

Dynamic Investment in Extraction Capacity of Exhaustible Resources

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Abstract

This paper applies real options theory into a resource extraction problem with capacity constraints and demand uncertainty. The producer decides on the optimal rate of extraction and the optimal time to build further capacity simultaneously. It is shown that previous results which suggest that extraction capacity should be built at the beginning, are not necessarily true under uncertainty. The model characterizes the optimal path of capacity investments. Using Monte Carlo simulation, the model generates term-structure of forward prices and their volatility. The results of this paper can contribute to better understanding of long-run energy and commodity price dynamics.

1 Introduction

The maximum extraction rate of exhaustible resources depends on the level of relevant investments. For example, daily production capacity of an oil field is a function of the number of wells drilled and the annual production of a copper mine depends on the size of tunnels and material handling equipments. This fact introduces new frictions to the standard model of Hotelling (1931) where

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the optimal production path of an exhaustible resource is determined in a way to make its shadow price grow with riskless interest rate.

To apply Hotelling's formulation into the real world problems, one needs to include different uncertainties (on the demand process, production and investment costs, time to build, geological properties and deposits, unexpected stops, etc.) in addition to the capacity constraints. It is also important to consider these two aspects simultaneously and as interrelated to each other. This motivates a natural stage to apply the concept of capacity expansion options in order to analyze the investment and production decisions of the producer.

Despite its importance, there are few works in the literature of exhaustible resources which study the issue of capacity constraints. Importantly, most of existing papers ignore uncertainties in resource extraction problems and solve them in a deterministic way.

My paper differs in several ways from the existing literature. First of all, demand uncertainty motivates the "option value" of waiting to make further investments and therefore the optimal investment rule would be different. The result from the model is in contrast with most of the previous ones where the investment in capacity takes place at $t=0$. Second, in an uncertain environment the capacity building is not a one-shot decision but happens in a dynamic way and one can talk about the "optimal dynamic path" of capacity instead.

There is also, a mature literature (for example, Brennan and Schwartz (1985), Dangl (1999)) on optimal investment timing in macroeconomics and real options theory. This paper is contributing to that literature by adding the feature of "exhaustibility" to the traditional models. The feature of exhaustibility imposes a finite sum of production volume and hence decreases the marginal value of extra capacity.

Although the option value for an inactive producer has been extensively examined in the literature, only one study, to the best of my knowledge, has studied the production and capacity building decisions simultaneously. Previous studies (Brennan and Schwartz (1985), Morck, Schwartz, and Stangeland (1989), for instance) usually calculate the real option value of investing in resources with an exogenously specified price process. Moreover, they usually assume a right to exercise an investment option for a currently inactive natural resource producer. My model differs from them through the important fact that the producer is active in both periods before and after exercising the capacity option. Therefore, there is a second dynamics in the model which is the level of remaining stock. This makes the capacity option problem considered here, a two-dimensional one.

The closest paper to the current one is Carlson, Khokher, and Titman (2007) which introduces continuous adjustment costs to the resource extraction problem in a competitive market and leaves the oligopoly case as an open question. In addition, their paper does not assume a rigid capacity constraint and only imposes adjustment costs. I use a lumpy investment model since this is more closer to the reality of mineral investments.

The insights from considering the impact of finite reserves in the real options problems can be extended to other applications; specially in environments where the underlying demand for the product is deteriorating. One example would be the optimal capacity choice for a producer introducing new durable goods whose market will shrink over time and the rate of drop in the demand is a function of the rate of supply in previous periods.

The problem is also relevant to understanding long-term behavior of prices. Commodity prices in the long-run are driven by the demand process and the capacity of supply side which is strongly bounded by net investment in production facilities. The oil market would be a good example of this case. For years, majority of oil producers have been producing very close to their maximum capacity. Although in short-term the wedge between demand and supply can be covered through inventories, in the long-run supply may not be able to respond to upward demand shifts if the production capacity is not expanded. Under this inelastic supply scenario, demand shocks would be directly transferred into price processes while with enough excess production capacity, high demand shocks can partly be absorbed by an increase in production rate. This paper discusses the optimal trigger level to build new capacity and hence can be useful to understand long-run market fundamentals.

2 Related Literature

With no doubt, Hotelling (1931) is the most seminal work in the whole literature of natural resource economics. This paper raised a critical question “What is the optimal rate of extraction of exhaustible resources?”. Hotelling answered this question by introducing the famous r -percent rule implying that price net marginal cost (marginal revenue) will increase with the interest rate for competitive (monopoly) market. As an immediate implication of this rule, one should observe the futures curves of commodities always in contango. Furthermore, the production rate should diminish over time if the demand has constant elasticity.

