Measuring Systematic Risk Using Implied Beta in Option Prices

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ABSTRACT

This paper provides a novel method to estimate β thoroughly based on option prices. Through combining the market model and the multivariate risk-neutral valuation relationship in Stapleton and Subrahmanyam (1984) and Câmara (2003), we develop a pricing model for individual stock options involving the volatility of the market index level and the levels of the β and the idiosyncratic risk of the underlying stock asset. Based on this option pricing model, it is possible to estimate β implicitly from the current prices of index options and individual stock options rather than from the historical stock prices in the traditional method. The proposed option pricing model can explain some aspects of volatility smiles and term structures. The empirical studies for the component stocks in Dow Jones Industrial Average (DJIA) show that the option-implied β from this novel method can provide reasonable estimates of β and perform better than historical β in predicting the realized value of β in future periods of time. Furthermore, the results of the competitive regression suggest that the option-implied β contains the information different from that in the historical β.

I. Introduction

In order to estimate the forward-looking β, this paper develops a pricing model for individual stock options involving the volatility of the market index level and the levels of the β and the idiosyncratic risk of the underlying stock asset. Equipped with this novel model, we are able to estimate β for a future period of time purely based on the current prices of index options and individual stock options. Since the seminal work of Capital Asset Pricing Model (CAPM) in Sharpe (1964) and Lintner (1965), the systematic risk and its measurement β are the standard textbook measure of risk. The CAPM and the information of β are commonly used in finance, for instance in calculating the cost of capital, pricing the value of investments, or constructing a portfolio with a desired level of the systematic risk. Although CAPM has been subject to some criticisms (e.g., Roll (1977) and Fama and French (1992)), CAPM remains well on the frontier of both academic research and industry applications. Thus, pursuing accurate estimates of β, in particular for future periods of time, is an important issue for so long.

Although the information of β should reflect the sensitivity of the individual stock return to the return of the market portfolio for a future period of time, it is unobservable and estimated from a backward viewpoint traditionally, i.e., using historical excess returns of the individual stock and the market portfolio to estimate the β of an asset based on the regression model. This method was first proposed by Jensen (1968), who noted that the Sharpe-Lintner version of the linear relation between expected excess returns and the β also implies a regression test. It is known as the market or single-index model which takes the following form:

$$R_{i,t} - R_{f,t} = \alpha_i + \beta_i\left(R_{m,t} - R_{f,t}\right) + \varepsilon_{i,t},$$

where $R_{i,t} - R_{f,t}$ and $R_{m,t} - R_{f,t}$ are the excess returns on the asset i and the market portfolio at time t, respectively. The random variable $\varepsilon_{i,t}$ is a white noise and independently identical distributed over time. The coefficients α_i and β_i are constant parameters to be estimated. Under the normal assumptions, the ordinary least squares method provides unbiased estimates for α_i and β_i. The level of β_i assesses the impact of the price changes in the market portfolio on the price changes in asset i. The CAPM implies that the intercept term α_i in the regression, also termed "Jensen's alpha," should be zero for any asset.

This traditional backward-looking method to estimate (or to extrapolate) the future β may not perform well unless the patterns of beta are known and stable in the near future. However, numerous studies find that violations of the assumption of stationarity and independently identical distributed returns are rules rather than exceptions, such as Blume (1971, 1975), Baesel (1974), Klemkosky and Martin (1975), Roenfeldt, Griepentrog, and Pflaum (1978), Fabozzi and Francis (1978), Theobald (1981), Bos and Newbold (1984), Collins, Ledolter, and Rayburn (1987) and Faff, Lee, and Fry (1992). Moreover, when firms are involved in some events, like mergers or acquisitions, undertaking large-scale projects, or changing their capital structure, historical returns cannot provide adequate information to estimate the future β reliably. Although there are several financial econometric models to remedy this problem,[1] it is difficult to assess the bias associated with the estimation of *ex ante* information using *ex post* historical data.

To estimate the future β, some authors combine the implied volatilities from option prices and the historical correlation of individual equity and equity index prices, e.g., French, Groth, and Kolari (1983). However, a radically different

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[1] For instance, the conditional beta or other time-varying models of beta are considered in Ferson, Kandel, and Stambaugh (1987), Jagannathan and Wang (1996), Ferson and Harvey (1999), Park (2004), Fraser, Hamelink, Hoesli, and Macgregor (2004), and Adrian and Franzoni (2009).

methodology to estimate a forward-looking beta based on option prices is first introduced by Siegel (1995). Given the implied volatilities of the individual stock and the market portfolio, he exploits the option valuation model for exchange options developed in Margrabe (1978) to price an option which exchanges the individual stock return to the market index return and thus can calibrate the β from option prices. The core of Siegel's model depends on the existence of that kind of exchange option traded in the market. However, that kind of exchange option is not generally available in the market, so Siegel's method to estimate β is not practical.

There are several empirical studies, such as Bates (1998), Buraschi and Jackwerth (2001), Dennis and Mayhew (2002), and Bakshi, Kapadia, and Madan (2003), demonstrating the existence of systematic risk factors in option prices. Therefore, some academics try to extract the market beta from option prices.

Husmann and Stephan (2007), extending the model in Jarrow and Madan (1997) to price stock index options in an incomplete market, introduce an option pricing formula for individual equity options based on the CAPM so that the value of β can be estimated implicitly from the current market prices of individual equity options. However, since their model do no rely on the risk neutral valuation method, in addition to the correlation (or the β) parameter, there are other risk preference parameters in their option pricing formula, such as the expected returns of the individual stock as well as the market portfolio. Chen, Kim, and Panda (2009) and Chang, Christoffersen, Jacobs, and Vainberg (2010) also estimate the β from current option prices. To estimate option-implied β for a future period of time, Chen et al. (2009) derive the counterpart of the Black-Scholes formula under the physical measure and then links the expected returns of the underlying stock and the option with a single-index model. It is unavoidable that they require the expected stock return, which is preference-dependent, in their option pricing formula. Chang et al.

(2010) extend the method in Bakshi, Kapadia, and Madan (2003), which studies the relationship between the option-implied variance and skewness of the underlying asset return. However, this stream of models ignores that the possible conflict between the assumptions of the normal distribution in CAPM and the existence of skewness in stock returns.

Building on Siegel (1995) and Husmann and Stephen (2007), we propose a new method to estimate β purely based on quoted option prices. The core of this new method is to develop an option pricing model by combining the market model with the multivariate risk-neutral valuation relationship (RNVR) in Stapleton and Subrahmanyam (1984) and Câmara (2003). The RNVR is a very useful technique for asset pricing, especially for derivatives whose payoffs are determined by one or several underlying variables, whether traded or non-traded. The RNVR-based models exploit simply the relationship between derivatives and their underlying assets to derive the preference-free derivative pricing formula. As a consequence, the preference parameters are not involved in the valuation equation of the derivatives, and the expected return on the underlying assets is the riskless return.

The RNVR was first developed by Rubinstein (1976) and Brennan (1979) for derivatives pricing when there is only one underlying variable. Under the sufficient conditions that there is a representative agent whose risk preference is with an exponential representation, and his period-end wealth and the underlying asset price are jointly normally distributed, the RNVR can generate a preference-free option pricing formula, which could be identical to the formula in Black and Scholes (1973). Stapleton and Subrahmanyam (1984) extend the model in Brennan (1979) by allowing for multiple underlying variables. More recently, Câmara (2003) provides a generalized RNVR framework to encompass a family of bivariate transformed normal distribution for the period-end wealth and the underlying asset price to price derivatives. The transformed normal distributions considered in Câmara (2003) include the normal, lognormal, displaced lognormal, negatively skew lognormal, and S_U distributions. Câmara (2005) expands Câmara (2003) to a multivariate setting. By considering the multivariate transformed normal distribution, Câmara's method contributes substantially to the literature to provide different risk-neutral option pricing formulae with different preferences and alternative joint distributions of the state variables.

In comparison with the continuous time option pricing model proposed in Black and Scholes (1973), the discrete time RNVR-based model is less popular. However, Black and Scholes only assume that investors prefer more wealth to less but assume nothing about the risk preference of an investor. In contrast, the RNVR approach is based on the market equilibrium rather than on the no-arbitrage argument in the Black and Scholes framework. Therefore, the RNVR approach is highly general and useful in asset pricing especially when additional assumptions are imposed on the model. Our option pricing model is based on the RNVR approach together with the assumption that returns of individual stocks follow the market model.

We consider a general exponential form of marginal utility function for the representative agent and transformed normal distributions for state variables. Additionally, according to the market model, the return of the individual stock is assumed to be the sum of the market risk premium and an idiosyncratic risk component. Thus, the two state variables in the RNVR are the market index return and the idiosyncratic risk component of the individual stock. Finally, a preference-free pricing model for individual stock options is derived, based on the volatility of the market index level and the levels of the β and the idiosyncratic risk of the underlying stock asset. This novel option pricing formula enables the estimation of β purely based on prices of stock index and individual stock options. This paper not only provides an alternative to estimating β but is the first model to estimate β in a purely forward-looking way based only on quoted option prices.

The remainder of this paper is organized as follows. Section II presents the framework of multivariate risk-neutral valuation relationship. Section III is devoted to derive an explicit risk-neutral option pricing model for European calls and discuss the relationship between the implied volatility smiles or term structures and the levels of the β and the idiosyncratic risk of the individual stock. Section IV presents the empirical results of this paper, including the description of the data of option prices, the introduction of the calibration procedures, and the analysis of the calibrated results. Section V concludes this paper.

II. The General Multivariate Risk-Neutral Valuation Relationship

In a one-period economy, for the case of multiple underlying processes, the standard pricing relationship for derivatives considered in Brennan (1979), Stapleton and Subrahmanyam (1984), and Câmara (2003) is

$$V = R_f^{-1} E^P[C(\mathbf{X}) \cdot Z(\mathbf{X})], \tag{1}$$

where V is the price of a derivative contract today, R_f is the gross return of the risk-free asset for the examined period, and $E^P[\cdot]$ stands for the expectation operator under the actual probability measure. The variable $\mathbf{X}$ is a vector of payoffs of n underlying processes and $\mathbf{X}' \equiv [X_1, X_2, \ldots X_n]$, $C(\mathbf{X})$ is the payoff function for the derivative contract, and $Z(\mathbf{X})$ is the asset-specific pricing kernel. Following Câmara (2003), the definition of $Z(\mathbf{X})$ is given by:

$$Z(\mathbf{X}) = \frac{E^P[U'(w)|\mathbf{X}]}{E^P[U'(w)]}, \tag{2}$$

where $U'(w)$ is the marginal utility function of the wealth of the representative agent on the maturity date.

In addition, the period-end wealth, *w*, is assumed to follow a transformed normal distribution:[2]

$$g(w) \sim N(\mu_w, \sigma_w^2),$$

where μ_w and σ_w^2 are the mean and variance of $g(w)$, and $g(\cdot)$ is a strictly monotonic and differentiable function. Following the method in Câmara (2003) to generalize the Brennan-Rubinstein approach for pricing derivatives, we assume that the representative agent's marginal utility is of the exponential form:

$$U'(w) = \exp[\alpha g(w)],$$

where α is a constant preference parameter. As a consequence, it is straightforward to infer that $\ln U'(w) \sim N(\alpha\mu_w, \alpha^2\sigma_w^2)$. Câmara (2003) argues that the exact functional form of $g(\cdot)$ is not critical as long as $g(w)$ is normally distribution. For example, if the period-end wealth is normally distributed, $U'(w) = \exp(\alpha w)$ given $g(w) = w$ should be considered, which implies that the representative agent has constant absolute aversion; it is a necessary condition to derive the RNVR for a bivariate normal distribution of the price of the underlying assets and period-end wealth. On the other hand, if the period-end wealth follows a lognormal distribution, the representative individual's marginal utility $U'(w)$ is a power function, that is $U'(w) = w^\alpha$ given $g(w) = \ln w$, which implies that the representative agent is with constant proportional aversion; it is a necessary condition to derive the RNVR for a

[2] The definition of the transformed normal random variable is that a random variable Y can be expressed as $Y = f^{-1}(Z\sigma + \mu)$, where $Z \sim N(0,1)$, and f is a strictly monotonic differentiable function.

bivariate lognormal distribution of the price of the underlying assets and period-end wealth.

As to the underlying assets or state variables, we assume the joint distribution of the underlying assets at the end of the period to be a multivariate transformed normal distribution:

$$\mathbf{h}(\mathbf{X}) = \big(h_1(X_1), h_2(X_2), \dots, h_n(X_n)\big)^T \sim N(\boldsymbol{\mu}, \boldsymbol{\Sigma}),$$

where $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$ are the mean vector and the variance-covariance matrix of the underlying multivariate normal distribution, and h_i's are arbitrarily strictly monotonic and differentiable functions. The probability density function of the underlying assets $X_1, X_2, \dots, X_n$ is:

$$\phi^P(X_1, X_2, \dots, X_n) = \frac{1}{(2\pi)^{n/2}|\boldsymbol{\Sigma}|^{1/2}} |\mathbf{h}'(\mathbf{X})| \exp\left[-\frac{1}{2}\big((\mathbf{h}(\mathbf{x}) - \boldsymbol{\mu})\big)^T \boldsymbol{\Sigma}^{-1}\big((\mathbf{h}(\mathbf{x}) - \boldsymbol{\mu})\big)\right]. \quad (3)$$

In addition, the wealth of the representative agent and the underlying asset prices at the end of the period are further assumed to follow a jointly transformed normal distribution. Therefore, the conditional distribution of the representative agent's marginal utility is

$$(\ln U'(w)|\mathbf{X}) \sim N(\alpha\mu_w + \boldsymbol{\Sigma}_{w\mathbf{X}}\boldsymbol{\Sigma}^{-1}(\mathbf{h}(\mathbf{X}) - \boldsymbol{\mu}), \alpha^2\sigma_w^2 - \boldsymbol{\Sigma}_{w\mathbf{X}}\boldsymbol{\Sigma}^{-1}\boldsymbol{\Sigma}_{w\mathbf{X}}^T),$$

where $\boldsymbol{\Sigma}_{w\mathbf{X}}$ is a row vector representing the covariances between $\ln U'(w)$ and $\mathbf{X}$. According to the property of the lognormally distribution, it is straightforward to drive $E^P[U'(w)|\mathbf{X}]$ and $E^P[U'(w)]$. As a result, the pricing kernel $Z(\mathbf{X})$ is expressed as

$$Z(\mathbf{X}) = \exp\left[\alpha\boldsymbol{\Sigma}_{w\mathbf{X}}\boldsymbol{\Sigma}^{-1}\big((\mathbf{h}(\mathbf{X}) - \boldsymbol{\mu})\big) - \frac{\alpha^2}{2}\boldsymbol{\Sigma}_{w\mathbf{X}}\boldsymbol{\Sigma}^{-1}\boldsymbol{\Sigma}_{w\mathbf{X}}^T\right]. \quad (4)$$

Substituting Equations (3) and (4) into Equation (1), the derivative valuation formula becomes as follows.

