Optimal Deposit Pricing: There is no 'One-Size-Fits-All' Valuation Approach*

Andreas Blöchlinger†
Zürcher Kantonalbank

Basile Maire‡
University of Zurich,
Zürcher Kantonalbank

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†**Correspondence Information:** Andreas Blöchlinger, Zürcher Kantonalbank, Josefstrasse 222, CH-8010 Zurich, Switzerland; tel: +41 44 292 45 80, `mailto:andreas.bloechlinger@zkb.ch`

‡**Correspondence Information:** Basile Maire, Zürcher Kantonalbank, Josefstrasse 222, CH-8010 Zurich, Switzerland; tel: +41 44 292 45 79, `mailto:basile.maire@zkb.ch`

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Abstract

We derive profit-optimal deposit rates in oligopolistic banking markets, considering the depositors' supply sensitivities to deposit rates, the competing banks' deposit rates and brand powers. The resulting Nash equilibria agree with the empirical finance literature on deposit pricing: size matters. Our approach may thus serve as a serious contender for models used in Industrial Organization theory. Extending the framework to a multi-period economy shows the relation to the common valuation models in the banking literature. We show that there is no 'one-size-fits-all' approach, that is, the valuation for deposit accounts must be bank-specific.

EFM Classification Codes: 510, 450

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INTRODUCTION

In the current literature, modeling deposit rates and deposit volumes follows two main objectives. In Industrial Organization theory, the objective is to provide explanations for observed institutional pricing behaviors, as e.g. in Chiappori, Perez-Castrillo, and Verdier (1995). In the banking literature, the objective is to determine the risk and value of deposit accounts to the bank, examples are Wilson (1994), Jarrow and van Deventer (1998), Frauendorfer and Schürle (2003), and Kalkbrener and Willing (2004). Surprisingly, these two streams have remained independent. Apparently, the bank's product managers set deposit rates considering local market conditions, as described in Industrial Organization theory. In contrast, deposit account valuation is accomplished as described in the banking literature, without considering local market conditions. Based on this valuation, the Asset-Liability management department of a bank manages interest rate risk and liquidity risk. Therefore, if the two streams are not consistent, the bank's risk taking decisions rest upon wrong numbers, exposing the bank to the risk of loss or, in extreme cases, threatening the survival of the institution. Hence, it is important to bring the two streams together – we provide insights into the link between them.

We consider non-maturing deposit accounts with managed rates. That is, the coupon rates can be adjusted by the bank at any time. The depositor can add or subtract balances without notice period and there is no contractual maturity of the account. According to the definition of The Federal Reserve Board (2009), this type of account best matches a NOW (negotiable order of withdrawal) account, but may also apply to other types of accounts, depending on the construction.

Depositors are attracted by the different banks in a deposit market by the coupon rate and by other bank-specific measures – the bank's *brand power* – such as the location of the branches or the number of relationship managers. A large bank has a greater brand power than a small bank, all else being equal. Depositors trade off the benefit of the coupon rate against brand power, depending on their *price sensitivity*. If the price sensitivity is high, depositors give more weight to the price than to the non-pecuniary return obtained from choosing a bank with a strong brand. Based on the depositor's supply behavior, we derive the profit-optimal coupon rate for a bank. In the resulting Nash price equilibrium the law of one price does not apply. That is, banks having a greater brand power offer lower deposit rates than smaller banks on the otherwise identical deposit product. The bank's profits stay strictly positive, if depositors are not fully price sensitive. If they are, the equilibrium coincides with that of a pure Bertrand economy. Our model provides a plausible theoretical basis for observed pricing patterns: Rosen (2007) observes that large banks typically offer lower deposit rates, and finds that the more similar the banks are in size, the higher are deposit rates. Martín-Oliver, Salas-Fumás, and Saurina (2007) test the law of one price for the Spanish deposit market and reject it. Neuberger and Zimmerman (1990) attribute lower rates to a higher market concentration.

We extend our analysis to a multi-period economy and find that profit-optimal coupon rates do not depend on past wholesale market rates. In the multi-period economy, we approximate our model, which yields two contributions. First, the approximation resembles those models in the banking literature that consider the risk and value of deposit accounts, hence we have an instrument to link the models in Industrial Organization theory with the corresponding models in the banking literature. Second, we find that valuation models must be specific to a bank, the explanation is as follows. The models in the banking literature regress deposit rates and deposit volumes on wholesale market rates, without considering the competing banks. The approximation of our model is a function of wholesale market rates in which the coefficients depend on the competing banks. By comparison of coefficients, the models in the banking literature implicitly consider the competitors through the estimation of the regression coefficients. The coefficients differ among banks. In consequence, models must be specific to a bank, i.e., there is no 'one-size-fits-all' approach to valuing deposit accounts. This contrasts the current view of some regulators, see the Office of Thrift Supervision (2001), who provide a specific model with fixed parameters.

We develop our model in the multinomial logit framework, see McFadden (1980), Manski and McFadden (1983), or Anderson and De Palma (1992). This model is often applied by marketing researchers, see McFadden (1986). One reason for its success in the marketing literature may be that the utility that enters the model provides a simple balancing interpretation for product attributes and price. In our context, it is thus well suited for product managers who need to trade off prices and marketing expenditures for deposit accounts. We provide an explicit price reaction function, i.e., the profit-optimal deposit rate, as a function of the bank's brand powers, the depositors' price sensitivity and the competing bank's deposit rates.

The paper is structured as follows. In the first Section, we give a brief literature review. In Section 2, we derive the bank's market share and the profit-optimal price reaction function. In

Section 3, we prove the existence and uniqueness of a Nash price equilibrium and derive equilibrium prices. In Section 4, we use our model to explain observed pricing patterns. In Section 5, we extend our analysis to multiple periods and develop approximations to our model. In Section 6, we compare the profit of different price strategists in a simulation study. As benchmarks we use two deposit rate models from the banking literature, representing today's approach to deposit pricing. In a second simulation study, we compute the error of present value calculations, arising when we use our approximated model that only considers wholesale market rates. The last Section concludes.

1. LITERATURE

As shown in several empirical studies, see e.g. Hannan and Berger (1991), or Gambacorta (2008), deposit rates differ from market rates on comparable securities. Industrial Organization theory provides explanations for these price deviations. Two important models in this field are the Monti-Klein model and the circular city model. The Monti (1972)-Klein (1971) model derives several empirical predictions, see e.g. Freixas and Rochet (1999). However, in this model, banks behave as quantity competitors, and hence it meets the same criticism as the Cournot model. As originally pointed out by Bertrand, prices may be the better strategic variable than quantity. The circular city model of Salop (1979) implements this, in which banks are located on a circle and compete on price. Transportation costs on the circle generate price dispersions. This model explains observed deposit pricing behavior, see Chiappori, Perez-Castrillo, and Verdier (1995), Pita Barros (1999), and Park and Pennacchi (2008).

We develop our theory in the multinomial logit framework, which has not been applied to deposit pricing so far. We have three motivations to do so. First, Anderson, De Palma, and Thisse (1989) show that, if the discrete choice approach such as the multinomial logit, is derived from an address approach such as Salop (1979), then the dimension of the characteristics space over which depositor's preferences are defined must be at least $K-1$, where K is the number of banks offering deposit accounts. In this sense, the discrete choice considered in this paper is very different from the one-dimensional circle model considered in Chiappori, Perez-Castrillo, and Verdier (1995). Not surprisingly, this affects the Nash price equilibrium. Second, in the circular city model all depositors must invest, whereas we introduce a non-purchase option. Third, as emphasized by Chiappori, Perez-Castrillo, and Verdier (1995), a multi-branch setting in the Salop

(1979) approach is enormously more complex than a single branch setting, whereas we have a brand power variable, which can be continuously adjusted to account for the number and location of branches.

2. THE MARKET STRUCTURE

We consider an economy with $K > 1$ different banks and a large number of depositors. The product that is marketed is the deposit account. The depositor has the right to withdraw at any time without notice period.

The economy consists of a treasury security market and a deposit market with "managed" rates (related but not indexed to the treasury security rates). The market volume to be invested by the depositors is N. We allow for an outside good, i.e., instead of investing in the deposit market, the depositor invests in the outside good. This represents a purchase outside the deposit market, for instance the depositor decides to invest in the stock market.

We use a market segmentation along the line of Jarrow and van Deventer (1998): There are two types of market participants, (i) banks and (ii) depositors. Banks can offer deposit accounts, depositors cannot. There are significant entry barriers associated with the deposit market.

Bank j pays a coupon rate c_j on its nominal deposit amount. The rates are published and there is no collusion. The return of a dollar invested for one period in the treasury security market is i. We refer to the treasury security market as the *wholesale market*.

2.1. The Deposit Volume Distribution

We assume the depositors' heterogeneity is not observable and use a stochastic utility approach. The depositor chooses the bank offering him the greatest utility. The utility for depositor d choosing bank j is

$$U_{jd} = u_j + \epsilon_{jd}, \tag{2.1}$$

where ϵ_{jd} is a random variable and u_j is the observable part of the utility. The depositor is attracted by the coupon rate, but may also be attracted by other bank-specific factors, such as the number of branches, or the brand value. In this sense, the utility derived from choosing to deposit at bank j depends on the coupon rate paid by the bank, c_j, and on other observable characteristics, x_j:

$$u_j = x_j'\gamma + c_j\beta. \tag{2.2}$$

The coupon rate c_j and the exogenous characteristics x_j are specific to a given bank. The scalar β determines the influence of the coupon rate c_j on the depositor's utility, we refer to it as the *price sensitivity*. The vector γ weights the components in x_j. The depositor d chooses bank j out of the K banks with probability $p_{jd} = \text{Prob}(U_{jd} = \max_{k=1,\ldots,K} U_{kd})$. We assume the ϵ_{jd} are identically independently Gumbel distributed with mean zero and scale-one[1]. Then the resulting choice probabilities are given by the multinomial logit model (Luce and Suppes 1965)

$$p_{jd} = \frac{e^{(x_j'\gamma + c_j\beta)}}{\sum_{k=1}^{K} e^{(x_k'\gamma + c_k\beta)}}, \quad \text{for all } d. \tag{2.3}$$

For simplicity we omit the arguments of the probability function. We summarize the observable characteristics besides the coupon that contribute to the utility and define the *brand power variable* as

$$g_j = e^{x_j'\gamma}. \tag{2.4}$$

By definition, the brand power variable g_j is positive. The variable g_j reflects the non-pecuniary part of the returns obtained by the depositor. Since there is a large number of depositors, each choosing bank j with the probability given by equation (2.3), the *market share* for bank j becomes

$$p_j = \frac{g_j e^{\beta c_j}}{\sum_{k=1}^{K} g_k e^{\beta c_k}}, \tag{2.5}$$

a function of the brand power variables, g_j, the coupon rates, c_j, and the depositors' price sensitivity β. This measure is known as the Boltzmann distribution or the Gibbs measure.

