Higher-moment Asset Pricing and Allocation in a Heterogeneous Market Equilibrium

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Abstract

Investors have different preferences for portfolio skewness and kurtosis, i.e. return asymmetry and tails' fatness. We build up a new model to describe this documented empirical fact and study how this type of preferences can impact equilibrium asset return and their optimal allocation. In our economy there are three types of investors whose preferences can be characterized by "MV", "MVS" and "$MVSK$". (M: Mean V: Variance S: Skewness K: Kurtosis) and they are named as ("Traditional", "Lotto" and "Kurtosis Aversion" investor), correspondingly. We also investigate how the change of investor fraction on the market influences the equilibrium properties. Furthermore, by using the weekly world stock market indices (MSCI) ranging from January 1988 till January 2010, we are able to recover the investor properties, such as investor preferences and investor fraction in both bullish and bearish markets. We find that during crashes, there are more Traditional and Lotto investors than Kurtosis Aversion investors existing on the market; during dot-com boom, Traditionaland Kurtosis Aversion investor are the majority; in Housing boom period, most of the investors are the Lotto investors. To our best knowledge, we are the first to investigate this problem in a partial equilibrium setting and the model prediction is supported by the empirical evidence.

1 Introduction

It has been long documented that financial returns are not well described by a normal distribution because they tend to exhibit asymmetry and leptokurticity, i.e. higher peak and fatter tails than they are expected from normal distribution. Taking this empirical fact into consideration, many researchers have shown that a **representative** agent with a quadratic utility function is naturally sensitive to the first four moments of his expected wealth distribution (Rubinstein, 1973; Kraus and Litzenberger, 1976; Levy and Sarnat, 1984; Harvey and Siddique, 2000; Jurczenko and Maillet, 2001, 2006). However, in the financial market, investors may have different preferences for higher order moments, e.g., Kumar (2008) shows that socioeconomic and psychological factors can influence investor preferences: "*poor, young men who live in urban, Republican dominated regions, prefer individual stock's skewness. Also poor people exhibit a stronger aversion to kurtosis*". Therefore, this current work aims to propose a new model of which investors are assumed to have heterogeneous preferences for higher moments (hereafter **HM**).

The paper that is closest to our modeling method is Mitton and Vorkink (2007 RFS). However, our model differs from theirs in several aspects: first, They explain the positive relation between asset's individual skewness and its equilibrium return based on a simple model where investors have heterogeneous preferences for skewness. Except skewness we also study_investor preference for **kurtosis**; second, in the theoretical part, investors' fraction and risk aversions are varied for the comparative analysis, which shows that how the heterogeneous HM preferences impact on equilibrium return and investors' optimal investment allocation; third, based on real data we directly apply our model to forecast asset returns and investigate investor properties, such as the investor's risk aversion parameter, the distribution of each investor group in the financial market and their optimal portfolio holdings. To our best knowledge, we are the first to address these issues. The motivations and main contributions from this current work are detailed below.

Theoretical Aspect

1. Heterogeneous preferences for HM affect the optimal allocation and explains the underdiversification problem, which is new to the existing literature. We study both skewness and kurtosis effects on investment decisions. Skewness represents the probability to gain or lose a lot of money in your portfolio; and absolute prudence implies kurtosis aversion (Hass, 2007; Madan and Yen, 2004). Previous studies show that individual investors require higher expected return to hold the asset with higher kurtosis or lower skewness[1]. Furthermore, our work explains an empirical fact that investors are holding underdiversified portfolios (see, e.g., Polkovnichenko, 2005; Goetzmann and Kumar, 2007; Kumar, 2008[2]). From a mean-variance optimizer's point of view, underdiversified portfolio holding is a irrational/suboptimal investment decision. In their portfolio, there are too few securities to eliminate idiosyncratic risk which can be done through full diversification, i.e. holding the market portfolio. However investors with a skewness preference tend to trade the mean-variance efficiency for a higher skewness. They will hold an underdiversified portfolio which is a mean-vairance-skewness efficient investment (Simkowitz and Beedles, 1978; Conine and Tamarkin, 1981; Barberis and Huang, 2005). Our model not only generates the same result for skewness preference, but also extends the analysis to the kurtosis aversion's impact on optimal portfolio construction. We motivate this investigation through figure 1 which summarizes the HM values for portfolios with varying asset weights. Each portfolio is made up of two assets. The portfolio moments are calculated from weekly Morgan Stanley Capital International (MSCI) data sets of North-American, European and

[1] Harvey *et al.* (2002) propose a method to handle higher moments and estimation error using a Bayesian decision framework. Guidolin and Timmermann (2003) investigate the optimal asset allocation while assuming the returns are driven by a regime switching model. Jondeau and Rockinger (2006) evaluate how non-normality affects the optimal asset allocation and measured the advantages of using a strategy based on higher-order moments.

[2] Kumar carries out a further investigation of the relation between underdiversification and investor properties, such as age, education level and preference for skewness, etc. This work also presents empirical evidence on how underdiversification affects the expected return.

Asian stock markets. The blue-circle line represents linear combinations of individual assets' moments; the red-dot line denotes portfolio moments. The diversification benefit can be viewed as higher portfolio mean and skewness or lower variance and kurtosis than the asset-moment linear combinations. The sample period changes from pre-subprime (before subprime crisis in Sep. 2007) to the whole time period (from Jan. 1988 to Jan. 2010). We can see that the portfolio skewness is not guaranteed to be larger or smaller than the linear combination. But the portfolio variance and kurtosis are always lower than the linear combinations, i.e. diversification can reduce these two risks. So it's reasonable to predict that kurtosis aversion investors will prefer to hold more diversified portfolios. On the other hand, the aversion to the kurtosis will result in less investment into assets with high kurtosis values, i.e. it may generate an underdiversified investment. Therefore, it make sense to study the interaction effects on investment from HM preferences.

Previous studies about the HM preference simply assume one representative agent (see, e.g. Telmer, 1993; Chan and Kogan, 2002; Basak, 2005). However, heterogeneity in HM preference should be a determinant factor for underdiversified portfolio because identical HM preference among investors still leads to fully diversified portfolios (Rubinstein, 1981; Mitton and Vorkink, 2007). Therefore, our model relies on the assumption of heterogeneous preferences for skewness and kurtosis. The investors' preference structure is derived from the Taylor series expansion for CARA utility and they care about different orders of moments for the asset return distribution. To do so the truncation order for the Taylor series expansion is selected differently among investors. In our economy there exist in total three types of agents whose utilities are characterized by "MV", "MVS" and "$MVSK$". (M: Mean V: Variance S: Skewness K: Kurtosis). They are named as "Traditional", "Lotto" and "Kurtosis Aversion" investor[3].

[3] Characterizing investors' utility by the preference (aversion) for moments does not suggest investors actively examine the moments of return distributions and making investment solely based on these moments. Rather, this is an common approach used by many researchers (Jondeau and

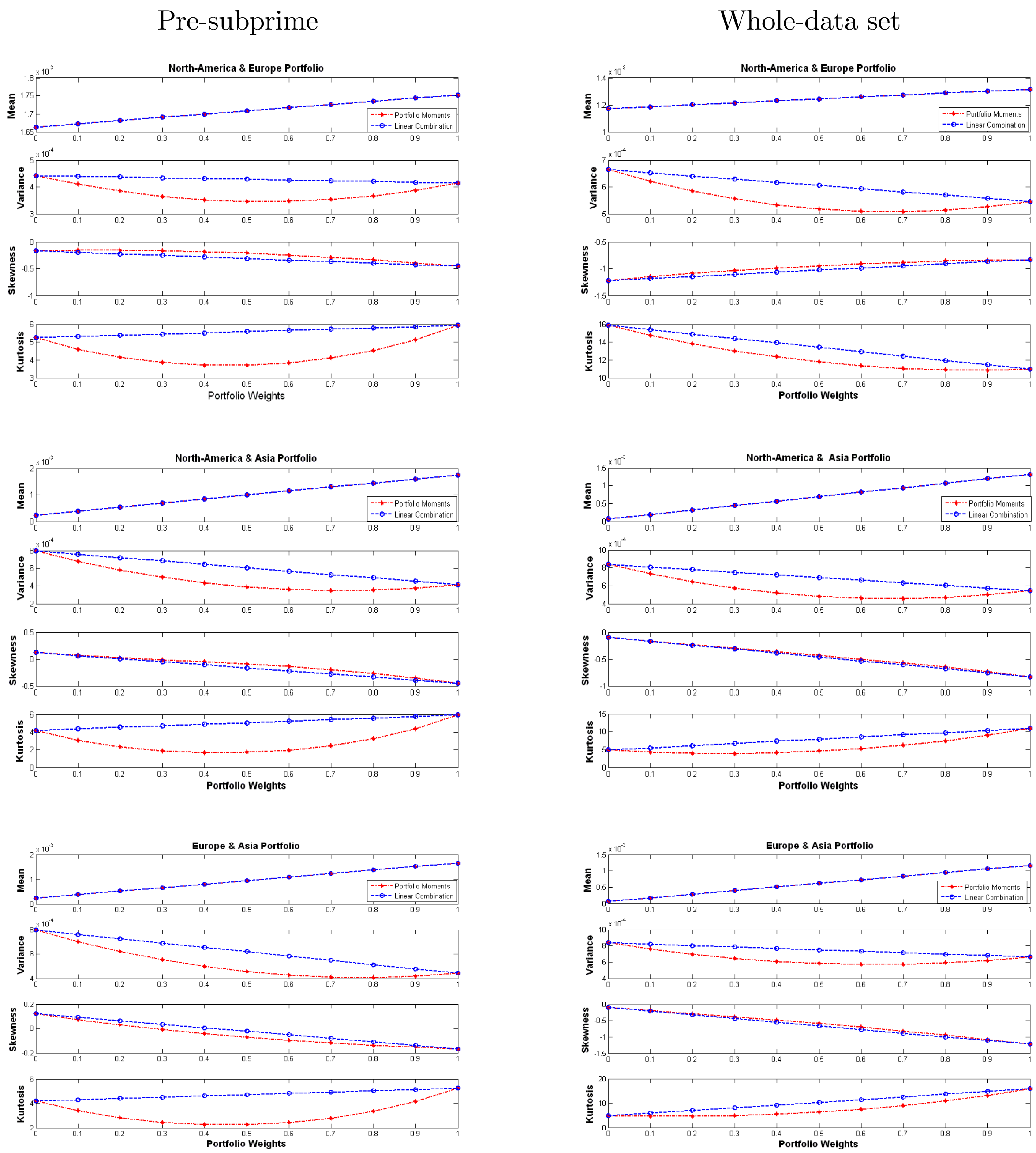

Figure 1: This figure contains plots of portfolio mean, variance, skewness and kurtosis and makes them compared with the linear combination of the asset returns' moments. The three plots on the right uses three pairs of MSCI returns: North-America, Europe and Asia, based on the whole data set ranging from January 1st, 1988 till January 26th, 2010. The three plots on the left are based on pre-crisis returns from January 1st, 1988 till December 29th, 2006. The blue circle-dash line represents the linear combination of the four moments. The portfolio mean, variance, skewness and kurtosis are represented by the red dot-dash line.

2. We propose a new method to forecast asset return based on an equilibrium setting with heterogeneous investors. To be more specific, we first assume that investors are able to estimate assets' moment values, such as variance, skewness and kurtosis, in the current period from historical data; then the heterogeneity in HM preference determines their optimal investment decisions. At equilibrium when market clears, the future expected returns arise from this mechanism. The results from simulation show that positive (negative) relation between expected asset return and kurtosis (skewness) arises from investors' preference for HM, and, suggest that heterogeneity in moment preference can predict subsequent equilibrium returns. This idea is not far from a strand of literature focusing on the equilibrium implied return [4]. These follow the path-breaking technique developed by Black and Litterman (1990). They created a framework in which every investor is a mean-variance optimizer, the equilibrium return forecast is obtained through combining all the equilibrium investors' views (the prior distribution for "implied return") and their active investment through a Bayesian approach (Meucci, 2006, 2008, 2009). However, they do not study how heterogeneity in investors' HM preferences affects the equilibrium return, which is one of the main focuses in our work.

3. Changing investors' risk aversion level and their fractions can also affect the equilibrium, i.e. higher risk aversion implies more significant equilibrium effect on returns and allocation. Fractions of heterogeneous investors can be varied which is consistent with previous studies such as papers related to market microstructure and evolutionary finance[5]. Our finding that investors with HM preferences can signifi-

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Rockinger, 2006; Mitton and Vorkink, 2007) in order to abstract away from a large number of behavioral mechanisms which are likely to generate a preference for higher order moments. −According to Kumar (2008, P4)

[4] The equilibrium return arising from Mitton and Vorkink's setting can also be referred as "Implied Return", which is a concept based on market equilibrium. The "equilibrium" means the market price is such that supply of assets equals their demand.

[5] For the market microstructure issue, see the survey "Heterogeneity, Market Mechanisms, and Asset Price Dynamics" from Chiarella, Dieci and He (2009); the issue in evolutionary finance can

cantly impact on the equilibrium is new to the literature.

Empirical Aspect

Our empirical analysis focuses on two distinct issues. First, we examine whether investor's heterogeneous preferences for HM determine the (ranking of) future expected returns. The result arising from our model is consistent with other previous empirical studies which find individual security's skewness and kurtosis are strongly related to subsequent returns(see e.g. Mitton and Vorkink, 2007; Barberis and Huang, 2005; Kumar, 2008; Conrad *et al.* 2009). The empirical result also supports the predication that HM preferences can affect investors' portfolio decisions.

Second, the significant impact from risk aversion level and investor fraction on the subsequent return (evidence shown in the simulation part) motivates us to rotate the point of view from investigating investor fraction effects on equilibrium to recovering the investor fraction on the market using the real posterior return. At the same time, we assume risk aversion level on the market is exogenously fixed[6]. More specifically, we apply our model using weekly MSCI data which covers three major financial markets: North America, Europe and Asia[7]. After conditioning on the bullish and bearish market, our model shows how HM preferences affect the equilibrium asset return through investors' asset allocation decision and how do they impact the investor structure in the international stock market. We find that Kurtosis Aversion investors are the majority in the Dot-com booming market; there are more Lotto investors exist during Housing boom and Dot-com crash period. The results imply that heterogeneous preferences' impact on equilibrium varies across different economic scenarios. This empirical test, to our best knowledge, is the first attempt to recover the investor

be referred to the survey by Hommes (2005). In behavior finance, Easley and O'Hara (2009) have proposed a model about ambiguity aversion investors with different fractions in the market.

[6] Many previous studies have successfully obtain measures of the risk aversion through the implied distribution based on option prices (Jackwerth, 2000; Kang and Kim, 2006; Bliss and Panigirtzoglow, 2003).

[7] We assume there are three types of investors with heterogeneous preference for higher moments in the international stock market.

structure in the market from a behavioral model.

The rest of this paper is organized as follows. Section two presents the theoretical model and shows how to obtain the equilibrium return and optimal allocation. Section three documents the numerical results from our equilibrium model using simulated data. Section four records the model's practical implementation based on actual data. Section five offers some concluding remarks. Section six is the technical appendix.

