Investment Policies under Semivariance

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Abstract

Downside deviation, semivariance, or the second lower partial moment are different names for the same risk measure, proposed in the literature for capturing the downside risk of investment decisions. This paper analyzes multiperiod decision making under such a risk measure, and finds that a V-shaped decision rule in wealth is prevalent for the most common assumptions on the distribution of uncertainty. Given that a V-shaped rule is acceptable nor desirable in practice, the use of semivariance needs a reconsideration in terms of its plausibility for guiding actual decision making in investment and elsewhere.

Keywords: semivariance, lower partial moment, downside deviation, decision making under risk, investment

JEL-Classification: D81, G11, G23.

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1 Introduction

A central issue in decision making under risk is the appropriate choice of utility function. In many instances, a mean-risk modeling of preferences is used, where risk is represented by the second lower partial moment of the distribution of outcomes. The current literature contains both theoretical and intuitive arguments for using this measure. As a result, LPM_2 , or, as it is also called, the semivariance, is used in models ranging from investing and risk management to problems in operations research.

To focus our ideas, consider the formulation of LPM_2 as a measure ρ on the distribution F:

$$\rho(F) = \int_{-\infty}^{t} (r - t)^2 dF, \tag{1}$$

where F is the outcome distribution of a risky prospect and t a target outcome level. The formulation in (1) has its roots in the earliest work on portfolio theory, and is used in academic work as well in practice by pension funds and banks. Arguments for its use are twofold. First, it captures an intuitive notion of risk by measuring downside deviations relative to a target. It preferes small losses over an occasional big loss. Second, it can be used to select efficient portfolios based on the criterion of second-order stochastic dominance. I.e., for any investor with a well-defined utility function, an LPM_2 -efficient portfolio is efficient.

In the literature, the above considerations have lead to selected use of the risk measure, especially in the area of investment management, although other examples from various fields in economics are provided below. We will argue that the use of LPM_2 in the context of sequential decisions under risk has some very undesirable properties in terms of the implied policy rules, that have not been noticed before. The resulting policy rules have a V-shaped relation with wealth, which means that risk is increasing in wealth above the target, and decreasing in wealth below the target. Compared to dynamic risk-taking under HARA preferences, this is highly undesirable. First, for below-target wealth the mean-LPM_2 formulation mimics a strategy that keeps the probability of ending at the target constant, at the cost of increasing risk at lower wealth levels. Second, risk is increased to undesirable levels at high wealth. The latter effect is only slightly affected by concavifying the above-target utility function.

Introducing the mean-LPM_2 criterion for decision making, Fishburn (1977) motivated a great deal of practical use of the semivariance. This paper argues that LPM_2 is not fit for decision making under uncertainty. Despite the intuitive attractiveness of "downside " risk measures, the perverse effects on policy rules cast a shadow over any practical use outside of (static) asset pricing tests.

The paper proceeds as follows. Section 2 discusses the roots of LPM_2 in the literature. Section 3 then introduces a multiperiod mean-LPM_2 model with its solution. Section 4 shows the robustness of the results to different distributions for the risky asset return. Section 5 discusses the implications of the V-shaped investment policy while Section 6 concludes.

2 Semivariance in the literature

The arguments for measuring risk using semivariance go back 50 years to the work of Markowitz (1959). As discusses in Fishburn (1977), the two attractive features of semivariance is that it is connected to second degree stochastic dominance, and that it corresponds to preferences of investment managers. We discuss each feature in turn.

First shown by Bawa (1975), the semivariance is a sufficient criterion for second-order and third-order stochastic dominance, abbreviated as SSD and TSD, respectively. Stochastic dominance is concerned with selection rules for uncertain prospects. The usefulness of SSD lies in its generality. For example, checking the SSD relation is sufficient to determine the relative attractiveness of any risky prospect, given that the utility function is monotonically increasing and concave. For any order of stochastic dominance (first, second, third, etc.), there is a corresponding class of utility functions U_1, U_2, U_3, etc., for which stochastic dominance implies optimal selection rules. The actual use of SD in economics is largely confined to decision theoretical considerations, since most selection rules are in the field of portfolio selection, where all linear combinations of securities are available prospects and comparing them becomes unwieldy. Moreover, stochastic dominance selection rules apply only when two distributions have the same mean. If that is not the case, an efficient "frontier" of alternatives results, like in Post (2003), who develops a test for the SSD efficiency of stock portfolios.

Besides the theoretical basis for semivariance, there is a widespread conviction that semivariance represents actual preferences for risk of investors. The arguments are in terms of "as reported by investment managers" (Fishburn (1977), from expirements with individual investors (Veld and Veld-Merkoulova (2008)) whose risk preferences are best modeled using semivariance, or by intuitive reasoning (Sortino and Van der Meer (1991)). An added justification for semivariance as given by Sortino and Van der Meer (1991) is that, because of the convexity in the loss domain, it leads to decisions that prefer several small losses over an occasional big loss. Price et al. (1982) argue that, compared to the variance, semivariance is better suited for return distributions with negative skewness.

