Equilibrium Effects of Liquidity Constraints

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Abstract

This paper analyzes the relationship between collateralized short-term debt and asset prices. Banks increasingly used short-term debt during the lending boom before the last financial crisis (Shin [2009]). This caused distortions in the asset price and triggered the excessive asset growth in the market. The increase in the short-term borrowing also raised the interconnectedness of the financial institutions and led to systemic risk. In order to achieve a stable financial system, the amount of short-term debt may need regulation. In this paper, I explore the welfare effects of a regulatory quantity limit on short-term debt. The results show that a quantity restriction on short-term borrowing is welfare improving during the expansion when the investors are optimistic. However, it is welfare decreasing during the recession when the investors are pessimistic. Therefore, the regulatory limit on short-term borrowing should be counter-cyclical: lower during expansions and higher during recessions.

keywords: Financial crisis, prudential regulation, Basel III, financial intermediaries

JEL Classifications: G01, G21, G28

1 Introduction

The fragility of bank liabilities was at the core of the last financial crisis. The excessive asset growth led to an extreme increase in the short-term funding compared to the retail deposits on the liability side of the banks' balance sheets. This makes banks more vulnerable to the liquidity of the capital market. A sudden freeze in the short-term funding market caused fire-sales of the assets and the bankruptcy of big banks and other financial institutions (Diamond and Rajan [2009]). The bankruptcies amplified the effects and led to one of the biggest financial crises.

In the post-crisis period, the new regulation Basel III is under construction to satisfy the needs of the financial system. The last financial crisis was a good indicator of the necessity for a macro-prudential regulation focusing not only on the solvency of individual financial institutions but also the resilience of the financial system. However, Basel III does not seem to satisfy this necessity under its current form. It concentrates on the increase of bank capital as a caution for losses, which is a solution for the solvency of an individual financial institution, but it is not adequate to support systemic stability.

Basel III, focusing on the amount of bank capital, takes too little account of the funding structure of banks. During booms, banks take on excessive short-term debt to finance excessive asset growth. Brunnermeier and Oehmke [2010] show that creditors have an incentive to shorten their loan maturity to be able to get their money back before the others in bad times. This contributes to the growth of the asset and they borrow more. The increase in short-term borrowing has two effects in the market. First, it causes a distortion in the asset price and exaggerates the increase in asset growth (Jeanne and Korinek [2010],Bengui [2010]). Second, it increases the interconnectedness of the financial institutions which leads to the systemic risk (Acharya and Viswanathan [2011],Acharya et al. [2010]). Thus, the liability side of the bank balance sheet needs to be regulated to achieve the goal of a stable financial system.

In this paper, I concentrate on the relationship between short-term debt accumulation and asset prices and how this relationship amplifies asset growth in the market during credit booms. The vast majority of short-term debt held by banks before the crisis was secured debt (e.g. asset backed commercial papers, overnight secured repos). During $2002-2007$, the overnight credit (repo) grew to a volume over ten trillion dollars(Gorton [2009]). Accordingly, I concentrate on short-term debt collateralized by an asset. So this paper is also related to the literature on how collateral debt amplifies the effects of credit cycles in the economy (Kiyotaki and Moore).

The modeling framework builds on the general equilibrium model of Geanakoplos [2009] with heterogeneous agents holding different beliefs about the future. Optimistic investors borrow using the asset as collateral and buy the asset, whereas pessimistic investors lend to the optimistic ones and sell the asset. If there is no limit on borrowing of the optimistic investor, they borrow at the maximum level and go bankrupt when a bad state occurs. This decreases the optimism in the market since optimistic investors are wiped out. This leads to a sharp decline in the price of the asset and the borrowing capacity of borrowers decreases and this amplifies the effect of bankruptcies of optimistic investors. Now apply a quantity limit on the borrowing capacity. When a bad state occurs, optimistic investors do not go bankrupt and this does not lead to a sharp decline in the optimism of the market. This mitigates the effects of the credit cycle. Thus, a quantity limit on borrowing capacity of investors increases the welfare of the economy during the normal times and credit booms. During busts, when the investors are so pessimistic about the future, any regulatory limit is welfare decreasing.

In the literature, there are papers on the advantages of short-term debt arguing that it can be used as a disciplining mechanism that gives correct incentives to the manager so it mitigates moral hazard (Calomiris and Kahn [1991], Diamond and Rajan [1999], Diamond and Dybvig [1983]). In contrast to this literature, this paper analyzes the disadvantages of having too much short-term debt and how this amplifies the effects of credit booms and busts.

