Understanding the Price Dynamics of Emission Permits: A Model for Multiple Trading Periods †

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Current Version: May 2011

JEL Classification: G13, Q56, Q58, C61

Keywords: emission permits, spot and futures price dynamics, carbon derivatives, stochastic optimal control

†We thank participants of the Energy Finance Conference 2009, Essen, the Campus for Finance Research Conference 2010, Vallendar, the World Congress of Environmental and Resource Economists 2010, Montreal, the Conference of the Swiss Society for Financial Market Research 2011, Zurich, as well as seminar participants at Universities of Zurich, Essen, and the industrial-academic forum on commodities, energy markets, and emission trading at the Fields Institute, Toronto, for their helpful comments and suggestions.

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Abstract

We develop a long-term equilibrium model for emission permit prices under uncertainty accounting for the main stylized facts of today's emission trading systems. By characterizing equilibrium outcomes, we show how the additional consideration of consecutive trading periods adds to and changes a finite period view. We argue that emission permits exhibit characteristics of investment assets within the single trading periods of an emission trading system, but within multiple period frameworks compliance at the end of each trading period leads to the very same option-like behavior that we know from commodities. Our model makes predictions about spot and futures price dynamics, volatility characteristics, and allows analyzing their dependency on important design elements and abatement measures.

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1 Introduction

The development of market-based approaches to environmental protection is one of the key contributions of economics (Stavins (2010)). The material increase in the attention given to market-based environmental policy instruments, both by policy-makers and regulators, has established emission permits as a new asset class in the recent years. Starting as exotic products traded only within voluntary systems in the beginning, they are by now the key element within the world's most significant climate policy, the European Emission Trading System (EU ETS).[1] The introduction of emission trading systems in several other countries[2] promises continued growth of this still new market.

The success of any emission trading program crucially depends on implementation and the details of its design. While the environmental economics literature, building on the seminal works by Coase (1960), Crocker (1966), Dales (1968), and Montgomery (1972),[3] has improved our understanding of how to design sensible, cost-effective policy solutions, it is silent when it comes to pricing and hedging carbon-linked contingent claims. General practice is to apply models of commodity prices such as Schwartz (1997), Routledge, Seppi, and Spatt (2000), Schwartz and Smith (2000), or Casassus and Collin-Dufresne (2005). However, there are reasonable doubts that these models adequately reflect emission permit price characteristics. Indeed, there is even disagreement in the literature whether to categorize them as commodities or as financial assets. Against this background this paper presents an equilibrium model of emission permits that captures their specifics originating from our market-based environmental policy instruments. The model reveals the hybrid nature of emission permits and allows us to study what kind of price dynamics is induced in a permit market by today's state-of-the-art cap-and-trade systems. We believe that knowledge of spot and forward price dynamics, volatility characteristics, and their dependency on important design elements and abatement measures is critical to

[1] In the current trading period from 2008 to 2012, an overall number of 10.4 billion allowances will be allocated, which corresponds to a volume of around 150 billion Euros at current market prices. Additionally, futures and options with different maturities are traded at several exchanges with steadily increasing liquidity.

[2] For example, the Australian Carbon Pollution Reduction Scheme (CPRS) is supposed to be launched in July 2011, and New Zealand has already started an emission trading scheme at the beginning of 2009. Japan has started its first obligatory cap-and-trade system in the area of Tokyo in 2010. Canada is preparing for a national emission trading system. In the US, a draft that would establish emissions trading has been approved by the House of Representatives, but not yet by the Senate.

[3] An overview of related literature is provided by Taschini (2009).

our understanding of emission markets in general and corresponding investment and risk-management strategies.

To shed light on these issues, we propose an equilibrium model for emission allowances that accounts for all main regulatory rules of current state-of-the art emission trading systems: a sequence of consecutive trading periods, the allowance of banking, penalties for non-compliance, and the later delivery of lacking permits. We are able to characterize an emission permit as a strip of European binary options written on global net cumulative emissions. In contrast to classical financial options, the dynamics of this non-tradable underlying is, however, no longer exogenously given, but derived endogenously through abatement measures. We exploit this option analogy to derive several general properties of the price dynamics of emission permits and document how prices and volatilities depend on upcoming trading periods. We show that each additional trading period leads to an additional value component in today's spot price with a relative share depending on current and future expected emissions. As a consequence, in economic downturns most of the value component comes from the possible use of permits in future periods which explains the still high price level in the EU ETS during the recent financial crisis. We also derive important implications for the most liquidly traded derivatives in emissions markets. On the one hand, we characterize properties of the futures price curve and show that standard convenience yield models prove to be inappropriate to represent these curves correctly. On the other hand, we analyze the probability distributions of permit prices which are essential for pricing carbon-related options and deduce characteristic properties of the volatility smile in this market.

Our paper proceeds as follows: Section 2 surveys the related literature and puts this work into perspective. In Section 3, we introduce our long-term equilibrium model and characterize equilibrium outcomes. Properties of the allowance price and volatility in general as well as within a concrete EU ETS model setting are given in Section 4. Section 5 derives implications for the most important carbon derivatives. Section 6 concludes.

2 Related Research

Our paper is related to three separate but related strands of literature. First, our study is naturally linked to the literature on environmental economics. In this literature, it is widely agreed that putting a price on carbon is a crucial step towards an efficient and effective climate policy. Both, theoretical and empirical work on the performance of market-based approaches shows that they achieve their environmental objectives at lower cost than conventional command-and-control systems (see Montgomery (1972), Cronshaw and Kruse (1996), Rubin (1996), as well as Carlson, Burtraw, Cropper, and Palmer (2000), Burtraw, Evans, Krupnick, Palmer, and Toth (2005)). While our paper is related to above research by virtue of studying market-based mechanisms to put a price on emissions, our paper focuses on the price dynamics within such systems.

Second, our work relates to the literature on commodity markets and the assessment of commodity-related securities and projects. In this literature (see e.g. Schwartz (1997), Routledge, Seppi, and Spatt (2000), Schwartz and Smith (2000), or Casassus and Collin-Dufresne (2005)) it is well recognized that economic decisions such as option valuation, investment decisions, and risk management depend critically on the assumed stochastic behavior of commodity prices. Recent models typically account for the mean reverting nature of commodity prices (under the pricing measure) induced via a stochastic instantaneous convenience yield.[4] It would stand to reason that these approaches are also a good choice for modeling the price dynamics of emission permits, which are typically classified as a commodity since they serve as an input to production for regulated companies. However, we show that within the single trading periods of an emission trading system, emission permits exhibit characteristics of investment assets. They are storable and transferable at no costs within trading periods and actual demand for compliance only occurs at the end of each trading period. Thus, within trading periods, the embedded timing option in storable spot commodities (the holder can decide when to consume) is never exercised and both, a mean-reverting behavior and a stochastic instantaneous convenience yield inappropriately account for the nature of emission permits. Still, when considering multiple consecutive trading periods, compliance at the end of each trading period leads to the very same option-like behavior that we know from the commodity literature. Routledge, Seppi, and Spatt (2000) highlight that a commodity's embedded timing option derives value due to the nonnegativity constraint in inventory and show how the option value depends on endogenous inventory and exogenous shocks to supply and demand. Analogous, we argue that endogenous abatement and exogenous demand shocks due to stochastic emission rates drive the embedded option in permit prices and the op-

[4]Often, a convenience yield dynamics is simply exogenously given either explicitly such as in the approach of Schwartz (1997) or implicitly as in the short-term/long-term model of Schwartz and Smith (2000). Routledge, Seppi, and Spatt (2000) among others endogenously derive the convenience yield as a result of inventory decisions in the presence of a physical nonnegativity constraint.

tion is valuable due to the presence of a physical nonnegativity constraint at compliance dates.[5] As a consequence, we can characterize the relationship between contemporaneous spot and forward prices[6] through an implied endogenous convenience yield within our equilibrium model. Obviously, this implied convenience yield crucially differs from the usual instantaneous convenience yields in continuous-time models of commodity prices.

Third, the literature on equilibrium models for permit markets is most closely related to our study. Equilibrium models for permit markets have been considered under certainty by Rubin (1996) and Schennach (2000). In the recent literature, equilibrium models under uncertainty have first been developed by Seifert, Uhrig-Homburg, and Wagner (2008), Carmona, Fehr, and Hinz (2009), and Carmona, Fehr, Hinz, and Porchet (2010). These models consider one finite trading period, in which different companies emit carbon according to an individual emission process and choose optimal trading and abatement strategies to minimize their overall costs. From the choice of the strategies, an equilibrium permit price results. The setting of Chesney and Taschini (2008) is similar as well, unless no abatement is possible in there, so that the whole emissions are exogenously given. Different to those models, in Kijima, Maeda, and Nishide (2009) or Cetin and Verschuere (2009) markets with two trading periods are constructed. The general equilibrium framework of Kijima, Maeda, and Nishide (2009) allows either both inter-period banking and borrowing or neither of them. Cetin and Verschuere (2009) analyze a setting without inter-period banking and derive the spot price dynamics from an exogenously given forward price process by relying on the no-arbitrage relation between spot- and forward prices. In contrast to this literature, our model accounts for all main regulatory rules of current state-of-the art emission trading systems: first and foremost a sequence of consecutive trading periods but also the allowance of banking, penalties for non-compliance, and the later delivery of lacking permits.

[5]By the characterization of Jarrow (2010), the usage option of emission permits is European, in contrast to an American usage option representing an ordinary convenience yield.

[6]Here, we assume the risk-free interest rate to be constant and abstract from differences between forward and future prices. See Casassus and Collin-Dufresne (2005) for a model with stochastic interest rates.

3 Theoretical Model

We develop a theoretical equilibrium model for the EU ETS accounting for multiple trading periods, allowance of banking into a following period but restriction of borrowing from these periods, penalties, and later delivery of lacking permits. In our basic version, we show that an emission permit is essentially equivalent to a strip of binary options written on net cumulative emissions. The full model incorporates abatement possibilities to take into account that the dynamics of the underlying is not exogenously given, but influenced by the abatement strategies of market participants.

3.1 Basic Model

To capture the current setting of the EU ETS we consider a finite set of companies I and n consecutive trading periods denoted as time intervals $[0, T_1], [T_1, T_2], \ldots, [T_{n-1}, T_n]$. At the beginning of each trading period $[T_{k-1}, T_k]$, a regulator allocates a certain amount of permits to each company $i \in I$, denoted by $e_{i,k-1}$.[7] The goal of an emission trading system is to enforce that companies stick to that amount of emissions. To reach this goal, penalties are imposed on companies that have realized more emissions than they can cover with allowances.

Trading takes place among all market participants, be they non regulated investors, risk intermediaries, or regulated companies aiming to adjust the number of allowed emissions to their demand. At the end of a trading period, a regulated company is penalized if cumulative emissions reduced by the allowances bought in addition (net realized emissions) exceed the amount of permits allocated. Additionally, lacking permits have to be delivered in the next trading period. If a company holds allowances not needed for compliance at the end of a trading period, they can be banked into the next period.

Above rules imply that the amount of emissions to be penalized at time T_k is determined by company i's net realized emissions x_{iT_k} (after trading) from 0 to time T_k and the cumulative amount $e_{iT_k} = \sum_{T_j < T_k} e_{i,j}$ of allowances allocated to company i before time T_k, no matter how many allowances have been remaining or penalized in each former trading period. The total penalty for the difference $R_k(x_{iT_k}) = e_{iT_k} - x_{iT_k}$ can be expressed by

[7]For non-regulated companies we simply set the allocation $e_{i,k-1}$ to zero. The same applies to parameters introduced later, e.g. the parameters of an emission process.

the penalty cost function

$$P_k(x_{iT_k}) = \min\left[0, p_k R_k(x_{iT_k})\right] \tag{1}$$

with cost coefficient p_k per ton of uncovered emissions. At T_n, the end of the last trading period, we do not consider any following trading period. To close the model, we set a fixed permit price for an imaginary following trading period and denote the time-0 value of this price by S_{end}.[8] Lacking allowances as well as leftover allowances at the end of the last period T_n are valued at $e^{rT_n} S_{end}$ with r being the constant interest rate.