The utility level associated with the non-purchase option, representing outside investment opportunities, is denoted by $u_0 = x_0'\gamma + c_0\beta$. The case where the whole market volume is invested in the deposit accounts, corresponds to the case where $u_0 \to -\infty$. The coupon rate c_0 represents an opportunity rate of investment. As an example, consider the case where the opportunity rate c_0 is set equal to the wholesale market rate minus a margin. Then, the depositors' utility of investing in the outside good, is greater in a high yield environment than in a low yield environment.

2.2. *Profit Maximizing Coupon*

We could incorporate the *location* of the bank's branches in the brand power variable g_i, by assuming a topology for the market, including a distance measure and transportation costs. Accordingly, banks compete in location. De Palma, Ginsburgh, Papageorgiou, and Thisse (1985) consider firms that compete in location using the multinomial logit demand system. Due to mobility barriers, prices are the most important instrument of competition between banks. In this paper, we thus investigate the case where banks compete on price for given – but potentially different – brand power variables.

The deposit volume of bank j is the product of market share, p_j, times market volume, N. The profit, π_j, of bank j is the product of deposit volume and margin between wholesale and retail market,

$$\pi_j = N p_j \left[i - c_j \right], \tag{2.6}$$

where c_j is bank j's coupon rate in the retail market, and i is the wholesale market rate. The market share p_j of bank j is given by equation (2.5). Maximizing the profit (2.6) results in the first-order condition

$$0 = \frac{\partial \pi_j}{\partial c_j} = N \frac{\partial p_j}{\partial c_j} \left[i - c_j \right] - N p_j. \tag{2.7}$$

Solving the first-order condition for the coupon rate yields

$$c_j^* = i - \frac{1}{\beta} \left[1 + W \left(\frac{g_j e^{\beta i - 1}}{\sum_{k \neq j} g_k e^{c_k \beta}} \right) \right], \tag{2.8}$$

where $W(.)$ is the Lambert W function, see Corless, Gonnet, Hare, Jeffrey, and Knuth (1996), which is the inverse function of $f(z) = z e^z$ with $z \in \mathbb{C}$. All proofs are in the Appendix. We call banks that set their prices according to this equation α-strategists.

If depositors are *perfectly price insensitive*, $\beta \to 0$, the market share p_j does not depend on the price anymore and hence the profit $\pi_j = N p_j \left[i - c_j \right]$ is maximized, when we set c_j to negative infinity. If the depositors' price sensitivity is low, they pay to hold their money at the bank.

If depositors are *perfectly price sensitive*, $\beta \to \infty$, they choose the bank paying the maximal coupon rate. If there is a bank paying above the wholesale market rate i, the α-strategist will not receive any volume anymore, since he does not offer coupon rates above i. If all banks pay below i, an α-strategist sets the coupon rate marginally above the maximal coupon rate in the market, to receive the whole market volume. In the next Section, we discuss the fixed point problem that arises in the presence of several α-strategists.

3. EQUILIBRIA

There is a unique Nash price equilibrium, i.e., every bank reaches a price that is optimal given the other bank's prices. Anderson, De Palma, and Thisse (2001) prove in the Appendix 7.10.1 the existence and uniqueness of profit-optimal prices for the multinomial logit demand system with different qualities. This corresponds to our model in a non-banking setting where the price negatively contributes to the utility. However, in the Appendix we prove the existence and uniqueness of equilibrium coupons in our setting, in which not all strategists' prices depend on the other banks prices and in the presence of an outside good.

3.1. Existence And Convergence

The number of α-strategists in the market with K banks is $M \leq K$. The α-strategists set prices, c_j^*, according to equation (2.8). The prices of the $K - M$ remaining strategists do not depend on the competing bank's prices. The opportunity rate of the outside good does not depend on the bank's prices either.

Proposition 3.1 (Unique Nash Equilibrium):
There exists a unique Nash equilibrium, i.e., the system of equations c_j^* has a unique solution for every c_j^*, $j = 1...M$.

The proof given in the Appendix also proves the convergence, that is, if banks simultaneously set prices, they converge to the unique Nash equilibrium.

3.2. Markets with α-Strategists

We now consider the case, where all banks in the market are α-strategists and allow for an outside good. The following Proposition shows that banks pay coupon rates below market rates if depositors are not perfectly price sensitive and that coupon rates are equal for two banks having the same brand power.

Proposition 3.2: *The profit-optimal coupon in a market with K α-strategists, and possibly an outside good, is a function of the wholesale market rate minus a margin. The margin depends on the depositor's price sensitivity β and the brand power variables $g = g_0, g_1, ..., g_K$:*

$$c_j^* = i - \frac{1}{\beta} m_j \tag{3.1}$$

with m_j the solution to the fixed point problem

$$m_j = 1 + W\left(\frac{g_j}{g_0 e^{\beta c_0} e^{1-\beta i} + \sum_{k\neq j} g_k e^{1-m_k}}\right), \; j = 1...K, \tag{3.2}$$

where $\frac{1}{\beta} m_j$ is the margin of bank j.

In the absence of an outside good ($g_0 = 0$), the m_j's do not depend on price sensitivity β and therefore, with perfect price sensitivity, $\beta \to \infty$, banks earn no margin income. This still holds in the presence of an outside good, as we explain now. It is easy to see for $c_0 > i$, here the perfectly price sensitive depositors all purchase the outside good and banks thus earn no margin income. If the wholesale market rate i exceeds the opportunity rate c_0, an increasing price sensitivity decreases the summand containing g_0 in (3.2) and hence, $\beta \to \infty$ corresponds to the situation where no outside good is present, $g_0 = 0$. Therefore, with perfect price sensitivity, the Nash equilibrium coincides with that of the pure Bertrand model.

If the utility of the outside good increases, the banks' margins decrease. To see this, note that equation (3.2) decreases with an increasing utility of the outside good, subsequently, all m_j decrease with decreasing arguments m_k.

If two banks have the same brand power, they pay the same coupon rates. This follows by contradiction. Assume bank k pays a higher coupon rate than bank j, but they have equal brand power $g_k = g_j$. Then from the pricing formula (2.8), the assumption $c_k > c_j$ implies

$$\frac{g_k e^{\beta i - 1}}{g_j e^{c_j \beta} + \sum_{s\neq j,k} g_s e^{c_s \beta}} < \frac{g_j e^{\beta i - 1}}{g_k e^{c_k \beta} + \sum_{s\neq j,k} g_s e^{c_s \beta}}.$$

But this inequality cannot hold if $c_k > c_j$ and $g_k = g_j$ and hence the assumption is wrong, so the contrary must be true $c_k \leq c_j$. We can rule out the strict inequality in the same way and thus prices must be equal.

From equation (3.1), we see that, if all prices are equal, the m_j must coincide. Hence, if all banks are equal in brand power and in the absence of an outside good, the price (3.1) reduces to

$$c_j^* = i - \frac{1}{\beta} \frac{K}{K-1}, \tag{3.3}$$

where K is the number of banks. We obtain (3.3) by setting all m_j in equation (3.2) to the same value and inverting the Lambert W function.

4. EMPIRICAL EVIDENCE

In this Section we present pricing patterns that have been observed in empirical work and that our model explains. We show why small banks often set higher prices than their larger competitors, how pricing depends on brand power, and we present the effect of market concentration on the average coupon paid.

4.1. Brand Power and Market Share

The marginal increase in market share with a marginal increase in the coupon rate is given by $\partial p_j / \partial c_j = \beta p_j (1 - p_j)$. Therefore, small banks with a low initial market share, can expect less changes in market share by adjusting their price than large banks. A bank having an initial market share of fifty percent experiences the maximal marginal increase. Hence, huge banks (with a market share above fifty percent) can expect less changes, than large banks.

This is consistent with Rosen (2007) and Bassett and Brady (2002). Rosen reports that small banks competed more aggressively than large banks in one part of his sample period. Bassett and Brady write that small, rapidly growing banks set aggressive prices.

4.2. How Brand Power influences Pricing

There is a stylized fact, that larger banks pay lower deposit rates than smaller banks, see Rosen (2007). According to our definition of brand power, larger banks have a greater brand power than their smaller rivals (all else being equal). Our model confirms this stylized fact: Equilibrium coupons are coincident in markets with equally strong banks, but there is a coupon rate dispersion if banks differ in brand power.

PROPOSITION 4.1: *A bank with a greater brand power has a lower optimal coupon rate, that is,* $g_k > g_j \Leftrightarrow c_k^* < c_j^*$.

The above result appears intuitive, we give two examples. If the branches' location is part of the econometrician's brand power variable g_j, then an increasing disutility of transportation costs leads to smaller values g_j and thus yields higher coupon rates for banks located more distant. If the econometrician adds creditworthiness to g_j, for instance in the form of credit default swap spreads, then a decreasing creditworthiness leads to higher coupon rates.

4.3. Market Concentration and the average Coupon

We are interested in the average coupon paid in the economy as a function of the market concentration, i.e., for a given price sensitivity β and wholesale market rate i we look at the weighted average coupon rate

$$\bar{c} = \sum_{k=1}^{K} p_k c_k^*, \tag{4.1}$$

where p_k is the market share of bank k and c_k^* its coupon rate. The left graph of Figure 1 shows the equilibrium coupons paid and the weighted average coupon rate $\bar{c}$ as a function of relative brand power, g_1/g_2, in a duopoly. The right graph of Figure 1 shows the weighted average coupon rate for three banks as a function of relative brand power g_1/g_2 and g_1/g_3. The price sensitivity β is fixed to 200, and the market rate to $i = 3\%$. In the left graph, the average coupon rises as the relative brand power g_1/g_2 approaches unity. In the three-dimensional plot, the maximal average coupon is at the upper right corner, where $g_1 = g_2 = g_3$. The average coupon has the slowest increase along the diagonal (the dashed line). On the diagonal, we have that $g_2 = g_3$, that is, two banks are equal in brand power. This gives evidence to the suggestion in Rosen (2007): the more similar the banks are, the higher is the average coupon paid.