2 The Model

2.1 Notations and assumptions

First we assume investors can only purchase N risky assets (with $N = 3$) and no short-sell is allowed. The return vector for N risky assets is denoted by $R = [R_1, ..., R_i, ...R_N]$. The expected return, covariance, skewness and kurtosis structure of the risky securities are, respectively, denoted as μ, V, S and K, which are fully defined in Appendix A.

In this economy, Q investors are assumed having CARA utility functions which can be written as $U(W) = -\exp(-\lambda_q W)$, where $q = [1, ..., Q]$ and λ_q is the risk aversion parameter for each agent q. They have the same preference for mean and variance, but they may have different preferences for HM. In particular, three types of investors exist in our economy. The first and second type, which we denote as "Traditional Investor" and "Lotto Investor" (same as in Mitton and Vorkink's paper); the third type is called "Kurtosis Aversion Investor" with preference for skewness and aversion to kurtosis. Investors are assumed to be fully informed, in the sense that they know the structure of the market: investor preference, investor fraction and asset returns' distribution; therefore, they are able to correctly anticipate the equilibrium outcome and optimize their investment accordingly.

Their CARA utility functions are approximated through Taylor series[8]:

Traditional Investor:

$$U(W) = -\exp(-\lambda_q \mu_p)\left[1 + \frac{\lambda_q^2}{2} V_p\right]$$

Lotto Investor:

$$U(W) = -\exp(-\lambda_q \mu_p)\left[1 + \frac{\lambda_q^2}{2} V_p - \frac{\lambda_q^3}{3!} S_p\right]$$

Kurtosis Aversion Investor:

$$U(W) = -\exp(-\lambda_q \mu_p)\left[1 + \frac{\lambda_q^2}{2} V_p - \frac{\lambda_q^3}{3!} S_p + \frac{\lambda_q^4}{4!} K_p\right]$$

For each investor q, the dollar amount invested in each of the N risky assets is denoted by an $N \times 1$ vector $\alpha_q = [\alpha_{q,1},\ \alpha_{q,2}, ... \alpha_{q,i}, ... \alpha_{q,N}]$.

2.2 Optimal portfolio allocation

Proposition 1 *(Equilibrium allocation) Investor q belonging to type $\bar{m} = T,\ L$ or K, maximizes her utility function $U(W_{\bar{m},q})$, subject to her own budget constraint.*

Traditional Investor:

$$\underset{\alpha'_{T,q}}{\arg\max} U(W_{T,q}) = -\exp(-\lambda_q \mu_{p,T,q})\left[1 + \delta_{2q} \alpha'_{T,q} V \alpha_{T,q}\right]$$

Lotto Investor:

$$\underset{\alpha'_{L,q}}{\arg\max} U(W_{L,q}) = -\exp(-\lambda_q \mu_{p,L,q})\left[1 + \delta_{2q} \alpha'_{L,q} V \alpha_{T,q} - \delta_{3q} \alpha'_{L,q} S\left(\alpha_{L,q} \otimes \alpha_{L,q}\right)\right]$$

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[8] Taylor expansion for the expected utility up to fourth order can be written as:

$$\begin{aligned} E\left[U(W)\right] &= U(\overline{W}) + U^{(1)}(\overline{W})E[W - W] + \frac{1}{2}U^{(2)}(\overline{W})E[(W - W)^2] \\ &\quad + \frac{1}{3}U^{(3)}(\overline{W})E[(W - W)^3] + \frac{1}{4}U^{(4)}(\overline{W})E[(W - W)^4] + \textit{Taylor remainder} \end{aligned}$$

where $\overline{W}$ is the expected end-of-period wealth. (See Jondeau and Rockinger, 2007 *European Financial management P34*)

Kurtosis Aversion Investor:

$$\arg\max_{\alpha'_{K,q}} U(W_{K,q}) = -\exp(-\lambda_q \mu_{p,K,q}) \left[\begin{array}{c} 1 + \delta_{2q}\alpha'_{K,q} V \alpha_{K,q} - \delta_{3q}\alpha'_{K,q} S \left(\alpha_{K,q} \otimes \alpha_{K,q}\right) \\ +\delta_{4q}\alpha'_{K,q} K \left(\alpha_{K,q} \otimes \alpha_{K,q} \otimes \alpha_{K,q}\right) \end{array} \right]$$

where $\delta_{iq} \equiv \frac{\lambda_q^i}{i!}$, $i = 2, 3$ *or* 4. *The final wealth budget constraint is*

$$W_q = W_{0,q}\left[\alpha'_q R\right]$$

where $W_{0,q}$ *is investor* q*'s initial wealth.*

Each type of investors' portfolio weights can be denoted by a $N \times 1$ *vector:* $\alpha_{\bar{m},q}$. *The Traditional investor's demand function is the same as the mean-variance demand function leading to the traditional Sharpe-Lintner-Mossin CAPM model.*

$$\mu = \frac{2\delta_{2q} V \alpha_{T,q}}{1 + \delta_{2q}\alpha'_{T,q} V \alpha_{T,q}} \tag{1}$$

Lotto investor's demand function is

$$\mu = \frac{2\delta_{2q} V \alpha_{L,q} - 3\delta_{3q} S \left(\alpha_{L,q} \otimes \alpha_{L,q}\right)}{1 + \delta_{2q}\alpha'_{L,q} V \alpha_{L,q} - \delta_{3q}\alpha'_{L,q} S \left(\alpha_{L,q} \otimes \alpha_{L,q}\right)}$$

The demand function for Kurtosis Aversion investor is

$$\mu = \frac{2\delta_{2q} V \alpha_{K,q} - 3\delta_{3q} S \left(\alpha_{K,q} \otimes \alpha_{K,q}\right) + 4\delta_{4q} K \left(\alpha_{K,q} \otimes \alpha_{K,q} \otimes \alpha_{K,q}\right)}{1 + \delta_{2q}\alpha'_{K,q} V \alpha_{K,q} - \delta_{3q}\alpha'_{K,q} S \left(\alpha_{K,q} \otimes \alpha_{K,q}\right) + \delta_{4q}\alpha'_{K,q} K \left(\alpha_{K,q} \otimes \alpha_{K,q} \otimes \alpha_{K,q}\right)} \tag{2}$$

Proof. See appendix B. ■

2.3 Market Clearing Condition

In this heterogeneous market, each agent makes the portfolio choice according to her own preference for moments. At equilibrium, the composition of the market portfolio will equal to the composition of the optimal portfolio Π (risky fund). In particular, Π is calculated by summing up the demand function for risky assets over all the investors. Therefore, to make sure that the market clearing condition is satisfied, demand for each asset i should equal to its supply[9].

Since there are in total Q agents in the market, we let $\phi_{\bar{m},q}$ be the fraction of agents who have the risk aversion parameter λ_q and belongs to type $\bar{m}$, where $\bar{m}$ $\in m = [T, L, K]$. By normalization, $\sum_{\bar{\bar{m}} \in m} \mathbf{I}_m \sum_{q=1}^{Q} \phi_{\bar{m},q} = 1$, $\bar{\bar{m}} \in m = [T, L, K]$ and $\mathbf{I}_m$ is the indicator function $\mathbf{I}_m = 1$ when $\bar{m} = \bar{\bar{m}}$.

Summing up the optimal asset allocation $\alpha_{\bar{m},q}$ for each agent q in group $\bar{m}$, we can get each group's demand for asset i:

$$D_i(\bar{m}) = \sum_{q=1}^{Q} \alpha_{\bar{m},q,i} \cdot \mathbf{I}$$

where indicator function $\mathbf{I} = 1$ when agent q belongs to group $\bar{m}$ and zero otherwise.

Thus the total demand for asset i is

$$\begin{aligned} D_i &= \sum_{\bar{m} \in m=[T,L,K]} Q \cdot \phi_{\bar{m},q} \cdot D_i(\bar{m}) \\ &= Q \sum_{\bar{m} \in m=[T,L,K]} \phi_{\bar{m},q} \cdot D_i(\bar{m}) \\ &= Q \sum_{\bar{m} \in m=[T,L,K]} \phi_{\bar{m},q} \sum_{q=1}^{Q} \alpha_{\bar{m},q,i} \cdot \mathbf{I} \end{aligned} \tag{3}$$

where Q is the total number of investors. Finally the total demand for all the assets

[9]To show the relationship between the composition of market portfolio and the composition of the optimal portfolio, we borrowed the proof from Malevergne and Sornette (2002 Appendix C.2) and followed the way they construct the equilibrium asset demand and supply. They derive the generalized efficient frontiers based on a new set of consistent measures of risk both in the case of homogeneous and heterogeneous markets. This method enables us to build up the market constraint and solve the equilibrium.

and for all agents

$$
\begin{aligned}
D &= \sum_i D_i \\
&= Q \sum_i \sum_{\bar{m} \in m} \phi_{\bar{m},q} \cdot \sum_{q=1}^{Q} \alpha_{\bar{m},q,i} \cdot \mathbf{I} \\
&= Q \sum_{\bar{m} \in m} \phi_{\bar{m},q} \cdot \sum_{q=1}^{Q} \alpha_{\bar{m},q} \cdot \mathbf{I} \qquad (4)
\end{aligned}
$$

since $\sum_i \sum_{q=1}^{Q} \alpha_{\bar{m},q,i} \cdot \mathbf{I} = \sum_{q=1}^{Q} \alpha_{\bar{m},q} \cdot \mathbf{I}$ for any agent q . We also define, for each group $\bar{m}$, the weight of asset i in the risky fund is

$$
\tilde{\omega}_{\bar{n},i} = \frac{\sum_{q=1}^{Q} \alpha_{\bar{m},q,i} \cdot \mathbf{I}}{\sum_{\bar{m} \in m} \phi_{\bar{m},q} \cdot \sum_{q=1}^{Q} \alpha_{\bar{m},q} \cdot \mathbf{I}}
$$

We also set w_i^C as asset i's market capitalization ratio

$$
w_i^C = \frac{S_{upplyi}}{\sum_i S_{upplyi}}
$$

At equilibrium when demand equals supply, we can get

$$
w_i^C = \frac{S_{upplyi}}{\sum_i S_{upplyi}} = \frac{D_i}{D} = \sum_{\bar{m} \in m} \phi_{\bar{m},q} \tilde{\omega}_{\bar{m},i}
$$

where $\frac{D_i}{D}$ is derived based on equation (4) and (5).

And we could see that the market portfolio is the weighted sum of optimal portfolios constructed by all the heterogeneous agents.

2.4 Equilibrium asset return

Proposition 2 *(Equilibrium return) Since the demand function implies that the investor's optimal allocation is a function of the equilibrium return, given market clearing condition, we can have:*

$$\phi_T \cdot \alpha_{T,i}(\mu^i) + \phi_L \cdot \alpha_{L,i}(\mu^i) + \phi_K \cdot \alpha_{K,i}(\mu^i) = w_i^C$$

Proof. At equilibrium, given that the market clearing condition is satisfied, the per capita demand for asset i must equal to its per capita supply. To simplify, we first assume all the agents have the same risk aversion parameter, no matter how different their preference towards moments is. So we can drop the notation q and denote the fractions of the Traditional, Lotto and Kurtosis Aversion investors by ϕ_T,ϕ_L, and ϕ_K, where $\phi_T + \phi_L + \phi_K = 1$. ■

The above relation implies that the larger the weight of one investor type on the market, the greater its impact on the equilibrium prices.

3 Simulation Analysis

Due to the nonlinear portion of the first order conditions[10], it's not possible to obtain the analytical solution for demand function. However, thanks to the advanced programming algorithm, we can numerically solve the model. The equilibrium effects from the skewness and kurtosis are investigated separately. Comparative analysis shows how investor fraction and risk aversion impact on equilibrium.

In the first part, we give a brief description for the optimization procedure used to generate the numerical results.

In the second part, we investigate the equilibrium effect from the third and fourth moment separately while assuming different fractions for different investors. In particular, for the skewness analysis, under similar model specification as Mitton and Vorkink's we study investor fractions' effect on equilibrium. The analysis on the (co-)kurtosis impact on equilibrium is carried out based on simulation.

[10] Such as $\left\{\alpha_{L,q}^{'} S_i \left[\left(\left(\alpha_{L,q}^{'} e_i\right) \otimes I_{nn}\right) + \alpha_{L,q}^{'} e_i^{'}\right] + e_i^{'} \alpha_{L,q} \alpha_{L,q}^{'} S_i\right\}$ in the equation from Appendix B.

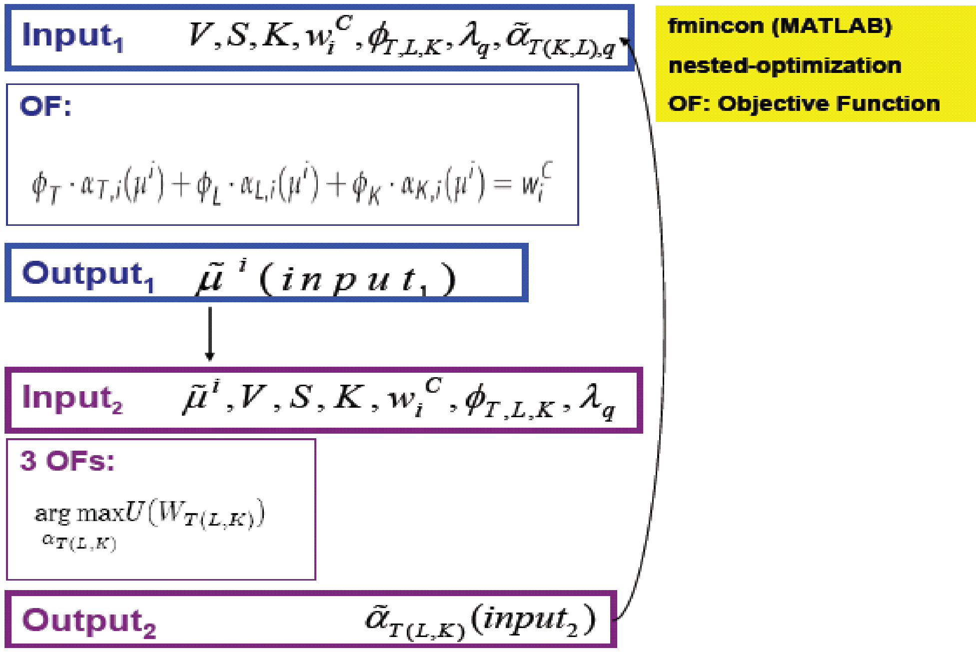


Figure 2: Optimization Procedure

3.1 Optimization procedure

At equilibrium, investors make their investment decisions in order to maximize their objective functions. They fully anticipate the asset's distribution and the equilibrium return since they know the investor structure, i.e. risk aversion and fraction of heterogeneous investors. Therefore, equilibrium return is a function of optimal allocations and meanwhile the market clearing condition needs to be satisfied. This intuition helps to complete the optimization procedure which is summarized in figure 2. The inputs for the optimization are V, S, K which have already been defined in Appendix A, and can be estimated through real data or based on simulation; W_i^C, $\phi_{\bar{m},q}$, λ_q defined in section 2 are assigned by fixed numbers. Each agent's portfolio weights $\alpha_{\bar{m},q}$ and asset equilibrium return μ are given by some values at the initial step; then, after the objective functions for investors' utility and market clearing condition are satisfied, we can obtain the $\alpha^*_{\bar{m},q}$ and μ^* as outputs from the optimization procedure.

