i}{2}\right)!}$$

if $k \in \Gamma$ and zero otherwise, and

$$\Gamma = \left\{ k \in \mathbb{N} : |i-j| \leq k \leq i+j; \quad \frac{i-j+k}{2} \in \mathbb{N} \right\}.$$

For instance, the values of $\gamma_k(\boldsymbol{\nu})$ when $m = 2$ are:

$$\begin{aligned}
&\gamma_0(\boldsymbol{\nu}) = 1, \\
&\gamma_1(\boldsymbol{\nu}) = 2\nu_1\left(\nu_0 + \sqrt{2}\nu_2\right)/\boldsymbol{\nu}'\boldsymbol{\nu}, \\
&\gamma_2(\boldsymbol{\nu}) = \sqrt{2}\left(\nu_1^2 + 2\nu_2^2 + \sqrt{2}\nu_0\nu_2\right)/\boldsymbol{\nu}'\boldsymbol{\nu}, \\
&\gamma_3(\boldsymbol{\nu}) = 2\sqrt{3}\nu_1\nu_2/\boldsymbol{\nu}'\boldsymbol{\nu}, \\
&\gamma_4(\boldsymbol{\nu}) = \sqrt{6}\nu_2^2/\boldsymbol{\nu}'\boldsymbol{\nu}.
\end{aligned}$$

2.1 Moments of x and z

The first four non-central moments of x, $\mu_x'(k)$, can be obtained by using the relationship between the powers of x and the Hermite polynomials:

$$\begin{aligned}
\mu_x'(1) &\equiv E_f(x) = E_f[H_1(x)], \\
\mu_x'(2) &\equiv E_f\left(x^2\right) = \sqrt{2}E_f[H_2(x)] + 1, \\
\mu_x'(3) &\equiv E_f\left(x^3\right) = \sqrt{3!}E_f[H_3(x)] + 3E_f[H_1(x)], \\
\mu_x'(4) &\equiv E_f\left(x^4\right) = \sqrt{4!}E_f[H_4(x)] + 6\sqrt{2}E_f[H_2(x)] + 3, \qquad (6)
\end{aligned}$$

where the operator $E_f[\cdot]$ takes the expectation of its argument with respect to the density function $f(x)$ in (1). Note that we can rewrite both $\mu_x'(3)$ and $\mu_x'(4)$ as

$$\begin{aligned}
\mu_x'(3) &= \sqrt{3!}E_f[H_3(x)] + 3\mu_x'(1), \\
\mu_x'(4) &= \sqrt{4!}E_f[H_4(x)] + 6\mu_x'(2) - 3.
\end{aligned}$$

Then, the corresponding central moments, $\mu_x(k)$, can be easily obtained from the relationships:

$$\begin{aligned}
\mu_x(2) &= \mu_x'(2) - \mu_x'^2(1), \\
\mu_x(3) &= \mu_x'(3) - 3\mu_x'(2)\mu_x'(1) + 2\mu_x'^3(1), \\
\mu_x(4) &= \mu_x'(4) - 4\mu_x'(3)\mu_x'(1) + 6\mu_x'(2)\mu_x'^2(1) - 3\mu_x'^4(1).
\end{aligned}$$

Finally, we can also compute the skewness and kurtosis coefficients, denoted by sk and ku, respectively, which are defined as

$$sk = \frac{\mu_x(3)}{\mu_x^{3/2}(2)}; \qquad ku = \frac{\mu_x(4)}{\mu_x^2(2)}. \qquad (7)$$

But since $\mu_x'(k)$ in (6) depends on $\{E_f[H_i(x)]\}_{i\in\mathbb{N}}$, we first need to find $E_f[H_k(x)]$:

Proposition 2 *Let $f(\cdot)$ denote the density function of the random variable x, which we assume is given by (1). Then*

$$E_f\left[H_k(x)\right] = \gamma_k(\boldsymbol{\nu}), \tag{8}$$

if $k \leq 2m$, and zero otherwise.

On this basis, we can easily compute the first four non-centred moments of x for the important special case of $m = 2$:

Lemma 1 *If the density function of the random variable x is given by (1) with $m = 2$, then*

$$\begin{aligned}
\mu'_x(1) &= \frac{2\nu_1}{\boldsymbol{\nu}'\boldsymbol{\nu}}\left(\nu_0 + \sqrt{2}\nu_2\right), \\
\mu'_x(2) &= \frac{2}{\boldsymbol{\nu}'\boldsymbol{\nu}}\left(\nu_1^2 + 2\nu_2^2 + \sqrt{2}\nu_2\nu_0\right) + 1, \\
\mu'_x(3) &= \frac{6\nu_1}{\boldsymbol{\nu}'\boldsymbol{\nu}}\left(\nu_0 + 2\sqrt{2}\nu_2\right), \\
\mu'_x(4) &= \frac{12}{\boldsymbol{\nu}'\boldsymbol{\nu}}\left(\nu_1^2 + 3\nu_2^2 + \sqrt{2}\nu_2\nu_0\right) + 3.
\end{aligned}$$

In general, we can show that:

Proposition 3 *The moment generating function corresponding to the SNP density (1) is*

$$E_f\left[e^{tx}\right] = \exp\left(\frac{t^2}{2}\right)\Lambda(\boldsymbol{\nu}, t),$$

while its characteristic function is

$$\psi_{SNP}(it) = \exp\left(\frac{-t^2}{2}\right)\Lambda(\boldsymbol{\nu}, it),$$

where i is the usual imaginary unit,

$$\Lambda(\boldsymbol{\nu}, t) = \sum_{k=0}^{2m}\gamma_k(\boldsymbol{\nu})\frac{t^k}{\sqrt{k!}},$$

and $\gamma_k(\boldsymbol{\nu})$ is defined in (5).

Since z is a linear transformation of x, it is trivial to find the non-central moments of z, $\mu'_z(k)$, as a function of those of x. Specifically,

$$\mu'_z(n) \equiv E_f\left[(a + bx)^n\right] = \sum_{i=0}^{n}\binom{n}{i}a^{n-i}b^i\mu'_x(i).$$

In this sense, notice that the skewness and kurtosis coefficients of z are exactly the same as those of x because the relation between x and z is affine. In addition, we can always choose the location and dispersion coefficients a and b such that z has zero mean and unit variance. In particular, if we denote by z^* the standardised variable

$$z^* = \frac{x - \mu_x'(1)}{\sqrt{\mu_x(2)}}, \tag{9}$$

then its density function can be directly obtained from (3) with

$$a(\boldsymbol{\nu}) = -\frac{\mu_x'(1)}{\sqrt{\mu_x(2)}} \tag{10}$$

and

$$b(\boldsymbol{\nu}) = 1/\sqrt{\mu_x(2)}. \tag{11}$$

We can also use Proposition 3 to derive the distribution of linear combinations of SNP variables (see Appendix B).

2.2 Gram-Charlier expansion of the semi-nonparametric density

Under certain regularity conditions, any density function $h(y)$ can be expressed as the product of a standard normal density times an infinite series of Hermite polynomials:

$$h(y) = \phi(y) \sum_{k=0}^{\infty} c_k H_k(y), \tag{12}$$

where the coefficients c_k are

$$c_k = \int_{-\infty}^{\infty} H_k(y)\, h(y)\, dy = E_h(H_k(y)). \tag{13}$$

This is the so-called Gram-Charlier series of Type A (see Stuart and Ord, 1977).

With this in mind, we will first determine the Gram-Charlier expansion of the SNP density of z, and then we will particularise it for the standardised random variable z^* in (9). In the case of z, we will use the fact that, according to (3) and (4), its density can be written as

$$g(z) = \frac{1}{b}\phi\left(\frac{z-a}{b}\right) \sum_{i=0}^{2m} \gamma_i(\boldsymbol{\nu})\, H_i\left(\frac{z-a}{b}\right), \tag{14}$$

where the coefficients γ_i are the functions of the vector $\boldsymbol{\nu}$ described in Proposition 1. Then, if we compare (13) and (14), we can write c_k for z as

$$c_k = \frac{1}{b} \sum_{i=0}^{2m} \gamma_i \int_{-\infty}^{\infty} \phi\left(\frac{z-a}{b}\right) H_i\left(\frac{z-a}{b}\right) H_k(z)\, dz, \qquad \forall\, k \geq 0,$$

which, with the simple change of variable $x = (z - a)/b$, becomes

$$c_k = \sum_{i=0}^{2m} \gamma_i E_\phi \left[H_i(x) H_k(a + bx) \right], \qquad \forall\, k \geq 0. \tag{15}$$

The following proposition gives a general formula for the expectations in (15):

Proposition 4

$$\begin{aligned} E_\phi \left[H_i(x) H_k(a + bx) \right] &= \int_{-\infty}^{\infty} H_i(x) H_k(a + bx) \phi(x)\, dx \\ &= \sqrt{\frac{k!}{i!}} \sum_{j=0}^{\left\lfloor \frac{k-i}{2} \right\rfloor} \frac{H_{k-i-2j}(a)}{j! \sqrt{(k-i-2j)!} 2^j} b^{i+2j} \end{aligned}$$

for $i \leq k$ and zero otherwise, where $\lfloor \cdot \rfloor$ rounds its argument to the nearest integer toward zero.

In consequence, the coefficients of z defined in (15) will be

$$c_k = \sum_{i=0}^{\min(k,2m)} \sum_{j=0}^{\left\lfloor \frac{k-i}{2} \right\rfloor} \frac{\gamma_i(\boldsymbol{\nu})}{j! 2^j} \sqrt{\frac{k!}{(i!)(k-i-2j)!}} H_{k-i-2j}(a)\, b^{i+2j} \tag{16}$$

where $\gamma_i(\boldsymbol{\nu})$ is defined in (5). Finally, we can easily find the coefficients of the Gram-Charlier expansion of z^* by substituting a and b by their respective values in (10) and (11). Given the results in Stuart and Ord (1977) on Gram-Charlier series of Type A for standardised random variables, it is not surprising that the coefficients of the expansion of z^* are precisely

$$\begin{aligned} & c_0 = 1, \\ & c_1 = c_2 = 0, \\ & c_3 = sk/\sqrt{3!}, \\ & c_4 = (ku - 3)/\sqrt{4!}, \\ & c_5 = \kappa(5)/\sqrt{5!} \\ & c_6 = \left[\kappa(6) + 10sk^2 \right]/\sqrt{6!} \\ & \vdots \end{aligned} \tag{17}$$

where $\kappa(k)$ is the k^{th}-order cumulant of z^*.

These higher order coefficients will generally be different from zero in view of (16). However, there is one instance in which the Gram-Charlier expansion of the SNP standardised density will be finite. Specifically, if $\nu_1 = \nu_2 = 0$ when $m > 2$, then it can be shown that $c_k = 0$ for $k > 2m$, since $a(\boldsymbol{\nu}) = 0$ and $b(\boldsymbol{\nu}) = 1$ in that case. Lim, Martin, and Martin (2005) have explored this restricted parametrisation with $m = 4$ for option pricing purposes. In this paper, though, we will not impose any restrictions on the parameters of the SNP density.

2.3 Comparison with a truncated Gram-Charlier expansion

Jondeau and Rockinger (2001) use as density function a truncated Gram-Charlier expansion of the form

$$h\left(z^{+}\right)=\phi\left(z^{+}\right)\left[1+\sum_{i=3}^{n} c_i H_i\left(z^{+}\right)\right]. \tag{18}$$

Notice that this density function has zero mean and unit variance by construction. In addition, if $n = 2m$, it involves exactly the same number of parameters as our standardised SNP variable z^*. However, Jondeau and Rockinger (2001) need to impose further restrictions on the parameters c_i $(i = 3, 4, \cdots, n)$ to ensure that the pdf in (18) is non-negative for all values of $z^+ \in (-\infty, \infty)$. Unfortunately, Jondeau and Rockinger (2001) only determined those restrictions for $n = 4$, because it becomes exceedingly difficult to find them for higher n. In contrast, we can leave the vector of parameters $\boldsymbol{\nu}$ free, except for a scale restriction, because positivity is always satisfied by a SNP density regardless of the expansion order.

Given that both z^* and z^+ have zero mean and unit variance, the natural question to ask is which of them leads to more general higher order moments. For the sake of concreteness, we will answer this question in terms of the values of skewness and kurtosis that each distribution can generate when $m = 2$ and $n = 4$. Specifically, we plot the envelope of all the combinations of skewness and kurtosis that these two distributions can generate in Figure 1.[1] In addition, we also represent the skewness-kurtosis frontier that no density function can surpass (see e.g. Stuart and Ord, 1977). As we can see in Figure 1, the combinations of skewness and kurtosis that the variable z^+ can generate are well within the combinations spanned by the SNP standardised variable z^* with exactly the same number of free parameters.[2] For instance, while z^+ could never be platykurtic, z^* can indeed have kurtosis coefficients lower than 3. More importantly, the differences in minimum and maximum skewness are also substantial. Finally, it is worth recalling that the SNP distribution guarantees positive densities regardless of m. In this sense, Figure 1 shows that we could achieve much more flexibility with just one additional parameter.

[1] We have used the procedure devised by Jondeau and Rockinger (2001) to obtain the frontier for a positive Gram-Charlier distribution with $n = 2$, while we rely on (7) to represent the frontier of SNP densities with $m = 2$ and $m = 3$.

[2] Although the SNP density is a function of the parameters ν_0, ν_1, and ν_2, only 2 of them are independent. To allow for $\nu_0 = 0$, we have used spherical coordinates in generating this graph.

To get a clearer sense of the underlying differences between the distributions of z^+ and z^*, we can compare their Gram-Charlier expansions.[3] Since both variables are standardised, both have $c_0 = 1$ and $c_1 = c_2 = 0$. The third and fourth coefficients are functions of the skewness and kurtosis of the distributions, which we have already compared in the previous paragraph. Still, the main difference between z^* and z^+ is found in the higher order coefficients. In particular, whereas (18) imposes that $c_k = 0$ for all $k > 4$, such a restriction no longer holds for z^*. In other words, while the Gram-Charlier expansion of z^+ is finite, the Gram-Charlier expansion of z^* is generally infinite as we can see from (16).

3 Option valuation

3.1 From the real to the risk neutral measure, and vice versa

Consider a frictionless market with a risk free asset and a risky asset with price S_t at time t. Assume that, for $T > t$, S_T can be written in terms of S_t under the real measure $\mathbb{P}$ as:

$$S_T = S_t \exp\left[\left(\mu - \sigma^2/2\right)\tau + \sigma\sqrt{\tau} z^*\right], \tag{19}$$

where $\tau = T - t$, while μ and σ represent the instantaneous drift and volatility, respectively, of S_T, and z^* is defined in (9). That is, z^* is the standardised version of a random variable $x^{\mathbb{P}}$ whose pdf is (1). In this context, we can write the log-return as $y_T = \log(S_T/S_t) = \delta_{\mathbb{P}} + \lambda_{\mathbb{P}} x^{\mathbb{P}}$, where $\delta_{\mathbb{P}} = (\mu - \sigma^2/2)\,\tau + \sigma\sqrt{\tau} a\,(\boldsymbol{\nu})$, $\lambda_{\mathbb{P}} = \sigma\sqrt{\tau} b\,(\boldsymbol{\nu})$, and $x^{\mathbb{P}}$ is a SNP variable with density function (1).

Following Bertholon, Monfort, and Pegoraro (2003), our solution to the option pricing problem will be based on the use of a stochastic discount factor with an exponential affine form:

$$M_{t,T} = \exp(\alpha y_T + \beta), \tag{20}$$

which is consistent with a constant relative risk aversion utility function where α is (minus) the coefficient of relative risk aversion and β the discount factor.

