A Market-Based Real-Asset Martingale Valuation Approach to Optimum Inventory Control

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ABSTRACT

We propose a novel approach to optimum inventory control by modeling a commodity trader’s inventory investment as a portfolio of forward commitments taking explicit account of the dynamics of demands, costs and prices in open markets. We apply the robust real-asset martingale valuation methodology to derive a closed-form solution for the inventory value and a simple and intuitive optimality condition. Numerical analysis verifies this condition and demonstrates that the resulting optimum policy has robust properties.
1. Introduction

In traditional inventory theory, the optimum inventory is determined by finding the optimal reorder point and order quantity with the assumptions that costs and prices are constant, and demand is the only source of uncertainty. While these assumptions are reasonable for common applications such as the inventory management of manufactured goods, service parts and many retail products, they are much less likely to be valid for the management of inventories of products that are sold in open markets, like agricultural products, minerals, and other commodities. In open markets, prices fluctuate depending on the instantaneous balance of supply and demand. So effective inventory management requires understanding how price and demand interrelate through trading in markets. This is a particular challenge if there are many suppliers to the market, and the relative size of these suppliers means that it is not possible for a very small number of large suppliers to dominate the market and thus determine prices. A further complication is that associated with many commodity markets are futures markets where futures contracts on commodities on traded, so some market participants have the choice of participating in either the spot or futures markets.

This paper examines inventory or purchase decisions that have to be made by a participant in a commodity market. The participant could be a commodity trader deciding what contracts to enter into with wheat farmers at planting time (although our model does not allow for harvest failure). Our approach would also have been relevant to a late 19th century wool broker who bought wool in Australia and then would sell it to mills in the UK and Europe after a ship voyage lasting at least 3 months. Note that there are two different but connected markets: the futures market where the product is bought and the spot market where the product is sold. Product sold in the spot market is available for use immediately, while product bought in the futures market is not available for use until after some time delay. When our commodity broker buys product in the futures market he does not know the demand that he will have to meet in the spot market. During the time delay until the product is available for use he will accumulate orders that he will meet once the product is available.
A classic inventory model that appears to be relevant to our commodity broker’s decision is the newsvendor model. The decision maker faces only uncertain demand and has to decide how much inventory to acquire. Too much and surplus inventory will have to be sold at a greatly reduced price; too little and the shortage will have to be met by acquiring additional product at a high price. Since these prices are assumed to be exogenously given by his supplier with certainty, he can make the decision in isolation from open markets. Our commodity supplier however has to deal with the additional price uncertainty that he does not know the price at which he will be able to sell the product - price will fluctuate in open markets over the period until his product is available for delivery. As such if the supplier does not have enough inventories to meet demand, he does not know how much more he will have to pay for additional product required to meet his customers’ demands, while if he has too much inventory, he does not know how much concession he will have to make to clear the surplus. Another complication arises from the need for our trader to finance his inventory investment. Interest rates can fluctuate over the period between making the inventory investment and realizing the cash receipts on sale of the inventory. These fluctuations in interest rates may be linked to macro economic conditions.

There have been a number of attempts to expand the newsvendor framework by incorporating certain non-stationary behaviors of the demand as well as by determining risk premiums in ways consistent with firm’s objective of shareholder wealth maximization. Kim and Chung (1989), Morries and Chang (1991), and Singhal et al. (1994) apply the finance theory of CAPM to factor risk premium into valuation. Stowe and Su (1997) apply the contingent-claim approach to the inventory-stocking decision. Khang and Fujiwara (2000) derive myopic optimum inventory policies under stochastic supply. Ritchken and Tapiero (1986) consider a periodic model with both stochastic demand and stochastic purchase price and address the issue of hedging inventory in stock with futures contracts. Berling and Rosling (2005) look at the problem in a diffusion framework with stochastic demand and purchase costs and address the issue of financial risks. Cheung (1998) develops a continuous review inventory model with a time discount to motivate customers to accept delayed deliveries and thus avoid the occurrence of

In this research, we develop a market-based approach to value the inventory investment at the instant when the commodity broker has to make a decision on what to order either from his supplier or in the futures market. The optimal decision will be that maximizing the value of the inventory in open markets taking into consideration of market risk-return equilibrium. To meet the task, we have to develop a model of the economy in order to determine the relationship between risk and return, and we also have to develop a model of the relationship between the futures market and the spot market for the commodity since they are related through no-arbitrage. Compared to prior newsvendor models, our model is thus unique in three ways: 1) it employs a robust framework to simultaneously address the many limitations of the prior models in terms of the non-stationary behaviors of the input and output variables, their connections, as well as means to determine risk premiums, 2) it is novel in viewing the inventory investment as a portfolio of forward commitments (or futures contracts), and 3) it applies the robust real asset martingale valuation paradigm to incorporate the market risk premiums and to drive the optimum results in a compact way.

Our paper is structured as follows: in Section 2, we formally describe our decision problem and then we briefly introduce the traditional newsvendor-type of valuation model with constant prices and costs but uncertain and unsystematic one-period demand, and discuss its shortcomings. To address these shortcomings and show why our market-based modeling approach is more general and powerful, we develop our model in Section 3. In Section 4, we use simulation to demonstrate our model’s robustness in addressing the inventory basis, the discount, the penalty, the transaction cost, the interest rate and the clustering effects. In Section 5, we offer concluding remarks and future research directions.
2. The Basic Model and the Newsvendor Solution

A commodity trading firm orders a quantity \( y \) of a single product at time \( t \) in the whole-sale (or futures) market at a market ordering cost of \((1-u)F_t\) payable at the time of delivery, where \( F_t \) and \( u \) denotes respectively the equilibrium ordering cost (or futures price) and the percent discount offered.

Over the interval \((t,t+L)\) the trader will receive a random number of orders \( D \) for the commodity with commitments to deliver an agreed quantity of the product to customers at the prevailing retail (or spot) market price of \( S_{t+L} \) at the time of delivery \( t+L \), where \( S_{t+L} \) is random with evolution determined in open markets through supply and demand.

