A dynamic limit order market with fast and slow traders*

[Preliminary Draft]

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Abstract

We extend Foucault's (1999) dynamic limit order market by allowing for heterogeneity in traders' ability to revise limit orders after the arrival if new information. Fast traders' limit orders do not risk being picked off when interaction with slow traders, resembling high-frequency traders speed advantage over human market participants. Depending on the magnitude of the winner's curse, this heterogeneity may increase or decrease trading volume and therefore social welfare. Overall, the presence of of fast traders leads to a welfare loss for slow traders unless it triggers a move from a low-trade to a high-trade equilibrium.

1 Introduction

While the proverb "time is money" applies to virtually all economic activities, the accelerated proliferation of electronic trading has taken this wisdom to the extreme. High-frequency trading (HFT), a variant of algorithmic trading, relies on computer programs to generate a vast amount of orders in very small time intervals. The propagation of such trading strategies has spurred an expensive arms race within the financial industry, where propietary trading desks, hedge funds and so-called pure-play HFT outlets invest large amounts of money in human (IT experts, mathematicians, linguists, etc.) and technological (co-location, data feeds and warehouses, etc.) resources in an effort to outpace the competition. Recent estimates suggest that HFTs are now responsible for more than 50% of trading in U.S. equities.[1]

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[1]See Financial Times, "High-frequency boom time hits slowdown", April 12, 2011.

This ongoing "race to zero"[2] is being accompanied by a heated debate among financial economists, practitioners, and regulators about the implications of an increasing computerization of the trading process. While proponents[3] argue that technology increases market efficiency via improved liquidity and price discovery, critiques[4] claim that HFTs make profits at the expense of other (slow) market participants and have the potential to destabilize markets.

This paper contributes to this debate by presenting a model of trading in a limit order market where agents differ in the speed with which they can react to the arrival of new information, which is thought to capture the difference between (fast) HFTs and (slow) human market participants. We build on the model of Foucault (1999), in which limit orders face the winner's curse because they cannot be cancelled once submitted and thus may become stale once new information hits the market. In this world, a high level of fundamental volatility leads to order shading and reduces the gains from trade. We provide a natural extension of this framework by assuming that fast traders (FTs) may revise their limit orders upon news arrivals and therefore avoid being "picked off" (Copeland and Galai (1983)), but only if the next agent is a slow trader (ST). In this stylized environment, a world with only FTs is identical to a world with only STs because speed only matters in relative terms.

We analyze the stationary equilibrium of this dynamic limit order market and compare it with the baseline case of identical traders studied by Foucault (1999). Our findings are as follows. In equilibrium, trader heterogeneity has two opposing effects on trading volume and hence social welfare. On the one hand, FTs' limit orders face a reduced winner's curse, which diminishes order shading and therefore allows more gains from trade to be exploited. On the other hand, STs may increase order shading in equilibrium as a response to FTs' improved outside option (the expected profits from limit orders are higher for FTs than for STs). For a high (low) level of fundamental volatility, the first effect dominates (is absent) and trading volume increases (decreases). We also analyze a measure of trading costs, and find that the presence of FTs can lead both to a decrease or an increase in the average expected trading costs. Importantly, their speed advantage allows FTs implicitly to price discriminate between FTs and STs, which implies that the latter incur higher trading costs because they have a lower endogenous reservation price. Hence, changes in trading costs for STs are entirely determined by changes in the aggressiveness of other STs' limit orders. Therefore, we find that FTs' presence leads to a welfare loss for STs except in the case where it triggers a move from a low-trade to a high-trade equilibrium.

Our work is closely related a number of recent papers that study the impact of HFT on market

[2]See the speech by Andrew Haldane in front of the IEA 16th World Congress, July 8th, 2011

[3]See e.g. Optiver, "High Frequency Trading", Position Paper, 2011, http://fragmentation.fidessa.com/wp-content/uploads/High-Frequency-Trading-Optiver-Position-Paper.pdf

[4]See SEC Chairman Mary Schapiro's speech in front of the Security Traders Association "Remarks Before the Security Traders Association", www.sec.gov/news/speech/2010/spch092210mls.htm.

quality and investor welfare. Biais and Woolley (2011) provide a comprehensive overview of recent findings. Closely related to this paper, Jovanovic and Menkveld (2011) study HFT as competitive middlemen that intermediate between early limit order traders and late market order traders. While in both this and their paper the presence of HFT may reduce order shading by updating quotes quickly, they offer another explanation on why HFT may also reduce trade (unawareness of information by STs when submitting market orders). A calibration exercise reveals a slight increase in welfare. In Biais et al. (2011), HFT helps to reap gains from trade by facilitating the search for trading opportunities, but at the same time increases adverse selection for STs. Because of this negative externality, the equilibrium investment in HFT is above its social optimum. They also show that fixed costs in HFT imply that large institutions are more likely to be informed. Finally, Cartea and Penalva (2011) propose a model where their increased speed allows HFT to impose a haircut on liquidity traders, which increases trading volume and price volatility, but lowers the welfare of liquidity traders.

There are several empirical studies of algorithmic and high frequency trading. In summary, this stream of the literature concludes that automated trading strategies improves liquidity (Hendershott et al. (2011), Hasbrouck and Saar (2011)), is highly profitable (e.g. Brogaard (2010) and Menkveld (2011)), and significantly contributes to price discovery (e.g. Hendershott and Riordan (2011a, 2011b)). Consistent with our results on trading costs, Hendershott and Riordan (2011a) find that automated traders "supply liquidity when it is expensive and consume liquidity when it is cheap". Chaboud et al. (2009)) study computer- and human-generated traders in the FX market and conclude that the trading strategies by automated traders are more correlated among each other than those of human market participants (see also Brogaard (2010)). Kirilenko et al. (2011) study the recent "flash crash" in U.S. equity markets and find that HFT may have exacerbated volatility during this brief liquidity crisis, although they are not to blame for the crash itself. In contrast, the results in Brogaard (2011) do not confirm the concerns that HFT activity leads to an increase in volatility. Finally, Moallemi and Sağlam (2011) estimate the cost of latency and find a dramatic increase between 1995 and 2005.

This paper is organized as follows. Section 2 provides the setup of the model, while we solve for the equilibrium in Section 3. Section 4 analyzes the order flow composition and trading costs, and welfare is examined in Section 5. Section 6 concludes. All proofs, tables and figures are relegated to the Appendix.

2 The model

2.1 The limit order market

We consider an infinite-horizon[5] version of Foucault's (1999) dynamic limit order market. There is a single risky asset whose fundamental value follows a random walk, i.e.

$$v_{t+1} = v_t + \varepsilon_{t+1}$$

where the innovations can take values of $+\sigma$ and $-\sigma$ with equal probability and are independent over time. Trading takes place sequentially at time points $t = 1, 2, \ldots$ and the order size is fixed at one unit. In this model, trading arises due to differences in private values. Specifically, we assume that at time t', the reservation price of a trader arriving at time $t \leq t'$ is given by

$$R_{t'} = v_{t'} + y_t$$

which is the sum of the asset's fundamental value and the time-invariant private valuation y_t. We assume that this private valuation can take two values $y_h = +L$ and $y_l = -L$ with equal probability, where $L > 0$. The y's are independent and identically distributed across traders, and moreover independent from the asset value innovations. All traders are risk-neutral and maximize their expected utility. The utility obtained by an agent purchasing or selling the asset is given by

$$U(y_t) = (v_{t'} + y_t - P_{t'})q_{t'}$$

where $t' \geq t$ denotes the time of the transaction, $P_{t'}$ is the transaction price, and $q_{t'}$ is a trade direction indicator that takes the value of $+1$ for buy transactions and -1 for sell transactions. The utility of an agent that does not trade is assumed to be equal to zero.

Trading is organized as a limit order market. Consider a buyer (i.e. an agent with private valuation y_h). Upon his arrival, he can either a) submit a market buy order or b) submit a buy limit order for one unit of the asset.[6] We assume that he decides to submit a limit order if he is indifferent between both choices. Similarly, sellers choose between market and limit sell orders. All limit orders are valid for one period, i.e. they expire unless being executed by the following agent.

[5] This assumption is merely for convenience, as it simplifies the algebra and in particular the calculation of trader welfare in Section 4. Foucault (1999) derives a stationary equilibrium by assuming that the terminal date is stochastic, as the trading process stops after each period with constant probability $1 - \rho > 0$. While assuming an infinite horizon implies that the asset never pays off, it can be readily interpreted as the limiting case where $\rho \to 1$.

[6] Foucault (1999) assumes that traders always submit a buy and sell limit order, which is without loss of generality as limit prices can always be chosen such that limit orders have a zero execution probability. In fact, in equilibrium, the ask (bid) quotes of buyers (sellers) are never executed, such that we directly assume that buyers (sellers) only submit buy (sell) limit orders.

Besides their private valuations, agents also differ in their trading technology, denoted θ_t. They can either be fast traders (FTs, $\theta_t = 1$) or slow traders (STs, $\theta_t = 0$). Let $\alpha \in [0, 1]$ denote the probability that an agent is a FT. Again, we assume that θ is identically distributed across traders and independent of y and ε. We call θ_t a trader's type.

Unless limit orders are monitored perfectly (Foucault, Röell and Sandås (2003) and Liu (2009) study the cost of monitoring limit orders), the arrival of new information may render them stale and thereby grant a free option to other market participants (Copeland and Galai (1983)). Clearly, the ability to react faster than others can greatly reduce this risk of being "picked off" and explains the ongoing "race to zero" within the financial industry. In order to model this difference between FTs and STs in the most parsimonious way we assume that FTs can revise (or update) their limit orders after the arrival of new information (i.e. the realization of ε_{t+1}) yet before the arrival of the next agent, but only provided he is a ST. If the next trader is a FT as well, the order cannot be revised. STs can never cancel their orders. Notice that for both $\alpha = 0$ and $\alpha = 1$ our model collapses to the model of Foucault (1999), where limit orders can never be revised once they are submitted. This implies that being fast is only an advantage as long as there is someone else that is slow, which is a natural assumption given our focus on the winner's curse problem in a limit order market.

Let $s_t = (B_t^m, A_t^m)$ denote the best bid and ask quote in the market. If there is no bid (ask) quote posted, we write $B_t^m = -\infty$ ($A_t^m = \infty$). Upon entering the market, a trader learns his type θ_t and private valuation y_t, and observes the state of the limit order book s_t as well as the current fundamental value of the asset v_t. Call $S_t = (s_t, v_t)$ the state of the market.

2.2 Payoffs

Consider a buyer that arrives at time t when the state of the market is S_t. If he chooses to submit a buy market order (which executes at the best available ask price), his payoff is equal to

$$U_{t,k}^{b,MO}(A_t^m) = v_t + L - A_t^m \qquad k \in \{ST, FT\} \tag{1}$$

Instead, he can choose to submit a buy limit order. The expected payoff of a slow buyer submitting a buy limit order with bid price $B_{t,ST}$ is given by

$$E(U_{t,ST}^{b,LO}(B_{t,ST})) = \eta_t^b(B_{t,ST}) E_{Ex}(v_{t+1} + L - B_{t,ST})$$

where $\eta_t^b(B_{t,ST})$ denotes the execution probability of a buy limit order with bid price $B_{t,ST}$ and $E_{Ex}(\cdot)$ is an expectation conditional on the execution of the respective limit order. Given that a FT may revise his limit order in case the next arriving trader is a ST, a fast buyer that decides to use a limit order chooses three different bid prices, $(B_{t,FT}, B_{t,FT}^{-\sigma}, B_{t,FT}^{+\sigma})$, and his payoff is given by

$$E(U_{t,FT}^{b,LO}(B_{t,FT},B_{t,FT}^{-\sigma},B_{t,FT}^{+\sigma}) = \alpha\eta_t^b(B_{t,FT}|\theta_{t+1}=1)E_{Ex}(v_{t+1}+L-B_{t,FT})$$
$$+(1-\alpha)[\frac{1}{2}\eta_t^b(B_{t,FT}^{+\sigma}|\theta_{t+1}=0,\varepsilon_{t+1}=+\sigma)(v_t+\sigma+L-B_{t,FT}^{+\sigma})$$
$$+\frac{1}{2}\eta_t^b(B_{t,FT}^{+\sigma}|\theta_{t+1}=0,\varepsilon_{t+1}=-\sigma)(v_t-\sigma+L-B_{t,FT}^{-\sigma})] \quad (2)$$

where the $\eta_t^B(\cdot|\theta_{t+1},\varepsilon_{t+1})$ denote execution probabilities conditional on the realization of next period's trader type and asset value innovation. Similarly, a seller submitting a sell market order obtains

$$U_{t,k}^{s,MO}(B_t^m) = B_t^m - (v_t - L) \qquad k \in \{ST, FT\} \quad (3)$$

while the expected payoffs for STs and FTs from posting sell limit orders with ask prices equal to $A_{t,ST}$ and $(A_{t,FT}, A_{t,FT}^{-\sigma}, A_{t,FT}^{+\sigma})$, respectively, are given by

$$E(U_{t,ST}^{s,LO}(A_{t,ST})) = \eta_t^s(A_{t,ST})E_{Ex}(A_{t,ST}-(v_{t+1}-L))$$

and

$$E(U_{t,FT}^{s,LO}(A_{t,FT},A_{t,FT}^{-\sigma},A_{t,FT}^{+\sigma}) = \alpha\eta_t^s(A_{t,FT}|\theta_{t+1}=1)E_{Ex}(A_{t,FT}-(v_{t+1}-L))$$
$$+(1-\alpha)[\frac{1}{2}\eta_t^s(A_{t,FT}^{+\sigma}|\theta_{t+1}=0,\varepsilon_{t+1}=+\sigma)(A_{t,FT}^{+\sigma}-(v_t+\sigma-L))$$
$$+\frac{1}{2}\eta_t^S(A_{t,FT}^{+\sigma}|\theta_{t+1}=0,\varepsilon_{t+1}=-\sigma)(A_{t,FT}^{-\sigma}-(v_t-\sigma-L))] \quad (4)$$

2.3 Equilibrium Definition

Let $B_{t,ST}^*$ denote the optimal bid price chosen by a slow buyer that decides to to place at limit order at time t. Thus, upon arrival, a slow buyer chooses between a) a buy market order at ask price A_t^m and b) a buy limit order with bid price $B_{t,ST}^*$. We call his choice the slow buyer's order placement strategy $O_{ST}^b(S_t) \in \{b_t^m, B_{t,ST}^*\}$ where b_t^m denotes a market buy order at time t. Similarly, let $(B_{t,FT}^*, B_{t,FT}^{-\sigma*}, B_{t,FT}^{+\sigma*})$ be the optimal bid prices for a fast buyer that opts for limit orders when arriving at time t. He then chooses between a) a buy market order at ask price A_t^m and b) a buy limit order with bid price $B_{t,FT}^*$, which, unless the next agent is a FT, is revised to $B_{t,AT}^{+\sigma*}$ ($B_{t,AT}^{-\sigma*}$) after the arrival of positive (negative) fundamental information. Hence, $O_{FT}^b(S_t) \in \left\{b_t^m, (B_{t,FT}^*, B_{t,FT}^{-\sigma*}, B_{t,FT}^{+\sigma*})\right\}$. The choices of slow and fast sellers are completely symmetric, i.e. they choose between a) a market sell at B_t^m and b) limit sell orders with ask prices equal to $A_{t,ST}^*$ and $(A_{t,FT}^*, A_{t,FT}^{-\sigma*}, A_{t,FT}^{+\sigma*})$, respectively, such that their order placement strategies are $O_{ST}^s(S_t) \in \{s_t^m, A_{t,ST}^*\}$ and $O_{FT}^s(S_t) \in \left\{s_t^m, (A_{t,FT}^*, A_{t,FT}^{-\sigma*}, A_{t,FT}^{+\sigma*})\right\}$, where s_t^m denotes a market sell. As in

Foucault (1999) and Colliard and Foucault (2011), we focus on stationary Markov-perfect equilibria, which is natural because trader's profits do not depend on the history of the game but only on the state of the market upon their arrival.