The paper is related to the literature on the regulation of liquidity risk. Shin [2011] proposes to have a levy on non-core liabilities (any liability of an intermediary held by another intermediary) to mitigate pricing distortions that lead to excessive asset growth and the systemic risk coming from the interconnectedness of financial institutions via short-term debt. Perotti and Suarez [2011] show that a pigovian tax on short-term borrowing (unsecured) can be an efficient solution to limit the banks' excessive short-term borrowing. They concentrate on the regulation of systemic externality associated with banks' short-term funding. This paper studies the effects of collateralized short-term debt on the price of the asset and how this interaction, between the amount of short-term debt and the collateral prices, feed each other during credit booms and busts.

In this paper, I apply a quantity limit on short-term borrowing in order to regulate the liability side of the banks' balance sheet. The results show that a quantity limit on short-term borrowing increases welfare when investors are optimistic about the market, but it decreases welfare when investors are pessimistic. In other words, limiting short-term borrowing is welfare improving during booms and welfare decreasing during busts. These results implythat a regulatory quantity limit on short-term debt is efficient if it is counter-cyclical.

2 Model

Consider a simple two-period economy with a continuum of heterogeneous investors. The economy contains one consumption good (used as numeraire) and one asset with payoffs shown in Figure 1. If the state is up (U) at date 1 then the asset pays off 1 unit of consumption good without risk at date 2. If the state is down (D) at date 1 then the asset pays off either 1 or d units of consumption good where $0 < d < 1$. At date 0, each atomistic investor holds one unit of consumption good and one unit of the asset. The investors are heterogeneous in their beliefs about the future. Each investor holds different belief about the probability of the good state which is denoted by h where $h \in [0, 1]$. I work on the general equilibrium model of Geanakoplos [2009] which allows collateralized short-term borrowing.

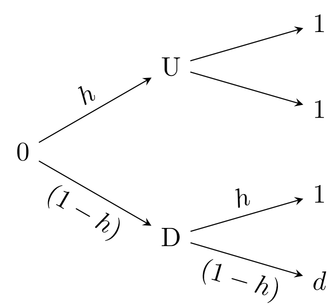

Figure 1: The two-period model.

Each investor is allowed to borrow using her asset as collateral. The borrowing capacity of the asset is limited by the worst payoff of the asset in the next period. For example, in state D at date 1, the investor who is holding y_1 assets can borrow at most dy_1 units of consumption good. I define the regulatory limit on the amount of short-term debt as a limit on the borrowing capacity of the asset with a coefficient α where $\alpha \in [0, 1]$. For example, the same investor holding y_1 assets can borrow at most αdy_1 units of consumption good when there exists a regulatory limit on borrowing. One can think this limit as the fraction of the asset that can be pledged to borrow using this asset as collateral.

In the following sections, I solve for the equilibrium of the model and analyse the comparative statics for both risk-neutral and risk-averse investors.

2.1 Equilibrium when the investors are risk-neutral

When investors are risk-neutral with heterogeneous beliefs h, where $h \in [0,1]$ and uniformly distributed, the date 0 utility for each agent is $C_0^h + hC_U^h + (1-h)C_D^h$ where C_0^h is the consumption at date 0, C_U^h and C_D^h are consumption in the good state and in the bad state at date 1, respectively.

At date 0, each investor chooses her consumption (C_0^h), the borrowing amount (φ_0), the amount that she warehouses (ω_0), and the amount of the asset she buys (y_0) by solving the following optimization problem

$$\max_{\{C_0, \varphi_0, \omega_0, y_0\}} \quad \{C_0^h + hC_U^h + (1-h)C_D^h\}$$

$$\begin{aligned}
C_U^h &= \omega_0 + y_0 - \varphi_0 \\
C_D^h &= \omega_0 + P_D y_0 - \varphi_0 \\
C_0^h + \omega_0 + P_0 y_0 &= P_0 + 1 + \varphi_0 \\
\varphi_0 &\leq \alpha P_D y_0 \\
\omega_0 &\geq 0 \\
y_0 &\geq 0
\end{aligned}$$

where P_0 is the price of the asset at date 0 and P_D is the price of the asset at date 1 in the state D.

In equilibrium, the investor holding the belief (marginal buyer)

$$h^*(P) = \frac{P_0 - P_D}{1 - P_D}$$

is indifferent between buying and selling the asset. The belief of the marginal buyer determines the price of the asset. Let P denote the vector of prices $P = (P_0, P_D)$.