The theoretical and practical arguments above leads many academics to propose the use of semivariance in an optimization setting of the form

$$\max_{x} \mathbb{E}[W] - \lambda \mathbb{E}[LPM_2(W)], \tag{2}$$

where W is wealth, and λ the level of risk aversion.

The above formulation is suggested by Markowitz (1959), Fishburn (1977) and Sortino and Van der Meer (1991) as being a natural expression of downside-risk preferences in the risk-return domain. Applications in the literature are in real estate (Coleman and Mansour (2005)), the newsvendor problem(Perakis and Roels (2008)), hedge fund investing (Adam et al. (2008)), Asset/Liability Management(Boender (1997), Consigli and Dempster (1998), Kouwenberg (2001), Gondzio and Kouwenberg (2001)), and even the hospitality industry (Madanoglu et al. (2008)).

The use of LPM_2 for constructing efficient portfolios gives little consideration to the choice of the actual portfolio, other than saying "the choice of portfolio on the efficient frontier depends on the risk preferences of the investor". However, taking the mean return and LPM_2 as criteria for efficiency implies that final utility should be captured by mean and LPM_2 . In terms of Fishburn (1977), the preference model presumes preferences that depend entirely on the utility of $\mu(F)$ and $\rho(F)$, where F is the risky prospect. A natural expression of these preferences, as suggested by Fishburn (1977) boils down to the mean-LPM_2 formulation given as in (2) above.

Following the theoretical arguments outlined above, downside deviation has become popular in studies of asset management or asset-liability management. The explicit penalty on downside risk fits with the notion of risk of institutional investors like pension funds, for whom a reference-dependent objective fits naturally with the situation of a given amount of liabilities that need to be covered. An extra benefit of LPM_2 is then that it gives a larger penalty to one sizeable loss than two small losses, which intuitively fits with preferences for risk for such institutions.

3 A Multi-period Mean-Downside Deviation model

Initial wealth[1] at time t is A_t. An investor can invest X_t in a risky project, fund, or asset class that has an uncertain return R_{t+1} over the next period. The remaining assets are invested in a safe asset that gives a fixed return with certainty. Without loss of generality we assume the

[1] An equivalent formulation exists in terms of returns.

simple return on the safe asset is 0%. The investor faces the following optimization problem:

$$\max_{X_0,\ldots,X_{T-1}} \mathbb{E}[A_T] - \lambda \cdot \mathbb{E}[(H - A_T)^+]^2, \tag{3}$$

where A_T represents the assets at the end of period T and H is a fixed reference point. With investment decisions X_t in each period $t = 0, \ldots, T-1$, (3) is a multi-period dynamic optimization problem. The second term in (3) is the semivariance of assets A_T below H, as in the definition of semivariance above. See equation (1), with r replaced by A_T and t by H. The model is the multiperiod equivalent of the formulation in (2), with decisions $X_0, \ldots, X_{T-1}$ at each time period and one target level H at the end of the final period.

In the following, we first analyze the ultimate payoffs by solving (3) numerically, assuming a normal distribution for the return on the risky asset. Then, we solve the optimal investment policy analytically and show that this results in a typical V-shaped policy rule.

3.1 Optimal Portfolio Payoffs

To obtain optimal payoffs, we solve a 5-period dynamic optimization numerically, using value function iteration, and then simulate optimal payoffs going forward. The value function iteration is obtained on a grid with cubic interpolation. The forward simulation uses 10000 simulations for the risky asset return for 5 periods. The risky asset return is drawn from a normal distribution with mean 0.06 and standard deviation of 0.18. Without loss of generality we put the riskfree rate at 0%.

We solve the 5-period problem for mean-LPM_2 preferences and for power utility. The power utility function for a payoff x is defined as

$$U(x) = \frac{1}{1-\gamma} x^{1-\gamma}, \tag{4}$$

where γ is the degree of relative risk aversion. Power utility belongs to the class of utility functions with Constant Relative Risk Aversion (CRRA), which has desirable theoretical properties such as having a positive third derivative, and implying a constant risk premium in asset market equilibrium, see Kimball (1990) and Campbell and Viceira (2002). In a dynamic setting it has a simple optimal policy rule, i.e., a fixed mix. See for example Merton and Samuelson (1992) and Ingersoll (1987). For different values of λ, Table 1 has the results for Mean-DD as well as the fixed mix that has equal mean return, measured over the 5 periods.

*** INSERT TABLE 1 HERE ***

The results in Table 1 show that given equal means, Mean-DD has between 2 and 4 times lower downside deviation than the CRRA-optimal strategy. The standard deviation, skewness, kurtosis, shortfall probability and expected loss are all higher for Mean-DD than for CRRA. This suggests that the optimal Mean-DD payoff is fine-tuned to a low LPM_2 , as intended, at the cost of a higher variance and probability of shortfall. The high skewness and kurtosis suggest a right-skewed payoff distribution, with more frequent, but smaller (in the square) shortfall below the reference point, compared to power utility. This is illustrated in the histogram of final payoffs in Figure 1.