Definition 1 *A Markov-perfect equilibrium of the limit order market consists of order placement strategies* $O^{b*}_{ST}(\cdot)$, $O^{s*}_{ST}(\cdot)$, $O^{b*}_{FT}(\cdot)$ *and* $O^{s*}_{FT}(\cdot)$ *such that, for each possible state of the market* S_t, *i)* $O^{b*}_{ST}(S_t)$ $(O^{b*}_{FT}(S_t))$ *maximizes the expected utility of a slow (fast) buyer arriving in state* S_t *if all other traders follow the strategies* $O^{b*}_{ST}(\cdot)$, $O^{s*}_{ST}(\cdot)$, $O^{b*}_{FT}(\cdot)$ *and* $O^{s*}_{FT}(\cdot)$ *and ii)* $O^{s*}_{ST}(S_t)$ $(O^{s*}_{FT}(S_t))$ *maximizes the expected utility of a slow (fast) seller arriving in state* S_t *if all other traders follow the strategies* $O^{b*}_{ST}(\cdot)$, $O^{s*}_{ST}(\cdot)$, $O^{b*}_{FT}(\cdot)$ *and* $O^{s*}_{FT}(\cdot)$.

Foucault (1999) shows that it is possible to characterize traders' optimal decisions by means of cutoff prices that depend on a trader's private valuation and the current fundamental value of the asset. The buy (sell) cutoff price is the highest (lowest) ask (bid) price at which an arriving buyer (seller) submits a market buy (sell) order instead of a buy (sell) limit order. Let $V^{LO*}_{ST}(y_t)$ and $V^{LO*}_{FT}(y_t)$ denote equilibrium expected profits from posting limit orders for STs and FTs, respectively, that is

$$V^{LO*}_{ST}(y_t) \equiv \begin{cases} E(U^{b,LO}_{t,ST}(B^*_{t,ST})) & if \quad y_t = y_h \\ E(U^{s,LO}_{t,ST}(A^*_{t,ST})) & if \quad y_t = y_l \end{cases}$$

$$V^{LO*}_{FT}(y_t) \equiv \begin{cases} E(U^{b,LO}_{t,FT}(B^*_{t,FT}, B^{-\sigma*}_{t,FT}, B^{+\sigma*}_{t,FT})) & if \quad y_t = y_h \\ E(U^{s,LO}_{t,FT}(A^*_{t,FT}, A^{-\sigma*}_{t,FT}, A^{+\sigma*}_{t,FT})) & if \quad y_t = y_l \end{cases}$$

Then, the equilibrium buy and sell cutoff prices are given by

$$C^{s*}_{ST}(v_t, y_t) - (v_t + y_t) = V^{LO*}_{ST}(y_t) \tag{5}$$

$$C^{s*}_{FT}(v_t, y_t) - (v_t + y_t) = V^{LO*}_{FT}(y_t) \tag{6}$$

$$(v_t + y_t) - C^{b*}_{ST}(v_t, y_t) = V^{LO*}_{ST}(y_t) \tag{7}$$

$$(v_t + y_t) - C^{b*}_{FT}(v_t, y_t) = V^{LO*}_{FT}(y_t) \tag{8}$$

Intuitively, the expected profits from submitting limit orders constitute an endogenous outside option for the arriving trader. Therefore, a buyer (seller) will only submit a market buy (sell) order when the best available ask (bid) price is below (above) his cutoff buy (sell) price. The above system of equations can be solved for the equilibrium cutoff prices, which in turn give rise to traders' equilibrium quotation strategy.

3 Equilibrium

Consider a buyer, i.e. a trader with private valuation y_h (symmetric arguments apply to sellers). It is easy to see that it is never optimal for a buyer to target another buyer, i.e. to submit a buy limit order that would be executed by a trader with private valuation y_h in the following period, because both agents will value the asset at $v_{t+1}+L$ and there are no gains from trade to be shared. Moreover, in understanding the construction of the equilibrium, it is crucial to notice that any optimal bid price must be such that it is marginally above some seller's cutoff price for a particular realization of the asset value innovation ε_{t+1}. Clearly, a slight increase in the bid price does not lead to a higher execution probability, while a small decrease in the bid price leads to a strictly lower execution probability. In the following, we will abuse notation by equating equilibrium quotes to cutoff prices as in Foucault (1999), because they can be made arbitrarily close.

Lemma 1 *In equilibrium, FTs' revised quotes are given by*

$$B_{t,FT}^{-\sigma*} = C_{ST}^{s*}(v_t - \sigma, -L) \qquad B_{t,FT}^{+\sigma*} = C_{ST}^{s*}(v_t + \sigma, -L)$$
$$A_{t,FT}^{-\sigma*} = C_{ST}^{b*}(v_t - \sigma, +L) \qquad A_{t,FT}^{+\sigma*} = C_{ST}^{b*}(v_t + \sigma, +L)$$

Proof. See the Appendix. ■

Due to their speed advantage, FTs may re-price their limit orders after observing the change in the fundamental value in case the next arriving agent is a ST. In essence, this enables FTs to discriminate between FTs and STs, because the former face the initial quotes while the latter face the updated quotes. Clearly, the updated limit prices incorporate both the innovation ε_{t+1} and the knowledge that the next agent is a ST. Then the optimally revised bid (ask) price is just equal to the cutoff sell (buy) price of a slow seller (buyer) given the fundamental asset value v_{t+1}. Consequently, FTs' decision boils down to choosing between initial quotes (which are only aimed at FTs) with a high fill rate and a low fill rate.[7] One the other hand, STs may employ four different quotation strategies in equilibrium, which reflects that their quotes always face both FTs and STs.

Lemma 2 *Let* $\alpha \in (0,1)$. *Then*

a) $C_{FT}^{s*}(v_t, -L) > C_{ST}^{s*}(v_t, -L)$ *and* $C_{FT}^{b*}(v_t, +L) < C_{ST}^{b*}(v_t, +L)$

b) $C_{ST}^{s*}(v_t, +L) > C_{FT}^{s*}(v_t, -L)$ *and* $C_{ST}^{b*}(v_t, -L) < C_{FT}^{b*}(v_t, +L)$

c) $C_{ST}^{s*}(v_t + \sigma, -L) > C_{FT}^{s*}(v_t - \sigma, -L)$ *and* $C_{ST}^{b*}(v_t - \sigma, +L) < C_{FT}^{b*}(v_t + \sigma, +L)$

[7] A fast buyer may either post a bid price equal to $C_{FT}^{s*}(v_t - \sigma, -L)$, which only executes after a decrease in the asset value (low fill rate), or a bid price equal to $C_{FT}^{s*}(v_t + \sigma, -L)$, which may execute both after an increase or a decrease of the asset value (high fill rate).

Proof. See the Appendix. ■

Unsurprisingly, the endogenous outside option of posting limit orders is more valuable for FTs than for STs, as the former face a lower risk of being picked off. This implies that fast sellers (buyers) have a higher (lower) sell (buy) cutoff price. Parts b) and c) of the Lemma state that there this advantage is naturally limited both by the gains from trade and the risk of being picked off. We are now ready to determine the equilibrium quotation strategies.

Proposition 1 *For fixed parameters (α, σ, L), there exists a unique Markov-perfect equilibrium. The type of equilibrium is as follows.*

Type 1: If $\alpha \leq \alpha_1^*$ and $\sigma \geq \sigma_1^*$, then
$B_{t,FT}^* = C_{FT}^{s*}(v_t - \sigma, -L)$ $\quad$ $B_{t,ST}^* = C_{ST}^{s*}(v_t - \sigma, -L)$
$A_{t,FT}^* = C_{FT}^{b*}(v_t + \sigma, +L)$ $\quad$ $A_{t,ST}^* = C_{ST}^{b*}(v_t + \sigma, +L)$

Type 2: If $\alpha_1^* < \alpha \leq \alpha_2^*$ and $\sigma \geq \sigma_4^*$ or $\alpha_2^* < \alpha$ and $\sigma \geq \sigma_5^*$, then
$B_{t,FT}^* = C_{FT}^{s*}(v_t - \sigma, -L)$ $\quad$ $B_{t,ST}^* = C_{FT}^{s*}(v_t - \sigma, -L)$
$A_{t,FT}^* = C_{FT}^{b*}(v_t + \sigma, +L)$ $\quad$ $A_{t,ST}^* = C_{FT}^{b*}(v_t + \sigma, +L)$

Type 3: If $\alpha \leq \alpha_1^*$ and $\sigma_3^* \leq \sigma < \sigma_1^*$ or $\alpha_1^* < \alpha \leq \alpha_2^*$ and $\sigma_3^* \leq \sigma < \sigma_*^4$, then
$B_{t,FT}^* = C_{FT}^{s*}(v_t - \sigma, -L)$ $\quad$ $B_{t,ST}^* = C_{ST}^{s*}(v_t + \sigma, -L)$
$A_{t,FT}^* = C_{FT}^{b*}(v_t + \sigma, +L)$ $\quad$ $A_{t,ST}^* = C_{ST}^{b*}(v_t - \sigma, +L)$

Type 4: If $\alpha \leq \alpha_2^*$ and $\sigma_2^* \leq \sigma < \sigma_3^*$, then
$B_{t,FT}^* = C_{FT}^{s*}(v_t + \sigma, -L)$ $\quad$ $B_{t,ST}^* = C_{ST}^{s*}(v_t + \sigma, -L)$
$A_{t,FT}^* = C_{FT}^{b*}(v_t - \sigma, +L)$ $\quad$ $A_{t,ST}^* = C_{ST}^{b*}(v_t - \sigma, +L)$

Type 5: If $\alpha \leq \alpha_2^*$ and $\sigma < \sigma_2^*$ or $\alpha_2^* < \alpha$ and $\sigma < \sigma_5^*$, then
$B_{t,FT}^* = C_{FT}^{s*}(v_t + \sigma, -L)$ $\quad$ $B_{t,ST}^* = C_{FT}^{s*}(v_t + \sigma, -L)$
$A_{t,FT}^* = C_{FT}^{b*}(v_t - \sigma, +L)$ $\quad$ $A_{t,ST}^* = C_{FT}^{b*}(v_t - \sigma, +L)$

The variables α_1^*, α_2^*, σ_1^*, σ_2^*, σ_3^*, σ_4^*, σ_5^* are defined in the Appendix.

Proof. See the Appendix. ■

In the following, we briefly describe the different equilibria, focusing on the type of limit orders chosen by STs and FTs. For brevity, we just focus on the behavior of buyers (sellers behave symmetrically). Figure 1 in Appendix B graphically depicts the regions in the (α, σ)-space that correspond to the different equilibria, where we have set $L = 1$ (which is without loss of generality).

Type 1: In the type-1 equilibrium, the buy limit orders posted by slow buyers are only executed by slow sellers in the case of a decrease in the fundamental asset value. The execution probability of such of such an order is $(1-\alpha)/4$. Fast buyers that decide to post a limit order initially set the bid price slightly above $C^{s*}_{FT}(v_t-\sigma,-L)$, such that the order is executed if the next agent is a fast seller and the asset value has decreased. If the next agent turns out to be a ST, the FT is able to revise the bid price according to the latest realization in the asset value process and posts a bid price that is equal to a slow seller's cutoff price (see Lemma 1). The execution probability of this quotation strategy is $(2-\alpha)/4$.

Type 2: In this type of equilibrium, slow buyers that post limit orders choose the bid price such that the order is executed if the next agent is either a slow or a fast seller and the asset value has decreased, such that these orders have an execution probability of 1/4. Fast buyers behave as in the type-1 equilibrium.

Type 3: Slow buyers submit buy limit orders whose bid price is slightly above the sell cutoff price of a slow seller after an asset value increase, that is $B^*_{t,ST} = C^{s*}_{ST}(v_t+\sigma,-L)$. Such orders are executed if a) the next arriving agent is slow seller independently of the asset value innovation, or b) the next trader is a fast seller and the asset value has decreased, and their equilibrium probability of execution is equal to $(2-\alpha)/4$. Fast buyers behave as in the type-1 equilibrium.

Type 4: In the type-4 equilibrium, slow buyers behave as in the type-3 equilibrium. Fast buyers employ a high fill rate strategy, i.e. they initially post a bid price slightly above $C^{s*}_{FT}(v_t+\sigma,-L)$, and then revise it according to Lemma 1 if the next agent turns out to be a ST. The execution probability of this quotation strategy is equal to 1/2.

Type 5: Slow buyers that opt for a limit order choose a bid price such that the order is executed if the next agent is a seller, such that their orders have an execution probability of 1/2. Fast buyers behave as in a type-4 equilibrium.