[Figure 1 about here.]

For more than three banks the above representation (plotting $\bar{c}$ as a function of relative market share) is not possible anymore. Therefore, Figure 2 plots $\bar{c}$ as a function of "similarity" of the brand power variables g_j for different numbers of banks. We use the Kullback-Leibler divergence to measure similarity. That is, we measure the Kullback-Leibler distance from the normalized brand power variables g_j (such that they sum up to 1) to the uniform distribution, i.e.,

$$KL(g) = \frac{1}{\sum_{j=1}^{K} g_j} \sum_{k=1}^{K} g_k \log\left(\frac{g_k}{\sum_{j=1}^{K} g_j} K\right).$$

The Kullback-Leibler divergence shows to have more discriminatory power than other distance measures (for instance the sum of the squared Euclidean distances).

[Figure 2 about here.]

Figure 2 confirms for 3, 5 and 7 banks, that the more similar the banks are, the higher is the average coupon paid. Neuberger and Zimmerman (1990) search an explanation to the persistently lower rates on deposits in the late eighties in California compared to the rest of the United States. They find by statistical analysis that the lower rates can partially be explained by higher market concentration. Our model supports this statistical finding, since higher market concentration leads to lower coupon rates.

5. MULTI-PERIOD

In this Section, we extend our model to a multi-period economy. A multi-period optimizer sets the same prices as a one-period optimizer. In the second part of this section, we approximate our model by means of Taylor series. These approximations make it possible to compare our model to the common models in the banking literature, on one hand, and to models used in Industrial Organization theory on the other hand.

We define a closed, discrete time economy with dates $t \in \{0, 1, ..., \tau\}$. We want to invest in a one-period riskless security, and rollover the investment. The riskless security worth 1 consumption unit today is worth $e^{r(0)}$ consumption units in the next period, where $r(t)$ is the risk-free rate of interest. At the end of the second period, the investment is worth $e^{r(0)+r(1)}$.

By the risk neutral valuation principle, the price of a security satisfies the no arbitrage condition if and only if there exists a probability measure $\mathbb{Q}$, equivalent to the real-world measure, such that under $\mathbb{Q}$, the price process is a martingale, see Harrison and Kreps (1979).

Let $(\Omega, \mathcal{F}, \{\mathcal{F}\}_{t=0}^{\tau}, \mathbb{Q})$ be a probability space, equipped with the finite filtration $\mathcal{F}_0 \subset \mathcal{F}_1 \subset ... \subset \mathcal{F}_\tau$. The price of the riskless security with remaining time to maturity T at time t is given by the expected value under $\mathbb{Q}$ of the discounted consumption unit:

$$B_T(t) = \mathbb{E}^Q[e^{-\sum_{j=t}^{t+T-1} r(j)} | \mathcal{F}_t]. \tag{5.1}$$

Analogous, the present value of the deposit account for bank j is the expected value of the sum of the discounted cash flows under the appropriate measure $\mathbb{Q}$. The basic structure is the same as in Jarrow and van Deventer (1998). The cash flows consist of volume increases minus coupon payments. As we show in the Appendix, the present value of the deposit account is given by the following Lemma.

LEMMA 5.1: *The present value of the deposit account of bank j at time 0 is*

$$P_j(0) = \mathbb{E}^Q \left[\sum_{t=0}^{\tau-1} \pi_j(t) e^{-\sum_{s=0}^{t} r(s)} \middle| \mathcal{F}_0 \right]. \tag{5.2}$$

The term $\pi_j(t)$ is the profit at time t, given by $\pi_j(t) = p_j(t)N[i(t) - c_j(t)]$, with $c_j(t)$ the coupon of bank j at time t, and $p_j(t)N$ the deposit volume of bank j at time t. Hence, the present value is the expected value under the risk-neutral measure of the sum of the discounted one-period profits $\pi_j(t)$.

5.1. Optimality in a Multi-Period Economy

The coupon rates of the α-strategists lead to an optimal present value. To prove the optimality, we use Lemma 5.1. Bank j faces the first-order conditions for profit-optimal coupons given by

$$\frac{\partial P_j(0)}{\partial c_j(t)} = \mathbb{E}^Q \left[\frac{\partial}{\partial c_j(t)} \sum_{t=0}^{\tau-1} \pi_j(t) e^{-\sum_{s=0}^{t} r(s)} \middle| \mathcal{F}_0 \right] = 0, \text{ for all } t. \tag{5.3}$$

This holds if the present value is finite, since in this case, we can exchange integration and differentiation by the dominated convergence theorem. The profit of a bank for time t is independent of the past (and future) coupon rates. Therefore, the t-th equation of the equation system (5.3) is

$$\mathbb{E}^Q \left[\frac{\partial}{\partial c_j(t)} \pi_j(t) e^{-\sum_{s=0}^{t} r(s)} \middle| \mathcal{F}_0 \right] = 0. \tag{5.4}$$

The solution to the one period optimization solves 5.4. Hence, the coupon rate that maximizes the present value of the deposit account corresponds to the optimal coupon rate of the one-period economy given by equation (2.8).

5.2. Approximating Market Share and Prices

We now approximate market share and the α-strategist's price by means of a Taylor series. We can thereby relate our approach to deposit pricing to today's practice in the banking literature, in which the dynamics of deposit rates and volume are expressed in terms of market rates. The models are often linear in market rates, see Hannan and Liang (1993), Wilson (1994), Hutchison and Pennacchi (1996), Jarrow and van Deventer (1998), the Office of Thrift Supervision (2001), and Kalkbrener and Willing (2004), for examples of both, linear and non-linear functions.

Using a Taylor series, the coupon rate c_j^* of the α-strategist is, up to a first-order approximation,

$$c_j^*(i, c_0, ..., c_{j-1}, c_{j+1}, c_K; g_0, ..., g_{j-1}, g_{j+1}, ..., g_K) \approx c_j^*(0) + i\frac{\partial c_j^*}{\partial i}(0) + \sum_{k \neq j} c_k \frac{\partial c_j^*}{\partial c_k}(0), \tag{5.5}$$

where the pricing formula $c_j^*(0)$ and the derivatives are evaluated at zero ($i = 0$, $c_k = 0$ for all banks). Since the function evaluations are constants, the approximated pricing formula (5.5) is a linear function of the wholesale market rate and the competing banks' prices. If the competing banks set prices based on moving averages of wholesale market rates with different maturities, as in Wilson (1994), we can replace the coupon rates c_k in (5.5) by these moving averages. Then, the α-strategist's approximated coupon rate can be written as a linear function of wholesale market rates as the only stochastic component.

Now, we consider the volume of the deposit account, which is Np_j, with p_j given by (2.5). Omitting the time index, the logarithm of the volume deposited at bank j is

$$\log V_j = \log N + \log g_j + \beta c_j - \log\left(\sum_k g_k e^{\beta c_k}\right). \tag{5.6}$$

Again, we approximate the above equation with a first-order Taylor series around zero and combine constant terms to get

$$\log V_j = \nu_{j0} + \sum_k \nu_{jk} c_k + u_j, \tag{5.7}$$

where ν_{jk} are constants specific to bank j, and u_j is an error term. If the competing banks' prices c_k are functions of wholesale market rates, then we replace the coupons c_k by these functions. In consequence, the resulting volume model (5.7) is a function of wholesale market rates only, as the common valuation models in the banking literature.

The coefficients of the volume model (5.7) and the coupon rate model (5.5) depend on the brand powers, pricing strategies, and the depositor's price sensitivity. To illustrate this, consider a market with K banks, banks $k = 1...n$ are short rate strategists, the remaining banks are replicating strategists. Now, we write the volume model (5.7) as a function of the two different coupon rates. The coefficient ν_{11} in the volume model is the coefficient for the coupon of the short rate strategist for bank 1. A Taylor approximation around a coupon rate $\tilde{c}$ (for both, the short rate strategist and the α-strategist) yields

$$\nu_{11} = \beta \left(1 - \frac{\sum_{k=1}^{n} g_k e^{\beta \tilde{c}}}{\sum_{k=1}^{K} g_k e^{\beta \tilde{c}}} \right) \tag{5.8}$$

for the coefficient. We demonstrate this in the Appendix. Feeding new market conditions into a Taylor approximation of our model shows how parameters of the regression model (according to e.g. Jarrow and van Deventer (1998)) might be adjusted. For instance, if product managers see an increasing price sensitivity of the depositors, then the parameter given by equation (5.8) can be adjusted accordingly. In this way, a bank can account for new market conditions without having historical data available yet.

We now summarize the above statements and conclude. The coefficients ν_{jk} and α_{jk} in the approximations (5.7) and (5.5) depend on the competing banks and are thus specific to a bank. We demonstrate this with a particular coefficient in equation (5.8). The coupon rates can be expressed as functions of different wholesale market rates, even for α-strategists, as shown in equation (5.5). Thus, the resulting volume model, equation (5.7), contains wholesale market rates as the only explanatory variables. corresponds to the approach in the banking literature, where the dynamics of deposit volume and deposit prices are expressed as functions of wholesale market rates. Therefore, these models, such as Jarrow and van Deventer (1998) or Kalkbrener and Willing (2004), must implicitly depend on the competing banks and price sensitivity. The wholesale market rates that have to be taken into account in equation (5.7) to express the competing banks' coupon rates, depend on the competing bank's pricing strategies. In brief, models must be specific to a bank. This contrasts the 'one-size-fits-all' approach of the Office of Thrift Supervision (2001) that adapts the policy that one model is appropriate for all banking institutions. Our model supports the view of Jarrow and van Deventer (1998) who write, without providing further rationale, that the parameters reflect local market characteristics and are firm-specific.

With the approximations above, we link Industrial Organization theory with the banking literature. We illustrate it in Figure 3.

[Figure 3 about here.]