$$V = R_f^{-1} \int_{X_n} \int_{X_{n-1}} \cdots \int_{X_1} \frac{C(\mathbf{X})}{(2\pi)^{n/2}|\mathbf{\Sigma}|^{1/2}} |\mathbf{h}'(\mathbf{X})|$$

$$\cdot \exp\left[-\frac{1}{2}[\mathbf{h}(\mathbf{X}) - (\boldsymbol{\mu} + \alpha\mathbf{\Sigma}_{w\mathbf{X}})]^T \mathbf{\Sigma}^{-1} [\mathbf{h}(\mathbf{X}) - (\boldsymbol{\mu} + \alpha\mathbf{\Sigma}_{w\mathbf{X}})]\right] dX_1 dX_2 \ldots dX_n, \quad (5)$$

and the current price of the underlying assets **P** is

$$\mathbf{P} = R_f^{-1} \int_{X_n} \int_{X_{n-1}} \ldots \int_{X_1} \frac{\mathbf{X}}{(2\pi)^{n/2}|\mathbf{\Sigma}|^{1/2}} |\mathbf{h}'(\mathbf{X})|$$

$$\cdot \exp\left[-\frac{1}{2}[\mathbf{h}(\mathbf{X}) - (\boldsymbol{\mu} + \alpha\mathbf{\Sigma}_{w\mathbf{X}})]^T \mathbf{\Sigma}^{-1} [\mathbf{h}(\mathbf{X}) - (\boldsymbol{\mu} + \alpha\mathbf{\Sigma}_{w\mathbf{X}})]\right] dX_1 dX_2 \ldots dX_n, \quad (6)$$

where the location parameter of the density of the underlying assets is a preference-related term $\boldsymbol{\mu} + \alpha\mathbf{\Sigma}_{w\mathbf{X}}$. Under the market equilibrium, if the expectation of Equation (6) has an inverse function for its location parameter $\boldsymbol{\mu} + \alpha\mathbf{\Sigma}_{w\mathbf{X}}$, then the resulting expression can be substituted into Equation (5), i.e., $\boldsymbol{\mu} + \alpha\mathbf{\Sigma}_{w\mathbf{X}}$ can be replaced by a function of $\mathbf{P} \cdot R_f^{-1}$. As a consequence, the probability density function of **X** without preference parameters can be derived, and thus the price of the derivative can be written as a preference-free option pricing equation:

$$V = R_f^{-1} E^Q[C(\mathbf{X})],$$

where $E^Q[\cdot]$ denotes the expectation under the risk-neutral probability measure Q. Since the location parameter is a function of R_f in the corresponding risk-neutral probability density function of **X**, it is consistent to the classic option pricing theory that to price derivatives in the risk-neutral world, the expected returns of all assets are equal to the riskless return. In the next section, we will apply this RNVR method to the pricing of individual stock options by incorporating the market model to formulate the return of the underlying stock price.

III. The Option Pricing Model Involving $\boldsymbol{\beta}$

III.1 Distributions of Underlying Variables and the Aggregate Wealth

In the case of pricing individual stock options, we consider two points of time–today is time 0 and the maturity date of the option is time T. Taking European call options as examples,[3] the payoff at the time point T is

$$\max(S_T - K, 0).$$

In addition, according to the market model, we can transform the payoff of the equity call option into a multivariate function of the market index return and an idiosyncratic risk component. That is

$$\begin{aligned}
&\max(S_T - K, 0) \\
&= S_0 \cdot \max\left[\frac{S_T - K}{S_0}, 0\right] \equiv S_0 \cdot \max(R_T - k, 0) \\
&= S_0 \cdot \max\left\{\left((1-\beta)R_f + \beta R_m + r_e\right) R_q^{-1} - k, 0\right\} \\
&\equiv C(R_m, r_e),
\end{aligned} \tag{7}$$

where S_T is the stock price at time T, S_0 is the stock price at time 0, and k is defined as K/S_0. In addition, $R_T \equiv S_T/S_0$ is the gross return on the underlying stock, R_f is the gross return of the risk-free asset between time 0 and time T, and R_m is the gross

[3] As for individual put options, it is straightforward to derive the counterpart option pricing formulae by simply considering the payoff of put options in Equation (7) and following the same procedure mentioned as follows.

return on the market portfolio from time 0 to time T and assumed to follow a lognormal distribution, i.e., $\ln R_m \sim N(\mu_m T, \sigma_m^2 T)$, where μ_m is the annualized expected return of the market portfolio, and σ_m^2 is the variance of the annualized market index return. The idiosyncratic (or firm-specific) risk component r_e associated with the underlying individual stock is assumed to follow $N(0, \sigma_e^2 T)$, where the expected return on the idiosyncratic risk component is zero by definition, and σ_e^2 is the corresponding annual variance for the idiosyncratic risk component. Finally, $R_q \equiv e^{qT}$, where q is the annualized dividend yield, reflecting the decline of the stock price due to dividend payments .

Following the RNVR approach introduced in Section II, we assume that the period-end wealth of the representative agent w, the gross return on the market portfolio R_m, and the idiosyncratic risk component r_e follow the trivariate transformed normal distribution as follows.

$$\begin{pmatrix} g(w) \\ h_1(R_m) = \ln(R_m) \\ h_2(r_e) = r_e \end{pmatrix} \sim N\left(\begin{pmatrix} \mu_w \\ \mu_m T \\ 0 \end{pmatrix}, \begin{pmatrix} \sigma_w^2 & \rho_{wm}\sigma_w\sigma_m\sqrt{T} & \rho_{we}\sigma_w\sigma_e\sqrt{T} \\ \rho_{wm}\sigma_w\sigma_m\sqrt{T} & \sigma_m^2 T & 0 \\ \rho_{we}\sigma_w\sigma_e\sqrt{T} & 0 & \sigma_e^2 T \end{pmatrix}\right),$$

where the idiosyncratic risk component is assumed to be independent of the return on the market portfolio according to the assumption of the market model, and the correlations between $g(w)$ and $\ln(R_m)$ or r_e are assumed to be ρ_{wm} and ρ_{we}, respectively.

Since the representative agent's marginal utility follows

$$U'(w) = \exp\big(\alpha g(w)\big),$$

where α is a constant preference parameter. Therefore, $U'(w)$ is lognormally distributed with the mean $\alpha\mu_w$ and the variance $\alpha^2\sigma_w^2$. Consequently, the mean of this lognormal random variable is

$$E^P[U'(w)] = \exp\left(\alpha\mu_w + \frac{1}{2}\alpha^2\sigma_w^2\right).$$

Furthermore, since

$$(\ln U'(w)|R_m, r_e) \sim N(\alpha\mu_w + \alpha\rho_{wm}\frac{\sigma_w}{\sigma_m\sqrt{T}}(\ln R_m - \mu_m T) + \alpha\rho_{we}\frac{\sigma_w}{\sigma_e\sqrt{T}}(r_e - 0),$$

$$\alpha^2\sigma_w^2(1 - \rho_{wm}^2 - \rho_{we}^2)\big),$$

we can obtain that

$$E^P[U'(w)|R_m, r_e] = \exp\left(\alpha\mu_w + \alpha\rho_{wm}\frac{\sigma_w}{\sigma_m\sqrt{T}}(\ln R_m - \mu_m T) + \alpha\rho_{we}\frac{\sigma_w}{\sigma_e\sqrt{T}}(r_e - 0)\right.$$
$$\left. + \frac{1}{2}\alpha^2\sigma_w^2(1 - \rho_{wm}^2 - \rho_{we}^2)\right).$$

Following Equation (2), we can obtain the asset-specific pricing kernel $Z(R_m, r_e)$ as follows.

$$Z(R_m, r_e) = \frac{E^P[U'(w)|R_m, r_e]}{E^P[U'(w)]} = \exp\left(\alpha\rho_{wm}\frac{\sigma_w}{\sigma_m\sqrt{T}}(\ln R_m - \mu_m T) + \alpha\rho_{we}\frac{\sigma_w}{\sigma_e\sqrt{T}}(r_e - 0)\right.$$
$$\left. - \frac{1}{2}\alpha^2\rho_{wm}^2\sigma_w^2 - \frac{1}{2}\alpha^2\rho_{we}^2\sigma_w^2\right) \qquad (8)$$

III.2 The Risk Neutral Valuation Relationship

According to the assumption about the return on the market portfolio and the idiosyncratic risk component following a bivariate transformed normal distribution by setting $h_1(R_m) = \ln(R_m)$ and $h_2(r_e) = r_e$, we can derive $h_1'(R_m) = 1/R_m$ and $h_2'(r_e) = 1$ and rewrite Equation (3) as follows.

$$\phi^P(R_m, r_e) = \frac{1}{\sqrt{2\pi}\sigma_m\sqrt{T}R_m}\exp\left[-\frac{1}{2\sigma_m^2 T}(\ln R_m - \mu_m T)^2\right]\cdot$$

$$\frac{1}{\sqrt{2\pi}\sigma_e\sqrt{T}}\exp\left[-\frac{1}{2\sigma_e^2 T}(r_e - 0)^2\right]. \qquad (9)$$

Given the density of $\phi^P(R_m, r_e)$ in Equation (9) and the asset-specific pricing kernel $Z(R_m, r_e)$ in (8), the option pricing formula (5) can be rewritten as:

$$V = R_f^{-1}\int_{-\infty}^{\infty}\int_0^{\infty}\frac{C(R_m, r_e)}{\sqrt{2\pi}\sigma_m\sqrt{T}R_m}\cdot\exp\left[-\frac{1}{2\sigma_m^2 T}\left(\ln R_m - \left(\mu_m T + \alpha\rho_{wm}\sigma_w\sigma_m\sqrt{T}\right)\right)^2\right]\cdot$$

$$\frac{1}{\sqrt{2\pi}\sigma_e\sqrt{T}}\exp\left[-\frac{1}{2\sigma_e^2 T}\left(r_e - \left(0 + \alpha\rho_{we}\sigma_w\sigma_e\sqrt{T}\right)\right)^2\right]dR_m dr_e. \qquad (10)$$

According to Equation (6), we can obtain the present value for the market portfolio return and the idiosyncratic risk component to be

$$\mathbf{P} = \begin{pmatrix} P_m \\ P_e \end{pmatrix}$$

$$= R_f^{-1}\int_{-\infty}^{\infty}\int_0^{\infty}\begin{pmatrix} R_m \\ r_e \end{pmatrix}\cdot\frac{1}{\sqrt{2\pi}\sigma_m\sqrt{T}R_m}\cdot\exp\left[-\frac{1}{2\sigma_m^2 T}\left(\ln R_m - \left(\mu_m T + \alpha\rho_{wm}\sigma_w\sigma_m\sqrt{T}\right)\right)^2\right]$$

$$\cdot\frac{1}{\sqrt{2\pi}\sigma_e\sqrt{T}}\exp\left[-\frac{1}{2\sigma_e^2 T}\left(r_e - \left(0 + \alpha\rho_{we}\sigma_w\sigma_e\sqrt{T}\right)\right)^2\right]dR_m dr_e.$$

The current prices of the underlying variables R_m and r_e, which are the return on the market portfolio and the idiosyncratic risk component for the individual stock, are 1 and 0 by definition. Hence

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} R_f = \begin{pmatrix} \exp\left(\mu_m T + \alpha\rho_{wm}\sigma_w\sigma_m\sqrt{T} + \frac{1}{2}\sigma_m^2 T\right) \\ 0 + \alpha\rho_{we}\sigma_w\sigma_e\sqrt{T} \end{pmatrix}.$$

Then we can obtain the following relationship:

$$\begin{pmatrix} \mu_m T + \alpha\rho_{wm}\sigma_w\sigma_m\sqrt{T} \\ 0 + \alpha\rho_{we}\sigma_w\sigma_e\sqrt{T} \end{pmatrix} = \begin{pmatrix} \ln R_f - \frac{1}{2}\sigma_m^2 T \\ 0 \end{pmatrix}. \tag{11}$$

Substituting Equation (11) into Equation (10), the current price of the option is given by

$$V = R_f^{-1} \int_{-\infty}^{\infty}\int_0^{\infty} \frac{C(R_m, r_e)}{\sqrt{2\pi}\sigma_m\sqrt{T}R_m} \cdot \exp\left[-\frac{1}{2\sigma_m^2 T}\left(\ln R_m - \left(\ln R_f - \frac{1}{2}\sigma_m^2 T\right)\right)^2\right] \cdot$$

$$\frac{1}{\sqrt{2\pi}\sigma_e\sqrt{T}} \cdot \exp\left[-\frac{1}{2\sigma_e^2 T}(r_e - 0)^2\right] dR_m dr_e$$

$$\equiv R_f^{-1} E^Q[C(R_m, r_e)], \tag{12}$$

with a preference-free probability density function

$$\phi^Q(R_m, r_e) = \frac{1}{\sqrt{2\pi}\sigma_m\sqrt{T}R_m} \cdot \exp\left[-\frac{1}{2\sigma_m^2 T}\left(\ln R_m - \left(\ln R_f - \frac{1}{2}\sigma_m^2 T\right)\right)^2\right] \cdot$$

$$\frac{1}{\sqrt{2\pi}\sigma_e\sqrt{T}} \exp\left[-\frac{1}{2\sigma_e^2 T}(r_e - 0)^2\right].$$

Finally, by substituting the payoff function in Equation (7) into Equation (12) and defining $\ln R_m = Z_m$, and $r_e = Z_e$, we can obtain:

$$V = R_f^{-1} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} S_0 \cdot \max\left[\left((1-\beta)R_f + \beta e^{Z_m} + Z_e\right) R_q^{-1} - k, 0\right] \cdot$$

$$\frac{1}{\sqrt{2\pi}\sigma_m\sqrt{T}} \cdot \exp\left[-\frac{1}{2\sigma_m^2 T}(Z_m - \mu_m^*)^2\right] \cdot$$

$$\frac{1}{\sqrt{2\pi}\sigma_e\sqrt{T}} \cdot \exp\left[-\frac{1}{2\sigma_e^2 T}(Z_e - \mu_e^*)^2\right] dZ_m dZ_e, \tag{13}$$

where $\mu_m^* = \ln R_f - \frac{1}{2}\sigma_m^2 T$ and $\mu_e^* = 0$. Finally, the RNVR expression for the call option price in a bivariate normal distribution can be presented as:

$$V = R_f^{-1} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} S_0 \cdot \max\left[\left((1-\beta)R_f + \beta e^{Z_m} + Z_e\right) R_q^{-1} - k, 0\right] \phi^*(Z_m)\phi^*(Z_e) dZ_m dZ_e, \tag{14}$$

where $\phi^*(Z_m)$ and $\phi^*(Z_e)$ are the normal probability density functions for $Z_m \sim$

$N\left(\ln R_f - \frac{1}{2}\sigma_m^2 T, \sigma_m^2 T\right)$ and $Z_e \sim (0, \sigma_e^2 T)$.