The constraints induced by the arbitrage free conditions are

$$\left.\begin{array}{l} E_{\mathbb{P}}\left[M_{t,T}\exp(r\tau)\right| I_t] = 1, \\ E_{\mathbb{P}}\left[M_{t,T}\exp(y_T)\right| I_t] = 1, \end{array}\right\} \tag{21}$$

[3]Note that (18) is already a proper Gram-Charlier expansion.

where r is the risk-free rate and I_t is the information available at time t. However, for ease of notation we will suppress the explicit dependence on I_t in what follows. Equations (21) give us the two restrictions that we need to find the values of α and β. In particular:

Proposition 5

$$\sum_{k=0}^{2m} \gamma_k(\boldsymbol{\nu}) \frac{(\alpha\lambda_{\mathbb{P}})^k}{\sqrt{k!}} = \exp\left[-\alpha\delta_{\mathbb{P}} - \frac{1}{2}\alpha^2\lambda_{\mathbb{P}}^2 - \beta - r\tau\right], \tag{22}$$

$$\sum_{k=0}^{2m} \gamma_k(\boldsymbol{\nu}) \frac{(1+\alpha)^k \lambda_{\mathbb{P}}^k}{\sqrt{k!}} = \exp\left[-(1+\alpha)\delta_{\mathbb{P}} - \frac{1}{2}(1+\alpha)^2\lambda_{\mathbb{P}}^2 - \beta\right]. \tag{23}$$

In this context, if $\mathbb{Q}$ denotes the risk neutral measure whose numeraire is the risk free asset, the real and risk-neutral measures can be easily related by means of the Radon-Nykodym derivative, which in this case is proportional to the discount factor

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \frac{M_{t,T}}{E_{\mathbb{P}}(M_{t,T})}.$$

Hence

$$E_{\mathbb{Q}}(\digamma(S_T)) = E_{\mathbb{P}}\left[\frac{d\mathbb{Q}}{d\mathbb{P}}\digamma(S_T)\right], \tag{24}$$

where $\digamma(\cdot)$ is an arbitrary function and $E_{\mathbb{P}}(M_{t,T}) = \exp(-r\tau)$, so that the discount factor correctly prices the risk-free asset. As a result, we can obtain the risk-neutral density from (24) as

$$f^{\mathbb{Q}}(y_T) = \exp(r\tau) M_{t,T} f^{\mathbb{P}}(y_T). \tag{25}$$

On this basis, we can fully characterise the risk-neutral measure as follows:

Proposition 6 *The asset price S_T can be written under the risk neutral measure $\mathbb{Q}$ as*

$$S_T = S_t \exp\left[\left(\mu^{\mathbb{Q}} - \frac{(\sigma^{\mathbb{Q}})^2}{2}\right)\tau + \sigma^{\mathbb{Q}}\sqrt{\tau}\kappa^*\right], \tag{26}$$

where κ^* is a standardised SNP variable of the same order as the real SNP variable z^*, whose parameters are:

$$\mu^{\mathbb{Q}} = \mu + \frac{\sigma^2}{2}\left[\left(\frac{b(\boldsymbol{\nu})}{b(\boldsymbol{\theta})}\right)^2 - 1\right] + \frac{\sigma}{\sqrt{\tau}}\left[a(\boldsymbol{\nu}) - a(\boldsymbol{\theta})\frac{b(\boldsymbol{\nu})}{b(\boldsymbol{\theta})}\right] + \alpha\sigma^2 b^2(\boldsymbol{\nu}), \tag{27}$$

$$\sigma^{\mathbb{Q}} = \sigma\frac{b(\boldsymbol{\nu})}{b(\boldsymbol{\theta})}, \tag{28}$$

and $\boldsymbol{\theta} = (\theta_0, \theta_1, \cdots, \theta_m)'$, with

$$\theta_i = \sum_{k=i}^{m} \frac{\nu_i}{(k-i)!}\sqrt{\frac{k!}{i!}}(\alpha\lambda_{\mathbb{P}})^{k-i}. \tag{29}$$

Therefore, in a SNP context the change of measure affects not only the mean and the variance of the log price, but also the higher moments, as can be seen from the difference between $\boldsymbol{\theta}$ and $\boldsymbol{\nu}$. For the case of $m = 2$, we can show that the relation between $\boldsymbol{\theta}$ and $\boldsymbol{\nu}$ is

$$\begin{aligned}
\theta_0 &= \nu_0 + \nu_1 \alpha \lambda_{\mathbb{P}} + \frac{\nu_2}{\sqrt{2}} \alpha^2 \lambda_{\mathbb{P}}^2, \\
\theta_1 &= \nu_1 + \nu_2 \sqrt{2} \alpha \lambda_{\mathbb{P}}, \\
\theta_2 &= \nu_2.
\end{aligned}$$

Obviously, our framework also allows us to value derivative assets by focusing on the risk-neutral measure directly without any reference to its relationship with the real measure, as in Jondeau and Rockinger (2001) or Jurczenko, Maillet, and Negrea (2002a,b). To follow this second approach, we just have to regard $\boldsymbol{\theta}$, $\mu^{\mathbb{Q}}$ and $\sigma^{\mathbb{Q}}$ as the structural parameters. The following proposition gives the expression that the risk-neutral drift must have to satisfy the martingale restriction (see Longstaff, 1995):

Proposition 7

$$\mu^{\mathbb{Q}} = r - \frac{1}{\tau} \left[\sigma^{\mathbb{Q}} \sqrt{\tau} a(\boldsymbol{\theta}) + \frac{1}{2} \left(\sigma^{\mathbb{Q}}\right)^2 \tau \left(b^2(\boldsymbol{\theta}) - 1\right) + \log \Lambda(\boldsymbol{\theta}, \lambda_{\mathbb{Q}}) \right], \tag{30}$$

where

$$\Lambda(\boldsymbol{\theta}, \lambda_{\mathbb{Q}}) = \sum_{k=0}^{2m} \gamma_k(\boldsymbol{\theta}) \frac{\left(\sigma^{\mathbb{Q}} b(\boldsymbol{\theta}) \sqrt{\tau}\right)^k}{\sqrt{k!}}. \tag{31}$$

Not surprisingly, we show in appendix D that (27) and (30) coincide, which implies that both strategies are indeed equivalent. This implies for instance that the value of an option obtained using the real-measure parameters μ, σ and $\boldsymbol{\nu}$ together with the stochastic discount factor $M_{t,T}$ in (20) will be the same as the option price computed in the risk-neutral world using the implied parameters $\boldsymbol{\theta}$, $\sigma^{\mathbb{Q}}$ and the drift $\mu^{\mathbb{Q}}$ in (30). This equivalence result has important computational advantages in empirical applications that only use option price data, because it allows us to estimate the option values from the risk neutral parameters without having to solve the nonlinear equations (22) and (23) within the optimisation algorithm. At the same time, we can always obtain the implied real-measure parameters if necessary. In particular, for a given drift μ, risk-free rate r and risk neutral parameters $\sigma^{\mathbb{Q}}$ and $\boldsymbol{\theta}$, we can recover the parameters of the real measure σ and $\boldsymbol{\nu}$, together with the coefficient of relative risk aversion α, from the following system of equations

$$\left(\mu-\sigma^2/2\right)\tau+\sigma\sqrt{\tau}a\left(\boldsymbol{\nu}\right)=\delta_{\mathbb{Q}}-\alpha\lambda_{\mathbb{Q}}^2,$$

$$\sigma\sqrt{\tau}b\left(\boldsymbol{\nu}\right)=\lambda_{\mathbb{Q}},$$

$$\nu_i=\sum_{k=i}^{m}\frac{\theta_i}{(k-i)!}\sqrt{\frac{k!}{i!}}(-1)^{k-i}\left(\lambda_{\mathbb{Q}}\alpha\right)^{k-i}. \tag{32}$$

Finally, the discount factor β can be obtained from either (22) or (23).

3.2 Option pricing

Let C_t be the value at time t of a European call option with strike price K and expiration at time T, and let S_t be the underlying asset value. We can express C_t as

$$C_t=\exp(-r\tau)E_{\mathbb{Q}}\left[(S_T-K)^+\right], \tag{33}$$

where $(\cdot)^+=\max(\cdot,0)$. If we denote the indicator function as $\mathbf{1}(\cdot)$, and define the region $A=\{S_T>K\}$, then we can rewrite (33) as

$$C_t=\exp(-r\tau)E_{\mathbb{Q}}\left[S_T\mathbf{1}(A)\right]-K\exp(-r\tau)E_{\mathbb{Q}}\left[\mathbf{1}(A)\right]. \tag{34}$$

Following Geman, Karouri, and Rochet (1995), we can simplify the calculations in the above formula by changing the numeraire. Specifically, we consider as additional numeraire the ratio of the risky asset prices S_T/S_t, which gives an alternative risk-neutral measure $\mathbb{Q}_1$. Then, if we use the Radon-Nikodym derivative:

$$\frac{d\mathbb{Q}}{d\mathbb{Q}_1}=\frac{B_T}{B_t}\frac{S_t}{S_T}=\exp(r\tau)\frac{S_t}{S_T}, \tag{35}$$

we can easily express any expectation under $\mathbb{Q}$ in terms of $\mathbb{Q}_1$. Specifically, we will have that

$$E_{\mathbb{Q}}\left[S_T\mathbf{1}(A)\right]=E_{\mathbb{Q}_1}\left[\frac{d\mathbb{Q}}{d\mathbb{Q}_1}S_T\mathbf{1}(A)\right]=S_t\exp(r\tau)E_{\mathbb{Q}_1}\left[\mathbf{1}(A)\right],$$

which, once introduced in (34), gives us the general formula

$$\begin{aligned}C_t &= S_tE_{\mathbb{Q}_1}\left[\mathbf{1}(A)\right]-K\exp(-r\tau)E_{\mathbb{Q}}\left[\mathbf{1}(A)\right]\\ &= S_t\Pr_{\mathbb{Q}_1}\left[S_T>K\right]-K\exp(-r\tau)\Pr_{\mathbb{Q}}\left[S_T>K\right]. \end{aligned}\tag{36}$$

The analytical tractability of the SNP distribution allows us to obtain closed form expressions for the probabilities in (36):

Proposition 8 *The price of a European call option C_t^{SNP} written on a stock S_t with strike price K can be expressed as:*

$$C_t^{SNP} = S_t \Pr_{\mathbb{Q}_1}[x > d] - K\exp(-r\tau)\Pr_{\mathbb{Q}}[x > d], \tag{37}$$

where

$$\begin{aligned} \Pr_{\mathbb{Q}}[x > d] &= \Phi(-d) + \phi(d)\sum_{k=1}^{2m}\frac{\gamma_k(\boldsymbol{\theta})}{\sqrt{k}}H_{k-1}(d), \\ \Pr_{\mathbb{Q}_1}[x > d] &= \exp(-r\tau + \delta_{\mathbb{Q}})\sum_{k=0}^{2m}\gamma_k(\boldsymbol{\theta})I_k^*, \end{aligned}$$

$$I_k^* = \frac{1}{\sqrt{k}}\exp(\lambda_{\mathbb{Q}} d)H_{k-1}(d)\phi(d) + \frac{\lambda_{\mathbb{Q}}}{\sqrt{k}}I_{k-1}^*; \quad I_0^* = \exp(\lambda_{\mathbb{Q}}^2/2)\Phi(\lambda_{\mathbb{Q}} - d), \tag{38}$$

$$\delta_{\mathbb{Q}} = \left(\mu^{\mathbb{Q}} - \frac{\sigma^{\mathbb{Q}2}}{2}\right)\tau + a(\boldsymbol{\theta})\sigma^{\mathbb{Q}}\tau,$$

$$d = \frac{\log(K/S_t) - \delta_{\mathbb{Q}}}{\lambda_{\mathbb{Q}}}; \ \lambda_{\mathbb{Q}} = b(\boldsymbol{\theta})\sigma^{\mathbb{Q}}\sqrt{\tau}$$

and $\Phi(\cdot)$ is the cumulative distribution function of the standard normal density.

As expected, (37) reduces to the Black and Scholes (1973) formula, denoted as C_t^{BS}, when $\theta_0 = 1$ and $\theta_k = 0\ \forall k \geq 1$, which corresponds to a lognormal distribution for the underlying asset. Specifically

$$C_t^{BS} = S_t\Phi(d_1) - K\exp(-r\tau)\Phi(d_2), \tag{39}$$

where

$$d_1 = \frac{\log(S_t/K) + (r + \sigma^2/2)\tau}{\sigma\sqrt{\tau}}, \tag{40}$$

and $d_2 = d_1 - \sigma^{\mathbb{Q}}\sqrt{\tau}$.

In Figure 2 we compare the range of call prices that the SNP density can produce with the corresponding range when the assumed distribution is the restricted Gram-Charlier expansion of Jondeau and Rockinger (2001). Not surprisingly, the higher flexibility of the SNP in modelling skewness and kurtosis that we saw in Figure 1 results in a wider range of call prices. Similarly, a larger value of m also leads to a broader range. Nevertheless, there is a close relationship between the different pricing models: the Gram-Charlier call price formula can be obtained as a fourth-order Taylor expansion of (37), while Black-Scholes corresponds to a second-order one (see appendix C for further details).

Option market participants often use the so-called "Greek" parameters, which measure the sensitivity of the option pricing formula to the different inputs, for hedging purposes. We will compute the three most important sensitivities: Delta, Gamma and Vega. As is well known, Delta measures the sensitivity of the option price to changes in the underlying asset price, Gamma measures the sensitivity of Delta to variations in the underlying asset price; and finally, Vega measures the sensitivity of the option price to volatility movements.

Proposition 9 *When the European call option price is given by (37), the Delta, Gamma and Vega can be written as:*

$$\Delta_c \equiv \frac{\partial C_t}{\partial S_t} = P_1,$$

$$\Gamma_c \equiv \frac{\partial \Delta_c}{\partial S_t} = -\frac{1}{\lambda S_t}\frac{\partial P_1}{\partial d},$$

and

$$\upsilon_c \equiv \frac{\partial C_t}{\partial \sigma} = S_t\sqrt{\tau}\frac{\partial P_1}{\partial \sigma_\tau} - Ke^{-r\tau}\sqrt{\tau}\frac{\partial P_2}{\partial d}\frac{\partial d}{\partial \sigma_\tau},$$

where $\sigma_\tau = \sigma^{\mathbb{Q}}\sqrt{\tau}$, and P_1 and P_2 denote the cumulative probabilities $\Pr_{\mathbb{Q}_1}[x > d]$ and $\Pr_{\mathbb{Q}}[x > d]$, respectively, detailed in Proposition 8.

4 Empirical performance of SNP option pricing

In this section, we apply the SNP option valuation formula (37) in Proposition 8 to S&P 500 index options using the same database as Dumas, Fleming, and Whaley (1998). Option prices were collected every Wednesday between 2:45 p.m. and 3:45 p.m. from June 1988 to December 1993, which makes a total number of 292 days. Options are European-style and expire on the third Friday of each contract month. We will focus on call options, and use the bid-ask mid price for estimation purposes. The riskless interest rate will be proxied by the T-bill rate implied by the average of the bid and ask discounts reported in the *Wall Street Journal*. To account for the presence of dividends, the implied forward price is computed as the current stock price S_t minus the present value of dividends $\bar{D}_t$ times the interest accrued until maturity, i.e. $F_{t,T} = (S_t - \bar{D}_t)\exp(r\tau)$ (see Dumas, Fleming, and Whaley, 1998, for further details).

We will compare the performance of the SNP option valuation framework with the standard Black and Scholes (1973) model. We will also consider the formulas provided by Jondeau and Rockinger (2001), which impose positivity restrictions on the Gram-Charlier density, and a variant of the Black-Scholes model in which volatility is assumed to be a quadratic function of moneyness. We will call this methodology Practitioners'

Black-Scholes, a name inspired by its wide use in the financial industry. In order to guarantee positivity, we will consider the parametrisation

$$\sigma(x) = c_0 + c_1(x - c_2)^2 \tag{41}$$

where $c_0 > 0, c_1 \geq 0$ and $x = F_{t,T}/K$. Finally, note that since we are using implied forward prices, an adjustment in the spirit of Black (1976) is needed in all cases.

We consider separate estimations for short and long maturities, both of which are carried out by minimising the sum of squared pricing errors. To select the short maturity group, we begin by considering call options that mature in 45 days for the first day in the sample. We track those options every week until two weeks before they expire. Then, we move to the next group of options that are 45 days away from expiration and start the tracking process again. At the end, we have data on 3,462 call option prices with median time to expiration of 24 days, and a number of options per day that ranges from 4 to 25, with a median of 11. In the long maturity group we follow an analogous selection process. In particular, we have selected 4,310 call option prices with a median time to maturity of 150 days. The number of prices per day also ranges between 4 and 25, but the median is now 15.

