The impact of benchmarking and portfolio constraints on a fund manager's market timing ability*

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Abstract

We study the effects that relative (to a benchmark) performance evaluation has on the provision of incentives for the search of private information when managers are exogenously constrained in their ability to sell short and purchase on margin. With these portfolio constraints we show that benchmarking the manager's incentive fee affects her timing ability. We analyze the optimal contract between the investor and the manager. With portfolio constraints but without moral hazard the optimal benchmark is, like in the absence of portfolio constraints, the risk free asset. In the presence of moral hazard, the incentive fee increases relative to the unconstrained case. The proportion of the optimal benchmark invested in the market portfolio increases with the manager's risk aversion and the investor's risk tolerance.

Keywords: Market Timing, Incentive Fee, Benchmarking, Portfolio Constraints

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1 Introduction

The design of fund management compensation schemes has elicited interest amongst both practitioners and researchers. The focus of the academic literature has been on how incentives affect performance and risk-taking behavior of managers. A number of theoretical papers have studied the effect of a performance-related incentive fee on managers' incentive to search for private information (see, for example, Bhattacharya and Pfleiderer (1985), Stoughton (1993), Heinkel and Stoughton (1994) and Gómez and Sharma (2006)). Another strand of literature addresses issues related to the design of incentive fee. Adamati and Pfleiderer (1997) and Dybvig, Farnsworth and Carpenter (2001), among others, have discussed the convenience of absolute versus relative (benchmarked to a given portfolio) incentive fees.[1]

With respect to risk, Roll (1992) was the first to illustrate the undesirable effect of relative (i.e., benchmarked) portfolio optimization in a partial equilibrium, single-period model. In particular, he shows that the active portfolio has systematically higher risk than the benchmark. Despite this adverse risk incentive, relative performance evaluation measures such as the Information Ratio have become standard in the industry. In a static framework, several papers have studied how different constraints on the portfolio's total risk (Roll (1992)), tracking error (Jorion (2003)), and Value-at-Risk (VaR) (Alexander and Baptista (2006)), may help to reduce excessive risk taking. In a dynamic setting, Basak, Shapiro, and Tepla (2006) study the optimal policies of an agent subject to a benchmarking restriction. Basak, Pavlova and Shapiro (2006) analyze the effect of an exogenous benchmark restriction on the manager's risk-taking behavior. Their model shows that an exogenous benchmark restriction may ameliorate the adverse risk incentives induced by the manager's compensation. Brennan (1993), Cuoco and Kaniel (1993) and Gómez and Zapatero (2003) study the asset pricing implication of relative incentive fees.

The extant literature discussed above investigates the issue of fund manager compensation in a setting where the manager is unrestricted in her portfolio choice (for an interesting exception see Gómez and Sharma (2006)). However, in practice, fund managers face various portfolio constraints. For example, Almazan, Brown, Carlson and Chapman (2004) document that approximately 70% of mutual funds explicitly state (in Form N-SAR submitted to the SEC) that short-selling is not permitted. This figure rises to above 90% when the restriction is on margin purchases. Surprisingly, given the widespread existence of constraints, the literature has not addressed the implication of such constraints on fund manager's incentives.[2]

This paper's contribution is to incorporate exogenous portfolio constraints into the analysis of linear incentive fees for effort inducement. This allows us to focus on how the provision of incentives to induce manager's effort are affected by the interaction between the benchmark composition and the manager's incentive fee. In our model, the manager's incentives are ex-

[1] A further line of discussion concerns whether, if benchmarked, the incentive fee should be "convex" (i.e. asymmetric), implying that the manager only participates in the upside and suffers no penalty for underperforming the benchmark, or, as prescribed by the Securities and Exchange Commission (SEC) for mutual funds, a "fulcrum" (symmetric) type of fee. See, for example, Das and Sundaram (2002) and Ou-Yang (2003).

[2] Portfolio constraints have been discussed in the literature in other contexts. For example, Almazán et al. (2004) present evidence that portfolio constraints are devices to monitor the manager's effort. Grinblatt and Titman (1989) and Brown et al. (1996) argue that cross-sectional differences in constraint adoption might be related to characteristics that proxy for managerial risk aversion.

plicit: they arise from the design of the optimal compensation contract.[3] We propose a simple two-period, two-asset (the market and a risk-less bond) model. The manager is offered a compensation package that includes a flat fee and a performance-tied incentive fee, possibly benchmarked to a given portfolio return. Both the incentive fee and the benchmark composition are determined endogenously.

A number of new insights arise after introducing portfolio constraints. First, looking at the manager's effort and portfolio choice problem, we show that her effort decision (hence, her timing ability) depends on both the incentive fee and the benchmark composition. The relationship between the manager's effort and the incentive fee has been documented by Gómez and Sharma (2006). The relationship between the effort decision and the benchmark composition, however, contrasts with the well-known "irrelevance result" in Admati and Pfleiderer (1997): the manager's effort is independent of the benchmark composition; it only depends on the manager's effort disutility. We derive explicitly the effort maximizing benchmark's composition as a function of the market moments, the portfolio constraints, and the manager's risk-aversion coefficient. The benchmark is shown to be independent of the manager's disutility of effort. The irrelevance result in Admati and Pfleiderer (1997) arises in the limit, when the portfolio constraints vanish.

To understand the model's intuition, consider a manager who is totally constrained in her ability to sell short and purchase at margin. Under moral hazard, the manager's optimal portfolio can be decomposed in two components: her unconditional risk-diversification portfolio plus her timing portfolio. The timing portfolio depends on the manager's costly effort to improve her timing ability through superior information. For a uninformed manager, this portfolio would be zero. For a hypothetical perfectly informed manager, it would push the optimal total portfolio to either boundary: 100% in the risky asset if the market risk premium is forecasted to be positive; 100% in the bond otherwise. Now, assume that the unconditional portfolio consists of 30% invested in the risky market portfolio. For this perfectly informed manager, any timing portfolio that involves shorting the market by more than 30% or investing more than 70% in the market will hit the portfolio boundaries. Anticipating this and taking into account her effort disutility, the manager will decide her optimal effort expenditure.

Imagine now that the same manager is given a benchmarked contract. The benchmark consists of 20% in the market portfolio and 80% in the bond. The manager adjusts her optimal portfolio. *Relative to the benchmark*, the unconditional optimal portfolio is still 30% long in the market. The manager has to beat the benchmark for the incentive fee to kick in. Therefore, her total market investment will be now 50% of her portfolio: 20% to replicate the benchmark plus the optimal risk-diversification 30%. Holding the portfolio constraints constant, this implies that if the market premium is predicted to be negative, the manager's timing portfolio can now go short up to 50% in the market, 20% *more* than in the absence of the benchmark. This will increase the manager's utility from effort, thereby improving the incentives for sharpening her timing ability. At the same time, if the market premium is predicted to be positive, the

[3] In our model, the fund's net asset value is given. We abstract from the *implicit incentives* arising from the convex flow-performance relation documented in the literature (see, for instance, Gruber (1996), Sirri and Tufano (1998), Chevalier and Ellison (1997), Del Guercio and Tkac (2000) and Basak, Pavlova and Shapiro (2007)).

manager's timing portfolio can go long in the market only 50%, 20% *less* than before the benchmark was introduced. This has the opposite effect on the effort inducement: the manager will have less incentives to exert costly effort. Taking into account this trade-off, the benchmark is chosen such that the manager's unconditional portfolio (benchmark replication plus optimal risk-return trade-off) is equally distant from both portfolio boundaries. Such a benchmark would provide the manager with the highest incentives for effort exertion. The intuition is simple: such a benchmark leaves the manager marginally indifferent between hitting the short-selling or the margin purchase constraint. When the portfolio space is unconstrained, so is the timing portfolio. Benchmarking the manager's incentive fee fails to provide any incentive for effort expenditure.

Second, looking at the investor's problem, he has to decide the benchmark composition and the fee structure. We obtain two conclusion. First, we show that in the absence of moral hazard between the investor and the fund manager, the optimal incentive fee coincides with the Pareto-efficient risk allocation fee. In addition, we show that the optimal benchmark is the risk-free asset. This is not totally surprising: in the absence of moral hazard, the manager's effort is independent of the incentive fee and the benchmark composition. The only role for the incentive fee is to split the risk between the investor (the principal) and the manager (the agent). Hence, the first best split remains optimal. As for the benchmark, any deviation from the risk-free asset (uncorrelated with the market portfolio) will distort the principals optimal portfolio. This result extends the unconstrained contract of Ou-Yang (2003) into the constrained scenario: with or without constraints, the investor's optimal benchmark when effort is publicly observable is the risk free asset.

This does not necessarily hold in the presence of moral hazard between the investor and the manager. Under portfolio constrains and moral hazard, the manager's effort depends on the incentive fee and the benchmark composition. On the one side, increasing the incentive fee gives the manager more incentives to improve her timing ability (by putting more effort); the downside is that the compensation becomes more onerous for the investor. With respect to the benchmark, the risk free asset may not be optimal anymore: making the benchmark more risky may induce higher effort on the manager. On the other side, any benchmark other than the risk-free asset will affect the investor's optimal risk-return tradeoff. Moreover, these double tradeoff considerations (for the incentive fee as well as for the benchmark composition), are interrelated.

We show analytically that, in presence of moral hazard and portfolio constraints, the incentive fee contract under no moral hazard is not optimal. Numerical results show that the optimal incentive fee is higher than in the no moral hazard case. Moreover, contrary to the unconstrained case in Ou-Yang (2003), the optimal benchmark is different from the risk-free asset. More concretely, the optimal benchmark proportion invested in the market increases with the managers risk aversion and decreases with the investor's risk aversion. In other words, when managers are constrained in their portfolio choice, tying their compensation to their portfolio performance and benchmarking their incentive fee relative to the market will result in higher effort expenditure and, hence, better timing ability. Benchmarked incentive fees are expected to be more prominent among more aggressive (less risk-averse) funds.

The model has readily testable empirical implications and, in this regard, our paper is related to the literature on mutual fund performance evaluation. Golec (1992) and Elton, Gruber and

Blake (2003) document that the number of mutual funds that explicitly use incentive fees is relatively small in comparison with the pervasive use of a "flat" fee (a fixed percentage of the fund's net asset value).[4] Further, Elton, Gruber and Blake (2003) find that funds which use incentive fees have superior performance relative to those that do not. In their conclusions, they claim that "while at this time funds with incentive fees seem to offer superior performance relative to other actively managed funds, we don't know whether this is true because of the motivation supplied by incentive fees or because skilled managers adopt incentive fees to advertise their skills to the public." Our model shows that under portfolio constraints and moral hazard, portfolio managers who are offered a benchmarked incentive fee are more motivated than equally skilled managers whose compensation is not performance-linked.

In a related paper, Becker et al. (1999) test for market timing ability and benchmarking. However, in their empirical model, the manager faces no portfolio constraints. According to our results, in such a setting, the optimal benchmark is the risk free asset (effectively, no benchmarking). Consistent with this, they find no support for the use of benchmarks in an unconditional setting. However, after conditioning for public information, they find an economic meaningful estimate for benchmarking, albeit the overall performance of the model remains quite poor. The empirical implications of our model offer guidance on how to extend the tests in Becker et al. (1999) into a framework that accounts explicitly for the presence of short selling and margin purchase constraints, prevalent across the mutual fund industry.

The rest of the paper is organized as follows. Next we introduce the model. The standard unconstrained results are refreshed in Section 2.1. The effect of portfolio constraints are analyzed in section 2.2. In section 3, we derive the composition of the effort-maximizing benchmark portfolio. Section 4 studies the principal's problem. A numerical solution to the constrained manager's effort is presented in Section 5. The paper concludes with Section 6. All proofs are presented in the Appendix. Tables and figures are to be found after the Appendix.

2 The model

A typical fund sponsor will inform the customer that managers (who are involved in investment research) are responsible for choosing each fund's investments. Customers are also informed about how the advisory management company (responsible for choosing and monitoring the managers) is compensated. This is known as the advisory fee. Customers typically ignore how managers are compensated. Given this information, the customer decides how much to invest in the fund. In this paper, we shall abstract from the decision problem of the consumer and the relationship between the fund sponsor and the management company.[5] Instead, we shall focus on the determination of the manager's compensation scheme by the fund management company. Slightly abusing terminology, we call the management company the investor.

The manager and the investor have preferences represented by exponential utility functions:

[4] Agarwal, Daniel and Naik (2006) find that even for hedge funds, the call-option-like incentive fee contract provides incentives to deliver superior performance. In particular, they find that funds with higher delta have better future performance.

[5] For recent empirical studies of fund advisory fees see, for instance, Deli (2002) and Warner and Wu (2005).

$U_a(W) = -\exp(-aW)$ and $U_b(W) = -\exp(-bW)$, respectively. Throughout the paper we will use $a > 0$ ($b > 0$) to denote the manager (investor) as well as her (his) *absolute* risk aversion coefficient. The investment opportunity set consists of two assets. A risk-free asset with gross return R and a stock with stochastic excess return x normally distributed with mean excess return $\mu > 0$ and volatility σ. These two assets can be interpreted as the usual "timing portfolios" for the active manager: the bond and the market portfolio (or any other stochastic timing portfolio).

The investment horizon is one period. Payoffs are expressed in units of the economy's only consumption good. All consumption takes place in period-end. The manager's compensation is set as a percentage of the fund's average net asset value over the period, W. The percentage has two components: a fixed basic fee F and an incentive (performance-tied) fee. The incentive fee is calculated as a percentage $\alpha \in (0, 1]$ of the fund's end of the period return, possibly net of a benchmark return.

After learning the contract, the manager decides whether to accept it or not. If rejected, the manager gets her reservation value. If she accepts the contract, then she puts some (unobservable) effort $e > 0$ in acquiring private information (not observed by the fund's investor) that comes in the form of a signal

$$y = x + \frac{\sigma}{\sqrt{e}}\epsilon,$$

partially correlated with the stock's excess return. The noise term has a standard normal distribution $\epsilon \sim \mathcal{N}(0, 1)$. For simplicity, we assume

Assumption (S1) $E(x\epsilon) = 0$.

The higher the effort the more precise the manager's timing information. Conditional on the manager's effort, the stock's excess return is normally distributed with conditional mean return $E(x|y) = \frac{\mu+ey}{1+e}$ and conditional precision $\text{Var}^{-1}(x|y) = \frac{1}{\sigma^2}(1 + e)$. Hence, e can also be interpreted as the percentage (net) increase in precision induced by the manager's private information. Notice that, in case $e = 0$, the conditional and unconditional distributions coincide: there is no relevant private information.

Effort is costly. The monetary cost of effort disutility is a percentage $V(D, e)$ of the fund's net asset value W. $D > 0$ represents a disutility parameter. The function V is increasing in D and homogenous of degree one with respect to D. Moreover, for all $e > 0$, V satisfies:[6]

Assumption (S2) $V(D, 0) = V_e(D, 0) = V(0, e) = 0$,

Assumption (S3) $V_e(D, e) > 0$,

Assumption (S4) $\frac{V_{ee}(D,e)}{V_e(D,e)} > \frac{1}{1+e}$.

2.1 Unconstrained Portfolio Choice

Based on the conditional moments, the manager makes her optimal portfolio decision: she will invest a percentage $\theta(y)$ in the stock and the remaining $1-\theta(y)$ in the risk-free bond. Therefore,

[6]The subscripts e and ee denote, respectively, first and second derivative with respect to effort.

the portfolio's return will be $R_p = R + \theta x$. Define the benchmark's return as $R_h = R + hx$ with h as the *benchmark's policy weight*: the proportion in the benchmark portfolio invested in the risky stock. The portfolio's *net return* is given by $R_p - R_h = \bar{\theta} x$ with $\bar{\theta} = \theta - h$, the net (over the benchmark) investment in the risky stock. If $h = 0$, the benchmarked return is $R_p - R_h = \theta x$, the excess return. Since the risk-free return is a constant, from the point of view of the manager, this case is equivalent to no benchmarking. Notice that h can be also interpreted as a the benchmark's beta on the market portfolio.

Given a contract (F, α, h), the conditional end-of-the-period wealth is given as a percentage φ_a, for the manager, and φ_b, for the investor, of the fund's net asset value, W:

$$
\begin{aligned}
\varphi_a(\bar{\theta}) &= F + \alpha \bar{\theta} x, &(1)\\
\varphi_b(\bar{\theta}) &= R_h + (1 - \alpha)\bar{\theta} x - F, &(2)
\end{aligned}
$$

with $\bar{\theta} = \bar{\theta}(y)$ and $x = x(y)$, functions of the signal realization y. If the manager chooses the benchmark portfolio then $\bar{\theta} = 0$; the manager receives no incentive fee (only the fixed fee F) and the investor's payoff is the benchmark's return net of the fixed fee.