Parameter	Variable	Value
Investor fraction	ϕ_T, ϕ_L and ϕ_K	$\phi_T = [.1\ .2.\ 3\ \frac{1}{3}\ .4\ .5\ .6\ .7]$
		$\phi_L = \phi_K = (1-\phi_T)/2$
Market capitalization ratio	$cap_{1\ (2,3)}$	$\text{cap}_1 = \text{cap}_2 = \text{cap}_3 = \frac{1}{3}$
Risk-aversion coefficient	λ_q	[6 16]
Variance of risky asset returns	$\sigma_n^2, n = 1, 2 \text{ or } 3$	0.54×10^{-3}
Correlation	$\rho_{12=}\rho_{13=}\rho_{23}$	0
Idiosyncratic skewness for asset 2	$S_2 \quad (S_{1\ or\ 3} = 0)$	(-0.7—-0.7) ;$K_{1,2,3} = 3$
or		
Idiosyncratic kurtosis for asset 3 K_3 $(K_{1\ or\ 2} = 3)$		
Kurtosis structures are simulated according to Jondeau (2008, Sec. 8.1)		
Assuming $S_{1,2,3} = 0$		

Table 1: Model Parameters for Skewness Analysis

3.2 Higher moments & investor fraction analysis

We directly specify the model parameters of which some are based on the real data set[11]. Here we assume three risky assets with the same variance. The changing idiosyncratic skewness (kurtosis) value for asset two (three) is exogenously given. Table 1 gives a general description. One thing needs to emphasize is that, the value for skewness and kurtosis value are specified for reader's convenience, however, in the analysis, investors preferences are about the returns' moments which will have a much smaller scale than those specified values. For example, when asset 2's idiosyncratic skewness increases from -0.7 to 0.7, its third moment will change from -1.49×10^{-4} to 1.49×10^{-4}. Even though the empirical results on expected return and allocation are still significant given such a small change in the moment values.

[11] We use the weekly MSCI data for the covariance structure calculation. Here we only use the variance value of the North-America index to represent three risky assets' variance.

3.2.1 Skewness preference

Because the main focus in this step is to investigate the equilibrium influences from idiosyncratic skewness and the investor fraction, we assume: 1. the coskewness is zero. 2. the cokurtosis value remains the same while the idiosyncratic skewness changes. Therefore, the (co-) kurtosis structure for asset three is calculated based on zero correlation between assets as specified in table 1. In the following part, we analyze how the changes in asset two's skewness, S_2, investor fraction and risk aversion influence the equilibrium properties. The numerical results show that heterogeneous preferences for skewness and investor fraction significantly affect equilibrium return and asset allocation. The increase in risk-aversion parameter from 6 to 16, measuring the weight put on the HM preference, makes the changes in the equilibrium return and allocation similar but more dramatic. Furthermore, this also implies that our model is robust under different risk aversion levels. To save the space, we only report the case with $\lambda_q = 16$.

risk aversion=16

Equilibrium return

The plots in the second row of figure 3 shows: when the majority of investors on the market are Lotto and Kurtosis Aversion types (in total 90%) who have preference for positive skewness, asset two's expected return decreases from 0.03 to 0.001, of which the difference 2.9% is greater than the adjustment in return, 1%, from the case where the majority is Traditional investor (70%). So the more Lotto and Kurtosis Aversion investors exist on the market, the more significantly the expected return changes; also for the same S_2, the more these two types of investors on the market the smaller the asset two's return is. When asset two's return decreases, the other two risky assets become less attractive on the market and their expected returns go up.

Equilibrium allocation

The last two plots in the second row of figure 4 show that, when 30% investors

Equilibrium Return

Investor fraction: $\phi_T = [.1\ .2.\ 3\ \frac{1}{3}\ .4\ .5\ .6\ .7], \phi_L = \phi_K = (1 - \phi_T)/2$

Risk aversion $\lambda_q = 16$, $S_2 = (-0.7 — 0.7)$, M_3 =(-1.49×10^{-4} — 1.49×10^{-4}), $K_3 = 3$.

Traditional Investor MV	Lotto Investor MVS	Kurtosis Aversion MVSK

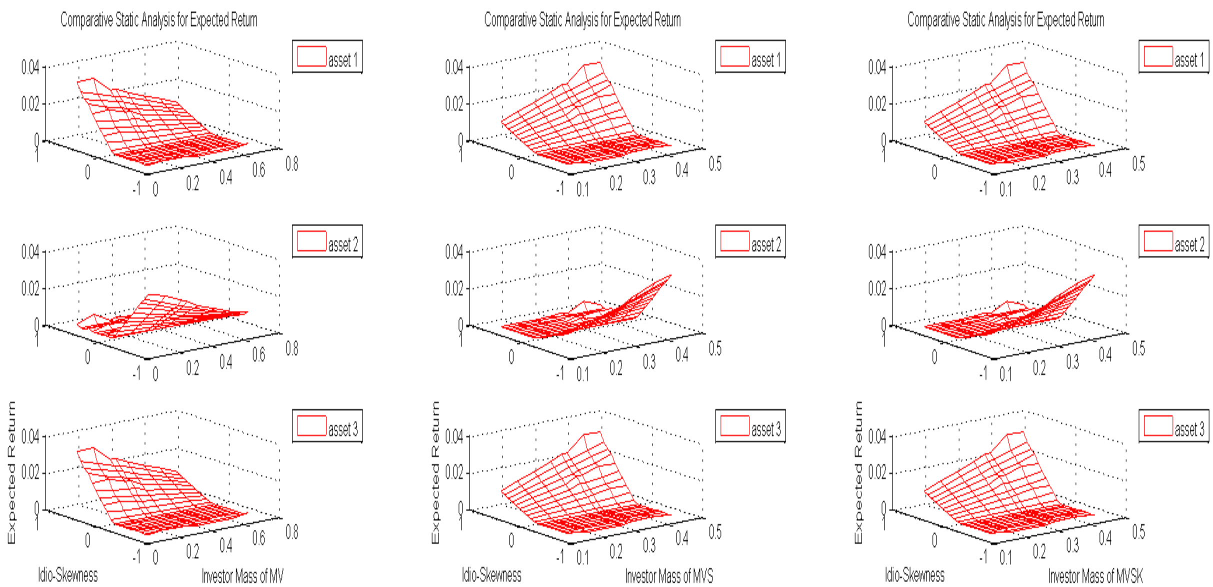


Figure 3: Expected return for asset 2 increases when its skewness decreases. When asset 2's is smaller than zero and the fraction of Lotto and Kurtosis Aversion investors increases, asset 2's expected return increases dramatically. This suggests that the fraction of Lotto and Kurtosis Aversion investors who also has preference for skewness plays an important role in the equilibrium return determination.

are Lotto and Kurtosis Aversion types, their investment in asset two increase from 1/3 to 34.9% and 34.5% when its skewness increases.

When Lotto and Kurtosis Aversion investors' total fraction reaches 90%, their investments in asset two are only increased by 0.5% and 0.4% of which the changes are less dramatic than in the previous case when their fraction is 30%. Because when there are more skewness preference investors in the market and S_2 increases, the competition to buy asset two becomes higher and drives up its price[12].

[12] This result is consistent with the one documented by Jondeau and Rockinger (2008, P19) who empirically showed that kurtosis aversion results in relatively more diversified portfolios.

Investor Allocation

Investor fraction: $\phi_T = [.1\ .2.\ 3\ \frac{1}{3}\ .4\ .5\ .6\ .7], \phi_L = \phi_K = (1 - \phi_T)/2$

Risk aversion $\lambda_q = 16$, $S_2 = (-0.7 — 0.7)$,M_3 =(-1.49×10^{-4} — 1.49×10^{-4}), $K_3 = 3$.

Traditional Investor MV	Lotto Investor MVS	Kurtosis Aversion MVSK

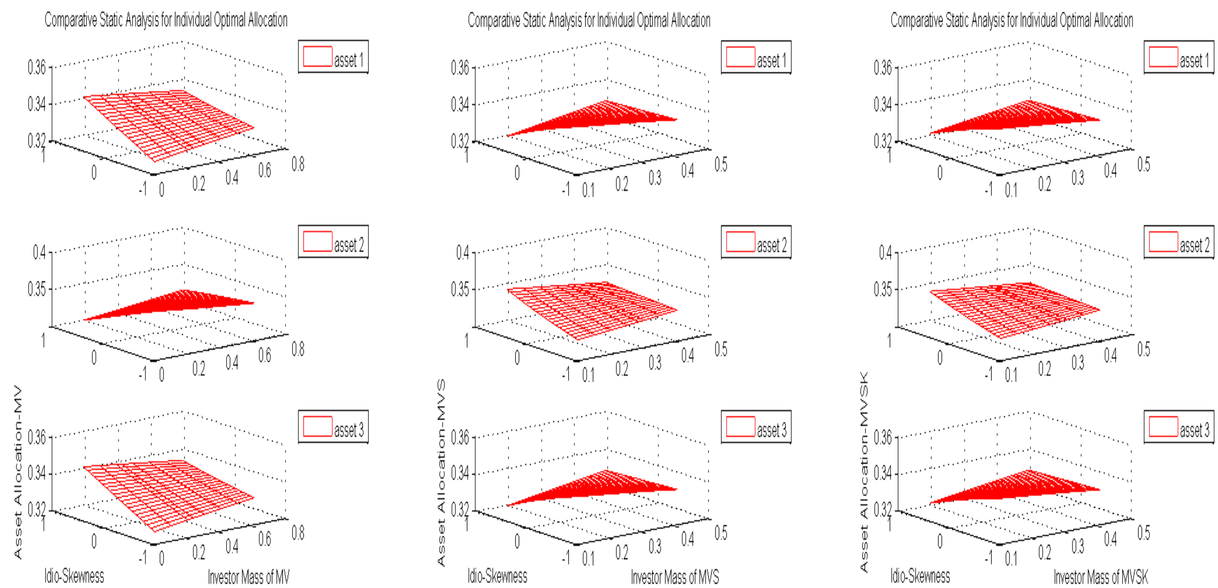


Figure 4: Individual investor allocation varies differently across the three types of investors. Lotto investor tends to hold less asset 2 when its skewness decreases. The underdiversification is more severe for the Lotto investor than the Kurtosis Aversion investor since she prefers to hold more diversified portfolio in order to decrease the portfolio kurtosis.

3.2.2 Analysis for (co-) kurtosis

Simulation Data Three risky assets' return series, 10^6 observations for each, are simulated following the procedure:

1. Simulate two standard-normal return series: r_1 , r_2.

2. Simulate asset three's return, r_3, which follows a student-t distribution with degree of freedom parameter v. The relationship $v = \frac{4K_3-6}{K_3-3}$ enables us to simulate student-t series and construct a set of r_3s, i.e. $\mathrm{r}_3 = [r_3^1, r_3^2, r_3^3, ...r_3^j]$, where j is the number of different values for v ($K_3 \in \{3, 3.5, 4, 5, 5.5, 6.5\}$).

3. Set covariance matrix as $\sum = \begin{pmatrix} 0.2 & 0.0212 & 0.0335 \\ 0.021 & 0.35 & 0.0296 \\ .0335 & 0.0296 & 0.25 \end{pmatrix}$ (covariance matrix from Mitton and Vorkink (2007)) and get its Cholesky decomposition $\sum^{\frac{1}{2}}$.

4. For each r_3 selected from set r_3 and the two standard normal returns, using $\sum^{\frac{1}{2}}$, we obtain one new set of return-series $R = [\tilde{r_1}\ \tilde{r_2}\ \tilde{r_3}]$ and the empirical moment structure $M2$—$M4$ which will change according to different v (K_3) specifications.

For simplicity, table C1 and C2 in appendix C report statistics on two selected simulations with lowest and highest kurtosis value for asset three. When K_3 increases, in most cases, cokurtosis K_{3ijk}, at least one from i, j and k assets also increases. In order to focus on the kurtosis effect, we assume all the elements in the skewness structure are zero.

Numerical results **Same** investor fraction

Figure 5 summarizes the fourth-moment effect on equilibrium properties and we assume same fractions for all types of investors and the risk aversion level is 2.5[13].

[13] In the unreported result we also investigate the case with higher risk aversion level, $\lambda_q = 4$. The result is consistent but with a more significant (co-) kurtosis effect on equilibrium.

The starting point is artificially made by assuming no effect from skewness and kurtosis[14]. From this special case in the figure denoted when M4=0, we can see that expected return difference purely arises from different variance value. And heterogeneous investors hold the equal-weighted portfolios, i.e. the market portfolio. Then the idiosyncratic kurtosis value, K_3 increases from 3 to 6.5, so asset three's excess fourth moment will increase from 0 to 1.023. Accordingly, the co-kurtosis can also be calculated as documented in table C1 and C2. Asset three's expected return will climb up from 23.7% to 30.36%. Kurtosis Aversion investor who dislike the increased kurtosis risk, reduces her investment in asset three from 31.8% to 28.7%.

Figure 6 shows the case when we assume zero co-kurtosis between different assets. The expected return is only slightly higher in the previous case due to the non-zero co-kurtosis effect. And the allocations in the two cases are quite close. So we can conclude that the effect on equilibrium from co-kurtosis is very small when there are small number of assets in the portfolio.

[14] We assume all the elements in skewness and kurtosis structure are equal to zero.