The investors who are holding beliefs (optimistic investors) higher than $h^*(P)$ are buyers of the asset and borrowers. They borrow at the maximum level $\varphi_0 = \alpha P_D y_0$. The sellers are the ones who are holding beliefs (pessimistic investors) lower than the marginal belief $h^*(P)$. In equilibrium, optimistic investors are buyers and borrowers, while pessimistic investors are sellers and lenders. The amount of the asset that is bought by optimistic investors is

$$y_0 = \frac{P_0 + 1}{P_0 - P_D \alpha}.$$

The market for the asset clears when the demand of the asset is equal to 1 since there is one asset in the market. The market clearing equation for the asset is then

$$\int_{h^*(P)}^{1} y_0 \,\mathrm{d}h = 1$$

and the market clearing condition for the borrowing amount is

$$\int_{h^*(P)}^{1} \alpha P_D y_0 \,\mathrm{d}h = \int_{0}^{h^*(P)} \varphi_0^L \,\mathrm{d}h$$

where the demand by borrowers (buyers of the asset) is equal to the supply by lenders (sellers of the asset).

Proposition 2.1 *In equilibrium at date* 0*, investors holding beliefs (optimistic investors) higher than* $h^*(P)$ *are borrowers and buyers of the asset while investors holding beliefs (pessimistic investors) lower than* $h^*(P)$ *are lenders and sellers of the asset where*

$$h^*(P) = \frac{P_0 - P_D}{1 - P_D}$$

is the belief of the marginal buyer who is indifferent between buying and selling the asset. The optimistic investors are willing to buy the amount

$$y_0 = \frac{P_0 + 1}{P_0 - P_D \alpha}$$

of the asset and borrow at the maximum level $\varphi_0 = \alpha P_D y_0$*, and pessimistic investors are willing to sell the asset and lend to optimistic investors.*

At date 1, if the good state occurs, all of the investors continue to hold their optimal choices till date 2. However, if the bad state occurs, optimistic investors go bankrupt (for $\alpha = 1$, where there is no regulatory limit on borrowing) and are wiped out of the market. If there exists a regulatory limit $\alpha < 1$, when they end up in state D (bad state), the optimistic investors pay back their debt and borrow again by using the amount of asset they keep as collateral. Remember that they borrow at the maximum level $\varphi_0 = \alpha P_D y_0$ so they continue holding $(1-\alpha)y_0$ assets after they pay their debt back. There are $1 - h^*(P)$ optimistic investors in the market so each one of them is holding $\frac{1-\alpha}{1-h^*(P)}$ assets. The budget constraint of an optimistic investor at date 1 is as follows

$$C_D^h + \omega_1 + P_D y_1 \leq \frac{P_D(1-\alpha)}{1-h^*(P)} + \varphi_1$$

where the borrowing limit becomes

$$\varphi_1 \leq d(\alpha y_1)$$

since the worst outcome of the asset is d at date 2.

In state D at date 1, optimistic investors solve the following optimization problem

$$\max_{\{C_D^h, \varphi_1, \omega_1, y_1\}} \quad \{C_D^h + hC_{DU}^h + (1-h)C_{DD}^h\}$$

$$\begin{aligned}
C_{DU}^h &= \omega_1 + y_1 - \varphi_1 \\
C_{DD}^h &= \omega_1 + dy_1 - \varphi_1 \\
C_D^h + \omega_1 + P_D y_1 &= \frac{P_D(1-\alpha)}{1-h^*(P)} + \varphi_1 \\
\varphi_1 &\leq d(\alpha y_1) \\
\omega_1 &\geq 0 \\
y_1 &\geq 0
\end{aligned}$$

where C_{DU}^h is the consumption at date 2 when the asset goes up and C_{DD}^h is the consumption in the worst state at date 2.

The optimistic investors with beliefs $h \geq h^*(P)$ continue to be the buyers of the asset at date 1. From now on, I will call these investors old buyers. The old buyers buy

$$y_1^O = \frac{P_D(1-\alpha)}{(1-h^*(P))(P_D - d\alpha)}$$

at date 1.

The pessimistic investors get the collateral back at date 1 and they hold all the consumption good in the market, so the budget constraint of a pessimistic investor becomes

$$C_D^h + \omega_1 + P_D y_1 \leq \frac{P_D\alpha + 1}{h^*(P)} + \varphi_1$$

since there are $h^*(P)$ pessimistic investors in the market.

In state D at date 1, pessimistic investors solve the following optimization problem

$$\max_{\{C_D^h, \varphi_1, \omega_1, y_1\}} \quad \{C_D^h + hC_{DU}^h + (1-h)C_{DD}^h\}$$

$$\begin{aligned}
C^h_{DU} &= \omega_1 + y_1 - \varphi_1 \\
C^h_{DD} &= \omega_1 + dy_1 - \varphi_1 \\
C^h_D + \omega_1 + P_D y_1 &= \frac{P_D\alpha + 1}{h^*(P)} + \varphi_1 \\
\varphi_1 &\leq d(\alpha y_1) \\
\omega_1 &\geq 0 \\
y_1 &\geq 0.
\end{aligned}$$