After these definitions, the conditional utility function for the manager and the investor can be expressed, respectively, as[7]

$$
\begin{aligned}
U_a\left(\varphi_a(\bar{\theta})\right) &= -\exp\left(-a\varphi_a(\bar{\theta})W + V(D, e)W\right), \\
U_b\left(\varphi_b(\bar{\theta})\right) &= -\exp\left(-b\varphi_b(\bar{\theta})W\right).
\end{aligned}
$$

In this setting, the Arrow-Pratt risk premium for the manager will be, $\alpha W \frac{a\alpha W}{2} \bar{\theta}^2 \sigma^2$. Thus, $a\alpha W$ represents the manager's *relative* risk aversion coefficient. For simplicity, and without loss of generality, we normalize $W = 1$.

We shall proceed backwards. First, we will obtain the optimal portfolio choice θ. Then, after recovering the manager's indirect utility function, we will tackle the manager's effort decision. The unconstrained manager's optimal net portfolio solves

$$
\bar{\theta}(y) = \arg\max_{\bar{\theta}} \left\{ E(\varphi_a(\bar{\theta})) - (a/2)\mathrm{Var}(\varphi_a(\bar{\theta})) \right\},
$$

which yields the optimal portfolio

$$
\theta(y) = h + \frac{\mu}{a\alpha\sigma^2} + \frac{ey}{a\alpha\sigma^2}. \tag{3}
$$

The manager's optimal portfolio has three components: the benchmark's investment in the risky stock, h; the unconditional optimal risk-return trade-off, $\frac{\mu}{a\alpha\sigma^2}$ and, depending on the manager's signal y and her effort expenditure, e, the *timing portfolio*, $\frac{ey}{a\alpha\sigma^2}$.

[7] Notice that, since V is homogenous of degree one with respect to D, we can always write $aV(D', e) = V(D, e)$ with $D = aD'$. Hence the parameter D is a (increasing) function of the manager's risk aversion among other factors.

Replacing $\theta(y)$ in the manager's expected utility function and integrating over the signal y we obtain the manager's (unconditional) expected utility:

$$EU(\varphi_a(e)) = -\exp\left(-(1/2)(\mu^2/\sigma^2) - aF + V(D,e)\right) g(e), \tag{4}$$

with $g(e) = \left(\frac{1}{1+e}\right)^{1/2}$. At the optimum, the effort marginal utility must be equal (first-order condition) to its marginal disutility:

$$V_e(D, e_{SB}) = \frac{1}{2(1+e_{SB})}. \tag{5}$$

We call this solution the *second best effort.*[8] Assumptions (S2) and (S3) guarantee that the necessary condition (5) is also sufficient for optimality. Clearly, the manager's second best effort choice (hence the quality of her private information) *is independent of the benchmark's composition, h.* This is the same result as in Admati and Pfleiderer (1997). Effort only depends on the manager's disutility coefficient, D.

2.2 Constrained Portfolio Choice

We now introduce the main theoretical contribution of the paper. Assume that the manager is constrained in her portfolio choice in that she cannot short-sell or purchase on margin. Let $m \geq 1$ denote the maximum trade on margin the manager is allowed: $m = 1$ means that the manager is not allowed to purchase the risky stock on margin; for any $m > 1$ the manager can borrow and invest in the risky stock up to $m - 1$ dollars per dollar of the fund's current net asset value. Let $s \geq 0$ denote the short-selling limit: $s = 0$ means that the manager cannot sell short the risky stock; for any $s > 0$ the manager can short up to s dollars per dollar of the fund's current net asset value. According to the SEC regulation, the maximum initial margin for leveraged positions is 50%, which implies that $m \leq 2$ and $s \leq 1$.[9] In terms of the manager's portfolio choice problem, this implies $m \geq \theta \geq -s$ or, equivalently, $m - h \geq \bar{\theta} \geq -(h+s)$.

The manager then solves the following constrained problem

$$\bar{\theta}(y) = \arg \max_{m-h \geq \bar{\theta} \geq -(h+s)} \left\{ E(\varphi_a(\bar{\theta})) - (a/2)\mathrm{Var}(\varphi_a(\bar{\theta})) \right\}.$$

Call $\lambda_m \leq 0$ and $\lambda_s \leq 0$ the corresponding Lagrange multipliers, such that $\lambda_m(m - h - \bar{\theta}) = \lambda_s(\bar{\theta} + h + s) = 0$. There are three solutions. If neither constraint is binding, $\lambda_m = \lambda_s = 0$, then the interior solution follows: $\bar{\theta}(y) = \frac{\mu + ey}{a\alpha\sigma^2}$. Alternatively, there are two possible corner solutions: first, if the short-selling limit is binding, $\lambda_m = 0$ and $\lambda_s = E(x|y) + a\alpha(h+s)\mathrm{Var}(x|y) < 0$. In such a case, $\bar{\theta} = -(h+s)$. In the second corner solution, the margin purchase bound is hit: $\lambda_s = 0$ and $\lambda_m = -E(x|y) + a\alpha(m-h)\mathrm{Var}(x|y) < 0$. In such a case, $\bar{\theta} = m - h$.

Solving for the optimal portfolio $\theta(y)$ as a function of the signal realization we obtain that, in the case of no timing ability ($e = 0$), $\theta = h + \frac{\mu}{a\alpha\sigma^2}$ provided $-\left(s + \frac{\mu}{a\alpha\sigma^2}\right) \leq h \leq m - \frac{\mu}{a\alpha\sigma^2}$.

[8] The first best effort is the effort the unconstrained manager would exert under no asymmetric information, that is, in the absence of moral hazard.

[9] Of course, investors can effectively leverage their portfolios above those limits by investing in derivatives.

For the case when $e > 0$ we obtain:

$$\theta(y) = \begin{cases} -s & \text{if } y < -\frac{\mu}{e}L_s \\ \\ h + \frac{\mu}{a\alpha\sigma^2} + \frac{ey}{a\alpha\sigma^2} & \text{otherwise} \\ \\ m & \text{if } y > \frac{\mu}{e}L_m. \end{cases} \tag{6}$$

We call

$$\begin{aligned} L_s(h) &= 1 + (h+s)\left(\frac{\mu}{a\alpha\sigma^2}\right)^{-1} \\ L_m(h) &= (m-h)\left(\frac{\mu}{a\alpha\sigma^2}\right)^{-1} - 1 \end{aligned}$$

the *leverage ratios*. These ratios represent the *net* (relative to the benchmark) maximum leverage from selling short $(h+s)$ or trading at margin $(m-h)$ as a proportion of the manager's optimal unconstrained portfolio when $e = 0$ and $h = 0$.

Looking at the way the leverage ratios change with benchmarking, we observe that $\frac{\partial}{\partial h}L_s = \left(\frac{\mu}{a\alpha\sigma^2}\right)^{-1} > 0$ and $\frac{\partial}{\partial h}L_m = -\left(\frac{\mu}{a\alpha\sigma^2}\right)^{-1} < 0$. That is, L_s (L_m) increases (decreases) with h. Moreover, given the (risk-adjusted) market premium μ/σ^2, the marginal change in L_s (L_m) increases (decreases) with the manager's relative risk aversion $a\alpha$.

Equation (6) shows how the constraints and benchmarking interact to provide incentives for effort expenditure. To see the intuition, let us focus first on the short-selling constraint. Let us assume for the moment that there exist no limit to margin purchases ($m \to \infty$) and that no short position can be taken ($s = 0$). Under these assumptions, and after putting some effort e, the manager receives a signal y and makes her optimal portfolio choice:

$$\theta(y) = \begin{cases} 0 & \text{if } y < -\frac{\mu}{e}L_s \\ \\ h + \frac{\mu+ey}{a\alpha\sigma^2} & \text{otherwise,} \end{cases}$$

with $L_s = 1 + h\left(\frac{\mu}{a\alpha\sigma^2}\right)^{-1}$. When $h = 0$, all signals $y < -\frac{\mu}{e}$ lead to short-selling. Imagine now that the manager is offered a benchmarked contract, with $h > 0$ the benchmark's proportion invested in the risky stock. In this case, the short-selling bound is only hit for *smaller* signals $y < -\frac{\mu}{e}L_s$. In general, increasing h leads to a "wider range" of implementable signals relative to the case of no benchmarking ($h = 0$). Since the effort decision is taken prior to the signal realization, the fact that more signals are implementable under benchmarking ($h > 0$) increases the marginal *expected* utility of effort. The size of this *incremental area* grows with $ha\alpha$. Hence, we expect the impact of benchmarking to be relatively higher for more risk averse investors.

Alternatively, assume there is no benchmarking ($h = 0$) but the short-selling limit is expanded from $s = 0$ to $s = h$. Figure 1 shows that, *ceteris paribus*, the effort choice of the manager will coincide with the effort put under benchmarking: given that $s = 0$, benchmarking the manager's portfolio return ($h > 0$) is, in terms of effort inducement, *equivalent to relaxing the short-selling bound from* 0 *to* h. In other words, in the absence of margin purchase constraints, the manager's effort depends on $s + h$; benchmarking the manager's performance and relaxing her short-selling constraints are *perfect substitutes for effort inducement.* The higher s the lower the marginal expected utility of effort induced by benchmarking. In the limit, when the short-selling bounds vanish ($s \to \infty$), we converge to the unconstrained scenario in Section 2.1 where benchmarking was shown to be irrelevant for the manager's effort decision.

Let us focus now on the margin purchase constraint. Assume $s \to \infty$ and $m = 1$. This implies that the manager can short any amount but cannot trade on margin: for "very good" signals the manager can only invest up to 100% of the fund's net asset value in the risky stock. Her optimal portfolio (as a function of the signal) will be:

$$\theta(y) = \begin{cases} 1 & \text{if } y > \frac{\mu}{e} L_m, \\ \\ h + \frac{\mu + ey}{a\alpha\sigma^2} & \text{otherwise,} \end{cases}$$

with $L_m = (1-h)\left(\frac{\mu}{a\alpha\sigma^2}\right)^{-1} - 1$. L_m is decreasing in h. Decreasing h in the manager's compensation just makes the portfolio constraint "less binding," i.e., binding for bigger signals. For instance, moving from a benchmarked contract ($h > 0$) to a non benchmarked contract ($h = 0$) would increase the manager's effort: signals that were not implementable under benchmarking become now feasible. Symmetrically to the short-selling constraint, the expected impact on effort expenditure would be analogous if benchmarking were not removed ($h > 0$) and the constraint on margin purchases made looser: from $m = 1$ to $m = 1 + h$. Therefore, in the absence of short selling constraints, the manager's effort depends on $m - h$: benchmarking the manager and tightening the margin purchase constraint are perfect substitutes for the manager's effort (dis)incentive. Again, the impact of benchmarking increases, in absolute terms, with the manager's relative risk aversion, $a\alpha$. In the limit, when the manager faces no margin purchase constraint ($m \to \infty$) the benchmark composition is irrelevant for the manager's effort decision.

In summary, by modifying the benchmark portfolio composition we observe two opposing effects: for the short selling constrained manager, increasing the benchmark's percentage invested in the risky stock (h) induces the manager to put more effort. On the other side, for the manager constrained in her ability to purchases at margin, increasing that percentage lowers the effort incentives. Thus, when (as for most mutual fund managers) both short selling and margin purchase are constrained, the trade-off between these two effects yields the *optimal benchmark composition.* This is the question we investigate in the next section.

3 The optimal benchmark portfolio composition

To address this question, we proceed as follows. Proposition 1 introduces the manager's unconditional expected utility under short selling ($0 \leq s < \infty$) and margin purchase ($1 \leq m < \infty$) constraints for all possible values of h in the real line. In Proposition 2 we show that Assumptions (S2)-(S4) are sufficient for the existence of a continuous and differentiable effort function, $e(h)$, that yields a unique effort choice for each value of h. The function attains a global maximum at $h^* = \frac{m-s}{2} - \frac{\mu}{a\alpha\sigma^2}$.

Before introducing the constrained manager's unconditional expected utility we need some notation. Let $\Phi(\cdot)$ denote the cumulative probability function of a Chi-square variable with one degree of freedom: $\Phi(x) = \int_0^x \phi(z)\,dz$, with

$$\phi(z) = \begin{cases} \frac{1}{\sqrt{2\pi}}\, z^{-1/2} \exp(-z/2) & \text{when } z > 0; \\ 0 & \text{otherwise.} \end{cases}$$

Proposition 1 *Given the finite portfolio constraints $s \geq 0$ and $m \geq 1$, the risk-averse manager's expected utility is $EU_a\left(\varphi_a(e)\right) = -exp(-(1/2)\mu^2/\sigma^2 - aF + V(D,e)) \;\times g(e, L_s, L_m)$ with $g(e, L_s, L_m) = (1/2)\times$*

$$\begin{aligned} & exp\left(\frac{\left(\frac{\mu}{\sigma}L_s\right)^2}{2}\right)\left[1 + \Phi\left(\tfrac{1+e}{e}\left(\tfrac{\mu}{\sigma}L_s\right)^2\right)\right] + \\ & \left(\tfrac{1}{1+e}\right)^{1/2}\left[\Phi\left(\frac{\left(\frac{\mu}{\sigma}L_m\right)^2}{e}\right) - \Phi\left(\frac{\left(\frac{\mu}{\sigma}L_s\right)^2}{e}\right)\right] + \\ & exp\left(\frac{\left(\frac{\mu}{\sigma}L_m\right)^2}{2}\right)\left[1 - \Phi\left(\tfrac{1+e}{e}\left(\tfrac{\mu}{\sigma}L_m\right)^2\right)\right] \end{aligned} \tag{7}$$

if $h < -\left(s + \frac{\mu}{a\alpha\sigma^2}\right)$;

$$\begin{aligned} & exp\left(\frac{\left(\frac{\mu}{\sigma}L_s\right)^2}{2}\right)\left[1 - \Phi\left(\tfrac{1+e}{e}\left(\tfrac{\mu}{\sigma}L_s\right)^2\right)\right] + \\ & \left(\tfrac{1}{1+e}\right)^{1/2}\left[\Phi\left(\frac{\left(\frac{\mu}{\sigma}L_s\right)^2}{e}\right) + \Phi\left(\frac{\left(\frac{\mu}{\sigma}L_m\right)^2}{e}\right)\right] + \\ & exp\left(\frac{\left(\frac{\mu}{\sigma}L_m\right)^2}{2}\right)\left[1 - \Phi\left(\tfrac{1+e}{e}\left(\tfrac{\mu}{\sigma}L_m\right)^2\right)\right] \end{aligned} \tag{8}$$

if $-\left(s + \frac{\mu}{a\alpha\sigma^2}\right) \leq h \leq m - \frac{\mu}{a\alpha\sigma^2}$;

$$\begin{aligned} & exp\left(\frac{\left(\frac{\mu}{\sigma}L_s\right)^2}{2}\right)\left[1 - \Phi\left(\tfrac{1+e}{e}\left(\tfrac{\mu}{\sigma}L_s\right)^2\right)\right] + \\ & \left(\tfrac{1}{1+e}\right)^{1/2}\left[\Phi\left(\frac{\left(\frac{\mu}{\sigma}L_s\right)^2}{e}\right) - \Phi\left(\frac{\left(\frac{\mu}{\sigma}L_m\right)^2}{e}\right)\right] + \\ & exp\left(\frac{\left(\frac{\mu}{\sigma}L_m\right)^2}{2}\right)\left[1 + \Phi\left(\tfrac{1+e}{e}\left(\tfrac{\mu}{\sigma}L_m\right)^2\right)\right] \end{aligned} \tag{9}$$

if $h > m - \frac{\mu}{a\alpha\sigma^2}$.

Equations (7), (8) and (9) are weighted sums of the manager's unconstrained expected utility (4), independent of h, and her expected utility function when the portfolio hits either the short-selling constraint bound, $\exp\left(\frac{\left(\frac{\mu}{\sigma}L_s\right)^2}{2}\right)$, or the margin purchase bound, $\exp\left(\frac{\left(\frac{\mu}{\sigma}L_m\right)^2}{2}\right)$. When the manager is constrained, the benchmark's composition (i.e., the value of the parameter h) affects the quality of the timing signal through the effort choice.

Corollary 1 *The first derivative* $g_e(e, L_s, L_m) = -\frac{1}{4}\left(\frac{1}{1+e}\right)^{3/2} \times$

$$\left[\Phi\left(\frac{\left(\frac{\mu}{\sigma}L_m\right)^2}{e}\right) - \Phi\left(\frac{\left(\frac{\mu}{\sigma}L_s\right)^2}{e}\right)\right] \quad \textit{if } h < -\left(s + \frac{\mu}{a\alpha\sigma^2}\right)$$

$$\left[\Phi\left(\frac{\left(\frac{\mu}{\sigma}L_s\right)^2}{e}\right) + \Phi\left(\frac{\left(\frac{\mu}{\sigma}L_m\right)^2}{e}\right)\right] \quad \textit{if } -\left(s + \frac{\mu}{a\alpha\sigma^2}\right) \leq h \leq m - \frac{\mu}{a\alpha\sigma^2}$$

$$\left[\Phi\left(\frac{\left(\frac{\mu}{\sigma}L_s\right)^2}{e}\right) - \Phi\left(\frac{\left(\frac{\mu}{\sigma}L_m\right)^2}{e}\right)\right] \quad \textit{if } h > m - \frac{\mu}{a\alpha\sigma^2},$$

is decreasing with respect to e.