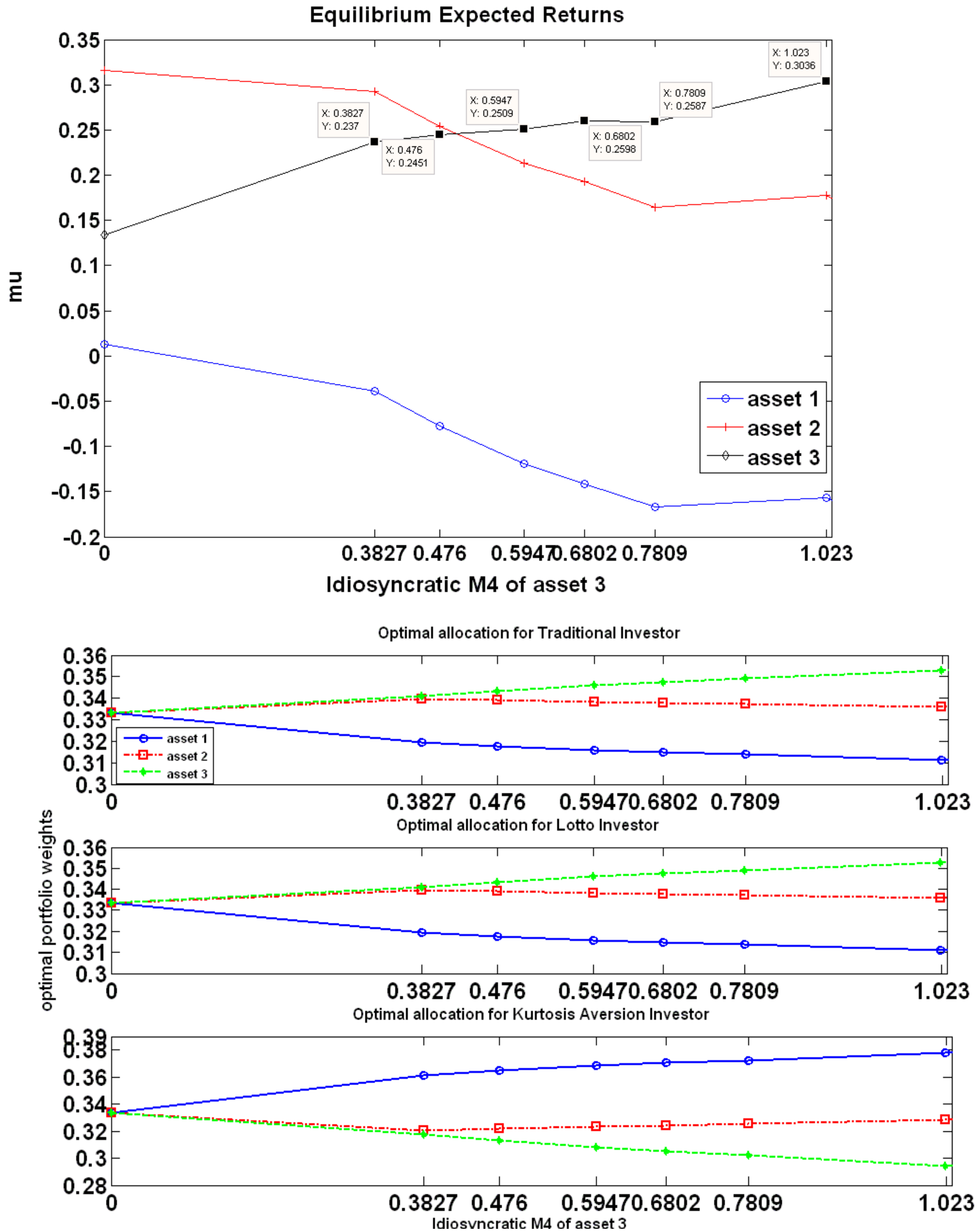


Figure 5: Non zero-4th-moment effect and changing idio-kurtosis K3

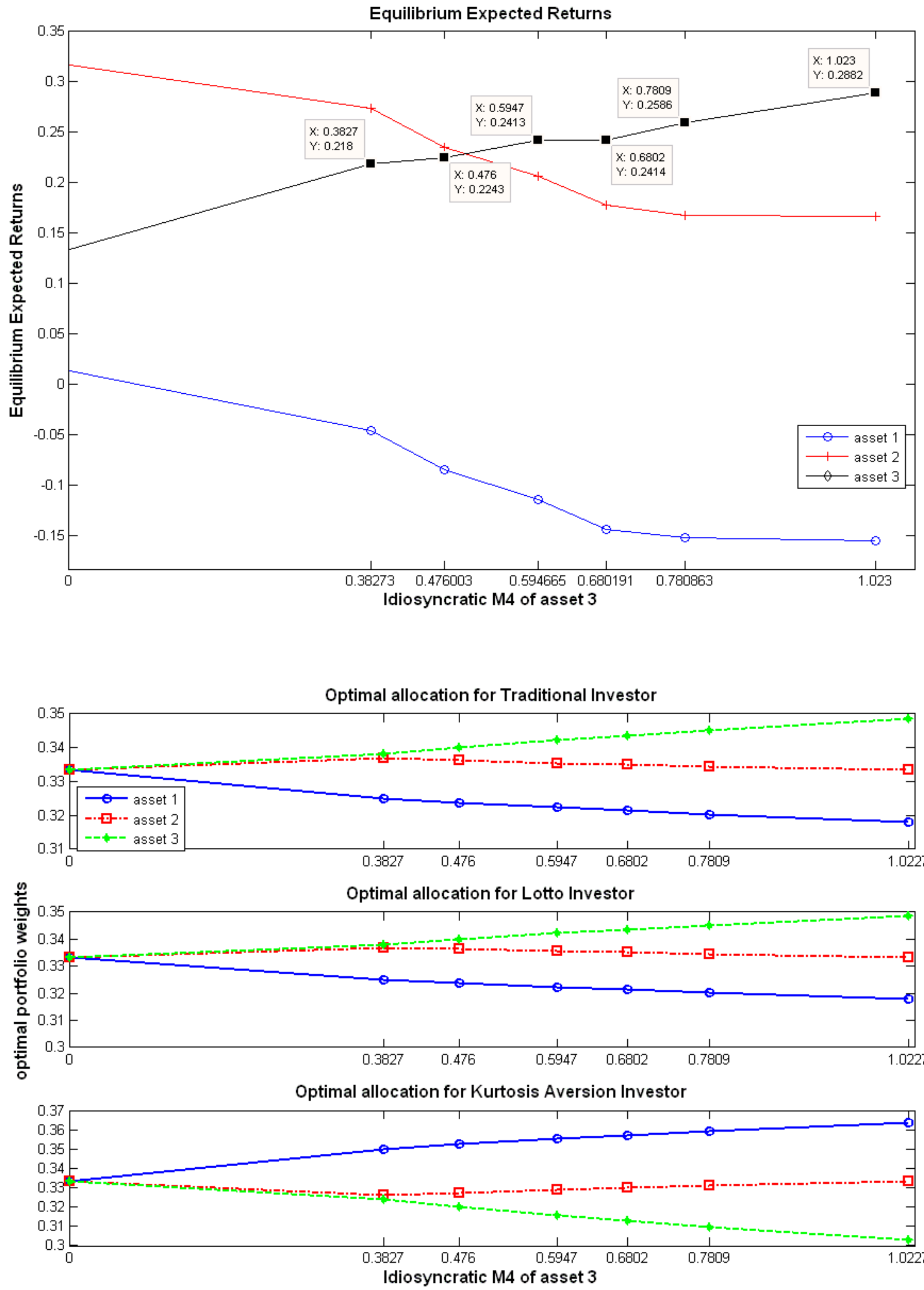


Figure 6: Zero fourth-co-moment and changing idio-kurtosis K3

Different investor fractions

Equilibrium return

Figure 7 shows that increase in K_3 can significantly raises asset three's expected return. For example, figure 7: plot 3 shows that when there are 1/3 investors are Kurtosis Aversion and K_3 rises, asset three expected return climbs up to 36.52%. Additionally, under the same K_3 level, when Kurtosis Aversion investor's fraction increases from 5% to 1/3, asset three's expected return also increases which shows that the larger the weight of Kurtosis Aversion on the market, the greater their impact on the equilibrium. Meanwhile, the other two risky assets become much more attractive and their expected returns are negatively related to K_3. Asset one's expected return even becomes negative because of its lowest variance and smallest kurtosis values among the three risky assets.

Equilibrium allocation

Figure 8 summarizes the optimal asset holdings of the three investor types. Figure 8: plot 3 tells that when K_3 increases, Kurtosis Aversion investor's investment in asset three declines and she starts to invest more in the other assets. Meanwhile, asset three's price drops due to the increased future expected return which attracts more Traditional and Lotto investors to hold this asset, because these two types don't have any particular preferences for kurtosis.

Investor fraction and (co-kurtosis) effect

Investor fraction: $\phi_T = [\frac{1}{3}\ .4\ .6\ .8\ .9], \phi_L = \phi_K = (1 - \phi_T)/2$

Risk aversion $\lambda_q = 2.5,\ K_3 = [4\ 5\ 5.5\ 6]$.

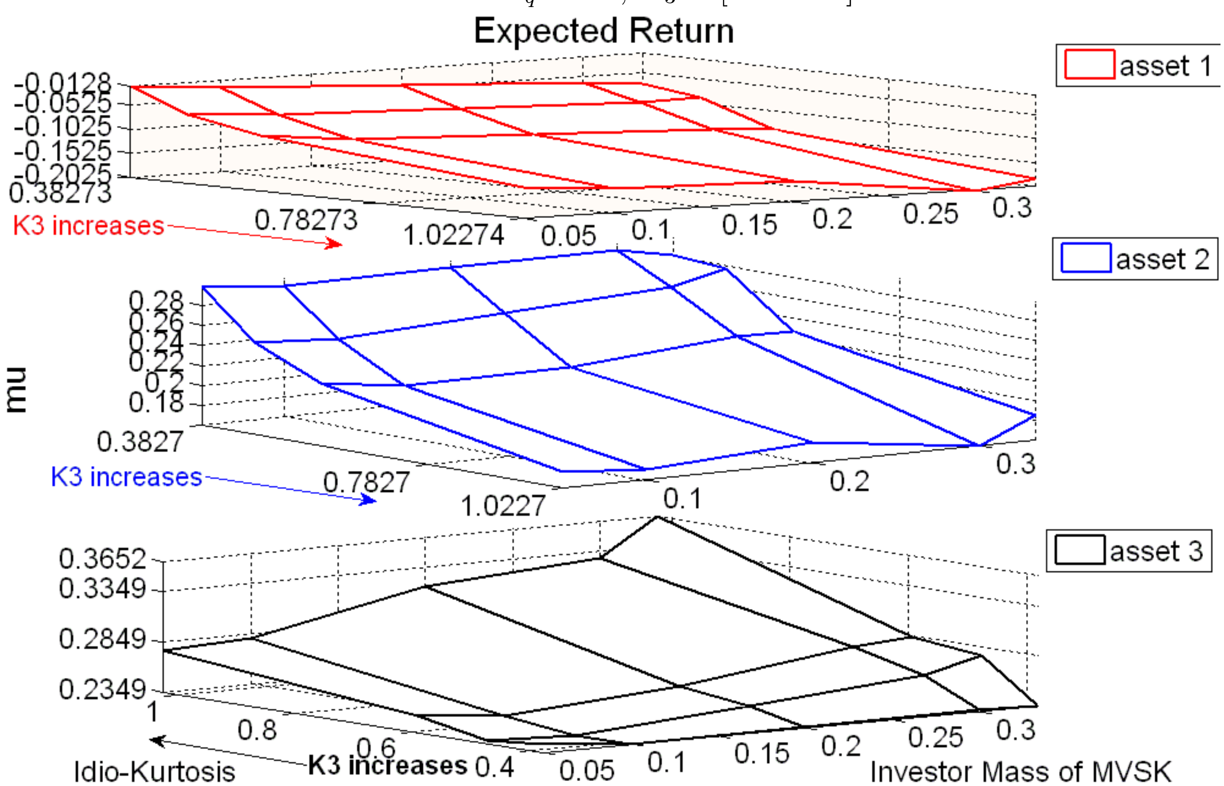


Figure 7: Asset 3's expected return increases when its idiosyncratic kurtosis becomes larger. Meanwhile, when the fraction of Kurtosis Aversion investors increases, asset 3's expected return increases dramatically. This suggests that the fraction of Kurtosis Aversion investor also plays a critical role in the equilibrium return determination.

Investor Allocation

Investor fraction: $\phi_T = [\frac{1}{3}\ .4\ .6\ .8\ .9], \phi_L = \phi_K = (1 - \phi_T)/2$

Risk aversion $\lambda_q = 2.5,\ K_3 = [4\ 5\ 5.5\ 6]$.

Traditional Investor MV	Lotto Investor MVS	Kurtosis Aversion MVSK

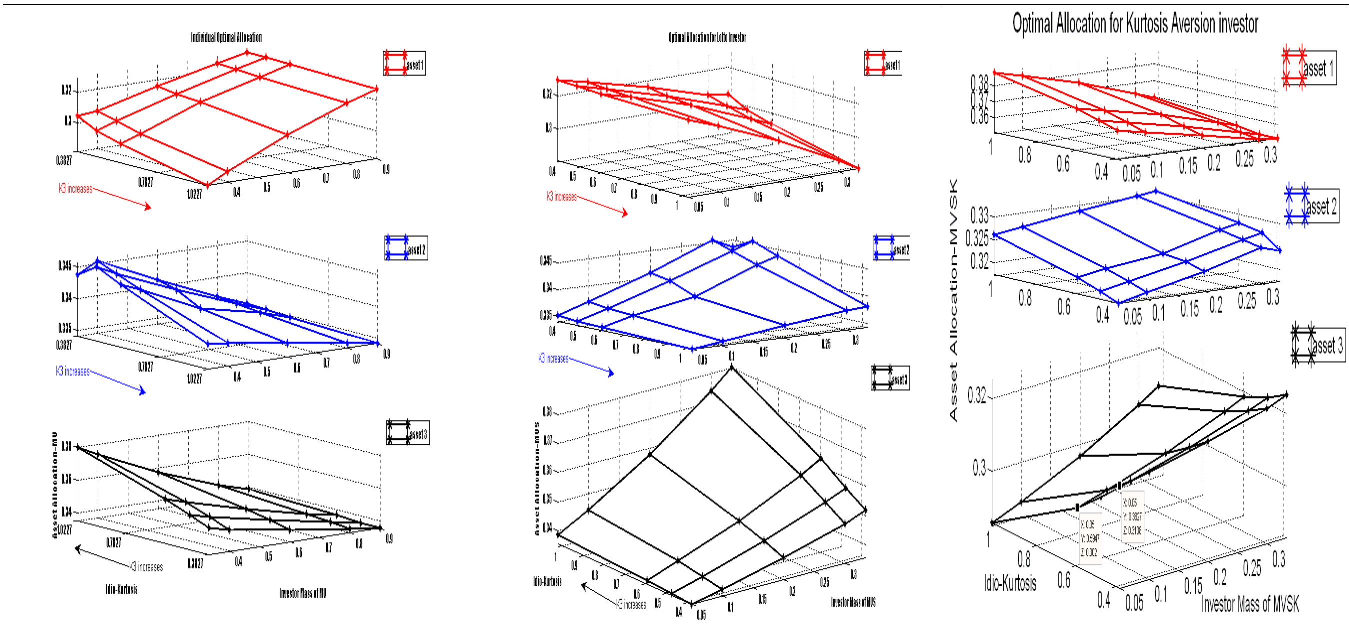


Figure 8: Investor allocation differs dramatically across three types of agents when the asset 3's idiosyncratic kurtosis changes. When asset 3's kurtosis increases, Kurtosis Aversion investor tends to hold less asset 3 than the other two types.

4 Empirical analysis

Section 4.1 and 4.2 focused on the equilibrium analysis based on real data the weekly Morgan Stanley Capital International (MSCI) indices. We investigate the equilibrium effect from HM preferences in the bullish and bearish markets. The previous simulation results imply a mapping among the fraction for each investor type, average level of risk aversion and HM preferences. This motivates the practical implementation of our model using real data in order to recover the investor property at equilibrium such as risk aversion and investor fraction which is done in section 4.2.3. Different from the previous analysis which studies skewness and kurtosis effects separately, this empirical part exams the equilibrium effects from all the heterogeneous preferences for HMs together.