Although Hotelling refers to the issue of limited production capacity, he does not explicitly incorporate it in his model. Therefore the results are valid only for an unbounded rate of extraction. Moreover, Hotelling does not consider any uncertainty in the problem contrary to the real world.

Pindyck (1980) was the first to formulate a version of Hotelling's problem with a continuous stochastic demand process. Assuming an Ito process for both demand and resource depletion dynamics, he derived a Hamilton Jacobi Bellman (HJB) equation which characterizes the optimal value function and extraction trajectory. He shows that if the marginal cost does not depend on stocks, the optimal extraction rate of the stochastic problem will be the same as the deterministic one. In other words, the r -percent rule for competitive and monopolistic case holds under expectations operator.

Mason (2000) considers a mine with costly switching from inactive to active mode and vice-versa. His model is partial equilibrium and the price process is exogenously given. This fact weakens the effect of exhaustibility and inter-temporal optimization since he assumes that the drift of price process is lower than interest rates and this forces the firm to produce at maximal capacity. Contrary to his work, current paper treats the inter-temporal effect as a key element of model.

Campbell (1980) added capital intensity to resource extraction problems. By this further assumption, the production at each instant is limited to a maximum level which is a function of the net accumulated capital. Solving the constrained dynamic optimization problem he showed that capacity constraints will be binding at some initial periods and then will be relaxed. Therefore it is optimal to build the capacity only at the very beginning of production. This comes from the fact that the optimal production trajectory is decreasing over time, hence the shadow price of extra capacity is also decreasing and becomes zero after some time. He derives the optimal level of investment at the beginning of the project's life-cycle by equating the marginal value of extra unit of capacity to the marginal cost of capital.

Davis and Moore (1998) revise the Hotelling valuation rule by adding capacity constraints. The older rule suggests that for a competitive market the present value of a natural resource with known quantity is just the product of net value (price less production costs) and the amount of resource. The empirical tests do not support this rule and conclude that the market value is lower than the predictions of Hotelling valuation rule. Davis and Moore justify this difference by introducing capacity constraint and show that as long as the con-

straint is binding, the value function is lower than the unlimited case. They however do not provide an explicit valuation formula for the constrained case.

Cairns (1998) looks into the microeconomics of capacity constraints. He discusses the effect of production function specification on the time path of optimal extraction path. He points out that the capacity investment will not be concentrated around the opening time if marginal cost is not fixed or if there is not a bound to investment. He also expresses that uncertainty has not entered into the capacity-constraint models of exhaustible resources.

Capacity constraints may also affect the optimal order of resource use. Holland (2003) shows that contrary to the previous results, if extraction from a low cost reserve is constrained, it is optimal to start extraction from a high-cost reserves simultaneously before the first one is fully depleted.

Litzenberger and Rabinowitz (1995) use the concept of real options existing in oil production to explain the seemingly odd backwardation in oil futures contracts. Backwardation is not consistent with the predictions of Hotelling rule as this rule expects the future prices to rise and therefore futures curve to be in contango. They argue that because of this option value, any decision to produce right now means to sacrifice the option to wait. As a result, current production will take place only if the present value of future incomes plus the option value exceeds the value of spot production. Since the option value is always positive, the present value of future income should be lower than the spot value which is equivalent to a strong backwardation.

3 The Model

The model of this paper is a partial equilibrium one, where the demand shocks, the price of investment good and the initial amount of reserves are given exogenously. I chose to focus on partial rather than general equilibrium, because I am mostly interested in studying the capacity building behavior of a single (or one with market power) resource producer. In reality, many of these factors are endogenous. The demand process will be affected by market's expectations regarding future supply of exhaustible resource. It can be argued that this depends on the probability of exercising capacity expansion options in future. Hence, if market expects that new investments in oil production will happen in a very conservative way, investments on alternative sources of energy (e.g solar or oil shale) will be accelerated, resulting in slower demand growth for oil. I

abstract from this interesting commitment effect.

At $t = 0$ the risk-neutral monopolist is endowed with an initial level R_0 of a homogeneous deposit of exhaustible resources. I assume no uncertainty regarding the resources and no exploration effort, therefore the dynamics of resource depletion is given by

$$dR_t = -q_t dt \tag{1}$$

where q_t is the instantaneous rate of production and R_t is the level of remaining stock. Since there is no storage in this version of model, the spot price of commodity is a function of instantaneous production rate q_t and an exogenously given random process X :

$$\begin{aligned} P_t &= P^{-1}(X, q) \\ \frac{\partial P^{-1}}{\partial q} &< 0 \end{aligned} \tag{2}$$

Extraction costs are assumed to be negligible. This is a realistic assumption for example for the production of oil at the Middle East where the variable cost is around 2 US\$ per barrel. Therefore the profit is determined by

$$\pi(X, q_t) = P^{-1}(X, q_t)q_t \tag{3}$$

The stochastic demand parameter X follows a Geometric Brownian Motion (GBM) with the following dynamics:

$$dX = \mu X dt + \sigma X dW \tag{4}$$

Where μ and σ are the drift and volatility parameters of the demand process.