Given company i's emission rate process $(y_{it})_{t\in[0,T_n]}$

$$dy_{it} = \mu_i(t, y_{it})dt + \sigma_i(t, y_{it})dW_{it}, \tag{2}$$

where dW_{it} are the increments of a standard Wiener process, and its trading strategy $\theta_i = (\theta_{it})_{t\in[0,T_n]}$ in emission allowances, we can specify the expected net cumulative emissions x_{it,T_k} at time t up to T_k as

$$x_{it,T_k} = -\int_0^t \theta_{is} ds + E_t\left[\int_0^{T_k} y_{is} ds\right]. \tag{3}$$

In the case $t = T_k$, x_{iT_k,T_k} is exactly the same as the net realized emissions x_{iT_k}.

Companies choose appropriate trading strategies to hedge the risk of paying penalties and trade at equilibrium prices $S(t)$. More formally, company i solves the optimization problem

$$\max_{(\theta_{is})_{s\in[0,T_n]}} E_0\left[-\int_0^{T_n} e^{-rs} S(s)\theta_{is} ds + \sum_{j=1}^{n} e^{-rT_j} P_j(x_{iT_j}) + R_n(x_{iT_n}) S_{end}\right]. \tag{4}$$

The first term of the optimization problem describes the costs of the trading strategy, discounted to 0. In the second term, the sum incorporates possible penalties at the end of each trading period in a multi-period setting. Finally, leftover or lacking allowances at the end of a setting have the present value S_{end}.

[8]This assumption should not be too critical, even if the emission trading system is intended to be continued infinitely. First, we argue that market participants most probably think about emission trading systems with a finite horizon, because possible trading periods in the remote future are associated with high uncertainty. Second, the influence of S_{end} on today's prices goes to zero when the number of trading periods n becomes large because it is discounted away. The value of S_{end} does not influence the results of this paper qualitatively. For a quantitative impression of the ratio of the spot price related to S_{end} we refer to Section 4.2.3.

3.2 Accounting for Abatement Opportunities

Extending the basic model, we include the companies' opportunity to abate CO_2, i.e. a company is able to reduce the amount of emitted CO_2 at each point in time by implementing operative abatement measures. We model this by an abatement strategy $\xi_i = (\xi_{it})_{t\in[0,T_n]}$, so that including abatement the cumulative net expected emissions from (3) can be stated as

$$x_{it,T_k} = -\int_0^t \xi_{is} ds - \int_0^t \theta_{is} ds + E_t \left[\int_0^{T_k} y_{is} ds \right]. \quad (5)$$

We specify the costs for abatement ξ_{it} at time t by an abatement cost function C_i. Having the opportunity to abate, a company has to find an optimal trade-off between implementing abatement measures, buying allowances, and exposing itself to the risk of penalty costs. Formally, the individual optimization problem (4) extends to

$$\max_{(\theta_{is},\xi_{is})_{s\in[0,T_n]}} E_0 \left[\int_0^{T_n} e^{-rs} C_i(s,\xi_{is}) ds - \int_0^{T_n} e^{-rs} S(s)\theta_{is} ds + \sum_{j=1}^{n} e^{-rT_j} P_j(x_{iT_j}) + R_n(x_{iT_n}) S_{end} \right]. \quad (6)$$

We assume in general that the companies' abatement cost functions C_i are differentiable and convex in ξ_{it},[9] representing (theoretically) infinitely large abatement capacities. Our choice was motivated by studies like Klepper and Peterson (2006) and Enkvist, Nauclér, and Rosander (2007) which point out that the abatement cost function of the whole economy approximately has this features, in particular linearly increasing marginal abatement costs.[10] It is important to note that C_i stands for the costs of operative abatement ξ_i. Investments into carbon-friendly technologies do not operatively lead to carbon abatement, but would rather flatten the abatement cost function C_i, similar to general technological progress. We have assessed that simple models for technological progress resulting in deterministically time-varying abatement cost functions do not change the results of this paper. Thus, we do not account for this in our model.

[9] For notational ease, we include the negative sign in the cost function itself. In the strict sense, the cost function then is concave, of course.

[10] Still, our choice is only an approximation to the companies' abatement cost functions, which rather are step-wise functions, each step standing for the capacity of a specific abatement measure. An alternative simplification is taken by Carmona, Fehr, and Hinz (2009) who concentrate on one single abatement measure.

3.3 Equilibrium Price Processes

Our aim is now to find equilibrium spot price processes that fulfill the individual optimality conditions (4) or (6), respectively, together with optimal trading and abatement strategies (θ_i, ξ_i), $i \in I$.[11] While this is naturally simple in the basic model, we have to spend much more effort on establishing a solution when abatement is taken into account. Technically, we first consider the last trading period $[T_{n-1}, T_n]$ and then proceed to the previous periods using dynamic programming. Within each trading period the problem can be settled by the dynamic programming approach.

3.3.1 Without Abatement

In the first step, we derive the equilibrium spot price process within the last trading period. Given the stochastic optimal control problem (4), it is intuitively clear that the marginal value of an emission allowance has two components. The first component equals the penalty payment saved weighted by the probability that penalties arise. The second component derives from the value one additional allowance can be sold for at T_n. Since the last period's allowance price equals this marginal value, it can be written as[12]

$$S(t) = e^{-r(T_n - t)} E_t \left[1_{\{R_n(x_{iT_n}) < 0\}} \right] p_n + e^{rt} S_{end} \tag{7}$$

The equilibrium condition implies that $E_t \left[1_{\{R_n(x_{iT_n}) < 0\}} \right]$ is equal for all companies $i \in I$ for all $t \in [T_{n-1}, T_n]$. Considering the end of the setting T_n, this condition means that either all companies have uncovered emissions or all companies have remaining allowances at T_n. In contrast, a situation with some companies having uncovered emissions while others have remaining allowances can not persist in equilibrium. We therefore take a global point of view and consider $x_{t,T_k} = \sum_{i \in I} x_{it,T_k}$. Clearly, $R(x_{T_n}) < (\geq)\ 0$ exactly corresponds to the case $R(x_{iT_n}) < (\geq)\ 0$ for all companies i and the equilibrium spot price results in

$$S(t) = e^{-r(T_n - t)} E_t \left[1_{\{R_n(x_{T_n}) < 0\}} \right] p_n + e^{rt} S_{end}. \tag{8}$$

Thus, the remaining allowances in the whole economy crucially drive the spot price. Due to the market clearing condition, the individual trading strategies cancel out in x_{t,T_k} and the spot price only depends on the emissions process.

[11] Thereby, we do not account for asymmetric information and thus do not differentiate between information sets of the single companies and the global information set.

[12] For a formal derivation, see Appendix A.

In the second step, we derive the equilibrium spot price for the prior trading periods. Using the dynamic programming algorithm for discounted cost problems (cf. Bertsekas (1976)), we step back one period and solve the period $n-1$ problem along the same lines as before by including the period n solution as if it were a given terminal condition. Finally, we end up in the first trading period with the equilibrium spot price

$$S(t) = \sum_{j=1}^{n} e^{-r(T_j - t)} E_t \left[1_{\{R_j(x_{T_j})<0\}} \right] p_j + e^{rt} S_{end} \tag{9}$$

for $t \in [0, T_1]$, and altogether

$$S(t) = \sum_{\substack{j=1, \\ T_j \geq t}}^{n} e^{-r(T_j - t)} E_t \left[1_{\{R_j(x_{T_j})<0\}} \right] p_j + e^{rt} S_{end} \tag{10}$$

for $t \in [0, T_n]$. These formula can be evaluated by simply taking expectations. To this end, note that with (2) and (3) the dynamics of x_{t,T_k} can be written as[13]

$$dx_{t,T_k} = G_k(t) dW_t. \tag{11}$$

3.3.2 With Abatement

We can show that the general structure of the equilibrium spot price (9) still holds even if we introduce abatement opportunities into the problem.[14] However, we cannot evaluate this formula by simply taking expectations as in the case without abatement because the dynamics of cumulative net expected emissions x_{t,T_k}

$$dx_{t,T_k} = -\xi_t dt + G_k(t) dW_t, \tag{12}$$

with $\xi_t = \sum_{i \in I} \xi_{it}$, depends on the individual abatement strategies ξ_i, $i \in I$, which are themselves endogenous. Each new control, the abatement strategy ξ_{it}, gives rise to another optimality condition

$$S(t) = -C_i^{(\xi_{it})}(t, \xi_{it}), \tag{13}$$

stating that the equilibrium spot price equals instantaneous marginal abatement costs of each company. Again, this is very intuitive since otherwise a company would prefer abating more CO_2 and buying less allowances or the other way round.

[13]See Seifert, Uhrig-Homburg, and Wagner (2008), p. 182.

[14]See Appendix A.

It is not clear in the first place whether a global optimal abatement strategy Θ accords with optimal strategies of the individual optimization problems. Fortunately we are able to show both that this holds indeed and that the global optimal strategy can be equivalently obtained by considering the optimal control problem

$$\max_{(\xi_s)_{s\in[0,T_n]}} E_0\left[\int_0^{T_n} e^{-rs}C(s,\xi_s)ds + \sum_{j=1}^{n} e^{-rT_j}P_j(x_{T_j}) + R_n(x_{T_n})S_{end}\right] \tag{14}$$

acting on aggregated volumes.[15] The equilibrium spot price for $t \in [0, T_1]$ is then

$$S(t) = -e^{rt}\sum_{j=1}^{n} V^{(x_{t,T_j})}(t, \overrightarrow{x_t}), \tag{15}$$

where V is the expected value of an optimal abatement strategy of (14), given the state vector $\overrightarrow{x_t} = (x_{t,T_1}, x_{t,T_2}, \ldots, x_{t,T_n})$.

To establish an explicit solution, we consider a quadratic abatement cost function

$$C(t,\xi_t) = -\frac{1}{2}c\xi_t^2 \tag{16}$$

with cost coefficient c and solve the problem by backward induction, starting at the last trading period $[T_{n-1}, T_n]$. The strategy value $V_n(t, x_{t,T_n})$ related to $[T_{n-1}, T_n]$[16] can be shown to solve the PDE[17]

$$V_n^{(t)} = -\frac{1}{2}(G_n(t))^2 V_n^{(x_{t,T_n},x_{t,T_n})} - \frac{1}{2c}e^{rt}(V_n^{(x_{t,T_n})})^2 \tag{17}$$

with boundary condition

$$V_n(T_n, x_{T_n}) = e^{-r(T_n-T_{n-1})}P_n(x_{T_n}) + e^{rT_{n-1}}R_n(x_{T_n})S_{end}. \tag{18}$$

Now step back to the trading period $[T_{n-2}, T_{n-1}]$ and include the solution of period $[T_{n-1}, T_n]$ into the terminal condition. The value $V_{n-1}(t, x_{t,T_{n-1}}, x_{t,T_n})$ of the optimal strategy from T_{n-2} to T_n again results as the solution of a characteristic PDE

$$\begin{aligned} V_{n-1}^{(t)} = -\frac{1}{2}((G_{n-1}(t))^2 V_{n-1}^{(x_{t,T_{n-1}},x_{t,T_{n-1}})} + (G_n(t))^2 V_{n-1}^{(x_{t,T_n},x_{t,T_n})}) \\ - G_{n-1}(t)G_n(t)V_{n-1}^{(x_{t,T_{n-1}},x_{t,T_n})} - \frac{1}{2c}e^{rt}(V_{n-1}^{(x_{t,T_{n-1}})} + V_{n-1}^{(x_{t,T_n})})^2 \end{aligned} \tag{19}$$

[15]See Appendix B.