Our trader is also assumed to be a typical small player, i.e. a price taker, but not a large supplier, i.e. a price setter. He buys from his supplier in the whole-sale (or futures) market and sells in the spot market to his retail customers with any imbalance disposed of only through his supplier. If \( D<y \), the trader will have surplus inventory, while if \( D>y \) the trader will have an inventory shortfall. He can not dispose of his imbalances at the retail price in the spot market otherwise he becomes a larger player whose action affects the retail price. We assume that our trader has made prior arrangements with his supplier to deal with surplus or shortfalls in inventory at a price that is related to the prevailing market wholesale price, or futures price \( F_{t+L} \) per unit, at time \( t+L \). If there is excess inventory at time \( t+L \), we assume he has entered into a short forward commitment with the supplier to instantly return the commodity at a net salvage value equaling to \( F_{t+L} \) minus a penalty of \( q \) percent. If there is insufficient inventory to meet the demand, we assume that he has entered into a long forward commitment with the supplier to place an emergency order for the shortfall at a price of \( F_{t+L} \) plus a penalty of \( p \) percent. Obviously, the presence of these costs to circumvent over-supply and -demand affect the trader’s optimum inventory decision. We also define the difference, \( S_t - F_t \), as the inventory or forward basis and the difference, \( S_t - (1-u)F_t \), as the discounted inventory or forward basis, where \( S_t \) is the initial spot price of the commodity in the retail market. Obviously, the dynamic behavior of this inventory basis (basis risk) as well as the size of \( u \) is also important considerations in the trader’s optimum decision. For example,
widening of the inventory basis coupled with a deeper initial purchase discount will mean more profit and therefore the trader would order a larger quantity of inventory and vice-versa. The trader's decision is to determine the optimum inventory by maximizing the net present value of this portfolio of three forward commitments – a long forward commitment to purchase the commodity at a preset price, a short forward agreement to return excess inventory at a net salvage value, and a long forward agreement to meet excess demand by placing an emergency order with a penalty.

Next we consider the classic newsvendor solution. If demand $D$ is the only source of uncertainty and is unsystematic with the probability of demand $D$ over a single period, $[t, t+L]$, to be $P(D)$ with expected demand $E(D)$, then the risk-adjusted discount rate is simply the market risk-free rate without the need to incorporate a market risk premium. The inventory decision at time $t$ to decide on the ordering decision with given purchase price $(1-u)F_t$ and selling price $S_{t+L}$ can thus be made in isolation from the rest of the economy. If $y$ is ordered then the quantity $(y-\min\{D,y\})$ is surplus and we assume it can be disposed of at a discounted price of $(1-q)F_{t+L}$. However, if demand is greater than $y$, the quantity $(D-\min\{D,y\})$ must be acquired at a penalty price of $(1+p)F_{t+L}$.

Then given $F_{t+L} = F_t / B_t$ and $S_{t+L} = S_t / B_t$ where $B_t$ denotes the current market price of a zero-coupon unit riskless bond, the net present value of the inventory position if $y$ is ordered is

$$NPV(y) = E(D)S_t - y(1-u)B_tF_t - \sum_{D=y+1}^{\infty} p(D)(D-y)(1+p)F_t + \sum_{D=0}^{y-1} p(D)(y-D)(1-q)F_t.$$  

Note in the classical newsvendor model, the purchase price is independent of $L$ but here we have $F_{t+L} = F_t / B_t$, i.e. the purchase price $F_t$ decreases as $L$ increases. The optimal order quantity obtained by maximizing the $NPV$ is then given by the $y$ satisfying the condition

$$\sum_{D=0}^{y-1} p(D) \leq \frac{1 + p - (1-u)B_t}{p + q} \leq \sum_{D=0}^{y} p(D).$$

A closer look at this formula however reveals that it can not really be implemented. Although the trader knows his inventory costs, i.e. $(1-u)F_t$, $(1-q)F_{t+L}$ and $(1+p)F_{t+L}$ through his supplier, he can not
really determine \( u, q, \) and \( p, \) the sizes of the discounts or penalty without knowing \( F, \) the equilibrium inventory ordering cost, which can only be determined in the economy. To circumvent this problem, we need to generalize the newsvendor framework to allow for interactions among uncertain demand, prices and costs in open markets in ways consistent with the empirical literature. We also need to develop a proper asset pricing relation in order to incorporate market risk premiums in the valuation.

To this end, next we employ the robust real asset martingale valuation methodology of Schwartz (1997) and Gibson and Schwartz (1990) in a market environment consistent with the empirical dynamics of commodity prices and interest rate in the finance literature (e.g. Schwartz (1997)) - a multi-variate Ornstein-Uhlenbeck process with stochastic interest rates and volatility, augmented with a doubly stochastic jump process. We first derive a four-factor asset pricing relation in a partial equilibrium setting by exogenously specifying a pricing kernel process, with which to risk-neutralize the underlying processes. Next we determine the equilibrium inventory ordering cost in this setting as a forward price with respect to the retail sale (output) price, taking into consideration of the finance charge and the net convenience yield (defined as benefits arising through holding inventory minus the warehouse costs) incurred over the lead time. Finally, we determine the net present value of the portfolio of forward commitments under no-arbitrage as a martingale and the corresponding optimum inventory policy by maximizing this value.

3. The Real Asset Martingale Valuation Approach

3.1 The Pricing Kernel Process and the Consumption-Based Jump-Diffusion Asset Pricing Relation

We extend Schwartz (1997) to allow for jumps and stochastic arrival intensity by assuming that the following jump-diffusion processes regarding the changes of aggregate consumption and commodity (asset) price are exogenously given:

\[
(1) \quad \frac{dc_t}{c_t} = \mu_c(\sigma, t)dt + \sigma_c(\sigma, t)dZ_c + (A-1)d\pi, \\
(2) \quad \frac{dS_t}{S_t} = \mu_s(\sigma, t)dt + \sigma_s(\sigma, t)dZ_s + (X-1)d\pi, 
\]
where $\mu_c$ and $\sigma_c$, and $\mu_s$ and $\sigma_s$ denote the respective instantaneous expected value and standard-deviation conditional on no jumps occurring for consumption change and asset return, respectively, and are functions of state variables $\sigma \in \mathcal{X}$, where $\mathcal{X}$ denotes the information set at time $t$; $Z_c$ and $Z_s$ denote the standard Wiener processes with $\sigma_{sc}=\sigma_s \sigma_c E[ \frac{dz_s}{dt} \frac{dz_c}{dt}] / dt$ denoting the instantaneous diffusion covariance between asset return and consumption change; $\pi$ is the Poisson process with stochastic intensity parameter $j$, representing the arrival process of consumption jump as well as asset demand; $A$ and $X$ denote the gross jump sizes of consumption and asset price triggered by this arrival with $\ln A \sim N(\mu_a(\omega,t),\sigma_a(\omega,t))$ and $\ln X \sim N(\mu_x(\omega,t),\sigma_x(\omega,t))$, respectively; and finally, $\pi$, $A$, and $X$ are assumed to be independent to $Z_c$ and $Z_s$. For ease of exposition, we will denote $c_t$ as $c$ and $S_t$ as $S$ in the remainder of the paper. We synchronize the jump arrivals of aggregate consumption and asset demand for notational convenience, otherwise the generalization is straightforward. For this reason, we will use the two terms, jump and demand, interchangeably in the remainder of the paper.

We assume that the following mean-reverting Ornstein-Uhlenbeck process governs the change of the jump arrival intensity:

\begin{equation}
\begin{aligned}
dj &= \kappa_j (m_j - j)dt + \sigma_j dZ_j,
\end{aligned}
\end{equation}

where $j$ denotes the jump arrival intensity, $\kappa_j$ the speed of adjustment, $m_j$ the long-run mean rate, $\sigma_j^2$ the instantaneous variance, and $Z_j$ the standard Wiener process.