The producer maximizes the expected sum of the present value of future profit streams:

$$\begin{aligned}
V(R, X) &= \max_{q_t} \mathbb{E} \left\{ \int_0^{\infty} \pi(X, q_t) e^{-\rho t} dt \right\} \\
\text{s.t} & \\
dR &= -q dt \\
dX &= \mu X dt + \sigma X dW
\end{aligned} \tag{5}$$

3.1 Optimal Extraction Rates

It is useful to first characterize optimal policies under different assumptions concerning the behavior of the demand.

3.1.1 Deterministically Growing Constant elasticity Demand

The simplest case would be the one where the demand is given by $P = X \cdot q^\gamma$, $-1 < \gamma < 0$ and the coefficient X has a deterministic dynamics as $dX = \mu X dt$.

$$\begin{aligned}
V(R, X) &= \text{Max}_{q_t} \mathbb{E} \left\{ \int_0^{\infty} \pi(X, q_t) e^{-\rho t} dt \right\} \\
\text{s.t} & \\
dR &= -q dt \\
dX &= \mu X dt \\
0 &\leq R_t
\end{aligned} \tag{6}$$

Basic Hotelling rule suggests that in the absence of production costs, the marginal revenue of the monopolist will grow with the interest rate. Using this rule:

$$\begin{aligned}
\frac{dMR}{MR} &= r dt, MR = (1 + \gamma) X q^\gamma, dMR = (1 + \gamma) [dX q^\gamma + X d(q^\gamma)] = (1 + \gamma) [X \mu dt q^\gamma + X \gamma q^{\gamma-1} dq] \\
\frac{dMR}{MR} &= \frac{(1 + \gamma) [X \mu dt q^\gamma + X \gamma q^{\gamma-1} dq]}{(1 + \gamma) X q^\gamma} = r dt \Rightarrow \gamma \frac{dq}{dt} = (r - \mu) q \Rightarrow \frac{dq}{q} = \frac{r - \mu}{\gamma} \Rightarrow q_t = q_0 e^{\frac{r - \mu}{\gamma} t} \\
\text{Since : } \int_0^{\infty} q_0 e^{\frac{r - \mu}{\gamma} t} dt &= R_0 \Rightarrow q_t = R_t \left(\frac{\mu - r}{\gamma} \right)
\end{aligned} \tag{7}$$

This functional form leads into a result where the production rate is always a linear function of the remaining stock and declines exponentially.

3.1.2 Stochastic Constant Elasticity Demand

Pindyck (1980) shows that even with stochastic demand shocks, the basic Hotelling rule for the expectation of marginal revenue holds. Therefore we have:

$$\begin{aligned}
\frac{E(dMR)}{MR} &= rdt, E(dMR) = (1 + \gamma)[X\mu dtq^\gamma + X\gamma q^{\gamma-1}dq] \\
\frac{E(dMR)}{MR} &= \frac{(1+\gamma)[X\mu dtq^\gamma + X\gamma q^{\gamma-1}dq]}{(1+\gamma)Xq^\gamma} = rdt \Rightarrow \gamma \frac{dq}{dt} = (r - \mu)q \Rightarrow \\
\frac{dq}{q} &= \frac{r-\mu}{\gamma} dt \Rightarrow q_t = q_0 e^{\frac{r-\mu}{\gamma}t} \\
\text{Since : } \int_0^\infty q_0 e^{\frac{r-\mu}{\gamma}t} dt &= R_0 \Rightarrow q_t = R_t \left(\frac{r-\mu}{\gamma}\right)
\end{aligned} \tag{8}$$

It is important to notice that the instantaneous production rate does not depend on X (meaning that it is independent of the realizations of the demand shocks) and is only the function of parameters of demand function.