4.1 Subprime crisis effect analysis

4.1.1 Data

We calculate weekly return on Morgan Stanley Capital International (MSCI) dollar-denominated indices of North America, Europe and Asia from January 1st, 1988 till January 26th, 2010. There are in total 1151 return observations. In order to analyze the subprime crisis effect on the market equilibrium structure, we extract a sub-sample ranging from January 1988 till December 2006 before the crisis happened (sample size: 992). North American, European and Asian markets, in total, represent almost 97% of the total market capitalization in the world MSCI index. Here we assume the market caps for North American, European and Asian markets are: 50%, 30% and 20% respectively[15]. We also assume there are three types of investors with heterogeneous preferences for HM investing into the three regional stock market indices.

Table 2 summarizes the ranking for the three risky assets' higher-moment values.

[15] "At the end of 1999, the North American, European, and Asian markets represented 47.2%, 30.3%, and 19.4% of total market capitalisation in the world MSCI index."-Jondeau and Rockinger, European Financial Management, Vol. 12, No. 1, 2006, 29–55

Data-set	***Idio-Skewness***	***Idio-Kurtosis***
pre-subprime	$S_3 > S_2 > S_1$	$K_3 > K_2 > K_1$
whole period (including subprime)	$S_3 > S_1 > S_2$	$K_2 > K_3 > K_1$

Table 2: Ranking of assets' higher moments for pre-subprime and whole data sets. Asset 1, 2 and 3 represent stocks from North-American, European and Asian markets.

To save space, a full description about the data can be found in table C3 and C4 in appendix C.

4.1.2 Empirical Result: Equilibrium Return and Allocation

Using the data sets covering the whole-time and the pre-subprime period, the fraction of different investors on the market can be recovered while fixing different levels of risk aversion parameter. Because all the investors have same the preference for the first two moments, different investment decisions are solely caused by heterogeneous preferences for HM.

Based on the pre-subprime data set, the case satisfying the market clearing condition is the one where Lotto type investors are the majority on the market. The average level of risk aversion is 15. The real mean return in the subsequent period for the three risky assets are: North America: -0.000968086, Europe: -0.001821788, Asia: -0.000851901 ($\overline{\mu_{AS}}>\overline{\mu_{EU}}>\overline{\mu_{NA}}$). Figure 9: plot 1 shows that, at equilibrium, our model generates the same ranking for the future predicted returns.

As shown in figure 9: plot 2, the optimal allocations differ among investors: Lotto investor invests the smallest amount (about 1.5% less than Traditional investor) in North America due to its lowest skewness value; on the other hand, she holds the largest amount in Asian and European stocks because of their relatively higher skewness values. Kurtosis Aversion investor who has preference for (positive) skewness and aversion to (positive) kurtosis invests in Asian market by 0.1% less than the Lotto investor. Her aversion to the highest kurtosis value of Asian stock decreases her demand rising from the preference for its highest skewness value, i.e. the aversion to kurtosis increases portfolio diversification. Meanwhile, Kurtosis Aversion investor invests almost 10% more than the Lotto investor in North American market due to its lowest kurtosis value.

Figure 10 shows the results based on the whole data set which includes the subprime crisis period. The risk aversion level rises up to 20 and Lotto investors are the minority on the international market. The predicted return for the after-crisis period follows such order: $\overline{\mu_{EU}}>\overline{\mu_{NA}}>\overline{\mu_{AS}}$. Figure 10: plot 2 shows when considering the crisis effect on allocation, Kurtosis Aversion investor holds the minimum amount of European stock due to its highest kurtosis and lowest skewness. During crisis the asset distribution exhibits much fatter tail than the normal case which leads to this extreme asset holdings of Kurtosis Aversion investor. During crisis, Lotto investor tends to hold relatively more diversified portfolio than in the pre-subprime period at equilibrium.

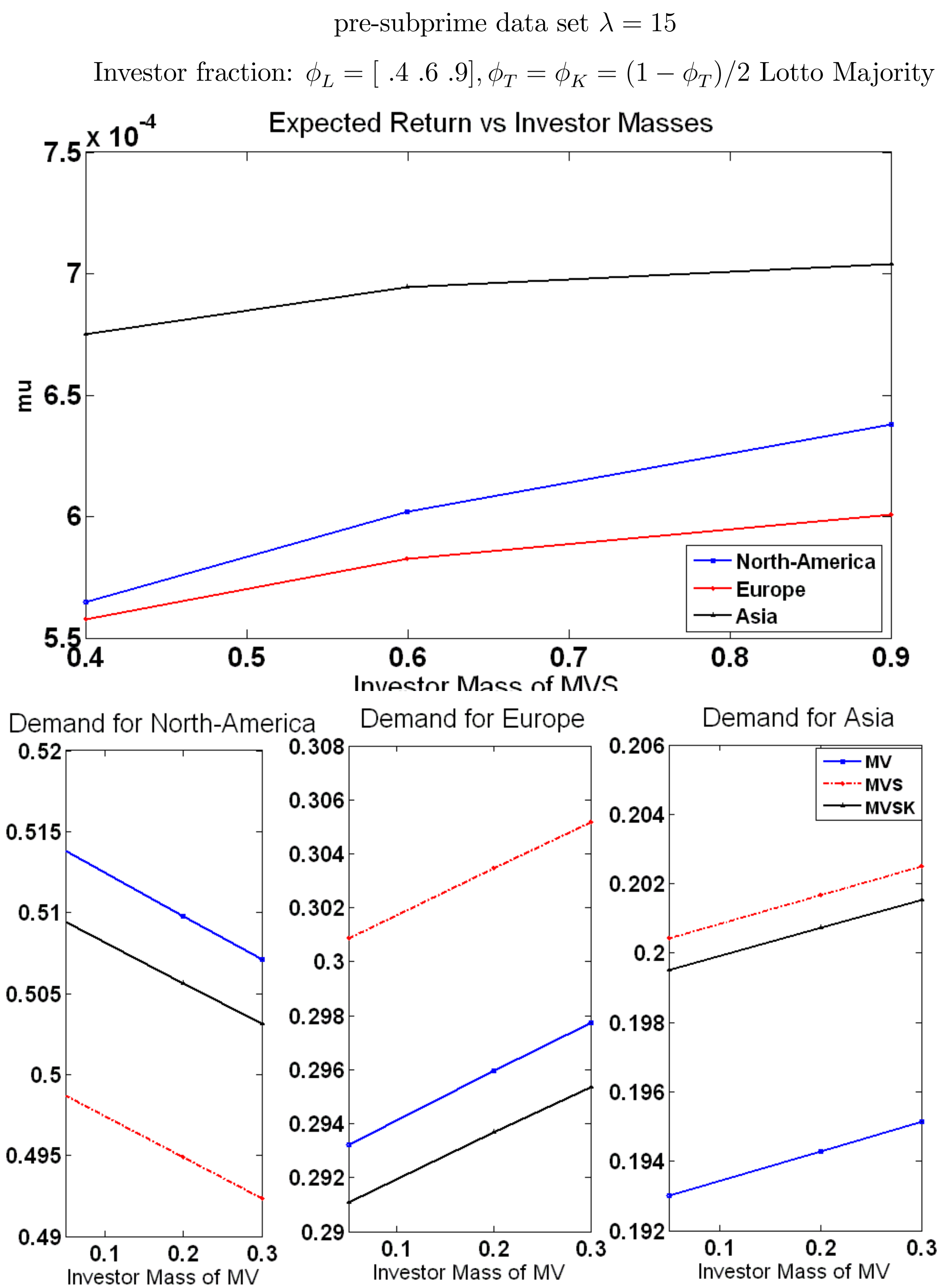


Figure 9: Return prediction and equilibrium allocation using pre-subprime data set.

Wholedata set (including Sub-prime crisis) $\lambda = 20$

Investor fraction: $\phi_L = [.1\ .25\ .3\ ], \phi_T = \phi_K = (1 - \phi_T)/2$ Lotto Minority

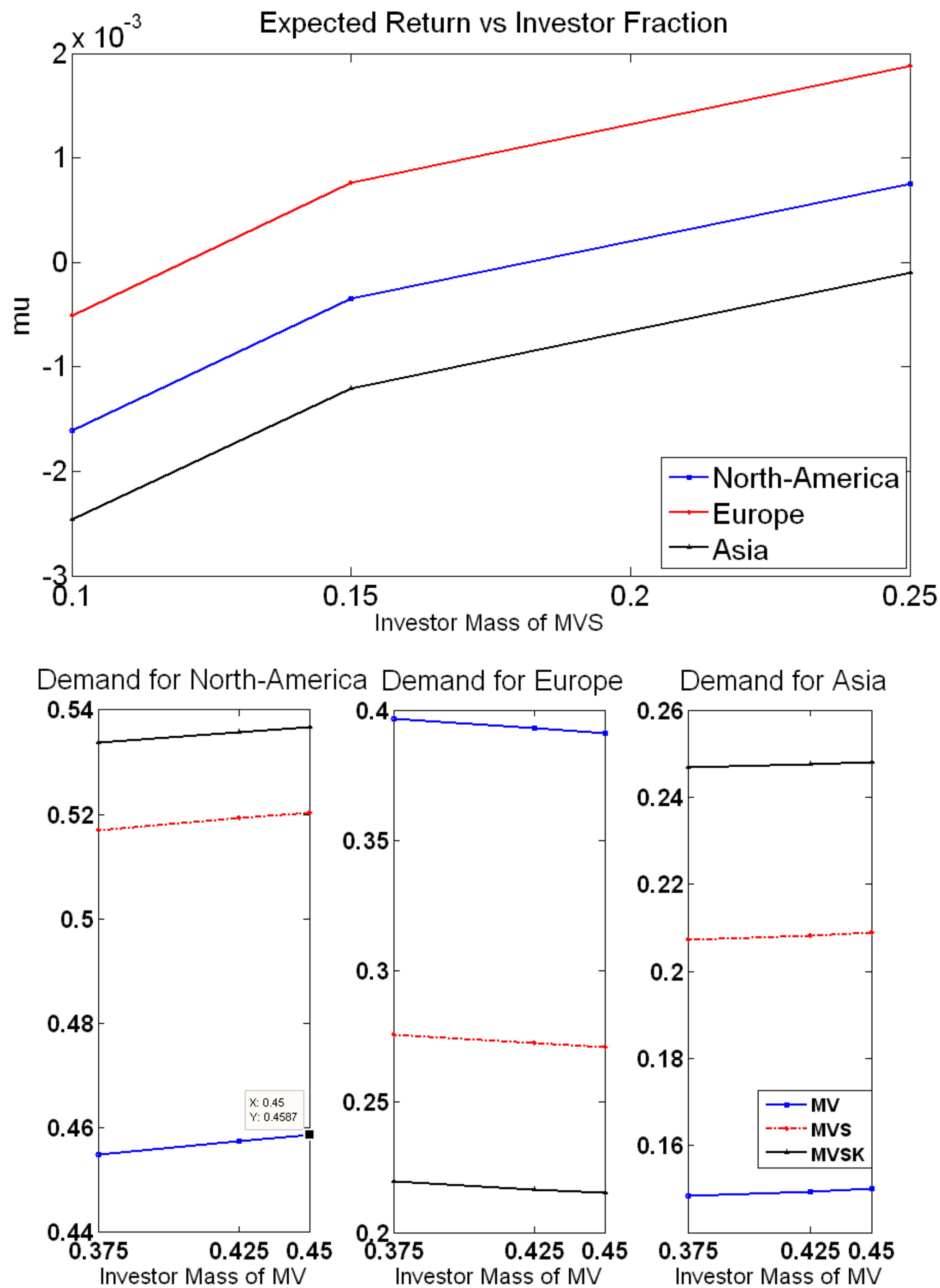


Figure 10: Return prediction and equilibrium allocation using Whole-data set.

Data-set	***Idio-Skewness***	***Idio-Kurtosis***
	($3rd$ central moment)	($4th$ central moment)
dot-com boom	$S_3 > S_2 > S_1$	$K_3 > K_2 > K_1$
dot-com crash	$S_3 > S_2 > S_1$	$K_1 > K_2 > K_3$
Housing boom	$S_1 > S_2 > S_3$	$K_3 > K_2 > K_1$
subprime crash	$S_3 > S_1 > S_2$	$K_2 > K_1 > K_3$

Table 3: Real higher-moment ranking under booms and crashes. Asset 1, 2 and 3 represent stocks from North-American, European and Asian markets.

4.2 Booms and Crashes

4.2.1 Data

To analyze how the heterogeneous HM preferences impact equilibrium allocation and return prediction under different economic scenarios, we select four samples representing major booms and crashes from the weekly MSCI data set. Then, we calculate each period's subsequent one year return in order to compare with the ranking for the model predicted future returns. The four subperiods are listed as following:

1. dot-com boom (6-March-1998 to 1-March-2000)
2. dot-com crash (17-March-2000 to 1-March-2002)
3. housing boom (3-May-2002 to 29-December-2006)
4. subprime crash (6-April-2007 to 10-April-2009)

To save space, table 3 only reports the rankings for the higher moments of the risky assets under different scenarios.

4.2.2 Empirical Result: Equilibrium Return and Allocation

Dot-com boom Figure 11 shows that, during the dot-com boom period, the case satisfying the market clearing condition is the one with Lotto and Kurtosis Aversion investors as the majority on the market (ranging from 2/3 to 90% in total). The average risk aversion level is 25 which induces the significant difference of portfolio holdings among investors (8% at most): Lotto investor holds the smallest amount in North American stock due to its lowest skewness value. However, Kurtosis Aversion investor invests the most in this market because of its lowest kurtosis value. And she also prefers Asian than European stocks because of her preference for higher positive skewness. Under such market structure where Lotto and Kurtosis Aversion investors constitutes the majority in the market, preference for kurtosis plays an important role which is implied by the dominant asset holdings of Kurtosis Aversion investor.

Dot-com crash There are much less Lotto and Kurtosis Aversion investors on the market during this crash (at most 40% in total) and the risk aversion level is 5 on average. Figure 12 reports that the aversion to negative skewness value from the Lotto and Kurtosis investors causes their under-investment in the North America market. Kurtosis Aversion investors prefer Asian stock because of its lowest kurtosis value. Since North-American stock market performs the worst due to its highest kurtosis and lowest skewness value, Lotto and Kurtosis investors both underinvest into this market during dot-com crisis.