*** INSERT FIGURE 1 HERE ***

Having found plausible outcomes for a Mean-LPM_2 decision problem, we now turn to the heart of the matter: what are the investment policies that lead to the payoffs in Figure 1?

3.2 Optimal Investment Policy

We cannot solve the multiperiod problem in (3) for an arbitrary return distribution, but it turns out that a tractable solution exists for a two-point distribution. Assume the excess return R on the risky asset is either equal to $e + s$ with probability p, or equal to $e - s$ with probability $(1 - p)$. We can interpret e as an exogenous positive drift and s as the size of a shock. We assume $s > e$ and $p > 0.5$ so that the risk premium is positive. The mean and variance of the return on the risky asset are $\mathbb{E}(R) = e + (2p-1)$ and $var(R) = 4s^2p(1-p)$.

Theorem 1 *Define a surplus variable as $S_t = A_t - H$, which is the asset position of the investor relative to the target H. The solution to the dynamic problem in* (3) *is given by*

$$X_t^* = \begin{cases} \frac{\mathbb{E}(R)}{M_2} \cdot (\frac{1}{2\lambda} - S_t) & \textit{if } S_t < S_t^m, \\ \\ \frac{B}{\lambda_t} - \frac{1}{e-s} \cdot (S_t - S_{t+1}^m) & \textit{if } S_t \geq S_t^m, \end{cases} \qquad \textit{for t=0,..., T-1,} \tag{5}$$

where

$$\begin{aligned} M_2 &= \mathbb{E}(R^2), \\ \lambda_t &= \frac{\lambda}{c^{T-t-1}}, \\ c &= \frac{-2ps}{e-s} \cdot \frac{M_2}{\sigma^2}, \\ S_t^m &= \frac{1}{2\lambda} - \frac{1}{2\lambda_t}, \qquad t = 0, \ldots, T-1, \\ S_T^m &= 0, \\ B &= \frac{\mathbb{E}(R)}{2(1-p)(e-s)^2}. \end{aligned}$$

Proof: See appendix.

To give intuition for the solution, consider that for any time t, all parameters except S_t do not depend on wealth. Given parameter values p, e, s and λ, (5) has a piecewise linear shape for X_t^* as a function of S_t. Moreover, since $\mathbb{E}(R)/M_2$ and $-1/(e+s)$ are both positive, the investment policy as a function of the surplus is V-shaped, with the kink at the surplus value S_t^m. For low surplus levels (S_t below S_t^m) the risky investment is decreasing in S_t, for high surplus levels it is increasing in S_t.

The sensitivity to parameter values is as expected: ceteris paribus, X_t^* is decreasing in risk aversion λ and the second moment of the return, and increasing in the expected return. The location of the kink in terms of the surplus, S_t^m, is decreasing in risk aversion. The evolution of λ_t, which determines the slope of the V-shape at each time t, depends on the parameter c which if fully determined by the properties of the return distribution. Figure 3 illustrates optimal risk taking as a function of the surplus, for several values of risk aversion.

*** INSERT FIGURE 2 HERE ***

A few properties of the V-shaped investment policy stand out from the figure. First, the left-hand side of the V is less steep than the right-hand side. With the surplus on the left-hand side range, the investor is more cautious is trading-off downside risk against expected return while for right-hand side (high) surpluses, there is ample room for risk taking. Second, the left-hand side is less sensitive to changes in risk aversion λ than the right-hand side. This is because for low surpluses risk aversion enters the expression for X_t^* through the constant term $1/\lambda$, while for high surpluses X_t^* is directly affected by λ through the term B/λ as well as indirectly through S_{t+1}^m.

As shown in the appendix, it turns out that for S_0 below the threshold S_0^m, the agent will end up in a shortfall situation with certainty. I.e., both up and down states lead to shortfall. On and above S_0^m, shortfall only occurs in the down state. The fact that shortfall is certain for low surpluses means that the optimal strategy is not a 'gamble to resurrect' or 'double or nothing' strategy, as the V-shaped pattern might suggest. Although risk taking increases with the extent of initial shortfall (the left side of the V-shape), shortfall remains unavoidable. At the same time, the left-hand side would look worrying for an investor who had initially adopted a mean-downside deviation objective. Figure 3 shows the investment policy for different time horizons, or put differently, in different points in time.

*** INSERT FIGURE 3 HERE ***

The figure shows that the left-hand side of the V-shaped investment policy remains fixed through time. This follows directly from the first line in (5), where for given parameters, X_t^* only depends on S_t and not on S_t^m. The right-hand side of the V moves up when the time horizon is further away, caused by the dependence on λ_t and S_{t+1}^m in the second line of (5).

4 Other probability distributions

To analyze the sensitivity of our results to the two-point distribution used, we numerically compute optimal decisions for a Normal and a student-T distribution. Figure 4 shows the results

*** INSERT FIGURE 4 HERE ***

The results in the figure show that for the Normal and the student-T distribution the V-shaped decision rules are clearly present. The left-hand side of the shapes are practically equal to that of the two-point distribution. The right-hand side differs, and is steepest for the two-point distribution, followed by the Normal and t-distribution.