3.1.3 Deterministic Growing Linear Demand

Unlike the constant elasticity case, the optimal production with linear demand will not be independent of demand shocks.

$$\begin{aligned}
\frac{dMR}{MR} &= rdt \Rightarrow MR(t) = MR(0)e^{rt} \\
(X(t) - 2q(t)) &= (X(0) - 2q(0))e^{rt} \Rightarrow q_t = \frac{1}{2}X(0)e^{\frac{r}{2}t} + q(0)e^{rt}
\end{aligned} \tag{9}$$

With linear demand the reserves will deplete in finite time (if the growth rate of demand is less than the interest rate). Therefore, there is a full depletion time T where the production stops and extraction rate is zero for any $t > T$. The total supply of reserves is given by R and the level of shocks at time 0, $X(0)$ is known. Using these facts one can write down two equations with two unknowns ($q(0)$ and T) in order to fully characterize the optimal production path.

$$\int_0^T q(t)dt = \int_0^T \left\{ \frac{1}{2}X(0)e^{\frac{r}{2}t} + q(0)e^{rt} \right\} dt = R \tag{10}$$

3.2 The Impact of Capacity Constraints

If the demand is of a constant elasticity type, the extraction rate is declining over time. Based on Campbell (1980), the capacity constraint is binding only for some initial periods of production horizon and once the resource reaches a

critical level it will never bind. This helps to write down the value function of resource production problem with a given capacity as the sum of two separate sub-problems: the problem for the period when the constraint is binding and the problem for the period right after that.

In the first period there will be a flat rate of production till the remaining reserves reach the critical level. It is possible to find this “non-binding” point for the constant elasticity demand function. The equations 7 gives the relation between the remaining available reserves and the (unconstrained) instantaneous optimal production. For a given production capacity \bar{Q} the capacity will not bind if and only if:

$$q_t = \left(\frac{\mu-r}{\gamma}\right)R \leq \bar{Q} \Rightarrow R_t \leq \bar{Q}\left(\frac{\gamma}{\mu-r}\right) \quad (11)$$

Using this result, one can calculate the time it takes for a producer with a given reserve R_0 to reach the non-binding region:

$$R_0 - T^*\bar{Q} = \bar{Q}\left(\frac{\gamma}{\mu-r}\right) \rightarrow T^* = \text{Max} \left\{ 0, \frac{R_0}{\bar{Q}} - \frac{\gamma}{r-\mu} \right\} \quad (12)$$

The value of first period problem is just the present value of a fixed income stream in a finite horizon.

$$\begin{aligned} V_1 &= E \int_0^{T^*} \pi(\bar{Q})e^{-rt} dt = \\ & \int_0^{T^*} E(X_t)\bar{Q}^{1+\gamma} e^{-rt} dt = X_0\bar{Q}^{1+\gamma} \int_0^{T^*} e^{(\mu-r)t} dt = \frac{X_0\bar{Q}^{1+\gamma}}{\mu-r} (e^{(\mu-r)T^*} - 1) \end{aligned} \quad (13)$$

The second period problem starts at T^* and its current value can be calculated by taking the sum of discounted future cash flows coming from the optimal control problem.

$$\begin{aligned} V_2 &= E \int_0^{\infty} \pi(q^*)e^{-rt} dt = \int_0^{\infty} E(X_t)q_t^{1+\gamma} e^{-rt} dt = \\ & X_0 \int_0^{\infty} (q_0 e^{\frac{r-\mu}{\gamma}t})^{1+\gamma} e^{(\mu-r)t} dt = \\ & X_0 q_0^{1+\gamma} \int_0^{\infty} e^{\frac{r-\mu}{\gamma}t} dt = X_0' q_0^{1+\gamma} \frac{\gamma}{r-\mu} \end{aligned} \quad (14)$$

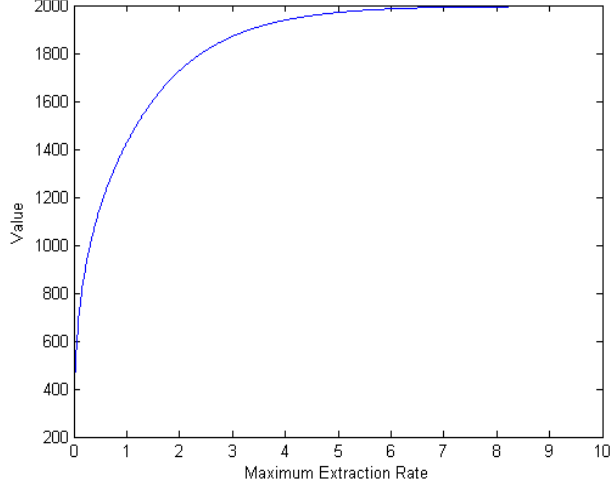


Figure 1: Effect of Capacity Constraints on Value Function

In the beginning of the second period (non-binding region), $q_0 = \bar{Q}$ and $E(X'_0) = X_0 e^{\mu T^*}$. By plugging the parameters back to the valuation equation one gets

$$\begin{aligned}
 V(X_0, R_0, \bar{Q}) &= \frac{X_0 \bar{Q}^{1+\gamma}}{\mu-r} (e^{(\mu-r)T^*} - 1) + e^{-rT^*} X_0 q_0^{1+\gamma} \frac{\gamma}{r-\mu} = \\
 &= \frac{X_0 \bar{Q}^{1+\gamma}}{\mu-r} ((1+\gamma)e^{(\mu-r)T^*} - 1) \\
 T^* &= \frac{R_0}{\bar{Q}} + \frac{\gamma}{r-\mu}
 \end{aligned} \tag{15}$$