Dot-com boom $\lambda = 25$

Investor fraction: $\phi_T = [.1\ .2\ \frac{1}{3}], \phi_L = \phi_K = (1 - \phi_T)/2$ Traditional investor Minority

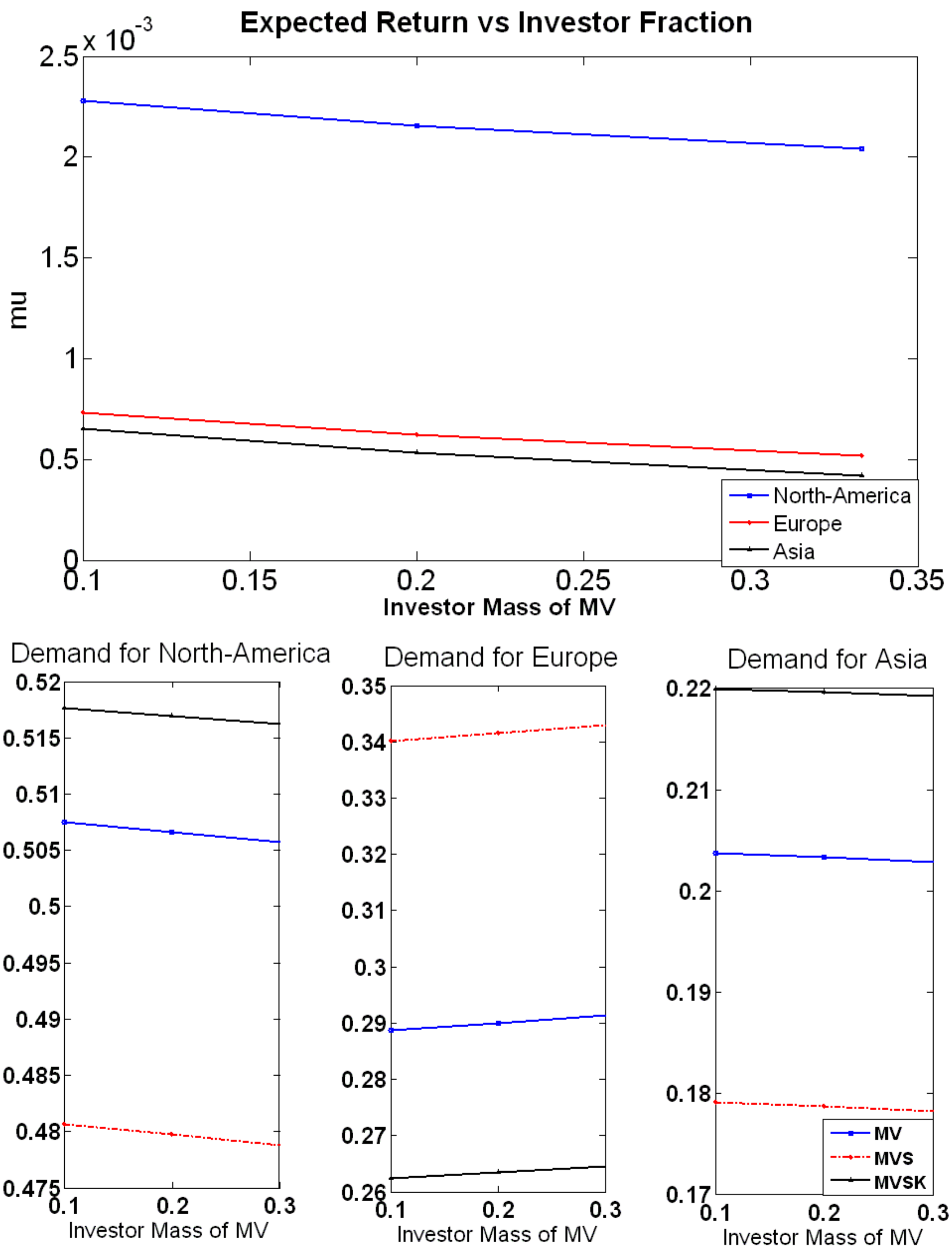


Figure 11: Return prediction and equilibrium allocation using "Dot-com boom data set"

Dot-com crash $\lambda = 5$

Investor fraction: $\phi_T = [0.6\ 0.7\ 0.8], \phi_L = \phi_K = (1 - \phi_T)/2$ Traditional investor Majority

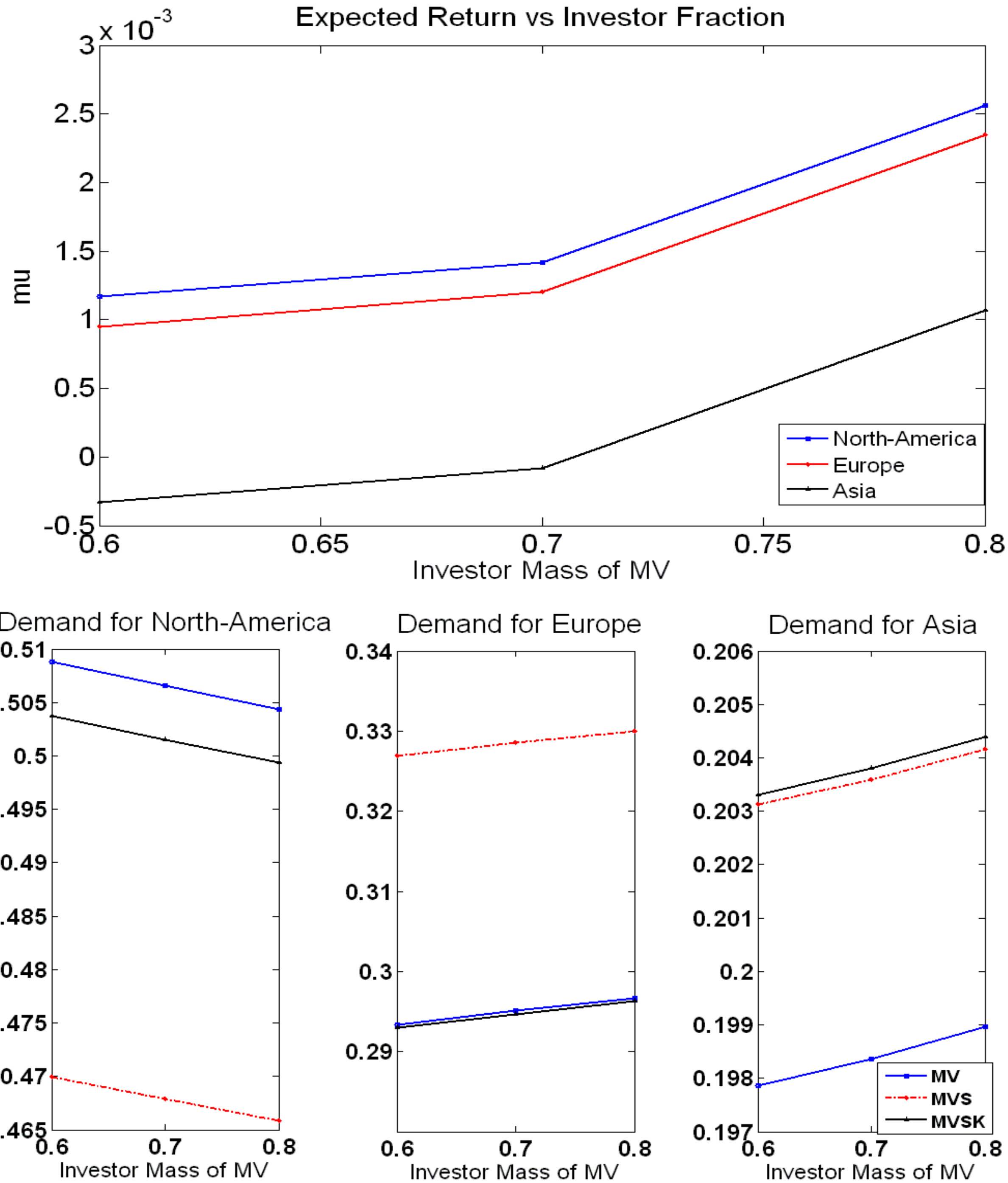


Figure 12: Return prediction and equilibrium allocation using "Dot-com crash data set"

Housing boom Figure 13 tells that during the housing boom period, the major investors are Lotto and Kurtosis Aversion types with preference for the positive skewness (10% or 30% Traditional investors on the market). Lotto investor invests the most in North American market because of its highest skewness value. This dramatic increased demand for the North American stock from Lotto investors drives up the equilibrium asset price in this market and causes the low demand from Traditional and Kurtosis Aversion investors. The holding from Lotto investors plays a dominant role at equilibrium in the sense that their high demand for North-American stock drives up its price and lead to the pricing decrease in the other two markets. Therefore, Kurtosis aversion investors tend to hold more in European and Asian stocks, even the latter's distribution exhibits lowest skewness and highest kurtosis.

Subprime crash Subprime crisis period is characterized by significant non-normal assets in the international finance market. Given an average risk aversion level as 5 and majority investors are Traditional type on the market, the market clearing condition can be satisfied. Figure 13: plot 2 shows that heterogeneous investors hold significant different portfolios (around 15% at most): Kurtosis Aversion investor invests most in the Asian stock market because of its lowest kurtosis and highest skewness. Lotto investor holds the smallest amount of European stock in her portfolio due to its lowest skewness value.

Housing boom $\lambda = 25$

Investor fraction: $\phi_T = [0.1 \quad 0.3], \phi_L = \phi_K = (1 - \phi_T)/2$ Traditional-investor Minority

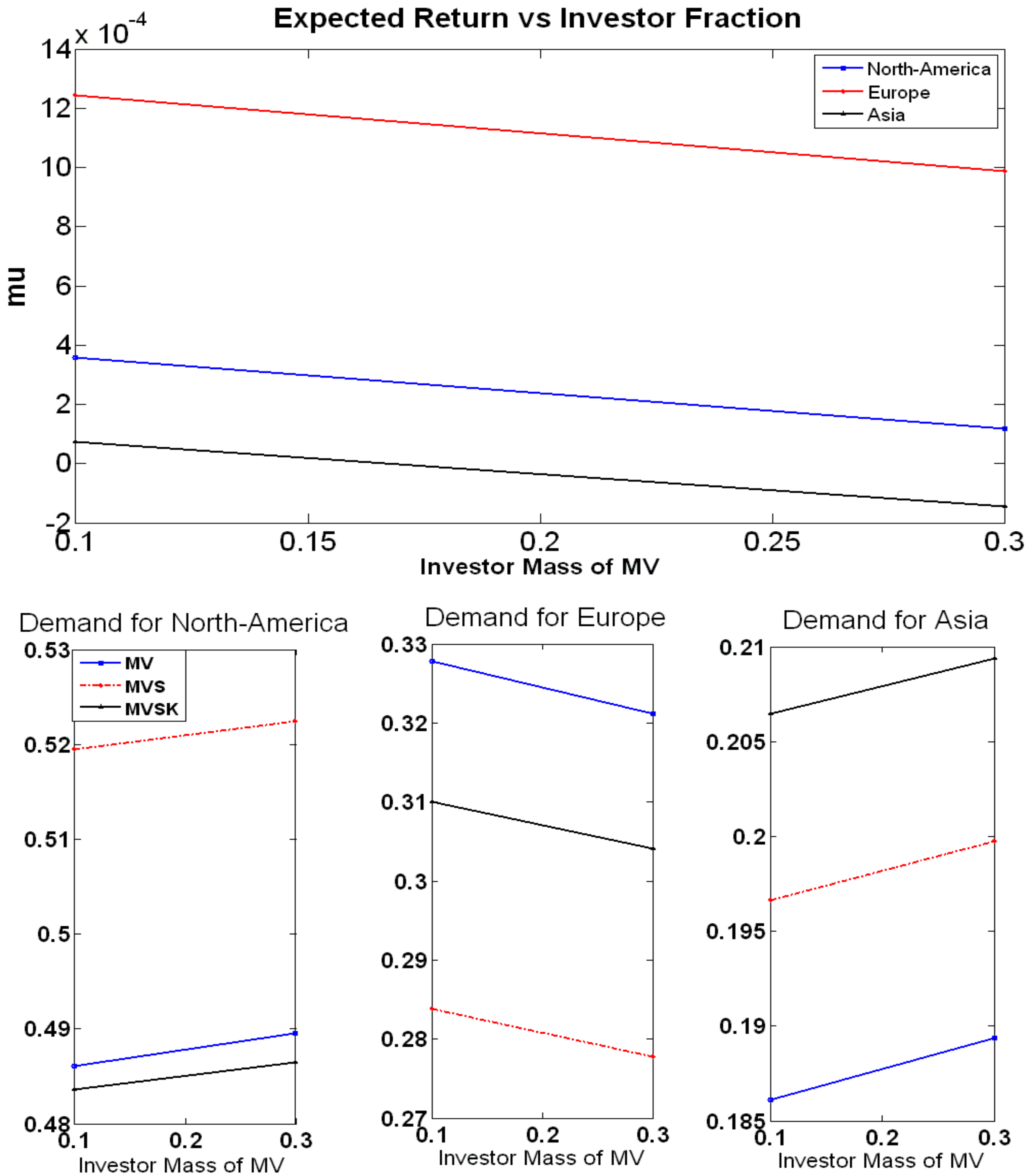


Figure 13: Return prediction and equilibrium allocation using " Housing boom data set"

Subprime crash $\lambda = 5$

Investor fraction: $\phi_T = [0.5\ 0.65\ 0.9], \phi_L = \phi_K = (1 - \phi_T)/2$ Traditional Investor Majority