Industrial Organization theory uses the circular city model of Salop (1979) to explain pricing patterns, e.g. by Chiappori, Perez-Castrillo, and Verdier. Salop's approach belongs to the address models, while we develop our model in the discrete choice framework. The address approach is connected to the discrete choice approach as shown by Anderson, De Palma, and Thisse (1989), thereby our model is linked to Industrial Organization theory. Approximations of our model provide the link to models employed in the banking literature (such as in Jarrow and van Deventer (1998)). This closes the link between Industrial Organization theory and the banking literature.

In the next Section, we compare the performance of the α-strategists to other strategists. After this, we compute the error emerging when present values are computed using the approximated volume model (5.7).

6. SIMULATION

To compute the present value of a savings deposit account according to equation (5.2), we need to describe the evolution of the interest rates. The short rate follows the time-discrete process

$$r(t) = a + br(t-1) + \epsilon(t), \text{ with } \epsilon(t) \sim \mathcal{N}\left(0, \sigma^2\right), \ |b| < 1, \tag{6.1}$$

a discrete version of the Vasicek (1977) one-factor short rate model. The discretization unit is one month, $t+1$ is one month from t. The parameters we choose for the short rate model are $a/(1-b) = 5.5\%$, $b = 0.95$, $\sigma = 0.007/\sqrt{12}$, $r(0) = 2.5\%$.

Banks are either α-strategists, equation (2.8), replicating strategists, or short rate strategists, as described in the sequel. The practitioner's approach to the replicating portfolio method is to assume 20 percent of the deposits is highly volatile, while the remaining 80 percent are more stable, see Wolff (2000). In this sense, we choose a replicating portfolio consisting of 20 percent 1-month tranches and of 80 percent 5-year tranches. The resulting coupon rate of the *replicating strategist* is then the weighted average of the one month rate and the moving average of $5 \cdot 12$ five year rates:

$$c_{\text{rep}}(t) = 0.2i(t) + 0.8\frac{1}{60}\left(\sum_{j=t-59}^{t} i_{60M}(j)\right) - M_1, \tag{6.2}$$

where the variable i_{60M} is the wholesale market rate with a maturity of five years, M_1 is a constant spread. To compute (6.2), we need the closed-form solution for the zero bond yield with a maturity of 60 months. We provide the solution in the Appendix. The *short rate strategists* price their deposits according to Jarrow and van Deventer (1998), equation 8b of their publication, where the coupon rate equals the wholesale market rate i minus a constant spread:

$$c_{\mathrm{sr}}(t) = i(t) - M_2. \tag{6.3}$$

To ensure that the replicating strategists and the short rate strategists have similar chances of profit, we choose the constant margins $M1$ and $M2$ such that both pay the same coupon rate in average (over time). The *α-strategists* use the profit maximizing coupon, i.e., the fixed point solution to equation (2.8).

6.1. Valuing different Strategists

We show the simulation results for different price strategists having identical brand power. To compare the performance of the competing banks, we compute the present value of the deposit accounts, integrated over state $r(0)$:

$$\bar{P} = \mathbb{E}^Q \left[\sum_{t=0}^{\tau-1} \pi_j(t) e^{-\sum_{s=0}^{t} r(s)} \right]. \tag{6.4}$$

To integrate over state, we allow the economy to adjust for 180 periods. We hold the market volume N constant, this implies that depositors consume the coupon payments. We integrate over time ($\tau = 5$ years) and state (5000 paths) to compute the present value $\bar{P}$. Figure 4 depicts the simulation results, it shows expected coupons, and the present value of margin income as a function of price sensitivity.

[Figure 4 about here.]

Present values of the short rate strategists and replicating strategists are close to the optimal present value for certain price sensitivities β. This confirms the statement of Section 5.2, that profit-optimal rates can be approximated with a function of market rates only. The coupons of the α-strategists are low, if the price sensitivity is low. For increasing price sensitivities the curve approaches the maximal coupon in the economy. In a market consisting of α-strategists only, the present values collapse for high price sensitivities, since with increasing β, margins tend to zero, as seen in Section 3.2. In the presence of only one α-strategist, his price reaches a maximum and then decreases as β increases. This is not possible in the presence of several α-strategist, where the price increases in β.

Figure 5 shows the market share and the coefficient of variation for the market share for different strategists. Comparing the corresponding graphs of Figure 4 and Figure 5 shows that α-strategists have a skimming strategy if price sensitivities remain low. That is, they have a low market share and pay low coupon rates. With high price sensitivities their strategy is different, they have a large market share and low margins.

[Figure 5 about here.]

The coefficient of variation for the market share stays relatively low for α-strategists for all price sensitivities, whereas the conservative replicating strategists suffer from the quick reactions of α-strategists. A short rate strategist can be viewed as the outside good in which the opportunity rate equals the wholesale market rate minus a margin. Interestingly, the short rate strategist's coefficient of variation is low in the presence of several α-strategists and few other strategists. This can possibly be explained by Proposition 3.2: α-strategists are short rate strategists if the market consists only of α-strategists. If there are many α-strategists in the market, the pricing behavior is dominated by them and their strategy is much like a short rate strategy. In consequence, the short rate strategist that pays the α-strategists' price plus a spread, has a coefficient of variation close to the α-strategist.

6.2. *Can we compute Present Values without considering the Competitors?*

In Section 5.2 we showed that the deposit volume Np_j of a bank, can be approximated in terms of wholesale market rates. In this Section, we calculate present values of a short-rate strategist, replacing the volume by the approximation given by (5.7). Omitting the index for bank j, the approximation for a market with short rate strategists, replicating strategists, and α-strategists is

$$\log(\widehat{V}(t)) = \nu_0 + \nu_1 c_{\mathrm{sr}}(t) + \nu_2 c_{\mathrm{rep}}(t) + u(t), \tag{6.5}$$

with c_{sr} the coupon rate of the short rate strategist given by equation (6.3), c_{rep} that of a replicating strategist given by equation (6.2). We assume $u(t)$ to be an autocorrelated error, as in Janosi, Jarrow, and Zullo (1999). That is, $u(t) = \rho u(t-1) + \varepsilon(t)$, with i.i.d. normal $\varepsilon(t)$, having mean 0 and variance σ_u^2, and $u(0) = 0$. In a first stage, we estimate the parameters ν_k, ρ, σ_u^2 of the volume model (6.5), with the market share p_j as dependent variable (using 6000 time steps and averaging over 3000 paths). In a second stage, we calculate the present value $\bar{P}$, equation (6.4), and the resulting present value with the approximated volume model (6.5), i.e.,

$$\widehat{P} = \mathbb{E}^Q \left[\sum_{t=0}^{\tau-1} \widehat{V}(t)(i(t) - c_{sr}(t))e^{-\sum_{s=0}^{t} r(s)} \right], \tag{6.6}$$

with $\widehat{V}(t)$ given by equation (6.5) and c_{sr} the coupon rate of the short rate strategist given by equation (6.3). The simulation horizon is $5 \cdot 12$ months. The error is $(\widehat{P} - \bar{P})/\bar{P}$, with $\bar{P}$ given by equation (6.4). We consider two different markets, both seen in Section 6.1. The first with 1 α-strategist, 3 short rate strategists, and 3 replicating strategists and the second with 3 α-strategist, 2 short rate strategists, and 2 replicating strategists. Table A displays the percentage error. Note that coupon rates, present values, market share, and the variation in market share for the scenarios presented in Table A are displayed in Figure 4 and Figure 5.

[Table 1 about here.]

A short rate strategist that calculates the present value using the approximated volume model given by equation (6.6) overestimates the value by about 2%. Table A also displays the estimated parameters for the volume model. The intercept ν_0 is decreasing with increasing price sensitivity. The parameter ν_3 is negative, because the short rate strategist's market share profits from low rates of the replicating strategist. The parameter ν_1 is positive, because the short rate strategist gets more volume if he pays higher coupon rates. We're left to explain the size of the parameters ν_1 and ν_2. In a market with 3 replicating strategists and 3 short rate strategists only, a Taylor approximation shows that the coefficients ν_1 and ν_2 for the coupon rates are increasing in β (in absolute terms). In fact we have already done this approximation for ν_1 in equation (5.8), $\nu_1 \approx \beta(1 - 3/6) = \beta/2$. With only one α-strategist, as in the upper part of the Table, this pattern still holds. If there are several α-strategists in the market, as in the lower part of the Table, the estimates show a u-shaped pattern and are less straight forward to explain. We can see in the corresponding graph of Figure 5 (middle row) that the short rate strategist's market share is almost zero for $\beta = 800$ and thus the volume does not depend on the coupon rates anymore, so ν_1 and ν_2 are close to zero. If the variation in market share is sufficiently large (see Figure 5), the error increases above 2%.

7. CONCLUSION

Industrial Organization theory provides models that explain banks' deposit pricing decisions, whereas in the banking literature, corresponding models are used to consider the risk and value of deposit accounts. So far, these two streams have remained unrelated. If Industrial Organization theory tells us that market conditions are crucial to the banks' profit-optimal prices and deposit volumes, this must somehow be incorporated in the valuation models. Otherwise the Asset-Liability Management's risk taking decisions, which are based on these valuations, are not as intended. We show that the current valuation models implicitly account for local market conditions, since the statistically estimated model parameters incorporate local market conditions. The consequence is that the Asset-Liability Manager's valuation models have to be reestimated on a regular basis to account for changing market conditions.

There is no 'one-size-fits-all' approach to valuing deposits, meaning that models must be specific to a bank. The Office of Thrift Supervision evaluates savings institutions' interest rate risk by estimating the sensitivity of their portfolio to changes in wholesale market rates. They provide a model with parameters that differ among the type of the deposit account[2] but not among banks. Models must be bank-specific, these risk valuations and the resulting recommendations for the thrift institutions may be seriously in error. The rationale for bank-specific models is that profit-optimal deposit prices and deposit volumes depend on the competing banks' prices and brand powers. These quantities differ among markets and banks in a given market. Greater banks have lower profit-optimal deposit rates and their pricing strategy has a large impact on the competing banks' optimal prices, hence: size matters.