By definition, equilibrium cutoff prices represent a trader's endogenous outside option of posting limit orders. Hence it is immediate from Lemma 2 that FTs obtain higher expected profits from limit orders than STs. This is a direct consequence of their speed advantage and understood best by looking at the type-3 equilibrium, where the quotation strategies of both types of agents have the same execution probability of $(2-\alpha)/4$. Nevertheless, STs' limit orders always execute at $B^*_{t,ST} = C^{s*}_{ST}(v_t+\sigma,-L)$, while FTs' limit orders may also execute at $C^{s*}_{ST}(v_t-\sigma,-L) < B^*_{t,ST}$ or $C^{s*}_{FT}(v_t-\sigma,-L) < B^*_{t,ST}$, which implies higher expected profits due to the avoidance of picking-off risks and the ability to price discriminate between FTs and STs. By symmetry, we have $V^{LO*}_{ST}(y_h) = V^{LO*}_{ST}(y_l) \equiv V^{LO*}_{ST}$ and $V^{LO*}_{FT}(y_h) = V^{LO*}_{FT}(y_l) \equiv V^{LO*}_{FT}$. Let V^{LO*}_0 denote the expected utility obtained from limit orders in a market with identical traders (i.e. in the Foucault (1999) model). Then we can additionally deduce the following.

Proposition 2 *Let $\alpha \in (0,1)$. Then, $V_{FT}^{LO*} > V_{0}^{LO*} > V_{ST}^{LO*}$.*

Proof. See the Appendix. ∎

In any equilibrium with a positive fraction of FTs, STs expected profits from posting limit orders are lower than in the absence of FTs. Intuitively, there are two ways in which STs can respond to the arrival of FTs with a higher outside option. They can either *i*) post more aggressive limit orders or *ii*) incur a decreased order execution probability (i.e. they decide to post quotes that are not (always) executed by FTs). It is immediate that the first reaction always harms STs expected profits from limit orders, as they simply offer better quotes, but the execution probability of these orders is as in the case where $\alpha = 0$. On the other hand, choosing *ii*) allows STs to post less aggressive quotes than in the absence of FTs, as the outside option of other STs has suffered. Nevertheless, the effect of a reduced execution probability dominates, such that their expected profits are also lower in this case.

4 Order flow composition and trading costs

4.1 Order flow composition

At each point in time, the arriving agent can be 1) a ST submitting a limit order, 2) a ST submitting a market order, 3) a FT submitting a limit order, or 4) a FT submitting a market order. Let $\varphi^i = (\varphi_1^i, \varphi_2^i, \varphi_3^i, \varphi_4^i)$ be the stationary probability distribution in a type-i equilibrium, where $i \in \{1,2,3,4,5\}$. Given this distribution, one can easily deduce the equilibrium order flow composition. Call the probability of a randomly arriving investor submitting a market (limit) order the trading (make) rate, which are given by

$$TR^i = \varphi_2^i + \varphi_4^i \tag{9}$$
$$MR^i = \varphi_1^i + \varphi_3^i = 1 - TR^i \tag{10}$$

Similarly, we can calculate the trading and make rates for ST and FT separately, which are given by

$$TR_{ST}^i = \varphi_2^i/(\varphi_1^i + \varphi_2^i) \tag{11}$$
$$MR_{ST}^i = \varphi_1^i/(\varphi_1^i + \varphi_2^i) = 1 - TR_{ST}^i \tag{12}$$
$$TR_{FT}^i = \varphi_4^i/(\varphi_3^i + \varphi_4^i) \tag{13}$$
$$MR_{FT}^i = \varphi_4^i/(\varphi_3^i + \varphi_4^i) = 1 - TR_{FT}^i \tag{14}$$

Let TR^*, TR^*_{ST} and TR^*_{FT} denote the equilibrium trading rates (e.g. $TR^* = TR^1$ if $\alpha \leq \alpha_1^*$ and $\sigma \geq \sigma_1^*$), and let TR_0^* be the equilibrium trading rate when all agents are identical (i.e. $\alpha = 0$ or $\alpha = 1$). Then, we obtain the following.

Proposition 3 *Let $\alpha \in (0,1)$. Then, $TR^*_{ST} \geq TR^* \geq TR^*_{FT}$. Moreover, if $\sigma \geq \sigma_1^*(0)$ ($\sigma < \sigma_1^*(0)$), then $TR^* > TR_0^*$ ($TR^* \leq TR_0^*$).*

Proof. See the Appendix. ■

In equilibrium, ST use market orders more frequently than FT because they are more likely to encounter a quote they find worth accepting when arriving at the market. This follows directly from FTs' comparative advantage at posting limit orders (their higher outside option), which diminishes the aggressiveness of STs limit orders (they sometimes submit limit orders that only "target" STs for a given realization of the asset value innovation).

To understand the results on the overall trading rate, it is important to understand that the introduction of trader heterogeneity in terms of trading speed has two opposing effects. First, FTs' speed advantage reduces their risk of being picked off and thereby the need for order shading, which leads to an increase in trading volume. Second, STs may decide to post less aggressive limit orders because FTs endogenous outside option is too high, thus leading to a decrease in trading volume.

If $\sigma \geq \sigma_1^*(0)$), the first effect dominates. Absent FTs ($\alpha = 0$), buyers (sellers) submit limit orders that only execute in case the asset value decreases (increases), i.e. order shading diminishes trading because the risk of being picked off is too severe. Now, the introduction of FTs increases trading volume, as their ability of updating quotes quickly after the arrival of new information allows buys (sales) to occur after price increases (decreases) as well. On the other hand, the first effect is absent for $\sigma < \sigma_1^*(0)$, because there is no order shading for $\alpha = 0$. Hence the presence of FTs leads to a decrease in trading volume because it is optimal for STs to post less aggressive limit orders for a low level of α. Once the proportion of FTs is sufficiently high, this effect also disappears, and the level of trading volume is as in the case with identical traders.

For illustration, Figure 2 in Appendix B depicts TR^* as a function of α for different values of σ. Notice that, for a fixed level of σ, different values of α may give rise to different equilibria (see Figure 1). Moreover, the exact values of α for which we move from one equilibrium to another depend on σ itself in most cases. Therefore, for each possible equilibrium combination that can arise as α increases from zero to one, we choose the mean level of σ that gives rise to the respective combination of equilibria. For example, for $0 \leq \sigma < \sigma_3^*(0)$, we may end up in a type-4 or a type-5 equilibrium, depending on α. Therefore, we plot the graph for $\sigma = \sigma_3^*(0)/2$. Table 1 lists the employed values for σ together with the associated values of α (we call them "switching points") where we move from one equilibrium to another.[8]

Our results on trading volume are similar to those derived in Jovanovic and Menkveld (2010) in the sense that the introduction of FTs may lead both to an increase or a decrease of the trading activity. In fact, both this and their paper suggest that the increased speed of FTs may reduce order shading because their quotes can reflect new fundamental information instantaneously. Nevertheless, the reason for a possible decrease in welfare in their paper (unawareness of hard information by STs when submitting market orders) differs considerably from the mechanism at work in this model (a higher endogenous outside option of FTs induces STs to post less aggressive orders). Empirically, Jovanovic and Menkveld (2011) find an increase in trading frequency, albeit an earlier version of their paper also reports a drop in trading volume. Cartea and Penalva (2011) predict an increase in trading volume through HFT because all trades get intermediated and the monopolistic HFT simply imposes an optimal "haircut".

4.2 Trading cost

The quoted or effective spread is frequently used as a measure of market liquidity and trading costs. While there is no explicit bid-ask spread in our model because limit order traders post only one quote, we directly follow Foucault (1999) and define the trading cost τ as the signed difference between the transaction price and the fundamental asset value. Thus, the cost incurred by a seller submitting a sell market order at time t is given by the fundamental value of the asset minus the best available bid price

$$\tau_t^s = v_t - B_t^m$$

It is important to note that the best available bid crucially depends on the type of both the market order trader and the limit order trader whose quote is executed. This is due to the assumption that FTs may revise their quotes, but only in the case the agent arriving after him is a ST. Additionally, the trading cost may also depend on the most recent realization of the fundamental asset value due to the picking off risk faced by the limit order trader. Let $\tau_{t,j,k}^s$ denote the trading cost for a type-j seller that arrives at time t and submits a sell market order that executes against a buy limit order posted by a type-k buyer at time $t-1$, where $j,k \in \{ST, FT\}$. In particular, consider a slow seller who arrives at time t and submits a sell market order that executes against the best available bid. If the bid stems from a ST, the trading cost is given by

$$\tau_{t,ST,ST}^{s,+\sigma} = v_{t-1} + \sigma - B_{t-1,ST} \tag{15}$$

[8] We use $\sigma = 1$ for the case where $\sigma \geq \sigma_1^*(\alpha_1^*)$, which is without loss of generality because the switching point from a type-1 to a type-2 equilibrium does not depend on σ.

for $\varepsilon_t = +\sigma$ and

$$\tau^{s,-\sigma}_{t,ST,ST} = v_{t-1} - \sigma - B_{t-1,ST} \tag{16}$$

for $\varepsilon_t = -\sigma$. In case the best available bid was posted by a fast buyer (and therefore was revised before the arrival of the market order trader), the trading cost is given by

$$\tau^{s,+\sigma}_{t,ST,FT} = v_{t-1} + \sigma - B^{+\sigma}_{t-1,FT} \tag{17}$$

for $\varepsilon_t = +\sigma$ and

$$\tau^{s,-\sigma}_{t,ST,FT} = v_{t-1} - \sigma - B^{-\sigma}_{t-1,FT} \tag{18}$$

for $\varepsilon_t = -\sigma$. In order to calculate the expected trading costs, we simply have to weight the trading costs for each possible event by its stationary probability. Let $\pi^{b,+\sigma}_{j,k}$ $(\pi^{b,-\sigma}_{j,k})$ denote the equilibrium probability that the asset value increases (decreases) and subsequently a buy limit order posted by a type-k trader is executed by a sell market order from a type-j trader, where $j, k \in \{ST, FT\}$. Then, submitting time subscripts, we have

$$E(\tau^s_{ST}) = \frac{\varphi_1(\pi^{b,+\sigma}_{ST,ST}\tau^{s,+\sigma}_{ST,ST} + \pi^{b,-\sigma}_{ST,ST}\tau^{s,-\sigma}_{ST,ST}) + \varphi_3(\pi^{b,+\sigma}_{ST,FT}\tau^{s,+\sigma}_{ST,FT} + \pi^{b,-\sigma}_{ST,FT}\tau^{s,-\sigma}_{ST,FT})}{\varphi_1(\pi^{b,+\sigma}_{ST,ST} + \pi^{b,-\sigma}_{ST,ST}) + \varphi_3(\pi^{b,+\sigma}_{ST,FT} + \pi^{b,-\sigma}_{ST,FT})} \tag{19}$$

Following exactly the same logic, the expressions for the trading costs of fast sellers are given by

$$\tau^{s,+\sigma}_{t,FT,ST} = v_{t-1} + \sigma - B_{t-1,ST} \tag{20}$$

$$\tau^{s,-\sigma}_{t,FT,ST} = v_{t-1} - \sigma - B_{t-1,ST} \tag{21}$$

$$\tau^{s,+\sigma}_{t,FT,FT} = v_{t-1} + \sigma - B_{t-1,FT} \tag{22}$$

$$\tau^{s,-\sigma}_{t,FT,FT} = v_{t-1} - \sigma - B_{t-1,FT} \tag{23}$$

and the expected trading cost for an AT seller is consequently given by

$$E(\tau^s_{FT}) = \frac{\varphi_1(\pi^{b,+\sigma}_{FT,ST}\tau^{s,+\sigma}_{FT,ST} + \pi^{b,-\sigma}_{FT,ST}\tau^{s,-\sigma}_{FT,ST}) + \varphi_3(\pi^{b,+\sigma}_{FT,FT}\tau^{s,+\sigma}_{FT,FT} + \pi^{b,-\sigma}_{FT,FT}\tau^{s,-\sigma}_{FT,FT})}{\varphi_1(\pi^{b,+\sigma}_{FT,ST} + \pi^{b,-\sigma}_{FT,ST}) + \varphi_3(\pi^{b,+\sigma}_{FT,FT} + \pi^{b,-\sigma}_{FT,FT})} \tag{24}$$

Finally, the average expected trading cost for sellers is given by

$$E(\tau^s) = \frac{\varphi_2}{\varphi_2 + \varphi_4}E(\tau^s_{ST}) + \frac{\varphi_4}{\varphi_2 + \varphi_4}E(\tau^s_{FT}) \tag{25}$$

By symmetry, we have $E(\tau^s_k) = E(\tau^b_k) \equiv E(\tau_k)$ for $k \in \{ST, FT\}$ and hence $E(\tau^s) = E(\tau^b) \equiv E(\tau)$. Let $E(\tau^*)$,$E(\tau^*_{ST})$ and $E(\tau^*_{FT})$ denote the equilibrium average expected trading cost, and let $E(\tau^*_0)$ be the equilibrium average expected trading cost if all agents are identical. One can show the following.

Proposition 4 *Let $\alpha \in (0,1)$. Then $E(\tau^*_{ST}) > E(\tau^*) > E(\tau^*_{FT})$. Moreover,*

a) For every $\sigma \geq \sigma^*_1(0)$ there exists α^+ such that we have $E(\tau^*) > E(\tau^*_0)$ ($E(\tau^*) < E(\tau^*_0)$) for $\alpha \leq \alpha^+$ ($\alpha > \alpha^+$).

b) For $\sigma^*_1(0) > \sigma \geq \sigma^*_2(1/3)$, we have $E(\tau^*) > E(\tau^*_0)$.

c) For every $\sigma < \sigma^*_2(1/3)$, there exists $\alpha^{++} < 1/3$ such that we have $E(\tau^*) < E(\tau^*_0)$ for $\alpha \in (\alpha^{++}, 1/3)$ and $E(\tau^*) \geq E(\tau^*_0)$ otherwise.

Proof. See the Appendix. ■

Naturally, FTs enjoy lower expected trading costs than STs because the latter only may pick off stale quotes when trading with other STs. This effect gets additionally amplified by the fact that FTs quotation strategies discriminate between FTs and STs. This result is consistent with the findings reported by Hendershott and Riordan (2011a), who study algorithmic trading on the German Stock Exchange and find that "algorithmic traders consume liquidity when it is cheap", i.e. they pay lower effective spreads than (slow) human traders. Additionally, the proposition states that equilibrium average expected trading costs in the presence of speed differences may increase or decrease in comparison to a market populated by identical traders. It is important to notice that $E(\tau^*_{ST}) > E(\tau^*) > E(\tau^*_{FT})$ implies that not all agents in the economy may benefit from a decline in trading costs. Hendershott et al. (2011) find empirically that algorithmic trading has caused a decrease in average trading costs for US equities. Our results indicate that this finding need not imply that all agents are better off, because some quotes are out of STs' reach. As shown by Hasbrouck & Saar (2009), the "lifetimes" of limit orders have decreased considerably over the last years, suggesting that a large proportion of quotes are in fact not accessible to for slower market participants. Similar concerns have been raised concerning particular order types such as flash-orders (see Skjeltorp et al. (2011) for details), which were discontinued recently after the SEC proposed a ban in 2009.