Notice that functions $g(e, L_s, L_m)$ and $g_e(e, L_s, L_m)$ are symmetric with respect to h around $h^* = \frac{m-s}{2} - \frac{\mu}{a\alpha\sigma^2}$, the center of the interval $[-(s + \frac{\mu}{a\alpha\sigma^2}), m - \frac{\mu}{a\alpha\sigma^2}]$. To see this, let δ represent the deviation in the benchmark portfolio's percentage invested in the risky asset above ($\delta > 0$) or below ($\delta < 0$) the reference value h^*. It can be shown that $L_s(h^* + \delta) = L_m(h^* - \delta)$ for all $\delta \in \Re$. Replacing the later equality in the functions g and g_e the symmetry is proved.

We call e_{TB} the *third best effort* that maximizes the constrained manager's expected utility function in Proposition 1:

$$e_{TB} = \text{argmax}_e -(1/2)\exp(-(1/2)\mu^2/\sigma^2 - aF + V(D,e)) \ \times g(e, L_s, L_m). \tag{10}$$

From the previous equation, it is obvious that, unlike in the unconstrained scenario, the manager's optimal effort depends on h (through L_s and L_m). We want to study how the third best effort changes with h, more concretely, whether there exists an optimal (effort maximizing) benchmark.

The following proposition presents general conditions on the effort disutility function and the range of the benchmark parameter h for which there exists a well behaved effort function, that is, a function that yields, for each benchmark portfolio h, the utility maximizing third best effort (10). More importantly, the same conditions are shown to be sufficient for the existence of a benchmark portfolio h^* that elicits the highest effort from the manager. The value of h^* is explicitly derived as a function of the manager's portfolio constraints on short selling, s, and margin purchase, m; her relative risk aversion, $a\alpha$; and the market portfolio moments, μ and σ^2.

Proposition 2 *Assume (S2)-(S4) hold. For all* $h \in [-(s+\frac{\mu}{a\alpha\sigma^2}), m-\frac{\mu}{a\alpha\sigma^2}]$ *there exists a unique function* $e(h)$*, continuous and differentiable, such that* $e(h) = e_{TB}$*. Let* $h^* = \frac{m-s}{2} - \frac{\mu}{a\alpha\sigma^2}$*. Then,* $e(h^*) > e(h)$ *for all* $h \neq h^* \in [-(s + \frac{\mu}{a\alpha\sigma^2}), m - \frac{\mu}{a\alpha\sigma^2}]$.

Corollary 2 *Assume (S2)-(S4) hold. Provided it exists, the effort function* $e(h)$ *is increasing in* h *for all* $h < -(s + \frac{\mu}{a\alpha\sigma^2})$ *and decreasing in* h *for all* $h > m - \frac{\mu}{a\alpha\sigma^2}$*. Moreover, the effort function is symmetric in* h *around* h^**, i.e.,* $e(h^* + \delta) = e(h^* - \delta)$ *for all* $\delta \in \Re$.

From proposition 2 and corollary 2, it is clear that the manager's effort function attains a global maximum at $h^* = \frac{m-s}{2} - \frac{\mu}{a\alpha\sigma^2}$. The intuition for this result is as follows: on the one hand, increasing benchmarking (i.e., higher h) lowers the likelihood of hitting the short selling constraint; on the other hand, it increases the probability of hitting the margin purchase constraint. The effect of decreasing benchmarking (i.e. lower h) is just symmetric. The trade-off of these two opposite effects yields the effort-maximizing value of the benchmark composition, h^*. In other words, the benchmark portfolio h^* makes the manager, in expected terms, *indifferent* between hitting either constraint (short selling and margin purchase).

This intuition is illustrated in figure 2. The manager has to decide her effort and her optimal portfolio. The graph represents the unconditional portfolio (independent of the signal y). Assume the manager is constrained. For instance, $0 \leq \theta \leq 1$ (zero leverage). If the benchmark coincides with the risk free asset, the manager will chose the tangent portfolio on the "absolute" capital market line that maximizes her expected utility. In the example, this portfolio holds less than 50% in the market. If the manager is given a benchmark $h^* > 0$ then she will choose a tangent portfolio $\bar{\theta} = \theta - h$ in the "relative" capital market line that trades off excess expected return $\bar{\theta}\mu$ against tracking error standard deviation. Notice that given the portfolio constraints, for $h = h^*$ the manager's optimal unconditional portfolio is equidistant from either boundary. The manager's effort choice maximizes her unconditional expected utility before receiving the signal. The benchmark composition h^* allows, ex-ante, more extreme signals to be implemented, increasing effort's marginal utility and, ultimately, the manager's effort choice.

The effort choice for the constrained manager is smaller than for the unconstrained manager. In the next corollary we formalize this intuition.

Corollary 3 *For any given contract (F, α, h) and finite manager's risk aversion, a, the constrained manager's third best effort $e_{TB} < e_{SB}$. In the limit, when the portfolio constraints vanish, the third best effort and the second best effort coincide.*

We conclude this section by studying to especial cases of the more general constrained problem. As illustrated in the examples in section 2.2, when the manager is only short selling constrained (i.e., unlimited margin purchases), increasing the benchmark investment in the risky asset, h, gives the manager more incentives to put higher effort. In the case of unlimited short selling and constrained margin purchases, the result is symmetric: effort decreases with h. In either case, there is no optimal benchmark composition. The following corollary summarizes these findings.

Corollary 4 *When the manager can purchase at margin with no limit but faces a short selling bound, the effort function is monotonous increasing in h. Symmetrically, when the manager can sell short with no restriction but faces limited margin purchase, the effort function is monotonous decreasing with h.*

4 The principal's problem

The investor's optimal contract (F, α, h) maximizes his expected utility subject to the manager's incentive compatibility and participation constraints. For simplicity, and without loss of generality, we normalize the manager's reservation value to $-\exp(-(1/2)\mu^2/\sigma^2)$. For a given contract (F, α, h), the manager's (conditional) wealth is given as a percentage, equation (2), of the fund's net asset value.

The constrained manager, after accepting the contract, decides how much effort to put. Then, she receives the signal y and invest a proportion $\theta(y)$ as in (6) in the risky asset.

Let $t(\alpha) = \frac{b(1-\alpha)}{a\alpha}$ and $T(\alpha) = (2 - t(\alpha))t(\alpha)$. The investor's expected utility is introduced in the following proposition.

Proposition 3 *Let $a - eb > 0$ and $a\alpha + eb\,(1 - \alpha(2 - t(\alpha))) > 0$. Given the portfolio constraints $s \geq 0$ and $m \geq 1$, the expected utility of the risk-averse investor is $EU_b(\varphi_b(e)) = -exp(b(F - R) - (1/2)\mu^2/\sigma^2) \times f(e, L_s, L_m)$ with $f(e, L_s, L_m) = (1/2)\times$*

$$\begin{aligned}
&exp\left(\frac{\left(\frac{\mu}{\sigma}(1+t(\alpha)(L_s-1)-bh\frac{\sigma^2}{\mu})\right)^2}{2}\right)\left[1+\Phi\left(\frac{1+e}{e}\left(\frac{\mu}{\sigma}(1+\frac{1+t(\alpha)e}{1+e}(L_s-1)-\frac{e}{1+e}bh\frac{\sigma^2}{\mu})\right)^2\right)\right]+\\
&exp\left(\frac{1}{2}\frac{1}{1+e}\left(\frac{\mu}{\sigma}\left(t(\alpha)-1+bh\frac{\sigma^2}{\mu}\right)\right)^2\right)exp\left(\frac{\mu^2}{2\sigma^2}\frac{e}{1+e}\frac{\left(T(\alpha)-1-bh(t(\alpha)-1)\frac{\sigma^2}{\mu}\right)^2}{1+eT(\alpha)}\right)\left(\frac{1}{1+T(\alpha)e}\right)^{1/2}\\
&\Bigg[\Phi\left(\frac{1+e}{e(1+T(\alpha)e)}\left(\frac{\mu}{\sigma}\left(\frac{1+T(\alpha)e}{1+e}(1+L_m)-1-\frac{e}{1+e}bh(t(\alpha)-1)\frac{\sigma^2}{\mu}\right)\right)^2\right)-\\
&\Phi\left(\frac{1+e}{e(1+T(\alpha)e)}\left(\frac{\mu}{\sigma}\left(1+\frac{1+T(\alpha)e}{1+e}(L_s-1)+\frac{e}{1+e}bh(t(\alpha)-1)\frac{\sigma^2}{\mu}\right)\right)^2\right)\Bigg]+\\
&exp\left(\frac{\left(\frac{\mu}{\sigma}(t(\alpha)(1+L_m)-1+bh\frac{\sigma^2}{\mu})\right)^2}{2}\right)\left[1-\Phi\left(\frac{1+e}{e}\left(\frac{\mu}{\sigma}(\frac{1+t(\alpha)e}{1+e}(1+L_m)-1+\frac{e}{1+e}bh\frac{\sigma^2}{\mu})\right)^2\right)\right].
\end{aligned}$$

if $h < -\left(s\frac{a\alpha+eb(1-\alpha)}{\alpha(a-eb)} + (1+e)\frac{\mu}{\alpha(a-eb)\sigma^2)}\right)$;

$$\begin{aligned}
&exp\left(\frac{\left(\frac{\mu}{\sigma}(1+t(\alpha)(L_s-1)-bh\frac{\sigma^2}{\mu})\right)^2}{2}\right)\left[1-\Phi\left(\frac{1+e}{e}\left(\frac{\mu}{\sigma}(1+\frac{1+t(\alpha)e}{1+e}(L_s-1)-\frac{e}{1+e}bh\frac{\sigma^2}{\mu})\right)^2\right)\right]+\\
&exp\left(\frac{1}{2}\frac{1}{1+e}\left(\frac{\mu}{\sigma}\left(t(\alpha)-1+bh\frac{\sigma^2}{\mu}\right)\right)^2\right)exp\left(\frac{\mu^2}{2\sigma^2}\frac{e}{1+e}\frac{\left(T(\alpha)-1-bh(t(\alpha)-1)\frac{\sigma^2}{\mu}\right)^2}{1+eT(\alpha)}\right)\left(\frac{1}{1+T(\alpha)e}\right)^{1/2}\\
&\Bigg[\Phi\left(\frac{1+e}{e(1+T(\alpha)e)}\left(\frac{\mu}{\sigma}\left(\frac{1+T(\alpha)e}{1+e}(1+L_m)-1-\frac{e}{1+e}bh(t(\alpha)-1)\frac{\sigma^2}{\mu}\right)\right)^2\right)-\\
&\Phi\left(\frac{1+e}{e(1+T(\alpha)e)}\left(\frac{\mu}{\sigma}\left(1+\frac{1+T(\alpha)e}{1+e}(L_s-1)+\frac{e}{1+e}bh(t(\alpha)-1)\frac{\sigma^2}{\mu}\right)\right)^2\right)\Bigg]+\\
&exp\left(\frac{\left(\frac{\mu}{\sigma}(t(\alpha)(1+L_m)-1+bh\frac{\sigma^2}{\mu})\right)^2}{2}\right)\left[1-\Phi\left(\frac{1+e}{e}\left(\frac{\mu}{\sigma}(\frac{1+t(\alpha)e}{1+e}(1+L_m)-1+\frac{e}{1+e}bh\frac{\sigma^2}{\mu})\right)^2\right)\right].
\end{aligned}$$

if $-\left(s\frac{a\alpha+eb(1-\alpha)}{\alpha(a-eb)} + \frac{(1+e)\mu}{\alpha(a-eb)\sigma^2)}\right) \leq h < -\left(s\frac{a\alpha+eb(1-\alpha)(2-t(\alpha))}{a\alpha+eb(1-\alpha(2-t(\alpha)))} + \frac{(1+e)\mu}{(a\alpha+eb(1-\alpha(2-t(\alpha))))\sigma^2}\right)$;

$$
\begin{aligned}
&exp\left(\frac{\left(\frac{\mu}{\sigma}(1+t(\alpha)(L_s-1)-bh\frac{\sigma^2}{\mu})\right)^2}{2}\right)\left[1-\Phi\left(\frac{1+e}{e}\left(\frac{\mu}{\sigma}(1+\frac{1+t(\alpha)e}{1+e}(L_s-1)-\frac{e}{1+e}bh\frac{\sigma^2}{\mu})\right)^2\right)\right]+\\
&exp\left(\frac{1}{2}\frac{1}{1+e}\left(\frac{\mu}{\sigma}\left(t(\alpha)-1+bh\frac{\sigma^2}{\mu}\right)\right)^2\right)exp\left(\frac{\mu^2}{2\sigma^2}\frac{e}{1+e}\frac{\left(T(\alpha)-1-bh(t(\alpha)-1)\frac{\sigma^2}{\mu}\right)^2}{1+eT(\alpha)}\right)\left(\frac{1}{1+T(\alpha)e}\right)^{1/2}\\
&\left[\Phi\left(\frac{1+e}{e(1+T(\alpha)e)}\left(\frac{\mu}{\sigma}\left(1+\frac{1+T(\alpha)e}{1+e}(L_s-1)+\frac{e}{1+e}bh(t(\alpha)-1)\frac{\sigma^2}{\mu}\right)\right)^2\right)+\right.\\
&\left.\Phi\left(\frac{1+e}{e(1+T(\alpha)e)}\left(\frac{\mu}{\sigma}\left(\frac{1+T(\alpha)e}{1+e}(1+L_m)-1-\frac{e}{1+e}bh(t(\alpha)-1)\frac{\sigma^2}{\mu}\right)\right)^2\right)\right]+\\
&exp\left(\frac{\left(\frac{\mu}{\sigma}(t(\alpha)(1+L_m)-1+bh\frac{\sigma^2}{\mu})\right)^2}{2}\right)\left[1-\Phi\left(\frac{1+e}{e}\left(\frac{\mu}{\sigma}(\frac{1+t(\alpha)e}{1+e}(1+L_m)-1+\frac{e}{1+e}bh\frac{\sigma^2}{\mu})\right)^2\right)\right].
\end{aligned} \tag{11}
$$

$if -\left(s\frac{a\alpha+eb(1-\alpha)(2-t(\alpha))}{a\alpha+eb(1-\alpha(2-t(\alpha)))}+\frac{(1+e)\mu}{(a\alpha+eb(1-\alpha(2-t(\alpha))))\sigma^2}\right)\le h<m\frac{a\alpha+eb(1-\alpha)(2-t(\alpha))}{a\alpha+eb(1-\alpha(2-t(\alpha)))}-\frac{(1+e)\mu}{(a\alpha+eb(1-\alpha(2-t(\alpha))))\sigma^2};$

$$
\begin{aligned}
&exp\left(\frac{\left(\frac{\mu}{\sigma}(1+t(\alpha)(L_s-1)-bh\frac{\sigma^2}{\mu})\right)^2}{2}\right)\left[1-\Phi\left(\frac{1+e}{e}\left(\frac{\mu}{\sigma}(1+\frac{1+t(\alpha)e}{1+e}(L_s-1)-\frac{e}{1+e}bh\frac{\sigma^2}{\mu})\right)^2\right)\right]+\\
&exp\left(\frac{1}{2}\frac{1}{1+e}\left(\frac{\mu}{\sigma}\left(t(\alpha)-1+bh\frac{\sigma^2}{\mu}\right)\right)^2\right)exp\left(\frac{\mu^2}{2\sigma^2}\frac{e}{1+e}\frac{\left(T(\alpha)-1-bh(t(\alpha)-1)\frac{\sigma^2}{\mu}\right)^2}{1+eT(\alpha)}\right)\left(\frac{1}{1+T(\alpha)e}\right)^{1/2}\\
&\left[\Phi\left(\frac{1+e}{e(1+T(\alpha)e)}\left(\frac{\mu}{\sigma}\left(1+\frac{1+T(\alpha)e}{1+e}(L_s-1)+\frac{e}{1+e}bh(t(\alpha)-1)\frac{\sigma^2}{\mu}\right)\right)^2\right)-\right.\\
&\left.\Phi\left(\frac{1+e}{e(1+T(\alpha)e)}\left(\frac{\mu}{\sigma}\left(\frac{1+T(\alpha)e}{1+e}(1+L_m)-1-\frac{e}{1+e}bh(t(\alpha)-1)\frac{\sigma^2}{\mu}\right)\right)^2\right)\right]+\\
&exp\left(\frac{\left(\frac{\mu}{\sigma}(t(\alpha)(1+L_m)-1+bh\frac{\sigma^2}{\mu})\right)^2}{2}\right)\left[1-\Phi\left(\frac{1+e}{e}\left(\frac{\mu}{\sigma}(\frac{1+t(\alpha)e}{1+e}(1+L_m)-1+\frac{e}{1+e}bh\frac{\sigma^2}{\mu})\right)^2\right)\right].
\end{aligned}
$$