A. PROOFS

Proof Of Equation (2.8), *Profit Maximizing Coupon Rate.* Maximizing the profit $\pi_j = Np_j\,[i - c_j]$ results in the first-order condition

$$0 = \frac{\partial \pi_j}{\partial c_j} = N\frac{\partial p_j}{\partial c_j}\,[i - c_j] - Np_j, \tag{A.1}$$

with p_j given by

$$p_j = \frac{g_j e^{\beta c_j}}{\sum_{k=1}^{K} g_k e^{\beta c_k}}. \tag{A.2}$$

Now, the marginal increase in market share with a marginal increase in the coupon rate is given by,

$$\frac{\partial p_j}{\partial c_j} = \frac{g_j e^{c_j\beta}\beta \sum_{k\neq j} g_k e^{c_k\beta}}{\left[\sum_{k=1}^{K} g_k e^{c_k\beta}\right]^2} = \beta p_j(1 - p_j),$$

so that the first-order condition (A.1) for an optimal profit becomes

$$e^{c_j\beta} = -\frac{1}{g_j}\beta\left(\sum_{k\neq j} g_k e^{c_k\beta}\right)\left[c_j - \left(i - \frac{1}{\beta}\right)\right]. \tag{A.3}$$

The second derivative of the profit is

$$\frac{\partial^2 \pi_j}{\partial c_j^2} = (i - c_j)N\beta^2(1 - 2p_j)p_j(1 - p_j) - 2N\beta p_j(1 - p_j). \tag{A.4}$$

Evaluating (A.4) at any point where the first-order condition (A.1) is met yields $-N\beta p_j$, which is negative. Therefore the solution to the first-order condition indeed maximizes profit. To solve equation (A.3) in c_j we resort to the following Lemma that uses the Lambert W function (Corless, Gonnet, Hare, Jeffrey, and Knuth 1996)

LEMMA A.1: *The solution to the transcendental algebraic equation in x of the form*

$$e^{-a_0 x} = a_1\,(x - b) \tag{A.5}$$

with $a_0, a_1, b \in \mathbb{R}$, has the solution $x = b + \frac{1}{a_0}W\left(\frac{a_0}{a_1}e^{-a_0 b}\right)$ where $W(.)$ is the Lambert W function, which is the inverse function of $f(z) = ze^z$ with $z \in \mathbb{C}$.

The proof of Lemma A.1 is straightforward, using the definition of the Lambert W function. Equation A.5 can be restated $a_0/a_1 e^{-a_0 x} = (x-b)a_0$, and then $a_0/a_1 e^{-a_0 b} = (x-b)a_0 e^{(x-b)a_0}$. Applying the definition of $W(.)$ yields the stated solution.

Hence, the solution to (A.3) is given by,

$$c_j^* = i - \frac{1}{\beta}\left[1 + W\left(\frac{g_j e^{\beta i - 1}}{\sum_{k \neq j} g_k e^{c_k \beta}}\right)\right], \tag{A.6}$$

when we replace a_0, a_1, and b in Lemma A.1 by

$$a_0 = -\beta,\ b = i - \frac{1}{\beta},\ \text{and } a_1 = -\frac{1}{g_j}\beta \sum_{k \neq j} g_k e^{c_k \beta}.$$

The arguments to $W(.)$ are always positive and thus the solution is in $\mathbb{R}$. ■

Proof Of Proposition 3.1, Unique Nash Equilibrium. Let $M \leq K$ be the number of α-strategists in the economy. We define the coupon vector as $c = (c_1, \ldots, c_M)^T$. If $M < K$ the coupons of the $K - M$ non α-strategists do not depend on the other banks' coupons. Let $C : \mathbb{R}^M \to \mathbb{R}^M$ be the mapping defined by applying c_j^*, equation (A.6), to every coupon that belongs to an α-strategist. Assume the initial coupon vector is $c^0 \in \mathbb{R}^M$. When every bank $j = 1, ..., M$ applies equation (A.6), the coupon is $c^1 = C(c^0)$. After two steps the coupon is $c^2 = C(c^1) = C(C(c^0))$, et cetera. The coupon vector obtained when t times applying the mapping C is c^t. We define the mapping C as

$$C : \begin{pmatrix} c_1 \\ \vdots \\ c_M \end{pmatrix} \mapsto \begin{pmatrix} i - \frac{1}{\beta}\left[1 + W\left(\frac{g_1 e^{\beta i - 1}}{\sum_{k=2}^{K} g_k e^{c_k \beta}}\right)\right] \\ \vdots \\ i - \frac{1}{\beta}\left[1 + W\left(\frac{g_j e^{\beta i - 1}}{\sum_{k=1, k \neq j}^{K} g_k e^{c_k \beta}}\right)\right] \\ \vdots \\ i - \frac{1}{\beta}\left[1 + W\left(\frac{g_M e^{\beta i - 1}}{\sum_{k=1, k \neq M}^{K} g_k e^{c_k \beta}}\right)\right] \end{pmatrix}. \tag{A.7}$$

The proof works as follows. We show that the infinity norm of the Jacobian of C is smaller than one. Then, we derive an upper and a lower bound for the equilibrium coupons. We then can apply Theorem 5.4.1 in Judd (1998).

We first show that the infinity norm of the Jacobian of C is smaller than one. The element (j,k) of the $M \times M$ Jacobian C' is $\partial c_j^*/\partial c_k$. It is zero for $j = k$. For $j \neq k$, we get

$$\frac{\partial c_j^*}{\partial c_k} = \frac{W\left(\frac{g_j e^{\beta i - 1}}{\sum_{l=1, l\neq j}^{K} g_l e^{\beta c_l}}\right)}{1 + W\left(\frac{g_j e^{\beta i - 1}}{\sum_{l=1, l\neq j}^{K} g_l e^{\beta c_l}}\right)} \frac{g_k e^{\beta c_k}}{\sum_{l=1, l\neq j}^{K} g_l e^{\beta c_l}}$$

Calculating the infinity norm of the Jacobian for $M \leq K$ yields

$$\|C'\|_\infty = \max_j \sum_{m=1}^{M} \frac{\partial c_j}{\partial c_m} = \max_j \frac{W\left(\frac{g_j e^{\beta i - 1}}{\sum_{k=1, k\neq j}^{K} g_k e^{\beta c_k}}\right)}{1 + W\left(\frac{g_j e^{\beta i - 1}}{\sum_{k=1, k\neq j}^{K} g_k e^{\beta c_k}}\right)} \frac{\sum_{l=1, l\neq j}^{M} g_l e^{\beta c_l}}{\sum_{l=1, l\neq j}^{K} g_l e^{\beta c_l}},$$

which is strictly smaller than 1.

Now, we derive bounds for the equilibrium coupons. Trivially, each bank will not offer a higher coupon rate than r, otherwise the profit is negative. The wholesale rate r is the upper bound for all K banks. We first consider the case of K α-strategists. To find the lower bound B_α, we assume $g_1 >= g_2 >= g_3 >= ... >= g_K$, without loss of any generality. In equilibrium bank 1 offers the lowest coupon, by Proposition 4.1. Hence in equilibrium we must have $c_1 < c_j$ if $g_1 > g_j$. Now, to find the lower bound B_α, we assume the other banks set the same coupon as bank 1. If the others optimized as well, they would offer a higher coupon rate. Thus, in equilibrium where all K banks follow an optimal strategy, each element of the equilibrium coupon vector is greater than B_α. To find B_α, we have therefore the equality

$$B_\alpha = i - \frac{1}{\beta}\left[1 + W\left(\frac{g_1 e^{\beta i - 1}}{\sum_{k=2}^{K} g_k e^{\beta B_\alpha}}\right)\right]. \tag{A.8}$$

Solving for B_α, we obtain

$$B_\alpha = i - \frac{1}{\beta}\left(1 + \frac{g_1}{\sum_{k=2}^{K} g_k}\right). \tag{A.9}$$

Now, if not all banks are α-strategists, this bound must not hold. Using the similar argumentation as above, a lower bound B in the presence of other strategists is

$$B = \min\left(B_\alpha, i - \frac{1}{\beta}\left[1 + W\left(\frac{g_j e^{\beta i - 1}}{\sum_{k=1, k\neq j}^{K} g_k e^{\beta c_{\min}}}\right)\right]\right), \tag{A.10}$$

where $c_{\min}$ is the smallest coupon rate of the $K - M$ other strategists and j is the bank having the largest g_j.

We can choose any $B_l, B_u \in \mathbb{R}$ such that $B_l < B$ and $B_u > i$. Altogether, we have a differentiable contraction map on $[B_l, B_u]^M$, since it is a closed, bounded, convex set and the matrix norm of the Jacobi matrix is strictly smaller than one, we can apply Theorem 5.4.1 in Judd (1998), the sequence defined by $c^{t+1} = C(c^t)$ converges to c^*.

■

Proof Of Proposition 3.2. Solving the first-order condition of the profit $N\partial p_j/\partial c_j\,[i - c_j] - Np_j = 0$, for $\beta(i - c_j)$ yields $1/(1 - p_j)$, or

$$\frac{\sum_k g_k e^{\beta c_k}}{\sum_{k \neq j} g_k e^{\beta c_k}} = \beta(i - c_j) \tag{A.11}$$

using $c_j^* = i - m_j/\beta$ for the α-strategists, we can state

$$\frac{g_0 e^{\beta c_0} + \sum_{k=1}^{K} g_k e^{\beta i - m_k}}{g_0 e^{\beta c_0} + \sum_{k=1, k \neq j}^{K} g_k e^{\beta i - m_k}} = m_j.$$

This can be rewritten as

$$e^{-m_j} = \left[\frac{g_0}{g_j} e^{\beta(c_0 - i)} + \sum_{k \neq j} \frac{g_k}{g_j} e^{-m_k}\right](m_j - 1).$$