The solution of this process is known to be

\begin{equation}
\begin{aligned}
j(\nu) &= m_j + (j(t) - m_j) e^{-\kappa_j (\nu - t)} + \sigma_j \int_t^\nu e^{\kappa_j (s - t)} dZ_j(s), \quad \nu \in [t, T].
\end{aligned}
\end{equation}

Integrating the expected value of (4) over $\tau$ yields the expected intensity over $\tau$:

\begin{equation}
\begin{aligned}
E[ \int_t^{t+\tau} j(\nu) d\nu ] = m_j \tau + [ j(t) - m_j ] H_j(\tau),
\end{aligned}
\end{equation}
where $H_j(\tau) = \frac{1 - e^{-\kappa_j \tau}}{\kappa_j}$.

This stochastic intensity specification addresses the clustering effect that around the time of important news announcements, the arrival intensity will be high to reflect the lumpiness of the jump (demand) arrival, but after the release of the news, it will revert back to a lower long run mean level $m_j$, and vice-versa. While $j$ itself governs the intensity of arrival, the speed of adjustment parameter $\kappa_j$ governs the level of persistence of the intensity process – higher values for $\kappa_j$ imply that the intensity process leaves the high state more quickly, and vice-versa. For hedging assets (commodity assets) we expect positive correlation between demand intensity and commodity price changes, but for cyclical assets (financial assets) that are traded in financial markets we expect the correlation to be negative to reflect the stylized leverage effect that when volatility goes up asset price goes down.

Given the aggregate consumption price dynamics and that the representative agent in the economy exhibits a preference of the CRRA type, it is well-known that, e.g. Duffie (2001), the pricing kernel is the agent’s intertemporal marginal rate of substitution, i.e.

$$q_{0,t} = e^{-\theta t} \frac{u'(c_t)}{u'(c_0)} = e^{-\theta t} \frac{(c_t - c_0)^{\varepsilon - 1}}{c_0^{\varepsilon}}$$

where $q_{0,t}$ is the pricing kernel, defined as the state price per unit of probability or the Radon-Nikodym derivative, $c_t$ is consumption at date $t$, $\theta$ is the time-discount factor, $u'(c_t)$ and $u'(c_0)$ denote the marginal utility with respect to the consumption level at time $t$ and zero, respectively, and $1 - \varepsilon$ is the coefficient of relative risk aversion with $\varepsilon < 1$. Furthermore, the pricing kernel process is

$$dq/q = \mu_q dt + \sigma_q dZ_c + (Y - 1) d\pi,$$

where

$$\mu_q = -(1 - \varepsilon) \mu_c - jE(A - 1)) + 1/2 \sigma_c^2 [(1 - \varepsilon)^2 + (1 - \varepsilon)] - \theta,$$

$$\sigma_q = -(1 - \varepsilon) \sigma_c,$$

$$Y = A^{\varepsilon - 1}.$$
Given the specifications of the asset price, the pricing kernel, and the intensity shift processes, it is straightforward to derive the following consumption-based jump-diffusion asset pricing relation:

**THEOREM 1**: Given the pricing kernel and the asset price dynamics, the fundamental Euler valuation equation dictates the following four-factor consumption-based jump-diffusion asset pricing relation:

\[
\begin{align*}
(8) \quad & \mu_S + \left[ j + \kappa_j(m_j - j) \right] E(X - 1) = r - (1 - \varepsilon) \sigma_{rc} + (1 - \varepsilon) \sigma_{sc} + \left[ j + \kappa_j(m_j - j) \right] E(X - 1) E(A^{\varepsilon - 1}) \\
& + \left[ j + \kappa_j(m_j - j) \right] \text{cov}(X, A^{\varepsilon - 1}),
\end{align*}
\]

where \( \mu_S \) and \( j + \kappa_j(m_j - j) \) denotes the instantaneous expected diffusion return and jump arrival intensity, respectively; \( E \) is the expected value operator and \( X \) the gross spot jump size; \( r \) is the instantaneous riskless rate, \((1 - \varepsilon)\) the CRRA measure, \( \sigma_{rc} = \sigma_c E[\frac{dz_r}{\sigma_c} | \mathcal{F}_t] / dt \) the instantaneous diffusion covariance between interest rate and consumption change, \( \sigma_{sc} = \sigma_s E[\frac{dz_s}{\sigma_s} | \mathcal{F}_t] / dt \) the instantaneous diffusion covariance between asset return and the consumption change, and \( A \) is the gross consumption jump size.

*Proof of Theorem 1*: See Appendix.

Equation (8) stipulates that the asset risk premium has four components: two diffusion risk premiums,

\[-(1 - \varepsilon) \sigma_{rc} \quad \text{and} \quad (1 - \varepsilon) \sigma_{sc},\]

and two jump risk premiums, \( [ j + \kappa_j(m_j - j)] E(X - 1) E(A^{\varepsilon - 1}) \), which compensates for the jump arrival uncertainty, and \( [ j + \kappa_j(m_j - j)] \text{cov}(X, A^{\varepsilon - 1}) \), which compensates for the jump size uncertainty. Since \( \sigma_c < 0 \) because when interest rates go up consumptions go down and \( 1 - \varepsilon > 0 \) because of risk-aversion, the interest rate risk premium is negative. Since the leverage effect dictates that \( \text{cov}(X A^{\varepsilon - 1}, j dZ_j) < 0 \), all other risk premiums are positive for cyclical assets (financial assets) and negative for hedging assets (commodity assets). When \( j \) is deterministic, both the interest rate and the intensity risk premiums shrink to zero, and consequently Eq. (8) shrinks to Chang’s (1995) three-
factor jump-diffusion asset pricing relation. Furthermore, when \( j \) is zero with no jump arrival, Eq. (8) shrinks to the one-factor diffusion asset pricing relation of Breeden (1979) and Ross (1989). Finally, when the asset price jump size is deterministic, i.e. \( \text{cov}(X,A^{\epsilon-1}) = 0 \), the bearing of jump arrival uncertainty and jump intensity uncertainty must still be rewarded.