$if\, m\frac{a\alpha+eb(1-\alpha)(2-t(\alpha))}{a\alpha+eb(1-\alpha(2-t(\alpha)))}-\frac{(1+e)\mu}{(a\alpha+eb(1-\alpha(2-t(\alpha))))\sigma^2}\le h<m\frac{a\alpha+eb(1-\alpha)}{\alpha(a-eb)}-\frac{(1+e)\mu}{\alpha(a-eb)\sigma^2};$

$$
\begin{aligned}
&exp\left(\frac{\left(\frac{\mu}{\sigma}(1+t(\alpha)(L_s-1)-bh\frac{\sigma^2}{\mu})\right)^2}{2}\right)\left[1-\Phi\left(\frac{1+e}{e}\left(\frac{\mu}{\sigma}(1+\frac{1+t(\alpha)e}{1+e}(L_s-1)-\frac{e}{1+e}bh\frac{\sigma^2}{\mu})\right)^2\right)\right]+\\
&exp\left(\frac{1}{2}\frac{1}{1+e}\left(\frac{\mu}{\sigma}\left(t(\alpha)-1+bh\frac{\sigma^2}{\mu}\right)\right)^2\right)exp\left(\frac{\mu^2}{2\sigma^2}\frac{e}{1+e}\frac{\left(T(\alpha)-1-bh(t(\alpha)-1)\frac{\sigma^2}{\mu}\right)^2}{1+eT(\alpha)}\right)\left(\frac{1}{1+T(\alpha)e}\right)^{1/2}\\
&\left[\Phi\left(\frac{1+e}{e(1+T(\alpha)e)}\left(\frac{\mu}{\sigma}\left(1+\frac{1+T(\alpha)e}{1+e}(L_s-1)+\frac{e}{1+e}bh(t(\alpha)-1)\frac{\sigma^2}{\mu}\right)\right)^2\right)-\right.\\
&\left.\Phi\left(\frac{1+e}{e(1+T(\alpha)e)}\left(\frac{\mu}{\sigma}\left(\frac{1+T(\alpha)e}{1+e}(1+L_m)-1-\frac{e}{1+e}bh(t(\alpha)-1)\frac{\sigma^2}{\mu}\right)\right)^2\right)\right]+\\
&exp\left(\frac{\left(\frac{\mu}{\sigma}(t(\alpha)(1+L_m)-1+bh\frac{\sigma^2}{\mu})\right)^2}{2}\right)\left[1+\Phi\left(\frac{1+e}{e}\left(\frac{\mu}{\sigma}(\frac{1+t(\alpha)e}{1+e}(1+L_m)-1+\frac{e}{1+e}bh\frac{\sigma^2}{\mu})\right)^2\right)\right].
\end{aligned}
$$

if $h > m\frac{a\alpha+eb(1-\alpha)}{\alpha(a-eb)} - \frac{(1+e)\mu}{\alpha(a-eb)\sigma^2}$;

The investor must choose the optimal linear contract, which includes the optimal fixed and incentive fees, F and α, respectively, and the optimal benchmark, h, subject to the participation constraint $-(1/2)\exp(-(1/2)\mu^2/\sigma^2 - aF + V(a,e)) \times g(e, L_s, L_m) \geq -\exp(-(1/2)\mu^2/\sigma^2)$. Clearly, neither effort nor h or α are a function of F. This, along with the fact that the left-hand side is increasing in F and the investor's utility is decreasing in F, implies that under the optimal contract the participation constraint is binding. So, the investor's expected utility can be expressed a function of the contract (α, h), and the manager's level of effort, e:

$$EU_b(\varphi_b(e)|\alpha, h) = -exp(-bR - (1/2)\mu^2/\sigma^2 + (b/a)V(D,e)) \times g(e, L_s, L_m)^{b/a} f(e, L_s, L_m). \quad (12)$$

We want to study how the portfolio constraints *and* the presence of moral hazard affect the investor's choice. We distinguish two cases: first, when effort is publicly observable; second, under moral hazard. Moreover, the manager could be constrained or unconstrained in her portfolio choice.

Assume first that the manager's effort decision is observable. In this case the investor maximizes his expected utility with respect to α, h and effort.

In the absence of short-selling and margin purchase constraints, we show that the optimal contract is given by the first best incentive fee, α_{FB}, and zero benchmarking, $h = 0$. The function $f(e, L_s, L_m)$ becomes $g(e)$. The investor chooses the *first best effort level*, e_{FB}, that maximizes $EU_b(\varphi_b(e)|\alpha_{FB}, 0) = -\exp\big(-(1/2)(\mu/\sigma)^2 + (b/a)V(D,e)\big)g(e)^{\frac{a+b}{a}}$:

$$V_e(D, e_{FB}) = \frac{1 + b/a}{2(1 + e_{FB})}.$$

When the manager's portfolio choice is constrained, we show that the contract $(\alpha_{FB}, 0)$ is still optimal. The function $f(e, L_s, L_m)$ becomes $g(e, L_s(0), L_m(0))$. In this *constrained first best scenario* the investor chooses the *constrained first best effort level*, e^c_{FB}, that maximizes $EU_b(\varphi_b(e)|\alpha_{FB}, 0) = -\exp\big(-(1/2)(\mu/\sigma)^2 + (b/a)V(D,e)\big)g(e, L_s(0), L_m(0))^{\frac{a+b}{a}}$:

$$V_e(D, e^c_{FB}) = (1 + b/a)\,\frac{g_e}{g}(e^c_{FB}, L_s(0), L_m(0)).$$

Notice that, as expected, portfolio constraints decrease the optimal effort choice: $e_{FB} > e^c_{FB}$.

On the other hand, when the manager's effort decision is not observable, the investor's problem consists in finding the optimal split that maximizes (12) subject to the manager's optimal effort condition.

Assume first that there exit no portfolio constraints. We call this scenario the second best. As shown in section 2.1, the manager's *second best effort*, e_{SB}, is independent of α and h. This result is in agreement with Stoughton (1993) and Admati and Pfleiderer (1997). The investor will choose the same contract than in the unconstrained, public information case: $(\alpha_{FB}, 0)$. The second best effort satisfies the optimality condition (5).

Assume now that the manager's portfolio choice is constrained. We call this scenario the third best. The manager's third best effort satisfies (10). Section 2.2 shows that effort is increasing in α and, given α, reaches an absolute maximum at h^*. The contract $(\alpha_{FB}, 0)$ is no longer optimal.

These results are presented in the following proposition.

Proposition 4 *Absent any portfolio constraint, the contract* $(\alpha_{FB}, 0)$ *is optimal, both for the public information case as well as under moral hazard.*

Under portfolio constraints and no moral hazard, the contract $(\alpha_{FB}, 0)$ *is still optimal. When the effort decision is not observable by the investor, the contract* $(\alpha_{FB}, 0)$ *is suboptimal.*

In spite of the simplifications, we cannot solve analytically for the general optimal contract under moral hazard and portfolio constraints. In the next section we present some numerical results.

5 A numerical solution of the third best contract

Due to the complexity of the manager's expected utility function in Proposition 1, we cannot solve analytically for the optimal third best contract. We can, however, solve the problem numerically. We propose the function $V(e) = \frac{D}{2}e^2$ with disutility parameter $D = 1$.

Throughout the numerical analysis, we take the market excess return $\mu = 6\%$ and the market volatility $\sigma = 18\%$, both on an annual basis. The principal's absolute risk aversion take values $b = 4$ and $b = 8$. The manager's absolute risk aversion parameter takes values $a = \{4, 8, 16, 20\}$. We consider different degrees of portfolio constraints: $s = 0$ and $m = 1$ is the zero leverage base-case. We then allow for short selling ($s = 1$ and $s = 2$) and margin purchases ($m = 2$ and $m = 3$). For each combination (a, b) and (m, s) we calculate the manager's effort and the investor's expected utility (12) for a grid of values for *alpha* and h around the first best contract $(\alpha_{FB}(a, b), 0)$. α changes from $70\% \times \alpha_{FB}(a, b)$ to $130\% \times \alpha_{FB}(a, b)$, at intervals of length $5\%\alpha_{FB}(a, b)$. Likewise, h changes from -30% to 30% at intervals of length 5%. In the absence of moral hazard, for each contract (α, h) the manager puts the constrained first best effort that maximizes the investor's expected utility in (12). Under moral hazard, for each contract (α, h) the manager puts the third best effort in (10).

Figures 3 and 4 introduce the base case under total constraints ($m = 1$ and $s = 0$) and for two values the investor's expected utility: $b = 4$ and $b = 8$, respectively. Two scenarios are considered: Panel A presents the optimal contract in the absence of moral hazard (the manager's effort decision is publicly observable), i.e., the constrained first best scenario; Panel B represents the optimal contract under moral hazard, i.e., the third best scenario. The investors expected utility is concave in α and h. In Panel A, we observe that, as predicted in proposition 4, in the absence of moral hazard, the investor's maximum expected utility is attained at the first best contract $(\alpha_{FB}, 0)$ with zero benchmarking.

The figures in Panel B show confirm the prediction in proposition 4: the first best contract is no longer optimal in the presence of moral hazard. In concrete, the incentive fee increases from 50% to 52.5% ($a = b = 4$) and 55% ($a = b = 8$); from 33% to 46% ($a = 8$,$b = 4$) and 41% ($a = 16$, $b = 8$). The benchmark becomes more risky: the percentage invested in the market portfolio rises from zero to 15% for $a = 8$ and $b = 4$; in the case $b = 8$ the percentage increases from zero to 5% for $a = 8$ and 10% for $a = 16$.

Figures 5 through 10 represent the investor's expected utility under moral hazard for different values of the portfolio constraints, m and s, and the risk aversion coefficients for the investor (b) and the manager (a). Table 1 summarizes the optimal contracts (α, h). We observe the following.

Holding the manager's short selling constraint at $s = 0$, the optimal benchmark composition h increases with m, the margin purchase limit. This is in agreement with the partial equilibrium intuition in section 2.2: as the margin purchase constraint is relaxed, increasing h induces higher effort on the manager. Moreover, the marginal effort increase is higher for more risk averse managers. This has to be traded-off against the optimal risk split between the principal and the agent. We observe that the increase in h is higher (relative to the unconstrained case) the lower the investor's risk aversion (b) and the higher the manager's risk aversion (a). In concrete, when the manager is more risk averse than the investor, the optimal contract substitutes a more risky benchmark (higher h) for a lower incentive fee (lower α) as m increases. For instance, when $b = 4$, for the total constrained case ($m = 1$ and $s = 0$), α decreases from 52.5% to 27.5% when a increases from 4 to 20. Simultaneously, h increases from 0 to 20%. When we relax the margin purchase constraint, for instance $m = 3$, α decreases from $50\% to 20\%$ while h increases from 15 to 20%. When $b = 8$, α decreases more than in the case of $b = 4$; on the other side, h increases less than in the case of $b = 4$.

The result is symmetric when the constraint on margin purchases is held constant at $m = 1$ and the bound on short sells is relaxed. For $b = 4$ and $a = 8$, moving from zero short selling ($s = 0$) to 100% of the original wealth endowment ($s = 1$) the optimal benchmark moves from $h = 15\%$ down to zero (the risk free asset). In the short selling bound increases by another 100%, the optimal benchmark should short the market by 5%. The benchmark sensitivity to changes in s decreases as the investor becomes more risk averse (from $b = 4$ to $b = 8$) or when the manager becomes more risk tolerant: from $a = 20$ to $a = 4$.

Finally, when both constraints are relaxed the optimal contract converges towards the first best contract $(\alpha_{FB}, 0)$. This convergence is faster the more risk averse the investor is. In concrete, when $b = 4$, $m = 3$ and $s = 2$, the first best contract is optimal for $a = 20$. When $b = 8$, $m = 2$ and $s = 1$ the first best contract is optimal both for $a = 8$ and $a = 16$.

6 Conclusions

This paper investigates the effort inducement incentives of (potentially benchmarked) linear incentive fee contracts. Incentives arise explicitly via the compensation of the manager. The investor has to decide simultaneously the incentive fee (the manager's participation in the delegated portfolio's return) and the benchmark composition.

The contribution of our paper to the literature on management compensation comes from the fact that we incorporate portfolio constraints in our model. These constraints are exogenous in our model and could be motivated by regulation or, as suggested by Almazan et al (2004), as alternative monitoring mechanism in a broader equilibrium model.

$b = 4$

$s \setminus m$	a	1	2	3
0	4	(52.5,0)	(55,10)	(55,15)
	8	(46,15)	(40,15)	(36,15)
	20	(27.5,20)	(21,20)	(20,20)
1	4	(55,0)	(55,5)	
	8	(40,0)	(35,5)	
	20	(21,0)	(19,5)	
2	4	(55,0)		(55,5)
	8	(38,-5)		(35,5)
	20	(20,-5)		(16,0)

$b = 8$

$s \setminus m$	a	1	2	3
0	8	(55,5)	(52.5,5)	(52.5,5)
	16	(41,10)	(38,10)	(38,10)
1	8	(52.5,0)	(50,0)	
	16	(35,0)	(33,0)	
2	8	(52.5,0)		(50,0)
	16	(35,0)		(33,0)

Table 1: Optimal contract (α, h) for different values of the maximum long (m) and short (s) position on the market portfolio allowed to the manager. α is the percentage incentive fee; h is the percentage of the benchmark portfolio invested in the market portfolio. $m = 1$ and $s = 0$ imply zero leverage. b (a) represents the investor's (manager's) risk aversion coefficient. The first best incentive fee is $\alpha_{FB} = 50\%$ for $a = b = 4$ and $a = b = 8$; $\alpha_{FB} = 33\%$ for $a = 8; b = 4$ and $a = 16; b = 8$; $\alpha_{FB} = 16\%$ for $a = 20; b = 4$.

Under portfolio constraints and moral hazard, our model predicts that portfolio manager's should be offered an incentive fee benchmarked against a portfolio that combines the risky market portfolio and the risky asset. Numerical exercises suggest that, in contrast with the predictions from the unconstrained setting in Ou-Yang (2003), the risk-free asset is not the optimal benchmark. When portfolio constraints are removed, the model predicts that the manager's effort is unrelated to the incentive fee and the benchmark composition, a well-known result in the literature.

These predictions are consistent with the prevalence of absolute return (non-benchmarked) compensation schemes among hedge fund managers, arguably much less constrained than mutual fund managers. Moreover, it offers a theoretical foundation for the observed out-performance of mutual funds who offer incentive fee compensation as documented by Elton, Gruber and Blake (2003). The model implies new empirically testable implications. Concretely:

1. When the manager is not constrained, the optimal benchmark is the risk free asset.

2. For constrained managers, the proportion of the optimal benchmark invested in the timing portfolio increases (decreases) when the margin purchase (short selling) constraint is removed. In other words, managers who cannot sell short but can purchase at margin should be compensated relative to higher beta benchmarks while manager's who cannot

purchase at margin but can sell sell short should be rewarded relative to lower beta benchmarks. When the manager is constrained both on selling short and margin purchases, the benchmark's beta should lie between to two partially constrained betas.

3. This pattern should be more evident among aggressive market timers than among more conservative funds. Additionally, benchmarking the manager is more effective when the manager is more risk averse.

4. If portfolio restriction are independent of effort incentives, effort (hence, timing ability) should be inversely related to constrains: more constrained managers should perform worse than unrestricted managers.