5 Implications for the use of the semivariance

We started out with a model that was motivated by its widespread use in pension fund modeling (ALM), risk measurement of hedge funds and investment management. The objective function has some attractive properties, such as prefering several small losses to one big loss, having a straightforward interpretation and incorporating a benchmark. All these properties would make downside deviation an ideal candidate for incorporation of a decision framework. However, having analyzed the dynamics of the solution above, we find that to achieve the payoffs illustrated in Table 1 an investor needs to follow a V-shaped investment policy in terms of assets.

The analysis in the preceding sections shows that the V-shaped policy is optimal under mean-LPM_2 preferences. The V-shaped might be seen as an expression of the idea that a zero-risk strategy in terms of the portfolio return is not equivalent to zero-risk for an investor. If a given reference point needs to be achieved, a zero-risk strategy when assets are low will certainly not achieve the reference point. The V-shaped policy makes this intuition explicit.

However, there are a number of reasons why a V-shaped policy is problematic. First, an agency problem. If an investment manager is given a mandate to increase risk after adverse returns, in order to achieve a downside-deviation optimal risk profile, it might be quite hard to distinguish such a strategy from 'gambling to resurrect'. Second, and more importantly, we cannot imagine a regulator or supervisory board agreeing on a strategy where after bad returns, the risk is deliberately increased. Third, communicating an investment strategy with a constant changing mix to stakeholders would be difficult if not impossible.

If the above considerations are taken into account, what are we to say of the use of mean-LPM_2 in theory and practice? With the outcomes in Table 1 depending on a strategy that is infeasible in practice, the theory and applications that start from such a mean-semivariance objective are in danger of becoming irrelevant. An investment strategy needs to have payoffs that match the required risk profile, with a feasible investment policy. On top of feasibility, for institutions like pension funds and university endowments, the investment policy needs to be communicated to stakeholders, who have low appreciation for sophisticated tactical strategies. Thus, a V-shaped investment policy rules out the use of mean-LPM_2 .

6 Conclusion

Theoretical arguments and practical reasoning have prevailed in the current literature in supporting the use of semivariance in a mean-risk optimization framework. However, the optimal policy rules show that the use of semivariance in a mean-LPM_2 framework suffers from a major drawback, namely an optimal investment policy that is V-shaped. Even if investment managers would be willing to follow such rules, the normative use of mean-LPM_2 is ill-advised. Hence, the results have implications for banks and institutional investors who use downside risk measures to guide strategic investment decisions. To start with, the widespread use of downside risk measure might have the adverse effect of leading to pro-cyclical investment decisions. I.e., under mean-LPM_2 , risk taking increases drastically when assets grow fast (the right-hand side of the V-shaped). Although weakened by institutional and regulatory restrictions, the effects might be observed in practice. Second, the use of mean-LPM_2 in theory as well as practice might be reconsidered, given that the resulting policy rules cannot be followed through. This is an issue of dynamic consistency: a strategy that is optimal for a given objective only remains optimal as long as future actions are consistent with the strategy. If the means (a V-shaped strategy) do not justify the ends (an optimal mean-LPM_2 payoff profile), the whole mean-LPM_2 formulation needs a reconsideration.

It is quite possible that the empirical observations on the fact that semivariance is connected to the practice of investment managers, is in fact a manifestation of loss aversion of the kind found by Kahneman and Tversky (1979). Such preferences fit experimental risk taking behavior well, but no institutional investor would be adviced to incentivize an investment manager to be risk-seeking in the domain of losses.

Getting rid of the V-shaped decision rules by going to higher order lower partial moments is not an option: numerical results (not included here) show the same effects. Moreover, the intuitive attractiveness of the semivariance is lost. We have to conclude, therefore, that the mean-LPM_2 formulation for strategic decision making under risk, has a serious problem.

A fruitful line of future research might be aimed at examining investment policies directly, relating the payoffs to downside measures of risk. I.e., instead of starting from an objective from which (in the case of mean-LPM_2) an infeasible investment policy results, one could start from sensible policies and examining how tweaking them leads to lower downside risk.

One example is a strategy with a minimum level of risk-taking, i.e., a hockey-stick instead of a V-shape.

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Appendix: Proof of the Theorem

The proof Theorem 1 proceeds in three steps. In the first step, we derive optimal decisions for the one-period problem. In step two, we derive the value function in the optimum. Thirdly, we proof by induction that the multiperiod problem is a sequence of one-period problems which have the value function of step two, and the solution of step one.

A1 The single-period problem

Consider the problem in (3) with T=1. Define

$$\alpha = \mu - 1 = \mathbb{E}[R] - 1 = e + (2p-1)\cdot s - 1,$$

and

$$M_2 = \mathbb{E}[R^2] = p\cdot(e+s)^2 + (1-p)\cdot(e-s)^2.$$

Also, define $S_0 = A_0 - H$, the measure of initial surplus. To solve model (3), we start from the premise that there are two possible regimes for an optimum, namely (i) shortfall only after a negative return (the 'down' state), or shortfall in both states (positive or negative return). We treat each case in turn and verify for which range of S_t it is applicable.