The impact of maximum capacity on the value function is presented in figure 1. The graph is first increasing and then becomes flat. The reason is that for a given level of reserves, the optimal extraction rate is finite and hence any maximum capacity above this level will not improve the value function.

3.3 Marginal Value of Capacity

A capacity expansion option for an active producer is an American call option. The strike price of the option is the investment cost I and the pay-off is the difference between value functions with new and old capacity levels. As mentioned before, the option considered here has two unique features which make it different from other capacity options. First, for a producer of exhaustible resources

the total supply of resources is fixed. This means that if the producer postpones the investment in the capacity option for one period, the profits of that period will not be lost completely but will contribute to the firm value in the next period. The second major difference relies on the fact that the reserves available for the producer are being constantly depleted. Therefore, at any time period in the future the level of reserves will be lower than today. *Ceteris Paribus*, it means that the value of option is declining over time

Formally, the marginal value of capacity Δ is given by

$$\begin{aligned} \Delta &= V(R, X, \bar{Q} + dQ) - V(R, X, \bar{Q}) \Rightarrow E(d\Delta) = E(dV(R, X, \bar{Q} + dQ)) - E(dV(R, X, \bar{Q})) = \\ &= V_R(\bar{Q} + dQ)E(dR) + V_X(\bar{Q} + dQ)E(dX) + \frac{1}{2}V_{XX}(\bar{Q} + dQ)(dX)^2 - \\ &= V_R(\bar{Q})E(dR) - V_X(\bar{Q})E(dX) - \frac{1}{2}V_{XX}(\bar{Q})(dX)^2 \end{aligned} \tag{16}$$

If the capacity constraint is binding $\frac{\partial R}{\partial t} = -\bar{Q}$. Also, notice that $V_{XX} = 0$ (since V is linear in X). Plug these into equation 16

$$\begin{aligned} E(d\Delta) &= E(dV(R, X, \bar{Q} + dQ)) - E(dV(R, X, \bar{Q})) \\ &= (-\bar{Q}dt)(V_R(R, X, \bar{Q} + dQ) - V_R(R, X, \bar{Q})) + \mu X dt (V_X(R, X, \bar{Q} + dQ) - V_X(R, X, \bar{Q})) \Rightarrow \\ &= \frac{E(dV)}{dt} = -\bar{Q}(V_R(R, X, \bar{Q} + dQ) - V_R(R, X, \bar{Q})) + \mu X (V_X(R, X, \bar{Q} + dQ) - V_X(R, X, \bar{Q})) \end{aligned} \tag{17}$$

Equation 17 tells us that the value of marginal unit of capacity is driven by two factors. Constant depletion of reserves is the first factor which lowers the value of extra capacity over time. The demand growth is the second term which can increase or decrease the value of extra capacity in the same way as demand evolves. Figure 2 presents sample time paths of marginal value of capacity for static, deterministic growing and stochastically growing demand processes. Since demand is constant in this case, only the first factor is active. Therefore, the marginal value of capacity under static demand is monotonically declining. This is the case which is already addressed in the literature. The cases for determinist and stochatic growing demand are more realistic and also more interesting since they offer a non-trivial optimal time to build capacity. One

important feature of option value in all three scenarios is that the life of option is finite and its value becomes zero after some finite periods. This happens because by continuing to extract, the level of reserves reaches a critical level where capacity constraint is not binding and hence the value of extra capacity is zero. This result may change if one introduces uncertainties about the level of reserves through adding for instance a jump process accounting for the discovery of new reserves. Under this new setting there is always a positive probability that the reserves becomes larger in future and therefore the value of extra unit of capacity will not die and remains always positive. I abstract from this aspect and leave it to the extensions of the paper.

It is an important observation to notice that although the changes in demand parameter X does not influence the instantaneous production rate, it still has a strong effect on the total value of the firm through changing the profits. Therefore, one sees that the changes of demand influences the value of an extra unit of capacity significantly.

3.4 The Firm's Problem under Capacity Constraints

A producer with a general demand function and a given non-depreciating capacity constraint solves the following stochastic optimal control problem where X and R are state variables, q is the control variable and \bar{Q} refers to the maximum extraction rate.