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Appendix

Proof of Proposition 1

Replacing (6) in the manager's utility function:

$$
\begin{aligned}
EU\left(\varphi_a(y)\right) = {} & -\exp(-aF + V(D,e)) \times \\
& \begin{cases}
\exp\left((h+s)a\alpha E(x|y) + (1/2)((h+s)a\alpha)^2 \mathrm{Var}(x|y)\right) & \text{if } y < -\frac{\mu}{e}L_s \\
\exp\left(-(1/2)E^2(x|y)/\mathrm{Var}(x|y)\right) & \text{otherwise} \\
\exp\left(-(m-h)a\alpha E(x|y) + (1/2)((m-h)a\alpha)^2 \mathrm{Var}(x|y)\right) & \text{if } y > \frac{\mu}{e}L_m.
\end{cases}
\end{aligned}
$$

Multiplying the previous expression by the density function of the signal variable, y, we obtain:

$$-\exp(-(1/2)(\mu^2/\sigma^2) - aF + V(D,e))\left(\frac{e}{1+e}\right)^{1/2}\frac{1}{\sqrt{2\pi}\sigma}\times$$
$$\begin{cases} \exp\left(\frac{\left(\frac{\mu}{\sigma}L_s\right)^2}{2}\right)\exp\left(-(1/2)\frac{e}{1+e}\left(\frac{y}{\sigma}-\frac{\mu}{\sigma}L_s\right)^2\right) & \text{if } y < -\frac{\mu}{e}L_s \\ \exp\left(-(1/2)e\left(\frac{y}{\sigma}\right)^2\right) & \text{otherwise} \\ \exp\left(\frac{\left(\frac{\mu}{\sigma}L_m\right)^2}{2}\right)\exp\left(-(1/2)\frac{e}{1+e}\left(\frac{y}{\sigma}+\frac{\mu}{\sigma}L_m\right)^2\right) & \text{if } y > \frac{\mu}{e}L_m. \end{cases}$$

Replace $k = \frac{e}{1+e}\left(\frac{y}{\sigma} - \frac{\mu}{\sigma}L_s\right)^2$ if $y < -\frac{\mu}{e}L_s$; $k = \frac{e}{1+e}\left(\frac{y}{\sigma} + \frac{\mu}{\sigma}L_m\right)^2$ if $y > \frac{\mu}{e}L_m$, and $k = e\left(\frac{y}{\sigma}\right)^2$ otherwise. Integrating over k and given the definition of $\Phi(\cdot)$, the unconditional utility function follows. *QED*

Proof of Corollary 1

By definition, $|L_m| > |L_s|$ for all $-\infty < h < -\left(s + \frac{\mu}{a\alpha\sigma^2}\right)$ such that $\left[\Phi\left(\frac{\left(\frac{\mu}{\sigma}L_m\right)^2}{e}\right) - \Phi\left(\frac{\left(\frac{\mu}{\sigma}L_s\right)^2}{e}\right)\right] > 0$; likewise $|L_s| > |L_m|$ for all $\infty > h > m - \frac{\mu}{a\alpha\sigma^2}$ such that $\left[\Phi\left(\frac{\left(\frac{\mu}{\sigma}L_s\right)^2}{e}\right) - \Phi\left(\frac{\left(\frac{\mu}{\sigma}L_m\right)^2}{e}\right)\right] > 0$. *QED*

Proof of Proposition 2

Let us define $\mathcal{J}(e, L_s, L_m) = V_e(D, e) \times g(e, L_s, L_m) + g_e(e, L_s, L_m)$. The function $\mathcal{J} \in \mathcal{C}^1$ for all (e, h). The third best effort in (10) satisfies:

$$\mathcal{J}(e_{TB}, L_s, L_m) = 0, \tag{A1}$$
$$\mathcal{J}_e(e_{TB}, L_s, L_m) > 0. \tag{A2}$$

The implicit function theorem allows us to solve "locally" the equation; that is, for all $(\hat{e}, \hat{h})$ that satisfy (A1) and (A2), effort e can be expressed as a function of h in a neighborhood of $(\hat{e}, \hat{h})$.

More formally: for all $(\hat{e}, \hat{h})$ that satisfy (A1) and (A2) there exists a function $e(h) \in \mathcal{C}^1$ and an open ball $B(\hat{h})$, such that $e(\hat{h}) = e_{TB}$ and $\mathcal{J}(e(h), L_s, L_m) = 0$ for all $h \in B(\hat{h})$.

Taking the derivative of $\mathcal{J}(e_{TB}, L_s, L_m)$ with respect to h:[10]

$$e_h(h) = -\mathcal{J}_h(e_{TB}, L_s, L_m) \times \mathcal{J}_e^{-1}(e_{TB}, L_s, L_m).$$

Taking the second derivative of (8) with respect to e:

[10] The subscript h denotes first derivative with respect to h. The subscript eh denotes cross derivative with respect to e and h.

$$g_{ee}(e,L_s,L_m)=\frac{1}{2}\left(\frac{1}{1+e}\right)^{3/2}\left\{\frac{3}{2}\left(\frac{1}{1+e}\right)\left[\Phi\left(\frac{\left(\frac{\mu}{\sigma}L_s\right)^2}{e}\right)+\Phi\left(\frac{\left(\frac{\mu}{\sigma}L_m\right)^2}{e}\right)\right]+\right.$$
$$\left.\frac{1}{e^2}\left[\phi\left(\frac{\left(\frac{\mu}{\sigma}L_s\right)^2}{e}\right)\times\left(\frac{\mu}{\sigma}L_s\right)^2+\phi\left(\frac{\left(\frac{\mu}{\sigma}L_m\right)^2}{e}\right)\times\left(\frac{\mu}{\sigma}L_m\right)^2\right]\right\}>0.$$

Condition (A2) can be written as $V_{ee}(D,e)>-\frac{g_e}{g}(e,L_s)\times V_e(D,e)-\frac{g_{ee}}{g}(e,L_s)$. $-\frac{g_e}{g}(e,L_s)<\frac{1}{2(1+e)}$ and $\frac{g_{ee}}{g}(e,L_s)\geq 0$. Then, (S4) implies (A2) for all $h\in[-\left(s+\frac{\mu}{a\alpha\sigma^2}\right),m-\frac{\mu}{a\alpha\sigma^2}]$.

The sign of $e_h(h)$, therefore, depends on the sign of $\mathcal{J}_h(e,L_s,L_m)=V_e(D,e)\times g_h(e,L_s,L_m)+g_{eh}(e,L_s,L_m)$.

From (S3), $V_e(D,e)>0$. From Corollary 1,

$$g_{eh}(e,L_s,L_m)=-\left(\frac{1}{1+e}\right)^{3/2}e^{-1/2}\frac{a\alpha\sigma}{\sqrt{2\pi}}\left[\exp\left(\frac{-\left(\frac{\mu}{\sigma}L_s\right)^2}{2e}\right)-\exp\left(\frac{-\left(\frac{\mu}{\sigma}L_m\right)^2}{2e}\right)\right]\tag{A3}$$

for all $h\in\Re$.

Let us define the gamma function $\Gamma(u)=\int_0^\infty t^{u-1}\exp(-t)dt$ for $u>0$. The *incomplete* gamma function is given by $\Gamma(u,v)=\int_v^\infty t^{u-1}\exp(-t)dt$ for $v>0$. From (8),

$$\begin{aligned}&g_h(e,L_s,L_m)=\\&\quad\frac{a\alpha\mu}{\sqrt{\pi}}\Gamma\left(\frac{1}{2},\frac{1+e}{e}\right)\left(L_s\exp\left(\frac{\left(\frac{\mu}{\sigma}L_s\right)^2}{2}\right)-L_m\exp\left(\frac{\left(\frac{\mu}{\sigma}L_m\right)^2}{2}\right)\right)-\\&\quad\left(\frac{e}{1+e}\right)^{1/2}\frac{2a\alpha\sigma}{\sqrt{2\pi}}\left[\exp\left(\frac{-\left(\frac{\mu}{\sigma}L_s\right)^2}{2e}\right)-\exp\left(\frac{-\left(\frac{\mu}{\sigma}L_m\right)^2}{2e}\right)\right].\end{aligned}\tag{A4}$$

By definition, $L_s(h^*+\delta)=L_m(h^*-\delta)$, for all $\delta\in\Re$. For all $0<\delta<\frac{m+s}{2}$, $L_s(h^*-\delta)<L_m(h^*-\delta)$ and $L_s(h^*+\delta)>L_m(h^*+\delta)$. Let $L_s^*=L_s(h^*)$ and $L_m^*=L_m(h^*)$. For $\delta=0$, $L_s^*=L_m^*$. Therefore, $e_h(h)>0$ for all $-\left(s+\frac{\mu}{a\alpha\sigma^2}\right)\leq h<h^*$ and $e_h(h)<0$ for all $h^*<h\leq m-\frac{\mu}{a\alpha\sigma^2}$; $e_h(h^*)=0$. Since the function $e(h)$ is continuous and differentiable, it follows that h^* is a local maximum in the interval $\left[-\left(s+\frac{\mu}{a\alpha\sigma^2}\right),m-\frac{\mu}{a\alpha\sigma^2}\right]$. *Q.E.D.*

Proof of Corollary 2

Let $h<-\left(s+\frac{\mu}{a\alpha\sigma^2}\right)$. Then, $L_s<0$ and $L_m>0$ and $|L_s|<|L_m|$. From (7),

$$
\begin{aligned}
&g_h(e, L_s, L_m) = \\
&\quad a\alpha\mu L_s \exp\left(\frac{\left(\frac{\mu}{\sigma}L_s\right)^2}{2}\right)\left[1+\Phi\left(\frac{1+e}{e}\left(\frac{\mu}{\sigma}L_s\right)^2\right)\right]- \\
&\quad a\alpha\mu L_m \exp\left(\frac{\left(\frac{\mu}{\sigma}L_m\right)^2}{2}\right)\left[1-\Phi\left(\frac{1+e}{e}\left(\frac{\mu}{\sigma}L_m\right)^2\right)\right]- \qquad \text{(A5)} \\
&\quad \left(\frac{e}{1+e}\right)^{1/2}\frac{2a\alpha\sigma}{\sqrt{2\pi}}\left[\exp\left(\frac{-\left(\frac{\mu}{\sigma}L_s\right)^2}{2e}\right)-\exp\left(\frac{-\left(\frac{\mu}{\sigma}L_m\right)^2}{2e}\right)\right]<0
\end{aligned}
$$

From (A3), $g_{eh}(e, L_s, L_m) < 0$. Given (S3), it follows that $e_h(h) > 0$ for all $h < -\left(s + \frac{\mu}{a\alpha\sigma^2}\right)$.

Let $h > m - \frac{\mu}{a\alpha\sigma^2}$. Then, $L_s > 0$ and $L_m < 0$ and $|L_s| > |L_m|$. From (9),

$$
\begin{aligned}
&g_h(e, L_s, L_m) = \\
&\quad a\alpha\mu L_s \exp\left(\frac{\left(\frac{\mu}{\sigma}L_s\right)^2}{2}\right)\left[1-\Phi\left(\frac{1+e}{e}\left(\frac{\mu}{\sigma}L_s\right)^2\right)\right]- \\
&\quad a\alpha\mu L_m \exp\left(\frac{\left(\frac{\mu}{\sigma}L_m\right)^2}{2}\right)\left[1+\Phi\left(\frac{1+e}{e}\left(\frac{\mu}{\sigma}L_m\right)^2\right)\right]- \\
&\quad \left(\frac{e}{1+e}\right)^{1/2}\frac{2a\alpha\sigma}{\sqrt{2\pi}}\left[\exp\left(\frac{-\left(\frac{\mu}{\sigma}L_s\right)^2}{2e}\right)-\exp\left(\frac{-\left(\frac{\mu}{\sigma}L_m\right)^2}{2e}\right)\right]>0.
\end{aligned}
$$

From (A3), $g_{eh}(e_{TB}, L_s, L_m) > 0$. Given (S3), it follows that $e_h(h) < 0$ for all $h > m - \frac{\mu}{a\alpha\sigma^2}$.
Q.E.D.

Proof of Corollary 3

Let $h \in \left[-\left(s + \frac{\mu}{a\alpha\sigma^2}\right), m - \frac{\mu}{a\alpha\sigma^2}\right]$. We re-write the function $\mathcal{J}(e, L_s, L_m)$ as:

$$
\begin{aligned}
\mathcal{J}(e, L_s, L_m) = &\left[V_e(D,e) - \frac{1}{2(1+e)}\right]\left(\frac{1}{1+e}\right)^{1/2}\left[\Phi\left(\frac{\left(\frac{\mu}{\sigma}(L_s)\right)^2}{e}\right)+\Phi\left(\frac{\left(\frac{\mu}{\sigma}(L_m)\right)^2}{e}\right)\right] \\
&+V_e(D,e)\left\{\exp\left(\frac{\left(\frac{\mu}{\sigma}(L_s)\right)^2}{2}\right)\times\left[1-\Phi\left(\frac{\left(\frac{\mu}{\sigma}(L_s)\right)^2}{e}(1+e)\right)\right]\right. \\
&\left.+\exp\left(\frac{\left(\frac{\mu}{\sigma}(L_m)\right)^2}{2}\right)\times\left[1-\Phi\left(\frac{\left(\frac{\mu}{\sigma}(L_m)\right)^2}{e}(1+e)\right)\right]\right\}.
\end{aligned}
$$

Evaluating this function at the second best effort and given (5) we obtain

$$\mathcal{J}(e_{SB}, L_s, L_m) =$$
$$V_e(D, e_{SB}) \left\{ \exp\left(\frac{\left(\frac{\mu}{\sigma}(L_s)\right)^2}{2} \right) \times \left[1 - \Phi\left(\frac{\left(\frac{\mu}{\sigma}(L_s)\right)^2}{e_{SB}} (1 + e_{SB}) \right) \right] \right. \tag{A6}$$
$$\left. + \exp\left(\frac{\left(\frac{\mu}{\sigma}(L_m)\right)^2}{2} \right) \times \left[1 - \Phi\left(\frac{\left(\frac{\mu}{\sigma}(L_m)\right)^2}{e_{SB}} (1 + e_{SB}) \right) \right] \right\} > 0.$$

This implies that $E_e U_a(\varphi_a(e_{SB})) = -\exp(-(1/2)\mu^2/\sigma^2 - aF + V(D, e_{SB})) \times \mathcal{J}(e_{SB}, L_s, L_m) < 0$.

Therefore, for the constrained manager, the marginal utility of effort at e_{SB} is negative. Since e_{TB} is unique and the function is continuous in e, given conditions (A1) and (A2), it follows that $e_{SB} > e_{TB}$ for all $h \in \left[-\left(s + \frac{\mu}{a\alpha\sigma^2}\right), m - \frac{\mu}{a\alpha\sigma^2} \right]$. Given Corollary 2 this result holds for all $h \in \Re$. Next we show that

$$\lim_{z \to \infty} \left[\exp(z/2) \times \left(1 - \Phi\left(z \frac{1+e}{e} \right) \right) \right] = 0. \tag{A7}$$

Re-writing (A7) and applying L'Hôpital's rule we get:

$$\lim_{z \to \infty} \frac{1 - \Phi\left(z \frac{1+e}{e}\right)}{\exp(-z/2)} = \lim_{z \to \infty} \frac{\exp(-z/e)}{z} = 0.$$

Therefore, given (A6) and (A7), $\mathcal{J}(e_{SB}, L_s, L_m)$ tends to zero when m and s tend to infinity. In the limit, the constrained manager's marginal expected utility of effort becomes zero at e_{SB}, $E_e U_a(\varphi_a(e_{SB})) = 0$. *Q.E.D.*

Proof of Corollary 4

Lemma 1 *For all* $0 < x < \infty$, $\frac{1}{2}(1 - \Phi(x)) - \phi(x) < 0$.

Proof: See Lemma 1 in Gómez and Sharma (2006)

Let $m \to \infty$ and $0 \leq s < \infty$. We call $g_h(e, L_s) = \lim_{m \to \infty} g_h(e, L_s, L_m)$ and $g_{eh}(e, L_s) = \lim_{m \to \infty} g_{eh}(e, L_s, L_m)$. From (A5), $g_h(e, L_s) < 0$ for $h < -\left(s + \frac{\mu}{a\alpha\sigma^2}\right)$. For $h > -\left(s + \frac{\mu}{a\alpha\sigma^2}\right)$, $g_h(e, L_s) = 2a\mu L_s \times \exp\left(\frac{\left(\frac{\mu}{\sigma} L_s\right)^2}{2} \right) \left\{ \frac{1}{2} \left[1 - \Phi\left(\frac{1+e}{e} \left(\frac{\mu}{\sigma} L_s\right)^2 \right) \right] \right.$
$\left. -\phi\left(\frac{1+e}{e} \left(\frac{\mu}{\sigma} L_s\right)^2 \right) \right\} < 0$, given Lemma 1.