III.3 Deriving the Pricing Formula for European Calls

In order to eliminate the max function in the payoff of the European call option in Equation (14), we need to figure out the constraints for Z_m and Z_e such that the call option is in the money at maturity. Three cases are considered as follows:

Case 1: When $\beta > 0$, we can infer that βe^{Z_m} is positive for any value of Z_m. Therefore, the payoff function of the call option in Equation (14) is in the money if $\left((1-\beta)R_f + Z_e\right) R_q^{-1} - k > 0$. On the contrary, if $\left((1-\beta)R_f + Z_e\right) R_q^{-1} - k \leq 0$, a constraint for Z_m is needed to ensure that the call option is in the money at maturity. Consequently, the following two situations can be derived.

i. $Z_e > R_q k - (1-\beta)R_f$ and $-\infty < Z_m < \infty$.

ii. $Z_e \leq R_q k - (1-\beta)R_f$ and $\beta e^{Z_m} - \left[R_q k - (1-\beta)R_f - Z_e\right] > 0$,

which implies $Z_m > \ln\left[\frac{R_q k - (1-\beta)R_f - Z_e}{\beta}\right]$.

As a result, the option price V can be expressed as

$$V = R_f^{-1} S_0 \int_{R_q k - (1-\beta)R_f}^{\infty} \int_{-\infty}^{\infty} \left[\left((1-\beta)R_f + \beta e^{Z_m} + Z_e\right) R_q^{-1} - k\right] \phi^*(Z_m)\, dZ_m\, \phi^*(Z_e)\, dZ_e$$

$$+R_f^{-1} S_0 \int_{-\infty}^{R_q k - (1-\beta)R_f} \int_{\ln(a - bZ_e)}^{\infty} \left[\left((1-\beta)R_f + \beta e^{Z_m} + Z_e\right) R_q^{-1} - k\right] \phi^*(Z_m)\, dZ_m\, \phi^*(Z_e)\, dZ_e, \quad (15)$$

where $a = \frac{R_q k - (1-\beta)R_f}{\beta}$ and $b = \frac{1}{\beta}$. The integration result of the above equation is shown as follows:

$$V = S_0 e^{-rT}\left[\left(e^{(r-q)T} - k\right) \cdot N(M_1) + e^{-qT} \sigma_e \sqrt{T} \cdot n(-M_1)\right]$$

$$+S_0 e^{-rT} \int_{-\infty}^{e^{qT}k - (1-\beta)e^{rT}} \{\left[\left((1-\beta)e^{rT} + Z_e\right) e^{-qT} - k\right] \cdot N(D_2)$$

$$+\beta e^{(r-q)T} \cdot N(D_1)\} \cdot \phi^*(Z_e)\, dZ_e, \tag{16}$$

where

$$M_1 = \frac{(1-\beta)e^{rT} - e^{qT}k}{\sigma_e\sqrt{T}},$$

$$D_1 = \frac{-\ln(a - bZ_e) + \left(rT + \frac{\sigma_m^2 T}{2}\right)}{\sigma_m\sqrt{T}},$$

$$D_2 = D_1 - \sigma_m\sqrt{T},$$

and $N(\cdot)$ and $n(\cdot)$ are the cumulative distribution function and the probability density function of the standard normal distribution. In addition, the definitions of $R_f \equiv e^{rT}$ and $R_q \equiv e^{qT}$ are introduced to further simplify the above equation. The details to derive Equation (16) are presented in Appendix II.

Case 2: When $\beta = 0$, the integral for Z_m is not needed and thus can be dropped. In addition, the option is in the money at maturity when $\left(R_f + Z_e\right)R_q^{-1} - k > 0$, i.e., $Z_e > R_q k - R_f$. As a consequence, we can obtain the option price V as follows. The details to derive Equation (17) are presented in Appendix II.

$$V = R_f^{-1} S_0 \int_{R_q k - R_f}^{\infty} \left[\left(R_f + Z_e\right)R_q^{-1} - k\right] \cdot \phi^*(Z_e)\, dZ_e$$

$$= S_0 e^{-rT}\left[\left(e^{(r-q)T} - k\right) \cdot N(M_2) + e^{-qT}\sigma_e\sqrt{T} \cdot n(-M_2)\right], \tag{17}$$

where

$$M_2 = \frac{e^{rT} - e^{qT}k}{\sigma_e\sqrt{T}}.$$

The option pricing formula in this case is consistent with the formula in the case of the jointly normal distribution for the underlying stock price and the aggregate wealth in Brennan (1979).

Case 3: When $\beta < 0$, as long as $Z_e > R_q k - (1-\beta)R_f$, and $\beta e^{Z_m} + [Z_e - R_q k + (1-\beta)R_f] \geq 0$, the option is in the money at maturity. Thus we can obtain the constraints for Z_e and Z_m as follows: $Z_e > R_q k - (1-\beta)R_f$ and $Z_m \leq \ln\left[\frac{R_q k-(1-\beta)R_f-Z_e}{\beta}\right] = \ln(a - bZ_e)$.

Consequently, we can express the option price as

$$V = R_f^{-1} S_0 \int_{R_q k-(1-\beta)R_f}^{\infty} \int_{-\infty}^{\ln(a-bZ_e)} \left[\left((1-\beta)R_f + \beta e^{Z_m} + Z_e\right)R_q^{-1} - k\right] \phi^*(Z_m)\, dZ_m\, \phi^*(Z_e)\, dZ_e.$$

As shown in Appendix II, the result of the above integration is as follows.

$$V = S_0 e^{-rT} \int_{e^{qT}k-(1-\beta)e^{rT}}^{\infty} \Big\{\left[\left((1-\beta)e^{rT} + Z_e\right)e^{-qT} - k\right] \cdot N(-D_2) + \beta e^{(r-q)T} \cdot N(-D_1)\Big\} \cdot \phi^*(Z_e)\, dZ_e. \qquad (18)$$

As a result, the option pricing formulae for individual call options are expressed in Equations (16) to (18).

Moreover, it is worth to note that in Case 1, if we restrict $\beta = 1$ and $\sigma_e = 0$, which implies $r_e = Z_e = \mu_e^* = 0$ with probability 1, Equation (7) reduces to the payoff function for the market index call option if S_0 represents the index level today. Since $Z_e = 0$ and $R_q k - (1-\beta)R_f = R_q k$ is positive due to $\beta = 1$, we can derive $-\infty < Z_e < R_q k - (1-\beta)R_f = R_q k$, and thus it is necessary to consider only the second term in Equation (15). Then the formula for the market index call option can be shown as follows, and the details to derive this formula are presented in Appendix III.

$$V = R_f^{-1} S_0 \int_{\ln(a-b\cdot 0)=\ln R_q k}^{\infty} \left(R_q^{-1} e^{Z_m} - k\right) \phi^*(Z_m)\, dZ_m$$

$$= S_0 R_q^{-1} \cdot N\left(\frac{\ln\left(\frac{S_0}{K}\right)+\ln R_f-\ln R_q+\frac{\sigma_m^2 T}{2}}{\sigma_m\sqrt{T}}\right) - R_f^{-1} \cdot K \cdot N\left(\frac{\ln\left(\frac{S_0}{K}\right)+\ln R_f-\ln R_q-\frac{\sigma_m^2 T}{2}}{\sigma_m\sqrt{T}}\right)$$

$$= S_0 \cdot e^{-qT} N\left(\frac{\ln\left(\frac{S_0}{K}\right)+(r-q)T+\frac{\sigma_m^2 T}{2}}{\sigma_m\sqrt{T}}\right) - Ke^{-rT} \cdot N\left(\frac{\ln\left(\frac{S_0}{K}\right)+(r-q)T-\frac{\sigma_m^2 T}{2}}{\sigma_m\sqrt{T}}\right), \quad (19)$$

which is identical to the Black-Scholes formula. The above result demonstrates the Black-Scholes formula is a special case of our model when $\beta = 1$ and $\sigma_e = 0$, and for pricing market index call options, our model is equivalent to the classic Black-Scholes model.

Finally, the pricing formulae in Equations (16) and (18) are semi-analytical solutions and should be computed with numerical techniques. For the remaining integral over Z_e, we exploit the Gaussian quadrature method to complete the computation of option prices. In our computer program, 2000 points are considered in the Gaussian quadrature method, and the option prices converge within 10^{-7}, which guarantees that option prices generated through our model converge to the theoretical ones. In addition, the speed of our computer program is also fast. It takes less than 0.1 second to compute each option price.

III.4 Implied Volatility Smiles Based on Our Model

Under the Black-Scholes model, the risk-neutral probability density function of the underlying price is a lognormal distribution with a constant volatility. However, this pattern has been convincingly rejected (see, for example, Macbeth and Merville (1979) and Rubinstein (1985)). In the literature, there are in general two ways to remedy this problem. The first way is to develop alternative stochastic processes, which in turn imply risk-neutral densities more similar to the implied risk-neutral density from the market prices of options. The other way is to extract implied risk-neutral densities from option prices in markets directly (e.g., Rubinstein (1994)). Many existing studies, like Jackwerth (2000), Dennis and Mayhew (2002), and Bakshi, Kapadia, and Madan (2003), have found that implied risk-neutral densities tend to be more negatively skewed than the lognormal density, which is the reason for the implied volatility smiles and term structures.

The model proposed in this paper provides an alternative explanation for the causation of the implied volatility smiles and term structures. We find that the levels of the β and the idiosyncratic risk can influence the implied volatility smile, and this effect diminishes for options with longer time to maturities.

Table 1 and its corresponding Figure 1 report the implied volatilities of option values calculated by our model given different combinations of the values of beta and the strike price. The values of other parameters in the base example in Table 1 are as follows: the current stock price S_0 is \$50, the risk-free rate r is 0.1, the dividend yield q is 0, the time to maturity T is 1, the volatility of the market index level σ_m is 0.15, the idiosyncratic volatility σ_e is 0.3, and the examined strike prices are from \$30 to \$70 with an increment of \$5. For each result in Table 1, we compute the option values via our model first, and next derive the corresponding implied volatilities based on the Black-Scholes model. The slope of the implied volatility curve is defined as $\left(IV_{K=30} - IV_{K=70}\right)$/ (30 – 70), where IV is the abbreviation of implied volatility. For example, the slope for β = 0.25 is –0.0031, which is obtained from (0.3682 – 0.2436) / (30 – 70). By observing the slope for the implied volatility curve of each β, we find that the implied volatility curves are with more negatively slopes for higher values of β than lower values of β. In other words, the implied volatility smile phenomenon is more significant for the underlying asset with a higher level of the systematic risk.

Our results are consistent with those found in Duan and Wei (2009). They find that a higher degree of systematic risk leads to higher implied volatilities and a steeper slope of the implied volatility curve. In fact, the option pricing model proposed in this paper is the first to incorporate this feature found in the empirical data.

Based on the same example in Table 1, we further analyze how the level of the idiosyncratic risk affects the implied volatility smile by examining σ_e to be 0.2 and 0.4. The corresponding implied volatility curves are shown in Table 2 and Figure 2. We find that the implied volatility smile is more pronounced for higher levels of the idiosyncratic risk, e.g., when β equals 1.5, the slopes of the implied volatilities curves are –0.00256 for $\sigma_e = 0.2$ and –0.00470 for $\sigma_e = 0.4$. To the best of our knowledge, we are the first to find that the level of the idiosyncratic risk could affect the slope of the implied volatility curve.

Based on the results in Tables 1 and 2, we conclude that the volatility smile phenomenon is more significant for higher levels of β and the idiosyncratic risk σ_e. Note that instead of focusing on studying the direct relationship between the negative skewness and the negative slope of the implied volatility curve, such as the methods in Jackwerth (2000), Dennis and Mayhew (2002), and Bakshi, Kapadia, and Madan (2003), this paper develops an option pricing model that explicitly incorporates the β and the idiosyncratic risk component under the lognormal (or normal) distribution and concludes that the higher levels of the β and the idiosyncratic risk could generate implied volatility curves with more negative slopes. If the negative skewness is the only reason for the negative slope of the implied volatility curve, then our results suggest that option traders may already consider the systematic risk and the idiosyncratic risk of the underlying stock when estimating option values, and this behavior may be the reason for the negative skewness of the risk-neutral distribution of the underlying stock price.

It is well-known that the implied volatility curve changes with the time to maturity. Based on the same example in Table 1, we further examine the implied volatility curves for the different time to maturity T. The results of $T = 0.25$ and $T = 2$ are presented in Table 3 and Figure 3, which suggest that the volatility grin effect decays with the increase of the time to maturity. Our results are reconciled with the results of many previous studies, such as Backus, Foresi, and Wu (2004) and Duque and Lopes (2003), which demonstrate empirically that the implied volatility skew dies out as the maturity becomes infinite.

In summary, the results in Tables and Figures 1 to 3 suggest that the higher the levels of the β and the idiosyncratic risk, the more negatively sloping implied volatility curves can be generated though our option pricing model. Furthermore, the effects decrease with the increase of the time to maturity. Since these phenomena are consistent with many empirical studies, the analyses based on Tables and Figures 1 to 3 demonstrate the ability of our option pricing model to capture some important aspects of individual stock options.

IV. Empirical Studies

This section presents the results of several empirical studies conducted in this paper. Based on our option pricing formula, it is possible to calibrate the levels of β and the idiosyncratic risk from prices of market index options and individual stock options. In this section, the data collected for the empirical studies is introduced first, followed by the introduction of our calibration procedures. Finally, we will show that the values of the β implied from option prices do provide a better performance on predicting the realized β in future periods of time.

IV.1 Data Description

In this paper, we collect the option data of the Dow Jones Industrial Average index (DJIA) and its component stocks as the examined sample, covering January 1, 2008 to December 31, 2008 for a total of 253 trading days. The Dow Jones Industrial Average index options (DJX) and the component individual stock options are traded on the Chicago Board of Trade (CME Group) and Chicago Board Options Exchange (CBOE), respectively. Daily data of the option prices are collected from the database of OptionMetrics. In addition, only call options are considered in our studies, and we use the average of the bid and ask quotes for each option contract as the option prices.

We use the continuously-compounded zero-coupon interest rates with different days to maturity provided in the database of OptionMetrics. Furthermore, the spline interpolation method is employed to generate the interest rates whose corresponding maturities are matched to the time to maturities of examined options.

However, since the component individual stock options traded on CBOE are American options, we need to convert these American option prices to their European counterparts. First, the binomial tree model is employed to find the implied dividend yield for each contract given its implied volatility provided in the database of OptionMetrics. Next, the Black-Scholes model is used with the input of the current stock price, matched risk free rate, implied dividend yield derived in the first step, implied volatility provided in OptionMetrics, and the strike price and time to maturity in the option contract to generate the corresponding European option price.

In addition, the option data are screened based on three criteria: (1) we filter out average quotes that are less than $0.025 because these option prices cannot reflect true option values due to the minimum tick problem; (2) we eliminate the average quotes if the implied volatility based on the Black-Scholes model does not exist[4]; (3) the minimum and maximum time to maturities are restricted to be 30 days and 180 days to ensure that time values represent a significant part of option values and the liquidity of option contracts is acceptable. In Table 4, the number of quotes and the means and standard deviations of the Black-Scholes implied volatilities for options on the DJX and its component individual stocks are reported. In the database of OptionMetrics, we find option prices of only 28 component stocks of DJIA in 2008. The OptionMetrics database does not provide the historical option prices for General Motors (GM) and American International Group Inc. (AIG), which are removed from the portfolio of the DJIA in June of 2009 and September of 2008.[5]

To construct the comparison benchmark for the implied beta generated from our model, we calculate the daily historical and realized betas based on the single-index model proposed by Jensen (1968). Since the total returns including the dividend yield should be employed in the single-index model, we collect the data of the total returns of the DJIA from the database of Dow Jones Company, and as to the individual stocks in the DJIA, the daily adjusted closing prices are obtained from the financial page at yahoo–finance.yahoo.com. For each date, we use the prior 90 daily returns to estimate the historical beta, and the next 90 daily returns following the examined date are employed to estimate the realized beta.