Next, given \( \mu_q = -((1-\epsilon)(\mu_c - jE(A^{-1}))+1/2\sigma_c^2 [(1-\epsilon)^2+(1-\epsilon)] - \theta \) and \( Y = A^{\epsilon-1} \), it is straightforward to show that applying the fundamental Euler valuation equation to a riskless asset yields

\[
(9) \quad r = -\mu_q + (1-\epsilon)\sigma_{rc} + (j + \kappa_j(m_j - j)E(A^{\epsilon-1})) = \varphi_1 + \varphi_2 j, \]

where

\[
(10) \quad \varphi_1 = (1-\epsilon)(\mu_c + \sigma_{rc}) - 1/2\sigma_c^2 [(1-\epsilon)^2+(1-\epsilon)] + \theta + \kappa_j m_j E(A^{\epsilon-1}), \quad \text{and} \]

\[
(11) \quad \varphi_2 = [(1-\kappa_j)E(A^{\epsilon-1}) - (1-\epsilon)E(A^{-1})].
\]

which indicates that, in our economy, uncertainty regarding jump (demand) arrival intensity induces stochastic interest rates. Since both \(-\mu_q\) and \(1-\epsilon\) are positive but \(\sigma_{rc}\) is negative, Eq. (9) reveals that the faster the diffusion mean of the pricing standard attenuates, the smaller the covariance between interest rate and consumption changes is, the faster demand arrives, and the larger the expected consumption jump size is, the higher is the interest rate. Furthermore, since \( r \) is linear in \( j \), an application of the generalized Ito’s lemma to Eq. (3) of \( r \) with respect to \( j \) and \( t \) leads to the following induced mean-reverting Ornstein-Uhlenbeck interest rate process:

\[
(12) \quad dr = \kappa_r(m_r - r)dt + \sigma_r dZ_j,
\]

where \( \kappa_r = \kappa_j \), \( m_r = \varphi_1 + \varphi_2 m_j \), \( \sigma_r = \sigma_j \phi_2 \), and \( \rho_{\kappa_r} = \rho_{\kappa_j} \).

Eq. (12) offers a trade-arrival interpretation of stochastic interest rates based upon the notion that demand or trade-arrival is what generates interest rates, with Eq. (9) offering the equilibrium supports to such a relation – aggregate consumption embeds a jump component, CRRA, and jump arrival intensity
follows a mean-reverting Ornstein-Uhlenbeck process. Replacing the interest rate risk premium with the intensity risk premium further condenses Eq. (8) to

\[
\mu_s + [j + \kappa_j(m_j - j)] E(X - 1) = r - (1 - \epsilon) \varphi_2 \sigma_{sc} + (1 - \epsilon) \sigma_{sc} + [j + \kappa_j(m_j - j)] E(X - 1) E(\epsilon - 1) \\
+ [j + \kappa_j(m_j - j)] \text{cov}(X, \epsilon - 1).
\]

Next, note that in our doubly-stochastic jump-diffusion specification, the total instantaneous variances of consumption change and asset return are \(\sigma_c^2 + [j + \kappa_j(m_j - j)]\sigma_a^2\) and \(\sigma_s^2 + [j + \kappa_j(m_j - j)]\sigma_x^2\), respectively. Since these variances are linear in \(j\), as in the case of \(r\), by Ito’s lemma they should also follow mean-reverting Ornstein-Uhlenbeck processes. Again this is because in our setting trade arrival is the underlying latent variable that generates volatility, i.e. a trade-arrival interpretation of stochastic volatility in the spirit of Andersen (1996). In sum, the above derivations show that our approach of modeling stochastic intensity in a doubly-stochastic jump-diffusion framework is an internally consistent and parsimonious way to simultaneously consider stochastic interest rates, stochastic volatility, and jumps with random heights and arrival intensity.

3.2 The Inventory Investment Problem

In our model of the small trader, the experienced demand will be a thinning of the process of jump arrivals in the model of the economy. The total number of jump arrivals during a period of length \(L\) is the total demand that must be met at the end of the period. At the beginning of the period, the trading firm must choose an optimum order quantity \(y\) with lead time of a single-period of \(L\) so as to bring the total initial position \((IP = rp + y, \text{where } rp \text{ denotes on-hand inventory})\) to an optimum level of \(IP\) for covering the forthcoming period’s stochastic demand. At the end of the period, the trading firm will liquidate excess inventory. Since on-hand inventory is a given constant, we can effectively assume it away without sacrificing the generality of the model. In this sense, \(y\) and \(IP\) are used interchangeably. We assume the inventory is replenished from an outside supplier with ample supply and the initial
inventory ordering cost quoted by this supplier is lower than its benchmark equilibrium value. An instant refill from the supplier to meet an excess demand incurs a penalty. An instant return to the supplier to liquidate an excess inventory incurs a discount to arrive at the net salvage value.

Effectively in this setting, the firm undertakes a long forward position by placing the inventory order at the beginning of the period to purchase the commodity from the supplier at a preset forward (purchase) price of \((1-u)F_t\) with a maturity of \(L\), where \(u\) denotes the percentage discount. In order to meet excess demand at the end of the period, the firm also enters into a long forward agreement to purchase the commodity from the supplier as an instant refill at a price of \((1+p)F_{t+L}\), where \(F_{t+L}\) is the prevailing equilibrium ordering cost and \(1+p\) is a preset penalty ratio. In order to liquidate excess inventory at the end of the period, the firm also enters into a short forward agreement to instantly return the surplus commodity back to the supplier at a price of \((1-q)F_{t+L}\) where \((1-q)\) is a preset cost ratio.

3.3 Determining the Equilibrium Purchase Price as a Forward Price in the Real Asset Martingale Framework

Since by placing an order with a preset purchase price and lead time a trader essentially enters in a long forward position, in the rest of the paper we will use the two terms, purchase price and forward price, and the two terms, sale price and spot price, interchangeably. By the same token, a backorder resembles a short forward position. Since the spot price as well as the financing charge change stochastically in our modeling, we should also expect the purchase price to be stochastic.

Past literature, e.g. Moinzadeh (1997), Golabi (1985) and Kalymon (1971) have exogenously specified the stochastic purchase prices according to certain processes. In this study we undertake a new and novel approach by endogenizing the stochastic purchase price based upon the seminal cost-of-carry forward pricing concept to link pricing in the input market to pricing in output market. Assuming that all participants in the market experience the same lead time \(L\), the seminal cost-of-carry concept in forward pricing dictates that, under no-arbitrage, the purchase price must equal the total cost of carrying the spot commodity over lead-time \(L\). These costs include the spot price and holding costs (financing and
warehouse costs), and also what we term “convenience yields,” which are benefits arising through holding inventory. They stem from two possible sources: 1) any flow of cost saving accruing to the firm from the supplier but not to the firm’s customers such as discounts on purchases, and 2) the flow of services accruing only to the holder of the spot commodity such as short term yields that can be earned by lending the physical inventory out to arbitrageurs and others. In this context, the convenience yields are the “profits” that the commodity trader earns while holding costs are the “costs” the trader incurs through embarking on the inventory investment. We define basis as spot price minus purchase price. Obviously, the basis becomes positive if convenience yields dominate but negative when inventory holding costs dominate. The dynamic behavior of the basis, i.e. basis risk, is thus dictated by the net change of these two factors over time.