[16]In this context, t and x_{t,T_n} are defined for the trading period $[T_{n-1}, T_n]$ and thus are different to the variables of (14).

[17]See Appendix C.

with boundary condition

$$V_{n-1}(T_{n-1}, x_{T_{n-1},T_{n-1}}, x_{T_{n-1},T_n}) = e^{-r(T_{n-1}-T_{n-2})}(P_{n-1}(x_{T_{n-1},T_{n-1}}) + V_n(T_{n-1}, x_{T_{n-1},T_n})). \quad (20)$$

If we continue iteratively, we finally get a solution for $V_1(t, \overrightarrow{x_t}) = V(t, \overrightarrow{x_t})$, the value of the optimal strategy from $t \in [0, T_1]$ to T_n. The equilibrium spot price for $t \in [0, T_1]$ directly follows by (15).

3.3.3 Comparative Statics Results

Both modeling versions reveal reasonable sensitivities towards the determinants of the equilibrium spot prices as is shown in Table 1. We obtain the sensitivities directly from the spot price formula (9) for the basic model without abatement and by comparative statics based on numerical calculations when abatement is taken into account.

[Insert Table 1 about here.]

Ceteris paribus, the permit price of the current trading period increases when the penalties for the current or one of the following trading periods are raised, when the expected net cumulative emissions increase, or when the abatement costs increase. The permit price decreases in the initial endowment for one of the trading periods and also in the discount rate. These results hold for an arbitrary number $n \geq 1$ of consecutive trading periods. Thus, our general sensitivity results prove to be robust to having more than one trading period.

4 Properties of the Allowance Price

4.1 General Properties

The theoretical model allows us to characterize the dynamics of permit prices in a long-term equilibrium model for environmental trading schemes. In this section, we derive general properties that are induced by the regulatory framework without presuming specific assumptions on emission rate processes (2) or abatement cost functions. These properties are not only useful in understanding the spot price characteristics within the EU ETS but are of value for up-to-date cap and trade systems more generally.

Proposition 1 (Intra-period martingale property)**.** *Discounted spot prices are martingales within each trading period.*

The proof of Proposition 1 is given in Appendix D. For our basic model without abatement possibilities the intra-period martingale property simply follows from (9) together with the risk-neutrality of actors. Interestingly, Proposition 1 implies that mean-reversion processes considered the natural choice for commodity prices are less suitable for allowance prices. The reason is that within a trading period there are no storability restrictions. As an example, take the current Kyoto commitment period from 2008 to 2012. An immediate consequence of Proposition 1 is that within this five-year period, EUA prices should not reflect any mean-reversion and seasonal behavior.

Proposition 2 (Option characteristics)**.** *Emission allowances can be considered as a strip of binary European call options.*

When abatement opportunities are not incorporated, each call is written on a non-tradable underlying, the net cumulative emissions until the end of a given trading period. Note that in this formulation we have a different underlying for each option of the strip. We might alternatively consider the emission rate as the underlying. In this case, we would have the same underlying for all options of the strip, but the options have a payout function similar to an Asian call option. When abatement is taken into account the same characterization applies in principle. The distinctive feature is that now market participants can and obviously do influence the state of the underlying through their abatement actions. It is this inherent element of an emissions trading system that leads to a clear distinction between emission allowances and classical financial options.

Proposition 2 is of interest for several reasons. First, it shows that each trading period leads to an additional value component, the current value of the binary option with non-negative payoff attributed to that trading period. From an economic perspective, an additional trading period on the one hand gives an additional chance to use left-over allowances for compliance, and on the other hand poses a risk of additional penalties. Both aspects lead to an additional value.
Second, the natural price bounds of binary options imply that the allowance price is bounded below and above. More precise, there is a time-dependent lower bound of

$$S_{lower}(t) = e^{rt} S_{end} \tag{21}$$

and a time-dependent upper bound of

$$S_{upper}(t) = \sum_{j=1}^{n} e^{-r(T_j - t)} p_j + e^{rt} S_{end}. \tag{22}$$

Obviously, the lower bound is independent of the number of trading periods. It represents a scenario where emissions are that low that there is no risk of paying penalties in any of the trading periods. Accordingly, left-over permits are banked until the end of the setting and finally pay the compounded value of S_{end}. The upper bound, on the contrary, depends on the number of trading periods considered. In a scenario of extremely high emissions where penalties are almost sure for all trading periods, one allowance less means an additional penalty in the current trading period plus later delivery and therefore an additional penalty in the next trading period and so on, which explains why the upper bound is dependent on the number of trading periods.[18]

Third, the binary part of the payoff at the end of each trading period gives rise to a discontinuity: To begin with, consider only the first term of the sum (9) as a function of x_{t,T_1} in the basic model. As this is a binary option, only two values are possible at T_1, depending on x_{t,T_1} being smaller or bigger than the initial amount of allocated allowances e_0. In other words, there is an x_{t,T_1}–directional increase around e_0 from the lower to the upper possible value, and this increase is infinitely steep in T_1. Over time this steepness around e_0 steadily increases the closer we approach T_1 while the steepness for very high and very low values of x_{t,T_1} decreases, ending up in this discontinuity. Clearly, this is because the range of expected net emission values x_{t,T_1} permitting a switch between penalty and non-penalty states becomes smaller when time passes and uncertainty decreases. These properties easily translate to the other binary options. Thus, we can conclude that during each trading period, the x_{t,T_k}–directional steepness of the permit price increases over time and finally shows a discontinuity at the end of the period T_k. By numerical calculations we can show that this feature carries forward to the full model including abatement.

The fourth point of interest of Proposition 2 is the induced transition from one trading period to the next. To explain what happens at the end of a trading period we evaluate equation (10). When proceeding from T_1^- to T_1^+, the change of the permit price is simply

$$S(T_1^-) - S(T_1^+) = 1_{\{R_1(x_{T_1})<0\}} p_1. \tag{23}$$

That means, if the economy is in permit surplus, there is a smooth transition from one trading period to the next one and the price does not change. Contrary, if the economy

[18]Note that the upper bound nevertheless converges to a finite value for positive interest rates when the number of trading periods goes to infinity.

is short of permits, prices decrease by the amount of the penalty when proceeding to the next trading period similar to the drop in value of a coupon bond after a coupon payment date.

Proposition 3 (Local volatility)**.** *Local volatility is state- and time-dependent.*

To prove Proposition 3, we apply Itô's lemma to the spot price S as a function of t and $\overrightarrow{x_t}$, and obtain as price dynamics for the basic model without abatement

$$dS = \left(S^{(t)} + \frac{1}{2} \sum_{i,j=1}^{n} G_i(t) G_j(t) S^{(x_{t,T_i}, x_{t,T_j})} \right) dt + \left(\sum_{i=1}^{n} G_i(t) S^{(x_{t,T_i})} \right) dW_t \tag{24}$$

and for the full model with abatement

$$dS = \left(S^{(t)} - \xi_t \sum_{i=1}^{n} S^{(x_{t,T_i})} + \frac{1}{2} \sum_{i,j=1}^{n} G_i(t) G_j(t) S^{(x_{t,T_i}, x_{t,T_j})} \right) dt + \left(\sum_{i=1}^{n} G_i(t) S^{(x_{t,T_i})} \right) dW_t. \tag{25}$$

Obviously, in both cases, (relative) local volatilities

$$\sigma_S(t, \overrightarrow{x_t}) = \frac{\sum_{i=1}^{n} G_i(t) S^{(x_{t,T_i})}(t, \overrightarrow{x_t})}{S(t, \overrightarrow{x_t})} \tag{26}$$

depend on time and the expected net cumulative emissions until the end of each trading period.

Considering the local volatilities with respect to the x_{t,T_k}–directional steepness of the allowance price described before, we find that the volatility has very low values if the x_{t,T_k} are all very low or very high since the $S^{(x_{t,T_k})}$ are very low then. In those cases, it is very unlikely to switch from penalty states to non-penalty states or the other way round. The volatility has a maximum at the point where the x_{t,T_k}–directional increase is the steepest, i.e. where $S^{(x_{t,T_k})}$ is the largest, for all k. The dependence on the $G_k(t)$, however, does not allow for more general statements on how the volatility evolves over time. At least, we can say that $S^{(x_{t,T_1})}$ goes to zero almost surely when t approaches T_1, since x_{t,T_1} takes the value e_0 with probability zero. Therefore the absolute volatility at the end of the trading period goes to $\sum_{i=2}^{n} G_i(t) S^{(x_{t,T_i})}(t, \overrightarrow{x_t})$ almost surely, which is the volatility coming from the next trading period. The following section will give more information about the volatility behavior over time for concrete model settings according to the EU ETS.

4.2 Concrete EU ETS Model Settings

As an example of how the price dynamics evolves, we consider concrete model settings in accordance with the current EU ETS. Two different dimensions are worth to be studied. The first is the effect of additional trading periods on today's price. Thus, we compare settings from one up to four consecutive trading periods. The second is the price impact of the equilibrium abatement strategies.

4.2.1 Regulatory Framework and Parameter Choice

Following the three year learning phase from 2005 to 2007 and the current five year trading period that coincides with the first commitment period of the Kyoto Protocol, future trading periods from 2013 onwards are extended to a length of eight years with the aim to achieve increased efficiency. Instead of national caps in Phase I and II, an EU-wide cap has been agreed on as of 2013, with a total number of allowances decreasing annually in a linear manner. Table 2 gives presumable figures for the future allocations according to MEMO/08/35 of the European Parliament.

[Insert Table 2 about here.]

All explicit calculations and graphical representations are based on allocations e_{k-1} according to these plans. For simplification, we take penalties of 100 EUR/t for each trading period, although in fact the penalties increase "in accordance with the European index of consumer prices" (see Directive 2009/29/EC). S_{end} as 14.11 is taken as an average discounted value of different market participants' long-term price expectations. Finally, we consider a risk-free rate of $r = 4\%$ and linear increasing marginal abatement costs with a cost parameter $c = 0.4$. Table 3 summarizes the parameter values chosen.

[Insert Table 3 about here.]

To keep the model tractable, we select the emission rate process as an arithmetic Brownian motion given by

$$y_t = y_0 + \sigma W_t. \tag{27}$$

By this choice, all expected net cumulative emissions $x_{t,T_1}, \dots, x_{t,T_n}$ at some point in time $t \in [0, T_1]$ are uniquely determined by the value of the expected net cumulative emissions for the current trading period, x_{t,T_1}, and the current emission rate, y_t.[19] For the parameters of the emission rate process we take an initial emission rate y_0 of $2,200$ tons per year and choose σ to be 80.

In our parameter choice an important caveat is that for many parameters of interest, there is little empirical evidence. Thus, any choice of parameters is inevitably ad hoc. To cope with this problem we perform various robustness checks showing that our economic insights and implications are qualitatively unaffected by parameter variation within a feasible range.

4.2.2 Spot Price and Volatility Functions

Within the modeling framework defined, we now illustrate the structure of the spot price functions and volatility surfaces. We consider settings of one and four trading periods and oppose the basic model to the general model incorporating abatement.[20] Table 4 shows equilibrium spot prices and volatilities as functions of expected net cumulative emissions x_{t,T_1} for the current trading period and the currently prevailing emission rate y_t. We report prices for three different points in time within the current five year trading period, one and a half ($t = 1.5$) and three years ($t = 3$) after the trading period's start and shortly before the period's end $t = 4.9$.

[Insert Table 4 about here.]