Therefore, $g_h(e, L_s) < 0$ for all $h \in \Re$. From (A3), $g_{eh}(e, L_s) < 0$ for all $h \in \Re$. Thus, $e_h(h) > 0$ for all $h \in \Re$. Following the same procedure, it is trivial to show that $e_h(h) < 0$ for all $h \in \Re$ when $s \to \infty$ and $1 \leq m < \infty$. *Q.E.D.*

Proof of Proposition 3

Replacing (6) in the investor's utility function:

$$EU\left(\varphi_b(y)\right) = -\exp(b(F-R)) \times$$
$$\begin{cases} \exp\left(-b(h-(1-\alpha)(s+h))E(x|y) + (b^2/2)(h-(1-\alpha)(s+h))^2 \mathrm{Var}(x|y)\right) & \text{if } y < -\frac{\mu}{e}L_s \\ \exp\left(-b\left(h+(1-\alpha)\frac{\mu+ey}{a\alpha\sigma^2}\right)E(x|y) + (b^2/2)\left(h+(1-\alpha)\frac{\mu+ey}{a\alpha\sigma^2}\right)^2 \mathrm{Var}(x|y)\right) & \text{otherwise} \\ \exp\left(-b(h+(1-\alpha)(m-h))E(x|y) + (b^2/2)(h+(1-\alpha)(m-h))^2 \mathrm{Var}(x|y)\right) & \text{if } y > \frac{\mu}{e}L_m. \end{cases}$$

Multiplying the previous expression by the density function of the signal variable y, we obtain:

$$EU\left(\varphi_b(y)\right) = -\exp(b(F-R) - (1/2)\mu^2/\sigma^2)\left(\frac{e}{1+e}\right)^{1/2}\frac{1}{\sqrt{2\pi}\sigma}\times$$
$$\begin{cases} \exp\left(\frac{\left(\frac{\mu}{\sigma}(1+t(\alpha)(L_s-1)-bh\frac{\sigma^2}{\mu})\right)^2}{2}\right)\times \\ \exp\left(-\frac{e}{2(1+e)}\left(\frac{y}{\sigma}-\frac{\mu}{\sigma}\left(1+t(\alpha)(L_s-1)-bh\frac{\sigma^2}{\mu}\right)\right)^2\right) & \text{if } y < -\frac{\mu}{e}L_s \\ \exp\left(\frac{1}{2}\frac{1}{1+e}\left(\frac{\mu}{\sigma}\left(t(\alpha)-1+bh\frac{\sigma^2}{\mu}\right)\right)^2\right)\exp\left(\frac{\mu^2}{2\sigma^2}\frac{e}{1+e}\frac{\left(T(\alpha)-1-bh(t(\alpha)-1)\frac{\sigma^2}{\mu}\right)^2}{1+eT(\alpha)}\right)\times \\ \left(\frac{1}{1+T(\alpha)e}\right)^{1/2}\exp\left(-\frac{e(1+eT(\alpha))}{2(1+e)}\left(\frac{y}{\sigma}+\frac{\mu}{\sigma}\frac{T(\alpha)-1-bh(t(\alpha)-1)\frac{\sigma^2}{\mu}}{1+eT(\alpha)}\right)^2\right) & \text{otherwise} \\ \exp\left(\frac{\left(\frac{\mu}{\sigma}(t(\alpha)(1+L_m)-1+bh\frac{\sigma^2}{\mu})\right)^2}{2}\right)\times \\ \exp\left(-\frac{e}{2(1+e)}\left(\frac{y}{\sigma}+\frac{\mu}{\sigma}\left(t(\alpha)(1+L_m)-1+bh\frac{\sigma^2}{\mu}\right)\right)^2\right) & \text{if } y > \frac{\mu}{e}L_m. \end{cases}$$

Replace $k = \frac{e}{1+e}\begin{cases} \left(\frac{y}{\sigma}-\frac{\mu}{\sigma}\left(1+t(\alpha)(L_s-1)-bh\frac{\sigma^2}{\mu}\right)\right)^2 & \text{if } y < -\frac{\mu}{e}L_s \\ (1+eT(\alpha))\left(\frac{y}{\sigma}+\frac{\mu}{\sigma}\frac{T(\alpha)-1-bh(t(\alpha)-1)\frac{\sigma^2}{\mu}}{1+eT(\alpha)}\right)^2 & \text{otherwise} \\ \left(\frac{y}{\sigma}+\frac{\mu}{\sigma}\left(t(\alpha)(1+L_m)-1+bh\frac{\sigma^2}{\mu}\right)\right)^2 & \text{if } y > \frac{\mu}{e}L_m. \end{cases}$

Integrating over k and given the definition of $\Phi(\cdot)$, the unconditional utility function follows. *Q.E.D.*

Proof of Proposition 4

We first need some partial derivatives of function f and g with respect to α and h. Taking the derivative of (11) with respect to h and evaluating it at the contract $(\alpha_{FB}, 0)$ yields:

$$
\begin{aligned}
&f_h(e, L_s, L_m|\alpha_{_{FB}}, 0) = -2\frac{b^2}{a+b}\mu \\
&\left\{\exp\left(\frac{\left(\frac{\mu}{\sigma}L_s(0)\right)^2}{2}\right)L_s(0)\times\right. \\
&\left(\frac{1}{2}\left[1-\Phi\left(\frac{1+e}{e}\left(\frac{\mu}{\sigma}L_s(0)\right)^2\right)\right]-\phi\left(\frac{1+e}{e}\left(\frac{\mu}{\sigma}L_s(0)\right)^2\right)\right)+ \\
&\exp\left(\frac{\left(\frac{\mu}{\sigma}L_m(0)\right)^2}{2}\right)L_m(0)\times \\
&\left.\left(\frac{1}{2}\left[1-\Phi\left(\frac{1+e}{e}\left(\frac{\mu}{\sigma}L_m(0)\right)^2\right)\right]-\phi\left(\frac{1+e}{e}\left(\frac{\mu}{\sigma}L_m(0)\right)^2\right)\right)\right\}.
\end{aligned}
$$

Equation (A4) evaluated at $(\alpha_{_{FB}}, 0)$ becomes:

$$
\begin{aligned}
&g_h(e, L_s, L_m|\alpha_{_{FB}}, 0) = 2\frac{ab}{a+b}\mu \\
&\left\{\exp\left(\frac{\left(\frac{\mu}{\sigma}L_s(0)\right)^2}{2}\right)L_s(0)\times\right. \\
&\left(\frac{1}{2}\left[1-\Phi\left(\frac{1+e}{e}\left(\frac{\mu}{\sigma}L_s(0)\right)^2\right)\right]-\phi\left(\frac{1+e}{e}\left(\frac{\mu}{\sigma}L_s(0)\right)^2\right)\right)+ \\
&\exp\left(\frac{\left(\frac{\mu}{\sigma}L_m(0)\right)^2}{2}\right)L_m(0)\times \\
&\left.\left(\frac{1}{2}\left[1-\Phi\left(\frac{1+e}{e}\left(\frac{\mu}{\sigma}L_m(0)\right)^2\right)\right]-\phi\left(\frac{1+e}{e}\left(\frac{\mu}{\sigma}L_m(0)\right)^2\right)\right)\right\}.
\end{aligned}
$$

Taking the derivative of (11) with respect to α and evaluating it at the contract $(\alpha_{_{FB}}, 0)$ we obtain:

$$
\begin{aligned}
&f_\alpha(e, L_s, L_m|\alpha_{_{FB}}, 0) = \\
&-2bs\mu\exp\left(\frac{\left(\frac{\mu}{\sigma}L_s(0)\right)^2}{2}\right)L_s(0)\times \\
&\left(\frac{1}{2}\left[1-\Phi\left(\frac{1+e}{e}\left(\frac{\mu}{\sigma}L_s(0)\right)^2\right)\right]-\phi\left(\frac{1+e}{e}\left(\frac{\mu}{\sigma}L_s(0)\right)^2\right)\right) \\
&-2bm\mu\exp\left(\frac{\left(\frac{\mu}{\sigma}L_m(0)\right)^2}{2}\right)L_m(0)\times \\
&\left(\frac{1}{2}\left[1-\Phi\left(\frac{1+e}{e}\left(\frac{\mu}{\sigma}L_m(0)\right)^2\right)\right]-\phi\left(\frac{1+e}{e}\left(\frac{\mu}{\sigma}L_m(0)\right)^2\right)\right).
\end{aligned}
$$

Equation (A4) at $(\alpha_{_{FB}}, 0)$ can be rewritten as follows:

$$
\begin{aligned}
&g_\alpha(e, L_s, L_m|\alpha_{FB}, 0) = \\
&\quad 2as\mu\exp\left(\frac{\left(\frac{\mu}{\sigma}L_s(0)\right)^2}{2}\right)L_s(0)\times \\
&\quad \left(\frac{1}{2}\left[1-\Phi\left(\frac{1+e}{e}\left(\frac{\mu}{\sigma}L_s(0)\right)^2\right)\right]-\phi\left(\frac{1+e}{e}\left(\frac{\mu}{\sigma}L_s(0)\right)^2\right)\right)+ \\
&\quad 2am\mu\exp\left(\frac{\left(\frac{\mu}{\sigma}L_m(0)\right)^2}{2}\right)L_m(0)\times \\
&\quad \left(\frac{1}{2}\left[1-\Phi\left(\frac{1+e}{e}\left(\frac{\mu}{\sigma}L_m(0)\right)^2\right)\right]-\phi\left(\frac{1+e}{e}\left(\frac{\mu}{\sigma}L_m(0)\right)^2\right)\right).
\end{aligned}
$$

Assume first that the manager's effort choice is publicly observable. The investor chooses the contract (α, h) that satisfies the first order optimality condition:

$$
\begin{aligned}
\frac{\partial}{\partial i}EU_b(\varphi_b(e)|\alpha, h) &= -\exp\big(-(1/2)(\mu/\sigma)^2 + (b/a)V(D, e)\big)\times \\
&\left(\frac{b}{a}g(e, L_s, L_m)^{b/a-1}g_i(e, L_s, L_m)f(e, L_s, L_m) + g(e, L_s, L_m)^{b/a}f_i(e, L_s, L_m)\right) = 0,
\end{aligned}
$$

for $i = \{\alpha, h\}$. We distinguish two cases: with and without portfolio constraints.

Without portfolio constraints, $s \to \infty$ and $m \to \infty$. The manager's expected utility (4) is independent of α and h. Given (A7) and the partial derivatives for f, it follows immediately that, $\lim_{s,m\to\infty} f_i(e, L_s, L_m|\alpha_{FB}, 0) = 0$, for $i = \{\alpha, h\}$. Hence, the contract $(\alpha_{FB}, 0)$ is optimal.

With portfolio constraints, notice first that $g(e, L_s, L_m|\alpha_{FB}, 0) = f(e, L_s, L_m|\alpha_{FB}, 0)$. Evaluating the optimality condition at $(\alpha_{FB}, 0)$ and given the partial derivatives for f and g, it follows that the contract $(\alpha_{FB}, 0)$ satisfies the first order optimality condition in the absence of moral hazard.

We turn now to the case of moral hazard. Without portfolio constraints (second best scenario), the manager's effort (5) is independent of α and h. Hence, the contract $(\alpha_{FB}, 0)$ is optimal. Under portfolio constraints (third best scenario), the third best effort, e_{TB}, is a function of α and h. The first order condition for optimality requires that

$$
\frac{\partial}{\partial i}EU_b(\varphi_b(e_{TB})|\alpha, h) = \frac{\partial}{\partial i}EU_b(\varphi_b(e)|\alpha, h) + \frac{\partial}{\partial e}EU_b(\varphi_b(e)|\alpha, h)\frac{\partial}{\partial i}e_{TB}(\alpha,h) = 0,
$$

for $i = \{\alpha, h\}$. Given the partial derivatives above, $\frac{\partial}{\partial i}EU_b(\varphi_b(e)|\alpha_{FB}, 0) = 0$. By definition, $\frac{\partial}{\partial e}\ EU_b(\varphi_b(e)|\alpha_{FB}, 0)|_{e=e_{TB}} =$

$$
-\exp\big(-(1/2)(\mu/\sigma)^2 + (b/a)V(D, e)\big)g(e_{TB}, L_s(0), L_m(0))^{b/a}g_e(e_{TB}, L_s(0), L_m(0)) > 0
$$

and $\frac{\partial}{\partial\alpha}e_{TB}(\alpha, 0) = -\mathcal{J}_\alpha(e_{TB}, L_s(0), L_m(0)) \times \mathcal{J}_e^{-1}(e_{TB}, L_s(0), L_m(0)) > 0$, for all $\alpha \in (0, 1]$.

From Proposition 2 and Corollary 2, for all $\alpha \in (0,1]$, $\frac{\partial}{\partial h} e_{TB}(\alpha, h) > 0$ (< 0) for $h < h^*$ $(h > h^*)$; $\frac{\partial}{\partial h} e_{TB}(\alpha, h) = 0$ for $h = h^*$. Hence, in general, the contract $(\alpha_{FB}, 0)$ is suboptimal. *Q.E.D.*

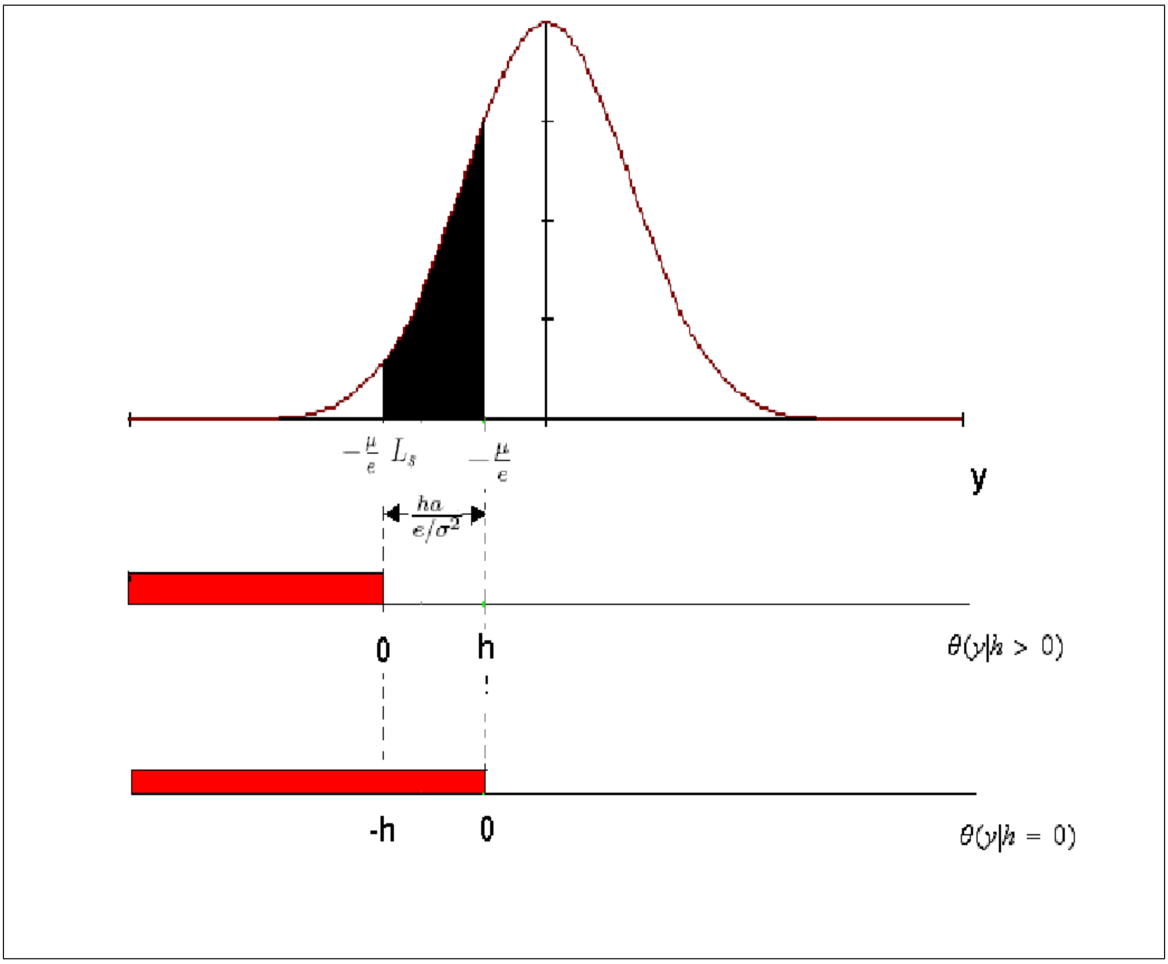

Figure 1: We assume that short-selling is totally forbidden ($s = 0$) and there is no limit to margin purchase ($m \to \infty$). For simplicity, let $\alpha = 1$. After putting effort e the manager receives a signal y and makes her optimal portfolio θ. When $h = 0$ (bottom portfolio line), all signals $y < -\frac{\mu}{e}$ lead to short-selling. When $h > 0$ (upper portfolio line), the short-selling bound is hit for signals $y < -\frac{\mu}{e}L_s$. In both cases, the region of these non-implementable portfolios is marked by the thick line. Under benchmarking ($h > 0$) there is an *incremental* area for implementable signals relative to the case of no benchmarking. The size of this area, $\frac{ha}{e/\sigma^2}$, increases with benchmarking (h) and the manager's risk aversion (a); it has probability mass equal to the shaded area in the density function plot.