Proposition 5 *Let $\alpha \in (0,1)$. Then,*

a) For every $\sigma \geq \sigma^*_1(0)$ there exists $\alpha^+_{ST}, \alpha^{++}_{ST}$ with $\alpha^+_{ST} < \alpha^{++}_{ST}$ such that we have $E(\tau^*_{ST}) < E(\tau^*_0)$ for $\alpha \in (\alpha^+_{ST}, \alpha^{++}_{ST})$ and $E(\tau^*_{ST}) \geq E(\tau^*_0)$ otherwise.

b) For $\sigma^*_1(0) > \sigma \geq \sigma^*_2(1/4)$, we have $E(\tau^*_{ST}) > E(\tau^*_0)$.

c) For every $\sigma < \sigma^*_2(1/4)$, there exist $\alpha^{+++}_{ST} < 1/4$ such that we have $E(\tau^*_{ST}) < E(\tau^*_0)$ for $\alpha \in (\alpha^{+++}_{ST}, 1/4)$ and $E(\tau^*_{ST}) \geq E(\tau^*_0)$ otherwise.

d) For every $\sigma \in [\sigma^*_2(\alpha^*_2), \sigma^*_1(0))$, there exist $\alpha^+_{FT}, \alpha^{++}_{FT}$ with $\alpha^+_{FT} < \alpha^{++}_{FT}$ such that we have $E(\tau^*_{FT}) > E(\tau^*_0)$ for $\alpha \in (\alpha^+_{FT}, \alpha^{++}_{FT})$. Otherwise, $E(\tau^*_{FT}) < E(\tau^*_0)$.

Proof. See the Appendix. ■

To gain the intuition on how the presence of FTs affects STs' trading costs, it is best to focus on the case where $\sigma \geq \sigma_1^*(\alpha_1^*)$. In this parameter region, only a type-1 or a type-2 equilibrium may arise. Without FTs ($\alpha = 0$), the equilibrium bid (ask) prices are set such that a buy (sell) limit order is only executed in case the next agent is a seller (buyer) and the asset value has decreased (increased). In other words, the picking off risk is sufficiently high to induce order shading by limit order traders. Now consider what happens if we introduce a small proportion of FTs ($\alpha \leq \alpha_1^*$). Given that α is small, it is optimal for STs to submit buy (sell) limit orders that are only executed by other STs, but not by FTs (their advantage over STs is very high for small α, as their limit orders can only be picked off by other STs). Because $V_{HT}^{LO*} < V_0^{LO*}$ (see Proposition 2), STs post lower (higher) bid (ask) prices as in the absence of FTs and the expected trading costs increase.

Now consider what happens if we increase α further. Once we have $\alpha > \alpha_1^*$, it becomes optimal for STs to post buy (sell) limit orders that are also executed by fast sellers (buyers) in the case of a price decrease (increase). This happens for two reasons. First, a higher level of α leads to a considerable increase in the execution probability of limit orders if they are also targeted at FTs. Second, FTs exert a negative externality on each other by increasing the picking off risk (note that FTs may not cancel their limit orders if the next trader is also a FT), such that it becomes "cheaper" for STs to target them. The increased aggressiveness of STs' limit orders benefits other STs and decreases their expected trading costs below the level obtained when $\alpha = 0$ (recall that $V_{FT}^{LO*} > V_0^{LO*}$). Nevertheless, this effect diminishes as α increases further. First, STs that arrive to the market are less and less likely to find a quote submitted by another ST (FTs can discriminate between STs and FTs, offering worse quotes to the former), and second, the aggressiveness of the STs quotes diminishes as FTs face an increasingly higher picking off risk and therefore are willing to accept worse quotes.

While the expected trading costs incurred by FTs are below those prevailing in a human-only market, there is one exception that is owed to the discrete nature of the model. Suppose that $\alpha \leq \alpha_1^*$ and $\sigma \in [\sigma_2^*(\alpha_2^*), \sigma_1^*(0))$, i.e. parameters are such that we are in a type-3 equilibrium. In this equilibrium, a slow buyer that opts for a limit order sets $B_{t,ST}^* = C_{ST}^{s*}(v_t + \sigma, -L)$, i.e. his limit order is executed if i) the next agent is either a slow seller or ii) the next agent is a fast seller and the asset value decreases. Now, if α increases, we move to a type-2 equilibrium and slow buyers lower their bid to $B_{t,ST}^* = C_{FT}^{s*}(v_t - \sigma, -L)$. Note that this limit order will still be executed if the next agent is a fast seller despite a lower bid price. It turns out that this order shading by STs increase FTs' expected trading costs above $E(\tau_0^*)$. For a sufficiently high value of α, we move to a type-5 equilibrium such that we are back to $E(\tau_{FT}^*) < E(\tau_0^*)$.

For illustration, we plot $E(\tau^*)$, $E(\tau_{ST}^*)$ and $E(\tau_{FT}^*)$ as a function of α for different values of σ in Figure 3 (see the Appendix).

5 Welfare

In our setting, the calculation of social welfare is extremely simple. Each transaction yields a welfare of $2L$, such that per-period social welfare is given by

$$W = TR \times 2L$$

Now let W^* denote equilibrium welfare and let W_0^* be equilibrium welfare in a market with identical agents. Then the following statement is a direct consequence of Proposition 3.

Corollary 1 *Let $\alpha \in (0,1)$. Then, we have $W^* > W_0^*$ ($W^* \leq W_0^*$) for $\sigma \geq \sigma_1^*(0)$ ($\sigma < \sigma_1^*(0)$).*

It is important to notice that whenever a transaction occurs, the gains from trade are not split evenly between market and limit order trader because of the latter's market power and the risk of non-execution. Let W_{ST}^* and W_{FT}^* denote the equilibrium welfare (expected utility) of STs and FTs, respectively. Adding and subtracting the average expected trading costs and rearranging, we can then decompose equilibrium per-period welfare as

$$\begin{aligned} W^* &= TR^* \times (L - E(\tau^*)) + TR^* \times (L + E(\tau^*)) \\ &= TR^* \times V^{MO*} + MR^* \times V^{LO*} \\ &= \varphi_1^* V_{ST}^{LO*} + \varphi_2^* V_{ST}^{MO*} + \varphi_3^* V_{FT}^{LO*} + \varphi_4^* V_{FT}^{MO*} \\ &= (1-\alpha) W_{ST}^* + \alpha W_{FT}^* \end{aligned}$$

where $V^{MO*} = L - E(\tau^*)$ and $V^{LO*} = \frac{TR^*}{MR^*}(L + E(\tau^*))$ are the average trader's equilibrium expected profits from posting market and limit orders, respectively, and we have used the fact that $V^{LO*} = \frac{\varphi_1^*}{\varphi_1^*+\varphi_3^*} V_{ST}^{LO*} + \frac{\varphi_3^*}{\varphi_1^*+\varphi_3^*} V_{FT}^{LO*}$ and $V^{MO*} = \frac{\varphi_2^*}{\varphi_2^*+\varphi_4^*} V_{ST}^{MO*} + \frac{\varphi_4^*}{\varphi_2^*+\varphi_4^*} V_{FT}^{MO*}$.

Equipped with the equilibrium expected profits from posting market and limit orders, it is straightforward to evaluate the welfare for each trader type separately. We have the following.

Proposition 6 *Let $\alpha \in (0,1)$. Then, we have $W_{FT}^* > W_{ST}^*$. Moreover,*

a) For every $\sigma \in [\sigma_1^*(0), \sigma_1^*(\alpha_1^*))$, there exist $\alpha_{ST}^{\times}, \alpha_{ST}^{\times\times}$ with $\alpha_{ST}^{\times} < \alpha_1^* < \alpha_{ST}^{\times\times}$ such that we have $W_{ST}^* > W_0^*$ for $\alpha \in (\alpha_{ST}^{\times}, \alpha_{ST}^{\times\times})$. Otherwise $W_{ST}^* < W_0^*$.

b) For every $\sigma \in [\sigma_2^*(\alpha_2^*), \sigma_1^*(0))$, there exist $\alpha_{FT}^{\times}, \alpha_{FT}^{\times\times}$ with $\alpha_{FT}^{\times} < \alpha_{FT}^{\times\times}$ such that we have $W_{FT}^* < W_0^*$ for $\alpha \in (\alpha_{FT}^{\times}, \alpha_{FT}^{\times\times})$. Otherwise, $W_{FT}^* > W_0^*$.

Proof. See the Appendix. ■

The conclusion that $W_{FT}^* > W_{ST}^*$ is trivial given the results in Propositions 2 and 4. Although we find that $W_{FT}^* > W_0^* > W_{ST}^*$ for most parameter constellations, there are two exceptions. First, we find that $W^*_{ST} > W^*_0$ for some values of α if $\sigma \in [\sigma^*_1(0), \sigma^*_1(\alpha^*_1))$. In this case, σ is such that in an equilibrium with identical traders (say all traders are slow, i.e. $\alpha = 0$) it is optimal for traders to engage in order shading, but they are almost indifferent to not doing so. Hence slow buyers (sellers) submit buy (sell) limit orders that are only executed in the case of a price decrease (increase). Now, introduce a small proportion of FTs, which triggers a decrease (increase) in STs sell (buy) cutoff prices. Then, if α increases sufficiently, it becomes optimal for STs to submit limit orders that are also executed if the asset value increases (decreases), albeit only by STs. These more aggressive quotes lead to a discrete drop in STs expected trading costs when moving from a low-trade equilibrium to a high-trade equilibrium. One the other hand, the change in the expected profits from limit orders is infinitesemal (we are at the point of indifference between different strategies), such that STs' welfare increases above W^*_0. The possibility of having $W^*_{FT} < W^*_0$ has a similar reason, only that here the relevant move is from a high-trade equilibrium (type-3) to a low-trade equilibrium (type-2). As explained in the previous section, this triggers a jump in expected trading costs for FTs, which may be sufficiently large to cause $W^*_{FT} < W^*_0$. For graphical illustration, Figure 4 plots W^*_{ST} and W^*_{FT} as a function of α for several levels of σ.

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6 Conclusion

This paper studies a dynamic limit order in which agents differ in their trading speed, and FTs may avoid the risk of being picked off when interacting with STs. This type of heterogeneity across agents has two opposing effects. First, FTs' speed advantage allows them to adjust their quotes quickly to new public information such that they do not need to shade their orders, which allows more gains from trade to be reaped. Second, they obtain higher expected profits from posting limit orders than STs as they face a reduced risk of being picked off. Because these profits effectively constitute an endogenous outside option, it is optimal for STs to post less aggressive limit orders as long as α is not too high. If the winner's curse is sufficiently high, the first effect dominates and trading volume (which is proportional to social welfare) is above its level in the benchmark case with identical traders studied in Foucault (1999). On the other hand, the first effect is absent when the risk of being picked off is small, such that we observe a lower trading volume as long as α is not too high.

While the average expected trading costs (measured as the signed difference between the transaction prices and the fundamental asset value) may decline with the introduction of FTs, this effect is entirely driven by the quotes of STs, who increase the aggressiveness of their quotes once the proportion of fast traders has surpassed a critical level. This in turn decreases slow traders' expected profits from posting limit orders, such that their overall expected utility (or welfare) is lower than in the case with identical traders, except for the case where the introduction of a small proportion of FTs leads to a switch from a low-trade equilibrium to a high-trade equilibrium. Consequently, our results indicate that although practices such as algorithmic and high-frequency trading may lead to an increase market liquidity (see e.g. Hendershott et al. (2011)), this does not necessarily imply that all agents are better off. In fact, their speed allows fast traders to price discriminate between fast and slow traders, i.e. the latter are not able to access the best quotes. Practices such as fleeting limit orders (see Hasbrouck and Saar (2009)) and flash orders (see Skjeltorp et al. (2011)) constitute real-world examples of prices that are not available to slow (human) market participants. Overall, our results are very much in line with the results in Hendershott and Riordan (2011a), who find that automated traders "supply liquidity when it is expensive and consume liquidity when it is cheap".

7 Appendix A: Proofs

7.1 Proof of Lemma 1

Consider a fast buyer that has placed a bid at time t. If he observes the innovation ε_{t+1} and can still modify his order, he knows that the next agent (provided he is a seller) is a ST with sell cutoff price $C^{s*}_{ST}(v_t + \varepsilon_{t+1}, -L)$. Clearly, the optimal bid price is slightly above this cutoff price, as a lower (higher) bid has a zero (the same) execution probability. A symmetric argument holds for fast sellers.