The belief of the marginal investor decreases to

$$h^{**}(P) = \frac{P_D - d}{1 - d}$$

in state D at date 1 since the optimistic investors lose their collateral value when they come to date 1. The pessimistic investors who hold beliefs $h^{**}(P) \leq h \leq h^*(P)$ become buyers of the asset and borrowers at date 1. From now on, I will call these investors the new buyers. The new buyers of the asset buy

$$y_1^N = \frac{P_D\alpha + 1}{(h^*(P))(P_D - d\alpha)}$$

of the asset. Pessimistic investors who hold lower beliefs $h \leq h^{**}(P)$ continue to be sellers of the asset and lenders. At date 1, we have three different investors: the old buyers ($h \geq h^*(P)$), the new buyers ($h^{**}(P) \leq h \leq h^*(P)$) and the sellers ($h \leq h^{**}(P)$) of the asset.

The market clearing condition for the asset is as following

$$\int_{h^{**}(P)}^{h^*(P)} y_1^N \, \mathrm{d}h + \int_{h^*(P)}^{1} y_1^O \, \mathrm{d}h = 1$$

where the intervals for the amount of the asset are given as above.

Proposition 2.2 *In equilibrium at date* 1 *in state* D*, investors holding beliefs (optimistic investors) above* $h^{**}(P)$ *are borrowers and buyers of the asset while investors holding beliefs (pessimistic investors) below* $h^{**}(P)$ *are lenders and sellers of the asset where*

$$h^{**}(P) = \frac{P_D - d}{1 - P_D}$$

is the belief of the marginal buyer who is indifferent between buying and selling the asset. Old buyers ($h \geq h^*(P)$) *are willing to buy*

$$y_1^O = \frac{P_D(1-\alpha)}{(1 - h^*(P))(P_D - d\alpha)}$$

of the asset while new buyers $(h^{**}(P) < h < h^{*}(P))$ *are willing to buy*

$$y_1^N = \frac{P_D\alpha + 1}{(h^*(P))(P_D - d\alpha)}$$

of the asset. Both buyers borrow at the maximum level $\varphi_1 = \alpha d y_1^i$ *where* $i = O, N$*.*

2.1.1 Comparative Statics

Firstly, we consider the effect of a change in the borrowing limit (α) on the price of the asset at date 0 and date 1. If we increase α, then the buyers will borrow more and buy more asset. This will increase the price of the asset and will increase the borrowing capacity of the borrowers and they will buy even more asset. So the expected effect of an increase in α on the price of the asset is positive.

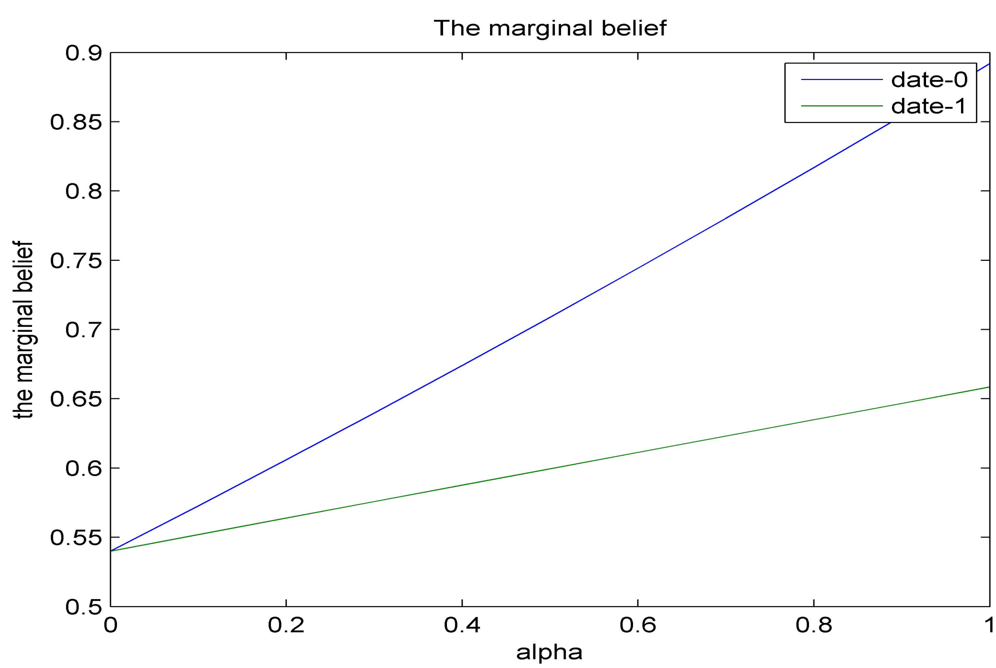

Figure 2: The belief in the market at date 0 and date 1 for $d = 0.3$.