Let $i$ to be the marginal net rate of convenience yield, defined as total convenience yield minus warehouse cost. Should this rate and the financing rate be constant, we would have the familiar cost of carry formula that $F_t = S_t e^{(r-i)L}$. In our dynamic setting however, we price the forward contract as martingale by assuming that $i$ follows a mean-reverting Ornstein-Uhlenbeck process. This means that any current basis, whether positive or negative, will revert back to its long-run mean at a given speed of adjustment:

$$d_i = \kappa_i (m_i - i) dt + \sigma_i dZ_i,$$

where $i$ is the net convenience yield rate, $\kappa_i$ denotes the speed of adjustment, $m_i$ the long-run mean rate, $\sigma_i^2$ the instantaneous variance, and $Z_i$ the standard Wiener process. Further, let $\text{cov}(dZ_s,dZ_i) = \rho_{s\bar{i}} dt$, where $\rho_{s\bar{i}}$ denotes the correlation coefficient.

The presence of convenience yield means that the spot commodity can be treated as an asset that pays a stochastic dividend yield $i$. Since jump risk, convenience yield risk, as well as intensity risk can not be hedged, the risk-neutralized processes for the three factors according to (13) should be

$$dS/S = (r - i - [j + \kappa_f(m_j - j)]E(A^{\gamma - 1})E(X - 1))dt + \sigma_g dZ_g + (X^* - 1)d\pi^*,$$
(16) \[ di = \left[ \kappa_i(m_i - i) - \lambda_i \sigma_i \right] dt + \sigma_i dZ_i, \]

(17) \[ dr = \left[ \kappa_r(m_r - r) - \lambda_r \sigma_r \right] dt + \sigma_r dZ_r, \]

where \( j + \kappa_j(m_j - j) \) is the risk-adjusted jump arrival intensity, \( d\pi^* \) is the corresponding Poisson process, \( X^* \) is the risk-adjusted gross jump size with \( \ln X^* \sim N(\mu_X - (1 - \varepsilon) \text{cov}(\ln X, \ln A) \sigma_X) \), and \( \lambda_i = (1 - \varepsilon) \rho_i c \sigma_c \) and \( \lambda_r = (1 - \varepsilon) \rho_j c \sigma_c \) are the respective convenience yield and diffusion risk premiums.

Under no-arbitrage and with \( B_t \), a zero-coupon unit bond with matching maturity \( L \), being the numeraire, the martingale valuation principal dictates that the normalized forward price must equal to the expected future spot price with respect to the risk-neutralized processes (15) - (17):

(18) \[ F(S_t, i_t, t, t + L) = S_t A(L) e^{-H_i(L) t} \frac{1}{B_t}, \]

\[ A(L) = \exp \left[ \frac{(H_i - L)(\kappa_i^2 m_i - \kappa_i \lambda_i \sigma_i - \sigma_i^2 / 2 + \rho_i \sigma_i \sigma_i \kappa_i) - \frac{\sigma_i^2 H_i^2}{4 \kappa_i^2}}{\kappa_i^2} \right] \]

\[ H_i(L) = \frac{1 - e^{-\kappa_i^2 L}}{\kappa_i}, \]

where \( B_t \) according to Vasicek (1977) is

(19) \[ B_t = e^{\frac{k(H_i(L) - L) - ((kH_i(L)/2)^2)}{\kappa_i^2} - p(r)}, \]

where \( H_i(L) = (1 - e^{-\kappa_i^2 L}) / \kappa_i, k = m_i + \sigma_i p(r) / \kappa_i - (\sigma_i / \kappa_i)^2 / 2, p(r) = (\kappa_i / \sigma_i) [R(\infty) - m_i + 1/2 \sigma_i^2 / \kappa_i^2] \), and \( p(r) \) and \( R(\infty) \) denote, respectively, the market price of risk and the bond’s limiting yield.

Close examination of (18) reveals that the jump components have disappeared. This happens because the occurrence of jumps only causes the distribution of the spot price to be more skewed and kurtotic than the log-normal; it does not affect the risk-neutralized expectation that is based upon the entire distribution.
To determine the risk-neutralized forward price process, let us assume the forward price function $F(S_i, r, t, t + L)$ is twice continuously differentiable in $S_i$, $r$, and $t$. Then applying generalized Ito’s lemma to $F(S_i, r, t, t + L)$ with respect to the risk-neutralized processes for the three factors gives the risk-neutralized process for forward price change,

\[
\frac{dF}{F} = (r - [j + \kappa_j(m_j - j)]E(A^{e-1}) E(X - 1))dt + \sigma_f dZ_f + (X^* - 1)d\pi^*,
\]

where

\[
\sigma^2_f = \sigma^2_s + \sigma^2_i H_i(\tau)^2 + \sigma^2_r H_r(\tau)^2 - 2\sigma_s \sigma_i \rho_{si} H_i(\tau) + 2\sigma_s \sigma_r \rho_{sr} H_i(\tau) - 2\sigma_r \sigma_i \rho_{ri} H_i(\tau)H_r(\tau),
\]

\[
\rho_{sf} = \frac{\sigma_s - H_i(\tau)\sigma_i \rho_{si} + H_r(\tau)\sigma_r \rho_{sr}}{\sigma_f},
\]

where $H_i(\tau) = \frac{1 - e^{\kappa_j \tau}}{\kappa_i}$ and $H_r(\tau) = \frac{1 - e^{\kappa_r \tau}}{\kappa_r}$; $\sigma_f$ denotes the forward return’s instantaneous standard-deviation and $\rho_{sf}$ denotes the correlation coefficient between spot return and forward return; the Poisson process $d\pi$ and random gross jump size $X^*$ are the same as in the spot price process because the Poisson event for the forward price occurs if and only if the Poisson event for the spot price occurs. Moreover, when the Poisson event occurs, the jump in the forward price should be equal to the jump in the spot price given the former is linear in the latter.

### 3.4 Risk-Neutral Valuation of the Inventory as a Portfolio of Forward Commitments in the Real Asset Martingale Framework

For simplicity, we assume that demand arrival intensity and asset price revision are uncorrelated in our risk-neutral world. The firm will meet the end-of-period demand at a cost of $(1-u)F_i$ up to the order quantity. Excess inventory will be liquidated at a price of $(1-q)F_{i+1}$ and excess demand will be met by placing an emergency order at a penalty cost $(1+p)F_{i+1}$, where $q$ and $p$ are transaction cost and penalty ratios, respectively. The expected end-of-period payoff conditional on an initial inventory position $y$ is
\( E[y] = \sum_{D = 0}^{\infty} P(D) E[y(L)] \{ D \{ E[\tilde{S}_{t+L}^-] - (1-u)F_t \} + (y-D) j(1-y)E[\tilde{F}_{t+L}^-] - (1-u)F_t \} + \sum_{D = y+1}^{\infty} P(D, E[y(L)]) (D - y) \}

where \( P(\cdot) \) is the Poisson p.d.f., \( E[y(L)] \) is the risk-adjusted expected demand arrival intensity over \( L \), i.e. \( E[y(L)] = \{ m_j L + \{ j(t) - m_j \} H_j(L) \} E(A^{e-1}) \) with \( E(A^{e-1}) \) being the risk-adjustment factor, and per Eqs. (15) and (20), the risk-neutralized spot and forward price changes over \( L \) are respectively,

\[
\ln \left( \frac{\tilde{S}_{t+L}}{S_t} \right) - N \left( r - \frac{1}{2} \sigma^2 \right) L + D \left[ \mu_X \sigma \right] - (1) \sigma \text{ cov} (\ln X, \ln \lambda) \sigma L + D \sigma^2 \]

and

\[
\ln \left( \frac{\tilde{F}_{t+L}}{F_t} \right) - N \left( r - \frac{1}{2} \sigma^2 \right) L + D \left[ \mu_X \sigma \right] - (1) \sigma \text{ cov} (\ln X, \ln \lambda) \sigma L + D \sigma^2 \).