7.2 Proof of Lemma 2

We prove only the statements for sell cutoff prices. Symmetry then establishes the corresponding arguemnts for buy cutoff prices.

a) Suppose that, in equilibrium, a slow buyer posts a buy limit order that executes only if the asset value decreases. Then, a fast trader can always do better by posting the same bid price and then revise it to according to Lemma 1 in case the next trader is a ST. Similarly, if a slow buyer posts a limit order that executes irrespectively of the asset value innovation, a fast trader can do better by revising his order according to Lemma 1 and obtaining higher profits by incorporating the latest innovation into his limit price whenever possible (i.e. when a slow seller follows). Hence we conclude

$C^{s*}_{FT}(v_t, -L) \geq C^{s*}_{ST}(v_t, -L)$.

b) By definition, we have $C^{s*}_k(v_t, y_t) - (v_t + y_t) = V^{LO*}_k(y_t)$ for $k \in \{ST, FT\}$ and $y_t \in \{-L, +L\}$. As the execution probability of any limit order is no greater than 1/2, we have $L \geq V^{LO*}_k(y_t) \geq 0$, which implies $v_t + y_t + L \geq C_k^{s*}(v_t, -L) \geq v_t + y_t$. Hence, we can write $v_t + 2L \geq C_{ST}^{s*}(v_t, +L) \geq v_t + L$ and $v_t \geq C_{FT}^{s*}(v_t, -L) \geq v_t - L$, which establishes the result.

c) First, suppose that that $\sigma \geq L/2$. From the previous step, we know that $v_t+\sigma \geq C_{ST}^{s*}(v_t+\sigma, -L) \geq v_t + \sigma - L$ and $v_t - \sigma \geq C_{FT}^{s*}(v_t - \sigma, -L) \geq v_t - \sigma - L$, which directly implies $C_{ST}^{s*}(v_t + \sigma, -L) \geq C_{FT}^{s*}(v_t - \sigma, -L)$. Now assume that $\sigma < L/2$ and consider an fast buyer's decision regarding his quotation strategy. He will opt for a high fill rate in equilibrium iff $\frac{\alpha}{4}[v - \sigma + L - C_{FT}^{s*}(v_t + \sigma, -L)] + \frac{\alpha}{4}[v + \sigma + L - C_{FT}^{s*}(v_t + \sigma, -L)] \geq \frac{\alpha}{4}[v - \sigma + L - C_{FT}^{s*}(v_t - \sigma, -L)]$, which is satisfied in our case as $v - \sigma + L \geq v + \sigma \geq C_{FT}^{s*}(v_t + \sigma, -L)$. Now consider a slow buyer and suppose he posts a buy limit order with bid price equal to $C_{FT}^{s*}(v_t + \sigma, -L)$. As this is not necessarily his equilibrium strategy we have that $V_{ST}^{LO*}(y_h) \geq \frac{1}{2}[v + L - C_{FT}^{s*}(v_t + \sigma, -L)]$. But we just concluded that $V_{FT}^{LO*}(y_h) = \frac{\alpha}{2}[v + L - C_{FT}^{s*}(v_t + \sigma, -L)] + \frac{1-\alpha}{4}[v - \sigma + L - C_{ST}^{s*}(v_t - \sigma, -L)] + \frac{1-\alpha}{4}[v + \sigma + L - C_{ST}^{s*}(v_t + \sigma, -L)]$, and therefore $V_{FT}^{LO*}(y_h) - V_{ST}^{LO*}(y_h) \leq \frac{1-\alpha}{4}[C_{FT}^{s*}(v_t + \sigma, -L) - C_{ST}^{s*}(v_t - \sigma, -L)] + \frac{1-\alpha}{4}[C_{FT}^{s*}(v_t + \sigma, -L) - C_{ST}^{s*}(v_t + \sigma, -L)] = \frac{1-\alpha}{2}[C_{FT}^{s*}(v_t, -L) - C_{ST}^{s*}(v_t, -L)] + \frac{1-\alpha}{2}\sigma$. Symmetry between buyers and seller implies $V_{FT}^{LO*}(y_h) - V_{ST}^{LO*}(y_h) = C_{FT}^{s*}(v_t, -L) - C_{ST}^{s*}(v_t, -L)$, such that we conclude $C_{FT}^{s*}(v_t, -L) - C_{ST}^{s*}(v_t, -L) \leq \frac{1-\alpha}{1+\alpha}\sigma$, which finally leads us to $C_{ST}^{s*}(v_t + \sigma, -L) - C_{FT}^{s*}(v_t - \sigma, -L) = C_{ST}^{s*}(v_t, -L) - C_{FT}^{s*}(v_t, -L) + 2\sigma > 0$ as desired.

7.3 Proof of Proposition 1

For each type of equilibrium, the proof procedes in three steps:
1) Conjecture an ordering of cutoff prices.
2) Conjecture equilibrium strategies and solve for the equilibrium cutoff prices.
3) Verify that
  a) the assumed strategies are best replies (i.e. deviations are not profitable) and
  b) the cutoff prices satisfy the assumed ordering.

Lemma 2 implies that it suffices to consider the following four orderings of cutoff prices.
Ordering 1: $C_{ST}^{s*}(v_t - \sigma, -L) \leq C_{FT}^{s*}(v_t - \sigma, -L) \leq C_{ST}^{s*}(v_t - \sigma, +L) \leq C_{FT}^{s*}(v_t - \sigma, +L) \leq C_{ST}^{s*}(v_t + \sigma, -L) \leq C_{FT}^{s*}(v_t + \sigma, -L) \leq C_{ST}^{s*}(v_t + \sigma, +L) \leq C_{FT}^{s*}(v_t + \sigma, +L)$

Ordering 2: $C_{ST}^{s*}(v_t - \sigma, -L) \leq C_{FT}^{s*}(v_t - \sigma, -L) \leq C_{ST}^{s*}(v_t - \sigma, +L) \leq C_{ST}^{s*}(v_t + \sigma, -L) \leq C_{FT}^{s*}(v_t - \sigma, +L) \leq C_{FT}^{s*}(v_t + \sigma, -L) \leq C_{ST}^{s*}(v_t + \sigma, +L) \leq C_{FT}^{s*}(v_t + \sigma, +L)$

Ordering 3: $C_{ST}^{s*}(v_t - \sigma, -L) \leq C_{FT}^{s*}(v_t - \sigma, -L) \leq C_{ST}^{s*}(v_t + \sigma, -L) \leq C_{ST}^{s*}(v_t - \sigma, +L) \leq C_{FT}^{s*}(v_t + \sigma, -L) \leq C_{FT}^{s*}(v_t - \sigma, +L) \leq C_{ST}^{s*}(v_t + \sigma, +L) \leq C_{FT}^{s*}(v_t + \sigma, +L)$

Ordering 4: $C_{ST}^{s*}(v_t - \sigma, -L) \leq C_{FT}^{s*}(v_t - \sigma, -L) \leq C_{ST}^{s*}(v_t + \sigma, -L) \leq C_{FT}^{s*}(v_t + \sigma, -L) \leq C_{ST}^{s*}(v_t - \sigma, +L) \leq C_{FT}^{s*}(v_t - \sigma, +L) \leq C_{ST}^{s*}(v_t + \sigma, +L) \leq C_{FT}^{s*}(v_t + \sigma, +L)$

For each ordering of sell cutoff prices, there is a corresponding ordering (due to symmetry) of buy cutoff prices. For example, for Ordering 1, we have $C^{b*}_{ST}(v_t + \sigma, +L) \geq C^{b*}_{FT}(v_t + \sigma, +L) \geq C^{b*}_{ST}(v_t + \sigma, -L) \geq C^{b*}_{FT}(v_t + \sigma, -L) \geq C^{b*}_{ST}(v_t - \sigma, +L) \geq C^{b*}_{FT}(v_t - \sigma, +L) \geq C^{b*}_{ST}(v_t - \sigma, -L) \geq C^{b*}_{FT}(v_t - \sigma, -L)$

The following four tables contain the conditional and unconditional execution probabilities of buy limit orders according to the position of the limit price relative to the cutoff sell prices, separately for each employed ordering.

Bid Price (Ordering 1)	Execution Probability	Execution Probability conditional on $\varepsilon_{t+1} = -\sigma$	Execution Probability conditional on $\varepsilon_{t+1} = +\sigma$
$\leq C^{s*}_{ST}(v_t - \sigma, -L)$	0	0	0
$\in (C^{s*}_{ST}(v_t - \sigma, -L), C^{s*}_{FT}(v_t - \sigma, -L)]$	$(1-\alpha)/4$	$(1-\alpha)/2$	0
$\in (C^{s*}_{FT}(v_t - \sigma, -L), C^{s*}_{ST}(v_t - \sigma, +L)]$	1/4	1/2	0
$\in (C^{s*}_{ST}(v_t - \sigma, +L), C^{s*}_{FT}(v_t - \sigma, +L)]$	$(2-\alpha)/4$	$(2-\alpha)/2$	0
$\in (C^{s*}_{FT}(v_t - \sigma, +L), C^{s*}_{ST}(v_t + \sigma, -L)]$	1/2	1	0
$\in (C^{s*}_{ST}(v_t + \sigma, -L), C^{s*}_{FT}(v_t + \sigma, -L)]$	$(3-\alpha)/4$	1	$(1-\alpha)/2$
$\in (C^{s*}_{FT}(v_t + \sigma, -L), C^{s*}_{ST}(v_t + \sigma, +L)]$	3/4	1	1/2
$\in (C^{s*}_{ST}(v_t + \sigma, +L), C^{s*}_{FT}(v_t + \sigma, +L)]$	$(4-\alpha)/4$	1	$(2-\alpha)/2$
$> C^{s*}_{FT}(v_t + \sigma, +L)$	1	1	1

Bid Price (Ordering 2)	Execution Probability	Execution Probability conditional on $\varepsilon_{t+1} = -\sigma$	Execution Probability conditional on $\varepsilon_{t+1} = +\sigma$
$\leq C^{s*}_{ST}(v_t - \sigma, -L)$	0	0	0
$\in (C^{s*}_{ST}(v_t - \sigma, -L), C^{s*}_{FT}(v_t - \sigma, -L)]$	$(1-\alpha)/4$	$(1-\alpha)/2$	0
$\in (C^{s*}_{FT}(v_t - \sigma, -L), C^{s*}_{ST}(v_t - \sigma, +L)]$	1/4	1/2	0
$\in (C^{s*}_{ST}(v_t - \sigma, +L), C^{s*}_{ST}(v_t + \sigma, -L)]$	$(2-\alpha)/4$	$(2-\alpha)/2$	0
$\in (C^{s*}_{ST}(v_t + \sigma, -L), C^{s*}_{FT}(v_t - \sigma, +L)]$	$(3-2\alpha)/4$	$(2-\alpha)/2$	$(1-\alpha)/2$
$\in (C^{s*}_{FT}(v_t - \sigma, +L), C^{s*}_{FT}(v_t + \sigma, -L)]$	$(3-\alpha)/4$	1	$(1-\alpha)/2$
$\in (C^{s*}_{FT}(v_t + \sigma, -L), C^{s*}_{ST}(v_t + \sigma, +L)]$	3/4	1	1/2
$\in (C^{s*}_{ST}(v_t + \sigma, +L), C^{s*}_{FT}(v_t + \sigma, +L)]$	$(4-\alpha)/4$	1	$(2-\alpha)/2$
$> C^{s*}_{FT}(v_t + \sigma, +L)$	1	1	1

Bid Price (Ordering 3)	Execution Probability	Execution Probability conditional on $\varepsilon_{t+1} = -\sigma$	Execution Probability conditional on $\varepsilon_{t+1} = +\sigma$
$\leq C^{s*}_{ST}(v_t - \sigma, -L)$	0	0	0
$\in (C^{s*}_{ST}(v_t - \sigma, -L), C^{s*}_{FT}(v_t - \sigma, -L)]$	$(1-\alpha)/4$	$(1-\alpha)/2$	0
$\in (C^{s*}_{FT}(v_t - \sigma, -L), C^{s*}_{ST}(v_t + \sigma, -L)]$	$1/4$	$1/2$	0
$\in (C^{s*}_{ST}(v_t + \sigma, -L), C^{s*}_{ST}(v_t - \sigma, +L)]$	$(2-\alpha)/2$	$1/2$	$(1-\alpha)/2$
$\in (C^{s*}_{ST}(v_t - \sigma, +L), C^{s*}_{FT}(v_t + \sigma, -L)]$	$(3-2\alpha)/4$	$(2-\alpha)/2$	$(1-\alpha)/2$
$\in (C^{s*}_{FT}(v_t + \sigma, -L), C^{s*}_{FT}(v_t - \sigma, +L)]$	$(3-\alpha)/4$	$(2-\alpha)/2$	$1/2$
$\in (C^{s*}_{FT}(v_t - \sigma, +L), C^{s*}_{ST}(v_t + \sigma, +L)]$	$3/4$	1	$1/2$
$\in (C^{s*}_{ST}(v_t + \sigma, +L), C^{s*}_{FT}(v_t + \sigma, +L)]$	$(4-\alpha)/4$	1	$(2-\alpha)/2$
$> C^{s*}_{FT}(v_t + \sigma, +L)$	1	1	1

Bid Price (Ordering 4)	Execution Probability	Execution Probability conditional on $\varepsilon_{t+1} = -\sigma$	Execution Probability conditional on $\varepsilon_{t+1} = +\sigma$
$\leq C^{s*}_{ST}(v_t - \sigma, -L)$	0	0	0
$\in (C^{s*}_{ST}(v_t - \sigma, -L), C^{s*}_{FT}(v_t - \sigma, -L)]$	$(1-\alpha)/4$	$(1-\alpha)/2$	0
$\in (C^{s*}_{FT}(v_t - \sigma, -L), C^{s*}_{ST}(v_t + \sigma, -L)]$	$1/4$	$1/2$	0
$\in (C^{s*}_{ST}(v_t + \sigma, -L), C^{s*}_{FT}(v_t + \sigma, -L)]$	$(2-\alpha)/4$	$1/2$	$(1-\alpha)/2$
$\in (C^{s*}_{FT}(v_t + \sigma, -L), C^{s*}_{ST}(v_t - \sigma, +L)]$	$1/2$	$1/2$	$1/2$
$\in (C^{s*}_{ST}(v_t - \sigma, +L), C^{s*}_{FT}(v_t - \sigma, +L)]$	$(3-\alpha)/4$	$(2-\alpha)/2$	$1/2$
$\in (C^{s*}_{FT}(v_t - \sigma, +L), C^{s*}_{ST}(v_t + \sigma, +L)]$	$3/4$	1	$1/2$
$\in (C^{s*}_{ST}(v_t + \sigma, +L), C^{s*}_{FT}(v_t + \sigma, +L)]$	$(4-\alpha)/4$	1	$(2-\alpha)/2$
$> C^{s*}_{FT}(v_t + \sigma, +L)$	1	1	1

Type 1 equilibrium:
Let $\sigma^*_1 = 4L/(5-\alpha)$ and $\alpha^*_1 = \sqrt{5} - 2$.