We very well recognize all the properties discussed in Section 4. Additionally, differences between permit prices or volatilities resulting from the basic model and the general model

[19] To see this, suppose a stochastic process (y_t) with stationary increments. Using definition (11) and (12), respectively, we can write

$$x_{t,T_k} = x_{t,T_1} + E_t\left[\int_{T_1}^{T_k} y_s ds\right] = x_{t,T_1} + \sum_{i=2}^{k} E_t\left[\int_{T_{i-1}}^{T_i} y_s ds\right] \qquad (28)$$

and since (y_t) has stationary increments, all terms of the sum may only depend on the current emission rate y_t. Contrary, x_{t,T_1} additionally depends on the whole realized path of the Wiener process as well as resulting abatement actions up to t. Thus, the current emission rate y_t determines the expected emissions belonging to all the following trading periods from T_2 on. Furthermore, it determines how x_{t,T_1} is composed of realized emissions up to t and future emissions from t to T_1. However, for given x_{t,T_1} and y_t, all relevant expectations are specified.

[20] Settings of two and three periods have been considered as well, but were omitted for a better overview. The plots for these cases look like a crossover from one to four periods, structurally more similar to the four period setting.

incorporating abatement allow us to study the implications of equilibrium abatement strategies. First, abatement opportunities consistently lead to lower allowance prices. This was expected, since market participants reduce the emissions by abatement actions and therefore decrease the probability of penalties to accrue, which is also the reason for the steep part of the x_{t,T_1}–directional increase to start at higher emission levels when abatement is modeled. While without abatement, non-penalty states switch to penalty states when x_{t,T_1} becomes greater than e_0, with abatement this only happens when abatement actions are not supposed to reduce the emissions below e_0. Second, for very low or high emissions, however, abatement does not make a difference because the permit price anyway approaches the lower or upper price bound. Third, proceeding in time, the price difference decreases as well due to the decreasing time left for abatement actions. Fourth, similarly the difference increases with the number of trading periods because of the additional value components that can be influenced by abatement measures.

Furthermore, Table 4 also sheds light into the development of the volatility surface over time in this setting. We make out that the x_{t,T_1}–directional maximum first increases, and then decreases to some basic value when t approaches T_1. The explanation is that first, the increasing x_{t,T_1}–directional steepness of the permit prices is also reflected by increasing volatilities, but at some point in time it becomes clear whether there will be an allowance surplus or a shortage at the end of the period because the amount of emissions cannot change too abruptly anymore. As discussed before, the remaining volatility can be attributed to the following trading periods, while the volatility caused by the current trading period goes to zero. This can explicitly be observed in the setting of only one trading period.

Additionally, we note that a higher number of trading periods leads to lower maximal volatility levels. This can be explained by the additional value components of the permit price that weaken the high relative volatility coming from the current trading period.

4.2.3 Value Components in Today's Prices

Each additional trading period results in an additional value component of a permit. Our model is well suited to partition today's price into its constituents attributable to the different trading periods. We accomplish this by calculating the spot price S for settings of four, three, two and one trading periods, so that we can subdivide the spot price into the ratios q_1, q_2, q_3, q_4 and $q_{S_{end}}$ attributable to the first, second, third, fourth period of the setting and to S_{end}. Table 5 shows the resulting ratios at $t = 3$ for different combinations of low, medium and high[21] values for x_{t,T_1} and y_t both for the basic model and one incorporating abatement measures.

[Insert Table 5 about here.]

We directly observe that the distribution of value components to the different trading periods is qualitatively quite similar in both model variations. At least, there are quantitative differences.

First, we turn to the case "medium/medium". This is the situation typically existing when designing an emission trading scheme today: The amount of allocated permits is set moderately lower than the business-as-usual expected emissions. Then, the ratio of the spot price attributable to a trading period decreases the further the period is in the future. This is on the one hand due to the discounting of future cash flows. If abatement is possible, there is an additional effect that more extensive abatement measures can be applied if the compliance date is in the remote future. This additional effect is the reason why the ratios decrease much more strongly for the model incorporating abatement opportunities.

With some differences, a similar distribution of the ratios can be observed in the cases "high/high" and "high/low". In the case "high/high", the discount effect strongly dominates even if abatement is allowed, because it is expected to pay penalty costs for all the periods due to the high expected emissions. In contrast, the effect of possible abatement measures is rather low in here. Thus, the distribution of ratios is also quantitatively very similar for both model variations. The case "high/low" is in a way a mixture of the situations "medium/medium" and "high/high": Due to the high emissions in the first trading period, penalty costs are expected with high probability for this one. After that, later delivery of permits together with low expected future emissions add up to a medium level then, comparable to the case "medium/medium". The way in which the ratios are distributed in these three cases might be called usual.

The case "low/high" differs from that, but can be explained in a similar way. Because of low emissions for the first trading period, we have no value coming from this period. After that, high future emissions generate a certain level of allowance scarcity again, so that the ratios are again distributed in the usual way from the second trading period on.

[21] We have set $x_{3,T1} = 11,000$ and $y_3 = 2,200$ in the medium scenario. Low values are 15% below those values, high values 15% above.

A totally different situation exists in the case "low/low". Here it is not unlikely that no penalty costs will have to be paid for either one of the trading periods, especially when abatement is allowed. So a relatively large part of the permit price comes from the price S_{end} one can sell allowances for at the end of the setting. The other part comes from the remaining probability of penalty costs in the future periods, and this probability increases with the time left to the end of a particular trading period. Only q_4 is smaller than q_3 in here because the discounting effect again dominates at that point. However, we may say that in principal, in the case "low/low" the ratios are distributed exactly opposite to the usual way.

This last case is comparable to the current situation of the EU ETS. Due to the recent economic downturn, we are expected to have more permits allocated in the current trading period than we are supposed to need for compliance. According to that, it is very likely that a large ratio of the currently traded permit prices is attributable to the coming trading periods in the future. In fact, the current empirical prices for European emission allowances are much lower than 49.04 Euros according to this model setting. This may be because market participants do not expect three following trading periods after 2012 or at least that they do not incorporate them into today's prices.

5 Derivative Pricing in a Multi Period Emissions Trading Framework

Carbon derivatives are becoming an increasingly important segment of the commodity derivatives market. Futures accounted for almost three quarters of the 90 billion Euro EUA transaction volume in 2009. Meanwhile, market players have also chosen the maturing carbon option market to manage risks or take positions. While trading volume in the option market mostly concentrates on options maturing in the current trading period (intra-period options), both futures contracts maturing in the current (intra-period futures) and in the next trading period (inter-period futures) are liquidly traded on several exchanges across Europe. In particular, December 2013 futures contracts are traded actively due to the demand from power companies to hedge their forward electricity sales.

Our long-term equilibrium model has meaningful implications for pricing, hedging, and risk-managing carbon derivatives. First, we characterize properties of the futures price curve and show that standard convenience yield models prove to be inappropriate to represent these curves correctly. Second, the pricing of non-linear carbon-related derivatives crucially depends on the probability distribution of emission permit prices under the pricing measure. The latter is directly given in our equilibrium model due to the intra-period martingale property. Thus, a simulation study based on our equilibrium model allows us to assess which kind of distribution is appropriate for different settings. Additionally, we use the price probability distributions to deduce characteristic properties of the volatility smile in this market.

5.1 Futures

Consider a futures contract for one emission permit delivered at $\bar{t}$. Due to the risk-neutrality of traders, its price F at $t < \bar{t}$ is given by

$$F(t, \overrightarrow{x_t}, \bar{t}) = E_t \left[S(\bar{t}, \overrightarrow{x_{\bar{t}}}) \right], \tag{29}$$

with S as the spot price according to Section 3.

5.1.1 Futures Price Curves

By exploiting equation (29), we are able to formally derive qualitative features of the permit futures price curve from our theoretical model. Therefore we refer to Litzenberger and Rabinowitz (1995) who define that a single futures contract is in strong backwardation if its price is lower than the current spot price ($F(t, \overrightarrow{x_t}, \bar{t}) < S(t, \overrightarrow{x_t})$) and it is in weak backwardation if its price is lower than the compounded spot price ($F(t, \overrightarrow{x_t}, \bar{t}) < e^{r(\bar{t}-t)} S(t, \overrightarrow{x_t})$). In all other cases, it is in contango. The whole futures curve is in backwardation (contango), if all single futures considered are in backwardation (contango).

Within a trading period, say for $t \in [0, T_1[$, $\bar{t} \in [t, T_1[$, applying (29) and the intra-period martingale property (Proposition 1) directly leads to the standard cost-of-carry relation

$$F(t, \overrightarrow{x_t}, \bar{t}) = e^{r(\bar{t}-t)} S(t, \overrightarrow{x_t}). \tag{30}$$

This corresponds to the fact that permits can only be used for compliance at T_1, so that spot and futures are perfect substitutes.[22] Contrary, this does not apply to inter-period

[22] The standard cost-of-carry relation within a trading period has been confirmed empirically by several studies, see e.g. Uhrig-Homburg and Wagner (2009).

futures, e.g. the case $t \in [0, T_1[$, $\bar{t} \in]T_1, T_2[$. Holding current permits has an additional benefit compared to holding futures maturing in the next trading period, namely the possible use for compliance at the end of the current trading period. Accordingly, (29) and Proposition 1 together with (23) give us

$$F(t, \overrightarrow{x_t}, \bar{t}) = e^{r(\bar{t}-t)} S(t, \overrightarrow{x_t}) - e^{r(\bar{t}-T_1)} E_t \left[1_{\{R_1(x_{T_1})<0\}} \right] p_1. \tag{31}$$

Therefore the properties of the futures price curve become obvious:

Proposition 4 (Futures Price Curves)**.**

a) Within a trading period, the futures price curve is in contango.

b) Futures with maturity in the following trading period are in weak backwardation if $E_t \left[1_{\{R_1(x_{T_1})<0\}} \right] > 0$. *They are in strong backwardation if* $E_t \left[1_{\{R_1(x_{T_1})<0\}} \right] > (e^{r(T_1-t)} - e^{-r(\bar{t}-T_1)}) \frac{S(t, \overrightarrow{x_t})}{p_1}$.

Particularly, this means that inter-period futures are in weak backwardation if the probability of permit shortage at the end of the trading period is not exactly zero. As an example, Figure 1 shows futures price curves at $t = 3$ plotted for the different scenarios from Section 4.2.3.

[Insert Figure 1 about here.]

The price curves have been constructed by Monte-Carlo simulation[23] of the expected spot price for different maturities according to (29). As expected, the backwardation of inter-period futures is the strongest for very high current emissions, i.e. large $E_t \left[1_{\{R_1(x_{T_1})<0\}} \right]$. Contrary, the backwardation is only marginal when current emissions are very low.

5.1.2 Convenience Yields

Futures contracts exhibiting backwardation are well-known in the commodity literature. It is common to express the benefit of holding the spot commodity rather than a futures contract as a convenience yield according to Brennan (1958). In general, a time-dependent

[23]For all Monte-Carlo simulations documented in this paper, we discretize processes to 260 trading days per year and simulate 10,000 price paths per setting.

stochastic instantaneous convenience yield δ_t changes the standard cost-of-carry relation to

$$F(t, \overrightarrow{x_t}, \bar{t}) = E_t \left[e^{r(\bar{t}-t) - \int_t^{\bar{t}} \delta_s ds} \right] S(t, \overrightarrow{x_t}). \tag{32}$$

For given $t \in [0, T_1[$ we define $D(\bar{t}) := \int_t^{\bar{t}} \delta_s ds$ as the convenience yield from t to $\bar{t}$. From (30) and (31) we can easily derive necessary properties of D.

Proposition 5 (Convenience Yields)**.**

a) $D(\bar{t}) = 0$ *for intra-period futures, i.e.* $\bar{t} \in [t, T_1[$.

b) For inter-period futures, i.e. $\bar{t} \in]T_1, T_2[$, *D fulfills the condition* $E_t \left[e^{-D(\bar{t})} \right] = 1 - \frac{e^{-r(T_1-t)} E_t \left[1_{\{R_1(x_{T_1})<0\}} \right] p_1}{S(t, \overrightarrow{x_t})}$.