Under no-arbitrage and with \( B_t \) being the numeraire, the martingale valuation principal dictates that the normalized net present value of the initial inventory position must equal to the expected value of its end-of-period payoff with respect to the risk-neutralized demand uncertainty and price processes, i.e. \( NPV(y)/B_t = E(y) \). Since in this risk-neutral world, \( E[\tilde{S}_{t+L}] = \frac{S_t}{B_t} \) and \( E[\tilde{F}_{t+L}] = \frac{F_t}{B_t} \), upon substitution and simplification we derive the NPV of the initial position as a function of the order quantity \( y \) as

\[
NPV(y) = E[y(L)] [S_t - (1-u)B_t F_t] + (1 - q - B_t (1-u)) F_t \sum_{D = 0}^{\infty} P(D, E[y(L)]) (y-D) + \]

\[
[(1-u)B_t - (1+p)] F_t \sum_{D = y+1}^{\infty} P(D, E[y(L)]) (D - y) \].

This expression dictates that in our risk-neutral world the present value of the inventory investment as a portfolio of forward commitments is composed of three terms: the first value stems from the expected cash inflow due to the initial under-pricing, the second value stems from the expected cash flow due to excess inventory, and the third value stems from the expected cash flow due to excess demand.

Next, examining the difference between \( NPV(y) \) and \( NPV(y+1) \) leads to
This difference is negative if

$$-(q + p) \sum_{D=0}^{\gamma} P(D, E[j(L)]) \leq (1-u)B_t - (1 + p).$$

Similarly \(NPV'(y) - NPV'(y-1) \geq 0\) if \(- (q + p) \sum_{D=0}^{\gamma} P(D, E[j(L)]) \geq (1-u)B_t - (1 + p).\)

Hence \(y\) is optimal if

$$\sum_{D=0}^{\gamma} P(D, E[j(L)]) \geq \frac{1 + p - (1-u)B_t}{p + q} \geq \sum_{D=0}^{\gamma-1} P(D, E[j(L)]),$$

where note that \(E[j(L)]\) is the risk-adjusted expected demand arrival intensity over \(L\). Note however that if \(\frac{1 + p - (1-u)B_t}{p + q} > 1\), i.e., \((1-q) > (1-u)B_t\), then the optimal \(y\) is the maximum possible demand bounded by the trader’s available investment capital.

To summarize, we have developed a market-based optimum inventory control model that works in the real world with risk-averse investors, explicitly incorporates the stylized dynamics of prices, costs and demands, and explicitly determines the equilibrium inventory ordering cost and market discount. It is straightforward to show that our valuation formula (Eq. (24)) and optimality condition (Eq. (27) collapse onto the corresponding results in the traditional newsvendor solution where demand is the only source of uncertainty as derived in Section 2 under 1) risk-neutrality, 2) constant demand arrival intensity, 3) constant interest rates, and 4) exogenously given equilibrium inventory ordering cost and market discount.

Since according to Eq. (18), \(F(S_l, i_t, I, i, t + L) = S_l A(L) e^{-H_i(L) j_i} \frac{1}{B_t}\), we can further condense Eq. (24) to become a function of the initial spot price as
This expression subsumes the equilibrium inventory order cost and relates the \( NPV \) of the inventory investment directly to the spot price and the net cost of carrying the spot commodity, among others.

### 3.5 Estimation and Calibration of the Model Parameters

We need to estimate a number of jump parameters in our model but it is well known that direct estimation of these parameters is difficult to gain great precision. Theoretically, since the probability density function corresponding to the jump-diffusion process is known, we can estimate jump parameters by using the maximum likelihood approach with time-series data as illustrated in Jorion (1988). However in actual implementation, two issues need to be addressed: first, it requires a very long time series of spot price to capture actual random and sporadic occurrences of jumps and second, since the jump arrival intensity may change substantially over a long period of time reverting between low and high state, it is necessary to estimate jointly the spot price and the intensity shift processes. To this end, Eraker (2000) has recently developed an alternative Markov chain Monte Carlo based approach that treats jumps as an additional state variable, making it convenient to draw inferences about jump occurrences.

Alternatively, one can calibrate the jump and mean-reversion parameters to traded market prices. For example, Hillard and Reis (1999) imply jump parameters from traded option prices in their study of commodity futures option pricing. Pan (2002) implies jump risk premia from a joint time series of the S&P 500 index and near-the-money short-dated option prices. In the real option literature, it is also common to calibrate the commodity process parameters to the traded commodity futures and option prices.
4. Numerical Validation

We demonstrate optimization and verify the model’s predictions numerically. We first examine the inventory basis effect, the discount effect, the penalty effect; the transactions cost effect, the interest rate effect, and then take a close look at clustering by examining the impacts of the mean-reverting behavior of the trade arrival intensity. We select the following parameter values for the common factors: $L = 1$, $m = 10$, $S = 100$, $E(A^{e-I}) = 1.0202$, where $L$ denotes the number of time period, $m$ denotes the long-run demand arrival intensity per period, $S$ denotes the current sale price of the commodity, and $E(A^{e-I})$ denotes the risk-adjustment factor for the expected demand arrival density. This value of 1.0202 is within the range of that of an average risk-averse investor per Jorion (1988) and Friend and Blume (1975).

Since the larger the basis (the difference between the current sale price of the commodity and the current purchase price or ordering cost of the inventory) the lower the initial purchase price of the inventory, we would expect the inventory value to be a positive function of the inventory basis. However since we fix the initial discount, we should expect the optimum order quantity to be a constant. Table 1 below reports the numerical result when the basis is allowed to increase from zero to 20 percent of the sale price of the commodity. With the initial demand arrival intensity at its long-run mean of 10 units per period, speed of adjustment at 5 units per period, discount at 0.2, transaction cost at 0.3, penalty at 0.2, and bond price at 0.9, the optimum order quantity is 16 units and the corresponding inventory value increase from 270.794 to 420.675. These results are consistent with the predictions.