Case A:
Step 1: Assume Ordering 1.
Step 2: Conjecture the following equilibrium strategies: HT buyers submit a buy limit order with a bid price slightly above $C^{s*}_{ST}(v_t - \sigma, -L)$, which has a probability of execution of $(1-\alpha)/4$ (see Table A.1, Panel 1). AT buyers submit a buy limit order with a bid price slightly above $C^{s*}_{FT}(v_t - \sigma, -L)$. If the next trader is not an AT, they cancel this order after observing the innovation in the fundamental value and set a new bid price slightly above $C^{s*}_{ST}(v_t - \sigma, +L)$ ($C^{s*}_{ST}(v_t + \sigma, +L)$) if $\varepsilon_{t+1} = -\sigma$

($\varepsilon_{t+1} = +\sigma$). The probability of execution for this strategy is $(1-\alpha)/4+(1-\alpha)/4+\alpha/4 = (2-\alpha)/4$ (see Table A.1, Panel 1). Moreover, conjecture the analogous strategies for HT and AT sellers, e.g. a HT seller submits a sell limit order with ask price slightly below $C^{b*}_{ST}(v_t+\sigma,+L)$ with probability of execution equal to $(1-\alpha)/4$. Thus, cutoff prices have to satisfy the following system of equations.

$$\begin{aligned}
v_t + L - C^{b*}_{ST}(v_t,+L) &= \frac{1-\alpha}{4}\left[v_t - \sigma + L - C^{s*}_{ST}(v_t-\sigma,-L)\right] \\
C^{s*}_{ST}(v_t,-L) - (v_t - L) &= \frac{1-\alpha}{4}\left[C^{b*}_{ST}(v_t+\sigma,+L) - (v_t+\sigma-L)\right] \\
v_t + L - C^{b*}_{FT}(v_t,+L) &= \frac{1-\alpha}{4}\left[v_t - \sigma + L - C^{s*}_{ST}(v_t-\sigma,-L)\right] \\
&\quad + \frac{1-\alpha}{4}\left[v_t + \sigma + L - C^{s*}_{ST}(v_t+\sigma,-L)\right] \\
&\quad + \frac{\alpha}{4}\left[v_t - \sigma + L - C^{s*}_{FT}(v_t-\sigma,-L)\right] \\
C^{s*}_{FT}(v_t,-L) - (v_t - L) &= \frac{1-\alpha}{4}\left[C^{b*}_{ST}(v_t+\sigma,+L) - (v_t+\sigma-L)\right] \\
&\quad + \frac{1-\alpha}{4}\left[C^{b*}_{ST}(v_t-\sigma,+L) - (v_t-\sigma-L)\right] \\
&\quad + \frac{\alpha}{4}\left[C^{b*}_{FT}(v_t+\sigma,+L) - (v_t+\sigma-L)\right]
\end{aligned}$$

Tedious, but straightforward algebra yields the following cutoff prices:

$$\begin{aligned}
C^{s*}_{ST}(v_t,-L) &= v_t - L + (2L)\frac{1-\alpha}{5-\alpha} \\
C^{s*}_{FT}(v_t,-L) &= v_t - L + (2L)\frac{8-\alpha(3+\alpha)}{(5-\alpha)(4+\alpha)} \\
C^{b*}_{ST}(v_t,+L) &= v_t + L - (2L)\frac{1-\alpha}{5-\alpha} \\
C^{b*}_{FT}(v_t,+L) &= v_t + L - (2L)\frac{8-\alpha(3+\alpha)}{(5-\alpha)(4+\alpha)}
\end{aligned}$$

Step 3: Due to symmetry, it suffices to analyze the strategies of buyers. As mentioned, the optimal bid price must be chosen such that it is slightly higher than the lower bound of any of the proposed intervals, because a higher bid can be decreased without reducing the execution probability. Moreover, it is easy to see that Ordering 1 implies that it is not optimal to post a bid price $B \in (C^{s*}_{ST}(v_t-\sigma,+L), C^{s*}_{ST}(v_t+\sigma,-L)]$. Such a limit order is only executed in the case of a price decrease, and therefore $B > C^{s*}_{ST}(v_t-\sigma,+L) \geq v_t - \sigma + L$. But then, the execution of this limit order cannot be profitable, because the bid price is above the reservation price of the trader posting the limit order. Moreover, it is clear that a bid price $B > C^{s*}_{ST}(v_t+\sigma,+L) \geq v_t+\sigma+L$ cannot be optimal, because it is higher the maximum valuation of any trader.[9] Thus, the proposed strategy

[9] Using the same kind of reasoning, we can reduce the possible equilibrium bid prices for other orderings of cutoff prices as well.

for HT buyers is a best reply iff:

$$\begin{aligned}
\frac{1-\alpha}{4}\left[v_t-\sigma+L-C_{ST}^{s*}(v_t-\sigma,-L)\right] &\geq \frac{1}{4}\left[v_t-\sigma+L-C_{FT}^{s*}(v_t-\sigma,-L)\right] \\
\frac{1-\alpha}{4}\left[v_t-\sigma+L-C_{ST}^{s*}(v_t-\sigma,-L)\right] &\geq \frac{1}{2}\left[v_t-\sigma+L-C_{ST}^{s*}(v_t+\sigma,-L)\right] \\
&\quad +\frac{1-\alpha}{4}\left[v_t-\sigma+L-C_{ST}^{s*}(v_t+\sigma,-L)\right] \\
\frac{1-\alpha}{4}\left[v_t-\sigma+L-C_{ST}^{s*}(v_t-\sigma,-L)\right] &\geq \frac{1}{2}\left[v_t-\sigma+L-C_{ST}^{s*}(v_t+\sigma,-L)\right] \\
&\quad +\frac{1}{4}\left[v_t-\sigma+L-C_{FT}^{s*}(v_t+\sigma,-L)\right]
\end{aligned}$$

Similarly, AT buyers have no incentives to deviate iff:

$$\begin{aligned}
\frac{\alpha}{4}\left[v_t-\sigma+L-C_{FT}^{s*}(v_t-\sigma,-L)\right] &\geq \frac{\alpha}{2}\left[v_t-\sigma+L-C_{FT}^{s*}(v_t+\sigma,-L)\right] \\
&\quad +\frac{\alpha}{4}\left[v_t+\sigma+L-C_{FT}^{s*}(v_t+\sigma,-L)\right]
\end{aligned}$$

Brute-force algebra reveals that these inequalities and the assumed ordering of cutoff prices are satisfied if and only $\alpha \leq \alpha_1^*$ and $\sigma \geq \frac{24+\alpha(1-\alpha)}{(5-\alpha)(4+\alpha)}L$.

Case B:

Step 1: Assume Ordering 2.

Step 2: Conjecture the same equilibrium strategies as in Case A, which implies identical cutoff prices.

Step 3: The proposed strategies are best replies (for buyers) iff

$$\begin{aligned}
\frac{1-\alpha}{4}\left[v_t-\sigma+L-C_{ST}^{s*}(v_t-\sigma,-L)\right] &\geq \frac{1}{4}\left[v_t-\sigma+L-C_{FT}^{s*}(v_t-\sigma,-L)\right] \\
\frac{1-\alpha}{4}\left[v_t-\sigma+L-C_{ST}^{s*}(v_t-\sigma,-L)\right] &\geq \frac{2-\alpha}{4}\left[v_t-\sigma+L-C_{ST}^{s*}(v_t+\sigma,-L)\right] \\
&\quad +\frac{1-\alpha}{4}\left[v_t-\sigma+L-C_{ST}^{s*}(v_t+\sigma,-L)\right] \\
\frac{1-\alpha}{4}\left[v_t-\sigma+L-C_{ST}^{s*}(v_t-\sigma,-L)\right] &\geq \frac{1}{2}\left[v_t-\sigma+L-C_{ST}^{s*}(v_t+\sigma,-L)\right] \\
&\quad +\frac{1}{4}\left[v_t-\sigma+L-C_{FT}^{s*}(v_t+\sigma,-L)\right] \\
\frac{\alpha}{4}\left[v_t-\sigma+L-C_{FT}^{s*}(v_t-\sigma,-L)\right] &\geq \frac{\alpha}{2}\left[v_t-\sigma+L-C_{FT}^{s*}(v_t+\sigma,-L)\right] \\
&\quad +\frac{\alpha}{4}\left[v_t+\sigma+L-C_{FT}^{s*}(v_t+\sigma,-L)\right]
\end{aligned}$$

These inequalities together with the conjectured ordering of cutoff prices are satisfied iff $\alpha \leq \alpha_1^*$ and $\frac{24+\alpha(1-\alpha)}{(5-\alpha)(4+\alpha)}L > \sigma \geq L$.

Case C:

Step 1: Assume Ordering 3.

Step 2: Conjecture the same equilibrium strategies as in Case A, which implies identical cutoff prices.

Step 3: The proposed strategies are best replies (for buyers) iff

$$\begin{aligned}
\frac{1-\alpha}{4}\left[v_t-\sigma+L-C_{ST}^{s*}(v_t-\sigma,-L)\right] &\geq \frac{1}{4}\left[v_t-\sigma+L-C_{FT}^{s*}(v_t-\sigma,-L)\right] \\
\frac{1-\alpha}{4}\left[v_t-\sigma+L-C_{ST}^{s*}(v_t-\sigma,-L)\right] &\geq \frac{1}{4}\left[v_t-\sigma+L-C_{ST}^{s*}(v_t+\sigma,-L)\right] \\
&\quad +\frac{1-\alpha}{4}\left[v_t-\sigma+L-C_{ST}^{s*}(v_t+\sigma,-L)\right] \\
\frac{1-\alpha}{4}\left[v_t-\sigma+L-C_{ST}^{s*}(v_t-\sigma,-L)\right] &\geq \frac{2-\alpha}{4}\left[v_t-\sigma+L-C_{ST}^{s*}(v_t+\sigma,-L)\right] \\
&\quad +\frac{1}{4}\left[v_t-\sigma+L-C_{FT}^{s*}(v_t+\sigma,-L)\right] \\
\frac{\alpha}{4}\left[v_t-\sigma+L-C_{FT}^{s*}(v_t-\sigma,-L)\right] &\geq \frac{\alpha}{4}\left[v_t-\sigma+L-C_{FT}^{s*}(v_t+\sigma,-L)\right] \\
&\quad +\frac{\alpha}{4}\left[v_t+\sigma+L-C_{FT}^{s*}(v_t+\sigma,-L)\right]
\end{aligned}$$

These inequalities together with the conjectured ordering of cutoff prices are satisfied iff $\alpha \leq \alpha_1^*$ and $L > \sigma \geq \sigma_1^*$. Combining Cases A, B and C, we conclude that the following quotation strategy and the associated order choice strategy constitute an equilibrium iff $\alpha \leq \alpha_1^*$ and $\sigma \geq \sigma_1^*$

$$\begin{aligned}
B_{t,HT}^* &= C_{ST}^{s*}(v_t-\sigma,-L) \qquad & B_{t,AT}^* = C_{FT}^{s*}(v_t-\sigma,-L) \\
A_{t,HT}^* &= C_{ST}^{b*}(v_t+\sigma,+L) \qquad & A_{t,AT}^* = C_{FT}^{b*}(v_t+\sigma,+L)
\end{aligned}$$

The proof for the remaining equilibria follows exactly the same logic. In order to conserve space, we will simply indicate the orderings that give rise to those equilibria and give the equilibrium bid quotes in terms of the equilibrium cutoff prices (the ask quotes follow by symmetry).

Type 2 equilibrium: Orderings 1 – 4

Type 3 equilibrium: Orderings 3 and 4

Type 4 equilibrium: Ordering 4

Type 5 equilibrium: Ordering 4

The remaining cutoff variables are defined as

$\alpha_2^* = \frac{\sqrt{33}-5}{2}$ $\qquad \sigma_2^* = L\frac{2\alpha(1+\alpha)}{3-4\alpha}$ $\qquad \sigma_3^* = L\frac{4(4+\alpha)}{26-\alpha^2}$ $\qquad \sigma_4^* = L\frac{2(1-\alpha)(4+\alpha)}{7+3\alpha}$
$\sigma_5^* = L\frac{4(1+\alpha)}{7+3\alpha}$

The following table gives closed form solutions for the sellers' sell cutoff prices in each type of equilibrium.

Equilibrium Type	$C_{ST}^{s*}(v_t, -L)$	$C_{FT}^{s*}(v_t, -L)$
1	$v_t - L + (2L)\frac{1-\alpha}{5-\alpha}$	$v_t - L + (2L)\frac{8-\alpha(3+\alpha)}{(5-\alpha)(4+\alpha)}$
2	$v_t - L + (2L)\frac{1+\alpha}{7+3\alpha}$	$v_t - L + (2L)\frac{3-\alpha}{7+3\alpha}$
3	$v_t - L + (2L)\frac{2-\alpha}{6-\alpha} - \frac{2}{6-\alpha}\sigma$	$v_t - L + (2L)\frac{8-\alpha(2+\alpha)}{(6-\alpha)(4+\alpha)} + \frac{4(1-\alpha)}{(6-\alpha)(4+\alpha)}\sigma$
4	$v_t - L + (2L)\frac{2-\alpha}{6-\alpha} - \frac{2}{6-\alpha}\sigma$	$v_t - L + (2L)\frac{4+\alpha(2-\alpha)}{(6-\alpha)(2+\alpha)} + \frac{2-\alpha(8-\alpha)}{(6-\alpha)(2+\alpha)}\sigma$
5	$v_t - L + (2L)\frac{1}{3} - \frac{5+3\alpha}{3(1+\alpha)}\sigma$	$v_t - L + (2L)\frac{1}{3} + \frac{1-3\alpha}{3(1+\alpha)}\sigma$

Finally, it is possible to show that there exist no other equilibria than the ones just obtained, which yields uniqueness. The involved calculations are very long and tedious, such they are omitted for brevity. ■

7.4 Proof of Proposition 2

It follows from equations (5) and (6) that $V_{ST}^{LO*} = C_{ST}^{s*}(v_t, -L) - (v_t - L)$ and $V_{FT}^{LO*} = C_{FT}^{s*}(v_t, -L) - (v_t - L)$. Then $V_{FT}^{LO*} > V_{ST}^{LO*}$ is a direct consequence of Lemma 2. Moreover, we know from Foucault (1999) that $V_0^{LO*} = 2L/5$ for $\sigma \geq \sigma_1^*(0)$ and $V_0^{LO*} = (2L - \sigma)/3$ for $\sigma < \sigma_1^*(0)$. Using the sell cutoff prices in Table x, it is straightforward to verify that we have $V_{FT}^{LO*} > V_0^{LO*} > V_{ST}^{LO*}$. ■