I. Shortfall only in the down state

When shortfall occurs only in the down state, the first-order condition becomes

$$\alpha - 2\lambda(1-p)(S_0 + X_0(e-s))(e-s) = 0. \tag{A1}$$

Solving (A1) for X_0^* gives

$$X_0^* = \frac{\alpha}{2\lambda(1-p)(e-s)^2} - \frac{S_0}{e-s} \tag{A2}$$

The second order condition for a local maximum is

$$-2\lambda(1-p)(e-s)^2 < 0,$$

which is indeed the case since $0 < p < 1$.

We assumed that shortfall only occurs in the down state, which we need to verify ex post. Given an expression for X_0^* as in (A2), shortfall only occurs in the down state when $X_0^* > -S_0/(e+s)$. This boils down to

$$S_0 > S_0^m \equiv \frac{\alpha\cdot(e+s)}{4\lambda s(1-p)(e-s)}. \tag{A3}$$

Hence, the condition in equation (A3) gives us a threshold value of S_0^m, above which the expression for X_0^* in (A2) is a (local) maximum.

II. Shortfall in both up and down state

When shortfall occurs in both the up and down state, the first-order condition is given by

$$\alpha - 2\lambda\cdot(1-p)(S_0 + X_0(e-s))(e-s) - 2\lambda p(S_0 + X_0(e+s))(e+s) = 0. \tag{A4}$$

Solving (A4) for X_0^* gives

$$X_0^* = \frac{\alpha}{M_2} \cdot (\frac{1}{2\lambda} - S_0),$$

with second-order condition for a local maximum

$$-2\lambda(1-p)(e-s)^2 - 2\lambda p(e+s)^2 < 0,$$

which is satisfied as $e+s > e-s$ and $p > 0.5$. In addition, the necessary conditions for ensuring shortfall in both up and down state are

$$\begin{aligned} X_0^* &> -S_0/(e-s), \quad \text{(SF in down state)} \\ \text{and } X_0^* &< -S_0/(e+s) \quad \text{(SF in up state).} \end{aligned}$$

The right-hand side of condition (A5) must be positive to ensure $X_0^* > 0$, which implies that the relevant values of S_0 are negative. This satisfies condition (A5), which has a negative right-hand side. To find the relevant range for (negative values of) S_0, we must find the value of S_0 that just satisfies (A5). This value is given by

$$S_0^m \equiv \frac{\alpha \cdot (e+s)}{4\lambda s(1-p)(e-s)},$$

and is equal to the earlier found S_0^m. Hence, for S_0 larger than S_0^m, case I determines the optimal solution, while for S_0 smaller than S_0^m, case II determines the optimum. This proves the theorem for T=1.

A2 The value function

Starting point for deriving the value function is the optimal decision X_0^* given by

$$X_0^* = \begin{cases} \frac{\alpha}{M_2} \cdot (\frac{1}{2\lambda} - S_0) & \text{if } S_0 < S_0^m, \\ B - \frac{S_0}{e-s} & \text{if } S_0 \geq S_0^m. \end{cases},$$

where

$$B = \frac{\alpha}{2\lambda(1-p)(e-s)^2}.$$

We analyze the value function for the case $S_0 \geq S_0^m$ and $S_0 < S_0^m$ separately.

For $S_0 \geq S_0^m$ there is only shortfall in the downstate, so we have

$$\begin{aligned} V^* &= A_0 + X_0^* \cdot \alpha - \lambda(1-p)\left[H - A_0 - X_0^*(e-s)\right]^2, \\ &= A_0 + X_0^* \cdot \alpha - \lambda(1-p)\left[-B(e-s)\right]^2, \\ &= A_0 + \alpha(B - S_0/(e-s)) - \lambda(1-p)B^2(e-s)^2, \\ &= A_0 - S_0\alpha/(e-s) + B\cdot\alpha - B\cdot\alpha/2, \\ &= A_0 - S_0\frac{\alpha}{e-s} + \frac{B\alpha}{2}, \\ &= \frac{-2ps}{e-s}\cdot A_0 + \frac{\alpha}{e-s}\cdot H + \frac{B\alpha}{2}. \end{aligned}$$

which uses the fact that $\alpha = e + (2p-1)s$.