Insert Table 1 About Here

Next we look at the discount effect. The prediction is that the larger the discount the more the firm should order and thus the larger should be the inventory value. Table 2 below reports the numerical result when the discount increases from zero to 0.5. With the initial demand arrival intensity at its long-run mean of 10 units per period, speed of adjustment at 5 units per period, basis at zero, transaction cost at
0.3, penalty at 0.2, and bond price at 0.9, the optimum order quantity increases from 11 units to above 30 units and the corresponding inventory value increases from 40.0141 to above 1056.06. These results are consistent with the predictions.

Next, we should also expect that 1) the larger the penalty ratio, the larger the optimum order quantity to prevent stock-outs and the smaller the initial inventory value to reflect the more severe penalty, and 2) the larger the transaction cost, the lower the optimum order quantity to prevent excess inventory and thus the smaller the inventory value to reflect the larger transaction cost when excess inventory are returned, and 3) the higher the bond price, the higher the present value of the purchase price of the inventory, and thus the lower the optimum order quantity and the corresponding initial inventory value. Table 3 below reports the numerical results regarding the penalty effect with the penalty increasing from 0 to 1.0. The results show that with the initial demand intensity at its long-run mean rate of 10 units per period, speed of adjustment at 5 units per period, basis at zero, discount at 0.2, transaction cost at 0.3, and bond price at 0.9, the optimum order quantity and its corresponding inventory value increases from above 15 units to 18 units and decreases from 272.417 to 267.905, respectively. These results are consistent with the predictions.

Table 4 below reports the numerical results regarding the transaction cost effect with the cost increasing from 0 to 0.5. The results show that with the initial demand intensity at its long-run mean rate of 10 units per period, speed of adjustment at 5 units per period, basis at zero, discount at 0.2, penalty at 0.2, and bond price at 0.9, the optimum order quantity and its corresponding inventory value decreases
from above 30 units to 12 units and from above 840 to 204.485, respectively. These results are consistent with the predictions.

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Insert Table 4 About Here
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Table 5 below reports the results regarding the interest rate effect with the bond price increasing from 0.8 to 1.0. The results show that with the initial demand intensity at its long-run mean rate of 10 units per period, speed of adjustment at 5 units per period, basis at 10 units, discount at 0.2, transaction cost at 0.3, and penalty at 1.2, the optimum order quantity and its corresponding inventory value decreases from above 30 units to 13 units and from above 486.06 to 157.733, respectively. These results are consistent with the predictions.

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Insert Table 5 About Here
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Finally we take a close look at the impacts of clustering. With the long-run demand arrival intensity at 10 units per period, basis at 10, penalty ratio at 1.2, and the bond price at 0.9, we consider two scenarios: when the initial intensity is low at 5 units and high at 15 units. In each scenario, we vary the speed of adjustment toward the mean from 2 to 10 units per period. Table 6 below reports the results. As the speed of adjustment increases, clustering weakens and mean-reversion toward the mean strengthens, which leads to increased expected demand intensity when the initial intensity is low and decreased expected demand intensity when the initial intensity is high. Consequently, the optimum inventory level jumps up from 13 units to 15 units and down from 19 units to 17 units, respectively, with the corresponding value of the initial inventory increasing from 269.515 to 327.995 and decreasing from 422.073 to 363.349, respectively. These results are also consistent with the predictions.
To summarize, we have found that 1) the lower the net carrying cost and the larger the initial discount, the higher is the initial inventory value and thus the more the firm should order, 2) the larger the penalty ratio, the larger should be the initial inventory to prevent from stock-outs but the smaller is the initial inventory value to reflect the more severe penalty, 3) the larger the discount in determining the net salvage value, the lower the optimum order quantity to prevent excess inventory and thus the smaller the initial inventory value, 4) the higher the bond price, the higher the present value of the purchase price of the inventory, and thus the lower the optimum order quantity and the initial inventory value, and 5) in the clustering effects, when the initial demand arrival intensity is low, as the speed of adjustment increases, clustering weakens and mean-reversion toward the higher mean strengthens, leading to increased expected demand intensity. Consequently, the optimum inventory level jumps up with the corresponding value of the initial inventory increasing to reflect the increased expected demand. Conversely, when the initial demand arrival intensity is high, the opposite happens.

5. Conclusions

We have modeled a commodity trader’s inventory as a portfolio of forward contracts in open markets by using the real asset martingale valuation methodology with the market risk premiums attained in an internally consistent jump-diffusion, stochastic interest rates, and stochastic volatility economy, where the underlying trade arrival intensity serves as the latent stochastic factor. Optimum inventory control is implemented by shareholders’ wealth maximization with respect to the inventory decision variables, leading to a simple and intuitive optimality condition and a closed-form solution for the inventory value. Numerical analysis demonstrates that the resulting optimum policy has robust properties. In comparison to the traditional approach the contribution of our model is three-fold: 1) it works in the real world but the traditional model only works in the risk-neutral world, 2) it explicitly
incorporates the stylized dynamics of prices, costs and demands but the traditional model only allows for demand uncertainty, and 3) it shows how to endogenize the inventory ordering cost but this cost is not known in the traditional model.

This new and novel real asset approach can be further generalized to a multi-period dynamic control setting, with inclusion of commodity futures trading to further fine-tune the topology contour. Applications to other operational management problems such as capacity expansion and supply chain are also potential topics. It will be informative to examine empirically how the optimum inventory decision varies across different commodities with particular characteristics, e.g. gold that is more sensitive to interest rate shocks vs. oil that is more sensitive to clustering. It will also be informative to examine how the size of a demand shock and the current level of inventory jointly impact the model. It is conceivable that a large positive demand shock when inventories are high would produce a different decision than when inventories are low – the decision should be time varying and partly driven by business cycles.

References


Working Paper, Graduate School of Business, University of Chicago.


Table 1: The Basis Effect

Parameters:

\[ L = 1.0, \quad \kappa = 5.0, \quad m = 10.0, \quad j = 10.0, \quad u = 0.2, \quad p = 0.2, \quad c = 0.3, \quad B = 0.90, \quad S = 100.0 \]

Table 1 below reports the numerical result when the inventory ordering cost is allowed to decrease from 100 to 80. With the initial demand arrival intensity at its long-run mean of 10 unit per period, speed of adjustment at 5 units per period, bond price at 0.9, discount at 0.2, penalty at 0.2, and transaction cost at 0.3, the optimum order quantity is 16 units and the corresponding inventory value increase from 270.794 to 420.675.
Table 2: The Discount Effect

Parameters:

\[ L = 1.0, \quad \kappa = 5.0, m = 10.0, j = 10.0, p = 0.2, c = 0.3, B = 0.9, S = 100.0, F = 100.0 \]