Tables 1a to 1d report root mean square pricing errors (RMSE) of the four competing models when we re-estimate all the parameters each Wednesday. We also provide information on the degree of fit achieved for different degrees of moneyness using the six categories proposed by Bakshi, Cao, and Chen (1997), together with the number of options in each category. Tables 1a and 1c report in-sample RMSE's based on the first four years of data. In contrast, Tables 1b and 1d report out-of-sample results based on pricing errors for each Wednesday in the last year of the sample using the parameters estimated on the previous Wednesday. We can observe in Table 1c that the SNP clearly dominates in the longer maturity group. In the short maturity group, though, the total RMSE of the SNP is slightly higher than the one obtained with the Practitioner's Black-Scholes method (see Table 1a). Nevertheless, we still find a better performance of the SNP for all but the more in the money options. Out-of-sample results also seem to favour our model. In addition, they provide strong evidence against the Practitioner's Black-Scholes approach. Indeed, it seems that this ad-hoc model tends to overfit option prices in-sample.

In Figures 3a and 3b we have plotted the skewness and kurtosis values implied by both the SNP and Joundeau and Rockinger's model. Several important patterns arise from these figures. First, there is high dispersion in the estimated higher order moments, although skewness is usually negative and kurtosis is typically higher than 3. Second, skewness and kurtosis tend to be lower when the time to expiration is longer. Furthermore, skewness and kurtosis in Gram-Charlier densities with positivity restrictions are usually on the frontier of values admitted by these densities. In particular, market prices often suggest a more (negative) skewness than the approach of Jondeau and Rockinger (2001) is able to account for. However, some SNP estimates are also located on the frontier, especially in the short maturity group. Although we could easily enlarge the SNP frontier by simply increasing the order m (see Figure 1), it is interesting to analyse in more detail the possible sources of the high sampling variability.

To do so, we have carried out the following bootstrap exercise. First, we group the SNP pricing errors obtained for short maturities in the six moneyness categories already considered. Then, we simulate prices for a specific but broadly representative day (November 13, 1991), by adding random pricing errors to the 19 prices of that day estimated with the SNP model. In this sense, we sample the errors that we add to each price from the same moneyness category to which that price belongs. In this way, we take into account possible distributional differences between pricing errors for, say, deep in the money and out of the money options. Finally, we re-estimate the SNP model on the simulated data. We plot the implied skewness and kurtosis for 1,000 such simulations in Figure 3c. As we can observe, the estimates are again highly disperse, and basically cover the whole region of negative skewness. Nevertheless, the true option prices have constant parameters by construction, which correspond to skewness of -1.5 and kurtosis of 7.7, approximately (see Figure 3c).

Therefore, it may well be the case that, even if the true parameters are constant, the high variation in skewness and kurtosis that we observe in Figures 3a and 3b simply results from the relatively low number of prices with which we are estimating the daily models. For that reason, we also study the performance of the different models under the assumption that their shape parameters (i.e. $\boldsymbol{\nu}$ in the SNP model, the skewness and kurtosis parameters in the Gram-Charlier density, and c_1 and c_2 in (41)) are time invariant, while volatility (or the intercept c_0 in Practitioner's Black-Scholes) is allowed to change over time as before. Again, we carry out an in-sample and an out-of-sample analysis, which are reported in Tables 2a to 2d. In order to estimate all the parameters for the five year long in-sample period, we use the following iterative zig-zag method. Initially, we estimate volatilities by fixing the remaining shape parameters to some reasonable values, a procedure which is easy to carry out since we can estimate volatility separately for each day. Then we fix those volatility estimates, and obtain the remaining parameters. We iterate this procedure until the change in RMSE between two successive iterations is lower than 10^{-5}. As Tables 2a to 2d show, the results are now even more in favour of the SNP model. Out-of-sample RMSE's are particularly supportive of our model. In addition, if we compare the SNP pricing errors in Tables 2b and 2d with those of Tables 1b and 1d, we can observe that the assumption of a constant $\boldsymbol{\nu}$ does indeed yield much better out-of-sample results. Importantly, the SNP with fixed parameters generally performs better out-of-sample than Practitioner's Black-Scholes with time varying parameters. In terms of skewness and kurtosis, Figure 3d shows that SNP estimations are no longer at the frontier. In contrast, Gram-Charlier estimates are pretty close (short maturities) or exactly on its frontier (longer maturities). In any case, the tendency to lower kurtosis and skewness for longer maturities is observed again, which is consistent with the flatter smiles typically found for longer maturities.

As a sanity check, we have also investigated whether the main differences between Black-Scholes and the SNP model are observed in the tails of the distribution, and not so much in the temporal evolution of the volatilities, which is confirmed by Figure 4.

Another interesting issue is whether the main reason for the rejection of the Black-Scholes model is skewness or excess kurtosis. To find out, we have re-estimated our SNP model with fixed parameters imposing zero skewness first, and then kurtosis equal to 3. Interestingly, it turns out that when we force the skewness to be zero we obtain the Black-Scholes special case. In contrast, if we fix the kurtosis to 3, we obtain substantial negative skewness for both the short and long maturity groups. Hence, it seems that negative skewness plays a more fundamental role in determining option prices than excess kurtosis, especially for longer maturities.

Finally, we compare the estimated risk-neutral densities in Figures 5a to 5d, having obtained the density implied by the Practitioner's Black-Scholes model from the second derivative of the call price with respect to the strike (see Breeden and Litzenberger, 1978).

All the models except Black-Scholes imply negative skewness and more peaked densities due to the presence of leptokurtosis, but they are reasonably similar at the centre. However, zooms of the left tails show that the Practitioner's Black-Scholes model attaches unreasonably high probabilities to extreme negative events. This result is consistent with the fact that the Practitioner's Black-Scholes method gives relatively good results in-sample but extremely unrealistic implications when we extrapolate it out-of-sample. In Figure 6 we provide an additional illustration of this phenomenon. In that figure, we compare the smiles that each model can generate with the bid, ask and mid-price quotes for the same day as in the bootstrap exercise before. Practitioner's Black-Scholes tries to fit a quadratic curve to the smile, at the cost of providing low reliable results at the extremes (see in particular the out-of-the money area). This picture also shows that the rather limited amount of skewness allowed by "positive" Gram-Charlier densities prevents their smile from reproducing the empirical smile as we get deeper in the money. However, lack of liquidity is stronger in deep in-the-money options, so the real importance of this result must be taken with some caution. In any case, the SNP density clearly provides a much better fit to the empirical smile.

5 Conclusions

In this paper we propose the use as a parametric model of the SNP distribution introduced by Gallant and Nychka (1987) for nonparametric estimation purposes. The SNP distribution shares the analytical tractability of truncated Gram-Charlier densities, but, unlike them, it has a density function which is always positive. From the statistical point of view, we give expressions for the moments and the distribution of linear combinations of SNP variables. We also construct a standardised SNP variable and compare it with the standardised Gram-Charlier random variable with positivity restrictions of Jondeau and Rockinger (2001). In this sense, we show that the SNP distribution provides more flexibility in terms of both skewness and kurtosis.

Next, we focus our attention on option pricing. In this respect, we show that if the log of the underlying asset price has a conditional SNP distribution under the real measure, and the stochastic discount factor is exponentially affine, which is consistent with constant relative risk aversion preferences, the log of the underlying asset price will also have a conditional SNP distribution of the same order under the risk-neutral measure. On this basis, we obtain closed form expression for European option prices, as well as their sensitivity to the model inputs (the "Greeks"). Alternatively, we can obtain equivalent option prices by directly assuming that the log of the underlying asset price follows a SNP distribution under the risk-neutral measure, although in this case we must first determine the risk-neutral drift which guarantees that the martingale restriction is satisfied. We also relate our pricing formulas to the ones of Black and Scholes (1973), Corrado and Su (1996, 1997), and Jurczenko, Maillet, and Negrea (2002a). In this respect, we show that their formulas can be obtained as second and fourth order Taylor expansions of our formulas, respectively.

Finally, we carry out an empirical application to the S&P 500 options data of Dumas, Fleming, and Whaley (1998) in which we evaluate the performance of our pricing formulas using the Black and Scholes (1973) model as a benchmark. We also compare our model with the so-called Practitioner's Black-Scholes procedure, which fits a quadratic polynomial to the volatility smile, as well as with Gram-Charlier densities with positivity restrictions. We find that the SNP model generally beats its competitors, both in and out of sample. Interestingly, we also find a high dispersion in the daily estimates of skewness and kurtosis. However, we show with a bootstrap exercise that this effect is probably due to sampling variability. In this sense, we find that the pricing performance of our model improves out-of-sample if we keep the shape parameters constant over time. It is also worth mentioning that although the empirical rejection of the Black-Scholes model is due to the presence of both negative skewness and excess kurtosis, skewness seems to be relatively more important than excess kurtosis, especially for longer maturities.

A fruitful avenue for future research would be to exploit the relationship between real and risk-neutral measures in the estimation of our option pricing model by combining data on the underlying asset price, which is informative about the real measure, with option price data, which contains information about the risk-neutral measure. It would also be interesting to explore possible time varying specifications for the parameters of the model, such as GARCH parametrisations for the volatility (see Heston and Nandi, 2000), and analogous extensions for the remaining shape parameters.

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Appendix

A Properties of Hermite polynomials

The j^{th} derivative of a Hermite polynomial of order k (see Stuart and Ord, 1977), is

$$\frac{d^j}{dx^j}H_k(x) = \sqrt{\frac{k!}{(k-j)!}}H_{k-j}(x)$$

if $j \leq k$, and zero otherwise. Using this result, $H_k(a+b)$ can be expressed as the following finite order Taylor expansion around a

$$\begin{aligned} H_k(a+b) &= \sum_{j=0}^{k} \frac{1}{j!}\,\frac{d^j}{dx^j}H_k(x)\bigg|_{x=a} b^j \\ &= \sum_{j=0}^{k} \frac{1}{j!}\sqrt{\frac{k!}{(k-j)!}}H_{k-j}(a)b^j \end{aligned} \tag{A1}$$

B Distribution of linear combinations

In this appendix, we study the distribution of linear combinations of SNP variables. To address this problem, we will rely on the following lemma, which gives the characteristic function of linear combinations of SNP variables.

Lemma 2 *The characteristic function of the distribution of the linear combination of* n *i.i.d. SNP variables* x_k *with weights* p_k, $(k = 1, 2, \cdots, n)$

$$q = \sum_{k=1}^{n} p_k x_k$$

is

$$\psi_q(t) = \prod_{k=1}^{n}\left\{\exp\left(\frac{-p_k^2 t^2}{2}\right)\sum_{j=0}^{2m}\frac{(ip_k t)^j}{\sqrt{j!}}\gamma_j\right\}.$$

Notice that $\psi_q(t)$ can always be written as

$$\psi_q(t) = \exp\left(\frac{-t^2\left\|p\right\|^2}{2}\right)\sum_{j=0}^{2mn}\frac{(it)^j}{\sqrt{j!}}\left\|p\right\|^j d_j \tag{B2}$$

for some $d_j = d_j(p,\gamma) \quad j = 1, 2, \cdots n$, $p = (p_1, \cdots, p_n)'$, $\gamma(\boldsymbol{\nu}) = (\gamma_0(\boldsymbol{\nu}), \cdots, \gamma_m(\boldsymbol{\nu}))'$, and $\|p\| = \sqrt{\sum_{k=1}^{n} p_k^2}$, where $d_0 = 1$. On this basis, it is easy to find the distribution of q:

Proposition 10 *The random variable* $q = \sum_{k=1}^{n} p_k x_k$ *follows a Gram-Charlier distribution with density function*

$$\varphi(q) = \frac{\phi\left(\frac{q}{\|p\|}\right)}{\|p\|} \sum_{j=0}^{2mn} d_j H_j\left(\frac{q}{\|p\|}\right). \tag{B3}$$

Thus, the distribution of a linear combination of SNP variables can also be written as the product of a standard normal density and a finite but longer sum of Hermite polynomials. Importantly, the resulting density is always positive by construction.

C Relationship with Gram-Charlier option pricing models

Although the closed-form solution for the call price in Proposition 8 is easy to implement in practice, it is not easy to compare it to the Gram-Charlier and Black Scholes pricing models. For this reason, we have obtained the following alternative expression:

Proposition 11 *The call price* C_t^{SNP} *in (37) can be rewritten as the following infinite expansion:*

$$C_t^{SNP} = \xi_0 + \xi_3 sk + \xi_4 (ku - 3) + \zeta, \tag{C4}$$

where

$$\zeta = e^{-r\tau} \sum_{k=5}^{\infty} c_k \int_{\omega}^{\infty} \left(S_T(\kappa^*) - K\right) H_k(\kappa^*) \phi(\kappa^*) \, d\kappa^*,$$

$$\begin{aligned}
\xi_0 &= S_t e^{\left(\mu^{\mathbb{Q}} - r\right)\tau} \Phi(d_1^*) - K e^{-r\tau} \Phi(d_1^* - \sigma_\tau), \\
\xi_3 &= \frac{1}{3!} \sigma_\tau S_t e^{\left(\mu^{\mathbb{Q}} - r\right)\tau} \left[\sigma_\tau^2 \Phi(d_1^*) + (2\sigma_\tau - d_1^*) \phi(d_1^*)\right], \\
\xi_4 &= \frac{1}{4!} \sigma_\tau S_t e^{\left(\mu^{\mathbb{Q}} - r\right)\tau} \left[\sigma_\tau^3 \Phi(d_1^*) + \left(3\sigma_\tau^2 - 3 d_1^* \sigma_\tau + d_1^{*2} - 1\right) \phi(d_1^*)\right],
\end{aligned}$$

$$\begin{aligned}
\omega &= \sigma_\tau - d_1^*, \\
\sigma_\tau &= \sigma^{\mathbb{Q}} \tau, \\
d_1^* &= \frac{\log(S_t/K) + (\mu^* + \sigma^2/2)\tau}{\sigma\sqrt{\tau}},
\end{aligned}$$

$S_T(\kappa^*)$ *is (26) when regarded as a function of the standardised random variable* κ^**, and the values of* c_k *are given in (16).*

Expression (C4) allows us to relate our result to Corrado and Su (1996, 1997), who proposed an option pricing model that accounts for departures from normality in skewness and kurtosis by means of a Gram-Charlier density like (12), truncated after the first five elements of the sum (i.e.: $c_k = 0$, for $k \geq 5$), and without imposing positivity restrictions. In this respect, it is important to mention that the original Corrado-Su formula, apart from containing a mistake in the definition of the Hermite polynomials, does not satisfy the martingale restriction (33). Both these problems are dealt with in Jurczenko, Maillet, and Negrea (2002b), who provide an expression for the drift that satisfies the martingale restriction under the risk-neutral measure, as well as the corresponding option prices. The following result shows that the martingale restriction in Jurczenko, Maillet, and Negrea (2002b) can be regarded as a truncated version of our drift model (30), if we disregard the terms in σ_τ^k for $k > 4$:

Lemma 3 *The drift of the risk neutral price model can be written as*

$$\mu^{\mathbb{Q}} = r - \frac{1}{\tau} \log \left[1 + \frac{sk}{3!}\sigma_\tau^3 + \frac{(ku-3)}{4!}\sigma_\tau^4 + o\left(\sigma_\tau^4\right) \right].$$

On this basis, it is easy to show that the modified Corrado-Su formula is an approximated version of our call formula in which we only retain the elements in σ_τ^k for $0 \leq k \leq 4$ in a Taylor expansion of the SNP call pricing formula:

Proposition 12 *Consider the call price C_t^{SNP} in (C4). Then, if we neglect the term ζ, C_t^{SNP} can be written as*

$$C_t^{SNP} = C_t^{*CS} + o\left(\sigma_\tau^4\right), \tag{C5}$$

where C_t^{*CS} is the modified Corrado-Su formula (see Jurczenko et al., 2002b)

$$C_t^{*CS} = C_t^{*BS} + sk\, Q_3^* + (ku-3)\, Q_4^*, \tag{C6}$$

$$C_t^{*BS} = S_t \Phi(d^*) - K e^{-r\tau} \Phi(d^* - \sigma_\tau),$$

$$d^* = \frac{\log\left(S_t/K\right) + \left(r + \frac{\sigma^2}{2}\right)\tau}{\sigma_\tau} - \frac{\log\left(1 + \frac{sk}{3!}\sigma_\tau^3 + \frac{(ku-3)}{4!}\sigma_\tau^4\right)}{\sigma_\tau},$$

$$Q_3^* = \frac{\sigma_\tau S_t \left(2\sigma_\tau - d^*\right)\phi\left(d^*\right)}{3!\left(1 + \frac{sk}{3!}\sigma_\tau^3 + \frac{(ku-3)}{4!}\sigma_\tau^4\right)},$$

and

$$Q_4^* = \frac{\sigma_\tau S_t \left(3\sigma_\tau^2 - 3d^*\sigma_\tau + d^{*2} - 1\right)\phi\left(d^*\right)}{4!\left(1 + \frac{sk}{3!}\sigma_\tau^3 + \frac{(ku-3)}{4!}\sigma_\tau^4\right)}.$$

The main difference between the SNP model and the modified Corrado-Su formula results from the fact that Corrado and Su do not impose positivity restrictions on the density. In fact, a statistically correct version of the Corrado-Su model should impose the positivity restrictions of Jondeau and Rockinger (2001) on the coefficients of the Gram-Charlier density. In this sense, our SNP assumption, which implicitly guarantees a non-negative density, leads to a slightly more complex formula for the same number of parameters (i.e., for $m = 2$). However, as Proposition 12 shows, if we eliminate the less important terms in (C4), the same fundamental effects of skewness and kurtosis emerge.