Where U is the value function of a problem with capacity constraints but no expansion option.

$$\begin{aligned}
 U(X, R, \bar{Q}) &= \text{Max}_{q_t} \mathbb{E} \left\{ \int_0^{\infty} P^{-1}(X, q_t) q_t e^{-rt} dt \right\} \\
 \text{s.t} & \\
 dR &= -q dt \\
 0 &\leq q_t \leq \bar{Q} \\
 0 &\leq R_t
 \end{aligned} \tag{18}$$

The standard steps lead into the HJB equation for this problem:

$$\text{Max}_{\{q \leq \bar{Q}\}} \left\{ \pi(q, X) - qU_R + \mu XU_X + \frac{1}{2} \sigma X^2 U_{XX} - rU \right\} = 0$$

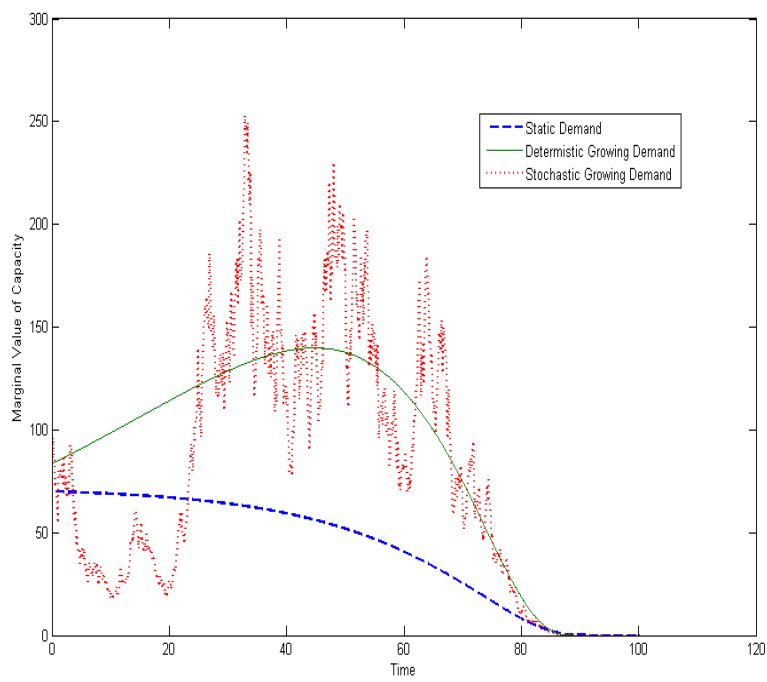


Figure 2: Dynamics of Marginal Value of Capacity

(19)

One solves the constrained HJB equation by forming a Lagrangian, where λ is the Lagrange coefficient (or shadow value) of capacity constraints.

$$\begin{aligned}
& \text{Max}_{\{q \leq \bar{Q}\}} \left\{ \pi(q, X) - qU_R + \mu XU_X + \frac{1}{2}\sigma X^2 U_{XX} - rU - \lambda(\bar{Q} - q) \right\} = 0 \\
& \frac{\partial \pi}{\partial q} = U_R, \text{ if } q \leq \bar{Q} \\
& \frac{\partial \pi}{\partial q} = U_R + \lambda \text{ Otherwise}
\end{aligned} \tag{20}$$

By adding capacity expansion option to the problem, the producer can increase the maximum production rate by spending a cost of $C(I)$ and gain a further capacity of dQ . The producer is simltonously chosing the instantaneous production rate and the optimal switching time T^* .

$$\begin{aligned}
V(X, R, \bar{Q}) &= \text{Max}_{\{q_t, T^*\}} \mathbb{E} \left\{ \int_0^{T^*} P^{-1}(X, q_t) q_t e^{-rt} dt + e^{-rT^*} (U(X, R, \bar{Q} + dQ) - C(I)) \right\} \\
&\text{s.t} \\
& dR = -q dt \\
& 0 \leq q_t \leq \bar{Q}_t \\
& 0 \leq R_t
\end{aligned} \tag{21}$$

Following the language of real options, we can see that the value function consists of two elements: the value of asset in place and the option value to move to a less constrained problem. The value of expansion option can be calculated using the standard no-arbitrage approach.

$$\begin{aligned}
\rho F dt &= E(dF) = E(F(X + dX, R + dR) - F(X, R)) = \\
& E(F_X(dX) + \frac{1}{2}F_{XX}(dX)^2 + F_R(dR)) = \\
& (F_x \mu X + \frac{1}{2}F_{xx}\sigma^2 X^2 - q^* F_R) dt \Rightarrow \\
& F_x \mu X + \frac{1}{2}F_{xx}\sigma^2 X^2 - q^* F_R - \rho F = 0 \\
& \text{The initial conditions:} \\
& F(R, 0, \bar{Q}) = 0 \\
& F(0, X, \bar{Q}) = 0
\end{aligned}$$

(22)

The first initial condition comes from the fact that $X=0$ is an absorbing state for the GBM process. The second condition suggests that when there is no reverses the option has no value.