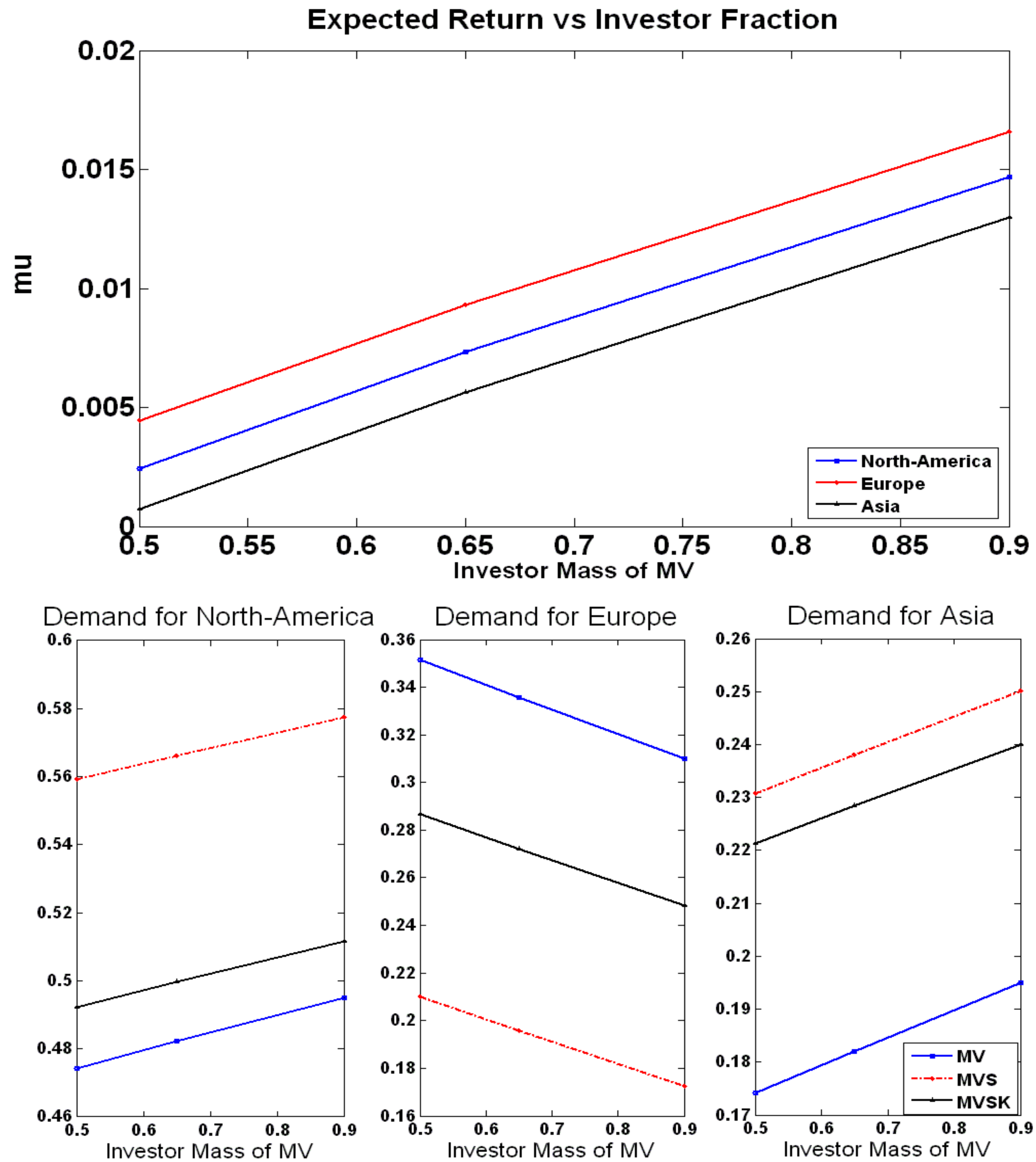


Figure 14: Return prediction and equilibrium allocation using "Subprime crash data set"

Data-set	midst (**NA EU AS**)	one year posterior(**NA EU AS**)
(*same order*)	model predicted return	real future return
dot-com boom	$\mu_{NA} > \mu_{EU} > \mu_{AS}$	
	0.00228 0.00073 0.00065	-0.0046 -0.0061 -0.0077
dot-com crash	$\mu_{NA} > \mu_{EU} > \mu_{AS}$	
	0.0012 0.00095 -0.00033	-0.00461 -0.00588 -0.00818
Housing boom	$\mu_{EU} > \mu_{NA} > \mu_{AS}$	
	0.000119 0.000986 -0.00015	0.000167 0.00101 -3.50796E-05
subprime crash	$\mu_{EU} > \mu_{NA} > \mu_{AS}$	
	0.00245 0.0045 0.00071	0.0047 0.0051 0.0037

Table 4: Compare model predicted return and real return during booms and crashes

The empirical results documented in table 4 imply that our model can make the correction prediction for the ranking of the future returns.

From figure 11 to 14 we can see that during financial booms, the more Traditional investors on the market, the smaller the expected returns will be. However, during crashes we get the opposite. The results can be interpreted in this way: during booms, Traditional investors who are lack of preference for skewness and kurtosis will be optimistic for the financial situation and require a lower risk compensation, i.e. expected return, for their investment. But during crisis, the financial stress makes the Lotto and Kurtosis aversion investors start to leave the market and their aversion to the current economic situation (low skewness and high kurtosis) drives up the expected return. The phase transition from a bull market phase to a bear market phase is considered as a stock market crash. The consistency between the predicted returns' ranking and the real posterior one implies that booms and crashes have their origin in the investment behavior of many heterogeneous investors[16].

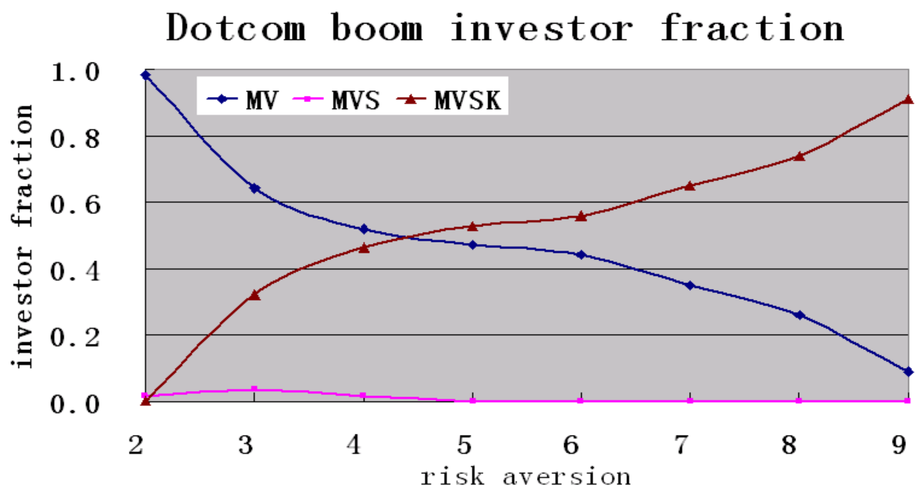
[16]Some literature related to behavior finance have studied the mapping between micro level of individual investor behavior and the macro level of aggregate market phenomena. For example,

4.2.3 Empirical Result: recover investor fraction

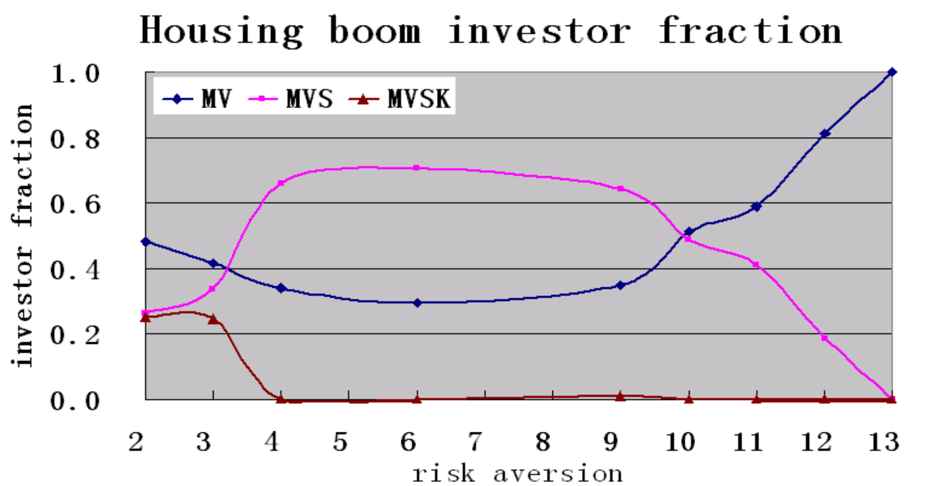
After fixing risk aversion at a reasonable range, we are able to recover the investor fraction for different economic situations, i.e. boom and crash. Figure 15: plot 1 and 3, show that when Dot-com bubble bursts, Kurtosis Aversion investors start to leave the market, and the Lotto investors begin to invest. This result implies that during dot-com boom, most of investors are aware of the high probability of gain (high skewness) and they are also concerned about the fat-tail risk due to their limited knowledge about the high-tech stocks. When the bubble bursts, they are so disappointed by the high kurtosis value (realized fat-tail risk) that they retreat their investment. Meanwhile, Lotto investors, who care only about the high potential to gain and compensated by the high expected return in the future, begin to invest more in the market.

Housing boom which is viewed as an economic bubble, has been claimed to be driven by several economic reasons such as Housing tax policy, lowered Mortgage interest rates etc. So it's not surprising for the majority of investors to choose ignoring the fat-tail risk in the return distribution. They start to invest into stock market pursuing a high skewness value in their portfolios. Therefore, figure 15: plot 2 shows that Lotto investors are the major player during this period. However, when their risk aversion increases, they might start to leave or turn into Traditional investor type, due to the aversion to the variance risk. This result is different from the case in section 4.2.2 where the risk aversion is much higher: 25.

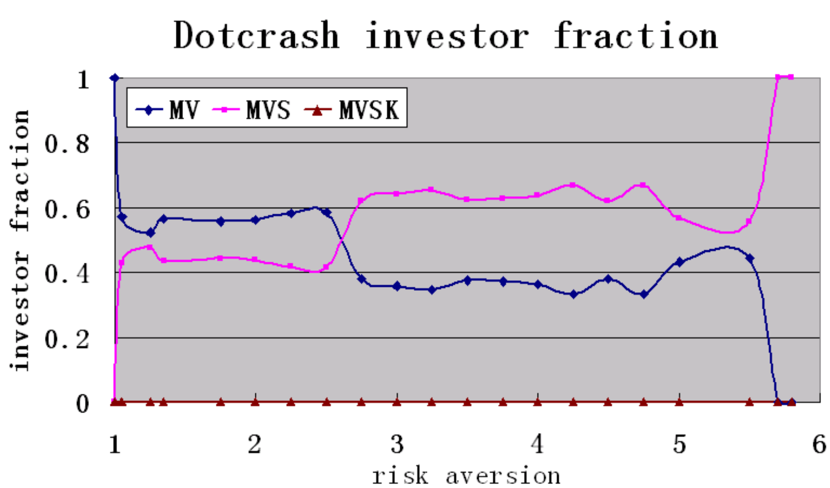
Kaizoji (2000) proposed an interacting-agent model of speculative activity in order to explain bubbles and crashes in stock markets.



λ =[2 3 4 5 6 7 8 9]



λ =[3 4 6 9 10 11 12]



λ =[1 5.8]

Figure 15: Estimated investor fraction during dot-com boom, housing boom and dot-

5 Conclusion

In this study, we address two main issues: first, we investigate how skewness and (co-) kurtosis preferences influence equilibrium properties: equilibrium asset return and asset allocation under different economic scenarios. We implement different order truncation for Taylor series approximation in order to represent heterogeneous investors on the market. To our knowledge, we are the first to introduce different investor fractions into this equilibrium model. In addition, we analyze several factors that can also have equilibrium impact, such as investor fraction and risk aversion.

Second, through our model's practical implementation using world MSCI index, we are able to recover the investor structure, i.e. investor fraction and risk aversion, on the market and generate the future equilibrium (expected) returns with the same ranking as the real ones'. Asset holdings predicted by our model are also consistent with the results from simulation. This empirical practise implies that this model can be used to forecast equilibrium asset return and, under certain specifications, the investor structure on the market can be also recovered.

There are several possible extensions can be made in the future. First, it will be interesting to build up a multi-period equilibrium model and consider the conditional asset allocation problem in a dynamic setting. Second, short-selling constraint for risky assets can be changed in order to investigate its effect on equilibrium. Third, in order to address the issue about regulation of financial institutions, it's also interesting to introduce institutional investors with Value-at-Risk (VaR) constraint into our model.

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6 Appendix

6.1 Appendix A

Define:

Covariance Structure:

$$V = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} & \dots & \sigma_{1N} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} & \dots & \sigma_{2N} \\ . & . & . & & \\ & & & \dots & \dots \\ . & . & . & & \\ \sigma_{N1} & \sigma_{N2} & \sigma_{N3} & \dots & \sigma_{NN} \end{bmatrix}$$

Skewness structure:

Element s_{ijk} is a triplet product moment of any asset 'i','j' and 'k', and can be used to construct the skewness structure S_i for asset i, where $i = [1, ..., N]$ (similar as Jondeau & Rockinger 2006). μ denotes the expected return.

$$s_{ijk} = E[(R_i - \mu_i)(R_j - \mu_j)(R_k - \mu_k)]$$

for each asset i, the corresponding skewness matrix S_i can be written

$$S_i = \begin{bmatrix} s_{i11} & s_{i12} & \cdots & s_{i1N} \\ s_{i21} & s_{i22} & \cdots & s_{i2N} \\ . & . & . & . \\ . & . & . & . \\ . & . & . & . \\ s_{iN1} & s_{iN2} & \cdots & s_{iNN} \end{bmatrix}$$

$$\cdots$$

$$S_N$$

Let S represent the N security portfolio's co-skewness matrix, it contains N assets' skewness matrix $S_{i(N\times N)}$. So S is a matrix of order $N \times N^2$. (Here subscript shows matrix dimension.)

$$S = [S_1 \quad S_2 \ ...S_i...S_N]$$

Kurtosis Structure:

Define each element in the kurtosis structure as following:

$$k_{ijkl} = E[(R_i - \mu_i)(R_j - \mu_j)(R_k - \mu_k)(R_l - \mu_l)]$$

For each asset i we can write its corresponding kurtosis structure: $K_{iikl(N\times N)}$, $K_{ijkl(N\times N)}$, $K_{ikkl(N\times N)}$, $K_{ilkl(N\times N)}$,

$$K_{iikl} = \begin{bmatrix} k_{ii11} & k_{i12} & \dots & k_{ii1N} \\ k_{ii21} & k_{ii22} & \dots & k_{ii2N} \\ . & . & . & . \\ . & . & . & . \\ . & . & . & . \\ k_{iiN1} & k_{iiN2} & \dots & k_{iiNN} \end{bmatrix}$$

$$\cdot\cdot$$

$$K_{jjkl} = \begin{bmatrix} k_{ij11} & k_{ij12} & \dots & k_{ij1N} \\ k_{ij21} & k_{ij22} & \dots & k_{ij2N} \\ . & . & . & . \\ . & . & . & . \\ . & . & . & . \\ k_{ijN1} & k_{ijN2} & \dots & k_{ijNN} \end{bmatrix}$$

$$\cdot\cdot$$

$$K_{NNkl}$$

For the 4 assets case, where $K_{11kl}(4 \times 4)_{k,l=1,2,3,4}$, we can write an $N \times N^3$ kurtosis matrix K for all the N risk assets.

$$K = [K_{11kl}\ K_{12kl}\ K_{13kl}\ K_{14kl}\,|K_{21kl}\ K_{22kl}\ K_{23kl}\ K_{24kl}|\ K_{31kl}\ K_{32kl}\ K_{33kl}\ K_{34kl}\,|K_{41kl}\ K_{42kl}\ K_{43kl}\ K_{44kl}|]$$

6.2 Appendix B

In order to show the derivation in a more refined way, some notation simplification has been made: for each type of investor, the notations L, T and K are dropped. The optimal allocation is denoted by the $N \times 1$ vector α.

Lotto Investor optimization problem

$$\begin{aligned}\arg\max_{\alpha'_{L,q}} U(W_{L,q}) &= -\exp(-\lambda_q \mu_{p,L,q}) \left[1 + \delta_{2q} \alpha'_{L,q} V \alpha_{T,q} - \delta_{3q} \alpha'_{L,q} S \left(\alpha_{L,q} \otimes \alpha_{L,q}\right)\right] \\ \text{s.t. } W_{L,q} &= W_{0,Lq}\left(1 + \alpha'_{L,q} R\right)\end{aligned}$$

$\otimes$ denotes the Kronecker product.