Proposition 2.3 *The price of the asset is increasing in* α

$$\frac{dP_0}{d\alpha} \geq 0 \text{ and } \frac{dP_D}{d\alpha} \geq 0$$

at both dates.

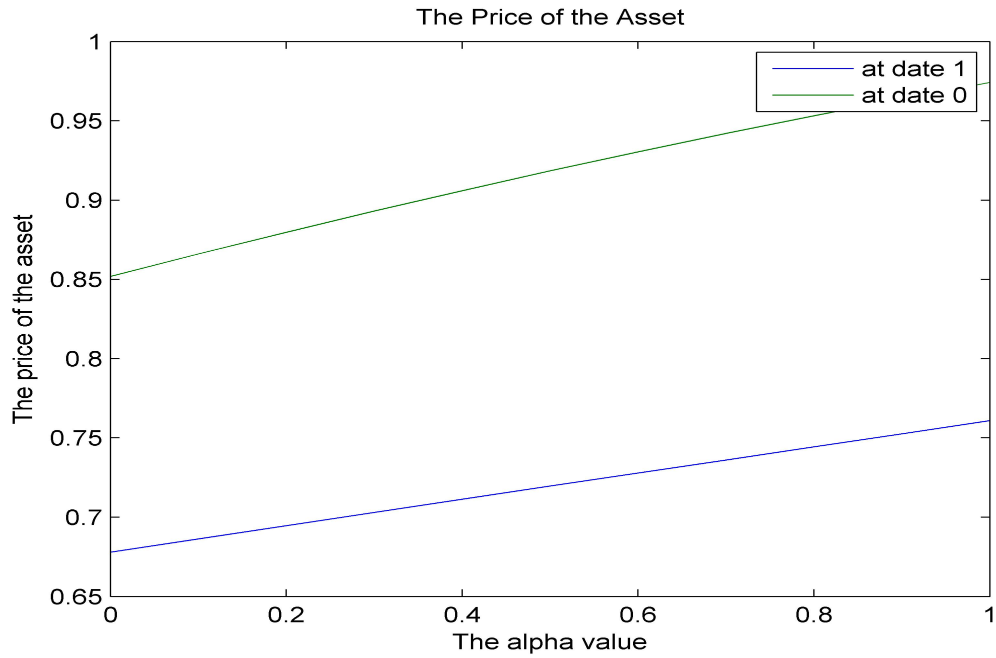

Figure 3: The price of the asset at date 0 and date 1 for $d = 0.3$.

Figure 3 shows the change in the price for different α values. The price increases at date 0 and date 1 as the investors are permitted to borrow more using their asset as collateral. This positive relationship shows the interaction between debt accumulation and the price of the collateral. The Figure illustrates how increases in borrowing capacity and asset price feed each other. Increases in borrowing capacity increases the price of the asset and this increases the borrowing capacity of the investor and this further increases the price of the asset. This amplifies the effect of a credit boom. However, as Figure 3 illustrates, if we limit the borrowing capacity of the investors, we can limit the distorting increase in the price of the asset which can mitigate the negative effects of the credit booms.

Secondly, we can calculate the effect of a change in the borrowing limit on the welfare of the economy. The question is whether a borrowing limit would decrease or increase the welfare. I use the CES aggregator (Constant Elasticity of Substitution) to calculate the welfare of the economy. Denote the date 2 welfare of the economy as $W_i(P)$ where $i = 1, 2, 3, 4$, from the best to the worst state respectively. Then the welfare of the four different states is calculated as

$$W_1(P,\alpha) = W_2(P) = \left(\int_0^{h^*(P)} \left(\frac{1+P_D\alpha}{h^*(P)} \right)^\rho \mathrm{d}h + \int_{h^*(P)}^1 \left(\frac{1-P_D\alpha}{1-h^*(P)} \right)^\rho \mathrm{d}h \right)^{\dfrac{1}{\rho}}$$

$$W_3(P,\alpha) = \left(\int_0^{h^{**}(P)} (\left(\frac{1+d\alpha}{h^{**}(P)} \right))^\rho \mathrm{d}h + \int_{h^{**}(P)}^{h^*(P)} (y_1^N(1-d\alpha))^\rho \mathrm{d}h + \int_{h^*(P)}^1 (y_1^O(1-d\alpha))^\rho \mathrm{d}h \right)^{\dfrac{1}{\rho}}$$

$$W_4(P,\alpha) = \left(\int_0^{h^{**}(P)} (\left(\frac{1+d\alpha}{h^{**}(P)} \right))^\rho \mathrm{d}h + \int_{h^{**}(P)}^{h^*(P)} (y_1^N d(1-\alpha))^\rho \mathrm{d}h + \int_{h^*(P)}^1 (y_1^O d(1-\alpha))^\rho \mathrm{d}h \right)^{\dfrac{1}{\rho}}.$$