To find the optimal time to exercise the capacity expansion option, the value matching and smooth pasting conditions are imposed.

$$\begin{aligned} F(R^*, X^*, \bar{Q}) &= U(R^*, X^*, \bar{Q} + dQ) - U(R^*, X^*, \bar{Q}) - C(I) \\ F_x(R^*, X^*, \bar{Q}) &= U_x(R^*, X^*, \bar{Q} + dQ) - U_x(R^*, X^*, \bar{Q}) \end{aligned} \quad (23)$$

3.5 Dynamic Investment

In a more general setting, capacity expansion will not take place only once. In other words, the firm will have an infinite set of options to exercise whenever the demand shock reaches the optimal threshold.

$$\begin{aligned} V &= \text{Max}_{\{q_t, Q_t\}} \mathbb{E} \left\{ \int_0^{T_1^*} \pi(q_t) dt e^{-rt} + \int_{T_1^*}^{T_2^*} \pi(q_t) dt e^{-rt} + \dots - \sum_{i=1}^{\infty} C(dQ_{T_i^*}) d^{-rT_i^*} \right\} \\ \text{s.t} & \\ dR &= -qdt \\ 0 &\leq q_t \leq \bar{Q}_t \\ 0 &\leq R_t \\ \bar{Q}_{T_i^*} &= \bar{Q}_{T_{i-1}^*} + dQ_i \end{aligned} \quad (24)$$

Since the number of expansion options and the time period are infinite, the problem is stationary. Therefore, we can use apply value-matching and smooth-pasting conditions at every stopping time.

$$\begin{aligned} V(\bar{Q}_{T^*}, R, X) &= V(\bar{Q}_{T^*} + dQ, R, X) - I(dQ) \\ V_x(\bar{Q}_{T^*}, R, X) &= V_x(\bar{Q}_{T^*} + dQ, R, X) \end{aligned} \quad (25)$$

These two conditions together with the HJB equation of the stationary value function can theoretically give the optimal free boundary.

$$\begin{aligned}
& \text{Max}_{\{q \leq \bar{Q}\}} \left\{ \pi(q, X) - qV_R + \mu XV_X + \frac{1}{2} \sigma X^2 V_{XX} - rV - \lambda(\bar{Q} - q) \right\} = 0 \\
& \frac{\partial \pi}{\partial q} = U_R, \text{ if } q \leq \bar{Q} \\
& \frac{\partial \pi}{\partial q} = U_R + \lambda \text{ Otherwise} \\
& V(0, R, \bar{Q}) = 0 \\
& V(X, 0, \bar{Q}) = 0 \\
& V(X, R, \bar{Q}) = V(X, R, \bar{Q} + dQ) - C(dQ) \\
& V_X(X, R, \bar{Q}) = V_X(X, R, \bar{Q} + dQ)
\end{aligned} \tag{26}$$

3.6 Numerical Solutions

The PDEs characterizing the value function and optimal extraction rate can not be solved analytically. Therefore, I use numerical methods of dynamic programming to characterize the solution. The time is discretized and the demand process is modeled using a binomial tree. In this tree, the process can move up or down in each period and the parameters of tree are determined in a way to match to the first and second moments of approximated process.

Following other authors (Brennan and Schwartz (1985), Chan and Young (2006)) the base-line simulation model uses a constant elasticity demand function $P = Xq^\gamma$, $\gamma = -0.7$, discount rate of $r = 0.1$, standard deviation of $\sigma = 0.3$ and time steps of one year.

The important question is to find the optimal stopping time or the demand level which triggers further investment. Figure 3 presents the set of optimal trigger points for combinations of demand shocks (X) and remaining reserves (R). The area to the upper right of plotted boundary is the exercise region and the area to the left is the waiting region. The plot suggests that there is a critical level of remaining reserves, below which the capacity option will not be exercised irrespective of demand shocks. This is because the optimal production rate is always below current capacity and therefore the value of extra unit of capacity is zero.