IV.2 Calibration Procedures

Since the option pricing formulae (16) to (18) for individual stock options are

[4] Generally speaking, the option prices violating arbitrage conditions and deeply in or out of the money are excluded by this screening process.

[5] On September 22, 2008, Kraft Foods substituted the American International Group (AIG) in the DJIA. Although the option prices of Kraft Foods (KFT) are provided in OptionMetrics, the number of observations is too small to derive reliable results. Thus, we do not take KFT into consideration in this paper.

functions of σ_m, it is necessary to estimate σ_m from the prices of the DJX index options on each trading day first. Equipped with the value of σ_m and the prices of the individual stock options, next we are able to calibrate the levels of β and σ_e for each individual stock on that trading day. The details associated with the calibration process are stated as follows.

IV.2.1 Calibration of σ_m

Since it is commonly believed that the volatilities of financial assets are not a constant for different time to maturities, we derive the whole term structure $\sigma_m(t,T)$ on each date t based on the prices of DJX index options.[6] To achieve that, we need to estimate the term structure of the dividend yield $q_m(t,T)$ for different maturity date T in advance because we need this information in our pricing formula (19) for market index options. Suppose the implied volatility of each option contract provided in the OptionMetrics database is representative enough to reflect the market consensus of the volatility of the underlying asset. Based on that information, we can extract the least squares estimations of $q_m(t,T)$ from market index option contracts with different maturity date T through the following equation.

$$\min_{q_m(t,T)} \sum_{K_T} \frac{\mathrm{OI}(K_T)}{\sum_{K_T} \mathrm{OI}(K_T)} \Big(M_{T,K_T} - \widehat{M}\big(T, K_T, S_m(t), R_f(t,T), \hat{\sigma}_{T,K_T}, q_m(t,T) \mid \beta = 1, \sigma_e = 0\big)\Big)^2,$$

where M_{T,K_T} and $\widehat{M}(\cdot)$ denote the market and theoretical prices for DJX index options, $S_m(t)$ denotes the current level of DJIA, K_T denotes the possible strike

[6] On each trading day, if there are, for example, 7 maturity dates for market index options, we will derive 7 $\sigma_m(t,T)$'s for that trading day.

prices of DJX index options with the maturity date *T*, $R_f(t,T)$ and $q_m(t,T)$ are the gross risk-free rate and estimated dividend yield from the current date *t* to the maturity date *T*, and $\hat{\sigma}_{T,K_T}$ is the implied volatility for the DJX index option with the strike price K_T on the maturity *T* in OptionMetrics. In addition, $\mathrm{OI}(K_T)$ is the open interest of the DJX index option with the strike price K_T, and $\mathrm{OI}(K_T)/\sum_{K_T}\mathrm{OI}(K_T)$ is the weight for each option contract. This weighted scheme is adopted because we believe that the prices of option contracts with higher open interests are monitored by more investors in the market and thus more efficient.

Next, for each date *t* in the examined period, we can estimate the term structure of $\sigma_m(t,T)$ across different strike prices given a maturity date *T*. Here the least squares approach is employed and thus $\sigma_m(t,T)$ can be calibrated from solving the following equation:

$$\min_{\sigma_m(t,T)} \sum_{K_T} \frac{\mathrm{OI}(K_T)}{\sum_{K_T}\mathrm{OI}(K_T)} \Big(M_{T,K_T} - \hat{M}\big(T, K_T, S_m(t), R_f(t,T), \sigma_m(t,T), q_m(t,T) \,|\, \beta = 1, \sigma_e = 0\big)\Big)^2, \quad (20)$$

IV.2 Calibration of β and σ_e

In the process to calibrate $\beta(t)$ and $\sigma_e(t)$ for each date *t*, we first calibrate $\beta(t, T)$ and $\sigma_e(t, T)$ for individual stock options with different maturity dates on each date *t*, and the estimations of $\beta(t)$ and $\sigma_e(t)$ for each date are derived through calculating the arithmetic average of the values of $\beta(t,T)$ and $\sigma_e(t,T)$ across different maturity dates on that date.

For each maturity date *T* of individual stock prices on date *t*, $\beta(t, T)$ and $\sigma_e(t, T)$

can be calibrated from minimizing the squares of differences between market and theoretical prices of the individual options with different strike prices given $\sigma_m(t,T)$, i.e.,

$$\min_{\beta(t,T),\sigma_e(t,T),q(t,T)} \sum_{K_T} \frac{\mathrm{OI}(K_T)}{\sum_{K_T}\mathrm{OI}(K_T)} \Big(V_{T,K_T} - \hat{V}\big(T, K_T, S(t), R_f(t,T), \beta(t,T), \sigma_e(t,T), q(t,T) | \sigma_m(t,T)\big)\Big)^2, \quad (21)$$

where V_{T,K_T} and $\hat{V}(\cdot)$ denote the market and theoretical call prices for individual stock options, $S(t)$ denotes the current stock price for the underlying stocks on the current date t, K_T represents the possible strike prices corresponding to maturity date T, $R_f(t,T)$ is the matched risk free rate, and the open interests for different strike prices are employed to decide the weights for each pair of the market and theoretical call prices. There are some other details about this process, which are stated as follows.

First, since we try to provide a thoroughly forward-looking estimation of the β, it is not appropriate to use the historical data to estimate the dividend yield q. Thus, in Equation (21), we calibrate not only the values of $\beta(t,T)$ and $\sigma_e(t,T)$ but also the value of $q(t,T)$ based on the prices of individual stock options simultaneously. Second, although we are equipped with the term structure of $\sigma_m(t,T)$ for each day, the maturity date of $\sigma_m(t,T)$ may not equal the maturity date of the individual stock option contracts. To deal with this problem, the spline interpolation method is employed to generate the $\sigma_m(t,T)$ whose horizon exactly matches the time to maturity of examined individual stock option contracts. Third, to solve the least squares problem in Equation (21), we combine the grid search method and nonlinear least squares procedure provided in Matlab. More specifically, the grid search method with an increment to be 0.01 in a proper range[7] is adopted for β, and for each examined value of the β, the function of lsqnonlin in Matlab is employed to find the optimal values of $\sigma_e(t,T)$ and $q(t,T)$ to minimize the least-squares errors between the market and theoretical option prices. Finally, among all examined values of β, find the one that can generate the smallest least-squares errors. Based on the above process, we can obtain the optimal $\beta(t,T)$, $\sigma_e(t,T)$, and $q(t,T)$ for different maturity date on each date.

IV.3 Empirical Results

The results of yearly averages and standard deviations of the implied $\beta(t)$ and $\sigma_e(t)$ for different individual stocks are shown in Table 5. In addition, the historical and realized β are also reported for comparison. The historical and realized β are computed based on the single-index model with prior and next 90-day returns, respectively. It can be observed first that the average values of implied β and σ_e for most stocks are significantly different from 0 by comparing the magnitudes of the average and the standard deviation. Second, the estimated results suggest that the option-implied β's of the individual stocks are in reasonable ranges. The scatter diagram of realized β's versus our implied β's is shown in Figure 4. From the regression results, the slope coefficient is 0.8876, which is very close to 1, and the R-squared value is also as high as 0.79. Both evidence that the values of implied β's are consistent with the general understanding of the distributions of β's for the companies in DJIA. Third, by comparing the implied β from our model with the historical and realized β, it is obvious to find that the implied β from our model is not far from the realized β and sometimes is closer to the realized β than the historical β.

[7] We derive the average value and standard deviation of the historical betas over prior 90 days to construct a proper range for the grid search method. The average value is adopted to be the central level of the range for β. The upper bound is the result of the average value plus six times the standard deviation, and the lower bound is the result of the average value minus six times the standard deviation. Moreover, if the range of six standard deviations is too narrow (< 0.7) or too wide (> 1.2), 0.7 or 1.2 is used to replace the six standard deviations to generate a proper range for β.

However, simply comparing the unconditional averages of the implied and historical β with the unconditional average of the realized β cannot clearly distinguish the prediction power between these two estimates. Further studies on this issue are conducted in the following subsections.

IV.3.1 Forecasting Performance of the Implied $\boldsymbol{\beta}$

To verify the superior forecasting power of the implied β over that of the historical β, for each firm i, we consider the following two regression equations:

$$\beta_{i,t}^{real} = \lambda_{1,i} + \lambda_{2,i}\beta_{i,t}^{impl} + \varepsilon_{i,t}^{impl}, t = 1,2,\dots,253, \tag{22}$$

$$\beta_{i,t}^{real} = \delta_{1,i} + \delta_{2,i}\beta_{i,t}^{hist} + \varepsilon_{i,t}^{hist}, t = 1,2,\dots,253, \tag{23}$$

where $\beta_{i,t}^{real}$, $\beta_{i,t}^{hist}$, and $\beta_{i,t}^{impl}$ denote the daily realized, historical, and implied β, respectively. The $\beta_{i,t}^{impl}$ is in essence another notation of the implied $\beta(t)$ of each firm i on date t, and the $\beta_{i,t}^{hist}$ and $\beta_{i,t}^{real}$ are computed based on the single-index model with the prior and next 90-day returns on date t. The $\varepsilon_{i,t}^{impl}$ and $\varepsilon_{i,t}^{hist}$ denote the residual error terms, which are independently and identically normally distributed. The regression results for each stock are reported in Table 6. For each firm i, theoretically speaking, if one estimate of β can provide a better prediction, the slope coefficient $\lambda_{2,i}$ or $\delta_{2,i}$ should be more positive and closer to 1. In the meanwhile, the correlation between the estimates of β and the realized β, which is denoted as $\rho_{I,R,i}$ (or $\rho_{H,R,i}$) for the implied β (or the historical β), should approach 1 ideally. Table 6 shows that there are 6 slope terms $\lambda_{2,i}$ significantly positive, and 5 slope terms $\delta_{2,i}$ significantly positive. Additionally, there are 13 positive correlation coefficients for the $\rho_{I,R,i}$, and only 8 positive for the $\rho_{H,R,i}$. Moreover, among the 28 individual stocks, there are 19 with the result of $\rho_{I,R,i} > \rho_{H,R,i}$. These results confirm the superiority of the implied β in forecasting realized β in the future.

Finally, we conduct the panel data analysis based on the following regression equations:

$$\beta_{i,t}^{real} = v_1 + v_2\beta_{i,t}^{impl} + \eta_{i,t}^{impl}, \tag{24}$$

$$\beta_{i,t}^{real} = \theta_1 + \theta_2\beta_{i,t}^{hist} + \eta_{i,t}^{hist}, \tag{25}$$

where i = 1, 2,…, 28 and t = 1, 2,…, 253, and the disturbances $\eta_{i,t}^{impl}$ and $\eta_{i,t}^{hist}$ are white-noise random variables. In Table 6, the corresponding statistic results are shown in the last row. The slope coefficients v_2 = 0.03 and θ_2 = –0.14 generated from the cross-section fixed-effect regression model confirm that the implied β has a better forecasting power.

The poor performance of the historical β in this empirical study might be due to the time varying or even mean reverting feature of beta, which is recognized in many papers, such as Levy (1971), Blume (1975), Klembosky and Martin (1975), Fabozzi and Francis (1978), Bos and Newbold (1984), and Collins, Ledolter, and Rayburn (1987). The results of the poor performance of the historical β in Table 6 again support the findings in the previous literature.

IV.3.2 Competitive Regression between the Implied and Historical $\boldsymbol{\beta}$

Inspired by the work of Jorion (1995) and Chen, Kim, and Panda (2009), a competitive regression analysis is performed to compare and identify the information sets of the implied and historical beta to explain the future realized beta. The examined multivariate regression equation for each firm i is

$$\beta_{i,t}^{real} = \omega_{0,i} + \omega_{1,i}\beta_{i,t}^{impl} + \omega_{2,i}\beta_{i,t}^{hist} + \varepsilon_{i,t}\,,\ \ t = 1,2,\dots,253, \tag{26}$$

where $\beta_{i,t}^{real}$, $\beta_{i,t}^{impl}$, and $\beta_{i,t}^{hist}$ denote the realized, implied, and historical β for each working day in 2008. Table 7 presents the results of this competitive regression between the implied and historical β. For each firm, we report the regression coefficients, the corresponding *t*-statistics, and the *F*-value of the above multivariate regression. Since the results of *F*-values suggests that almost all cases are significant, it can be inferred that combining these two types of betas is capable of explaining the dynamics of the future beta. By analyzing $\omega_{1,i}$ and $\omega_{2,i}$ separately, there are twenty-one significant cases for the implied β and twenty-three significant cases for the historical β among the examined 28 individual stocks, and furthermore, the significant cases for different types of betas are not exactly the same. These results demonstrate that the implied and historical β's contain different information sets and each has its own explanation power in predicting future betas. In addition, there are 19 cases in which both betas are significant, which suggest that the implied β could be complementary to the historical β for enhancing the prediction of the realized β in the future.

Finally, the panel data regression based on i = 1, 2,…, 28 and t = 1, 2,…, 253 in following equation is considered.

$$\beta_{i,t}^{real} = u_0 + u_1\beta_{i,t}^{impl} + u_2\beta_{i,t}^{hist} + \eta_{i,t} \,.$$

The regression results for the coefficients of u_1 and u_2 are 0.06 and –0.19, both of which are significant. These results again support that implied and historical β's contain different information sets, and the implied β is more positively correlated with the realized β.

IV.3.3 Forecasting Performance of the Implied σ_e

Traditionally, the systematic risk is the only risk that investors concern. Some recent literature, however, suggests that idiosyncratic risk might be actually driving a risk-return relation by examining the cross-sectional relationship between equity returns and idiosyncratic risk. For example, Lehmann (1990), Merton (1987). Barberis and Huang (2001) develop asset pricing models and find that future expected returns are a positive function of idiosyncratic risk. In addition, both the autoregressive model in Chua, Goh, and Zhang (2006) and the EGARCH models in Fu (2009) and Spiegel and Wang (2005) find empirically that future expected returns are positively related to expected idiosyncratic volatilities. In contrast, some other empirical evidences show different conclusions, such as Ang, Hodrick, Xing, and Zhang (2006) find a negative cross-sectional relationship between returns and idiosyncratic risk. In addition, Bali and Cakici (2008) believe that there is no robustly significant relation between idiosyncratic volatilities and the cross-section of expected equity returns.