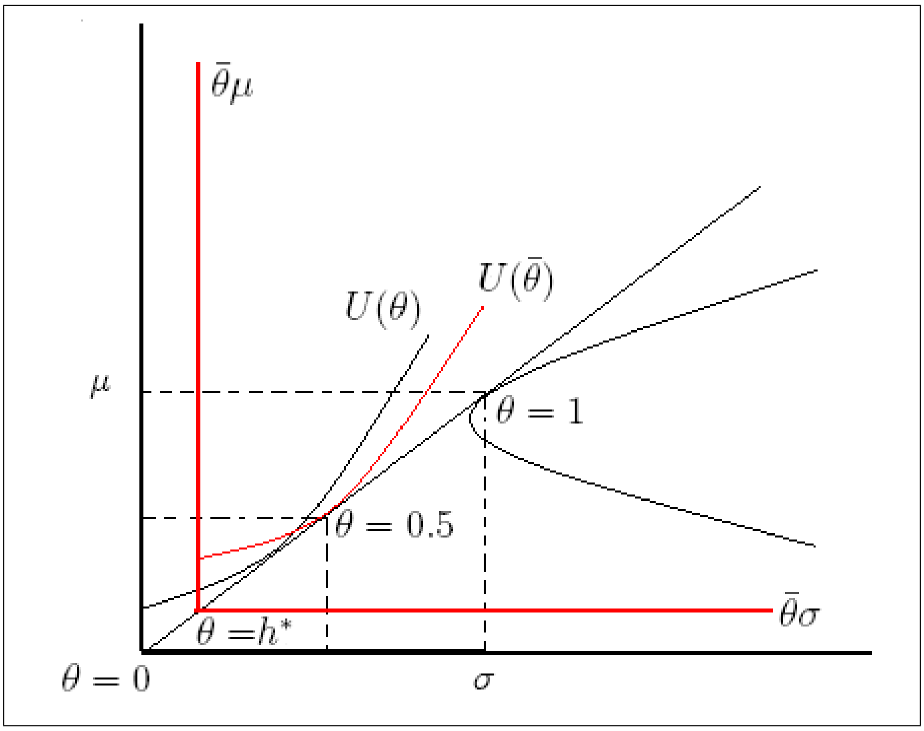

Figure 2: The graph represents the unconditional portfolio (independent of the signal y). Assume the manager is constrained. For instance, $0 \leq \theta \leq 1$ (zero leverage). If the benchmark coincides with the risk free asset, the manager will chose the tangent portfolio on the "absolute" capital market line that maximizes her expected utility. In the example, this portfolio holds less than 50% in the market. If the manager is given a benchmark $h^* > 0$ then she will choose a tangent portfolio $\bar{\theta} = \theta - h$ in the "relative" capital market line that trades off excess expected return $\bar{\theta}\mu$ against tracking error standard deviation. Notice that given the portfolio constraints, for $h = h^*$ the manager's optimal unconditional portfolio is equidistant from either boundary. The manager's effort choice maximizes her unconditional expected utility before receiving the signal. The benchmark composition h^* allows, ex-ante, more extreme signals to be implemented, increasing effort's marginal utility and, ultimately, the manager's effort choice.

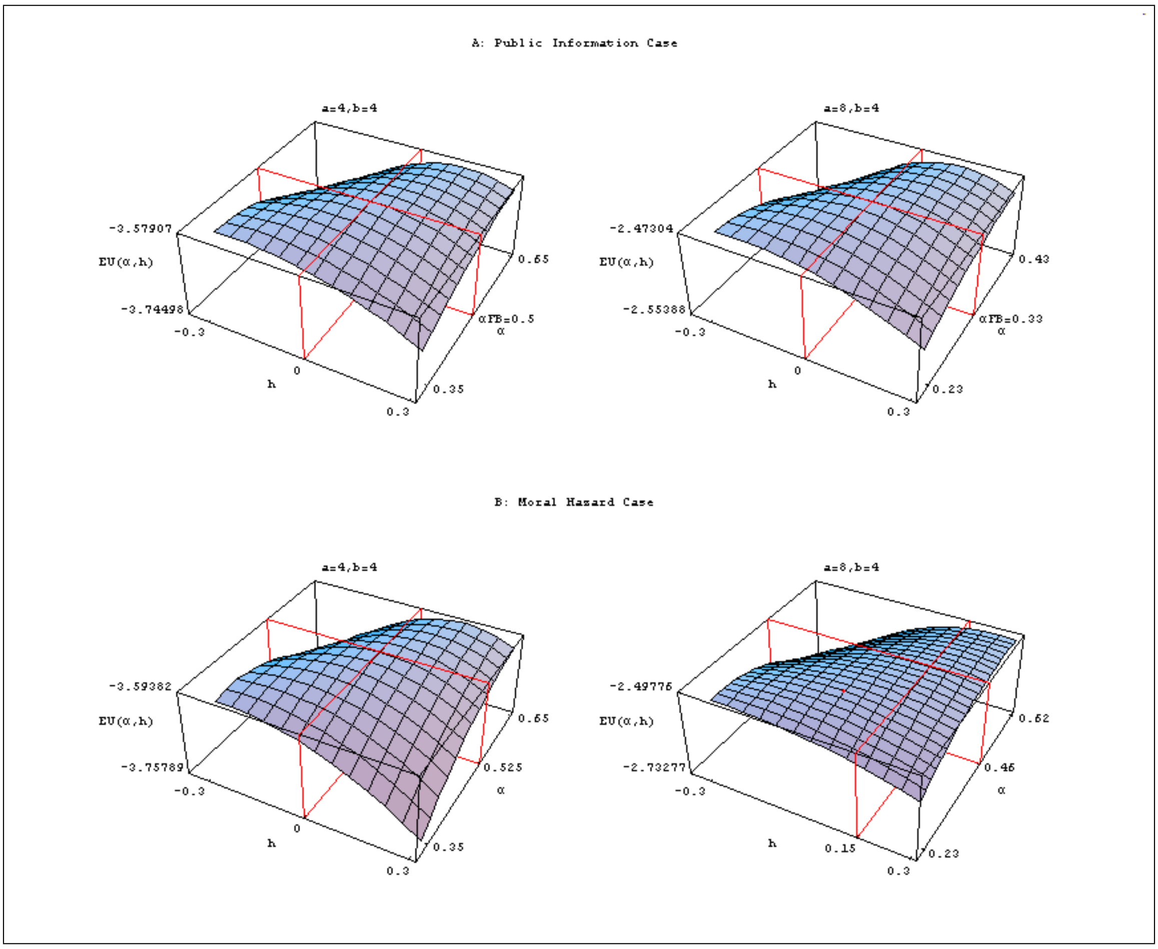

Figure 3: The manager is totally constrained in her portfolio choice: $m = 1$ and $s = 0$. The vertical axis in each figure represents the investor's expected utility when the manager's effort choice is observable (Panel A) and under moral hazard (Panel B). The maximum and minimum expected utility within the values of the contract represented are reported. The horizontal axes represent the incentive fee, α, and the percentage in the benchmark portfolio invested in the market, h, respectively. The three-dimensional cross identifies the optimal contract. b (a) denotes the investor's (manager's) risk aversion coefficient.

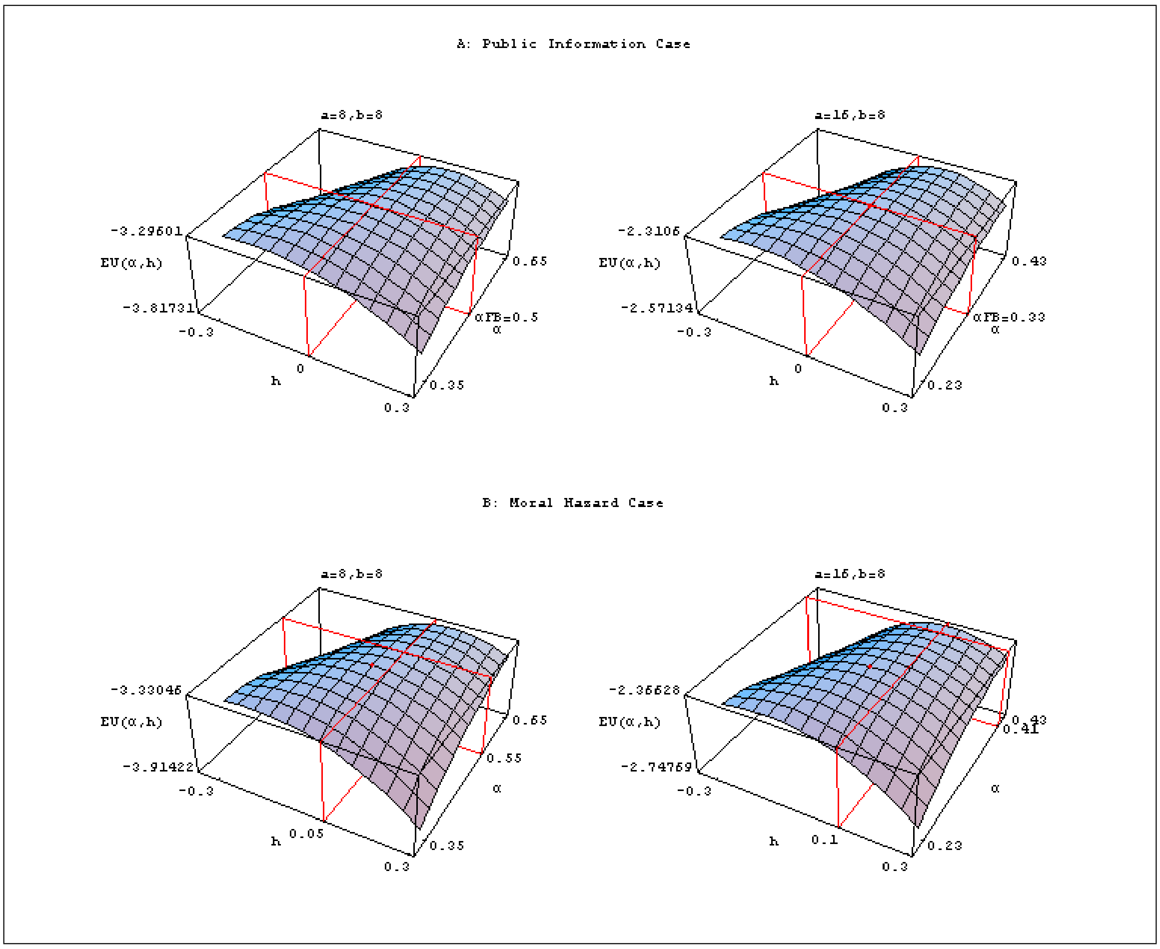

Figure 4: The manager is totally constrained in her portfolio choice: $m = 1$ and $s = 0$. The vertical axis in each figure represents the investor's expected utility when the manager's effort choice is observable (Panel A) and under moral hazard (Panel B). The maximum and minimum expected utility within the values of the contract represented are reported. The horizontal axes represent the incentive fee, α, and the percentage in the benchmark portfolio invested in the market, h, respectively. The three-dimensional cross identifies the optimal contract. b (a) denotes the investor's (manager's) risk aversion coefficient.

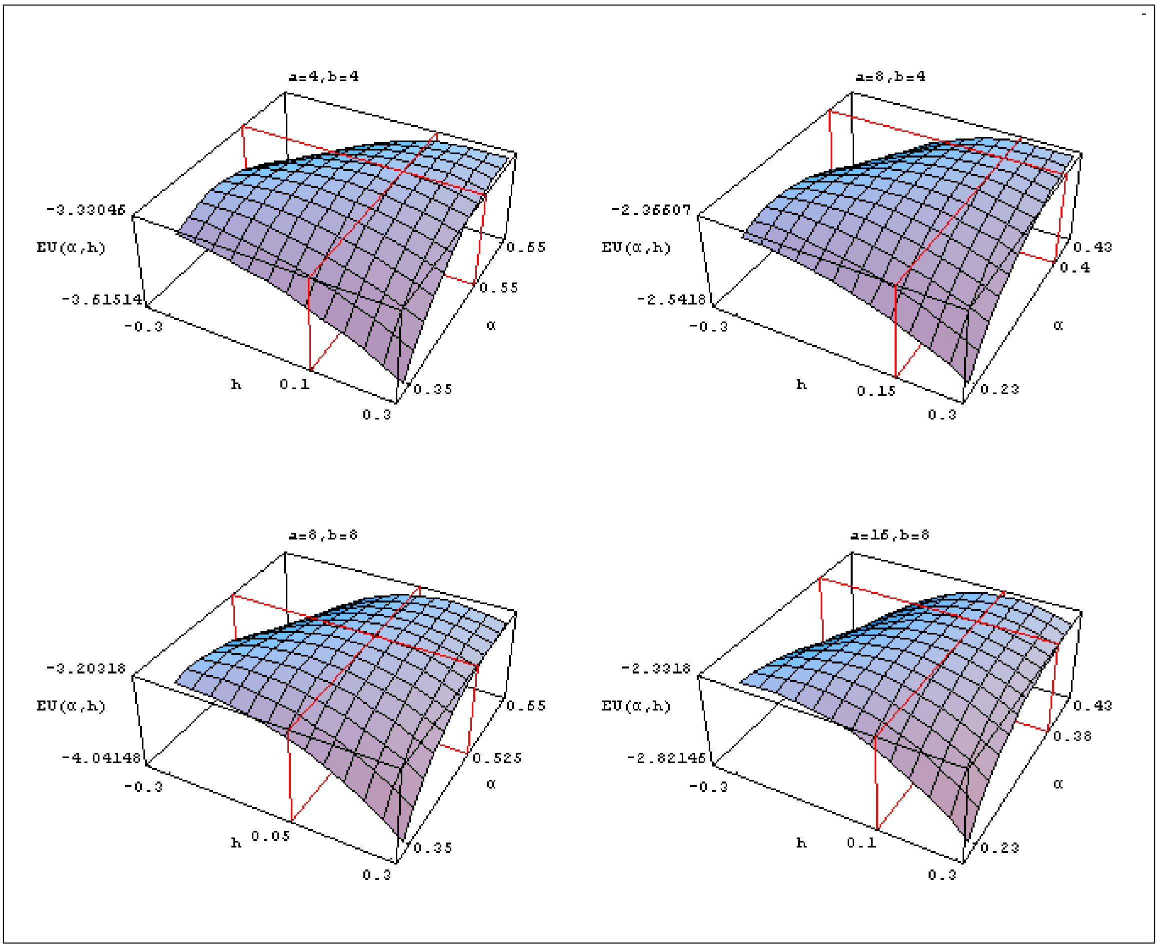

Figure 5: The manager's portfolio constraints are $m = 2$ and $s = 0$. The vertical axis in each figure represents the investor's expected utility under moral hazard. The maximum and minimum expected utility within the values of the contract represented are reported. The horizontal axes represent the incentive fee, α, and the percentage in the benchmark portfolio invested in the market, h, respectively. The three-dimensional cross identifies the optimal contract. b (a) denotes the investor's (manager's) risk aversion coefficient.

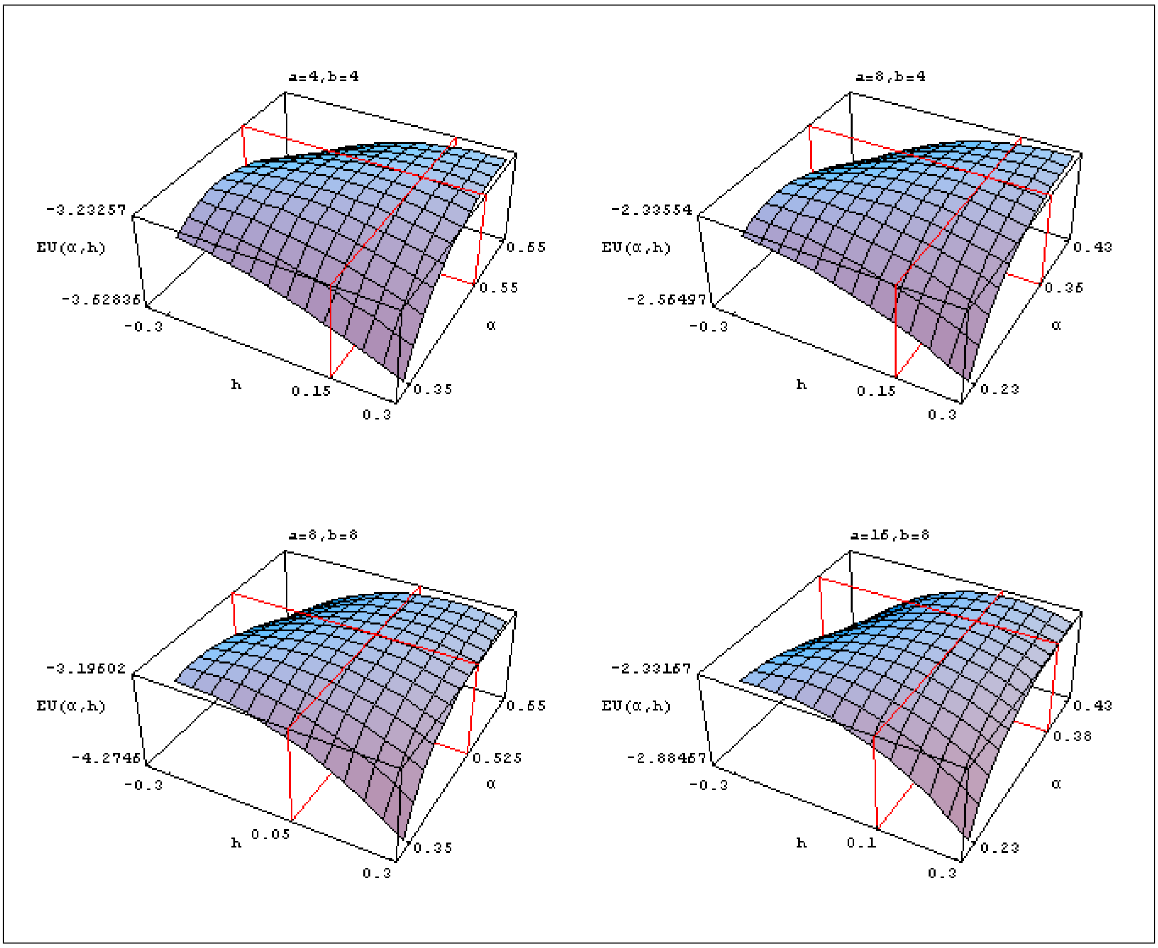

Figure 6: The manager's portfolio constraints are $m = 3$ and $s = 0$. The vertical axis in each figure represents the investor's expected utility under moral hazard. The maximum and minimum expected utility within the values of the contract represented are reported. The horizontal axes represent the incentive fee, α, and the percentage in the benchmark portfolio invested in the market, h, respectively. The three-dimensional cross identifies the optimal contract. b (a) denotes the investor's (manager's) risk aversion coefficient.