We can further explore the insights of our formula by neglecting the terms in σ_τ^k for $k \geq 3$ in a Taylor expansion of (C6). In this way, we can relate the SNP and the Black-Scholes model. A similar result is provided in Jurczenko, Maillet, and Negrea (2002b) for the modified Corrado-Su formula, under the name of "*Simplified Corrado-Su formula*". However, we will not obtain exactly the formula since Jurczenko, Maillet, and Negrea (2002b) approximate d^* by d_1, which implies that they are effectively discarding some terms in σ_τ^2.

Proposition 13 *We can write C_t^{SNP} as*

$$C_t^{SNP} = C_t^{BS} + \beta_3 sk + \beta_4(ku - 3) + o(\sigma_\tau^2), \tag{C7}$$

where C_t^{BS} is the Black-Scholes formula in (39), d_1 is defined in (40) and

$$\begin{aligned} \beta_3 &= \frac{1}{3!} S_t \sigma_\tau \left(\sigma_\tau - d_1\right) \phi\left(d_1\right) \\ &\quad + \frac{1}{3!} K \exp(-r\tau) \phi\left(d_1\right) \sigma_\tau^2, \end{aligned}$$

and

$$\beta_4 = \frac{1}{4!} S_t \sigma_\tau \left(d_1^2 - 3 d_1 \sigma_\tau - 1\right) \phi\left(d_1\right).$$

We can also provide an approximate expression for the implied volatility in the SNP model:

Proposition 14 *Let C_t^{SNP} denote the market price on a European call option. Then the implied volatility Ψ for a given moneyness and time to maturity can be written as*

$$\Psi \simeq \sigma\sqrt{\tau} + \widetilde{\beta}_3 sk + \widetilde{\beta}_4(ku - 3), \tag{C8}$$

where

$$\widetilde{\beta}_3 = \frac{1}{3!} \sigma_\tau \left(2\sigma_\tau - d_1\right) + \frac{1}{3!} \frac{K}{S_t} \exp(-r\tau) \sigma_\tau^2,$$

and

$$\tilde{\beta}_4 = \frac{1}{4!} \sigma_\tau \left(d_1^2 - 3 d_1 \sigma_\tau - 1\right).$$

D Proofs

Proposition 1

We know that

$$\frac{1}{\boldsymbol{\nu}'\boldsymbol{\nu}}\left[\sum_{i=0}^{m}\nu_i H_i(x)\right]^2 = \sum_{i=0}^{m}\sum_{j=0}^{m}\frac{\nu_i\nu_j}{\boldsymbol{\nu}'\boldsymbol{\nu}}H_i\left(x\right)H_j\left(x\right) = \sum_{k=0}^{2m}\gamma_k\left(\boldsymbol{\nu}\right)H_k\left(x\right), \tag{D9}$$

where it is verified that $\forall i, j$

$$H_i\left(x\right)H_j\left(x\right) = \sum_{q\in\Gamma}\frac{1}{\sqrt{q!}}\left(\begin{array}{c} q \\ \frac{i-j+q}{2}\end{array}\right)\left(\prod_{s=0}^{(i-j+q)/2-1}\left(i-s\right)\prod_{s=0}^{(j-i+q)/2-1}\left(j-s\right)\right)^{1/2}H_q\left(x\right), \tag{D10}$$

with

$$\Gamma = \left\{q \in \mathbb{N} : |i-j| \leq q \leq i+j;\ \ \frac{i-j+q}{2} \in \mathbb{N}\right\}.$$

We can rewrite (D10) as

$$\begin{aligned} H_i\left(x\right)H_j\left(x\right) &= \sum_{q\in\Gamma}\frac{\left(i!j!q!\right)^{1/2}}{\left(\frac{i+j-q}{2}\right)!\left(\frac{i+q-j}{2}\right)!\left(\frac{q+j-i}{2}\right)!}H_q\left(x\right) \\ &= \sum_{q\in\Gamma}a_{ij,q}H_q\left(x\right) \end{aligned}$$

after verifying that $a_{ij,q} = a_{iq,j} = a_{ji,q} = a_{jq,i} = a_{qi,j} = a_{qj,i}$ by using some properties of the binomial coefficients. Hence, we will have that

$$\sum_{i=0}^{m}\sum_{j=0}^{m}\frac{\nu_i\nu_j}{\boldsymbol{\nu}'\boldsymbol{\nu}}H_i\left(x\right)H_j\left(x\right) = \sum_{i=0}^{m}\sum_{j=0}^{m}\sum_{k\in\Gamma}\frac{\nu_i\nu_j}{\boldsymbol{\nu}'\boldsymbol{\nu}}a_{i,j,k}H_k\left(x\right). \tag{D11}$$

Finally, if we equate (D9) and (D11), we obtain the desired result.

Proposition 2

Consider the expanded SNP density function (4). Then

$$\begin{aligned} E_f\left[H_k\left(x\right)\right] &= \int_{-\infty}^{\infty}\phi\left(x\right)H_k\left(x\right)\left(\sum_{i=0}^{2m}\gamma_k\left(\boldsymbol{\nu}\right)H_i\left(x\right)\right)dx \\ &= \sum_{i=0}^{2m}\gamma_k\left(\boldsymbol{\nu}\right)E_\phi\left[H_i\left(x\right)H_k\left(x\right)\right] \end{aligned}$$

We can easily obtain (8) by using the property that $E_\phi\left[H_i\left(x\right)H_k\left(x\right)\right] = 1$ if $i = k$ and zero otherwise.

Lemma 1

By using Proposition 2 we can directly obtain the matrices:

$$A_k = \begin{pmatrix} a_{00,k} & & \\ a_{10,k} & a_{11,k} & \\ a_{20,k} & a_{21,k} & a_{22,k} \end{pmatrix}$$

for $k = 1, \ldots, 4$ and $m = 2$. Specifically,

$$\begin{aligned} A_1 &= \begin{pmatrix} 0 & & \\ 1 & 0 & \\ 0 & \sqrt{2} & 0 \end{pmatrix}; \qquad A_2 = \begin{pmatrix} 0 & & \\ 0 & \sqrt{2} & \\ 1 & 0 & 2\sqrt{2} \end{pmatrix}, \\ A_3 &= \begin{pmatrix} 0 & & \\ 0 & 0 & \\ 0 & \sqrt{3} & 0 \end{pmatrix}; \qquad A_4 = \begin{pmatrix} 0 & & \\ 0 & 0 & \\ 0 & 0 & \sqrt{6} \end{pmatrix}. \end{aligned}$$

On this basis, we can directly compute $E_f [H_k (x)]$ in (8). Finally, we can apply the equations in (6) to obtain the values of $\mu'_x (k)$.

Proposition 3

Note that

$$\begin{aligned} E_f \left(e^{tx}\right) &= \sum_{k=0}^{2m} \gamma_k(\boldsymbol{\nu}) \int_{-\infty}^{+\infty} e^{tx} H_k (x) \phi (x) \, dx \\ &= \sum_{k=0}^{2m} \gamma_k(\boldsymbol{\nu}) E_\phi \left[e^{tx} H_k (x)\right], \end{aligned} \tag{D12}$$

and that

$$\int H_k (x) \phi (x) \, dx = \frac{-1}{\sqrt{k}} H_{k-1} (x) \phi (x). \tag{D13}$$

If we consider (D13), and integrate by parts (D12), we obtain:

$$\begin{aligned} E_\phi \left[e^{tx} H_k (x)\right] &= \left[e^{tx} H_{k-1} (x) \phi (x) / \sqrt{k}\right]_{-\infty}^{+\infty} + \frac{t}{\sqrt{k}} E_\phi \left[e^{tx} H_{k-1} (x)\right] \\ &= \frac{t}{\sqrt{k}} E_\phi \left[e^{tx} H_{k-1} (x)\right]. \end{aligned}$$

where the subindex ϕ denotes integration with respect to the standard normal density. By l'Hospital rule, we can then verify that $e^{tx} H_{k-1} (x) \phi (x) \to 0 \ \ \forall k \geq 1$ when $x \to \pm\infty$. Hence,

$$E_\phi \left[e^{tx} H_k (x)\right] = \frac{t^k}{\sqrt{k!}} e^{t^2/2}. \tag{D14}$$

In addition, given (D12) and (D14), we will have that:

$$\begin{aligned} E\left(e^{\lambda x}\right) &= e^{t^2/2}\sum_{k=0}^{2m}\gamma_k(\boldsymbol{\nu})\frac{t^k}{\sqrt{k!}} \\ &= e^{\lambda^2/2}\Lambda(\boldsymbol{\theta},t). \end{aligned}$$

On the other hand, the characteristic function can be written as

$$\begin{aligned} \psi_{snp}(t) &= \int_{-\infty}^{+\infty}\exp\left(itx\right)\phi\left(x\right)\sum_{j=0}^{2m}\gamma_j(\boldsymbol{\nu})H_j\left(x\right)dx \\ &= \sum_{j=0}^{2m}\gamma_j(\boldsymbol{\nu})\int_{-\infty}^{+\infty}\exp\left(itx\right)\phi\left(x\right)H_j\left(x\right)dx, \end{aligned}$$

where

$$\int_{-\infty}^{+\infty}\exp\left(itx\right)\phi\left(x\right)H_0\left(x\right)dx = \exp\left(\frac{-t^2}{2}\right)$$

coincides with the characteristic function of a standard normal variable. Then, using integration by parts we will have that

$$\begin{aligned} \int_{-\infty}^{+\infty}\exp\left(itx\right)\phi\left(x\right)H_1\left(x\right)dx &= -\exp\left(itx\right)\phi\left(x\right)\Big]_{-\infty}^{+\infty} + it\int_{-\infty}^{+\infty}\exp\left(itx\right)\phi\left(x\right)dx \\ &= it\exp\left(\frac{-t^2}{2}\right). \end{aligned}$$

Finally, we can combine the relationships in (2) with

$$H_k'(x) = \sqrt{k}H_{k-1}(x),$$

to show by induction that

$$\int_{-\infty}^{+\infty}\exp\left(itx\right)\phi\left(x\right)H_k\left(x\right)dx = \frac{(it)^k}{\sqrt{k!}}\exp\left(\frac{-t^2}{2}\right).$$

Proposition 4

Consider the generating function of Hermite polynomials (see Bontemps and Meddahi, 2005):

$$\exp\left(zt - \frac{t^2}{2}\right) = \sum_{k=0}^{\infty}\frac{H_k(z)}{\sqrt{k!}}t^k. \tag{D15}$$

Notice that, using both the relation $z = a + bx$ and (D15), we can write the generating function as

$$\begin{aligned} \exp\left(zt - \frac{t^2}{2}\right) &= \exp\left(\frac{b^2t^2}{2}\right)\exp\left(btx - \frac{b^2t^2}{2}\right)\exp\left(at - \frac{t^2}{2}\right) \\ &= \exp\left(\frac{b^2t^2}{2}\right)\left\{\sum_{s=0}^{\infty}\frac{H_s(x)}{\sqrt{s!}}(bt)^s\right\}\left\{\sum_{m=0}^{\infty}\frac{H_m(a)}{\sqrt{m!}}t^m\right\}. \end{aligned} \quad \text{(D16)}$$

If we compute the expected value of the product of the generating function in (D15) times the Hermite polynomial of order i, both with argument x, where x is a standard normal variable, we get:

$$E_\phi\left[\exp\left((a+bx)t - \frac{t^2}{2}\right)H_i(x)\right] = \sum_{k=0}^{\infty}\frac{E_\phi\left[H_k(a+bx)H_i(x)\right]}{\sqrt{k!}}t^k. \quad \text{(D17)}$$

Analogously, we can obtain from (D16) that

$$\begin{aligned} E_\phi\left[\exp\left((a+bx)t - \frac{t^2}{2}\right)H_i(x)\right] &= \exp\left(\frac{b^2t^2}{2}\right)\left\{\sum_{m=0}^{\infty}\frac{H_m(a)}{\sqrt{m!}}t^m\right\} \\ &\quad \times\left\{\sum_{s=0}^{\infty}\frac{E_\phi\left[H_s(x)H_i(x)\right]}{\sqrt{s!}}(bt)^s\right\}. \end{aligned}$$

If we then combine the orthogonality property of the Hermite polynomials with the Taylor expansion for the above exponential function, we obtain

$$\begin{aligned} E_\phi\left[\exp\left((a+bx)t - \frac{t^2}{2}\right)H_i(x)\right] &= \frac{(bt)^i}{\sqrt{i!}}\exp\left(\frac{b^2t^2}{2}\right)\sum_{m=0}^{\infty}\frac{H_m(a)}{\sqrt{m!}}t^m \\ &= \frac{b^i}{\sqrt{i!}}\sum_{j=0}^{\infty}\sum_{m=0}^{\infty}\frac{H_m(a)}{j!2^j\sqrt{m!}}b^{2j}t^{2j+i+m}. \end{aligned}$$

Finally, if we define $l = 2j + i + m$, we can write the above equation as

$$E_\phi\left[\exp\left((a+bx)t - \frac{t^2}{2}\right)H_i(x)\right] = \frac{b^i}{\sqrt{i!}}\sum_{j=0}^{\infty}\sum_{l=i+2j}^{\infty}\frac{H_{l-i-2j}(a)}{j!2^j\sqrt{(l-i-2j)!}}b^{2j}t^l. \quad \text{(D18)}$$

Next, we can find the coefficients that multiply t^k for $k = 0, 1, 2, \cdots$, by comparing (D17) and (D18):

- When $i > k$:

$$E_\phi\left[H_k(a+bx)H_i(x)\right] = 0.$$

- When $i = k$:

$$E_\phi\left[H_i(a+bx)H_i(x)\right] = b^i.$$

- When $k > i$ and $k-i$ is an even number:

$$E_\phi\left[H_k(a+bx)H_i(x)\right] = b^i\sqrt{\frac{k!}{i!}}\sum_{j=0}^{\frac{k-i}{2}}\frac{H_{k-i-2j}(a)}{j!\sqrt{(k-i-2j)!2^j}}b^{2j}.$$

- When $k > i$ and $k-i$ is an odd number:

$$E_\phi\left[H_k(a+bx)H_i(x)\right] = b^i\sqrt{\frac{k!}{i!}}\sum_{j=0}^{\frac{k-i-1}{2}}\frac{H_{k-i-2j}(a)}{j!\sqrt{(k-i-2j)!2^j}}b^{2j}.$$

Proposition 5

Since we can write y_T as $y_T = \delta_{\mathbb{P}} + \lambda_{\mathbb{P}} x^{\mathbb{P}}$, the arbitrage free conditions become

$$\begin{aligned} E_{\mathbb{P}}\left[\exp\left(\alpha\lambda_{\mathbb{P}} x^{\mathbb{P}}\right)\right] &= \exp\left[-\alpha\delta_{\mathbb{P}} - \beta - r\tau\right], \\ E_{\mathbb{P}}\left[\exp\left(\left(1+\alpha\right)\lambda_{\mathbb{P}} x^{\mathbb{P}}\right)\right] &= \exp\left[-\left(1+\alpha\right)\delta_{\mathbb{P}} - \beta\right]. \end{aligned}$$

Then, using Proposition 3, we can easily obtain (22) and (23) from the previous two equations.