To study this issue, for each firm i, we examine the following regression equation for realized risk-adjusted excess returns over the implied levels of the idiosyncratic risk $\sigma_{e,i}(t)$.

$$\bar{R}_{i,t} - \beta_{i,t}^{real}\bar{R}_{m,t} = \alpha_{0,i} + \alpha_{1,i}\sigma_{e,i}(t) + \varepsilon_{i,t}^{e}, t = 1,2,\ldots,253, \qquad (27)$$

where $\bar{R}_{i,t}$ denotes the future 90-day average excess return of the individual stock i, $\bar{R}_{m,t}$ denotes the future 90-day average excess market return, $\beta_{i,t}^{real}$ is the realized beta for the individual stock i, $\sigma_{e,i}(t)$ is the idiosyncratic risk of the individual stocks, and $\varepsilon_{i,t}^{e}$ is the firm-specific residual. Table 8 shows the regression results of Equation (27) for each individual stock. The slope coefficients in bold represent the regression results that are positive and statistically significant. These cases support the hypothesis that the idiosyncratic risk is able to contribute to future excess return.

In contrast, we also find that some of the slope coefficients are negative and statistically significant. These mixed results are in accordance with that it is still under debate about whether the idiosyncratic risk can influence future stock returns.

Finally, the panel data regression for all individual stocks and all dates t is also conducted: $\bar{R}_{i,t} - \beta_{i,t}^{real}\bar{R}_{m,t} = \gamma_0 + \gamma_1\sigma_{e,i}(t) + \eta_{i,t}^{e}$, for i = 1, 2,…, 28, and t = 1, 2, …, 253. The results of the estimated coefficients, t-statistics, and R-squared are shown in the last row of Table 8. The slope coefficient is very close to 0, which inclines to support Bali and Cakici (2008) that the relationship between the idiosyncratic risk and cross-section of the future excess returns is uncertain and ambiguous.

V. Conclusion

It is commonly believed that option prices can provide additional information on the underlying stock prices, particularly about the volatility of the underlying stock prices in the future. Although option prices are informative about future volatility, there is little research on using option prices to infer future values of β, which, by definition, should be determined according to the volatilities of the market index level and the individual stock prices.

In this paper, a novel method is proposed to estimate the levels of β and the idiosyncratic risk purely from the prices of market index and individual stock options. Building on Siegel (1995) and Husmann and Stephen (2007), we develop a semi-analytical pricing model for individual stock options involving the volatility of the market index level and the levels of the β and the idiosyncratic risk of the underlying stock asset through combining the market model and the multivariate risk-neutral valuation relationship (RNVR) developed in Stapleton and

Subrahmanyam (1984) and Câmara (2003). Our analysis demonstrates the superior ability of this option pricing model to explain the price behavior of individual stock options found in previous literature, i.e., the higher the levels of the β and the idiosyncratic risk, the more negatively-sloping implied volatility curves can be generated, and both these effects diminish with the increase of the time to maturity.

Moreover, we conduct empirical studies to calibrate the β and the idiosyncratic risk from the prices of index options of DJIA and individual stock options of the DJIA components, and demonstrate that the values of the β implied from our model not only provide reasonable estimations for the realized β of firms in DJIA, but also contain different information sets and provide a better performance versus the historical β in predicting the realized β for future periods of time. In addition, we also analyze the relationship between the level of the idiosyncratic risk implied from the option prices and the future underlying stock returns, and the results support that there is no robustly significant relation between idiosyncratic risk and the cross-section of future excess return in the underlying stock.

The option pricing formula incorporating the market model in this paper enables derivation of forward-looking β of individual stock assets purely based on prices of stock index and individual stock options. Furthermore, our model has much potential. For instance, it is possible to apply our estimates of implied β and σ_e to test the validity of the CAPM, and it is possible to incorporate RNVR with more general option pricing models that take the effects of the skewness and the kurtosis into account. In addition, one might conduct the analyses for different periods of time and different markets to further understand the behavior of this forward-looking β estimation in detail.

Table 1

Implied Volatility Curves Given Different β (for σ_e = 0.3)

This table presents the implied volatility smile curves under different values of β. The examined parameter values are as follows. The current stock price is \$50, the risk-free rate r is 0.1, the dividend yield q is 0, the time to maturity T is 1 (one year), the volatility of the market index level σ_m is 0.15, the idiosyncratic risk σ_e is 0.3, and the examined strike prices are from \$30 to \$70. Based on the option values generated by our model, the values of the implied volatilities are derived according to the Black-Scholes model. It is evident that the implied volatility decreases as the strike price increases in our model, which demonstrates that our option pricing model could explain the phenomenon of volatility smiles properly. In addition, the values of slopes suggest that the volatility smile phenomena for higher β stocks are more pronounced.

implied vol.	**Strike price (K)**									
	30	**35**	**40**	**45**	**50**	**55**	**60**	**65**	**70**	**slope***
beta = 0.25	0.3682	0.3430	0.3220	0.3043	0.2890	0.2756	0.2637	0.2531	0.2436	-0.00311
beta = 0.5	0.3779	0.3521	0.3307	0.3126	0.2970	0.2833	0.2712	0.2604	0.2507	-0.00318
beta = 0.75	0.3930	0.3665	0.3444	0.3258	0.3098	0.2957	0.2834	0.2723	0.2624	-0.00326
beta = 1	0.4129	0.3853	0.3625	0.3433	0.3267	0.3123	0.2996	0.2882	0.2781	-0.00337
beta = 1.25	0.4369	0.4082	0.3844	0.3644	0.3473	0.3323	0.3192	0.3075	0.2971	-0.00349
beta = 1.5	0.4646	0.4345	0.4097	0.3887	0.3708	0.3553	0.3417	0.3296	0.3188	-0.00364

* The slope of the line in the last column is defined as ($IV_{K=30}$ – $IV_{K=70}$) / (30 – 70), where IV is the abbreviation of implied volatility. For example, the slope for β = 0.25 is (0.3682 – 0.2436) / (30 – 70) = –0.00311.

Table 2

Implied Volatility Curves Given Different β (for σ_e = 0.2 and σ_e = 0.4)

Based on the numerical example in Table 1, this table reports the implied volatilities smile curves under different β for a lower and higher levels of the idiosyncratic risk σ_e. The level of the idiosyncratic risk σ_e is 0.2 in Panel A and 0.4 in Panel B. After generating option prices via our model, the values of the implied volatilities are derived based on the Black-Scholes model. Comparing with Table 1, it is obvious that the implied volatility smile becomes more pronounced as σ_e increases in our model. In other words, the phenomenon of the volatility smile is more pronounced for stocks with higher idiosyncratic risk.

implied vol.	Strike price (K)									slope*
	30	35	40	45	50	55	60	65	70	
Panel A. Implied volatilities for σ_e = 0.2										
beta = 0.25	0.2473	0.2305	0.2165	0.2047	0.1945	0.1855	0.1776	0.1706	0.1642	-0.00208
beta = 0.5	0.2604	0.2430	0.2286	0.2164	0.2059	0.1968	0.1887	0.1815	0.1751	-0.00213
beta = 0.75	0.2800	0.2618	0.2469	0.2342	0.2234	0.2141	0.2059	0.1986	0.1922	-0.00220
beta = 1	0.3050	0.2859	0.2702	0.2570	0.2458	0.2361	0.2277	0.2203	0.2138	-0.00228
beta = 1.25	0.3345	0.3143	0.2976	0.2838	0.2720	0.2618	0.2530	0.2453	0.2385	-0.00240
beta = 1.5	0.3680	0.3462	0.3284	0.3136	0.3010	0.2902	0.2808	0.2726	0.2654	-0.00256
Panel B. Implied volatilities for σ_e = 0.4										
beta = 0.25	0.4911	0.4572	0.4291	0.4053	0.3847	0.3668	0.3509	0.3367	0.3240	-0.00418
beta = 0.5	0.4988	0.4644	0.4359	0.4117	0.3909	0.3727	0.3566	0.3423	0.3293	-0.00424
beta = 0.75	0.5110	0.4759	0.4468	0.4221	0.4009	0.3824	0.3660	0.3513	0.3382	-0.00432
beta = 1	0.5274	0.4913	0.4615	0.4362	0.4144	0.3955	0.3787	0.3637	0.3503	-0.00443
beta = 1.25	0.5476	0.5104	0.4796	0.4536	0.4312	0.4117	0.3945	0.3791	0.3654	-0.00456
beta = 1.5	0.5712	0.5326	0.5008	0.4739	0.4508	0.4307	0.4129	0.3972	0.3830	-0.00470

* The slope of the line in the last column is defined as ($IV_{K=30} - IV_{K=70}$) / (30 – 70), where IV means implied volatility. For example, the slope for $\beta = 0.25$ in Panel A is (0.2473 – 0.1642) / (30 – 70) = –0.00208.

Table 3
Implied Volatility Curves Given Different β (for T = 0.25 and T = 2)

This table studies the effect of different time to maturities on the implied volatilities. Under the same parameter values in Table 1, the volatilities smile become more significant as the time to maturity T changes from 1 to 0.25. In contrast, the volatilities smile diminishes as the time to maturity T changes from 1 to 2. Thus, it can be concluded that the volatility smile effect decays with the increase of the time to maturity, which is consistent with the results of many empirical studies in the literature.

implied vol.	Strike price (K)									slope*
	30	**35**	**40**	**45**	**50**	**55**	**60**	**65**	**70**	
Panel A. Implied volatilities for T = 0.25										
beta = 0.25	0.3815	0.3553	0.3335	0.3149	0.2990	0.2850	0.2726	0.2616	0.2516	-0.00325
beta = 0.5	0.3902	0.3635	0.3412	0.3224	0.3061	0.2919	0.2793	0.2681	0.2580	-0.00331
beta = 0.75	0.4039	0.3764	0.3537	0.3343	0.3177	0.3032	0.2903	0.2789	0.2686	-0.00338
beta = 1	0.4221	0.3937	0.3702	0.3503	0.3332	0.3183	0.3051	0.2934	0.2829	-0.00348
beta = 1.25	0.4441	0.4146	0.3903	0.3697	0.3521	0.3367	0.3232	0.3112	0.3005	-0.00359
beta = 1.5	0.4695	0.4388	0.4135	0.3922	0.3739	0.3580	0.3441	0.3317	0.3207	-0.00372
Panel B. Implied volatilities for T = 2										
beta = 0.25	0.3506	0.3268	0.3071	0.2903	0.2759	0.2632	0.2521	0.2421	0.2330	-0.00294
beta = 0.5	0.3617	0.3373	0.3171	0.2999	0.2851	0.2722	0.2607	0.2505	0.2413	-0.00301
beta = 0.75	0.3789	0.3536	0.3326	0.3149	0.2996	0.2862	0.2745	0.2640	0.2546	-0.00311
beta = 1	0.4011	0.3747	0.3529	0.3344	0.3185	0.3047	0.2926	0.2818	0.2721	-0.00323
beta = 1.25	0.4279	0.4001	0.3772	0.3579	0.3413	0.3269	0.3143	0.3031	0.2931	-0.00337
beta = 1.5	0.4587	0.4293	0.4051	0.3848	0.3674	0.3523	0.3390	0.3273	0.3169	-0.00355

* The slope of the line in the last column is defined as ($IV_{K=30} - IV_{K=70}$) / (30 – 70), where IV represents the implied volatility. For example, the slope for $\beta = 0.25$ is (0.3815 – 0.2516) / (30 – 70) = –0.00325 as the time to maturity T = 0.25.

Table 4
Descriptive Statistics of Implied Volatilities of Options

This table reports summary statistics for the implied volatilities of the call option contracts used in this paper. We collect index call options for DJIA as well as the call option contracts for its component firms in 2008. Option prices are collected from OptionMetrics, but only 28 component stocks are examined due to the availability problem of the option prices of General Motors (GM) and American International Group Inc. (AIG), which are removed from the portfolio of the DJIA in June of 2009 and September of 2008, respectively. In addition, the American option prices of individual stock options are converted to the counterpart European option prices in a proper procedure introduced in Subsection IV.1. Implied volatilities are calculated through the Black-Scholes model, and we report the number of quotations and the means and standard deviations of the implied volatilities for individual stock call options and market index call options.

DJIA Components	# of Qutoes	Mean	s.d.
Alcoa Incorporated	6144	0.54	0.18
American Express Company	8988	0.61	0.31
Boeing Company	7773	0.43	0.18
Bank of American	8866	0.68	0.35
Citigroup Incorporated	6426	0.71	0.33
Caterpillar Incorporated	8945	0.45	0.19
Chevron Corporation	8133	0.43	0.25
DuPont	5846	0.38	0.16
Walt Disney Company	5084	0.39	0.17
General Electric Company	9573	0.45	0.27
Home Depot Incorporated	5989	0.51	0.19
Hewlett-Packard Company	7832	0.41	0.16
International Business Machines	8715	0.37	0.16
Intel Corporation	6065	0.45	0.14
Johnson & Johnson	3873	0.28	0.14
J.P. Morgan Chase & Company	8874	0.63	0.30
Coca-Cola Company	7220	0.31	0.14
McDonald’s Corporation	7820	0.35	0.14
3M Company	6249	0.35	0.16
Merck & Company, Incorporated	6561	0.40	0.13
Microsoft Corporation	7645	0.39	0.15
Pfizer Incorporated	2978	0.40	0.18
Procter & Gamble Company	6497	0.30	0.16
AT&T Incorporated	6324	0.40	0.16
United Technologies Corporation	7020	0.40	0.18
Verizon Communications Inc.	4489	0.38	0.17
Wal-Mart Stores Incorporated	7710	0.37	0.18
Exxon Mobil Corporation	7336	0.43	0.31
DJX index option	102842	0.30	0.15

Table 5
Estimation Results for Implied β and σ_e

For each component stock in DJIA, the average and standard deviation of implied β and σ_e as well as the average and standard deviation of the historical β and realized β in 2008 are reported in this table. The implied β and σ_e are calculated based on the results of Equation (21), and the historical and realized β are estimated by the single-index model with the prior and next 90-day returns on each date. Comparing with the realized β's, the implied β's from our model can provide reasonable estimations for the future β.