Table 2 below reports the numerical results regarding the discount effect with the discount increasing from 0 to 0.5. The results show that with the initial demand intensity at its long-run mean rate of 10 units per period, speed of adjustment at 5 units per period, basis at zero, bond price at 0.9, penalty at 0.2, and transaction cost at 0.3, the optimum order quantity and its corresponding inventory value increases from 11 units to above 30 units and from 40.0141 to above 1,056.06, respectively.
Table 3: The Penalty Effect

**Parameters:**
L = 1.0, \( \kappa = 5.0, m = 10.0, j = 10.0, u = 0.2, c = 0.3, B = 0.9, S = 100.0, F = 100.0 \)

Table 3 below reports the numerical results regarding the penalty effect with the penalty increasing from 0 to 1. The results show that with the initial demand intensity at its long-run mean rate of 10 units per period, speed of adjustment at 5 units per period, basis at zero, bond price at 0.9, discount at 0.2, and transaction cost at 0.3, the optimum order quantity and its corresponding inventory value increases from 15 units to 18 units and decreases from 272.417 to 267.905, respectively.
Table 4: The Transaction Cost Effect

Parameters:
\[ L = 1.0, \ k = 5.0, m = 10.0, j = 10.0, u=0.2, p=0.2, B = 0.9, S = 100.0, F = 100.0 \]

Table 4 below reports the numerical results regarding the transaction cost effect with the cost increasing from 0 to 0.5. The results show that with the initial demand intensity at its long-run mean rate of 10 units per period, speed of adjustment at 5 units per period, basis at zero, bond price at 0.9, discount at 0.2, and penalty at 0.2, the optimum order quantity and its corresponding inventory value decreases from above 30 units to 12 units and from above 840 to 224.485, respectively.
Table 5: The Interest Rate Effect

Parameters:
L = 1.0, κ = 5.0, m = 10.0, j = 10.0, u = 0.2, p = 0.2, c = 0.3, S = 100.0, F = 90.0

Table 5 below reports the results regarding the interest rate effect with the bond price increasing from 0.8 to 1.0. The results show that with the initial demand intensity at its long-run mean rate of 10 units per period, speed of adjustment at 5 units per period, basis at 10 units, discount at 0.2, transaction cost at 0.3, and penalty at 0.2, the optimum initial inventory position and its corresponding inventory value decreases from above 30 units to 13 units and from above 486.06 to 157.733, respectively.
Table 6: The Clustering Effects

Parameters:
L = 1.0, m = 10.0, j = 5.0 and 15.0, p = 1.2, B = 0.90, S = 100.0, F = 90.0

We take a close look at the impacts of clustering with high and low initial intensities by varying the speed of adjustment toward the mean ($\kappa$) from 2 to 10 units per period. As this speed increase, clustering weakens and mean-reversion toward the mean strengthens, which leads to increased expected demand intensity when the initial intensity is low and decreased expected demand intensity when the initial intensity is high. Consequently, the optimum inventory level jumps up from 13 units to 15 units and down from 19 units to 17 units, respectively, with the corresponding value of the initial inventory increasing from 269.515 to 327.995 and decreasing from 422.073 to 363.349, respectively.
Appendix: Proof of Theorem 1

**THEOREM 1:** Given the pricing kernel and the asset price dynamics, the fundamental Euler valuation equation dictates the following four-factor consumption-based jump-diffusion asset pricing relation:

\[
\begin{align*}
\mu_s + \left[ j + \kappa_j (m_j - j) \right] E( X - 1) &= r - (1 - \epsilon) \sigma_{rc} + (1 - \epsilon) \sigma_{sc} + \left[ j + \kappa_j (m_j - j) \right] E( X - 1) E( A^F - 1 ) \\
+ \left[ j + \kappa_j (m_j - j) \right] \text{cov}( X, A^F - 1 ),
\end{align*}
\]

where \( \mu_s \) and \( j + \kappa_j (m_j - j) \) denotes the instantaneous expected diffusion return and jump arrival intensity, respectively; \( E \) is the expected value operator and \( X \) the gross spot jump size; \( r \) is the instantaneous riskless rate, \( (1 - \epsilon) \) the CRRA measure, \( \sigma_{rc} = \sigma_r \sigma_c E( dZ_r dZ_c ) \) the instantaneous diffusion covariance between interest rate and consumption change, \( \sigma_{sc} = \sigma_s \sigma_c E( dZ_s dZ_c ) \) the instantaneous diffusion covariance between asset return and the consumption change, and \( A \) is the gross consumption jump size.

**Proof of Theorem 1:**

Applying the generalized Ito’s lemma to the pricing-kernel-scaled spot price \( q_s \), we obtain

\[
(A2-1) \quad d(qs)/qs = (\mu_q + \mu_s + \sigma_{sq}) dt + \sigma_q dZ_c + \sigma_s dZ_s + (XY - 1) d\pi,
\]

where \( dp = k_j (m_j - j) dt + \sigma_j dZ_j \) and \( \sigma_{sq} = \sigma_s \sigma_q E( dZ_s dZ_q ) \) is the instantaneous diffusion covariance between spot return and pricing kernel change.

Applying the fundamental Euler valuation equation that \( E[d(qs)/qs] = 0 \) to (A2-1) by taking the expectation of (A2-1), we obtain

\[
(A2-2) \quad \mu_s = \mu_q - \sigma_{sq} - E( j + dp / dt ) E( XY - 1 ) - \text{cov}( XY, \sigma_j dZ_j / dt ),
\]

where \( E( j + dp / dt ) = j + \kappa_j (m_j - j) \) denotes the instantaneous expected jump arrival intensity.

Applying this equation to a riskless fund that continuously earns an instantaneous riskless return \( r \) whose change follows a diffusion process yields

\[
(A2-3) \quad r = - \mu_q - \sigma_{rq} - E( j + dp / dt ) E( Y - 1 ),
\]
where \( \sigma_{rq} = \sigma_r \sigma_q \mathbb{E}[d\xi_r \, d\xi_q | \mathcal{F}_t] / dt \) is the instantaneous diffusion covariance between the interest rate and the pricing kernel change, and \( Y = A^{\varepsilon^{-1}} \) because of the CRRA preference assumption.

Recall from Eq. (7), \( \sigma_q = -(1-\varepsilon)\sigma_r \), and thus with this substitution,

\[
(A2-4) \quad \sigma_{sq} \, dt = -(1-\varepsilon)\sigma_r \mathbb{E}[d\xi_q \, d\xi_r | \mathcal{F}_t] / dt = -(1-\varepsilon)\sigma_r \, dt, \quad \text{and}
\]

\[
(A2-5) \quad \sigma_{rq} \, dt = -(1-\varepsilon)\sigma_r \mathbb{E}[d\xi_r \, d\xi_r | \mathcal{F}_t] / dt = -(1-\varepsilon)\sigma_r \, dt.
\]

Finally, substituting (A2-3) - (A2-5) into (A2-2) and rearranging yields Theorem 1.

Q.E.D.