For $S_0 < S_0^m$ there is shortfall in both down and up-state.

$$\begin{aligned} V^* &= A_0 + X_0^*\alpha - \lambda p\left[H - A_0 - X_0^*(e+s)\right]^2 - \lambda(1-p)\left[H - A_0 - X_0^*(e-s)\right]^2, \\ &= A_0 + X_0^*\alpha - \lambda S_0^2 - \lambda X_0^* M_2 - 2\lambda S_0 X_0^*\alpha, \\ &= \ldots \\ &= [1 - \frac{\alpha^2}{M_2}]S_0 - \lambda[1 - \frac{\alpha^2}{M_2}]S_0^2 + \frac{\alpha^2}{4\lambda M_2}, \\ &= (1-k)S_0 - \lambda(1-k)S_0^2 + \frac{k}{4\lambda}, \end{aligned}$$

using $k = \alpha^2/M_2$. To check that V^* is continuous around S_0^m, consider the derivative of V^* with respect to S_0. We need to prove that it is the same when coming from below or above S_0^m. The partial derivative $\partial V^*/\partial S_0$ for $S_0 \uparrow S_0^m$ is given by

$$\begin{aligned} (1-k)(1-2\lambda S_0^m) &= (1-k)\left(1 - \frac{\alpha(e+s)}{2s(1-p)(e-s)}\right) \\ &= -(1-k)\frac{p(e+s)^2 + (1-p)(e-s)^2}{2s(1-p)(e-s)} \\ &= -\frac{\sigma^2}{M_2} \cdot \frac{M_2}{2s(1-p)(e-s)} \\ &= \frac{-2ps}{e-s}, \end{aligned}$$

which is the precisely the derivative of V^* with respect to S_0 for $S_0 \downarrow S_0^m$. Hence, the derivative of V^* with respect to S_0 is continuous at S_0^m.

A3 The multi-period problem

To prove the theorem we first show that the value function of the static problem can be scaled in such a way that it takes the form of a mean-downside deviation objective.

The one-period value function is given by

$$V^* = \begin{cases} (1-k)S_0 - \lambda(1-k)S_0^2 + \frac{k}{4\lambda} & \text{if } S_0 < S_0^m, \\ \frac{-2ps}{e-s}S_0 + \frac{\alpha}{2}B & \text{if } S_0 \geq S_0^m. \end{cases} \tag{A5}$$

Noting that S_0^m is a constant that only depends on the return properties and λ, we divide both lines by the constant term $-2ps/(e-s) = (1-k)(1-2\lambda S_0^m)$ to get

$$\hat{V} = \begin{cases} \frac{S_0}{1-2\lambda S_0^m} - \frac{\lambda}{1-2\lambda S_0^m}S_0^2 + c & \text{if } S_0 < S_0^m, \\ S_0 + d & \text{if } S_0 \geq S_0^m, \end{cases}$$

where c and d are normalizing constants. Now, we have that

$$\begin{aligned} \frac{S_0}{1-2\lambda S_0^m} - \frac{\lambda}{1-2\lambda S_0^m}S_0^2 &= S_0 + \frac{\lambda}{1-2\lambda S_0^m} \cdot (2S^m S_0 - S_0^2), \\ &= S_0 - \frac{\lambda}{1-2\lambda S_0^m} \cdot (S_0^m - S_0)^2 + \frac{\lambda (S_0^m)^2}{1-2\lambda S_0^m}. \end{aligned} \tag{A6}$$

Since the third term of (A6) does not depend on S_0, an equivalent function $\hat{V}$ is

$$\hat{V} = S_0 - \frac{\lambda}{1 - 2\lambda S_0^m} \cdot \left((-S_0 + S_0^m)^+\right)^2.$$

To simplify, define

$$c = \frac{-2ps}{e - s} \cdot \frac{M_2}{\sigma^2},$$

so that we can use

$$\begin{aligned} S_0^m &= \frac{1}{2\lambda}(1 - c), \\ \lambda_0 &= \frac{\lambda}{1 - 2\lambda S_0^m} = \frac{\lambda}{c}, \end{aligned}$$

to write

$$\hat{V} = S_0 - \lambda_0 \cdot \left((-S_0 + S_0^m)^+\right)^2.$$

(By induction)

So, for a T-period model, the value function at time $T - 1$ is given by

$$V_{T-1} = S_{T-1} - \lambda_{T-1} \cdot \mathbb{E}[(-S_{T-1} + S_{T-1}^m)^+]^2,$$

where

$$\begin{aligned} S_{T-1}^m &= \frac{1}{2\lambda}(1 - c), \\ \lambda_{T-1} &= \frac{\lambda}{c}. \end{aligned}$$

Now suppose that for some $t < T$,

$$V_t \sim S_t - \lambda_t \cdot \mathbb{E}[(-S_t + S_t^m)^+]^2, \tag{A7}$$

where $S_t = A_t - H$, $A_t = A_{t-1} + X_{t-1}R$, $\lambda_t = \lambda / c^{T-t-1}$, and

$$S_t^m = \frac{1}{2\lambda} - \frac{1}{2\lambda_t}. \tag{A8}$$

(This holds surely for $t = T - 1$.)