To study the dynamic path of capacity and price, I use a set of parameters which are close to the reality of copper industry. The total reservers are 150 unit (reserve base of Chile), $\sigma=0.3$, $r = 0.1$ and the cost of one unit of capacity

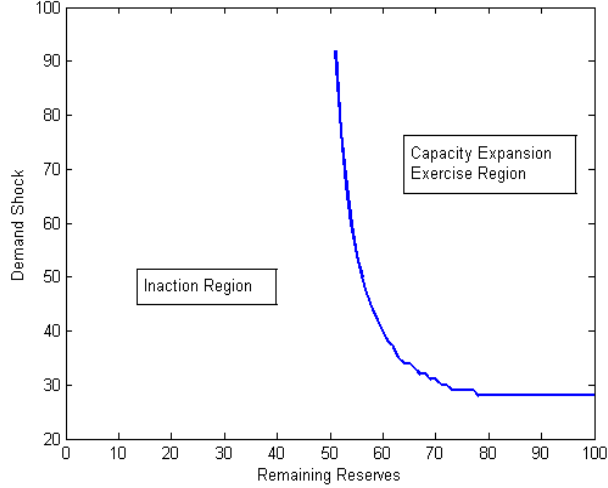


Figure 3: Optimal Boundary of Option Exercise Region

$C(dQ) = 10$. Moreover, Monte-Carlo simulation with 10000 iterations is used to generated expected values of variables of interest. Figure 4 depicts the expected capacity path over a 25 years period.

3.7 Forward Prices

At each moment, there are two possibilities for the next period. Either the capacity expansion option is exercised and the production rate jumps up or capacity remains as before. In the first case, the price drops significantly because of new supply (on contrast to traditional Hotelling model) and in the later, the price is purely driven by the evolution of demand.

Figure 5 shows two paths of forward prices for low and high capacity investment costs. The plot suggests that contrary to Hotelling’s model, the forward price path can be decreasing at the early periods since there is a sequence of capacity buildings. When the remaining reserves is low enough, capacity expansion stops and the price follows Hotelling rule.

Comparing two plots also give intuition on the effect of investment costs. Low capacity cost producers invest more in the early periods and extract most of their reserves while the high-cost producers build less capacity but supply more in the later periods.

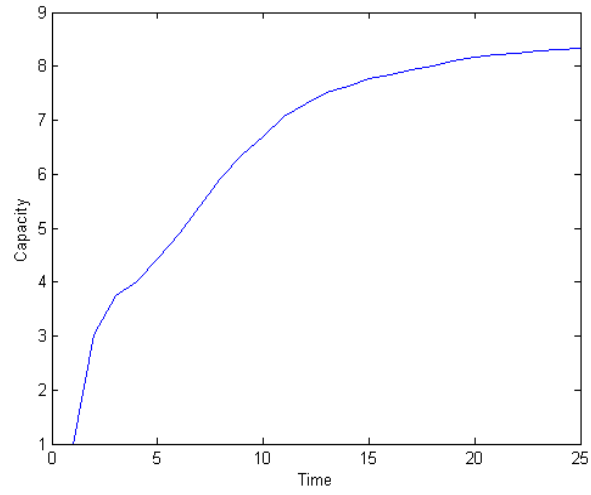


Figure 4: Optimal Capacity Path

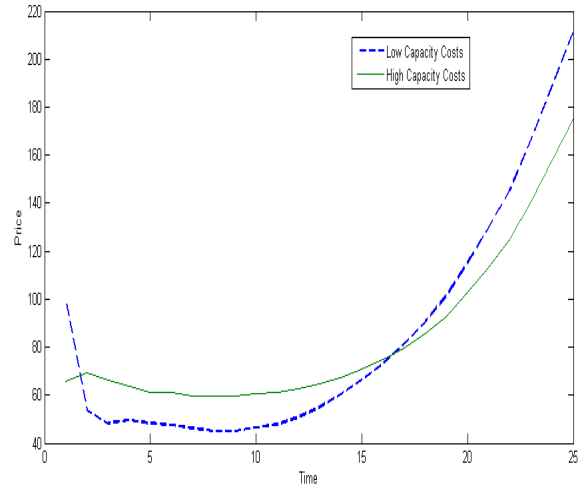


Figure 5: Forward Price Path

3.8 Conclusion

This paper has examined the problem of exhaustible resource extraction with capacity expansion options. Numerical solutions show that the value of expansion option goes to zero in finite time. Moreover, there is a lower bound on the level of reserve to exercise the option. Forward price curves suggest that a higher investment cost leads into more flatter forward price curve.

Several aspects can be added to the current model to make it more realistic. Uncertainties regarding production costs and reserve level are important in this regard. The dependency of production costs on the reserve level and the existence of learning effect in marginal cost will affect optimal stopping time. Finally, an important contribution would be to solve the problem for duopoly case. This requires solving a differential game and in this case the solution is not trivial to obtain.

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