Let the objective probability of the good state be p and the same for the two dates and independent. The expected welfare of the economy at date 0 is

$$E(W_0(P,\alpha)) = p^2 W_1(P,\alpha) + p(1-p)W_2(P,\alpha) + (1-p)pW_3(P,\alpha) + (1-p)^2 W_4(P,\alpha)$$

and the welfare for the two good states is the same. So the expected welfare of the economy becomes

$$E(W_0(P,\alpha)) = pW_1(P,\alpha) + (1-p)pW_3(P,\alpha) + (1-p)^2 W_4(P,\alpha).$$

By defining the objective probability, I will be able to discuss recessions and expansions by comparing the belief of the economy and the objective probability. When the objective probability is higher than the marginal belief of the market, investors are pessimistic about the economy and this can be thought of as a recession. On the other hand, when the objective probability is lower than the belief of the market then I will call this an expansion since the investors are more optimistic about the economy than they should be.

Now we can discuss the effect of a change in the regulatory limit on the date 2 welfare of the economy. We can discuss the two extreme cases as they end up in the best state and in the worst state. When they end up in the best state at date 2, the welfare of the economy is W_1. In this case, an increase in α increases the price of the asset and this increases the amount that the investors can borrow by using the asset as collateral. The borrowers can always pay their debt back without any problem. So an increase in α will increase the welfare of the economy. In other words, a limit on the borrowing capacity of the investors is welfare decreasing if the investors end up in the best state with probability 1. If the economy is doing better than what the investors believe, then a limit on borrowing capacity is welfare decreasing.

Proposition 2.4 *The date* 2 *welfare of the economy when the asset goes up at date* 1 *(in the best state) is increasing in* α

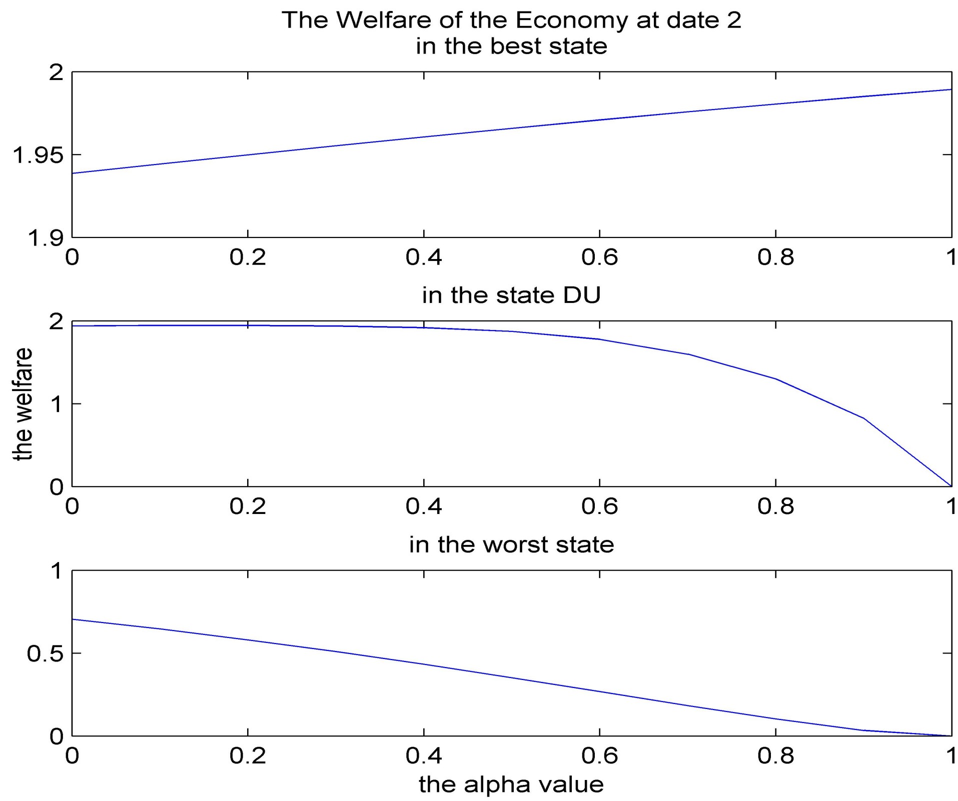

Figure 4: The welfare of the economy in the best state, in the state DU and in the worst state for $d = 0.3$

$$\frac{\partial W_1}{\partial \alpha} \geq 0.$$

This implies that $\alpha^* = 1$ is the optimal α for $p = 1$. If we know that the probability of going up is one then no regulatory limit on borrowing is optimal and we can let the investors borrow and lend as much as they want.