DJIA Components	**Implied β**		σ_e		**Historical β**		**Realized β**	
	mean	**s.d.**	**mean**	**s.d.**	**mean**	**s.d.**	**mean**	**s.d.**
Alcoa Incorporated	2.006	0.224	0.218	0.143	1.372	0.202	1.590	0.399
American Express Company	1.755	0.339	0.198	0.151	1.840	0.201	1.850	0.301
Boeing Company	0.834	0.457	0.244	0.105	0.902	0.164	1.006	0.140
Bank of America Corporation	1.597	0.641	0.208	0.160	2.053	0.553	2.460	0.644
Citigroup Incorporated	2.330	0.503	0.156	0.149	2.137	0.349	2.454	0.540
Caterpillar Incorporated	0.886	0.287	0.284	0.128	1.089	0.070	1.146	0.174
Chevron Corporation	0.858	0.449	0.206	0.149	0.904	0.311	0.984	0.358
DuPont	0.886	0.293	0.179	0.105	1.052	0.063	1.139	0.171
Walt Disney Company	0.943	0.339	0.175	0.145	0.951	0.127	1.099	0.176
General Electric Company	1.079	0.259	0.158	0.135	1.161	0.157	1.274	0.214
Home Depot Incorporated	1.183	0.422	0.250	0.097	1.284	0.184	1.207	0.200
Hewlett-Packard Company	0.728	0.312	0.260	0.136	0.971	0.148	0.910	0.090
International Business Machines	0.436	0.317	0.255	0.136	0.852	0.173	0.786	0.087
Intel Corporation	1.261	0.304	0.227	0.124	1.320	0.238	1.189	0.177
Johnson & Johnson	0.335	0.408	0.099	0.099	0.432	0.141	0.536	0.139
J.P. Morgan Chase & Company	1.580	0.377	0.167	0.145	1.846	0.311	1.970	0.450
Coca-Cola Company	0.318	0.484	0.192	0.117	0.520	0.115	0.555	0.138
McDonald's Corporation	0.350	0.337	0.211	0.087	0.679	0.106	0.697	0.089
3M Company	0.779	0.288	0.148	0.102	0.866	0.086	0.852	0.083
Merck & Company, Incorporated	0.980	0.428	0.234	0.113	0.709	0.150	0.803	0.176
Microsoft Corporation	0.754	0.373	0.251	0.113	0.990	0.085	1.049	0.082
Pfizer Incorporated	1.169	0.270	0.084	0.102	0.836	0.071	0.840	0.071
Procter & Gamble Company	0.280	0.362	0.139	0.100	0.542	0.093	0.615	0.108
AT&T Incorporated	0.980	0.323	0.161	0.116	0.986	0.057	0.965	0.076
United Technologies Corporation	0.994	0.284	0.112	0.087	1.045	0.055	1.034	0.066
Verizon Communications Inc.	0.927	0.304	0.162	0.104	0.948	0.052	0.901	0.064
Wal-Mart Stores Incorporated	0.450	0.305	0.209	0.093	0.746	0.070	0.664	0.132
Exxon Mobil Corporation	0.764	0.417	0.199	0.126	0.947	0.222	0.954	0.241

Table 6

Forecasting Regression of Realized β on Implied β and Historical β

This table displays the results from the regression of realized β on both the implied and historical β. The left-hand side reports the estimated parameters of $\beta_{i,t}^{real} = \lambda_{1,i} + \lambda_{2,i}\beta_{i,t}^{impl} + \varepsilon_{i,t}^{impl}$, and the right-hand side shows the regression results of $\beta_{i,t}^{real} = \delta_{1,i} + \delta_{2,i}\beta_{i,t}^{hist} + \varepsilon_{i,t}^{hist}$. The $\beta_{i,t}^{real}$, $\beta_{i,t}^{hist}$, and $\beta_{i,t}^{impl}$ denote the daily realized, historical, and implied β, and the definitions of them and $\varepsilon_{i,t}^{impl}$ and $\varepsilon_{i,t}^{hist}$ are mentioned in Subsection IV.3. For each firm, we report the regression coefficients, the related *t*-statistics, and the correlation and determination coefficients. The *t*-values in bold are significant at the 10% level or higher based on the two-tailed tests. The coefficient of correlation $\rho_{I,R,i}$ denotes the correlation between on the implied beta and realized beta. Similarly, the coefficient of correlation $\rho_{H,R,i}$ denotes the correlation between on the historical beta and realized beta. The results of $\lambda_{2,i}$ or $\delta_{2,i}$ and $\rho_{I,R,i}$ or $\rho_{H,R,i}$ that is more positive and closer to 1 indicate better prediction performance for the realized β in the future. The last row shows the estimated results for the cross-sectional fixed effect panel regression. All evidences suggest that the forward-looking β implied from our model is more able to predict the realized β in the future.

Ticker	Regression of Realized Beta on Implied beta						Regression of Realized Beta on Historical beta						
	λ_1		λ_2		R^2	$\rho_{I,R}$	δ_1		δ_2		R^2	$\rho_{H,R}$	
	coeff	*t*	coeff	*t*			coeff	*t*	coeff	*t*			
AA	0.30	1.43	0.64	**6.12**	0.13	0.36	1.38	7.99	0.15	1.24	0.01	0.078	*
AXP	1.98	19.82	-0.08	-1.35	0.01	-0.08	3.63	27.32	-0.97	-13.49	0.42	-0.648	*
BA	1.01	54.73	0.00	-0.07	0.00	0.00	0.97	19.71	0.04	0.66	0.00	0.042	
BAC	1.89	18.56	0.35	**5.98**	0.12	0.35	3.01	19.79	-0.27	-3.73	0.05	-0.229	*
C	2.26	14.02	0.08	1.24	0.01	0.08	4.28	24.26	-0.86	-10.50	0.31	-0.552	*
CAT	1.19	33.35	-0.04	-1.17	0.01	-0.07	1.82	10.95	-0.62	-4.06	0.06	-0.248	*
CVX	1.01	20.69	-0.03	-0.57	0.00	-0.04	1.36	20.94	-0.41	-6.10	0.13	-0.359	*
DD	1.03	30.47	0.13	**3.50**	0.05	0.22	1.53	8.49	-0.37	-2.17	0.02	-0.136	*
DIS	1.11	33.83	-0.01	-0.32	0.00	-0.02	0.15	2.59	1.00	**16.37**	0.52	0.719	
GE	1.33	23.05	-0.05	-1.01	0.00	-0.06	2.31	30.13	-0.89	-13.63	0.43	-0.652	*
HD	1.53	50.20	-0.28	-11.36	0.34	-0.58	1.38	15.67	-0.14	-2.00	0.02	-0.125	
HPQ	0.95	67.26	-0.06	-3.12	0.04	-0.19	1.02	27.54	-0.11	-2.97	0.03	-0.185	
IBM	0.81	90.03	-0.06	-3.76	0.05	-0.23	0.94	36.94	-0.19	-6.32	0.14	-0.371	*
INTC	1.16	24.31	0.02	0.66	0.00	0.04	1.05	16.88	0.10	**2.20**	0.02	0.138	
JNJ	0.53	46.75	0.01	0.39	0.00	0.02	0.45	16.11	0.20	**3.35**	0.04	0.208	
JPM	2.08	17.07	-0.07	-0.95	0.00	-0.06	3.63	27.22	-0.90	-12.63	0.39	-0.623	*
KO	0.56	54.21	-0.02	-1.24	0.01	-0.08	0.80	21.59	-0.47	-6.79	0.16	-0.394	*
MCD	0.73	94.53	-0.08	-5.30	0.10	-0.32	0.93	27.80	-0.34	-7.06	0.17	-0.407	*
MMM	0.84	55.62	0.01	0.58	0.00	0.04	1.09	21.23	-0.28	-4.66	0.08	-0.282	*
MRK	0.78	28.22	0.02	0.74	0.00	0.05	0.84	15.54	-0.05	-0.66	0.00	-0.041	*
MSFT	1.04	89.33	0.01	0.89	0.00	0.06	1.30	22.11	-0.25	-4.25	0.07	-0.259	*
PFE	0.65	40.55	0.16	**12.11**	0.37	0.61	1.20	24.63	-0.44	-7.48	0.19	-0.431	*
PG	0.60	71.28	0.06	**3.09**	0.04	0.19	0.39	10.35	0.42	**6.19**	0.13	0.364	
T	0.85	64.96	0.12	**9.75**	0.27	0.52	1.68	23.96	-0.72	-10.19	0.29	-0.541	*
UTX	1.06	69.87	-0.03	-1.73	0.01	-0.11	1.43	18.77	-0.38	-5.22	0.10	-0.313	*
VZ	0.90	68.92	0.00	0.34	0.00	0.02	0.74	10.13	0.17	**2.27**	0.02	0.142	
WMT	0.72	50.18	-0.12	-4.39	0.07	-0.27	0.65	7.31	0.02	0.16	0.00	0.010	
XOM	1.06	34.25	-0.14	-3.85	0.06	-0.24	1.53	27.85	-0.61	-10.80	0.32	-0.563	*
Number of positive and significant				6						5			
Number of positive correlation						13						8	19
Panel	1.10	131.79	0.03	3.46	0.80		1.28	64.34	-0.14	-7.74	0.80		

Table 7
Competitive Regression between the Implied and Historical $\boldsymbol{\beta}$

The table presents the results of the competitive regression between the implied and historical β. The examined multivariate regression equation for each firm i in 2008 is $\beta_{i,t}^{real} = \omega_{0,i} + \omega_{1,i}\beta_{i,t}^{impl} + \omega_{2,i}\beta_{i,t}^{hist} + \varepsilon_{i,t}$, where $\beta_{i,t}^{impl}$ is the implied beta based on option prices in our model, and $\beta_{i,t}^{real}$ and $\beta_{i,t}^{hist}$ are the real and historical betas computed based on the next and the prior 90-day returns. For each firm, we report the regression coefficient, the corresponding t-statistic, and the F-value of the above multivariate regression. Since almost cases are with high enough F-values and thus significant, it can be inferred that combining these two types of betas is able to provide better description of the dynamics of the future beta. In addition, there are twenty-one significant cases for the implied beta and twenty-three significant cases for the historical beta among 28 examined component stocks, and furthermore, the significant cases for different types of betas are not exactly the same. These results demonstrate that the implied and historical betas contain different information set and each has its own explanation power in predicting future beta. Finally, there are 19 cases in which both betas are significant, which suggests that the implied beta could be with complementary information to the historical beta for predicting future betas.

DJIA Components	ω_0		ω_1		ω_2		*F-value*
	coeff	t	coeff	t	coeff	t	
Alcoa Incorporated	0.19	0.75	0.63	**6.03**	0.09	0.79	**19.03**
American Express Company	3.90	**24.89**	-0.13	**-3.08**	-0.99	**-13.95**	**98.85**
Boeing Company	0.98	**18.90**	0.00	-0.09	0.04	0.66	0.22
Bank of America Corporation	2.84	**23.26**	0.72	**12.05**	-0.74	**-10.73**	**83.54**
Citigroup Incorporated	3.87	**21.62**	0.35	**6.08**	-1.04	**-12.66**	**81.43**
Caterpillar Incorporated	1.98	**11.00**	-0.09	**-2.25**	-0.70	**-4.51**	**10.89**
Chevron Corporation	1.71	**16.88**	-0.23	**-4.44**	-0.58	**-7.70**	**29.82**
DuPont	1.34	**7.19**	0.12	**3.23**	-0.29	**-1.71**	**7.65**
Walt Disney Company	0.18	**3.09**	-0.05	**-2.19**	1.01	**16.64**	**138.47**
General Electric Company	2.25	**29.28**	0.15	**3.68**	-0.98	**-14.38**	**104.31**
Home Depot Incorporated	1.60	**21.38**	-0.27	**-11.13**	-0.05	-0.98	**64.94**
Hewlett-Packard Company	1.03	**27.92**	-0.05	**-2.59**	-0.09	**-2.41**	**7.87**
International Business Machines	0.96	**37.73**	-0.05	**-3.41**	-0.18	**-6.09**	**26.63**
Intel Corporation	1.04	**14.21**	0.01	0.37	0.10	**2.13**	2.49
Johnson & Johnson	0.42	**13.18**	0.03	1.53	0.24	**3.67**	**6.81**
J.P. Morgan Chase & Company	3.46	**24.97**	0.23	**3.74**	-1.00	**-13.43**	**90.93**
Coca-Cola Company	0.89	**21.69**	-0.08	**-4.37**	-0.59	**-8.16**	**34.25**
McDonald's Corporation	0.94	**29.33**	-0.07	**-5.00**	-0.32	**-6.81**	**39.79**
3M Company	1.10	**19.16**	-0.01	-0.44	-0.28	**-4.64**	**10.93**
Merck & Company, Incorporated	0.81	**11.50**	0.02	0.55	-0.03	-0.44	0.37
Microsoft Corporation	1.29	**20.69**	0.00	0.12	-0.25	**-4.14**	**9.01**
Pfizer Incorporated	0.88	**16.69**	0.14	**10.00**	-0.24	**-4.52**	**89.34**
Procter & Gamble Company	0.39	**10.48**	0.04	**2.38**	0.40	**5.82**	**22.33**
AT&T Incorporated	1.40	**18.71**	0.09	**7.02**	-0.53	**-7.54**	**86.51**
United Technologies Corporation	1.46	**18.88**	-0.03	**-1.89**	-0.38	**-5.27**	**15.54**
Verizon Communications Inc.	0.73	**9.90**	0.00	0.36	0.17	**2.27**	2.63
Wal-Mart Stores Incorporated	0.69	**8.00**	-0.12	**-4.39**	0.03	0.30	**9.64**
Exxon Mobil Corporation	1.67	**28.85**	-0.15	**-5.38**	-0.63	**-11.65**	**79.25**
Panel	1.27	**64.08**	0.06	**6.55**	-0.19	**-9.55**	**996.49**

Table 8

Forecasts of Underlying Stock Return on implied σ_e

This table displays the results from the 90 day-ahead predictive regressions of the risk-adjusted excess returns on the idiosyncratic risk. This table reports the results of the following regression:

$$\bar{R}_{i,t} - \beta_{i,t}^{real}\bar{R}_{m,t} = \alpha_0 + \alpha_1\sigma_{e,i}(t) + \varepsilon_{i,t}^{e},\ t = 1,2,\ldots,253, i = 1,2,\ldots 28.$$

The average of future 90 daily excess returns of firm *i* on the date *t* is denoted as $\bar{R}_{i,t}$, the average of future 90 daily excess market returns on the date *t* is represented by $\bar{R}_{m,t}$, $\beta_{i,t}^{real}$ denotes the realized beta for the individual stock *i* over the future 90 days on each date *t*, and $\varepsilon_{i,t}^{e}$ is the individual-specific residual. The *t*-statistics in bold type are significant at the 10% level or higher based on the two-tailed tests. The last column reports the adjusted R^2 value. In addition, the last row shows the estimate results of pooled cross-sectional fixed effect panel regression for all DJIA component stocks. Generally speaking, the results of this table, especially the zero α_1 for the pooled cross-sectional regression, are inclined to support that the idiosyncratic risk has no robustly significant relationship with the future risk-adjust excess return.