Proposition 6

Using (3) and (25) we can write

$$\begin{aligned} f^{\mathbb{Q}}\left(y_T\right) &= \exp(r\tau)\exp(\alpha y_T + \beta) \\ &\quad \times \frac{\phi\left(\frac{y_T-\delta_{\mathbb{P}}}{\lambda_{\mathbb{P}}}\right)}{\boldsymbol{\nu}'\boldsymbol{\nu}\lambda_{\mathbb{P}}}\left[\sum_{i=0}^{m}\nu_i H_i\left(\frac{y_T-\delta_{\mathbb{P}}}{\lambda_{\mathbb{P}}}\right)\right]^2. \end{aligned} \tag{D19}$$

We can rearrange the elements in (D19) as

$$\begin{aligned} f^{\mathbb{Q}}\left(y_T\right) &= \exp(r\tau+\beta)\exp\left(\alpha\delta_{\mathbb{P}} + \frac{\alpha^2\lambda_{\mathbb{P}}^2}{2}\right) \\ &\quad \times \frac{\phi\left(\frac{y_T-\left(\delta_{\mathbb{P}}+\alpha\lambda_{\mathbb{P}}^2\right)}{\lambda_{\mathbb{P}}}\right)}{\boldsymbol{\nu}'\boldsymbol{\nu}\lambda_{\mathbb{P}}}\left[\sum_{i=0}^{m}\nu_i H_i\left(\frac{y_T-\delta_{\mathbb{P}}}{\lambda_{\mathbb{P}}}\right)\right]^2 \qquad \text{(D20)} \\ &= \frac{\phi\left(\frac{y_T-\delta_{\mathbb{Q}}}{\lambda_{\mathbb{Q}}}\right)}{\boldsymbol{\theta}'\boldsymbol{\theta}\lambda_{\mathbb{Q}}}\left[\sum_{i=0}^{m}\theta_i H_i\left(\frac{y_T-\delta_{\mathbb{Q}}}{\lambda_{\mathbb{Q}}}\right)\right]^2, \qquad \text{(D21)} \end{aligned}$$

where $\delta_{\mathbb{Q}} = \delta_{\mathbb{P}} + \alpha\lambda_{\mathbb{P}}^2$, $\lambda_{\mathbb{Q}} = \lambda_{\mathbb{P}}$. The parameters in the vector $\boldsymbol{\theta} = (\theta_0, \theta_1, \cdots \theta_m)$ can be easily obtained by noting that we can always rewrite (D20) in terms of a squared sum of

Hermite polynomials in $(y_T - \delta_{\mathbb{Q}})/\lambda_{\mathbb{Q}}$. That is, we can always find the value of $\boldsymbol{\theta}$ such that

$$\sum_{i=0}^{m} \theta_i H_i \left(\frac{y_T - \delta_{\mathbb{Q}}}{\lambda_{\mathbb{Q}}} \right) = \sum_{i=0}^{m} \nu_i H_i \left(\frac{y_T - \delta_{\mathbb{P}}}{\lambda_{\mathbb{P}}} \right). \tag{D22}$$

Starting from the right-hand side, we can write

$$\sum_{i=0}^{m} \nu_i H_i \left(\frac{y_T - \delta_{\mathbb{P}}}{\lambda_{\mathbb{P}}} \right) = \sum_{i=0}^{m} \nu_i H_i \left(\frac{y_T - \delta_{\mathbb{Q}}}{\lambda_{\mathbb{Q}}} + \alpha\lambda_{\mathbb{P}} \right). \tag{D23}$$

Then, using (A1), we can show that (D23) equals

$$\sum_{k=0}^{m} \sum_{j=0}^{k} \nu_k \frac{1}{j!} \sqrt{\frac{k!}{(k-j)!}} H_{k-j} \left(\frac{y_T - \delta_{\mathbb{Q}}}{\lambda_{\mathbb{Q}}} \right) (\alpha\lambda_{\mathbb{P}})^j ,$$

which, through the change of indices $i = k - j$ becomes

$$\sum_{k=0}^{m} \sum_{i=0}^{k} \nu_k \frac{1}{(k-i)!} \sqrt{\frac{k!}{i!}} H_i \left(\frac{y_T - \delta_{\mathbb{Q}}}{\lambda_{\mathbb{Q}}} \right) (\alpha\lambda_{\mathbb{P}})^{k-i} . \tag{D24}$$

Now, if we compare (D24) with (D22), it is straightforward to find (29). Finally, we only need to check that the integrating constants are equal, i.e.

$$\boldsymbol{\theta}'\boldsymbol{\theta} = \boldsymbol{\nu}'\boldsymbol{\nu} \exp \left(-r\tau - \beta - \alpha\delta_{\mathbb{P}} - \frac{\alpha^2\lambda_{\mathbb{P}}^2}{2} \right). \tag{D25}$$

We have already shown that both (D20) and (D21) are proportional. Since both expressions are well defined densities in the sense that both integrate to one, (D25) must necessarily be satisfied. In consequence, y_T can be written under the risk neutral measure as

$$y_T = \delta_{\mathbb{Q}} + \lambda_{\mathbb{Q}} x_T^{\mathbb{Q}}, \tag{D26}$$

where $x_T^{\mathbb{Q}}$ is a non-standardised SNP variable with parameters $\boldsymbol{\theta}$. Hence, both the real and the risk-neutral measures have a SNP distribution of the same order. In particular, if we express the asset price S_T under the risk-neutral measure as in (26), where $\kappa_T = a(\boldsymbol{\theta}) + b(\boldsymbol{\theta}) x_T^{\mathbb{Q}}$, then we can easily relate the risk-neutral drift and volatility by the following relations

$$\left(\mu^{\mathbb{Q}} - \frac{\left(\sigma^{\mathbb{Q}}\right)^2}{2} \right) \tau + \sigma^{\mathbb{Q}} \sqrt{\tau} a(\boldsymbol{\theta}) = \delta_{\mathbb{Q}}, \tag{D27}$$

$$\sigma^{\mathbb{Q}} \sqrt{\tau} b(\boldsymbol{\theta}) = \lambda_{\mathbb{Q}}. \tag{D28}$$

From (D28), it is straightforward to obtain (28), while the relationship for the drift can easily be found by replacing (28) in (D27).

Proposition 7

Let us start with (27). As we know, (21) implies

$$\begin{aligned} 1 &= E_{\mathbb{P}}\left[M_{t,T}\exp(y_T)\right] \\ &= \exp(-r\tau)E_{\mathbb{Q}}\left[\exp(y_T)\right]. \end{aligned}$$

Hence, since y_T can be written as (D26) in the risk neutral measure, we can use (D14) to show that

$$\exp\left(r\tau - \delta_{\mathbb{P}} - \alpha\lambda_{\mathbb{P}}^2 - \frac{1}{2}\lambda_{\mathbb{P}}^2\right) = \Lambda\left(\boldsymbol{\theta}, \lambda_{\mathbb{Q}}\right), \tag{D29}$$

where $\Lambda\left(\boldsymbol{\theta}, \lambda_{\mathbb{Q}}\right)$ is given in (31). From (D29), we can write

$$\alpha\sigma^2 b^2\left(\boldsymbol{\nu}\right) = r - \mu - \frac{\sigma^2}{2}\left(b^2\left(\boldsymbol{\nu}\right) - 1\right) - \frac{\sigma}{\sqrt{\tau}}a\left(\boldsymbol{\nu}\right) - \log\Lambda\left(\boldsymbol{\theta}, \lambda_{\mathbb{Q}}\right),$$

which, once substituted in (27), yields (30).

Proposition 8

Consider the general option formula (36) and equation (19), and express the set corresponding to $\{S_T > K\}$, denoted as A for brevity, as $\{x > d\}$, where d is given in Proposition 8. Then, (36) can be rewritten as

$$C_t^{SNP} = S_t \Pr_{\mathbb{Q}_1}\left[x > d\right] - Ke^{-r\tau}\Pr_{\mathbb{Q}}\left[x > d\right].$$

If we apply the limits of integration $+\infty$ and d to the indefinite integral (D13), taking into account that $H_k\left(x\right)\phi\left(x\right) \to 0$ when $x \to +\infty$ (use L'Hospital rule), then

$$\int_d^{\infty} H_k\left(x\right)\phi\left(x\right)dx = \frac{1}{\sqrt{k}}H_{k-1}\left(d\right)\phi\left(d\right), \qquad k \geq 1. \tag{D30}$$

Given (4), (D30) and the fact that $\gamma_0 = 1$, we can easily compute:

$$\begin{aligned} \Pr_{\mathbb{Q}}\left[x > d\right] &= \sum_{k=0}^{2m}\gamma_k(\boldsymbol{\theta})\int_d^{+\infty} H_k\left(x\right)\phi\left(x\right)dx \\ &= \Phi\left(-d\right) + \sum_{k=1}^{2m}\frac{\gamma_k(\boldsymbol{\theta})}{\sqrt{k}}H_{k-1}\left(d\right)\phi\left(d\right). \end{aligned}$$

Next, we will solve $E_{\mathbb{Q}_1}\left[\mathbf{1}(A)\right] = \Pr_{\mathbb{Q}_1}\left[x > d\right]$ by working under the $\mathbb{Q}$-measure, for which we must apply the Radon-Nikodym derivative, which in this case is just the inverse of (35), i.e.

$$\frac{d\mathbb{Q}_1}{d\mathbb{Q}} = e^{-r\tau}\frac{S_T}{S_t} = e^{-r\tau + \delta_{\mathbb{Q}} + \lambda_{\mathbb{Q}}x}.$$

Then,

$$
\begin{aligned}
E_{\mathbb{Q}_1}\left[\mathbf{1}(A)\right] &= E_{\mathbb{Q}}\left(\frac{d\mathbb{Q}_1}{d\mathbb{Q}}\mathbf{1}(A)\right) \\
&= e^{-r\tau+\delta_{\mathbb{Q}}}\sum_{k=0}^{2m}\gamma_k(\boldsymbol{\theta})\int_d^{\infty} e^{\lambda x}H_k\left(x\right)\phi\left(x\right)dx \\
&= e^{-r\tau+\delta_{\mathbb{Q}}}\sum_{k=0}^{2m}\gamma_k(\boldsymbol{\theta})E_{\phi}\left[e^{\lambda_{\mathbb{Q}}x}H_k\left(x\right)\mathbf{1}(A)\right]. \qquad \text{(D31)}
\end{aligned}
$$

For the sake of brevity, define I_k^* as $E_{\phi}\left[e^{\lambda_{\mathbb{Q}}x}H_k\left(x\right)\mathbf{1}(A)\right]$. The next step consists in computing I_k^* for each k. When $k=0$, the integral is easy to obtain, namely, $I_0^* = e^{\lambda_{\mathbb{Q}}^2/2}\Phi\left(\lambda - d\right)$. But since $\gamma_0 = 1$, we can rewrite (D31) as

$$
\Pr_{\mathbb{Q}_1}\left[x > d\right] = e^{-r\tau+\delta_{\mathbb{Q}}}\left[e^{\lambda_{\mathbb{Q}}^2/2}\Phi\left(\lambda_{\mathbb{Q}} - d\right) + \sum_{k=1}^{2m}\gamma_k(\boldsymbol{\theta})I_k^*\right].
$$

Now, we will obtain the value of I_k^* when $k \geq 1$. To do so, we will integrate by parts taking (D13) into account, which results in

$$
\begin{aligned}
I_k^* &= \int_d^{\infty} e^{\lambda_{\mathbb{Q}}x}H_k\left(x\right)\phi\left(x\right)dx \qquad \text{(D32)} \\
&= -\left[e^{\lambda_{\mathbb{Q}}x}H_{k-1}\left(x\right)\phi\left(x\right)/\sqrt{k}\right]_d^{\infty} + \frac{\lambda_{\mathbb{Q}}}{\sqrt{k}}\int_d^{\infty} e^{\lambda_{\mathbb{Q}}x}H_{k-1}\left(x\right)\phi\left(x\right)dx \\
&= -\left[e^{\lambda_{\mathbb{Q}}x}H_{k-1}\left(x\right)\phi\left(x\right)/\sqrt{k}\right]_d^{\infty} + \frac{\lambda_{\mathbb{Q}}}{\sqrt{k}}I_{k-1}^*.
\end{aligned}
$$

Since it is verified by applying L'Hospital rule that $e^{\lambda x}H_{k-1}\left(x\right)\phi\left(x\right) \to 0 \;\; \forall k \geq 1$ when $x \to \infty$, then

$$
I_k^* = \frac{1}{\sqrt{k}}e^{\lambda_{\mathbb{Q}}d}H_{k-1}\left(d\right)\phi\left(d\right) + \frac{\lambda_{\mathbb{Q}}}{\sqrt{k}}I_{k-1}^*.
$$

Finally, we can recursively obtain the formula for I_k^* given in (38).

Proposition 9

The following derivatives are easily obtained:

$$
\frac{\partial P_2}{\partial d} = \phi\left(d\right)\left[-1 - d\sum_{k=1}^{2m}\frac{\gamma_k(\boldsymbol{\theta})}{\sqrt{k}}H_{k-1}\left(d\right) + \sum_{k=2}^{2m}\gamma_k(\boldsymbol{\theta})\sqrt{\frac{k-1}{k}}H_{k-2}\left(d\right)\right],
$$

$$
\frac{\partial P_1}{\partial d} = e^{-r\tau+\delta_{\mathbb{Q}}}\sum_{k=0}^{2m}\gamma_k(\boldsymbol{\theta})\frac{\partial I_k^*}{\partial d},
$$

$$
\frac{\partial P_1}{\partial \sigma_\tau} = \frac{\partial \delta_{\mathbb{Q}}}{\partial \sigma_\tau}P_1 + e^{-r\tau+\delta_{\mathbb{Q}}}\sum_{k=0}^{2m}\gamma_k(\boldsymbol{\theta})\frac{\partial I_k^*}{\partial \sigma_\tau},
$$

and

$$\frac{\partial d}{\partial \sigma_\tau} = -\frac{1}{\lambda_{\mathbb{Q}}}\left(\frac{\partial \delta_{\mathbb{Q}}}{\partial \sigma_\tau} + b(\boldsymbol{\theta})d\right).$$

Let $\varphi(d) = e^{\lambda d}\phi(d)$ and consider the following results: $\phi'(x) = -x\phi(x)$, $H_k'(x) = \sqrt{k}H_{k-1}(x)$ and $\varphi'(d) = (\lambda - d)\varphi(d)$, where the superscript "′" denotes the first derivative of a function. The next step is to obtain the remaining derivatives: $\frac{\partial I_k^*}{\partial d}$, $\frac{\partial I_k^*}{\partial \sigma_\tau}$,and $\frac{\partial \delta}{\partial \sigma_\tau}$.

- Let $I_0^* = e^{\lambda_{\mathbb{Q}}^2/2}\Phi(\lambda_{\mathbb{Q}} - d)$, then $\partial I_0^*/\partial d = -e^{\lambda_{\mathbb{Q}}^2/2}\phi(\lambda_{\mathbb{Q}} - d)$. Given this result, the derivative of I_k^*, with respect to d when $k \geq 1$ is:

$$\begin{aligned}\frac{\partial I_k^*}{\partial d} &= \frac{\lambda_{\mathbb{Q}}^k}{\sqrt{k!}}\frac{\partial I_0^*}{\partial d} + \frac{(\lambda_{\mathbb{Q}} - d)\varphi(d)}{\sqrt{k!}}\sum_{j=0}^{k-1}\sqrt{j!}\lambda_{\mathbb{Q}}^{k-j-1}H_j(d)\\ &\quad + \frac{\varphi(d)}{\sqrt{k!}}\sum_{j=1}^{k-1}\sqrt{j}\sqrt{j!}\lambda_{\mathbb{Q}}^{k-j-1}H_{j-1}(d).\end{aligned}$$

- The derivative of I_0^* with respect to σ_τ is:

$$\frac{\partial I_0^*}{\partial \sigma_\tau} = e^{\lambda_{\mathbb{Q}}^2/2}\left[b(\boldsymbol{\theta})\lambda_{\mathbb{Q}}\Phi(\lambda_{\mathbb{Q}} - d) + \phi(\lambda_{\mathbb{Q}} - d)\left(b - \frac{\partial d}{\partial \sigma_\tau}\right)\right],$$

  and the derivative of I_k^* with respect to σ_τ when $k \geq 1$ is:

$$\begin{aligned}\frac{\partial I_k^*}{\partial \sigma_\tau} &= \frac{\lambda_{\mathbb{Q}}^k}{\sqrt{k!}}\left(\frac{b(\boldsymbol{\theta})k}{\lambda_{\mathbb{Q}}}I_0^* + \frac{\partial I_0^*}{\partial \sigma_\tau}\right)\\ &\quad - \left[\left(\frac{\partial \delta_{\mathbb{Q}}}{\partial \sigma_\tau} + d\frac{\partial d}{\partial \sigma_\tau}\right)\left(I_k^* - \frac{\lambda^k}{\sqrt{k!}}e^{\lambda^2/2}\Phi(\lambda_{\mathbb{Q}} - d)\right)\right]\\ &\quad + \frac{b(\boldsymbol{\theta})\varphi(d)}{\sqrt{k!}}\sum_{j=0}^{k-1}\sqrt{j!}(k-j-1)\lambda^{k-j-2}H_j(d)\\ &\quad + \frac{\varphi(d)}{\sqrt{k!}}\frac{\partial d}{\partial \sigma_\tau}\sum_{j=1}^{k-1}\sqrt{j}\sqrt{j!}\lambda_{\mathbb{Q}}^{k-j-1}H_{j-1}(d).\end{aligned}$$

- Finally, given μ^* in (30), we can express $\delta_{\mathbb{Q}}$ as

$$\delta_{\mathbb{Q}} = r\tau - \frac{b^2(\boldsymbol{\theta})\sigma_\tau^2}{2} - \log\Lambda(\boldsymbol{\theta}, \lambda_{\mathbb{Q}}).$$

  Then

$$\frac{\partial \delta_{\mathbb{Q}}}{\partial \sigma_\tau} = -b(\boldsymbol{\theta})\lambda_{\mathbb{Q}} - \frac{1}{\Lambda(\boldsymbol{\theta}, \lambda_{\mathbb{Q}})\sigma_\tau}\sum_{k=1}^{2m}\frac{\gamma_k(\boldsymbol{\theta})\lambda_{\mathbb{Q}}^k k}{\sqrt{k!}}.$$

Lemma 2

Due to the independence of the variables $x_1, x_2, \cdots, x_n$, we can transform the n-dimensional integral

$$\psi_q(t) = \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} \exp(itq_1)\, dF(x_1) \cdots dF(x_n)$$

into n one-dimensional integrals, each of which corresponds to the characteristic function of a SNP variable:

$$\begin{aligned}
\psi_q(t) &= \int_{-\infty}^{+\infty} \exp(itp_1 x_1)\, dF(x_1) \cdots \int_{-\infty}^{+\infty} \exp(itp_n x_n)\, dF(x_n) \\
&= \prod_{k=1}^{n} \psi_{SNP}(p_k t),
\end{aligned}$$

which yields the required result.