Using Lemma A.1 with $a_0 = 1$, $a_1 = \frac{g_0}{g_j} e^{\beta(c_0 - i)} + \sum_{k \neq j} \frac{g_k}{g_j} e^{-m_k}$, $b = 1$, we obtain

$$m_j = 1 + W\left(\frac{g_j}{g_0 e^{\beta c_0} e^{1 - \beta i} + \sum_{k \neq j} g_k e^{1 - m_k}}\right),\ j = 1 \ldots K,$$

completing the proof. ■

Proof Of Proposition 4.1. A bank with a greater brand power has a lower optimal coupon. This can be proven by contradiction. Bank j has brand power g_j, bank m brand power g_m and $g_j > g_m$. Assume bank j offers a lower coupon than bank m. From $g_m > g_j$ and $c_m \geq c_j$, it follows

$$\frac{g_m e^{\beta i - 1}}{g_j e^{\beta c_j} + G} > \frac{g_j e^{\beta i - 1}}{g_m e^{\beta c_m} + G}, \tag{A.12}$$

with $G = \sum_{k, k \neq j, m}^{K} g_k e^{\beta c_k}$. Applying the pricing formula (A.6) we obtain $c_m < c_j$ which contradicts our earlier assumption that in equilibrium $c_m \geq c_j$. Hence in equilibrium we must have $c_m < c_j$ if $g_m > g_j$. ■

Proof Of Lemma 5.1, Present Value Of The Deposit Account. The present value of the deposit account is the sum of the discounted cash flows under the risk neutral measure $\mathbb{Q}$. The deposit volume of bank j at time t is abbreviated by $V(t)$.

$$
\begin{aligned}
P_j(0) =& \mathbb{E}^Q\left[V(0)+\sum_{t=0}^{\tau-2}(V(t+1)-V(t))e^{-\sum_{s=0}^{t}r(s)}-V(\tau-1)e^{-\sum_{s=0}^{t}r(s)}\middle|\mathcal{F}_0\right]\\
&-\mathbb{E}^Q\left[\sum_{t=0}^{\tau-1}c_j(t)V(t)e^{-\sum_{s=0}^{t}r(s)}\middle|\mathcal{F}_0\right]\\
=&\mathbb{E}^Q\left[V(0)+\sum_{t=0}^{\tau-2}V(t+1)e^{-\sum_{s=0}^{t+1}r(s)}e^{r(t+1)}-\sum_{t=0}^{\tau-1}V(t)e^{-\sum_{s=0}^{t}r(s)}\middle|\mathcal{F}_0\right]\\
&-\mathbb{E}^Q\left[\sum_{t=0}^{\tau-1}c_j(t)V(t)e^{-\sum_{s=0}^{t}r(s)}\middle|\mathcal{F}_0\right]\\
=&\mathbb{E}^Q\left[\sum_{t=0}^{\tau-1}V(t)e^{-\sum_{s=0}^{t}r(s)}e^{r(t)}-\sum_{t=0}^{\tau-1}V(t)e^{-\sum_{s=0}^{t}r(s)}\middle|\mathcal{F}_0\right]\\
&-\mathbb{E}^Q\left[\sum_{t=0}^{\tau-1}c_j(t)V(t)e^{-\sum_{s=0}^{t}r(s)}\middle|\mathcal{F}_0\right]
\end{aligned}
$$

We use the equivalence $e^{r(t)}=1+i(t)$, to get

$$
\begin{aligned}
P_j(0) =& \mathbb{E}^Q\left[\sum_{t=0}^{\tau-1}V(t)(i(t)-c_j(t))e^{-\sum_{s=0}^{t}r(s)}\middle|\mathcal{F}_0\right] &\text{(A.13)}\\
=&\mathbb{E}^Q\left[\sum_{t=0}^{\tau-1}\pi_j(t)e^{-\sum_{s=0}^{t}r(s)}\middle|\mathcal{F}_0\right]. &\text{(A.14)}
\end{aligned}
$$

Hence, the present value of the deposit account reduces to the expected value of the sum of the discounted one period profits. ■

Derivation Of The Bond Price. If the short rate follows the time-discrete process

$$r(t)=a+br(t-1)+\epsilon(t),\ \text{with } \epsilon(t)\sim\mathcal{N}\left(0,\sigma^2\right),\ |b|<1, \text{ and } t\in\mathbb{N}, \tag{A.15}$$

the closed-form solutions for the price of the zero bond and the zero bond yield at time t with remaining time to maturity $T\in\mathbb{N}$ are given by

$$B_T(t)=\exp\left\{-m_T(t)+\frac{1}{2}v_T\right\} \text{ and } r_T(t)=\frac{1}{T}\left\{m_T(t)-\frac{1}{2}v_T\right\}, \tag{A.16}$$

where

$$m_T(t) = r(t)\frac{1-b^T}{1-b} + a\left(\frac{T-1}{1-b} - b\frac{1-b^{T-1}}{(1-b)^2}\right) \tag{A.17}$$

$$v_T = \sigma^2\left(\frac{T-1}{(1-b)^2} - 2b\frac{1-b^{T-1}}{(1-b)^3} + b^2\frac{1-b^{2(T-1)}}{(1+b)(1-b)^3}\right). \tag{A.18}$$

We obtain by iteration of equation (A.15),

$$r(t) = \sum_{j=0}^{t-1} b^j\left(a + \epsilon_{t-j}\right) + b^t r(0).$$

Today's price ($t=0$) of a bond paying a sure dollar in time T is given by $B_T(0) = \mathbb{E}^Q[e^{-\sum_{j=0}^{T-1} r(j)}]$. Since $|b| < 1$ by the properties of a geometric series,

$$\begin{aligned}
\sum_{t=0}^{T-1} r(t) &= \sum_{t=1}^{T-1}\sum_{j=0}^{t-1}\left\{b^j\left(a+\epsilon_{t-j}\right)\right\} + r(0)\sum_{t=0}^{T-1} b^t \\
&= \sum_{n=1}^{T-1}\frac{1-b^{T-n}}{1-b}\left(a+\epsilon_n\right) + r(0)\frac{1-b^T}{1-b} \\
&= \sum_{n=1}^{T-1}\frac{1-b^{T-n}}{1-b}\epsilon_n + a\left(\frac{T-1}{1-b} - b\frac{1-b^{T-1}}{(1-b)^2}\right) + r(0)\frac{1-b^T}{1-b}.
\end{aligned}$$

The first term on the last line is a sum of independent Gaussian variables and therefore Gaussian as well. The mean of the first term is zero and the variance can be determined via expansion of a geometric series,

$$\sum_{n=1}^{T-1}\left(\frac{1-b^{T-n}}{1-b}\right)^2 = \frac{T-1}{(1-b)^2} - 2b\frac{1-b^{T-1}}{(1-b)^3} + b^2\frac{1-b^{2(T-1)}}{(1+b)(1-b)^3}.$$

By the definitions of $m(T)$ and $v(T)$ in the Equations (A.17) and (A.18), we therefore have

$$\sum_{t=0}^{T-1} r(t) \sim \mathcal{N}\left(m(T), v(T)\right).$$

The definition of the zero bond yield $r_T(0) := -\frac{1}{T}\log B_T(0) = \frac{1}{T}\left\{m_T(0) - \frac{1}{2}v_T\right\}$ as well as the mean of the log-normal distribution, $B_T(t) = \mathbb{E}\left[\exp(-\sum_{j=t}^{t+T-1} r(j))\right]$, complete the proof.

■

Taylor Approximation of the Volume Model, Derivation of Equation (5.8). Consider a market with K banks. The first n banks are short rate strategists, the remaining banks are replicating strategists. The logarithm of the volume of bank 1 is given by the logarithm of market share p_1, equation (2.5), times market volume N,

$$\log N + \log p_1 = \log N + \log g_1 + \beta c_{\mathrm{sr}} - \log\left(\sum_{k=1}^{n} g_k e^{\beta c_{\mathrm{sr}}} + \sum_{k=n+1}^{K} g_k e^{\beta c_{\mathrm{rep}}}\right), \tag{A.19}$$

where c_{sr} is the coupon rate of the short rate strategist, and c_{rep} the coupon rate of the replicating strategist. The Taylor series approximation of the logarithm (containing the two sums) evaluated at c_{sr} and c_{rep} is, up to first-order terms,

$$\log(.) \approx f_0 + \frac{(c_{\mathrm{sr}} - \tilde{c}_{\mathrm{sr}})\beta \sum_{k=1}^{n} g_k e^{\beta \tilde{c}_{\mathrm{sr}}}}{\sum_{k=1}^{n} g_k e^{\beta \tilde{c}_{\mathrm{sr}}} + \sum_{k=n+1}^{K} g_k e^{\beta \tilde{c}_{\mathrm{rep}}}} + \frac{(c_{\mathrm{rep}} - \tilde{c}_{\mathrm{rep}})\beta \sum_{k=1}^{n} g_k e^{\beta \tilde{c}_{\mathrm{rep}}}}{\sum_{k=1}^{n} g_k e^{\beta \tilde{c}_{\mathrm{sr}}} + \sum_{k=n+1}^{K} g_k e^{\beta \tilde{c}_{\mathrm{rep}}}},$$

where $\tilde{c}_{\mathrm{sr}}$ and $\tilde{c}_{\mathrm{rep}}$ are nearby coupon rates, f_0 is the logarithm evaluated at $\tilde{c}_{\mathrm{rep}}$, $\tilde{c}_{\mathrm{sr}}$. Substituting this Taylor approximation into (A.19), combining constant terms and rearranging, we can write the equation as:

$$\begin{aligned}
\log N + \log p_1 \approx{} & \nu_{10} && \text{(A.20)}\\
& + \beta\left(1 - \frac{\sum_{k=1}^{n} g_k e^{\beta \tilde{c}_{\mathrm{sr}}}}{\sum_{k=1}^{n} g_k e^{\beta \tilde{c}_{\mathrm{sr}}} + \sum_{k=n+1}^{K} g_k e^{\beta \tilde{c}_{\mathrm{rep}}}}\right) c_{\mathrm{sr}} && \text{(A.21)}\\
& + \beta \frac{\sum_{k=1}^{n} g_k e^{\beta \tilde{c}_{\mathrm{rep}}}}{\sum_{k=1}^{n} g_k e^{\beta \tilde{c}_{\mathrm{sr}}} + \sum_{k=n+1}^{K} g_k e^{\beta \tilde{c}_{\mathrm{rep}}}} c_{\mathrm{rep}}. && \text{(A.22)}
\end{aligned}$$

The coefficient of the short rate strategist's coupon rate is given by (A.21). We obtain equation (5.8), when replacing $\tilde{c}_{\mathrm{sr}}$ and $\tilde{c}_{\mathrm{rep}}$ by $\tilde{c}$. ■

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Notes

[1] The density function is $f_{\text{Gum}}(z;\kappa,1) = e^{(\kappa-z)-e^{\kappa-z}}$, where κ is Euler's constant ($\kappa \approx 0.5772$). The scale-one assumption of the Gumbel distribution is not crucial to our analysis. One could instead assume scale μ. The weights γ and β in (2.3) would then be replaced by γ/μ and β/μ. This would not bring further insights into our analysis and we thus choose $\mu = 1$.