Then, conform the solution for a one-period model derived in step one for an objective as (A7), the optimal decision at time $t - 1$ is

$$X_{t-1}^* = \begin{cases} \frac{\alpha}{M_2} \cdot \left(\frac{1}{2\lambda_t} - (S_{t-1} - S_t^m)\right) & \text{if } S_{t-1} < S_{t-1}^m, \\ B - \frac{S_m}{e-s} & \text{if } S_{t-1} \geq S_{t-1}^m, \end{cases} \tag{A9}$$

where

$$S_{t-1}^m = \frac{1}{2\lambda_t} \cdot (1 - c) + S_t^m.$$

Given the definition of S_t^m in (A8), the first line in (A9) can be simplified to $\frac{\alpha}{M} \cdot (\frac{1}{2\lambda} - S^t)$. With X^* and S_{t-1}^m above, we get a value function at $t-1$ of

$$V_{t-1} \sim S_{t-1} - \lambda_{t-1} \cdot \mathbb{E}[(-S_{t-1} + S_{t-1}^m)^+]^2,$$

where $\lambda_{t-1} = \lambda_t / c$ and

$$\begin{aligned} S_{t-1}^m &= \frac{1}{2\lambda_{t-1}}(1-c) + S_t^m \\ &= \frac{1}{2\lambda} - \frac{1}{2\lambda_{t-1}}. \end{aligned}$$

Hence the expressions for S_t^m, λ_t and X_t^* hold for all $t < T$. ∎

Appendix B: Tables and Figures

Table 1: Ultimate payoffs for Mean-DD and CRRA utility

For a 5-period model, this table reports descriptive statistics of the simulation results at time 5 for a mean-downside deviation optimal strategy and a power utility optimal strategy, respectively. The model is solved numerically using value function iteration assuming normally distributed returns (μ = 0.05, σ = 0.18) and a riskfree rate of 0%. Starting wealth is 1, which is also the reference point at time 5 for measuring downside deviation. 10,000 simulations are used. α%-VAR measures the $(1 - \alpha)$ quantile of the wealth distribution, PD is the probability of shortfall below the reference point, EL is expected loss, DD is downside deviation.

	Mean-DD	CRRA	Mean-DD	CRRA	Mean-DD	CRRA
Risk aversion	20	3.4	30	5.0	50	7.8
Initial fraction	0.526	0.550	0.344	0.360	0.198	0.230
Mean	1.138	1.140	1.091	1.090	1.054	1.057
Median	1.024	1.128	1.018	1.086	1.014	1.055
St.dev	0.316	0.219	0.208	0.138	0.122	0.086
Skewness	2.697	0.402	2.702	0.252	2.677	0.148
Kurtosis	11.904	0.312	11.940	0.142	11.855	0.069
Min	0.342	0.476	0.574	0.627	0.753	0.748
Max	4.675	2.231	3.430	1.723	2.424	1.429
99%-VAR	0.759	0.694	0.840	0.794	0.907	0.866
99.9%-VAR	0.554	0.567	0.707	0.698	0.831	0.799
PD	0.429	0.274	0.427	0.265	0.413	0.261
EL	0.068	0.116	0.045	0.076	0.028	0.049
DD	0.009	0.021	0.004	0.009	0.001	0.004

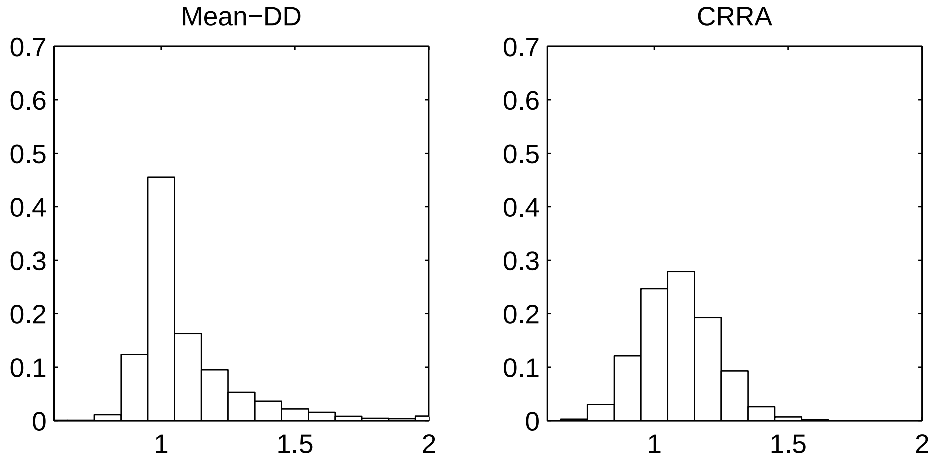

Figure 1: Payoff to Mean-DD and CRRA utility for $\lambda = 30$

For a 5-period model this figure shows the payoffs at time 5 for an investor with mean-downside deviation ($\lambda = 30$) and Constant Relative Risk Aversion preferences ($\gamma = 5.0$), respectively. The model is solved numerically, with normally distributed returns (μ = 0.05, σ = 0.18) and a riskfree rate of 0%. Starting wealth is 1, which is also the reference point for measuring downside deviation.