DJIA Components	α_0		α_1		R^2
	coeff	*t*	coeff	*t*	
Alcoa Incorporated	0.25	3.03	-3.72	-11.84	0.36
American Express Company	-0.16	-3.60	-0.23	-1.25	0.01
Boeing Company	-0.27	-7.54	-0.02	-0.18	0.00
Bank of America Corporation	-1.09	-10.01	2.80	**6.75**	0.15
Citigroup Incorporated	-1.04	-8.93	0.81	1.48	0.01
Caterpillar Incorporated	-0.23	-3.24	-0.18	-0.76	0.00
Chevron Corporation	0.49	14.97	-0.63	-4.89	0.09
DuPont	-0.21	-4.67	0.58	**2.68**	0.03
Walt Disney Company	0.19	10.73	-0.62	-7.87	0.20
General Electric Company	-0.38	-13.48	-0.72	-5.33	0.10
Home Depot Incorporated	0.32	5.63	-0.14	-0.67	0.00
Hewlett-Packard Company	0.15	6.17	-0.05	-0.56	0.00
International Business Machines	-0.23	-4.67	1.58	**9.32**	0.26
Intel Corporation	-0.21	-4.66	1.30	**7.65**	0.19
Johnson & Johnson	0.25	14.30	-1.25	-10.13	0.29
J.P. Morgan Chase & Company	0.21	4.41	0.95	**4.47**	0.07
Coca-Cola Company	-0.10	-5.39	0.38	**4.76**	0.08
McDonald’s Corporation	0.40	12.64	-0.45	-3.27	0.04
3M Company	-0.04	-2.27	0.18	1.61	0.01
Merck & Company, Incorporated	-0.11	-1.76	0.18	0.78	0.00
Microsoft Corporation	-0.16	-4.82	0.62	**5.03**	0.09
Pfizer Incorporated	0.15	5.35	-1.31	-6.06	0.13
Procter & Gamble Company	0.25	7.26	-1.51	-7.47	0.18
AT&T Incorporated	0.18	6.13	-0.67	-4.60	0.08
United Technologies Corporation	0.01	1.03	0.08	0.84	0.00
Verizon Communications Inc.	0.17	5.40	0.10	0.58	0.00
Wal-Mart Stores Incorporated	0.54	16.39	-1.41	-9.90	0.28
Exxon Mobil Corporation	0.21	4.28	0.62	**3.00**	0.03
Panel	0.19	132.38	0.00	0.11	0.14

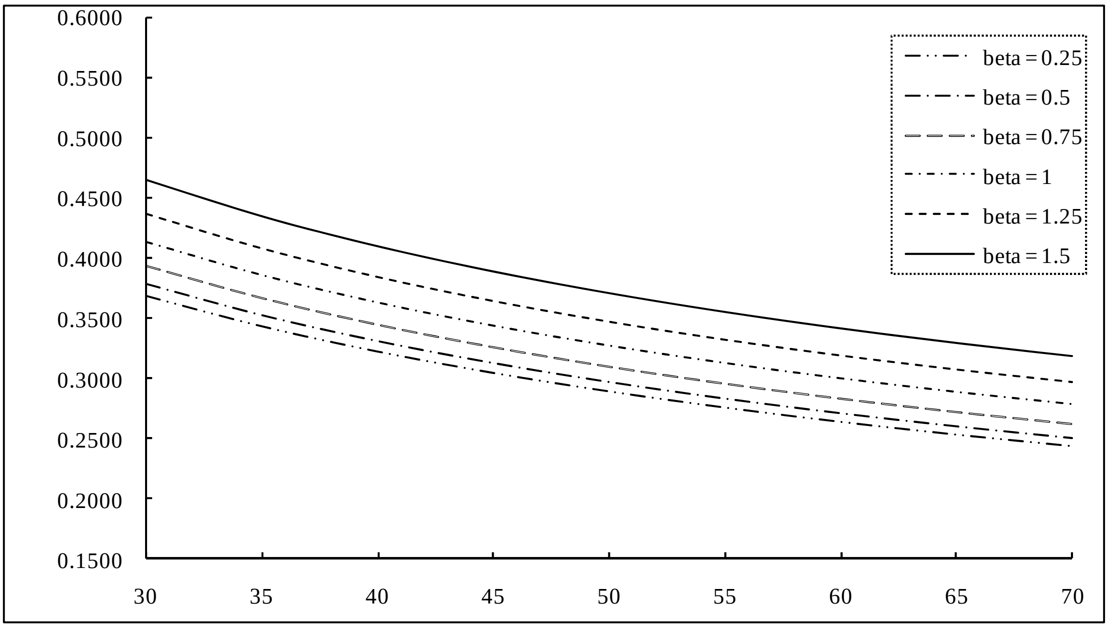

Figure 1. Implied Volatility Curves Given Different β (for σ_e = 0.3). This figure is based the result in Table 1. The x-axis represents the examined strike price, and y-axis presents the implied volatilities based on the Black-Scholes formula. The figure apparently shows that the higher β stocks smile more given the same σ_e. For example, for β equal to 1.5, the implied volatility increases from 0.3188 to 0.4646 when the examined strike price changes from 70 decreasing to 30. In contrast, for β equal to 0.25, the implied volatility changes from 0.2436 to 0.3682 when the examined strike price is from 70 decreasing to 30.

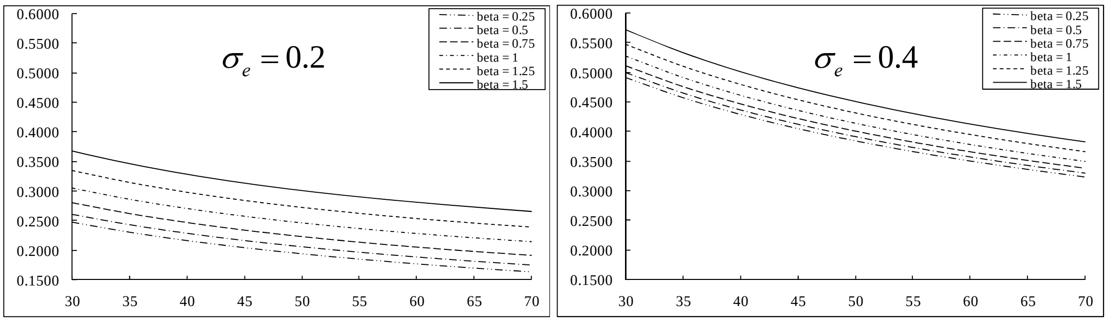

Figure 2. Implied Volatility Curves Given Different β (for σ_e = 0.2 and σ_e = 0.4). This figure is based on the results in Table 2. The *x*-axis represents the examined strike price, and the implied volatility based on the Black-Shoes model is on the *y*-axis. This figure demonstrates that the volatility smile phenomenon for higher idiosyncratic risk stocks is more pronounced. For example, when β equals 1.5, the slope of the volatilities smile curve (defined as $(IV_{K=30} - IV_{K=70})$ / (30 – 70) in Table 2), is −0.00256 for σ_e = 0.2 and −0.00470 for σ_e = 0.4.

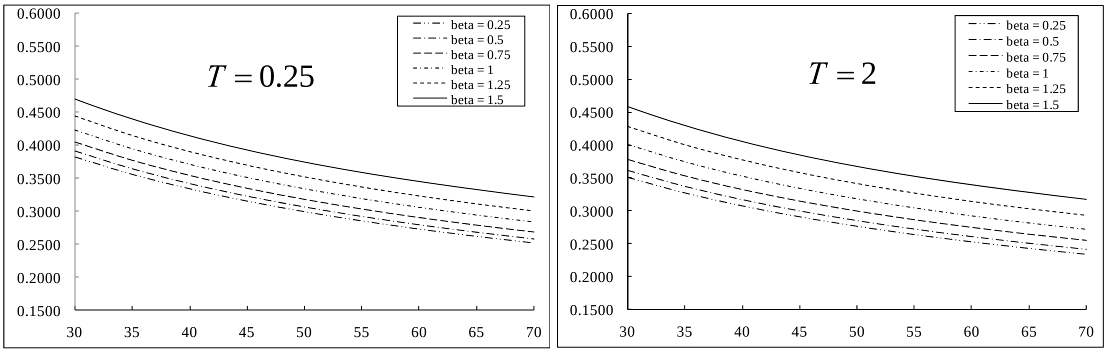

Figure 3 Implied Volatility Curves Given Different β **(for** T **=** 0.25 **and** T **=** 2) The above diagrams report the relationship between volatility smile and the time to maturity. The left-side diagram presents the volatility smile as the time to maturity $T = 0.25$, and the right-side diagram presents the volatility smile as the time to maturity $T = 2$. Comparing these two figures, it is able to conclude that the volatilities smile phenomenon decays as the time to maturity T increases.

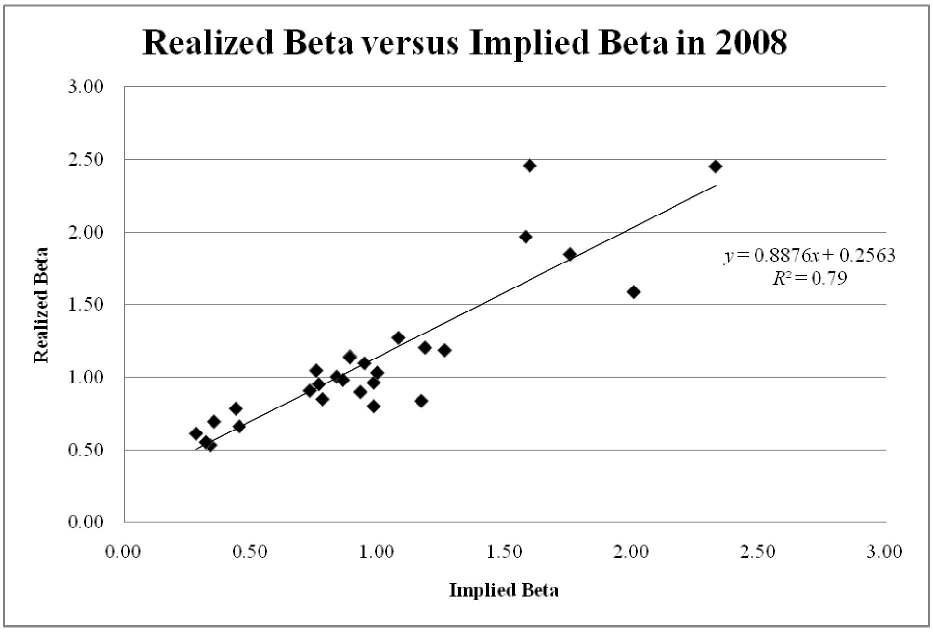

Figure 4 The Scatter Diagram for Realized β versus Implied β. This figure shows the scatter diagram of the average realized β and implied β of the component stocks in DJIA. In addition, the least regression results are also reported. The slope coefficient is 0.8876, which is close to 1. The R-squared value is also as high as 0.79. Both evidence that our model can generate reasonable levels of β for the component stocks of DJIA belonging to various industries.

Appendix I

Before deriving the option pricing formula based on the market model, we first demonstrate some results of the integrals based upon the normal distributions. The definition of the probability density function of a normal distributed random variable $x \sim N(\mu_x, \sigma_x^2)$ is

$$\phi(x) = \frac{1}{\sqrt{2\pi}\sigma_x} \cdot e^{\frac{-(x-\mu_x)^2}{2\sigma_x^2}}, \tag{A.1}$$

where μ_x and σ_x are the mean and the standard deviation of x, respectively. The following three results of integrals are useful in the following derivation of the option price formulae.

Result 1. The integral of $\phi(x)$:

$$\int_l^u \phi(x)\,dx = \int_{\frac{l-\mu_x}{\sigma_x}}^{\frac{u-\mu_x}{\sigma_x}} \phi(z)\,dz = \int_{-\infty}^{\frac{u-\mu_x}{\sigma_x}} \phi(z)\,dz - \int_{-\infty}^{\frac{l-\mu_x}{\sigma_x}} \phi(z)\,dz = N\left(\frac{u-\mu_x}{\sigma_x}\right) - N\left(\frac{l-\mu_x}{\sigma_x}\right), \tag{A.2}$$

where $N(z^*)$ is the cumulative distribution function for the standard normal distribution from $-\infty$ to z^*, that is, $N(z^*) = \int_{-\infty}^{z^*} (2\pi)^{-1/2} \cdot e^{-z^2/2}\,dz$.

Result 2. The integral of the product of $x\phi(x)$:

$$\int_l^u x \cdot \phi(x)\,dx = \sigma_x \cdot \left[n\left(\frac{l-\mu_x}{\sigma_x}\right) - n\left(\frac{u-\mu_x}{\sigma_x}\right)\right] + \mu_x \cdot \left[N\left(\frac{u-\mu_x}{\sigma_x}\right) - N\left(\frac{l-\mu_x}{\sigma_x}\right)\right]. \tag{A.3}$$

The proof for Equation (A.3) is as follows. Define $z = (x - \mu_x) / \sigma_x$, and we can obtain

$$\int_l^u x \cdot \phi(x)\,dx = \int_{\frac{l-\mu_x}{\sigma_x}}^{\frac{u-\mu_x}{\sigma_x}} \sigma_x \cdot z \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} dz + \mu_x \int_{\frac{l-\mu_x}{\sigma_x}}^{\frac{u-\mu_x}{\sigma_x}} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} dz$$

$$= \sigma_x \left(\frac{-1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} \Bigg|_{\frac{l-\mu_x}{\sigma_x}}^{\frac{u-\mu_x}{\sigma_x}} \right) + \mu_x \cdot \left[N\left(\frac{u-\mu_x}{\sigma_x}\right) - N\left(\frac{l-\mu_x}{\sigma_x}\right) \right]$$

$$= \sigma_x \cdot \left[n\left(\frac{l-\mu_x}{\sigma_x}\right) - n\left(\frac{u-\mu_x}{\sigma_x}\right) \right] + \mu_x \cdot \left[N\left(\frac{u-\mu_x}{\sigma_x}\right) - N\left(\frac{l-\mu_x}{\sigma_x}\right) \right],$$

where $n(z) = (2\pi)^{\frac{-1}{2}} \cdot e^{\frac{-z^2}{2}}$, which is the probability density function of the standard normal distribution.

Result 3. The integral of the product of $e^x \phi(x)$:

$$\int_l^u e^x \phi(x)\, dx = e^{\mu_x + \frac{\sigma_x^2}{2}} \cdot \left[N\left(\frac{u-(\mu_x+\sigma_x^2)}{\sigma_x}\right) - N\left(\frac{l-(\mu_x+\sigma_x^2)}{\sigma_x}\right) \right]. \tag{A.4}$$

The proof for the above formula is as follows.

$$\int_l^u e^x \cdot \phi(x)\, dx = \int_l^u \frac{1}{\sqrt{2\pi}\sigma_x} \cdot e^{\frac{-(x-\mu_x)^2}{2\sigma_x^2}+x} dx = \int_l^u \frac{1}{\sqrt{2\pi}\sigma_x} \cdot e^{\frac{-1}{2\sigma_x^2}[(x-\mu_x)^2 - 2x\sigma_x^2]} dx$$

$$= \int_l^u \frac{1}{\sqrt{2\pi}\sigma_x} \cdot e^{\frac{-1}{2\sigma_x^2}[x^2 - 2x\mu_x + \mu_x^2 - 2x\sigma_x^2]} dx = \int_l^u \frac{1}{\sqrt{2\pi}\sigma_x} \cdot e^{\frac{-1}{2\sigma_x^2}[x^2 - 2x(\mu_x+\sigma_x^2) + \mu_x^2]} dx$$

$$= \int_l^u \frac{1}{\sqrt{2\pi}\sigma_x} \cdot e^{\frac{-1}{2\sigma_x^2}\left[x^2 - 2x(\mu_x+\sigma_x^2) + (\mu_x+\sigma_x^2)^2 - 2\mu_x\sigma_x^2 - \sigma_x^4\right]} dx$$

$$= e^{\mu_x + \frac{\sigma_x^2}{2}} \left[\int_l^u \frac{1}{\sqrt{2\pi}\sigma_x} \cdot e^{\frac{-1}{2\sigma_x^2}[x-(\mu_x+\sigma_x^2)]^2} dx \right]$$

$$= e^{\mu_x + \frac{\sigma_x^2}{2}} \int_{\frac{l-(\mu_x+\sigma_x^2)}{\sigma_x}}^{\frac{u-(\mu_x+\sigma_x^2)}{\sigma_x}} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} dz \quad \text{(by defining } z = [x-(\mu_x+\sigma_x^2)]/\sigma_x\text{)}$$

$$= e^{\mu_x + \frac{\sigma_x^2}{2}} \cdot \left[N\left(\frac{u-(\mu_x+\sigma_x^2)}{\sigma_x}\right) - N\left(\frac{l-(\mu_x+\sigma_x^2)}{\sigma_x}\right) \right].$$

Appendix II

According to Equation (14), the current option value can be expressed as

$$V = R_f^{-1} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} S_0 \max\left[\left((1-\beta)R_f + \beta e^{Z_m} + Z_e\right) R_q^{-1} - k, 0\right] \cdot \phi^*(Z_m)\phi^*(Z_e) dZ_m dZ_e,$$

where $\phi^*(Z_m)$ and $\phi^*(Z_e)$ are the risk-neutral standard normal probability density functions for $Z_m \sim N\left(\ln R_f - \frac{1}{2}\sigma_m^2 T, \sigma_m^2 T\right)$ and $Z_e \sim (0, \sigma_e^2 T)$, and $\phi^*(Z_m) = \frac{1}{\sqrt{2\pi}\sigma_m \sqrt{T}} \cdot \exp\left\{-\frac{1}{2\sigma_m^2 T}\left[Z_m - \left(\ln R_f - \frac{1}{2}\sigma_m^2 T\right)\right]^2\right\}$ and $\phi^*(Z_e) = \frac{1}{\sqrt{2\pi}\sigma_e \sqrt{T}} \cdot$

$\exp\left[-\frac{1}{2\sigma_e^2 T}(Z_e - 0)^2\right]$.