Proposition 10

If we compute the density function of (B3), we will have that

$$\int_{-\infty}^{+\infty} \exp(itq) \frac{\phi\left(\frac{q}{\|p\|}\right)}{\|p\|} \sum_{j=0}^{2mn} d_j H_j\left(\frac{q}{\|p\|}\right) dq$$

which, with the change of variable $x = q/\|p\|$, becomes

$$\int_{-\infty}^{+\infty} \exp(it\,\|p\|\, x)\, \phi(x) \sum_{j=0}^{2mn} d_j H_j(x)\, dx.$$

Then, we can use Proposition 3 to conclude that the result of the previous integral is (B2).

Proposition 11

Given (26) for S_T where κ^* has a pdf defined in (14), and considering (17), we have that

$$\begin{aligned}
g(\kappa^*) &= \phi(\kappa^*) \sum_{k=0}^{\infty} c_k H_k(\kappa^*) \\
&= \phi(\kappa^*) \left[1 + \frac{sk}{\sqrt{3!}} H_3(\kappa^*) + \frac{ku-3}{\sqrt{4!}} + \sum_{k=5}^{\infty} c_k H_k(\kappa^*)\right].
\end{aligned}$$

Therefore, the call price C_t^{SNP} can be rewritten as:

$$\begin{aligned} C_t^{SNP} &= \xi_0 + \xi_3 sk + \xi_4 (ku-3) + \zeta \\ &= e^{-r\tau} \int_\omega^\infty \left(S_T\left(\kappa^*\right) - K\right) \phi\left(\kappa^*\right) d\kappa^* \\ &\quad + \frac{sk}{\sqrt{3!}} e^{-r\tau} \int_\omega^\infty \left(S_T\left(\kappa^*\right) - K\right) H_3\left(\kappa^*\right) \phi\left(\kappa^*\right) d\kappa^* \\ &\quad + \frac{ku-3}{\sqrt{4!}} e^{-r\tau} \int_\omega^\infty \left(S_T\left(\kappa^*\right) - K\right) H_4\left(\kappa^*\right) \phi\left(\kappa^*\right) d\kappa^* \\ &\quad + e^{-r\tau} \sum_{k=5}^\infty c_k \int_\omega^\infty \left(S_T\left(\kappa^*\right) - K\right) H_k\left(\kappa^*\right) \phi\left(\kappa^*\right) d\kappa^*, \end{aligned}$$

where ω is such that $S_T(\omega) = K$. Next, we will compute the values of ξ.

- For ξ_0:

$$\begin{aligned} \xi_0 &= e^{-r\tau} \int_\omega^\infty \left(S_T\left(\kappa^*\right) - K\right) \phi\left(\kappa^*\right) d\kappa^* \\ &= S_t e^{-r\tau + \overline{\mu}_\tau} \int_\omega^\infty e^{\sigma_\tau \kappa^*} \phi\left(\kappa^*\right) d\kappa^* - K e^{-r\tau} \Phi\left(-\omega\right) \\ &= S_t e^{\left(\mu^{\mathbb{Q}} - r\right)\tau} \Phi\left(d_1^*\right) - K e^{-r\tau} \Phi\left(d_1^* + \sigma_\tau\right), \end{aligned}$$

where $\overline{\mu}_\tau = \left(\mu^{\mathbb{Q}} - \sigma^2/2\right)\tau$ and $d_1^* = \sigma_\tau - \omega$.

To obtain ξ_3 and ξ_4, we will use (38) and (D30). Specifically:

- For ξ_3:

$$\begin{aligned} \xi_3 &= \frac{1}{\sqrt{3!}} e^{-r\tau} \int_\omega^\infty \left(S_T\left(\kappa^*\right) - K\right) H_3\left(\kappa^*\right) \phi\left(\kappa^*\right) d\kappa^* \\ &= \frac{1}{\sqrt{3!}} \left\{ S_t e^{-r\tau + \overline{\mu}_\tau} \int_\omega^\infty e^{\sigma_\tau \kappa^*} H_3\left(\kappa^*\right) \phi\left(\kappa^*\right) d\kappa^* - K e^{-r\tau} \int_\omega^\infty H_3\left(\kappa^*\right) \phi\left(\kappa^*\right) d\kappa^* \right\} \\ &= \frac{1}{\sqrt{3!}} \left\{ S_t e^{-r\tau + \overline{\mu}_\tau} I_3^*\left(\sigma_\tau, \omega\right) - \frac{1}{\sqrt{3}} K e^{-r\tau} H_2\left(\omega\right) \phi\left(\omega\right) \right\}, \end{aligned} \tag{D33}$$

where $I_3^*\left(\sigma_\tau, \omega\right)$ denotes the value of I_k^* for $k = 3$ as a function on (σ_τ, ω) instead of (λ, d). Since

$$e^{\sigma_\tau \omega} = \frac{K e^{-\overline{\mu}_\tau}}{S_t},$$

then

$$I_3^*\left(\sigma_\tau, \omega\right) = \frac{e^{\sigma_\tau^2/2}}{\sqrt{3!}} \left[\sigma_\tau^3 \Phi\left(\sigma_\tau - \omega\right) + \frac{K e^{-\mu^* \tau}}{S_t} \phi\left(\omega\right) \sum_{j=0}^{2} \sqrt{j!} \sigma_\tau^{2-j} H_j\left(\omega\right) \right].$$

Plugging $I_3^*(\sigma_\tau, \omega)$ into equation (D33), we finally obtain

$$\begin{aligned}
\xi_3 &= \frac{e^{\left(\mu^{\mathbb{Q}}-r\right)\tau}}{3!}\left[S_t\sigma_\tau^3\Phi\left(\sigma_\tau-\omega\right)+Ke^{-\mu^*\tau}\phi\left(\omega\right)\sum_{j=0}^{2}\sqrt{j!}\sigma_\tau^{2-j}H_j\left(\omega\right)\right] \\
&\quad -\frac{1}{\sqrt{3!}}\frac{1}{\sqrt{3}}Ke^{-r\tau}H_2\left(\omega\right)\phi\left(\omega\right) \\
&= \frac{e^{\left(\mu^{\mathbb{Q}}-r\right)\tau}}{3!}S_t\sigma_\tau^3\Phi\left(\sigma_\tau-\omega\right)+\frac{K}{3!}e^{-r\tau}\phi\left(\omega\right)\left[\sigma_\tau^2+\sigma_\tau\omega\right]. \qquad \text{(D34)}
\end{aligned}$$

Following the same idea as Jurczenko, Maillet, and Negrea (2002a), we can write:

$$\left(\sigma_\tau-\omega\right)^2=\omega^2+2\log\left(S_te^{\mu^*\tau}/K\right),$$

so that

$$\phi\left(\sigma_\tau-\omega\right)=\left(K/S_te^{\mu^*\tau}\right)\phi\left(\omega\right),$$

which implies that

$$K\phi\left(\omega\right)=S_te^{\mu^*\tau}\phi\left(\sigma_\tau-\omega\right).$$

If we substitute the above equation into (D34), we obtain:

$$\begin{aligned}
\xi_3 &= \frac{\sigma_\tau}{3!}S_te^{(\mu^*-r)\tau}\left[\sigma_\tau^2\Phi\left(\sigma_\tau-\omega\right)+\left(\sigma_\tau+\omega\right)\phi\left(\sigma_\tau-\omega\right)\right] \\
&= \frac{\sigma_\tau}{3!}S_te^{(\mu^*-r)\tau}\left[\sigma_\tau^2\Phi\left(d_1^*\right)+\left(2\sigma_\tau-d_1^*\right)\phi\left(d_1^*\right)\right].
\end{aligned}$$

- For ξ_4:

$$\begin{aligned}
\xi_4 &= \frac{1}{\sqrt{4!}}e^{-r\tau}\int_\omega^\infty\left(S_T\left(\kappa^*\right)-K\right)H_4\left(\kappa^*\right)\phi\left(\kappa^*\right)d\kappa^* \\
&= \frac{1}{\sqrt{4!}}\left\{S_te^{-r\tau+\overline{\mu}_\tau}I_4^*\left(\sigma_\tau,\omega\right)-\frac{1}{\sqrt{4}}Ke^{-r\tau}H_3\left(\omega\right)\phi\left(\omega\right)\right\}.
\end{aligned}$$

Following the same procedure as in ξ_3, we can show that:

$$\xi_4=\frac{\sigma_\tau}{4!}S_te^{(\mu^*-r)\tau}\left[\sigma_\tau^3\Phi\left(d_1^*\right)+\left(3\sigma_\tau^2-3d_1^*\sigma_\tau+d_1^{*2}-1\right)\phi\left(d_1^*\right)\right].$$

Lemma 3

From (30), we have

$$\mu^{\mathbb{Q}}=r-\frac{1}{\tau}\log\left[\exp\left(\sigma_\tau a(\boldsymbol{\theta})+\frac{1}{2}\sigma_\tau^2\left(b^2(\boldsymbol{\theta})-1\right)\right)\sum_{k=0}^{2m}\gamma_k(\boldsymbol{\theta})\frac{(\sigma_\tau b(\boldsymbol{\theta}))^k}{\sqrt{k!}}\right], \qquad \text{(D35)}$$

where

$$\exp\left(\sigma_\tau a(\boldsymbol{\theta}) + \frac{1}{2}\sigma_\tau^2\left(b^2(\boldsymbol{\theta})-1\right)\right) = 1 + a(\boldsymbol{\theta})\sigma_\tau + \frac{a^2(\boldsymbol{\theta})+b^2(\boldsymbol{\theta})-1}{2}\sigma_\tau^2$$
$$+\frac{a^3(\boldsymbol{\theta})+3a(\boldsymbol{\theta})b^2(\boldsymbol{\theta})-3a(\boldsymbol{\theta})}{6}\sigma_\tau^3$$
$$+\frac{3b^4(\boldsymbol{\theta})-6b^2(\boldsymbol{\theta})+3+6a^2(\boldsymbol{\theta})b^2(\boldsymbol{\theta})-6a^2(\boldsymbol{\theta})+a^4(\boldsymbol{\theta})}{24}\sigma_\tau^4$$
$$+o(\sigma_\tau^4).$$

Then, from Proposition 1 we obtain that

$$\begin{aligned}
\gamma_0(\boldsymbol{\theta}) &= 1,\\
\gamma_1(\boldsymbol{\theta}) &= \mu'_x(1) = \frac{-a(\boldsymbol{\theta})}{b(\boldsymbol{\theta})},\\
\gamma_2(\boldsymbol{\theta}) &= \frac{\mu'_x(2)-1}{\sqrt{2}} = \frac{a^2(\boldsymbol{\theta})-b^2(\boldsymbol{\theta})+1}{b^2(\boldsymbol{\theta})\sqrt{2}},\\
\gamma_3(\boldsymbol{\theta}) &= \frac{\mu'_x(3)-3\mu'_x(1)}{\sqrt{3!}},\\
&= \frac{sk(\boldsymbol{\theta})-a^3(\boldsymbol{\theta})-3a(\boldsymbol{\theta})+3a(\boldsymbol{\theta})b^2(\boldsymbol{\theta})}{b^3(\boldsymbol{\theta})\sqrt{3!}},\\
\gamma_4(\boldsymbol{\theta}) &= \frac{\mu'_x(4)-6\mu'_x(2)+3}{\sqrt{4!}}\\
&= \frac{6a^2(\boldsymbol{\theta})-6a^2(\boldsymbol{\theta})b^2(\boldsymbol{\theta})-6b^2(\boldsymbol{\theta})+3b^4(\boldsymbol{\theta})+3}{b^4(\boldsymbol{\theta})\sqrt{4!}}\\
&\quad+\frac{6a^2(\boldsymbol{\theta})-6a^2(\boldsymbol{\theta})b^2(\boldsymbol{\theta})-6b^2(\boldsymbol{\theta})+3b^4(\boldsymbol{\theta})+3}{b^4(\boldsymbol{\theta})\sqrt{4!}}
\end{aligned}$$

Next, if we use the property that $o(n^p)o(n^q) = o(n^{p+q})$ (see Davidson and MacKinnon, 1993), we will have

$$\exp\left(\sigma_\tau a(\boldsymbol{\theta}) + \frac{1}{2}\sigma_\tau^2\left(b^2(\boldsymbol{\theta})-1\right)\right)\underbrace{\sum_{k=0}^{2m}\gamma_k(\boldsymbol{\theta})\frac{(\sigma_\tau b(\boldsymbol{\theta}))^k}{\sqrt{k!}}}_{o(\sigma_\tau^0)} = \left[\sum_{k=0}^{4}\gamma_k(\boldsymbol{\theta})\frac{(\sigma_\tau b(\boldsymbol{\theta}))^k}{\sqrt{k!}}\right]$$
$$\times\left[1 + a(\boldsymbol{\theta})\sigma_\tau + \frac{a^2(\boldsymbol{\theta})+b^2(\boldsymbol{\theta})-1}{2}\sigma_\tau^2 + \frac{a^3(\boldsymbol{\theta})+3a(\boldsymbol{\theta})b^2(\boldsymbol{\theta})-3a(\boldsymbol{\theta})}{6}\sigma_\tau^3\right.$$
$$\left.+\frac{3b^4(\boldsymbol{\theta})-6b^2(\boldsymbol{\theta})+3+6a^2(\boldsymbol{\theta})b^2(\boldsymbol{\theta})-6a^2(\boldsymbol{\theta})+a^4(\boldsymbol{\theta})}{24}\sigma_\tau^4\right] + o(\sigma_\tau^4).$$

Finally, we can use tedious but otherwise straightforward algebraic operations to show that a Taylor expansion of the argument in the logarithm of (D35) around $\sigma_\tau = 0$ yields the proposed result.