The FOCs for all the moment structure of the optimal portfolio return are

$$\begin{aligned}\frac{\partial \mu_p}{\partial \alpha'} &= \mu, \\ \frac{\partial V_p}{\partial \alpha'} &= 2V\alpha, \\ \frac{\partial S_p}{\partial \alpha'} &= 3S\left(\alpha \otimes \alpha\right),\end{aligned}$$

Consequently the Lotto demand function can be written as

$$\mu = \frac{2\delta_{2q} V\alpha - 3\delta_{3q} S\left(\alpha \otimes \alpha\right)}{1 + \delta_{2q} \alpha' V\alpha - \delta_{3q} \alpha' S\left(\alpha \otimes \alpha\right)}$$

A more detailed way to make the first differentiation for skewness used by Mitton and Vorkink is presented below. In order to obtain demand function in formula 2, the skewness can be expressed in the following formula (using the notation simplification):

$$S_p = \sum_{i=1}^{N} e_i' \alpha \alpha' S_i \alpha,$$

where e_i is a $n \times 1$ vector of zeros with a one in the ith element .

$$\frac{\partial S_p}{\partial \alpha'} = \frac{\partial \sum_{i=1}^{N} e_i' \alpha \alpha' S_i \alpha}{\partial \alpha'} = \frac{\partial \sum_{i=1}^{N} f(\alpha)' S_i g(\alpha)}{\partial \alpha'},$$

define $f(\alpha) = \alpha \alpha' e_i$ and $g(\alpha) = \alpha$. According to the chain rule,

$$\frac{\partial S_p}{\partial \alpha'} = \frac{\sum_{i=1}^{N} \partial f(\alpha)' S_i g(\alpha)}{\partial \alpha'}$$

using the chain rule for the matrix quadratic forms (Mangus and Neudecker 1999),

$$\frac{\partial f(\alpha)^{'}S_i g(\alpha)}{\partial \alpha^{'}} = g(\alpha)^{'}S_i^{'}\frac{\partial f(\alpha)}{\partial \alpha^{'}} + f(\alpha)^{'}S_i\frac{\partial g(\alpha)}{\partial \alpha^{'}}$$

where $\frac{\partial g(\alpha)}{\partial \alpha^{'}} = I_{NN}$, an identity matrix of order $N \times N$ which can also help us to get the following result:

$$\frac{\partial f(\alpha)}{\partial \alpha^{'}} = \frac{\partial \alpha\alpha^{'}e_i}{\partial \alpha^{'}} = \left(\alpha^{'}e_i\right) \otimes I_{NN} + \alpha e_i^{'}$$

Therefore, we could get

$$\frac{\partial e_i^{'}\alpha\alpha^{'}S_i\alpha}{\partial \alpha^{'}} = \alpha_{L,q}^{'}S_i\left[\left(\left(\alpha_{L,q}^{'}e_i\right) \otimes I_{NN}\right) + \alpha_{L,q}^{'}e_i^{'}\right] + e_i^{'}\alpha_{L,q}\alpha_{L,q}^{'}S_i$$

and we could finally get equation (2).

Kurtosis Aversion Investor optimization problem

The optimization problem for Kurtosis Aversion investors is

$$\begin{aligned} \arg\max_{\alpha_{K,q}^{'}} U(W_{K,q}) &= -\exp(-\lambda_q\mu_{p,K,q})\left[\begin{array}{c} 1+\delta_{2q}\alpha_{K,q}^{'}V\alpha_{K,q}-\delta_{3q}\alpha_{K,q}^{'}S\left(\alpha_{K,q}\otimes\alpha_{K,q}\right) \\ +\delta_{4q}\alpha_{K,q}^{'}K\left(\alpha_{K,q}\otimes\alpha_{K,q}\otimes\alpha_{K,q}\right) \end{array}\right] \\ \text{s.t. } W_{k,q} &= W_{0,kq}(1+\alpha_{k,q}^{'}R) \end{aligned}$$

the FOCs for all the moment structure of the optimal portfolio return are

$$\begin{aligned} \frac{\partial \mu_p}{\partial \alpha^{'}} &= \mu, \\ \frac{\partial V_p}{\partial \alpha^{'}} &= 2V\alpha, \\ \frac{\partial S_p}{\partial \alpha^{'}} &= 3S\left(\alpha \otimes \alpha\right), \\ \frac{\partial K_p}{\partial \alpha^{'}} &= 4K\left(\alpha \otimes \alpha \otimes \alpha\right), \end{aligned}$$

Consequently the Lotto demand function can be written as

$$\mu = \frac{2\delta_{2q}V\alpha - 3\delta_{3q}S\left(\alpha\otimes\alpha\right) + 4\delta_{4q}K\left(\alpha\otimes\alpha\otimes\alpha\right)}{1+\delta_{2q}\alpha^{'}V\alpha - \delta_{3q}\alpha^{'}S\left(\alpha\otimes\alpha\right) + \delta_{4q}\alpha^{'}K\left(\alpha\otimes\alpha\otimes\alpha\right)}$$

To write the optimal solution in a more detailed way, we additionally use the chain rule's application for the Kronecker product (Mangus and Neudecker 1999).

The kurtosis for the end-of-period wealth can be written as below (simplified notation as defined before):

$$K_p = \sum_{i=1}^{N} e_i^{'}\alpha\alpha^{'} K_i(\alpha \otimes \alpha),$$

where

$$\begin{aligned}
K_1 &= [K_{11kl}\ K_{12kl}\ K_{13kl}\ K_{14kl}...K_{1Nkl}],\\
K_2 &= [K_{21kl}\ K_{22kl}\ K_{23kl}\ K_{24kl}...K_{2Nkl}],\\
&.\\
K_i &= [K_{i1kl}\ K_{i2kl}\ K_{i3kl}\ K_{i4kl}...K_{iNkl}],\\
&.\\
K_N &= [K_{N1kl}\ K_{N2kl}\ K_{N3kl}\ K_{N4kl}...K_{NNkl}], k, l \in [1, ..N]
\end{aligned}$$

K_i is a $N \times N^2$ matrix which can be used to calculate K_p.

$$\frac{\partial K_p}{\partial \alpha^{'}} = \sum_{i=1}^{N} e_i^{'}\alpha\alpha^{'} K_i(\alpha \otimes \alpha) = \sum_{i=1}^{N} (\alpha \otimes \alpha)^{'} K_i^{'} \left[\left(\alpha^{'} e_i\right) \otimes I_{NN} + \alpha e_i^{'}\right] + e_i^{'}\alpha\alpha^{'} K_i \left[I_{N^2N^2}(I_{NN} \otimes \alpha)\right]$$

Define $f(\alpha) = \alpha\alpha^{'} e_i$ and $h(\alpha) = (\alpha \otimes \alpha)$, and according to the chain rule, we can write

$$\frac{\partial K_p}{\partial \alpha^{'}} = \frac{\sum_{i=1}^{N} \partial f(\alpha)^{'} K_i h(\alpha)}{\partial \alpha^{'}} = \frac{\partial \sum_{i=1}^{N} e_i^{'}\alpha\alpha^{'} K_i(\alpha \otimes \alpha)}{\partial \alpha^{'}} = \frac{\partial \sum_{i=1}^{N} f(\alpha)^{'} S_i h(\alpha)}{\partial \alpha^{'}},$$

Then using the chain rule for the matrix quadratic forms (Mangus and Neudecker 1999),

$$\frac{\partial f(\alpha)^{'} K_i h(\alpha)}{\partial \alpha^{'}} = h(\alpha)^{'} K_i^{'} \frac{\partial f(\alpha)}{\partial \alpha^{'}} + f(\alpha)^{'} K_i \frac{\partial h(\alpha)}{\partial \alpha^{'}}$$

given

$$\frac{\partial f(\alpha)}{\partial \alpha^{'}} = \frac{\partial \alpha\alpha^{'} e_i}{\partial \alpha^{'}} = \left(\alpha^{'} e_i\right) \otimes I_{NN} + \alpha e_i^{'}$$

which has order of $N \times N$. Then make differentiation for $h(\alpha)$

$$\frac{\partial h(\alpha)}{\partial \alpha'} = \frac{\partial (\alpha \otimes \alpha)}{\partial \alpha'} = (I_{NN} \otimes I_{NN})(I_{NN} \otimes \alpha) = I_{N^2N^2}(I_{NN} \otimes \alpha)$$

Then we could write

$$\frac{\partial f(\alpha)' K_i h(\alpha)}{\partial \alpha'} = (\alpha \otimes \alpha)' K_i' \left[\left(\alpha' e_i \right) \otimes I_{NN} + \alpha e_i' \right] + e_i' \alpha \alpha' K_i \left[I_{N^2N^2}(I_{NN} \otimes \alpha) \right]$$

and get

$$\frac{\partial K_p}{\partial \alpha'} = \frac{\sum_{i=1}^{N} \partial f(\alpha)' K_i h(\alpha)}{\partial \alpha'} = \sum_{i=1}^{N} (\alpha \otimes \alpha)' K_i' \left[\left(\alpha' e_i \right) \otimes I_{NN} + \alpha e_i' \right] + e_i' \alpha \alpha' K_i \left[I_{N^2N^2}(I_{NN} \otimes \alpha) \right]$$

which can be used to generate the demand function for the Kurtosis Aversion investor shown in formula 3. Since these FOCs from "Kurtosis Aversion" have order of $1 \times N$, in order to integrate them into the final optimization condition, a simple transformation can be made as below. After specifying the notation for Kurtosis Aversion investor we could obtain her demand function.

$$\left[\frac{\partial K_p}{\partial \alpha'} \right]' = \sum_{i=1}^{N} \left\{ (\alpha \otimes \alpha)' K_i' \left[\left(\alpha' e_i \right) \otimes I_{NN} + \alpha e_i' \right] + e_i' \alpha \alpha' K_i \left[I_{N^2N^2}(I_{NN} \otimes \alpha) \right] \right\}'$$

6.3 Appendix C

Detailed information about simulated series

	r_1	r_2	r_3	
Panel A: Univariate statistics				
Moments				
Mean	7.8920×10^{-4}	-9.0796×10^{-4}	-1.0888×10^{-3}	
Variance	0.20004	0.34925	0.24957	
3rd-moments (assumed)	0	0	0	
4th-moments	0.12019	0.36580	0.18663	
Panel B: Multivariate statistics				
Correlation	stat.	stat.	stat.	stat.
	x_1	x_2	x_3	
x_1	1	0.080515	0.14945	
x_2	0.080515	1	0.10033	
x_3	0.14945	0.10033	1	
Co-3rd moments (assumed)	x_1^2	x_2^2	x_3^2	
$x_1(x_1, x_2)$	0	0	0	
Co-4th moments				
	x_1^3	x_2^3	x_3^3	
x_1	0.12019	0.022046	0.024876	
x_2	0.012953	0.36580	0.021909	
x_3	0.020096	0.030996	0.18663	
	$x_1x_2^2$	$x_1x_3^2$	$x_2x_3^2$	$x_1^2x_2$
x_1	0.070861	0.052071		
x_2		0.0072306	0.088842	
x_3	0.012738			0.0073208

Table C1: Statistics on the 1st simulation

	r_1	r_2	r_3	
	Panel A: Univariate statistics			
Moments				
Mean	7.8920×10^{-4}	-9.0796×10^{-4}	-5.9131×10^{-4}	
Variance	0.20004	0.34925	0.38064	
3rd-moments (assumed)	0	0	0	
4th-moments	0.12019	0.36580	1.0227	
	Panel B: Multivariate statistics			
Correlation	stat.	stat.	stat.	stat.
	x_1	x_2	x_3	
x_1	1	0.080515	0.12025	
x_2	0.080515	1	0.82673	
x_3	0.12025	0.82673	1	
Co-3rd moments (assumed)	x_1^2	x_2^2	x_3^2	x_1x_2
$x_1(x_1, x_2)$	0	0	0	
Co-4th moments				
	x_1^3	x_2^3	x_3^3	
x_1	0.12019	0.022046	0.024876	
x_2	0.012953	0.36580	0.021909	
x_3	0.019780	0.031476	1.0227	
	$x_1x_2^2$	$x_1x_3^2$	$x_2x_3^2$	$x_1^2x_2$
x_1	0.070861	0.078799		
x_2		0.010237	0.13438	
x_3	0.012889			0.0074985

Table C2: Statistics on the last simulation

Detailed information about real data series				
T=992	North-America	Europe	Asia	
		Panel A: Univariate statistics		
Moments				
Mean	0.0017528	0.0016640	2.3917×10^{-4}	
Std* (annual)	0.14681	0.15176	0.20348	
Skewness	-0.45093	-0.16671	-0.12094	
Kurtosis	5.9414	5.2438	4.2106	
		Panel B: Multivariate statistics		
Correlation	stat.	stat.	stat.	stat.
	x_1	x_2	x_3	
x_1	1	0.61457	0.30336	
x_2	0.61457	1	0.48902	
x_3	0.30336	0.48902	1	
Co-skewness	x_1^2	x_2^2	x_3^2	x_1x_2
x_1	-0.45093	-0.11409	-0.066927	
x_2	-0.23906	-0.16671	-0.050247	
x_3	-0.072422	-0.1257	-0.12094	-0.087478
Co-kurtosis				
	x_1^3	x_2^3	x_3^3	
x_1	5.9414	3.4153	0.90725	
x_2	3.6890	5.2438	1.8281	
x_3	1.4311	2.1131	4.2106	
	$x_1x_2^2$	$x_1x_3^2$	$x_2x_3^2$	$x_1^2x_2$
x_1	3.2901	1.3423		
x_2		1.0039	1.8345	
x_3	1.2526			1.2553

Table C3 Statistics on weekly MSCI returns – pre-subprime data set: Jan. 1988-Dec. 2006

T=1151	North-America	Europe	Asia	
	Panel A: Univariate statistics			
Moments				
Mean	0.0013164	0.0011725	6.7990×10^{-5}	
Std* (annual)	0.1685	0.1859	0.20871	
Skewness	-0.82876	-1.2209	-0.094939	
Kurtosis	10.965	15.889	4.9265	
	Panel B: Multivariate statistics			
Correlation	stat.	stat.	stat.	stat.
	x_1	x_2	x_3	
x_1	1	0.71454	0.40055	
x_2	0.71454	1	0.55557	
x_3	0.40055	0.55557	1	
Co-skewness	x_1^2	x_2^2	x_3^2	x_1x_2
x_1	-0.82876	-0.97686	-0.34921	
x_2	-0.85543	-1.2209	-0.39295	
x_3	-0.49875	-0.73011	-0.094939	-0.60732
Co-kurtosis				
	x_1^3	x_2^3	x_3^3	
x_1	10.965	12.912	2.7650	
x_2	10.484	15.889	3.7075	
x_3	5.6406	8.5490	4.9265	
	$x_1x_2^2$	$x_1x_3^2$	$x_2x_3^2$	$x_1^2x_2$
x_1	11.363	3.9816		
x_2		4.1619	5.2932	
x_3	6.9123			6.0965

Table C4: Statistics on weekly MSCI returns – whole data set