[2] The Office of Thrift Supervision (2001) provides models (including parameters) for transaction accounts, NOW, Super NOW, and other interest-bearing transaction accounts.

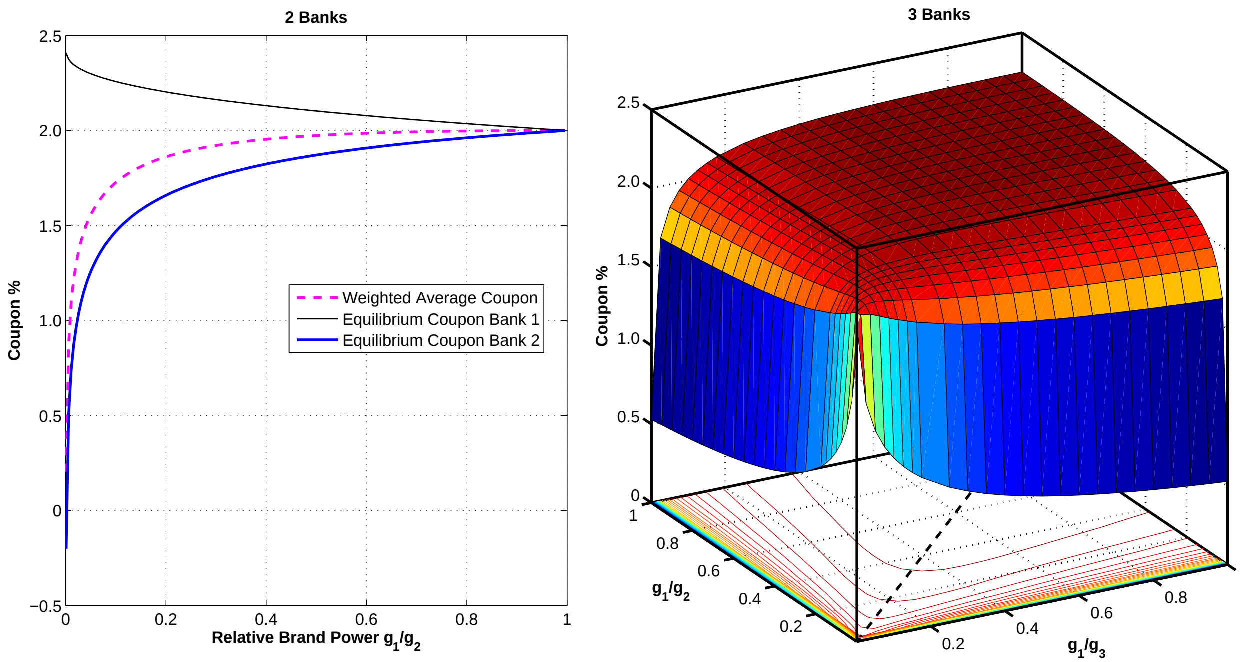

Figure 1: Equilibrium Coupon Rates as a Function of Relative Brand Power
This Figure shows the average coupon weighted by market share as a function of relative brand power for two banks (left) and three banks (right). Hence, the dashed line for two banks on the left plot corresponds to the surface for three banks on the right plot. The maximal average coupon rate is paid if banks have an equal brand power. The more the banks differ, the lower is the average coupon paid. In the three-dimensional plot, the average coupon has the slowest decrease along the dashed diagonal. On the diagonal, bank 2 and bank 3 are equal in brand power.

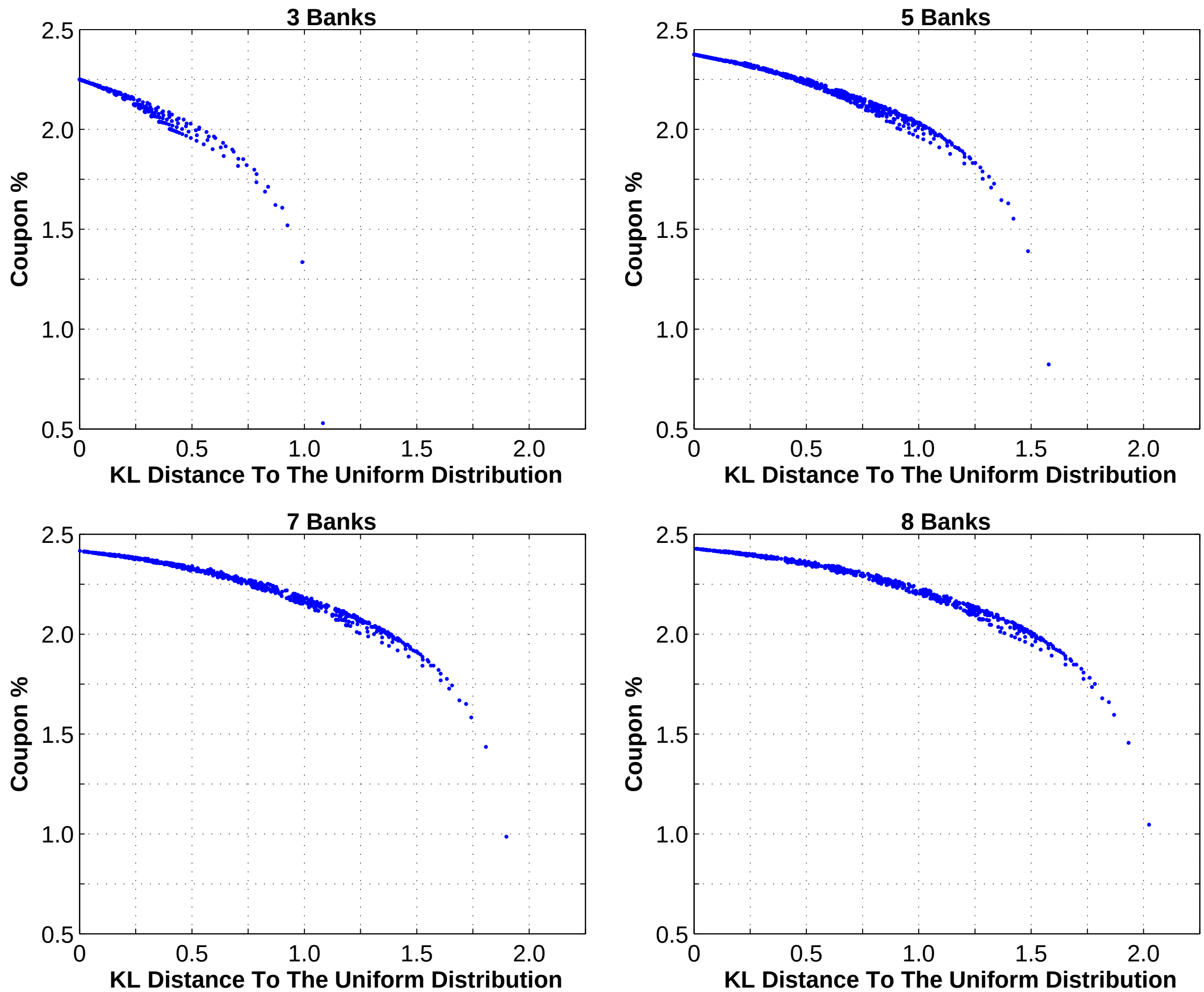

FIGURE 2: EQUILIBRIUM COUPON RATES AS A FUNCTION OF SIMILARITY OF THE BANKS
This Figure shows the average coupon weighted by market share, as in Figure 1, but here as a function of similarity for 3, 5, 7, and 8 banks. The similarity is measured by means of the Kullback-Leibler distance between the brand power variables and the uniform distribution. The distance is zero if all banks are equal in brand power. The graphs show that the more similar the banks are, the higher is the average coupon rate. We can directly compare the graph for 3 banks to the three-dimensional graph of Figure 1.

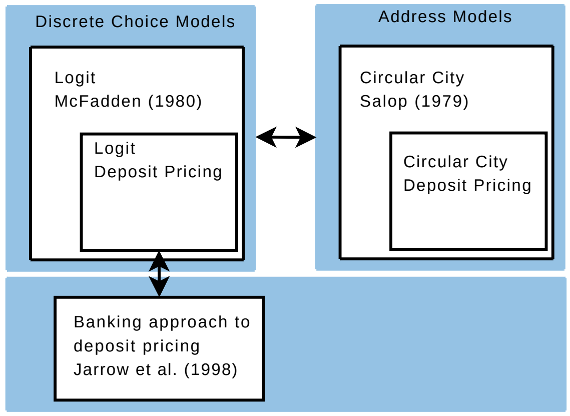

Figure 3: The Link between the Banking Literature and Industrial Organization Theory
Industrial Organization theory uses the circular city model to explain the banks' deposit pricing behavior, see e.g. Chiappori, Perez-Castrillo, and Verdier (1995). We develop our model in the multinomial logit framework. Anderson, De Palma, and Thisse (1989) show that discrete choice models, such as the multinomial logit, can be derived from an address approach. The circular city model belongs to the address models. This connection is represented by the horizontal arrow in the graph. Taylor approximations of our model yield expressions used in the banking literature, as e.g. in Jarrow and van Deventer (1998). This link is represented by the vertical arrow. We conclude that the approach to deposit pricing in Industrial Organization theory is – via our model – related to the approach in the banking literature.