Proposition 12

We can rewrite (C4) as

$$\begin{aligned} C_t^{SNP} &= S_t e^{\left(\mu^{\mathbb{Q}}-r\right)\tau}\Phi\left(d_1^*\right)\left[1+\frac{sk}{3!}\sigma_\tau^3+\frac{(ku-3)}{4!}\sigma_\tau^4\right]-Ke^{-r\tau}\Phi\left(d_1^*-\sigma_\tau\right) \\ &+\frac{sk}{3!}\sigma_\tau S_t e^{\left(\mu^{\mathbb{Q}}-r\right)\tau}\left(2\sigma_\tau-d_1^*\right)\phi\left(d_1^*\right) \\ &+\frac{(ku-3)}{4!}\sigma_\tau S_t e^{\left(\mu^{\mathbb{Q}}-r\right)\tau}\left(3\sigma_\tau^2-3d_1^*\sigma_\tau+d_1^{*2}-1\right)\phi\left(d_1^*\right), \end{aligned} \quad \text{(D36)}$$

where we have neglected ζ. From lemma 3, we finally have that

$$\begin{aligned} \exp[(\mu^{\mathbb{Q}}-r)\tau] &= \frac{1}{1+\frac{sk}{3!}\sigma_\tau^3+\frac{(ku-3)}{4!}\sigma_\tau^4+o(\sigma_\tau^4)} \\ &= \frac{1}{1+\frac{sk}{3!}\sigma_\tau^3+\frac{(ku-3)}{4!}\sigma_\tau^4}+o(\sigma_\tau^4) \end{aligned}$$

because as $o(n^0)+o(n^p)=o(n^0)$ (see Davidson and MacKinnon, 1993), which, substituted into (D36), gives

$$\begin{aligned} C_t^{SNP} &= S_t\Phi\left(d_1^*\right)-Ke^{-r\tau}\Phi\left(d_1^*-\sigma_\tau\right) \\ &+\frac{sk}{3!}\sigma_\tau S_t\frac{\left(2\sigma_\tau-d_1^*\right)\phi\left(d_1^*\right)}{1+\frac{sk}{3!}\sigma_\tau^3+\frac{(ku-3)}{4!}\sigma_\tau^4} \\ &+\frac{(ku-3)}{4!}\sigma_\tau S_t\frac{\left(3\sigma_\tau^2-3d_1^*\sigma_\tau+d_1^{*2}-1\right)\phi\left(d_1^*\right)}{1+\frac{sk}{3!}\sigma_\tau^3+\frac{(ku-3)}{4!}\sigma_\tau^4}. \end{aligned} \quad \text{(D37)}$$

Then, using again lemma 3, we can obtain the relationship

$$d_1^* = d^* + o\left(\sigma_\tau^4\right),$$

which, once introduced in (D37), yields the Corrado-Su modified formula after neglecting the terms $o\left(\sigma_\tau^4\right)$.

Proposition 13

Expanding d_1^* around d_1, we have

$$\begin{aligned} d_1^* &= d_1-\frac{1}{\sigma\sqrt{\tau}}\log\left(1+\frac{sk}{3!}\sigma_\tau^3+\frac{(ku-3)}{4!}\sigma_\tau^4+o\left(\sigma_\tau^4\right)\right) \\ &= d_1-\frac{sk}{3!}\sigma_\tau^2-\frac{(ku-3)}{4!}\sigma_\tau^3+o\left(\sigma_\tau^3\right), \end{aligned}$$

$$\begin{aligned} \Phi\left(d_1^*\right) &= \Phi\left(d_1\right)-\phi\left(d_1\right)\frac{sk}{3!}\sigma_\tau^2+o\left(\sigma_\tau^2\right) \\ \Phi\left(d_1^*-\sigma_\tau\right) &= \Phi\left(d_1-\sigma_\tau\right)-\phi\left(d_1-\sigma_\tau\right)\frac{sk}{3!}\sigma_\tau^2+o\left(\sigma_\tau^2\right) \\ &= \Phi\left(d_1-\sigma_\tau\right)-\phi\left(d_1\right)\frac{sk}{3!}\sigma_\tau^2+o\left(\sigma_\tau^2\right), \end{aligned}$$

$$\frac{sk}{3!}\sigma_\tau S_t\frac{(2\sigma_\tau - d_1^*)\,\phi\,(d_1^*)}{1+\frac{sk}{3!}\sigma_\tau^3+\frac{(ku-3)}{4!}\sigma_\tau^4} = \frac{sk}{3!}\frac{\sigma_\tau S_t\,(2\sigma_\tau - d_1)\,\phi\,(d_1)+o(\sigma_\tau^2)}{1+o(\sigma_\tau^2)}$$

$$= \frac{sk}{3!}\sigma_\tau S_t\,(2\sigma_\tau - d_1)\,\phi\,(d_1)+o\left(\sigma_\tau^2\right),$$

and

$$\frac{(ku-3)}{4!}\sigma_\tau S_t\frac{(3\sigma_\tau^2-3d_1^*\sigma_\tau+d_1^{*2}-1)\,\phi\,(d_1^*)}{1+\frac{sk}{3!}\sigma_\tau^3+\frac{(ku-3)}{4!}\sigma_\tau^4} = \frac{(ku-3)}{4!}\sigma_\tau S_t\left(d_1^2-3d_1\sigma_\tau-1\right)\phi\,(d_1)\,.$$

Then, we can easily take a Taylor series expansion of (D37) around $\sigma_\tau = 0$. If we only retain the terms in σ_τ^k, for $k = 0, 1, 2$, we finally obtain the desired result.

Proposition 14

Ψ is the implied volatility that equates the call market price C_t to the Black-Scholes formula, i.e. $C_t = C_t^{BS}(\Psi)$ where $C_t^{BS}(.)$ is the Black-Scholes formula in (39). Following Jurczenko, Maillet, and Negrea (2002a), we can take a linear approximation of (39) around the true volatility σ_τ of the underlying asset

$$C_t = C_t^{BS}(\Psi) = C_t^{BS}(\sigma_\tau) + \left.\frac{\partial C_t^{BS}(x)}{\partial x}\right|_{x=\sigma_\tau}(\Psi - \sigma_\tau)$$

Since

$$\left.\frac{\partial C_t^{BS}(x)}{\partial x}\right|_{x=\sigma_\tau} = K\phi\,[d_1(\sigma_\tau)-\sigma_\tau] = S_t e^{r\tau}\phi\,[d_1(\sigma_\tau)]\,,$$

then

$$C_t \simeq C_t^{BS}(\sigma_\tau) + S_t\phi\,[d_1(\sigma_\tau)]\,(\Psi - \sigma_\tau)\,. \tag{D38}$$

Finally, if the call market price follows the SNP model, i.e. $C_t = C_t^{SNP}$, we can equate (C7) and (D38) to obtain the approximation to Ψ given in (C8).

Table 1a

In-sample RMSE for the short maturity group with time-varying parameters.

Moneyness	Black-Scholes	Pract. Black-Scholes	Jondeau-Rockinger	SNP	N
< 0.94	0.488	0.127	0.216	**0.116**	65
0.94-0.97	0.542	0.137	0.209	**0.111**	287
0.97-1.00	0.489	0.143	0.193	**0.126**	450
1.00-1.03	0.291	0.176	0.156	**0.139**	439
1.03-1.06	0.662	0.160	0.166	**0.131**	434
>1.06	0.732	**0.284**	0.432	0.338	1,176
Total	0.611	**0.218**	0.310	0.239	2,851

Table 1b

Out-of-sample RMSE for the short maturity group with time-varying parameters.

Moneyness	Black-Scholes	Pract. Black-Scholes	Jondeau-Rockinger	SNP	N
< 0.94	0.637	0.079	0.391	**0.039**	2
0.94-0.97	0.855	**0.238**	0.759	0.431	40
0.97-1.00	1.044	**0.531**	0.841	0.665	91
1.00-1.03	0.836	**0.721**	0.770	0.748	107
1.03-1.06	1.041	1.141	0.677	**0.760**	106
>1.06	1.069	5.956	0.882	**0.851**	259
Total	1.008	3.944	0.815	**0.768**	605

Table 1c

In-sample RMSE for the long maturity group with time-varying parameters.

Moneyness	Black-Scholes	Pract. Black-Scholes	Jondeau-Rockinger	SNP	N
< 0.94	1.877	0.326	0.837	**0.258**	360
0.94-0.97	1.634	0.284	0.626	**0.209**	366
0.97-1.00	1.196	0.244	0.376	**0.191**	458
1.00-1.03	0.634	0.222	0.332	**0.215**	475
1.03-1.06	0.974	0.247	0.459	**0.166**	440
>1.06	1.661	0.381	0.452	**0.287**	1,600
Total	1.464	0.320	0.502	**0.245**	3,699

Table 1d

Out-of-sample RMSE for the long maturity group with time-varying parameters.

Moneyness	Black-Scholes	Pract. Black-Scholes	Jondeau-Rockinger	SNP	N
< 0.94	2.045	**0.402**	0.853	0.597	36
0.94-0.97	2.165	0.944	0.960	**0.750**	58
0.97-1.00	1.666	0.984	1.000	**0.936**	65
1.00-1.03	1.130	**0.858**	1.106	1.094	89
1.03-1.06	1.367	**0.770**	0.985	0.923	93
>1.06	1.841	1.948	0.918	**1.170**	253
Total	1.715	1.422	0.968	**1.033**	594

Notes: In-sample analysis uses different parameters for each Wednesday from 1988 to 1992, while Out-of-sample tables use the parameters estimated on the previous Wednesday during 1993. Moneyness is defined as the ratio of the implicit forward price of the underlying asset to the strike price. N denotes the number of option prices per moneyness category.

Table 2a

In-sample RMSE for the short maturity group with fixed shape parameters.

Moneyness	Black-Scholes	Pract. Black-Scholes	Jondeau-Rockinger	SNP	N
< 0.94	0.488	**0.211**	0.450	0.231	65
0.94-0.97	0.542	0.297	0.456	**0.257**	287
0.97-1.00	0.489	0.286	0.403	**0.246**	450
1.00-1.03	0.291	0.218	0.274	**0.206**	439
1.03-1.06	0.662	0.294	0.283	**0.293**	434
>1.06	0.732	0.503	**0.303**	0.442	1,176
Total	0.611	0.384	0.357	**0.343**	2,851

Table 2b

Out-of-sample RMSE for the short maturity group with fixed shape parameters.

Moneyness	Black-Scholes	Pract. Black-Scholes	Jondeau-Rockinger	SNP	N
< 0.94	0.637	**0.129**	0.225	0.132	2
0.94-0.97	0.855	0.554	0.407	**0.427**	40
0.97-1.00	1.044	2.697	**0.668**	0.695	91
1.00-1.03	0.836	3.088	0.724	**0.719**	107
1.03-1.06	1.035	1.742	0.638	**0.632**	106
>1.06	1.064	1.138	0.882	**0.814**	259
Total	1.005	1.967	0.759	**0.728**	605

Table 2c

In-sample RMSE for the long maturity group with fixed shape parameters.

Moneyness	Black-Scholes	Pract. Black-Scholes	Jondeau-Rockinger	SNP	N
< 0.94	1.877	0.582	0.851	**0.553**	360
0.94-0.97	1.634	0.526	0.636	**0.454**	366
0.97-1.00	1.196	0.406	0.406	**0.356**	458
1.00-1.03	0.634	0.270	0.343	**0.265**	475
1.03-1.06	0.974	0.312	0.449	**0.253**	440
>1.06	1.661	0.587	0.530	**0.514**	1,600
Total	1.464	0.501	0.540	**0.444**	3,699

Table 2d

Out-of-sample RMSE for the long maturity group with fixed shape parameters.

Moneyness	Black-Scholes	Pract. Black-Scholes	Jondeau-Rockinger	SNP	N
< 0.94	2.045	1.635	0.765	**0.419**	36
0.94-0.97	2.153	4.179	0.898	**0.720**	58
0.97-1.00	1.654	9.822	1.012	**0.987**	65
1.00-1.03	1.102	10.088	1.047	**0.988**	89
1.03-1.06	1.358	7.003	0.923	**0.882**	93
>1.06	1.838	3.541	**0.848**	0.905	253
Total	1.703	6.405	0.912	**0.887**	594

Notes: In-sample analysis uses different volatility parameters for each Wednesday, but all the other parameters are kept fixed, from 1988 to 1992. Out-of-sample tables use for each week in 1993 the volatility estimated in the previous week while the remaining parameters are those estimated for the first five years. Moneyness is defined as the ratio of the implicit forward price of the underlying asset to the strike price. N denotes the number of option prices per moneyness category.

Figure 1
Regions of skewness and kurtosis

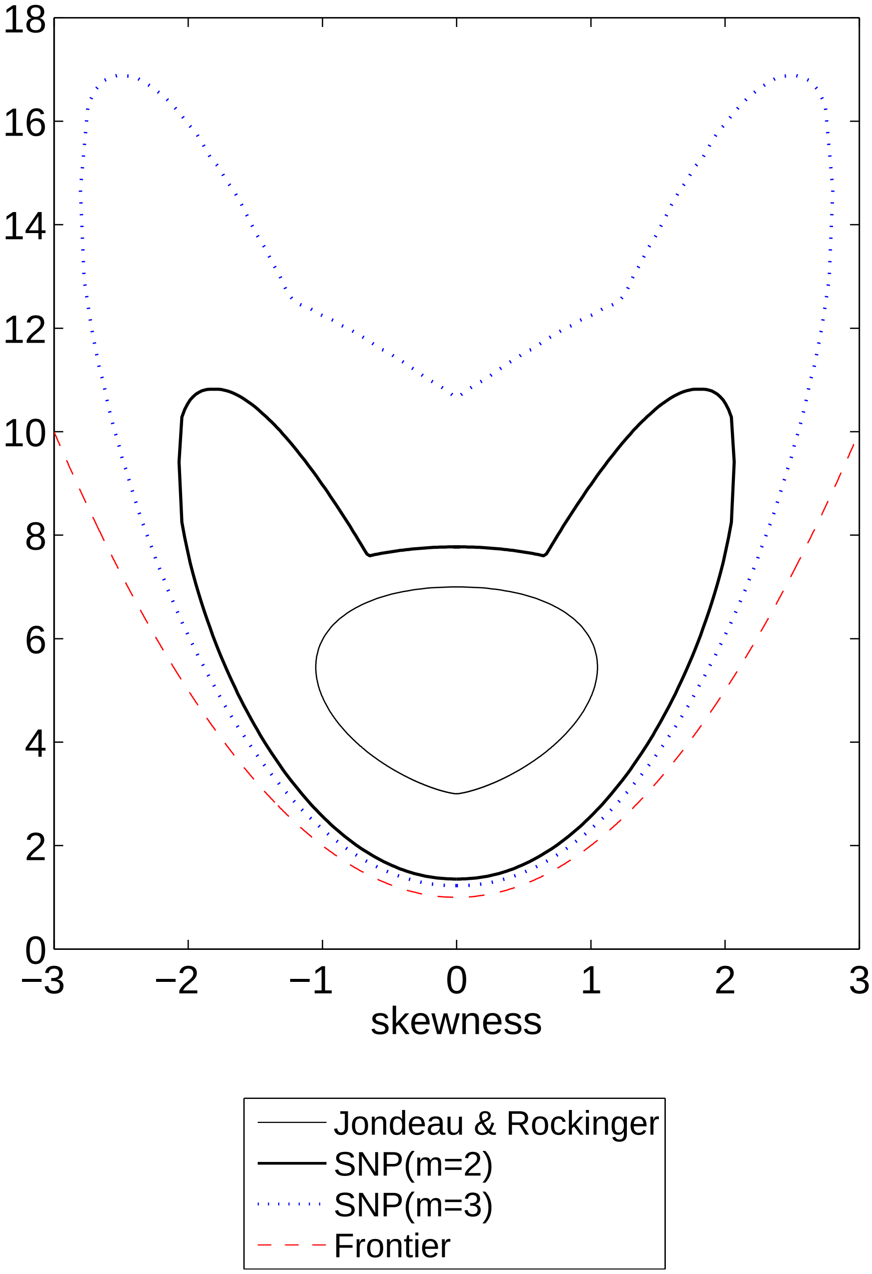
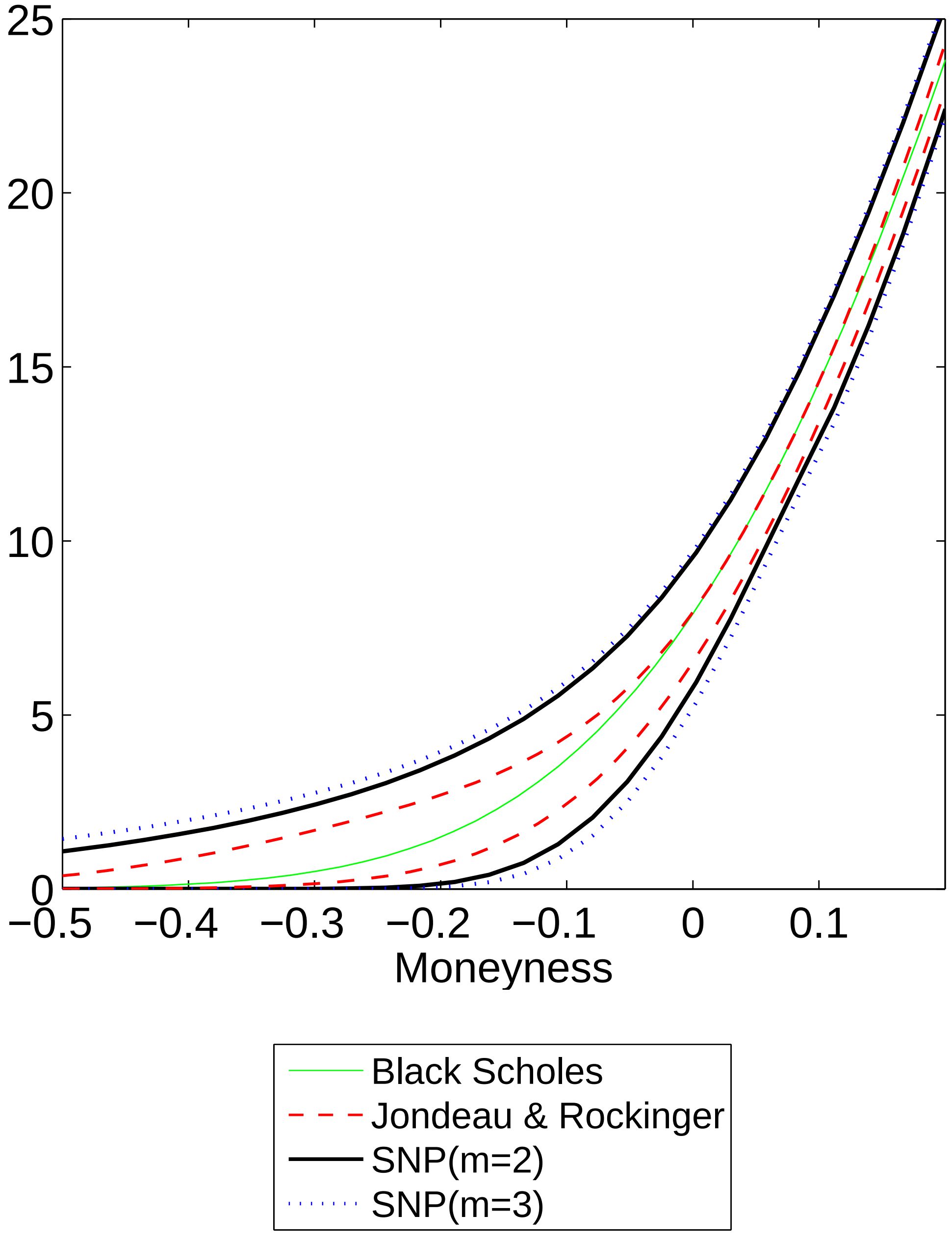