Secondly, we can discuss the effect of an increase in α when they end up in the worst state at date 2. In this case, an increase in α will increase the borrowing capacity of the buyers and this will increase their loss when they come to the worst state. As α increases, their loss will increase. So, an increase in α leads to a decrease in the welfare of the economy if the probability of the good state is zero. A regulatory limit on the borrowing capacity of investors is increasing the welfare if the probability of the good state is 0. In other words, if the economy is doing badly, then limiting the borrowing capacity of investors is welfare increasing.

Proposition 2.5 *The date 2 welfare of the economy, when the probability of the good state is zero ($p = 0$), is decreasing in α*

$$\frac{\partial W_4}{\partial \alpha} \leq 0.$$

This proposition implies that the optimal regulatory limit is $\alpha^* = 0$ if we have the probability of the good state $p = 0$.

Figure 4 shows the change in the welfare of the economy as α changes. It is increasing in the best state and decreasing in the bad states.

The expected welfare $E(W_0(P, \alpha))$ is increasing in α when $p = 1$ and it is decreasing in α when $p = 0$. This means that the optimal α is zero ($\alpha^* = 0$) for $p = 0$ and it is one ($\alpha^* = 1$) for $p = 1$. We can find the optimal α for p values in between by taking the derivative of the expected welfare with respect to α so $\frac{\partial E(W_0)}{\partial \alpha} = p\frac{\partial W_1}{\partial \alpha} + (1-p)p\frac{\partial W_3}{\partial \alpha} + (1-p)^2\frac{\partial W_4}{\partial \alpha}$. Remember that $\frac{\partial W_1}{\partial \alpha} \geq 0$ and $\frac{\partial W_4}{\partial \alpha} \leq 0$ so for lower p values this derivative is negative and for higher p values this derivative is positive. We can define these boundaries as p' and p'' then $\frac{\partial E(W_0(P))}{\partial \alpha} \leq 0$ for $p \leq p'$ and $\frac{\partial E(W_0(P))}{\partial \alpha} \geq 0$ for $p \geq p''$. This derivative is zero $\frac{\partial E(W_0)}{\partial \alpha} = 0$ for $p' \leq p \leq p''$ which implies that there are optimal α values between 0 and 1 for these p values. Now the question is whether the optimal α is increasing in p.

Now let's denote this partial derivative as $F(\alpha^*, p) = \frac{\partial E(W_0)}{\partial \alpha} = 0$. The derivative of this function with respect to p is

$$\frac{\partial F}{\partial \alpha}\frac{d\alpha^*}{dp} + \frac{\partial F}{\partial p} = 0.$$

The derivative of the optimal α^* with respect to p can be calculated as $\frac{d\alpha^*}{dp} = \frac{-\frac{\partial F}{\partial p}}{\frac{\partial F}{\partial \alpha}}$ where $\frac{\partial F}{\partial p} = \frac{\partial W_1}{\partial \alpha} + (1-2p)\frac{\partial W_3}{\partial \alpha} - 2(1-p)\frac{\partial W_4}{\partial \alpha}$ and $\frac{\partial F}{\partial \alpha} = \frac{\partial^2 E(W_0)}{\partial \alpha^2} \leq 0$. As p is increasing, the optimal α^* that gives the maximum expected welfare is expected to be increasing. The increase in the probability of the good state decreases the probability that the borrowers end up in the bad state. This encourages the borrowers to borrow more and this increases the price of the asset and increases the welfare of the economy.

Proposition 2.6 *The optimal regulatory limit that maximizes the welfare is increasing in p*