7.5 Proof of Proposition 3

For each type of equilibrium, the transitions from one state to another follow a Markov chain with transition matrix P^i, $i \in \{1, 2, 3, 4, 5\}$. Using the conditional equilibrium execution probabilities (they are easily read off the tables contained in the proof of Proposition 1), it is straightforward to obtain

$$P^1 = \begin{bmatrix} \frac{3(1-\alpha)}{4} & \frac{1-\alpha}{4} & \alpha & 0 \\ 1-\alpha & 0 & \alpha & 0 \\ \frac{1-\alpha}{2} & \frac{1-\alpha}{2} & \alpha & 0 \\ 1-\alpha & 0 & \alpha & 0 \end{bmatrix} \qquad P^2 = \begin{bmatrix} \frac{3(1-\alpha)}{4} & \frac{1-\alpha}{4} & \frac{3\alpha}{4} & \frac{\alpha}{4} \\ 1-\alpha & 0 & \alpha & 0 \\ \frac{1-\alpha}{2} & \frac{1-\alpha}{2} & \frac{3\alpha}{4} & \frac{\alpha}{4} \\ 1-\alpha & 0 & \alpha & 0 \end{bmatrix}$$

$$P^3 = \begin{bmatrix} \frac{1-\alpha}{2} & \frac{1-\alpha}{2} & \frac{3\alpha}{4} & \frac{\alpha}{4} \\ 1-\alpha & 0 & \alpha & 0 \\ \frac{1-\alpha}{2} & \frac{1-\alpha}{2} & \frac{3\alpha}{4} & \frac{\alpha}{4} \\ 1-\alpha & 0 & \alpha & 0 \end{bmatrix} \qquad P^4 = \begin{bmatrix} \frac{1-\alpha}{2} & \frac{1-\alpha}{2} & \frac{3\alpha}{4} & \frac{\alpha}{4} \\ 1-\alpha & 0 & \alpha & 0 \\ \frac{1-\alpha}{2} & \frac{1-\alpha}{2} & \frac{\alpha}{2} & \frac{\alpha}{2} \\ 1-\alpha & 0 & \alpha & 0 \end{bmatrix}$$

$$P^5 = \begin{bmatrix} \frac{1-\alpha}{2} & \frac{1-\alpha}{2} & \frac{\alpha}{2} & \frac{\alpha}{2} \\ 1-\alpha & 0 & \alpha & 0 \\ \frac{1-\alpha}{2} & \frac{1-\alpha}{2} & \frac{\alpha}{2} & \frac{\alpha}{2} \\ 1-\alpha & 0 & \alpha & 0 \end{bmatrix}$$

Given these transition matrices, the stationary probability distribution $\varphi^i = (\varphi_1^i, \varphi_2^i, \varphi_3^i, \varphi_4^i)$ is given by the left eigenvector associated with the unit eigenvalue. Straightforward calculations reveal

$$\begin{aligned}
\varphi^1 &= \left(\frac{4(1-\alpha)(4-\alpha)}{(4+\alpha)(5-\alpha)}, \frac{(1-\alpha)(4+5\alpha-\alpha^2)}{(4+\alpha)(5-\alpha)}, \frac{4\alpha}{4+\alpha}, \frac{\alpha^2}{4+\alpha}\right) \\
\varphi^2 &= \left(\frac{4(1-\alpha)(4-\alpha)}{20-\alpha+\alpha^2}, \frac{(1-\alpha)(4+3\alpha+\alpha^2)}{20-\alpha+\alpha^2}, \frac{16\alpha}{20-\alpha+\alpha^2}, \frac{\alpha(4-\alpha+\alpha^2)}{20-\alpha+\alpha^2}\right) \\
\varphi^3 &= \left(\frac{(1-\alpha)(4-\alpha)}{6-\alpha}, \frac{2(1-\alpha)}{6-\alpha}, \frac{\alpha(5-\alpha)}{6-\alpha}, \frac{\alpha}{6-\alpha}\right) \\
\varphi^4 &= \left(\frac{8(1-\alpha)}{12+\alpha-\alpha^2}, \frac{(1-\alpha)(4+\alpha-\alpha^2)}{12+\alpha-\alpha^2}, \frac{2\alpha(5-\alpha)}{12+\alpha-\alpha^2}, \frac{\alpha(2+3\alpha-\alpha^2)}{12+\alpha-\alpha^2}\right) \\
\varphi^5 &= \left(\frac{2(1-\alpha)}{3}, \frac{1-\alpha}{3}, \frac{2\alpha}{3}, \frac{\alpha}{3}\right)
\end{aligned}$$

Using the definition of the trading rates in equations (9), (11) and (13), we obtain

$$\begin{array}{lll}
TR^1 = \frac{4+\alpha(1-\alpha)}{(4+\alpha)(5-\alpha)} & TR^1_{ST} = \frac{4+\alpha(5-\alpha)}{(4+\alpha)(5-\alpha)} & TR^1_{FT} = \frac{\alpha}{4+\alpha} \\
TR^2 = \frac{4+3\alpha(1-\alpha)}{20-\alpha+\alpha^2} & TR^2_{ST} = \frac{4+\alpha(3+\alpha)}{20-\alpha+\alpha^2} & TR^2_{FT} = \frac{4-\alpha(1-\alpha)}{20-\alpha+\alpha^2} \\
TR^3 = \frac{2-\alpha}{6-\alpha} & TR^3_{ST} = \frac{2}{6-\alpha} & TR^3_{FT} = \frac{1}{6-\alpha} \\
TR^4 = \frac{4-\alpha(1-\alpha)}{12+\alpha-\alpha^2} & TR^4_{ST} = \frac{4+\alpha(1-\alpha)}{12+\alpha-\alpha^2} & TR^4_{FT} = \frac{2+\alpha(3-\alpha)}{12+\alpha-\alpha^2} \\
TR^5 = \frac{1}{3} & TR^5_{ST} = \frac{1}{3} & TR^5_{FT} = \frac{1}{3}
\end{array}$$

It is immediate that we have $TR^i_{ST} \geq TR^i_{FT}$ for all i, such that $TR^*_{ST} \geq TR^*_{FT}$ follows. Foucualt (1999) shows that $TR^*_0 = 1/5$ for $\sigma \geq \sigma_1^*(0)$ and $TR^*_0 = 1/3$ otherwise. If $\sigma \geq \sigma_1^*(0)$, a type-1, type-2 or type-3 equilibrium may arise. It is straightforward to verify that $TR^i > 1/5$ for $i = 1, 2, 3$. Similarly, if $\sigma < \sigma_1^*(0)$, a type-2, type-3, type-4 and type-5 equilibrium may arise. It is easy to check that in this case, we have $TR^i \leq 1/3$ for $i = 2, 3, 4, 5$ as required. ■

7.6 Proof of Proposition 4

Using the closed-form equilibrium quotes (see the proof of Proposition 1), it is straightforward to calculate the trading costs for each combination of limit order trader and market order trader via equations (15) - (18) and (20) – (23). They are collected in the following table.

Trading Cost	Type 1	Type 2	Type 3	Type 4	Type 5
$\tau^{S,+\sigma}_{ST,ST}$	n/a	n/a	$\frac{2+\alpha}{6-\alpha}L + \frac{2}{6-\alpha}\sigma$	$\frac{2+\alpha}{6-\alpha}L + \frac{2}{6-\alpha}\sigma$	$\frac{1}{3}L - \frac{1-3\alpha}{3(1+\alpha)}\sigma$
$\tau^{S,-\sigma}_{ST,ST}$	$\frac{3+\alpha}{5-\alpha}L$	$\frac{1+5\alpha}{7+3\alpha}L$	$\frac{2+\alpha}{6-\alpha}L - \frac{10-2\alpha}{6-\alpha}\sigma$	$\frac{2+\alpha}{6-\alpha}L - \frac{10-2\alpha}{6-\alpha}\sigma$	$\frac{1}{3}L - \frac{7+3\alpha}{3(1+\alpha)}\sigma$
$\tau^{S,+\sigma}_{FT,ST}$	n/a	n/a	n/a	n/a	$\frac{1}{3}L - \frac{1-3\alpha}{3(1+\alpha)}\sigma$
$\tau^{S,-\sigma}_{FT,ST}$	n/a	$\frac{1+5\alpha}{7+3\alpha}L$	$\frac{2+\alpha}{6-\alpha}L - \frac{10-2\alpha}{6-\alpha}\sigma$	$\frac{2+\alpha}{6-\alpha}L - \frac{10-2\alpha}{6-\alpha}\sigma$	$\frac{1}{3}L - \frac{7+3\alpha}{3(1+\alpha)}\sigma$
$\tau^{S,+\sigma}_{ST,FT}$	$\frac{3+\alpha}{5-\alpha}L$	$\frac{5+\alpha}{7+3\alpha}L$	$\frac{2+\alpha}{6-\alpha}L + \frac{2}{6-\alpha}\sigma$	$\frac{2+\alpha}{6-\alpha}L + \frac{2}{6-\alpha}\sigma$	$\frac{1}{3}L + \frac{2}{3(1+\alpha)}\sigma$
$\tau^{S,-\sigma}_{ST,FT}$	$\frac{3+\alpha}{5-\alpha}L$	$\frac{5+\alpha}{7+3\alpha}L$	$\frac{2+\alpha}{6-\alpha}L + \frac{2}{6-\alpha}\sigma$	$\frac{2+\alpha}{6-\alpha}L + \frac{2}{6-\alpha}\sigma$	$\frac{1}{3}L + \frac{2}{3(1+\alpha)}\sigma$
$\tau^{S,+\sigma}_{FT,FT}$	n/a	n/a	n/a	$\frac{4+\alpha^2}{(6-\alpha)(2+\alpha)}L - \frac{2-\alpha(8-\alpha)}{(6-\alpha)(2+\alpha)}\sigma$	$\frac{1}{3}L - \frac{1-3\alpha}{3(1+\alpha)}\sigma$
$\tau^{S,-\sigma}_{FT,FT}$	$\frac{4+\alpha(7+\alpha)}{(5-\alpha)(4+\alpha)}L$	$\frac{1+5\alpha}{7+3\alpha}L$	$\frac{8+\alpha(6+\alpha)}{(6-\alpha)(2+\alpha)}L - \frac{4(1+\alpha)}{(6-\alpha)(2+\alpha)}\sigma$	$\frac{4+\alpha^2}{(6-\alpha)(2+\alpha)}L - \frac{26-\alpha^2}{(6-\alpha)(2+\alpha)}\sigma$	$\frac{1}{3}L - \frac{7+3\alpha}{3(1+\alpha)}\sigma$

Then, the expected trading costs for ST and FT, respectively, are obtained by straightforward substitution into equations (19) and (24). We obtain

$$E(\tau^1_{ST}) = \frac{3+\alpha}{5-\alpha}L$$
$$E(\tau^1_{FT}) = \frac{4+\alpha(7+\alpha)}{(5-\alpha)(4+\alpha)}L$$
$$E(\tau^2_{ST}) = \frac{(4-\alpha)(1-\alpha)(1+5\alpha)+8\alpha(5+\alpha)}{(7+3\alpha)((4-\alpha)(1-\alpha)+8\alpha)}L$$
$$E(\tau^2_{FT}) = \frac{1+5\alpha}{7+3\alpha}L$$
$$E(\tau^3_{ST}) = \frac{2+\alpha}{6-\alpha}L + \frac{2\alpha(5-\alpha)-(1-\alpha)(4-\alpha)^2}{4(6-\alpha)}\sigma$$
$$E(\tau^3_{FT}) = \frac{(1-\alpha)(4-\alpha)(4+\alpha)(2+\alpha)+\alpha(5-\alpha)(8+\alpha(6+\alpha))}{(4+\alpha)(6-\alpha)}L + \frac{2(5-\alpha)((4+\alpha)(4-\alpha)+2\alpha)}{(4+\alpha)(6-\alpha)}\sigma$$
$$E(\tau^4_{ST}) = \frac{2+\alpha}{6-\alpha}L + \frac{2\alpha(5-\alpha)-(1-\alpha)(4-\alpha)^2}{4(6-\alpha)}\sigma$$
$$E(\tau^4_{FT}) = \frac{8(1-\alpha)(2+\alpha)^2+4\alpha(5-\alpha)(4+\alpha^2)}{(2+\alpha)(6-\alpha)(8+4\alpha(3-\alpha))}L + \frac{2(5-\alpha)(8(1-\alpha)(2+\alpha)+\alpha(28-8\alpha)}{(2+\alpha)(6-\alpha)(8+4\alpha(3-\alpha))}\sigma$$
$$E(\tau^5_{ST}) = \frac{1}{3}L - \frac{4-6\alpha}{3(1+\alpha)}\sigma$$
$$E(\tau^5_{FT}) = \frac{1}{3}L - \frac{4}{3(1+\alpha)}\sigma$$

Average expected trading costs for each type of equilibrium can then be computed using equation (25). It is easy to see that expected overall trading costs in the absence of AT are given by $E(\tau^*_0) = 3L/5$ for $\sigma \geq \sigma^*_1(0)$ and $E(\tau^*_0) = (L - 2\sigma)/3$ for $\sigma < \sigma^*_1(0)$.

a) For $\sigma \geq \sigma^*_1(0)$, the only equilibria that may arise are of type 1, 2 or 3. Some tedious calculations reveal $E(\tau^1) > 3L/5 > E(\tau^2)$ for all $\alpha \in (0, 1)$. Moreover, a type-3 equilibrium can only arise for $\sigma < \sigma^*_1(\alpha^*_1)$, and it is straightforward, albeit cumbersome, to conclude that we have $3L/5 > E(\tau^3)$

in this case. Finally, we note that for every level of $\sigma \geq \sigma_1^*(0)$, a type-1 equilibrium arises only for $\alpha \leq \alpha_1^*$.

b) For $\sigma_1^*(0) > \sigma \geq \sigma_2^*(1/3)$, equilibria of types 2, 3 or 5 may arise. Brute force algebra reveals that, for all $\alpha \in (0,1)$ and $i \in \{1,2,3\}$, we have $E(\tau^i) > (L-2\sigma)/3$ in the range considered for σ.

c) For $\sigma_2^*(1/3) > \sigma$, the only equilibria that can arise are of type 4 or type 5. Clearly, $E(TC^4) > (L-2\sigma)/3$ for all $\alpha \in (0,1)$. Moreover, it is easy to see that $E(\tau^5) < (L-2\sigma)/3$ for $\alpha < 1/3$ and $E(\tau^5) \geq (L-2\sigma)/3$ otherwise. Finally, we notice that for every level of $\sigma < \sigma_2^*(1/3)$, a type-4 equilibrium arises for a sufficiently small level of α. ■

7.7 Proof of Proposition 5

While the involved calculations are quite tedious, the proof follows along the lines of that of Proposition 4 and is therefore omitted for brevity. ■

7.8 Proof of Proposition 6

First, we note that $W_0^* = 2L/3$ for $\sigma < \sigma_1^*(\alpha_1^*)$ and $W_0^* = 2L/5$ otherwise. The equilibrium welfare for each trader type is obtained by combining the equilibrium expected profits from posting limit orders, stationary probability distributions and trading costs from the Proofs of Propositions 2, 3 and 4. While the calculations are very cumbersome, they are straightforward. ■

8 Appendix B: Figures

Table 1: This table contains the equilibrium combinations for fixed levels of σ (details are described at the end of Section 4.1). We have set $L = 1$. The switching point levels of α are denoted $\widetilde{\alpha}_1$ and $\widetilde{\alpha}_2$.