It is obvious that D has to "jump" in T_1 in order to fulfill the conditions of Proposition 5. Therefore it is not possible to define an instantaneous convenience yield δ_t fitting these properties. In particular, standard models like a mean-reverting stochastic convenience yield or a simple AR(4) process as used by Daskalakis, Psychoyios, and Markellos (2009) and Chevallier (2009) for inter-period futures do not satisfy them. These models inevitably lead to relative mispricing when futures of two or more different maturities are considered.

5.2 European Options

Besides futures contracts, the most important exchange-traded derivatives related to the EU ETS are European options.[24] As non-linear derivatives, option prices depend on the whole probability distribution of spot prices implied by underlying spot price dynamics. Empirical studies like Paolella and Taschini (2008), Benz and Trück (2009), and Daskalakis, Psychoyios, and Markellos (2009) analyze standard models for the spot price dynamics such as geometric Brownian motion, jump-diffusion, regime-switching, and different types of GARCH models. Recently, Grüll and Taschini (2009) and Carmona and Hinz (2009) have derived reduced forms of existing theoretical equilibrium models to capture the specific properties of emission permits.

[24] Usually, these options are written on carbon permit futures. However, the maturity dates of options and its underlying futures typically coincide apart from few days. Thus one can assume that option payoffs approximately depend on $F(\bar{t}, \overrightarrow{x_t}, \bar{t})$, which is equal to $S(\bar{t}, \overrightarrow{x_t})$.

Our aim is to characterize the volatility smile shapes of European carbon options within our model. We have seen that that log-returns are heteroskedastic according to Proposition 3. Since carbon options are written on a strip of binary European call options by Proposition 2, the pricing problem is structurally similar to compound options first studied by Geske (1979). One can expect in direct accordance that if all of these binary options are deep in-the-money (deep out-of-the-money), i.e. for very high (low) current and future emissions, the distribution of future permit prices is strongly left-skewed (right-skewed), inducing a downward-sloping (upward-sloping) volatility smile. However, there is no general intuition if some of these binary options are in- and some are out-of-the-money.

In our risk-neutral setting the price of a European call option with strike K and maturity $\bar{t}$ written on futures with the same maturity is given by

$$C(t, \overrightarrow{x_t}, \bar{t}, K) = e^{-r(\bar{t}-t)} E_t \left[\left(F(\bar{t}, \overrightarrow{x_{\bar{t}}}, \bar{t}) - K \right)^+ \right]. \tag{33}$$

We calculate call option prices for several emission level scenarios of a four trading period setting by simulating the probability distributions of permit prices. The related Black (1976) implied volatility at strike K is denoted by $IV(K)$ and we write IV_{ATM} for $IV(F(t, \overrightarrow{x_t}, \bar{t}))$. To capture the shape of volatility smiles, we follow Ederington and Guan (2011): We calculate implied volatilities for 9 strikes above and 9 strikes below $F(t, \overrightarrow{x_t}, \bar{t})$, given by $K_j := F(t, \overrightarrow{x_t}, \bar{t})(1 + \frac{1}{10} j IV_{ATM} \sqrt{\bar{t} - t})$ for $j \in \{-9, \ldots, 9\}$. Then we standardize implied volatilities according to $SIV_j = \frac{IV(K_j)}{IV_{ATM}}$. By regressing SIV_j on $\frac{j}{100}$ for $j \in \{-9, \ldots, 0\}$, we obtain the slope of the smile for strikes below $F(t, \overrightarrow{x_t}, \bar{t})$, denoted by LS. Doing the same for $j \in \{0, \ldots, 9\}$, we get HS, the slope of the smile for strikes above $F(t, \overrightarrow{x_t}, \bar{t})$.

[Insert Table 6 about here.]

The results for different scenarios and time parameters are given in Table 6. It is eye-catching that the volatility smile is downward-sloping in most cases. This corresponds to our intuition for scenarios of very high current and future emission levels. Still, the low emission scenarios do not translate to all binary options being deep out-of-the money due to the tightening of target emission levels in future trading periods. For medium emission scenarios, the single binary options can add up to various shapes of the price probability distributions. In fact, this makes all kinds of volatility smiles possible.

Proposition 6 (Volatility Smiles)**.** *Volatility smiles can be smile-shaped, upward-sloping, downward-sloping, and even hump-shaped, dependent on emission levels, maturity date, and time to maturity.*

To illustrate this, Table 7 gives an example for each kind of possible volatility smile shape and shows the volatility smile as well as the related log price probability distribution for a corresponding scenario.

[Insert Table 7 about here.]

Altogether, our model predicts that the volatility smile shape of European carbon options heavily depends on the current and future emission levels. This is an exceptional feature in comparison to other asset classes where usually one characteristic shape of the volatility smile exists.[25] In our model, the volatility smile shapes for emission permits are induced by the characteristics of permit price distributions under the real measure. It is left for future research to analyze the influence of risk premia on carbon volatility smiles.

6 Conclusion

In this paper, we develop a stochastic equilibrium model accounting for the main stylized facts of today's emission trading systems: multiple trading periods, allowance of banking, not-allowance of borrowing, penalties, and later delivery of lacking permits. We show that an emission permit is essentially equivalent to a strip of binary options written on net cumulative emissions and exploit this option analogy to derive several general properties of the price dynamics. Even so, attention should be paid to the price impact of the equilibrium abatement strategies, a crucial difference to classical financial options. Within a concrete model setting in accordance with the world's largest carbon market, the EU ETS, we analyze the resulting consequences for the allowance price and volatility properties and decompose current permit prices into value components coming from different trading periods. Furthermore, we apply our modeling framework to show implications for carbon-related derivatives, such as futures or European options.

Our results enhance the understanding of the price dynamics in emission markets and the characteristic properties of carbon-related derivatives. First, in a multiple trading period market each additional trading period leads to an additional component in the current

[25] In empirical literature, Rubinstein (1994) documents that equity option smiles are usually downward-sloping. Contrary, the volatility smiles of standard commodities tend to be upward-sloping, see e.g. Geman (2005). Ederington and Guan (2011) analyze the volatility smile shapes for numerous different assets.

price of emission allowances. As a consequence, recessions do not necessarily induce very low prices today if allowances are expected to be scarce in coming trading periods. Our results show that the inherent option value attributable to these future periods can be considerable. Second, within each of these trading periods, discounted spot prices should not reflect any mean-reversion or seasonal behavior. This is a major difference to most commodity markets, where a mean-reversion process is considered the natural choice. Third, our model predicts, when considering futures price curves, that inter-period futures should be in backwardation, a feature typically known from commodity markets. These properties highlight the hybrid character of emission permits: While exhibiting characteristics of investment assets within the single trading periods, compliance at the end of each trading period leads to the very same option-like behavior within multiple trading period frameworks that we know from commodities. Consequently, standard instantaneous convenience yield models prove to be inappropriate for pricing inter-period futures since they do not account for this hybrid character and lead to an inconsistent pricing of futures with different maturities. Finally, option pricing turns out to be complicated by special features induced by the regulatory setting.

A Optimality Conditions for Stochastic Optimal Control Problems

We derive necessary optimality conditions for the relation between the allowance price, the probability of paying penalties, and the marginal abatement costs for the individual optimization problem (6) incorporating abatement opportunities.[26] For that, we apply the dynamic programming principle and exploit the relationship to the corresponding adjoint equations (see Yong and Zhou (1999), Ch. 5).

We start at the last trading period, i.e. $t \in [T_{n-1}, T_n]$. By proceeding along the same lines as Seifert, Uhrig-Homburg, and Wagner (2008), we arrive at the Hamilton-Jacobi-Bellman (HJB) equation

$$0 = \max_{(\theta_{it},\xi_{it})} E_0 \Big[e^{-rt}(C_i(t,\xi_{it}) - S(t)\theta_{it}) + V_{in}^{(t)} - (\xi_{it} + \theta_{it}) V_{in}^{(x_{it,T_n})} + \frac{1}{2}(G_{in}(t))^2 V_{in}^{(x_{it,T_n},x_{it,T_n})} \Big] \quad (34)$$

with boundary condition

$$V_{in}(T_n, x_{iT_n}) = e^{-r(T_n - T_{n-1})} P_n(x_{iT_n}) + e^{rT_{n-1}} R_n(x_{iT_n}) S_{end}, \quad (35)$$

where V_{in} is the expected value of an optimal strategy.[27] Deriving the right-hand side of (34) with respect to θ_{it} and ξ_{it} and setting the derivative to zero yields the optimality conditions

$$C_i^{(\xi_{it})}(t, \xi_{it}) = e^{rt} V_{in}^{(x_{it,T_n})} \quad (36)$$

and

$$S(t) = -e^{rt} V_{in}^{(x_{it,T_n})}. \quad (37)$$

In addition, according to Yong and Zhou (1999), Ch. 5, Theorem 4.1, it has to be fulfilled that

$$V_{in}^{(x_{it,T_n})} = -\rho_i(t), \quad (38)$$

[26] For the optimization problem of the basic model (4), one can apply the same steps and simply leave out all terms containing ξ_i.

[27] We assume that all V_{ik}, $k \in \{1, \ldots, n\}$, are, within the single trading periods, three times differentiable, and especially three times continuously differentiable with respect to all state variables x_{t,T_j}, $j \in \{k, \ldots, n\}$.

where $\rho_i(t)$ is the adjoint process of an optimal strategy (θ_i, ξ_i) given by the adjoint equations corresponding to the stochastic optimal control problem. In our case, these are

$$d\rho_i(t) = \pi_i(t) dW_t \tag{39}$$

$$\rho_i(T_n) = -\left(e^{-rT_n} \begin{cases} -p_n, & \text{if } R_n(x_{iT_n}) < 0; \\ 0, & \text{otherwise.} \end{cases} - S_{end} \right) \tag{40}$$

and we can directly identify the solution

$$\rho_i(t) = e^{-rT_n} E_t[1_{R_n(x_{iT_n})<0}] p_n + S_{end}. \tag{41}$$

The adjoint process can be interpreted as the shadow price of carbon emissions, i.e. the value that can be attributed to having one unit of emissions less. Here, this is S_{end} plus the discounted penalty weighted by the probability of penalties to accrue, which makes perfect economic sense.

Combining (36), (37), (38), and (41), we finally obtain

$$S(t) = -C_i^{(\xi_{it})}(t, \xi_{it}) = e^{-r(T_n - t)} E_t[1_{R_n(x_{iT_n})<0}] p_n + e^{rt} S_{end} \tag{42}$$

for $t \in [T_{n-1}, T_n]$, which is identical to equation (7) and (13).