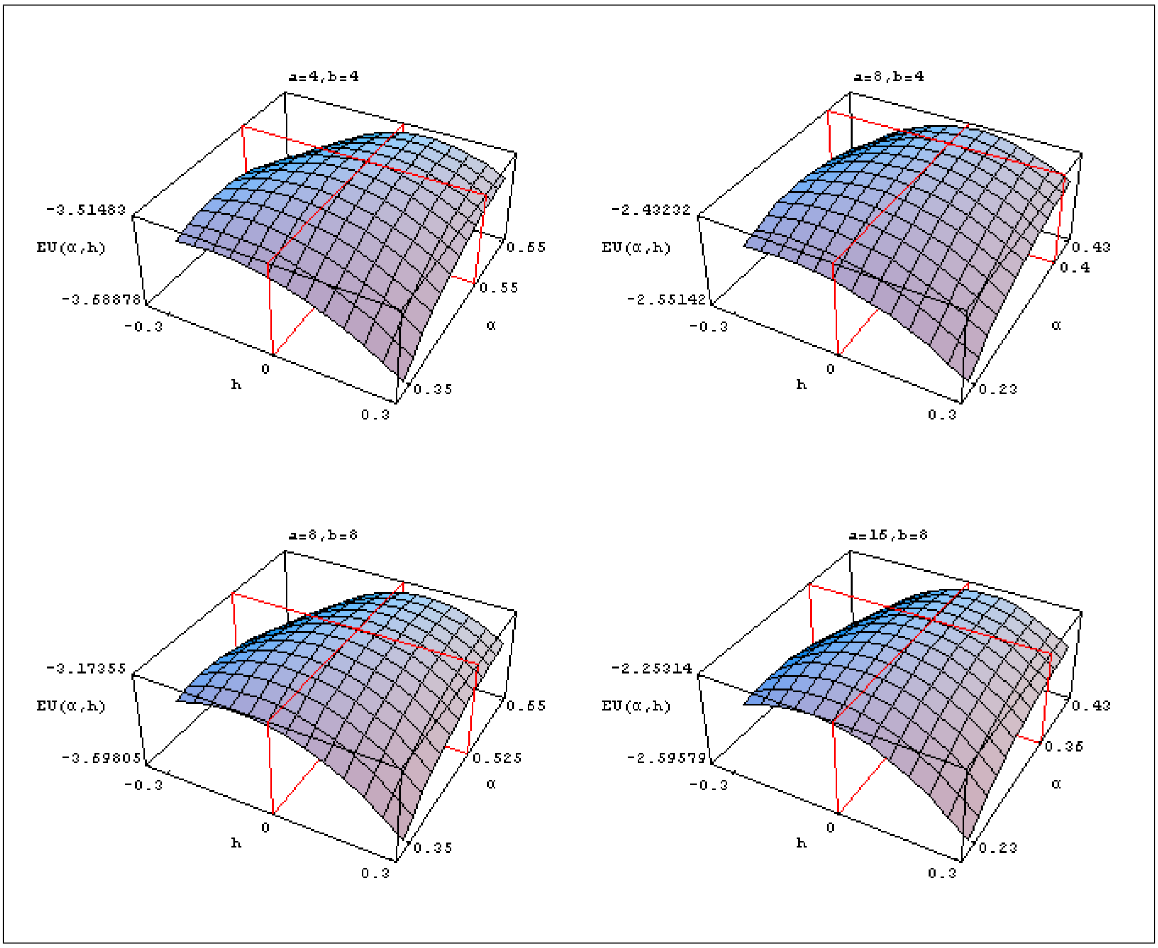

Figure 7: The manager's portfolio constraints are $m = 1$ and $s = 1$. The vertical axis in each figure represents the investor's expected utility under moral hazard. The maximum and minimum expected utility within the values of the contract represented are reported. The horizontal axes represent the incentive fee, α, and the percentage in the benchmark portfolio invested in the market, h, respectively. The three-dimensional cross identifies the optimal contract. b (a) denotes the investor's (manager's) risk aversion coefficient.

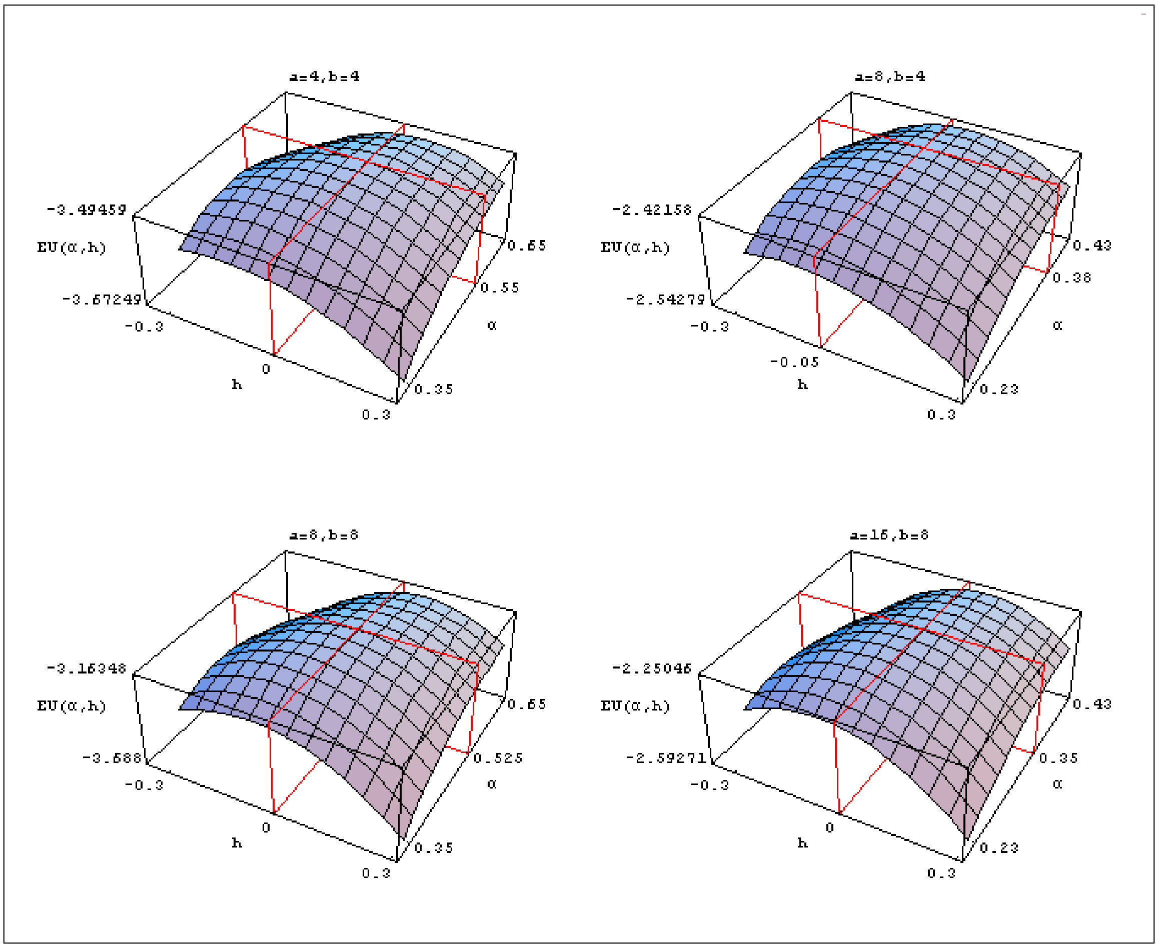

Figure 8: The manager's portfolio constraints are $m = 1$ and $s = 2$. The vertical axis in each figure represents the investor's expected utility under moral hazard. The maximum and minimum expected utility within the values of the contract represented are reported. The horizontal axes represent the incentive fee, α, and the percentage in the benchmark portfolio invested in the market, h, respectively. The three-dimensional cross identifies the optimal contract. b (a) denotes the investor's (manager's) risk aversion coefficient.

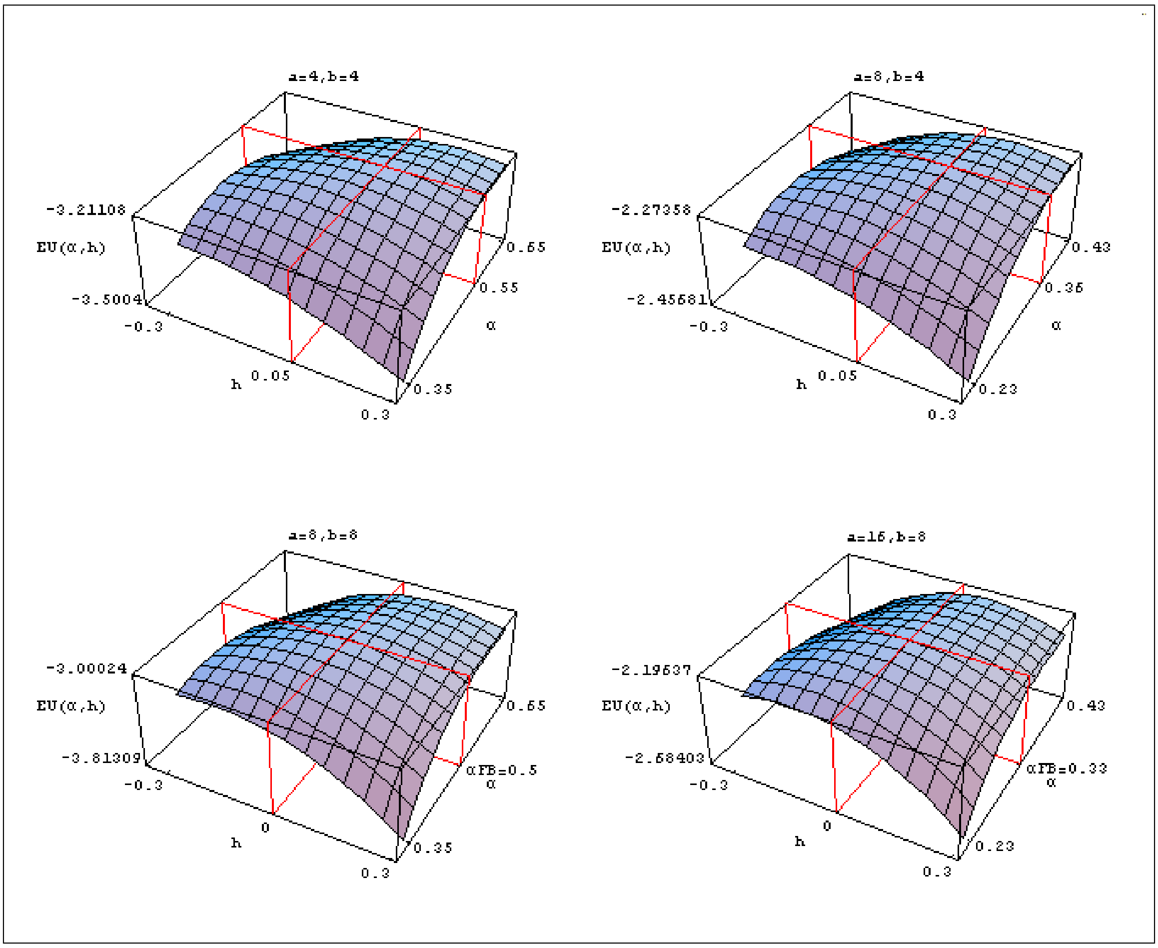

Figure 9: The manager's portfolio constraints are $m = 2$ and $s = 1$. The vertical axis in each figure represents the investor's expected utility under moral hazard. The maximum and minimum expected utility within the values of the contract represented are reported. The horizontal axes represent the incentive fee, α, and the percentage in the benchmark portfolio invested in the market, h, respectively. The three-dimensional cross identifies the optimal contract. b (a) denotes the investor's (manager's) risk aversion coefficient.

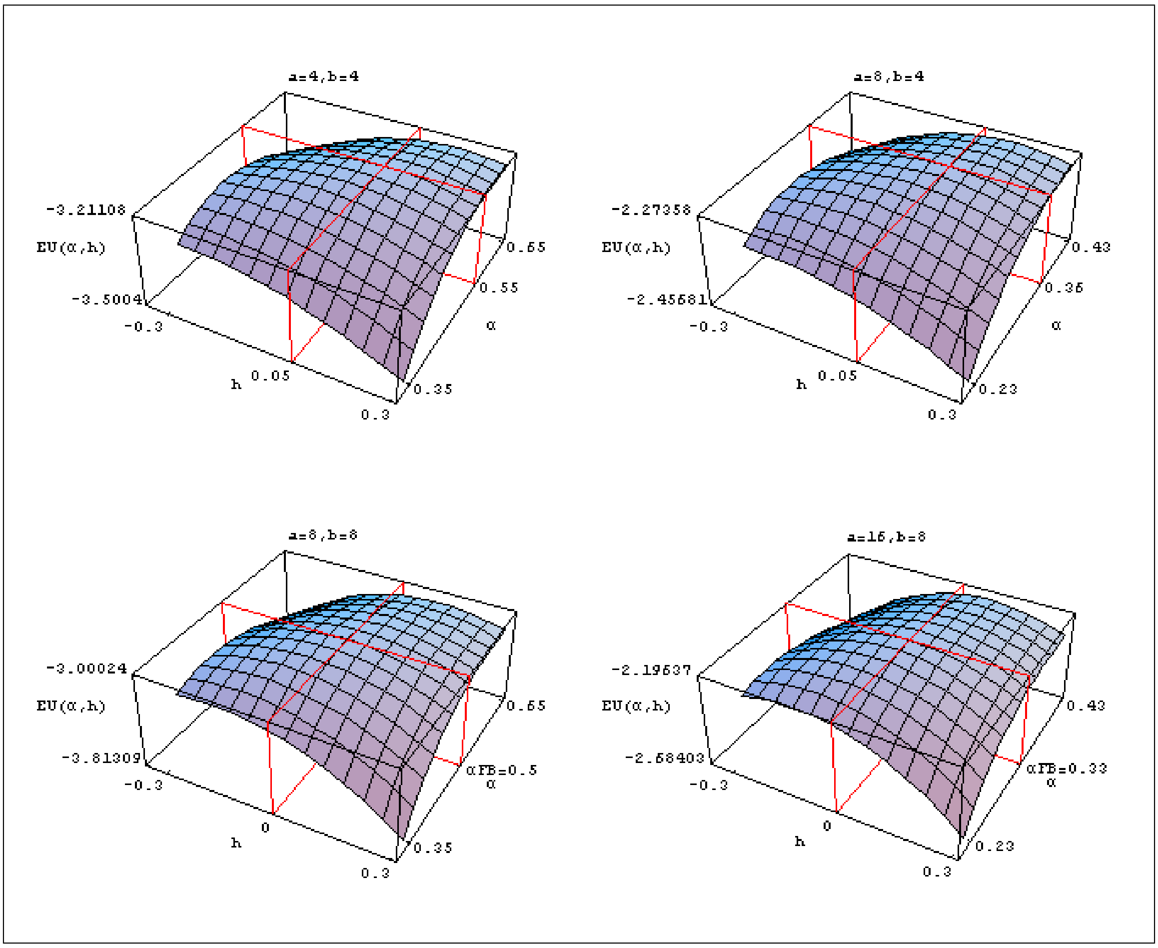

Figure 10: The manager's portfolio constraints are $m = 3$ and $s = 2$. The vertical axis in each figure represents the investor's expected utility under moral hazard. The maximum and minimum expected utility within the values of the contract represented are reported. The horizontal axes represent the incentive fee, α, and the percentage in the benchmark portfolio invested in the market, h, respectively. The three-dimensional cross identifies the optimal contract. b (a) denotes the investor's (manager's) risk aversion coefficient.