Case 1. For $\beta > 0$, the option value in Equation (15) is as follows.

$$V = R_f^{-1} S_0 \int_{R_q k-(1-\beta)R_f}^{\infty} \int_{-\infty}^{\infty} \left[\left((1-\beta)R_f + \beta e^{Z_m} + Z_e\right) R_q^{-1} - k\right] \phi^*(Z_m)\, dZ_m\, \phi^*(Z_e)\, dZ_e$$

$$+ R_f^{-1} S_0 \int_{-\infty}^{R_q k-(1-\beta)R_f} \int_{\ln(a-bZ_e)}^{\infty} \left[\left((1-\beta)R_f + \beta e^{Z_m} + Z_e\right) R_q^{-1} - k\right] \phi^*(Z_m)\, dZ_m\, \phi^*(Z_e)\, dZ_e, \tag{A.5}$$

where $\ln(a - bZ_e) = \ln\left(\frac{R_q k-(1-\beta)R_f - Z_e}{\beta}\right)$, which implies $a = \frac{R_q k-(1-\beta)R_f}{\beta}$ and $b = \frac{1}{\beta}$. For the first integral in Equation (A.5),

$$R_f^{-1}S_0\int_{R_qk-(1-\beta)R_f}^{\infty}\int_{-\infty}^{\infty}\left[\left((1-\beta)R_f\ +\beta e^{Z_m}+Z_e\right)R_q^{-1}-k\right]\cdot\phi^*(Z_m)\,dZ_m\,\phi^*(Z_e)\,dZ_e$$

$$=R_f^{-1}S_0\int_{R_qk-(1-\beta)R_f}^{\infty}\left[\left(R_f\ +Z_e\right)R_q^{-1}-k\right]\phi^*(Z_e)dZ_e$$ [8]

$$=R_f^{-1}S_0\left\{\int_{R_qk-(1-\beta)R_f}^{\infty}\left(R_f\,R_q^{-1}-k\right)\cdot\phi^*(Z_e)\,dZ_e+\int_{R_qk-(1-\beta)R_f}^{\infty}R_q^{-1}Z_e\cdot\phi^*(Z_e)dZ_e\right\}$$

$$\overset{\text{by (A.2) \& (A.3)}}{=}\quad R_f^{-1}S_0\left[\left(R_f\,R_q^{-1}-k\right)\cdot N\left(\frac{(1-\beta)R_f-R_qk}{\sigma_e\sqrt{T}}\right)+R_q^{-1}\sigma_e\sqrt{T}\cdot n\left(\frac{R_qk-(1-\beta)R_f}{\sigma_e\sqrt{T}}\right)\right]$$

$$=S_0e^{-rT}\left[\left(e^{(r-q)T}-k\right)\cdot N(M_1)+e^{-qT}\sigma_e\sqrt{T}\cdot n(-M_1)\right], \qquad \text{(A.6)}$$

where $M_1=\frac{(1-\beta)e^{rT}-e^{qT}k}{\sigma_e\sqrt{T}}$, and the last equation is derived based on the definitions of $R_f\equiv e^{rT}$ and $R_q\equiv e^{qT}$.

As to the second integral in Equation (A.5),

$$R_f^{-1}S_0\int_{-\infty}^{R_qk-(1-\beta)R_f}\int_{\ln(a-bZ_e)}^{\infty}\left[\left((1-\beta)R_f\ +\beta e^{Z_m}+Z_e\right)R_q^{-1}-k\right]\cdot\phi^*(Z_m)\,dZ_m\,\phi^*(Z_e)\,dZ_e$$

$$=R_f^{-1}S_0\int_{-\infty}^{R_qk-(1-\beta)R_f}\left\{\int_{\ln(a-bZ_e)}^{\infty}\left[\left((1-\beta)R_f\ +Z_e\right)R_q^{-1}-k\right]\cdot\phi^*(Z_m)\,dZ_m\right.$$

$$\left.+\int_{\ln(a-bZ_e)}^{\infty}\beta\,R_q^{-1}e^{Z_m}\cdot\phi^*(Z_m)\,dZ_m\right\}\phi^*(Z_e)\,dZ_e$$

$$\overset{\text{by (A.2) \& (A.4)}}{=}\ R_f^{-1}S_0\int_{-\infty}^{R_qk-(1-\beta)R_f}\left\{\left[\left((1-\beta)R_f\ +Z_e\right)R_q^{-1}-k\right]N\left(\frac{-\ln(a-bZ_e)+\left(\ln R_f-\frac{\sigma_m^2T}{2}\right)}{\sigma_m\sqrt{T}}\right)\right.$$

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[8] Suppose the value of Z_e is given. The result of the integral $\int_c^{\infty}\{[(1-\beta)R_f\ +\beta e^{Z_m}+Z_e]R_q^{-1}-k\}\phi^*(Z_m)\,dZ_m$ is $\int_c^{\infty}\{[(1-\beta)R_f\ +Z_e]R_q^{-1}-k\}\cdot\phi^*(Z_m)\,dZ_m+\int_c^{\infty}\beta R_q^{-1}e^{Z_m}\phi^*(Z_m)\,dZ_m$ $\overset{\text{by (A.2) \& (A.4)}}{=}$ $\left[\left((1-\beta)R_f\ +Z_e\right)R_q^{-1}-k\right]\cdot N\left(\frac{-c+\ln R_f-\frac{\sigma_m^2T}{2}}{\sigma_m\sqrt{T}}\right)+\beta R_q^{-1}R_f\cdot N\left(\frac{-c+\ln R_f+\frac{\sigma_m^2T}{2}}{\sigma_m\sqrt{T}}\right)$. Therefore, when $c\to\infty$, the result for $\int_c^{\infty}\left[\left((1-\beta)R_f\ +\beta e^{Z_m}+Z_e\right)R_q^{-1}-k\right]\cdot\phi^*(Z_m)\,dZ_m$ will converge to $\left[\left((1-\beta)R_f\ +Z_e\right)R_q^{-1}-k\right]\cdot 1+\beta R_q^{-1}\cdot 1=\left(R_f\ +Z_e\right)R_q^{-1}-k$.

$$+\beta R_q^{-1} \cdot R_f \cdot N\left(\frac{-\ln(a-bZ_e)+\left(\ln R_f + \frac{\sigma_m^2 T}{2}\right)}{\sigma_m\sqrt{T}}\right)\Bigg\}\phi^*(Z_e)\,dZ_e$$

$$= S_0 e^{-rT}\int_{-\infty}^{e^{qT}k-(1-\beta)e^{rT}}\left\{\left[\left((1-\beta)e^{rT}+Z_e\right)e^{-qT}-k\right]\cdot N(D_2)\right.$$

$$\left.+\beta e^{(r-q)T}\cdot N(D_1)\right\}\cdot\phi^*(Z_e)\,dZ_e, \tag{A.7}$$

where $D_1 = \frac{-\ln(a-bZ_e)+\left(rT+\frac{\sigma_m^2 T}{2}\right)}{\sigma_m\sqrt{T}}$ and $D_2 = \frac{-\ln(a-bZ_e)+\left(rT-\frac{\sigma_m^2 T}{2}\right)}{\sigma_m\sqrt{T}} = D_1 - \sigma_m\sqrt{T}$. As a result, the sum of Equations (A.6) and (A.7) is the result for the integral in Equation (A.5).

Case 2. For $\beta = 0$, the option value in Equation (17) is as follows.

$$V = R_f^{-1}S_0\int_{R_q k-R_f}^{\infty}\left[\left(R_f+Z_e\right)R_q^{-1}-k\right]\cdot\phi^*(Z_e)\,dZ_e$$

$$= R_f^{-1}S_0\left[\int_{R_q k-R_f}^{\infty}\left(R_f R_q^{-1}-k\right)\cdot\phi^*(Z_e)\,dZ_e+\int_{R_q k-R_f}^{\infty}R_q^{-1}Z_e\cdot\phi^*(Z_e)\,dZ_e\right]$$

$$\overset{\text{by (A.2) \& (A.3)}}{=} R_f^{-1}S_0\left[\left(R_f R_q^{-1}-k\right)\cdot N\left(\frac{R_f-R_q k}{\sigma_e\sqrt{T}}\right)+R_q^{-1}\sigma_e\sqrt{T}\cdot n\left(\frac{R_q k-R_f}{\sigma_e\sqrt{T}}\right)\right].$$

$$= S_0 e^{-rT}\left[\left(e^{(r-q)T}-k\right)\cdot N(M_2)+e^{-qT}\sigma_e\sqrt{T}\cdot n(-M_2)\right], \tag{A.8}$$

where $M_2 = \frac{e^{rT}-e^{qT}k}{\sigma_e\sqrt{T}}$.

Case 3. For $\beta < 0$, the option price formula is as follows.

$$V = R_f^{-1}S_0\int_{R_q k-(1-\beta)R_f}^{\infty}\int_{-\infty}^{\ln(a-bZ_e)}\left\{\left[(1-\beta)R_f+\beta e^{Z_m}+Z_e\right]R_q^{-1}-k\right\}\phi^*(Z_m)\,dZ_m\,\phi^*(Z_e)\,dZ_e$$

$$= R_f^{-1}S_0\int_{R_q k-(1-\beta)R_f}^{\infty}\left\{\int_{-\infty}^{\ln(a-bZ_e)}\left\{\left[(1-\beta)R_f+Z_e\right]R_q^{-1}-k\right\}\cdot\phi^*(Z_m)\,dZ_m\right.$$

$$+\int_{-\infty}^{\ln(a-bZ_e)} \beta R_q^{-1} e^{Z_m} \cdot \phi^*(Z_m)\, dZ_m \Bigg\} \cdot \phi^*(Z_e)\, dZ_e$$

$$\overset{\text{by (A.2) \& (A.4)}}{=} R_f^{-1} S_0 \int_{R_q k-(1-\beta)R_f}^{\infty} \left\{\left[\left((1-\beta)R_f + Z_e\right) R_q^{-1} - k\right] N\left(\frac{\ln(a-bZ_e)-\left(\ln R_f - \frac{\sigma_m^2 T}{2}\right)}{\sigma_m\sqrt{T}}\right)\right.$$

$$\left. +\beta R_q^{-1} R_f \cdot N\left(\frac{\ln(a-bZ_e)-\left(\ln R_f + \frac{\sigma_m^2 T}{2}\right)}{\sigma_m\sqrt{T}}\right)\right\} \phi^*(Z_e)\, dZ_e$$

$$= S_0 e^{-rT} \int_{e^{qT}k-(1-\beta)e^{rT}}^{\infty} \left\{\left[\left((1-\beta)e^{rT} + Z_e\right)e^{-qT} - k\right] \cdot N(-D_2)\right.$$

$$\left. +\beta e^{(r-q)T} \cdot N(-D_1)\right\} \cdot \phi^*(Z_e)\, dZ_e. \tag{A.9}$$

Appendix III

In Equation (7), if $\beta = 1$ and the normally distributed random variable r_e is fixed at zero, the payoff function becomes the one for a market index call option if S_0 represents the market index level today. If $\sigma_e = 0$, we can derive $r_e = Z_e = \mu_e^* = 0$ in probability 1. Since Z_e is fixed at zero, the integral over Z_e can be dropped. Furthermore, due to the fact that $R_q k - (1-\beta)R_f = R_q k$ is larger than zero, we can obtain $-\infty < Z_e = 0 < R_q k - (1-\beta)R_f$ and thus only the second term in Equation (15) needed to be considered.

$$V = R_f^{-1} S_0 \int_{\ln(a-b\cdot 0)=\ln R_q k}^{\infty} \left(R_q^{-1} e^{Z_m} - k\right) \cdot \phi^*(Z_m)\, dZ_m$$

$$= R_f^{-1} S_0 \int_{\ln R_q k}^{\infty} R_q^{-1} e^{Z_m} \cdot \phi^*(Z_m)\, dZ_m - R_f^{-1} S_0 \int_{\ln R_q k}^{\infty} k \cdot \phi^*(Z_m)\, dZ_m$$

$$\overset{\text{by (A.2) \& (A.4)}}{=} R_f^{-1} \cdot S_0 \cdot R_q^{-1} \cdot R_f \cdot N\left(\frac{-\ln R_q k + \ln R_f + \frac{\sigma_m^2 T}{2}}{\sigma_m \sqrt{T}}\right)$$

$$- R_f^{-1} \cdot S_0 \cdot k \cdot N\left(\frac{-\ln R_q k + \ln R_f - \ln R_q - \frac{\sigma_m^2 T}{2}}{\sigma_m \sqrt{T}}\right)$$

$$= S_0 \cdot R_q^{-1} \cdot N\left(\frac{\ln\left(\frac{S_0}{K}\right) + \ln R_f - \ln R_q + \frac{\sigma_m^2 T}{2}}{\sigma_m \sqrt{T}}\right) - R_f^{-1} \cdot K \cdot N\left(\frac{\ln\left(\frac{S_0}{K}\right) + \ln R_f - \ln R_q - \frac{\sigma_m^2 T}{2}}{\sigma_m \sqrt{T}}\right).$$

By substituting R_f and R_q with e^{rT} and e^{qT}, the above equation can be rewritten to be identical to the Black-Scholes formula for the market index call options:

$$V = S_0 e^{-qT} \cdot N\left(\frac{\ln\left(\frac{S_0}{K}\right) + (r-q)T + \frac{\sigma_m^2 T}{2}}{\sigma_m \sqrt{T}}\right) - K \cdot e^{-r_f T} \cdot N\left(\frac{\ln\left(\frac{S_0}{K}\right) + (r-q)T - \frac{\sigma_m^2 T}{2}}{\sigma_m \sqrt{T}}\right).$$

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