Now we consider $t \in [T_{k-1}, T_k]$, $k \leq n-1$, and suppose that V_{ik+1} is given and fulfills

$$\sum_{j=k+1}^{n} V_{ik+1}^{(x_{it,T_j})} = -\left(\sum_{j=k+1}^{n} e^{-rT_j} E_t \left[1_{\{R_j(x_{iT_j})<0\}} \right] p_j + S_{end} \right) \tag{43}$$

for $t \in [T_k, T_{k+1}]$. We include V_{ik+1} into the terminal condition of the optimization problem related to period $t \in [T_{k-1}, T_k]$. The HJB equation can be derived similarly, but one has to account for the multi-dimensional state vector $\overrightarrow{x_{it}} = (x_{it,T_k}, x_{it,T_{k+1}}, \ldots, x_{it,T_n})$. The principle of optimality for stochastic optimal control requires that

$$V_{ik}(t, \overrightarrow{x_{it}}) = \max_{(\theta_{it}, \xi_{it})} E_0 \left[e^{-rt}(C_i(t, \xi_{it}) - S(t)\theta_{it}) dt + V_{ik}(t + dt, \overrightarrow{x_{it} + dx_{it}}) \right] \tag{44}$$

and the multi-dimensional version of Itô's Lemma gives us

$$dV_{ik} = V_{ik}^{(t)} dt + \sum_{j=k}^{n} V_{ik}^{(x_{it,T_j})} dx_{it,T_j} + \frac{1}{2} \sum_{j_1, j_2 = k}^{n} V_{ik}^{(x_{it,T_{j_1}}, x_{it,T_{j_2}})} dx_{it,T_{j_1}} dx_{it,T_{j_2}}. \tag{45}$$

Together with (12) we get

$$E_0[dV_{ik}] = V_{ik}^{(t)} dt - (\xi_{it} + \theta_{it}) \left(\sum_{j=k}^{n} V_{ik}^{(x_{it,T_j})} \right) dt + \frac{1}{2} \sum_{j_1, j_2 = k}^{n} G_{ij_1}(t) G_{ij_2}(t) V_{ik}^{(x_{it,T_{j_1}}, x_{it,T_{j_2}})} dt. \tag{46}$$

Using this in (44) it finally leads to the HJB equation

$$0 = \max_{(\theta_{it},\xi_{it})} E_0 \Big[e^{-rt}(C_i(t,\xi_{it}) - S(t)\theta_{it}) + V_{ik}^{(t)} \\ -(\xi_{it}+\theta_{it}) \left(\sum_{j=k}^{n} V_{ik}^{(x_{it,T_j})} \right) + \frac{1}{2} \sum_{j_1,j_2=k}^{n} G_{ij_1}(t) G_{ij_2}(t) V_{ik}^{(x_{it,T_{j_1}}, x_{it,T_{j_2}})} \Big] \quad (47)$$

with boundary condition

$$V_{ik}(T_k, \overrightarrow{x_{iT_k}}) = e^{-r(T_k - T_{k-1})} (P_k(x_{iT_k}) + V_{ik+1}(T_k, \overrightarrow{x_{iT_k}})). \quad (48)$$

We are now able to derive the optimality conditions as before and obtain altogether

$$S(t) = -C_i^{(\xi_{it})}(t,\xi_{it}) = \sum_{j=k}^{n} e^{-r(T_j - t)} E_t[1_{R_j(x_{iT_j})<0}] p_j + e^{rt} S_{end} \quad (49)$$

for $t \in [T_{k-1}, T_k]$.

Since (43) is fulfilled for period $[T_{n-1}, T_n]$, (49) follows for all trading periods $[0, T_1], [T_1, T_2], \dots, [T_{n-1}, T_n]$ by backward induction.

B Relation to Global Optimization Problem

We consider a global joint cost problem and show that it induces an equilibrium spot price process. Afterwards we observe that the global joint cost problem can be simplified to the global optimization problem (14). Our approach partly builds on Carmona, Fehr, and Hinz (2009) as well as on the appendix of Seifert, Uhrig-Homburg, and Wagner (2008).[28] The global joint cost problem we consider is given by

$$\max_{(\Theta_s,\Xi_s)_{s\in[0,T_n]}} E_0 \Big[\int_0^{T_n} \sum_{i\in I} e^{-rs} C_i(s,\xi_{is}) ds + \sum_{j=1}^{n} \sum_{i\in I} e^{-rT_j} P_j(x_{iT_j}) \\ + \sum_{i\in I} R_n(x_{iT_n}) S_{end} \Big], \quad (50)$$

[28] In the appendix of Seifert, Uhrig-Homburg, and Wagner (2008) it is shown that the joint cost problem of a social planner acting on aggregated volumes is equal to the sum of all companies' cost problems, under the crucial assumption that all emission processes are driven by the same Wiener process. It follows that the global optimal solution is equal to the sum of all single companies' optimal solutions. By using a more general approach, based on Carmona, Fehr, and Hinz (2009), it is even possible to find a solution of the global optimization problem that is also optimal for the individual problems without this assumption. We do not need linearity of the single cost problems in the global cost problem.

with global trading and abatement strategy (Θ, Ξ) and dynamics of x_{it,T_k} given in vector notation by

$$\begin{aligned} dX_{t,T_k} &= -(\Theta_t + \Xi_t)dt + HdW_t \\ &= -\left(\begin{pmatrix} \theta_{1t} \\ \vdots \\ \theta_{|I|t} \end{pmatrix} + \begin{pmatrix} \xi_{1t} \\ \vdots \\ \xi_{|I|t} \end{pmatrix}\right) dt + \begin{pmatrix} G_{1k}(t) & 0 & 0 \\ \vdots & \ddots & \vdots \\ 0 & 0 & G_{|I|k}(t) \end{pmatrix} dW_t. \end{aligned} \tag{51}$$

For the next steps, we need to observe that a solution (Θ, Ξ) of the global joint cost problem fulfills two characteristic properties: First, market clearing requires costs for buying and revenues from selling allowances to cancel out, and the sum of remaining allowances at the end of the setting is not influenced by trading action as well. Therefore, the control variable Θ_t is only relevant for the penalties in T_j, $j = 1, \ldots, n$, and for an optimal trading strategy Θ penalties only apply in T_j if $R_j(x_{T_j}) < 0$.[29] Second, by using this first property and applying Appendix A analogously, we obtain that an optimal abatement strategy Ξ has to fulfill

$$-C_i^{(\xi_{it})}(t, \xi_{it}) = e^{-r(T_n - t)} E_t \left[1_{\{R_n(x_{T_n})<0\}} \right] p_n + e^{rt} S_{end}, \tag{52}$$

for all companies i.

Now we consider the individual optimization problems (6). The expected value can be split in two parts according to

$$\begin{aligned} &E_0 \left[\int_0^{T_n} e^{-rs} C_i(s, \xi_{is}) ds - \int_0^{T_n} e^{-rs} S(s) \theta_{is} ds + \sum_{j=1}^{n} e^{-rT_j} P_j(x_{iT_j}) + R_n(x_{T_n}) S_{end} \right] \\ &= E_0 \left[\int_0^{T_n} e^{-rs} (C_i(s, \xi_{is}) + S(s) \xi_{is}) ds \right] \\ &+ E_0 \left[- \int_0^{T_n} e^{-rs} S(s) \left(\theta_{is} + \xi_{is} \right) ds + \sum_{j=1}^{n} e^{-rT_j} P_j(x_{iT_j}) + R_n(x_{iT_n}) S_{end} \right]. \end{aligned} \tag{53}$$

We will show that the second expectation value of the right-hand side is independent of the decision variables. For that, recall from Section 3.3 that any strategy (Θ, Ξ) which solves the individual optimization problems fulfills

$$1_{\{R_j(x_{T_j})<0\}} = 1_{\{R_j(x_{iT_j})<0\}} \tag{54}$$

[29]That means, if the number of allowances in the whole market is not sufficient to cover overall emissions at T_j, companies distribute the permits in such way that none of the companies individually has left-over allowances. If the market is long, companies distribute the permits so that none of the companies has to pay penalties.

for all $i \in I$, $j = 1, \ldots, n$ and the spot price process fulfills (10). Thus, we can insert (10) and (54) into (53) and rewrite the second expectation value as

$$E_0 \left[-\int_0^{T_n} e^{-rs} \left(\sum_{\substack{j=1,\\ T_j \geq s}}^{n} e^{-r(T_j - s)} E_s \left[1_{\{R_j(x_{iT_j})<0\}} \right] p_j + e^{rs} S_{end} \right) (\theta_{is} + \xi_{is}) \, ds \right.$$
$$\left. + \sum_{j=1}^{n} e^{-rT_j} 1_{\{R_j(x_{iT_j})<0\}} R_j(x_{iT_j}) p_j + R_n(x_{iT_n}) S_{end} \right]. \quad (55)$$

Because of $R_j(x_{iT_j}) = e_{iT_j} + \int_0^{T_j} (\xi_{is} + \theta_{is}) ds - \int_0^{T_j} y_{is} ds$, reordering the sums shows that the control variables θ_i and ξ_i completely cancel out, so that this term is irrelevant for the optimization problem.

We now choose a spot price process by

$$S(t) = -C_i^{(\xi_{it})}(t, \xi_{it}), \quad (56)$$

setting the spot price equal to the marginal abatement costs of one (and every) company with regard to the optimal abatement strategy Ξ. We have seen that condition (54) holds for an optimal strategy Θ and by (52), the spot price process is well-defined and fulfills condition (10). After the previous discussion, for (Θ, Ξ) to fulfill the maximum property (6) of the individual optimization problems it is left to show that

$$\xi_i \to E_0 \left[\int_0^{T_n} e^{-rs} (C_i(s, \xi_{is}) + \xi_{is} S(s)) ds \right] \quad (57)$$

is maximized by ξ_i. Since the functional

$$\xi_{is} \to C_i(s, \xi_{is}) + \xi_{is} S(s) \quad (58)$$

is maximized by ξ_i for each $i \in I$ because the derivative with respect to ξ_{is} is zero at ξ_{is} then and the functional has no minimum, this maximizes the whole integral in (57) as well.

Altogether, the price process $S(t) = -C_i^{(\xi_{it})}(t, \xi_{it})$ solves the individual maximization problems (6) with chosen strategies Ξ and Θ.

Finally, it is easy to see that the problem (50) corresponds to the simplified aggregated volume problem (14) by choice of the aggregated cost functions. The aggregate abatement cost function C can be defined by

$$C(s, \xi_s) = \sum_{j \in I} C_j(s, C_j^{(\xi_{jt})^{-1}}(s, C_i^{(\xi_{it})}(s, \xi_{is}))), \quad (59)$$

where C_j are the cost functions of the single companies, C_i is the cost function of one arbitrarily chosen single company, $C_j^{(\xi_{jt})^{-1}}$ is the inverse function of $C_j^{(\xi_{jt})}$ with respect to ξ_{js}, and ξ_{is} is given by $\xi_s = \sum_{j \in I} C_j^{(\xi_{jt})^{-1}}(s, C_i^{(\xi_{it})}(s, \xi_{is}))$. C is well-defined then, because $C_i^{(\xi_{it})}(s, \xi_{is})$ is equal for all companies according to (52) and C_i, C_j are convex, so that $C_i^{(\xi_{is})}$, $C_j^{(\xi_{js})}$ as well as $C_j^{(\xi_{js})^{-1}}$ are strictly increasing and thus invertible.
C is convex and differentiable as well and fulfills the relation

$$C(s, \xi_s) = \sum_{j \in I} C_j(s, \xi_{js}). \tag{60}$$

Defining $R_k(x_{t,T_k})$ as

$$R_k(x_{t,T_k}) = \sum_{j \in I} e_{jT_k} - x_{jt,T_k} \tag{61}$$

and the aggregated penalty cost function P as

$$P_k(x_{T_k}) = \min[0, p_k R_k(x_{T_k})], \tag{62}$$

leads to

$$P_k(x_{T_k}) = \sum_{j \in I} P_{ik}(x_{iT_k}) \text{ and } R_k(x_{T_k}) = \sum_{j \in I} R_{ik}(x_{iT_k}). \tag{63}$$

Finally, $x_{t,T_k} = \sum_{i \in I} x_{it,T_k}$ by definition, so that altogether each solution of (14) is directly related to a solution of (50).

C Derivation of the Characteristic PDE

The HJB equations for the global optimization problem (14) related to the different trading periods can be derived along the lines of Appendix A.