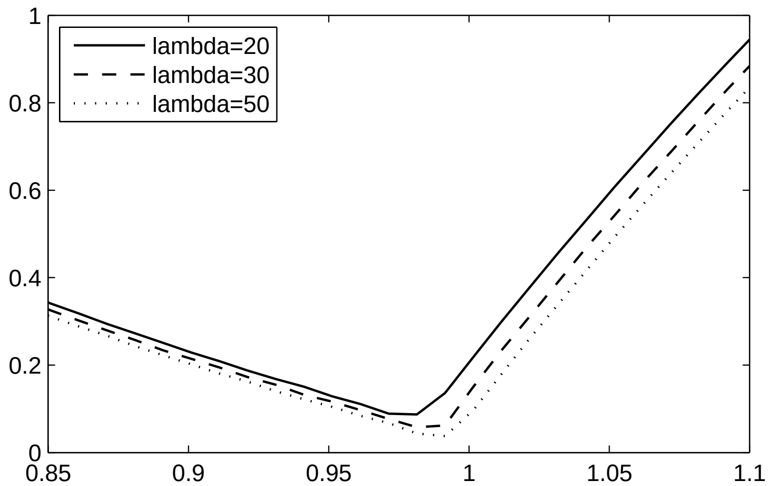

Figure 2: Static solution

This figure shows the optimal amount invested in the risky asset for the one-period model, for different degrees of risk aversion. The x-axis gives the level of the assets A_0 at time 0, with the threshold level H set at 1. Parameter values are $\lambda = 30, p = 0.5$, $e = 0.06$, $s = 0.18$.

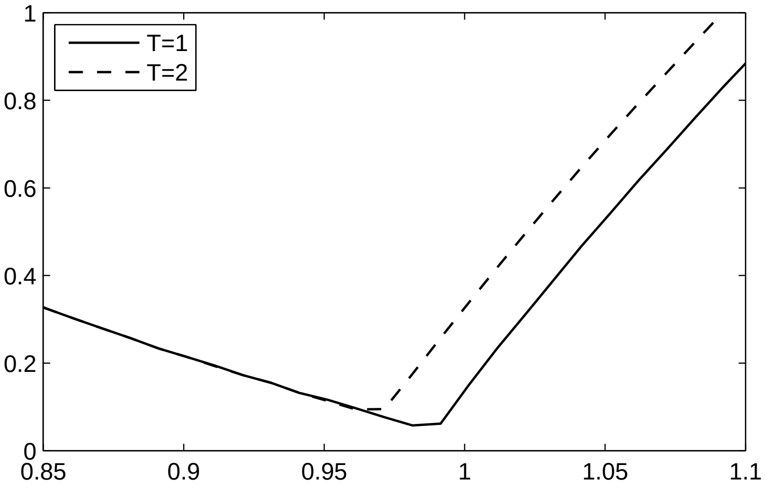

Figure 3: Dynamic solution

This figure shows the optimal amount invested in the risky asset for the one- and two-period model. The x-axis gives the level of the assets A_t at time t, with the threshold level H set at 1. Parameter values are $\lambda = 30$, $p = 0.5$, $e = 0.06$, $s = 0.18$.

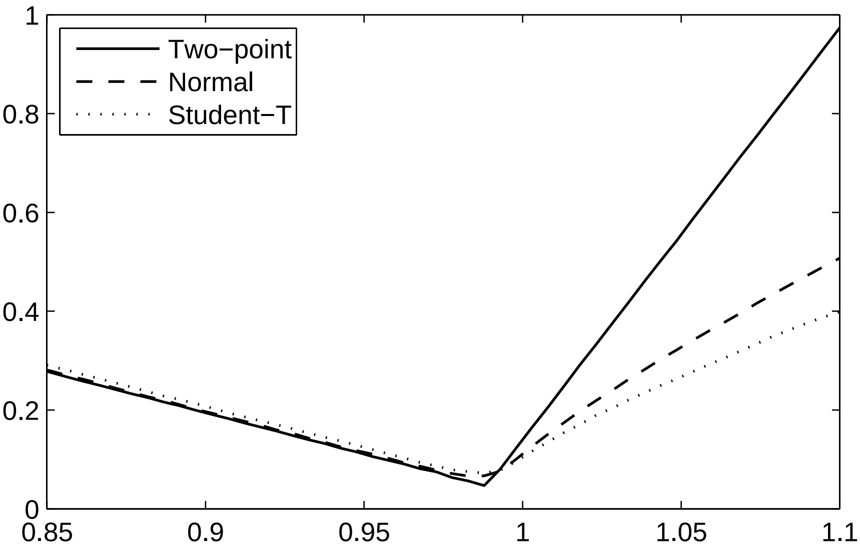

Figure 4: Comparison with the Normal and T-distribution

This figure shows the optimal amount invested for three probability distributions for the return on the risky asset. The x-axis gives the level of the assets A_t at time t, with the threshold level H set at 100. Parameter values are $\lambda = 0.5$, $p = 0.5$, $e = 0.06$, $s = 0.18$. For the Normal distribution, $\alpha = 0.06$ and $\sigma = 0.18$ is used. For the t-distributed returns, 30 random returns from the T-distribution with n=2.5 are drawn, and scaled to have a mean and standard deviation of 0.06 and 0.18, respectively.