Figure 2
Flexibility to model departures from Black-Scholes



Note: This figure shows the minimum and maximum European call prices that each distribution can yield for a strike price of 100, a maturity of 3 months and a risk free interest rate of 3%.

Figure 3a
Skewness and kurtosis for the short maturity group with time-varying parameters

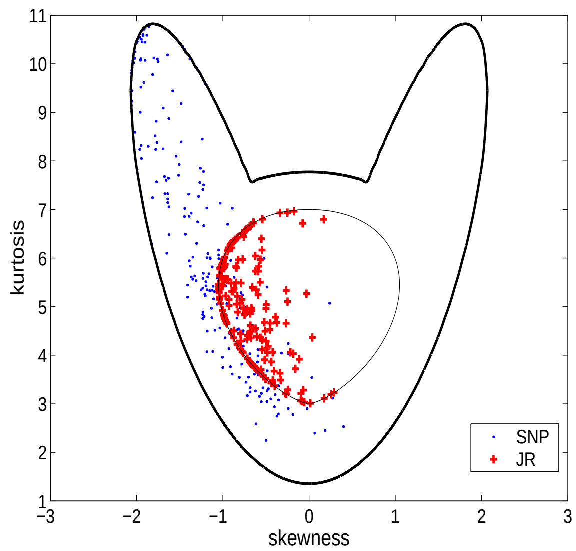


Figure 3b
Skewness and kurtosis for the long maturity group with time-varying parameters

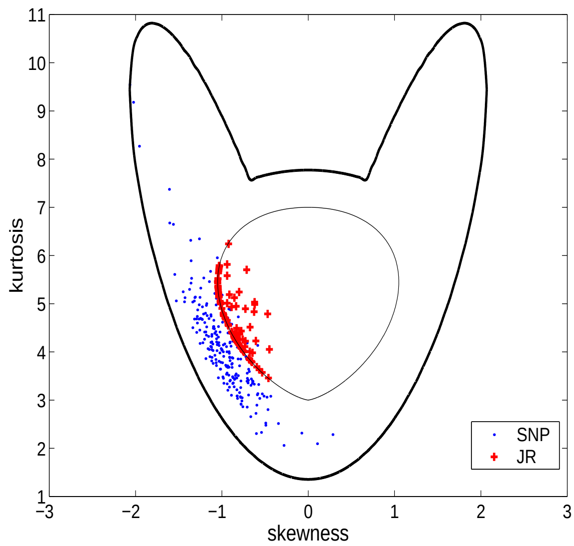


Figure 3c
Skewness and kurtosis of the bootstrapped call prices

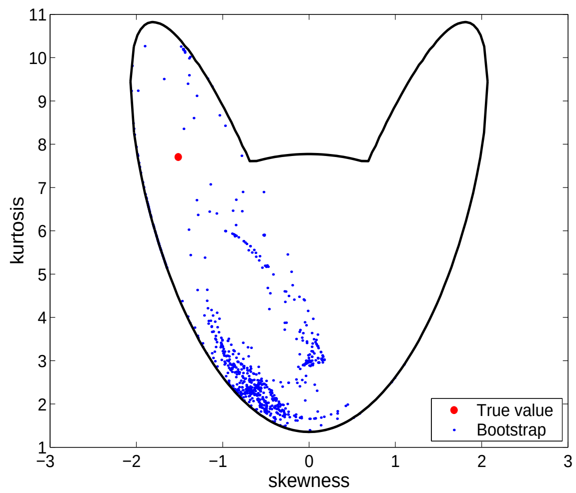
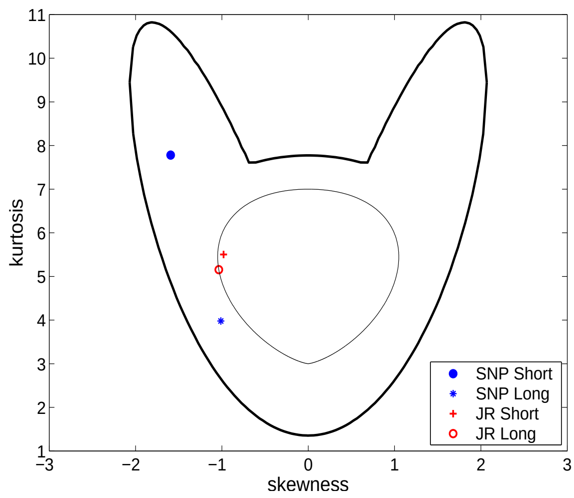
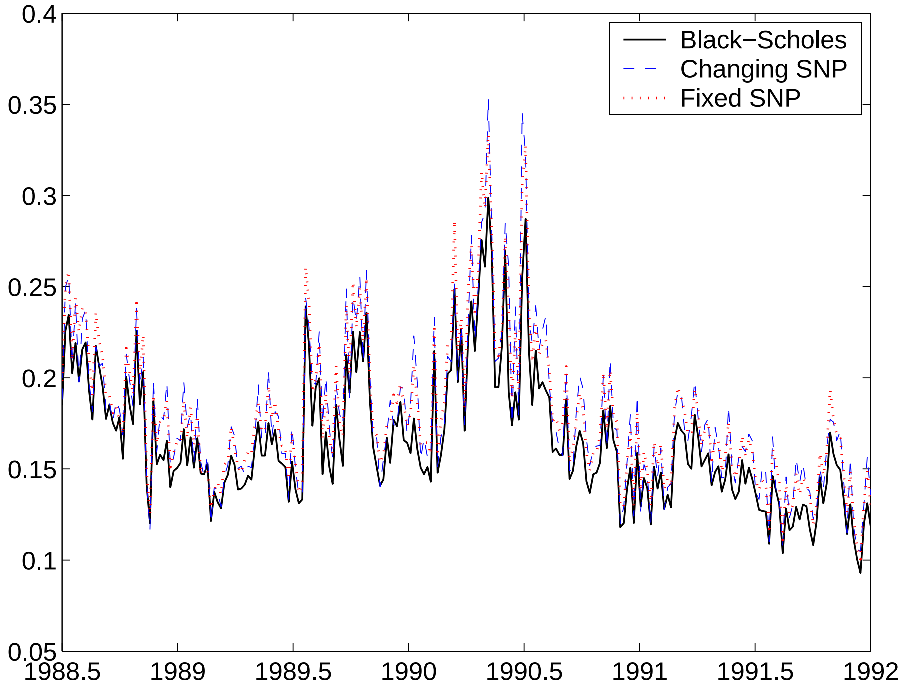


Figure 3d
Skewness and kurtosis for fixed parameters



Notes: The results in figures 3a and 3b correspond to separate estimations for each Wednesday in-sample, while to obtain figure 3d all parameters except volatility are assumed to be the same in the whole sample. SNP refers to a semi-nonparametric distribution of order 2. JR denotes Jondeau and Rockinger's option pricing model, and "Short" and "Long" denote the short and long maturity groups.

Figure 4
Volatility estimates for the short maturities



Note: "Fixed SNP" assumes $\boldsymbol{\nu}$ to be constant, while "Changing SNP" allows it to be time varying. Both cases refer to a semi-nonparametric distribution of order 2.

Figure 5a
Risk-neutral density of $\log(S_T/S_t)$ for the short maturity group

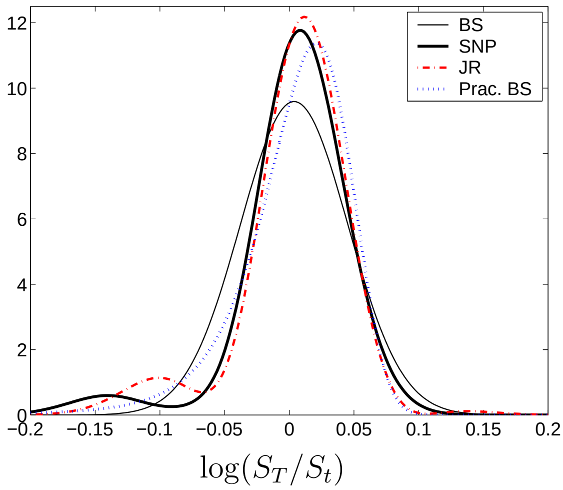


Figure 5b
Left tail of the risk-neutral density of $\log(S_T/S_t)$ for the short maturity group

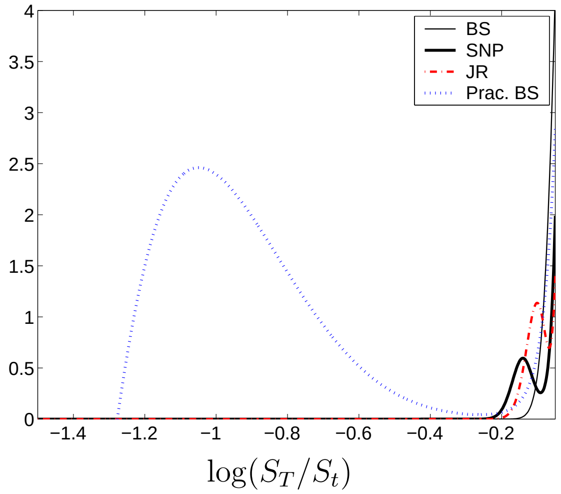


Figure 5c
Risk-neutral density of $\log(S_T/S_t)$ for the long maturity group

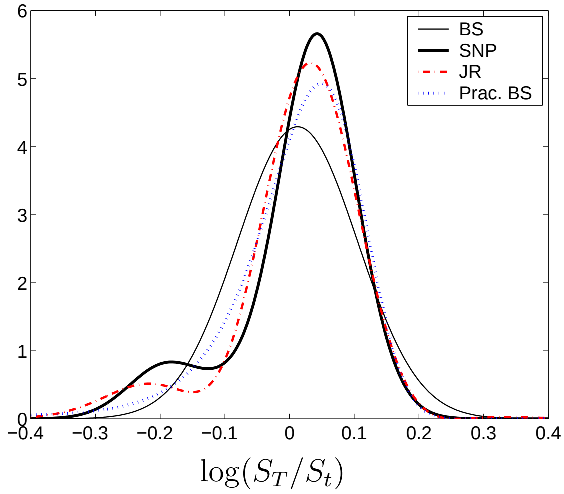
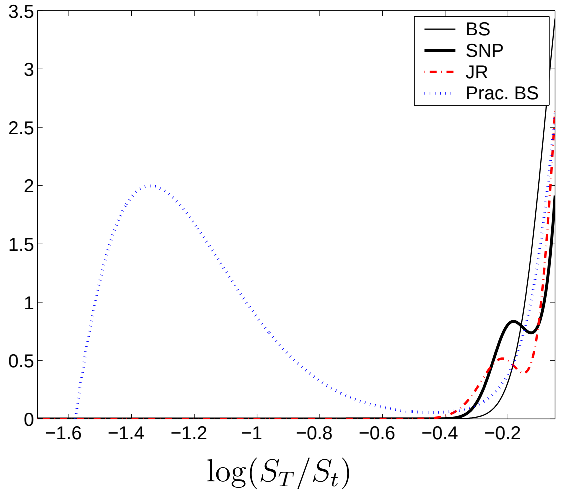
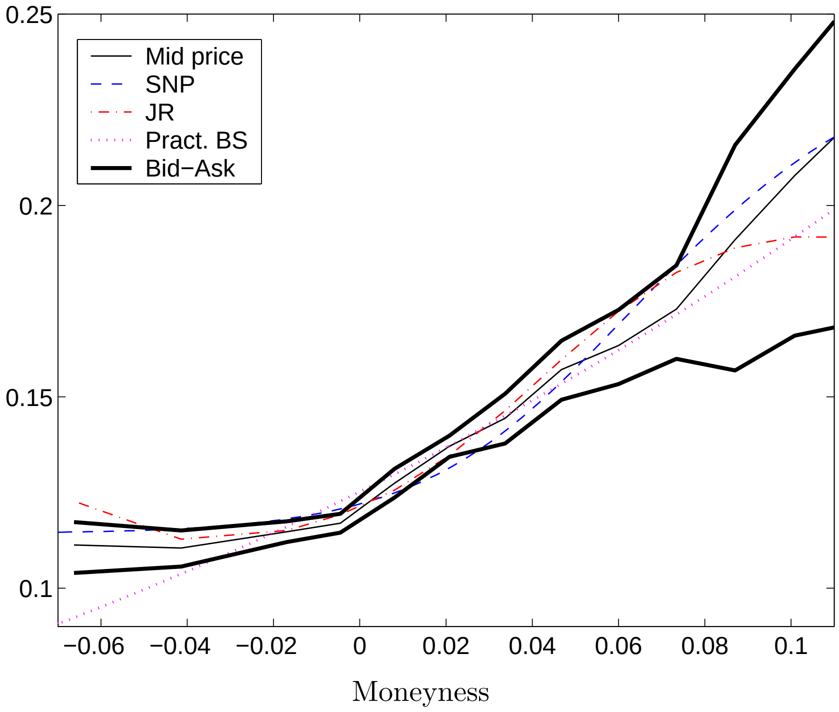


Figure 5d
Left tail of the risk-neutral density of $\log(S_T/S_t)$ for the long maturity group



Notes: These results are based on the volatility estimated on November 13, 1991, but the shape parameters are estimated using data between 1988 and 1992. Pract. BS denotes a model in which volatility is a quadratic function of moneyness. SNP refers to a seminonparametric distribution of order 2.

Figure 6:
Implied volatility on November 13, 1991



Note: All models used in Figure 4 assume time varying volatilities but constant shape parameters. Moneyness defined as $\log(S_t/K) + r(T - t)$. SNP refers to a seminonparametric distribution of order 2.