For the last trading period $[T_{n-1}, T_n]$, this yields

$$0 = \max_{\xi_t} E_0 \left[e^{-rt} C(t, \xi_t) + V_n^{(t)} - \xi_t V_n^{(x_{t,T_n})} + \frac{1}{2}(G_n(t))^2 V_n^{(x_{t,T_n}, x_{t,T_n})} \right] \tag{64}$$

with boundary condition

$$V_n(T_n, x_{T_n}) = e^{-r(T_n - T_{n-1})} P_n(x_{T_n}) + e^{rT_{n-1}} R_n(x_{T_n}) S_{end}. \tag{65}$$

By deriving the right-hand side of (64) by ξ_t, we obtain the optimal solution

$$\xi_t = C^{(\xi_t)^{-1}} \left(e^{rt} V_n^{(x_{t,T_j})} \right). \tag{66}$$

Inserting this in (64) again, we arrive at the characteristic PDE

$$V_n^{(t)} = C^{(\xi_t)^{-1}}\left(e^{rt}V_n^{(x_{t,T_n})}\right)V_n^{(x_{t,T_n})} - e^{-rt}C\left(t, C^{(\xi_t)^{-1}}\left(e^{rt}V_n^{(x_{t,T_n})}\right)\right) - \frac{1}{2}(G_n(t))^2 V_n^{(x_{t,T_n},x_{t,T_n})}. \quad (67)$$

For period $[T_{k-1}, T_k]$, $k \leq n-1$, with given V_{k+1}, the HJB equation is

$$0 = \max_{\xi_t} E_0\left[e^{-rt}(\,C(t,\xi_t)) + V_k^{(t)} - \xi_t\left(\sum_{j=k}^{n} V_k^{(x_{t,T_j})}\right) + \frac{1}{2}\sum_{j_1,j_2=k}^{n} G_{j_1}(t)G_{j_2}(t)V_k^{(x_{t,T_{j_1}},x_{t,T_{j_2}})}\right] \quad (68)$$

with boundary condition

$$V_{ik}(T_k, \overrightarrow{x_{iT_k}}) = e^{-r(T_k-T_{k-1})}(P_k(x_{iT_k}) + V_{ik+1}(T_k, \overrightarrow{x_{iT_k}})). \quad (69)$$

Here, the optimal solution can be obtained as

$$\xi_t = C^{(\xi_t)^{-1}}\left(e^{rt}\sum_{j=k}^{n} V_k^{(x_{t,T_j})}\right), \quad (70)$$

and by insertion we get the characteristic PDE

$$V_k^{(t)} = C^{(\xi_t)^{-1}}\left(e^{rt}\sum_{j=k}^{n} V_k^{(x_{t,T_j})}\right)\left(\sum_{j=k}^{n} V_k^{(x_{t,T_j})}\right) - e^{-rt}\left(C\left(t, C^{(\xi_t)^{-1}}\left(e^{rt}\sum_{j=k}^{n} V_k^{(x_{t,T_j})}\right)\right)\right) - \frac{1}{2}\sum_{j_1,j_2=k}^{n} G_{j_1}(t)G_{j_2}(t)V_k^{(x_{t,T_{j_1}},x_{t,T_{j_2}})}. \quad (71)$$

D Intra-period Martingale Property

We show the martingale property within a trading period $[T_{k-1}, T_k]$ for discounted spot prices given by

$$e^{-rt}S(t) = -\sum_{j=k}^{n} V_k^{(x_{t,T_j})}(t, \overrightarrow{x_t}) \quad (72)$$

according to (15). Following the idea of Seifert, Uhrig-Homburg, and Wagner (2008), it is enough to show $E[dV_k^{(x_{t,T_l})}] = 0$ for $l = k, \ldots, n$.

For $k = n$, the last trading period, the expected change equals

$$E_0[dV_n^{(x_{t,T_n})}] = \left(V_n^{(x_{t,T_n},t)} - \xi_t V^{(x_{t,T_n},x_{t,T_n})} + \frac{1}{2}(G(t))^2 V^{(x_{t,T_n},x_{t,T_n},x_{t,T_n})}\right)dt. \quad (73)$$

Inserting (70) we arrive at

$$E_0[dV_n^{(x_{t,T_n})}] = \left(V_n^{(x_{t,T_n},t)} - C^{(\xi_t)^{-1}} \left(e^{rt}\sum_{j=k}^{n} V_k^{(x_{t,T_j})}\right) V^{(x_{t,T_n},x_{t,T_n})} + \frac{1}{2}(G(t))^2 V^{(x_{t,T_n},x_{t,T_n},x_{t,T_n})}\right) dt. \quad (74)$$

Since V_n is assumed to be three times differentiable we directly obtain that this is equal to zero by deriving (67) with respect to x_{t,T_n}.

For $k \leq n-1$, the expected change of V_k with respect to x_{t,T_l}, $l \in \{k,\dots,n\}$ is equal to

$$E_0[dV_k^{(x_{t,T_l})}] = V_k^{(x_{t,T_l},t)} dt - \xi_t \left(\sum_{j=k}^{n} V_k^{(x_{x_{t,T_l}},t,T_j)}\right) dt + \frac{1}{2}\sum_{j_1,j_2=k}^{n} G_{j_1}(t)G_{j_2}(t)V_k^{(x_{t,T_l},x_{t,T_{j_1}},x_{t,T_{j_2}})} dt \quad (75)$$

according to (46). Inserting (70) again we get

$$E_0[dV_k^{(x_{t,T_l})}] = V_k^{(x_{t,T_l},t)} dt - C^{(\xi_t)^{-1}} \left(e^{rt}\sum_{j=k}^{n} V_k^{(x_{t,T_j})}\right)\left(\sum_{j=k}^{n} V_k^{(x_{x_{t,T_l}},t,T_j)}\right) dt + \frac{1}{2}\sum_{j_1,j_2=k}^{n} G_{j_1}(t)G_{j_2}(t)V_k^{(x_{t,T_l},x_{t,T_{j_1}},x_{t,T_{j_2}})} dt. \quad (76)$$

This is, by the assumption that V_k is three times differentiable, exactly zero, which can be shown by deriving (71) with respect to x_{t,T_l}. Therefore, $E[dV_k^{(x_{t,T_l})}]$ is zero for all $l = k,\dots,n$ and we have shown the intra-period martingale property.

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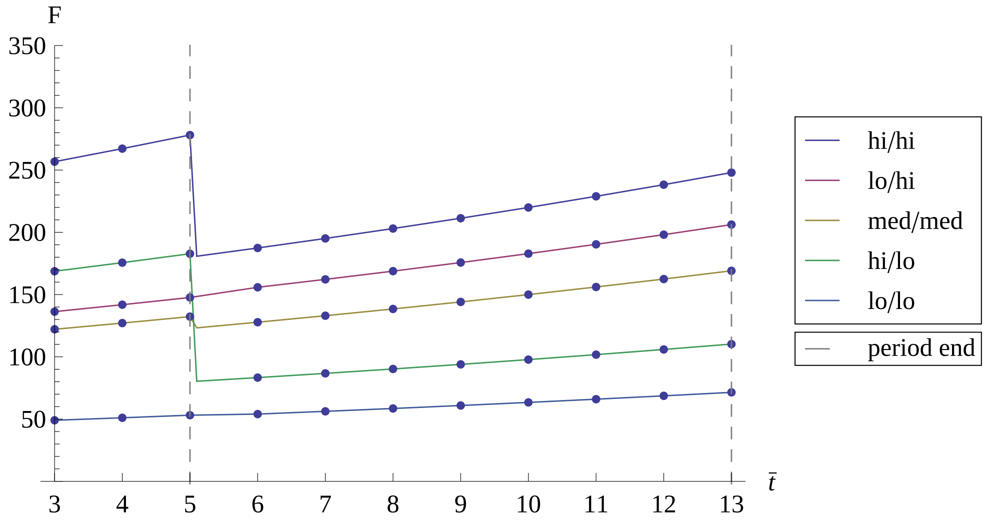


Figure 1: Futures price curves (futures prices F dependent on maturity $\bar{t}$) at $t = 3$ for five different scenarios of current and future emissions. The dashed vertical lines mark the end of the first and of the second trading period.

Table 1: Sensitivities of the spot price

Changing variable	Direction of spot price change
Penalty: p_k	$\uparrow$
Initial endowment: e_{k-1}	$\downarrow$
Expected total emissions: x_{t,T_k}	$\uparrow$
Marginal abatement costs: c	$\uparrow$
Interest rate: r	$\downarrow$

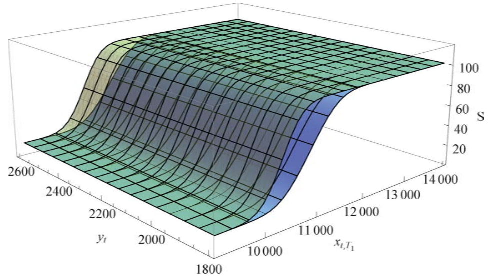
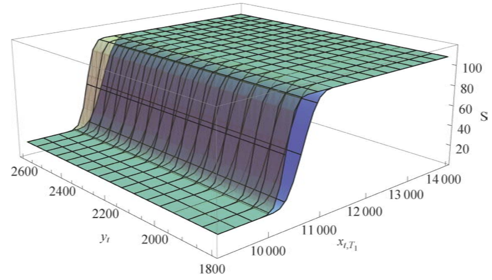
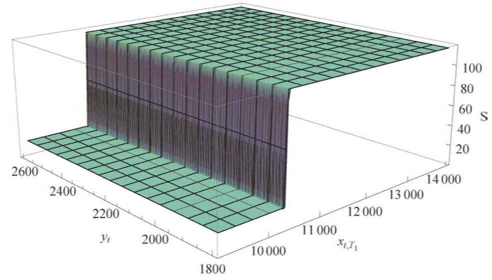
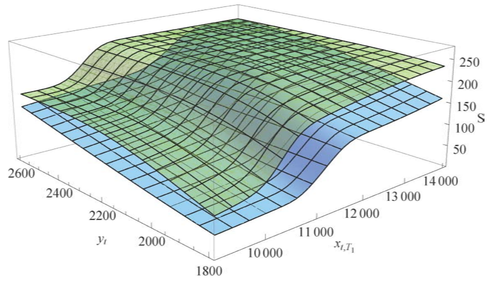
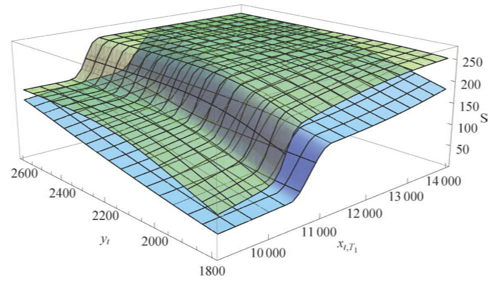
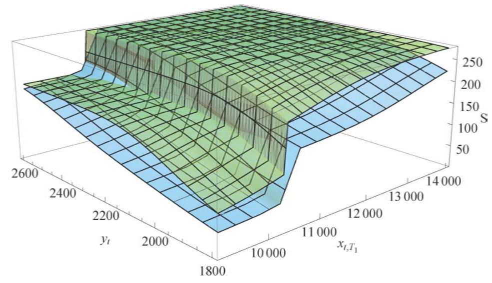
Table 2: Presumable allocated allowances for the EU ETS trading periods

Trading Period	Number of Allowances (Mio.)
Phase II (2008-2012)	10,400
Phase III (2013-2020)	14,775
Phase IV (2021-2028)	12,455
Phase V (2029-2036)	10,135

Table 3: Model parameters

Parameter	Value
p_k	100
S_{end}	14.11
σ	80
r	0.04
c	0.4
y_0	$2,200$

Table 4: Spot price S (upper panel) and relative local volatility σ_S (lower panel) of the first trading period dependent on expected net cumulative emissions x_{t,T_1} and current emission rate y_t in settings of one and four trading periods. The green plots are without abatement, the blue plots including abatement.