Equilibrium Combination	σ range	σ	$\widetilde{\alpha}_1$	$\widetilde{\alpha}_2$
1: EQ 1, EQ 2	$\sigma \geq \sigma_1^*(\alpha_1^*)$	1	0.2361	n/a
2: EQ 1, EQ 3, EQ 2	$\sigma_1^*(\alpha_1^*) > \sigma \geq \sigma_1^*(0)$	0.8198	0.1220	0.2521
3: EQ 3, EQ 2, EQ 5	$\sigma_1^*(0) > \sigma \geq \sigma_3^*(\alpha_2^*)$	0.7381	0.3198	0.6544
4: EQ 3, EQ 4, EQ 5	$\sigma_3^*(\alpha_2^*) > \sigma \geq \sigma_3^*(0)$	0.6458	0.1930	0.3647
5: EQ 4, EQ 5	$\sigma_3^*(0) > \sigma$	0.3077	0.2477	n/a

Figure 1: Equilibrium Map
This graph depicts the different regions in the (α, σ)-space that give rise to the respective equilibria. We have set $L = 1$.

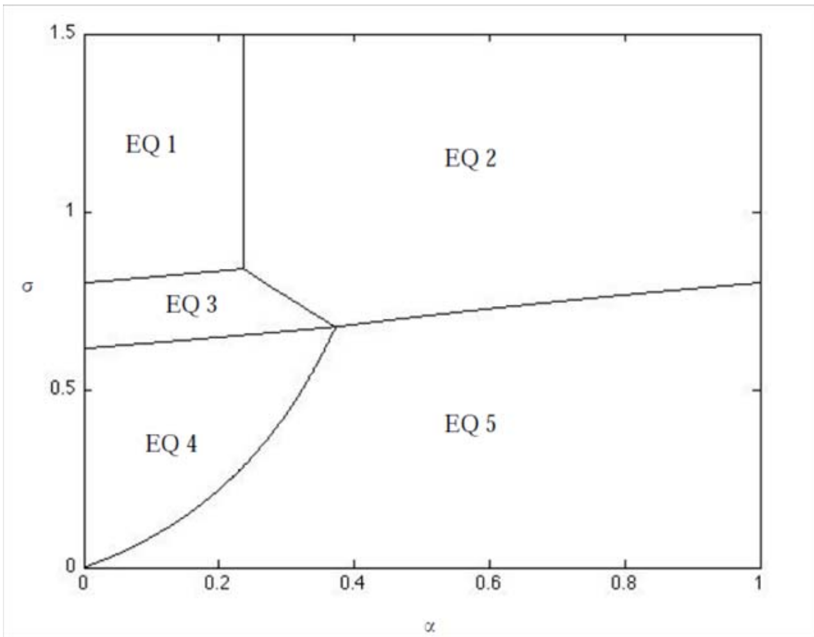
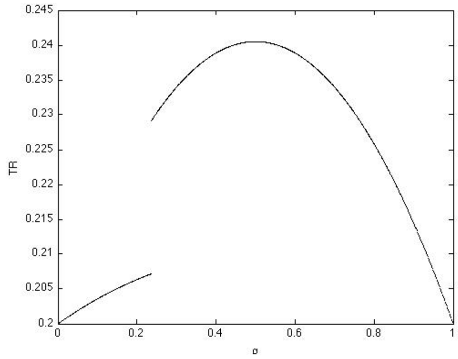
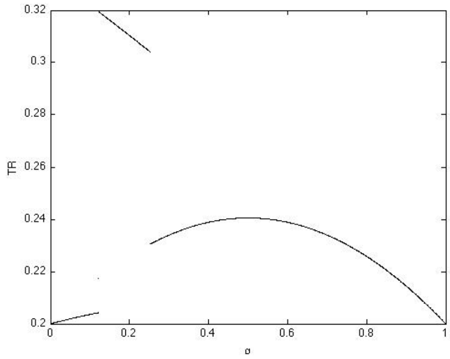


Figure 2: Trading Rate
These figures depict the equilibrium trading rates for different levels of σ (descending from top to bottom, see Table 1) as a function of α.



1)



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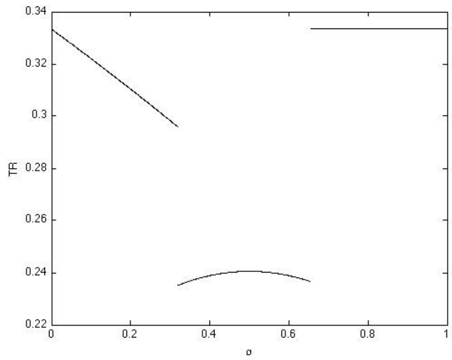

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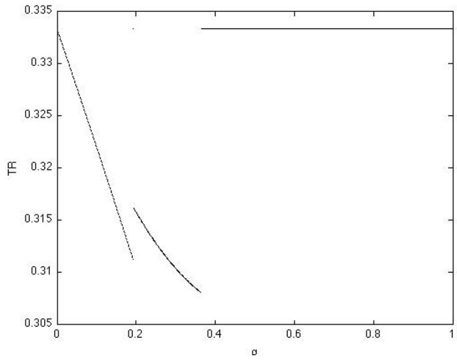

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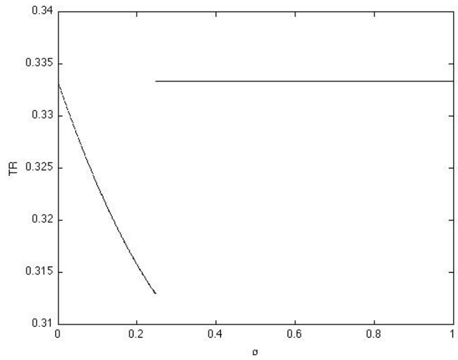
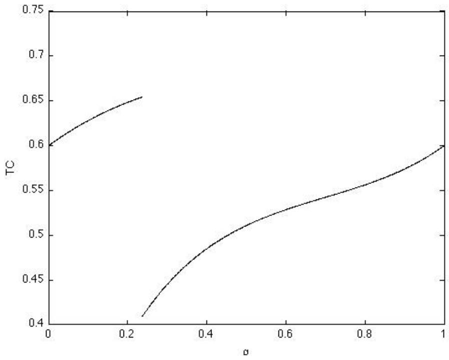


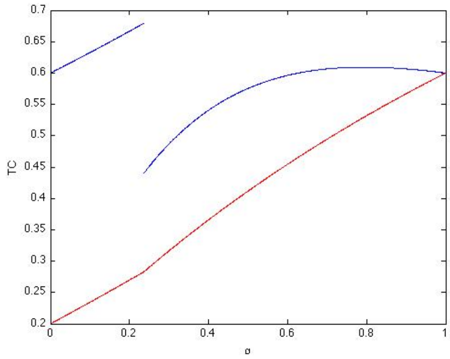
5)

Figure 3: Trading Costs

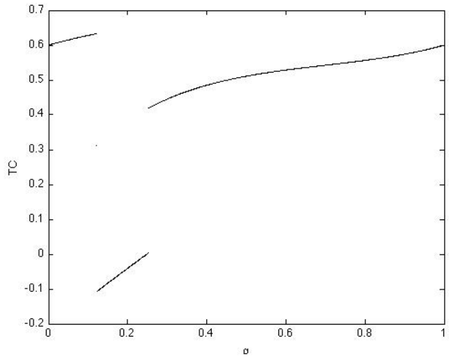
The figures in the left column (a) depict the the equilibrium average expected trading cost for different levels of σ (descending from top to bottom, see Table 1) as a function of α. The figures in the right column (b) show the equilibrium expected trading costs for slow (blue) and fast (red) traders.



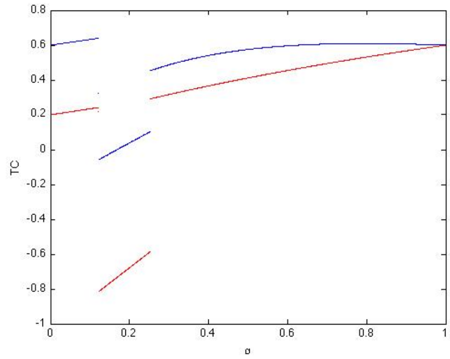
1a)



1b)



2a)



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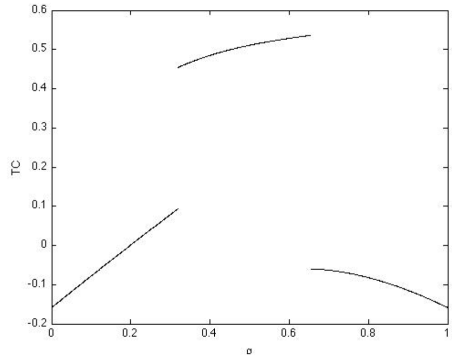
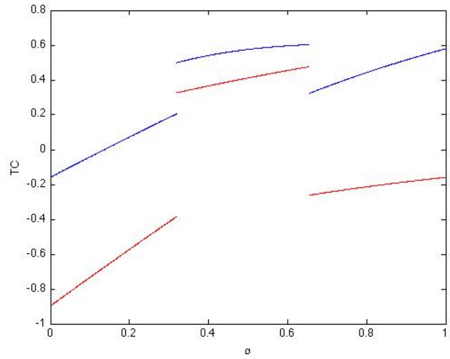
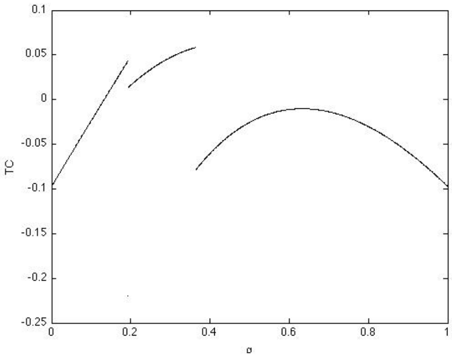


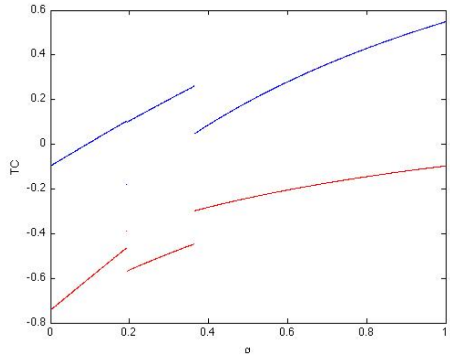
3a)



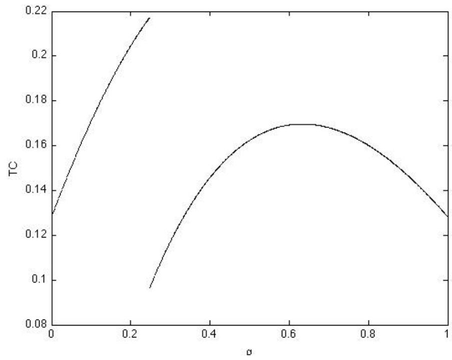
3b)



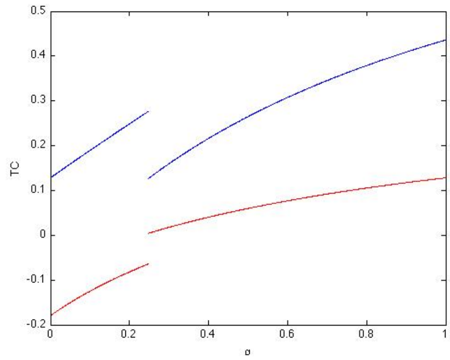
4a)



4b)



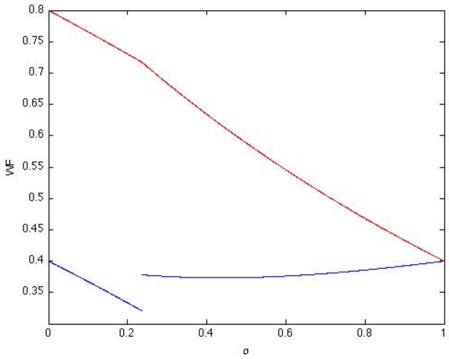
5a)



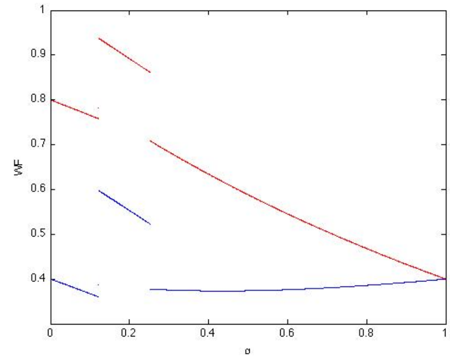
5b)

Figure 4: Welfare

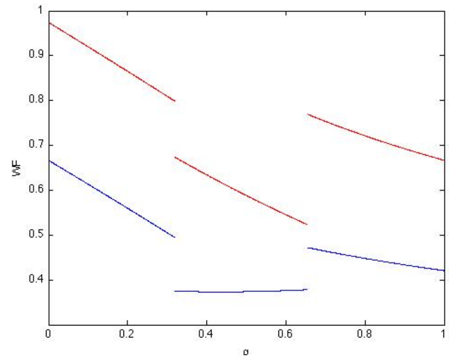
These graphs depict the equilibrium welfare (expected utility) for slow (blue) and fast (red) traders for different levels of σ (descending from top left to bottom right, see Table 1) as a function of α. Notice that welfare for the average trader is proportional to the equilibrium trading rate (Figure 2).



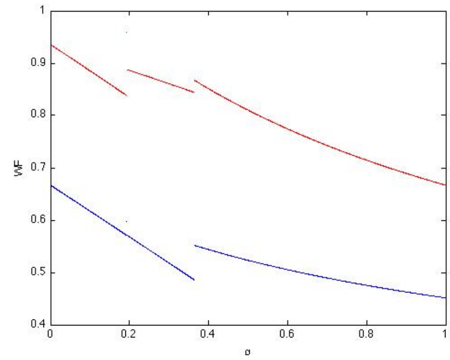
1)



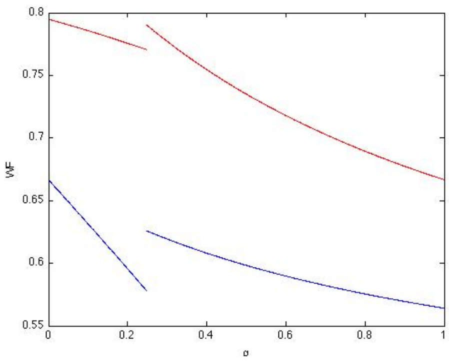
2)



3)



4)



4.5)

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