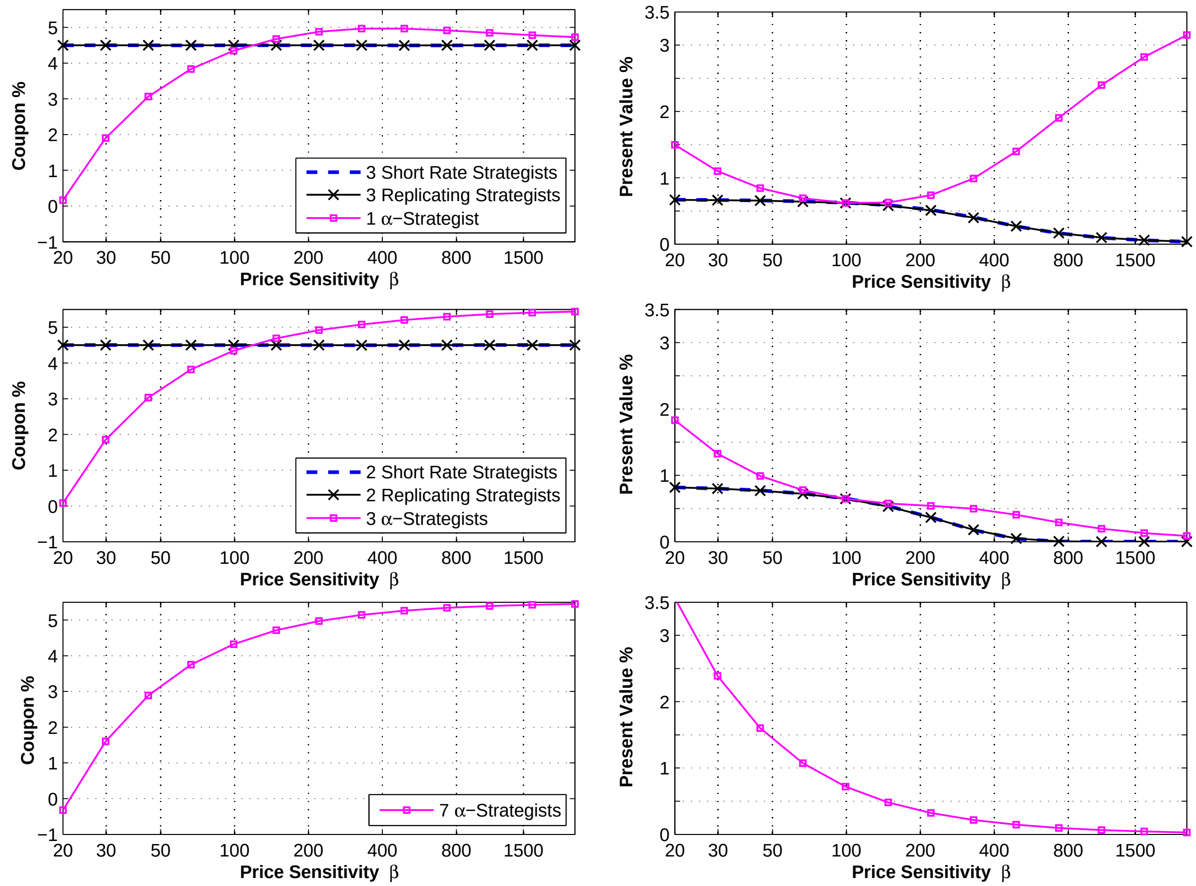

FIGURE 4: PRESENT VALUES AND COUPONS AS A FUNCTION OF PRICE SENSITIVITY
This graph shows coupon rates (left) and present values (right) for three different markets. There are three strategists with equal brand power: A replicating strategist, whose coupon consists of a moving average of different wholesale market rates, a short rate strategist, whose coupon equals the wholesale market rate minus a margin, and an α-strategist whose coupon is given by the profit-optimal rates derived in this paper. In the top row, the market consists of 3 replicating strategists, 3 short rate strategists, and one α-strategist, in the middle row of 2, 2 and 3 respectively, and in the bottom row of 7 α-strategists. The present values are shown in percent of the total volume to be deposited. The α-strategists have a larger present value than the other strategists for all price sensitivities.

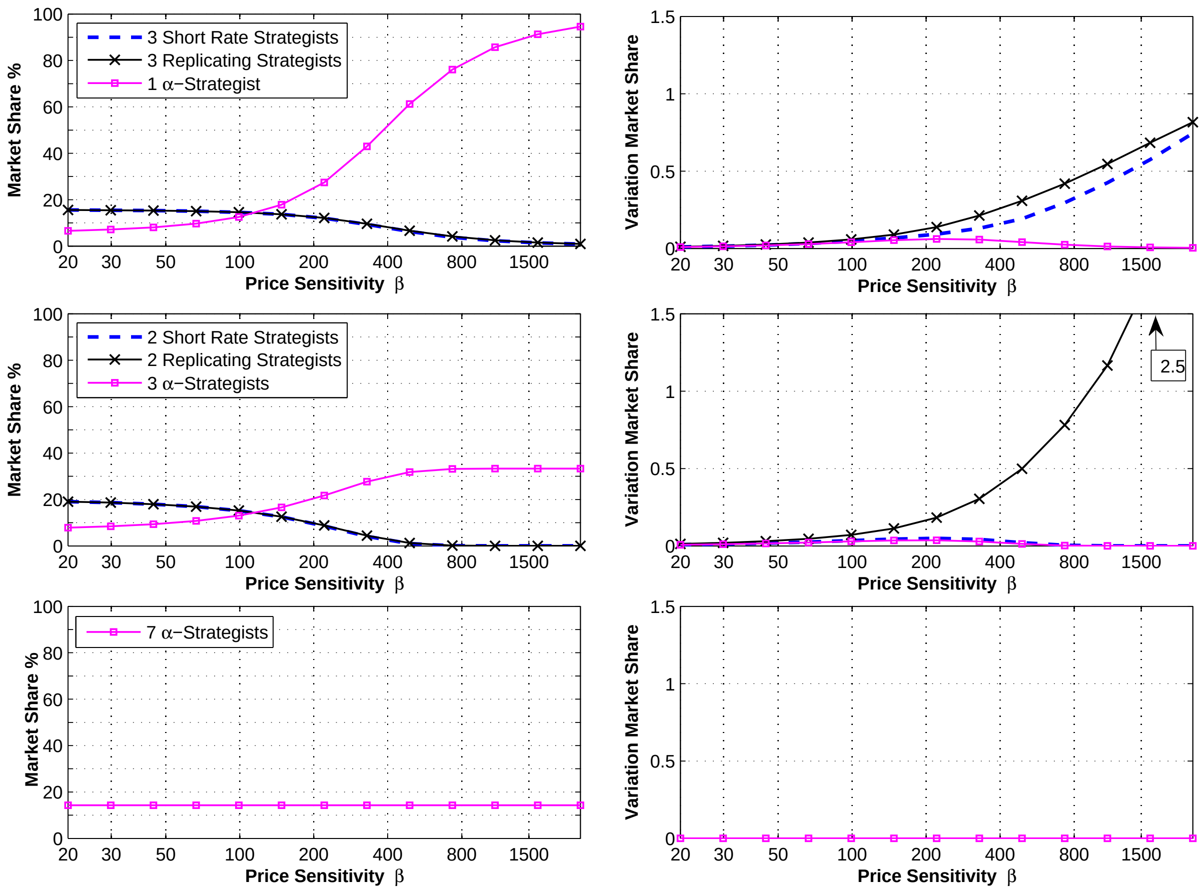

FIGURE 5: MARKET SHARE AS A FUNCTION OF PRICE SENSITIVITY
This graph shows market shares (left) and the coefficient of variation for the market share (right) for three different markets, corresponding to Figure 4. If the price sensitivity is large, the α-strategists have a high market share. If the price sensitivity is low, α-strategists have a skimming strategy and pay a low coupon rate having a low market share. For them, the variation in market share stays below the values of the competitors. With several α-strategists present in the market (middle row), the variation of a short rate strategist stays low too. In contrast, the variation for the market share becomes large (middle row) for the replicating strategist in the presence of several α-strategists. We cropped part of the replicating strategist's variation, having a maximum at 2.5 for $\beta = 3000$, at the right end of the graph.

TABLE A

Present Value Errors in two different Markets.

Market Structure	Price Sensitivity β	Parameter Estimates of the Volume Model ν_0	ν_1	ν_2	ρ	σ_u	Relative Error
1 α, 3 Short Rate, 3 Replicating	20	5.05	9.34	−9.35	0.88	0.01%	1.9%
	30	5.04	13.94	−13.95	0.88	0.01%	1.9%
	50	5.03	22.95	−22.98	0.88	0.04%	1.9%
	100	4.98	44.28	−44.33	0.88	0.15%	1.9%
	200	4.82	81.25	−81.23	0.88	0.54%	2.0%
	400	4.34	149.82	−149.88	0.88	1.74%	2.1%
	800	3.38	329.72	−329.04	0.88	6.46%	3.7%
	1500	2.28	678.70	−677.37	0.88	19.30%	22.2%
3 α, 2 Short Rate, 2 Replicating	20	5.25	7.73	−7.76	0.88	0.01%	1.9%
	30	5.23	11.38	−11.41	0.88	0.01%	1.9%
	50	5.18	18.19	−18.23	0.88	0.03%	1.9%
	100	5.02	32.08	−32.13	0.88	0.13%	1.9%
	200	4.57	45.77	−45.83	0.88	0.40%	1.9%
	400	3.13	30.88	−30.98	0.88	0.67%	1.9%
	800	−0.69	2.32	−2.46	0.85	0.16%	1.9%
	1500	−7.69	−0.10	−0.02	0.75	0.01%	1.9%

Notes: This Table shows the present value errors for a short rate strategist in two different markets. Errors arise when we use an approximated volume model that ignores the competing banks. The error is $(\widehat{P}-\bar{P})/\bar{P}$, where $\bar{P}$ is the present value and $\widehat{P}$ the approximation of the present value. We obtain $\widehat{P}$ by replacing the volume of the replicating strategist by a Taylor approximation. The approximated volume is given by $\widehat{V}(t)=\exp\left(\nu_0+\nu_1 c_{\mathrm{sr}}(t)+\nu_2 c_{\mathrm{rep}}(t)+u(t)\right)$ with $u(t)=\rho u(t-1)+\varepsilon(t)$. The error $\varepsilon(t)$ is normally distributed with mean zero and variance σ_u^2. The variable c_{rep} is the coupon rate of a replicating strategist, c_{sr} the coupon rate of a short rate strategist. For both markets and every price sensitivity, we estimate the parameters of the volume model (ν_0, ν_1, ν_2, σ_u, and ρ). In a second simulation stage, we calculate $\bar{P}$, and $\widehat{P}$ using these estimates. The error is around 2%. With few α-strategists in the market and very high price sensitivities, the error increases.