trading periods	t=1.5	t=3	t=4.9
one			
four			

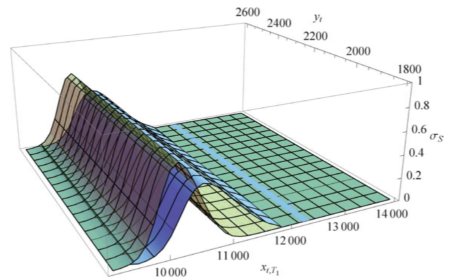
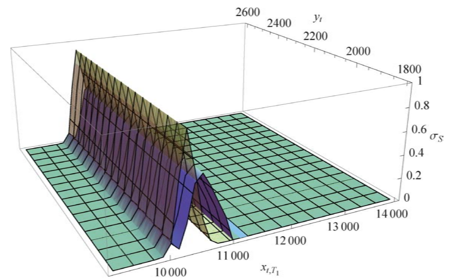
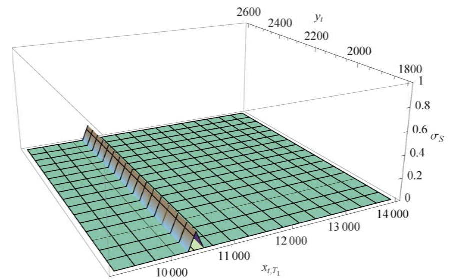
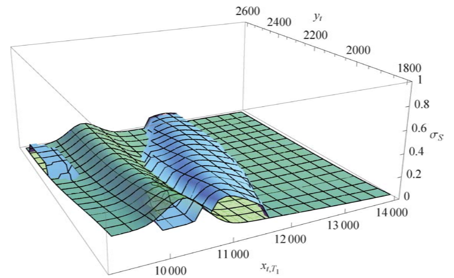
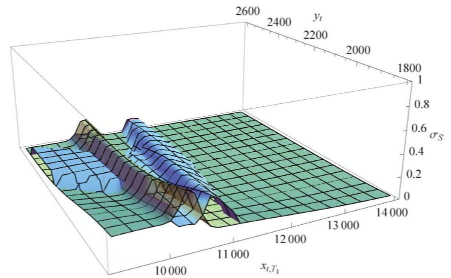
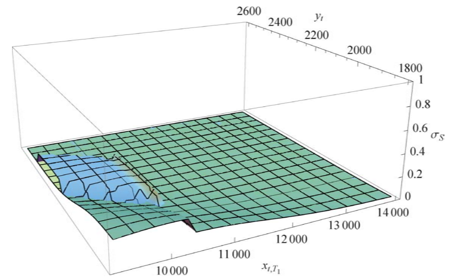
trading periods	t=1.5	t=3	t=4.9
one			
four			

Table 5: Ratios of the spot price coming from different trading periods and S_{end}. The upper table is without abatement, the lower table including abatement.

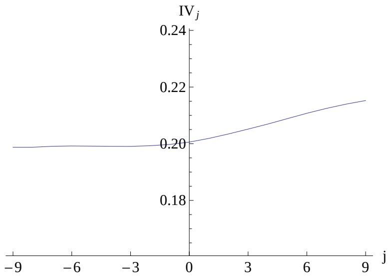
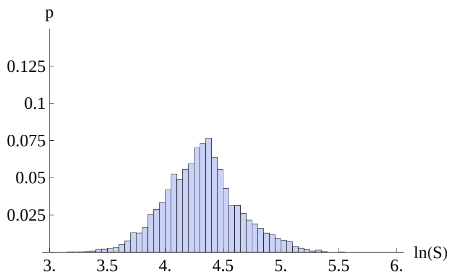
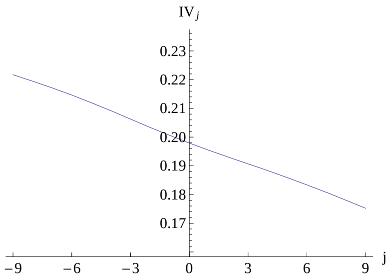
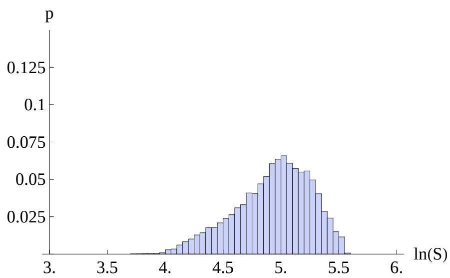
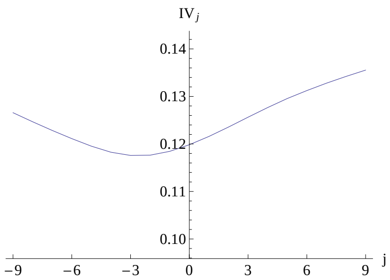
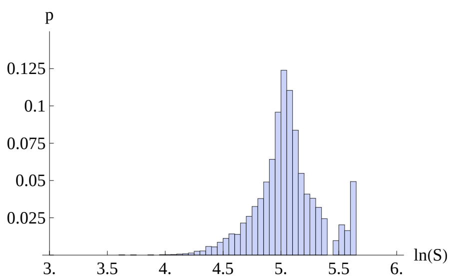
x_{t,T_1}	y_t	S	q_1	q_2	q_3	q_4	$q_{S_{end}}$
low	low	97.18	0.00%	18.70%	33.88%	31.04%	16.37%
medium	medium	258.07	35.77%	25.72%	18.71%	13.64%	6.16%
high	high	259.27	35.60%	25.85%	18.77%	13.63%	6.13%
low	high	166.85	0.00%	40.12%	29.16%	21.18%	9.53%
high	low	250.00	36.92%	25.49%	17.88%	13.35%	6.36%

x_{t,T_1}	y_t	S	q_1	q_2	q_3	q_4	$q_{S_{end}}$
low	low	49.04	0.00%	10.96%	30.15%	26.45%	32.44%
medium	medium	122.12	68.47%	13.51%	2.87%	2.13%	13.02%
high	high	256.72	35.96%	26.10%	18.79%	12.95%	6.20%
low	high	136.29	0.00%	46.91%	27.20%	14.21%	11.67%
high	low	168.73	54.71%	23.55%	8.04%	4.27%	9.43%

Table 6: Slope of volatility smiles represented by LS and HS for different time parameters t, $\bar{t}$ and emission levels x_{t,T_1} and y_t in a setting of four trading periods.

		x_{t,T_1}											
		9,000		10,000		11,000		12,000		13,000		14,000	
	y_t	LS	HS	LS	HS	LS	HS	LS	HS	LS	HS	LS	HS
$t = 0.5, \bar{t} = 1$	1,800	−0.11	−0.19	0.06	−0.11	−0.01	−0.21	−1.27	−1.27	0.37	0.23	0.22	0.03
	2,000			−0.16	−0.11	0.25	0.25	−0.86	−0.92	−0.11	−0.10	−0.29	−0.50
	2,200			−0.22	−0.21	−0.33	−0.31	−0.46	−0.59	−1.21	−1.14	−0.64	−0.99
	2,400					−0.72	−0.74	0.11	−0.03	−1.28	−1.64	−1.25	−1.71
	2,600							0.44	0.63	−1.19	−1.51	−2.66	−3.25
$t = 0.5, \bar{t} = 3$	1,800	0.16	−0.30	−0.05	−0.17	1.63	−0.67	−3.27	−2.84	0.97	0.44	0.59	−0.01
	2,000			−0.55	−0.37	0.07	0.85	−2.12	−1.86	−0.00	−0.26	−0.50	−1.11
	2,200			−0.55	−0.46	−0.68	−0.90	−1.36	−1.26	−2.25	−1.83	−1.60	−2.47
	2,400					−1.51	−2.08	0.53	0.16	−2.72	−4.91	−2.83	−5.09
	2,600							0.40	2.22	−2.28	−4.80	−4.66	−11.41
$t = 0.5, \bar{t} = 5$	1,800	0.34	−0.48	0.10	−0.45	2.38	−0.15	−5.43	−3.32	1.60	0.62	1.29	−0.09
	2,000			−0.88	−0.56	−0.52	0.22	−2.61	−3.49	0.43	−0.37	−0.45	−1.63
	2,200			−0.65	−0.47	−0.92	−1.39	−0.76	−0.53	−2.05	−2.28	−1.86	−3.86
	2,400					−1.63	−3.17	−0.72	1.49	−4.46	−11.36	−3.80	−11.21
	2,600							−1.53	4.48	−2.01	−11.42	−5.78	−13.37
$t = 2.5, \bar{t} = 3$	1,800	−0.03	−0.18	−0.07	−0.05	−0.98	−0.95	0.47	0.39	0.26	0.15	−0.10	−0.21
	2,000	−0.24	−0.30	−0.22	−0.11	−0.54	−0.66	0.08	−0.06	−0.25	−0.35	−0.78	−0.90
	2,200	−0.32	−0.05	−0.09	−0.15	0.53	0.47	−0.52	−0.45	−0.89	−1.13	−1.53	−1.88
	2,400	−0.11	−0.10	−0.63	−0.70	−0.97	−1.19	−1.78	−2.45	−1.77	−2.32	−2.51	−3.26
	2,600	−0.70	−0.93	−1.30	−1.55	−1.90	−2.35	−1.58	−2.45	−3.05	−4.12	−3.55	−5.04
$t = 2.5, \bar{t} = 5$	1,800	0.14	−0.25	−0.17	−0.31	−0.83	−1.87	1.19	0.87	1.10	0.38	0.06	−0.69
	2,000	−0.75	−0.70	−0.74	−0.36	−0.76	−0.75	0.42	−0.08	−0.25	−1.06	−1.47	−2.35
	2,200	−0.74	−0.26	−0.35	−0.37	−0.95	−0.13	−0.58	−1.50	−1.83	−3.01	−2.96	−5.22
	2,400	0.04	−0.34	−0.84	−1.63	−2.12	−3.32	−4.37	−9.24	−3.69	−9.15	−4.99	−11.65
	2,600	−1.35	−2.26	−2.59	−4.64	−4.01	−6.58	−4.51	−12.81	−6.00	−13.34	−6.71	−13.18
$t = 4.5, \bar{t} = 5$	1,800	−0.01	−0.18	−0.19	−0.16	0.36	0.23	0.38	0.31	−0.06	−0.20	−0.58	−0.68
	2,000	−0.42	−0.31	−0.31	−0.23	0.25	0.23	−0.10	−0.36	−0.84	−0.96	−1.40	−1.57
	2,200	−0.14	−0.07	−0.11	−0.26	−0.42	−0.66	−1.24	−1.46	−1.86	−2.42	−2.02	−2.55
	2,400	−0.21	−0.33	−1.03	−1.18	−1.60	−2.18	−2.67	−3.83	−3.05	−4.00	−3.08	−4.18
	2,600	−1.42	−1.70	−2.53	−3.34	−3.52	−5.03	−4.30	−8.89	−4.46	−12.25	−7.37	−10.96

Table 7: Implied volatility smiles (volatility IV_j dependent on strike K_j) of European futures options with maturity $\bar{t}$ (left-hand side) and probabilities p of log permit prices $\ln(S)$ at $\bar{t}$ (right-hand side) for different $\bar{t}$ and emission levels x_{t,T_1} and y_t in a setting of four trading periods, dependent on information at at $t = 0.5$.

	volatility smile	log permit price distribution
$t = 0.5$ $\bar{t} = 3$ $x_{t,T_1} = 11{,}000$ $y_t = 2{,}000$		
$t = 0.5$ $\bar{t} = 3$ $x_{t,T_1} = 12{,}000$ $y_t = 2{,}200$		
$t = 0.5$ $\bar{t} = 5$ $x_{t,T_1} = 12{,}000$ $y_t = 2{,}400$		
$t = 0.5$ $\bar{t} = 5$ $x_{t,T_1} = 13{,}000$ $y_t = 2{,}000$