$$\frac{\partial \alpha^*}{\partial p} \geq 0.$$

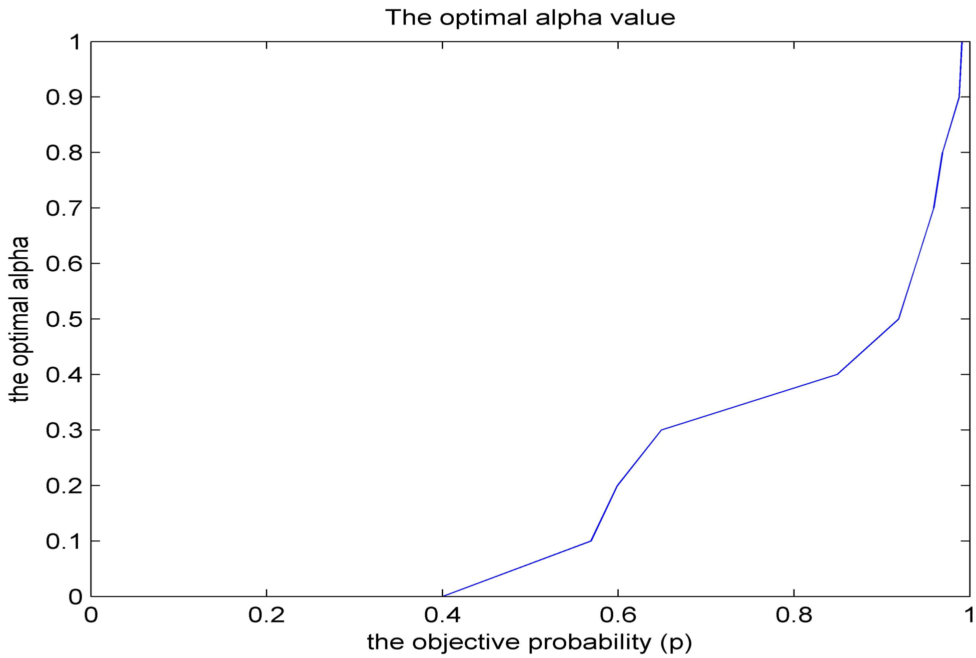

Figure 5: The optimal alpha for different objective probabilities for $d = 0.3$.

As the probability of the good state is increasing, the optimal borrowing limit is also increasing. The optimal borrowing limit is zero for lower probabilities, $\alpha^* = 0$ for $p \leq p'$, then it is increasing in the probability for intermediate probabilities, $\dfrac{\partial \alpha^*}{\partial p} \geq 0$ for $p' \leq p \leq p''$, and it is one for higher probabilities, $\alpha^* = 1$ for $p \geq p''$. In other words, as the economy is doing better then the optimal regulatory borrowing limit that maximizes the welfare of the economy is increasing.

Figure 5 shows the optimal α that maximizes the expected welfare of the economy as the objective probability p changes. It is seen that the optimal α is zero for lower probability values, it is increasing in α for intermediate probability values and it is one for higher probability values so close to one. This concludes that only when the objective probability of the economy is so close to one, then the optimal α is one. In other words, it is optimal not to regulate (limit) the borrowing capacity of the investors only when the success probability is so close to one. Moreover, if the objective probability of the success is lower than one, it is optimal to regulate the borrowing capacity of the investors. The optimal regulation amount depends on the objective probability of the economy.

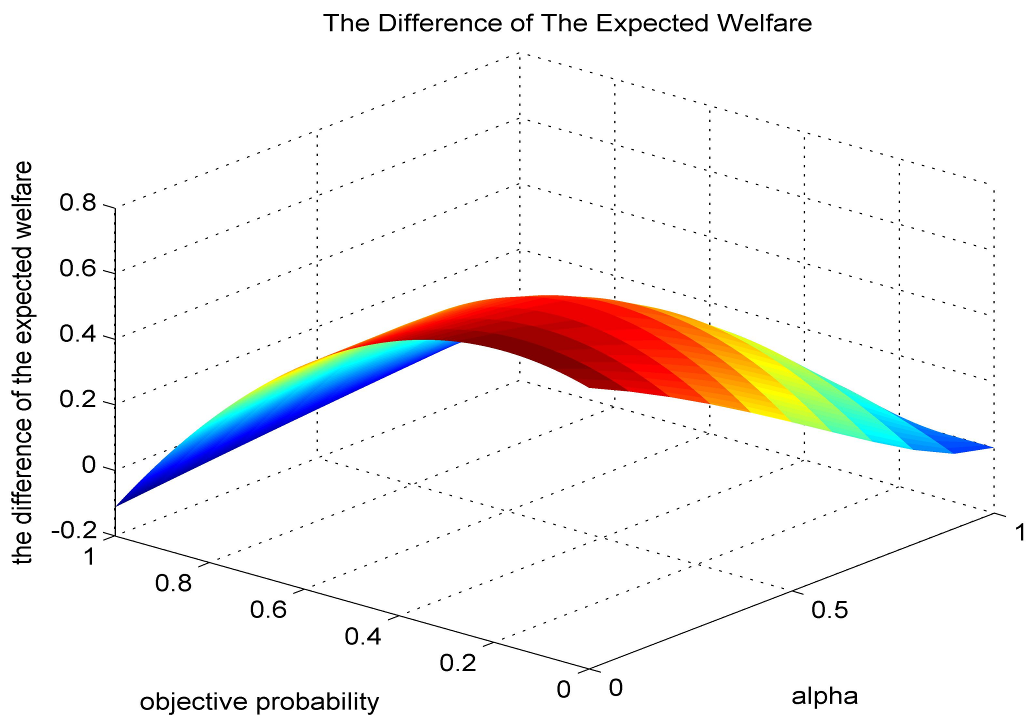

Figure 6: The difference of the welfare of the economy when there is full borrowing and when there is limited borrowing with α. The welfare calculation is done by using $\rho = -100$ (Leontieff) to have the distributional effect of the investors.

Now, I want to compare the expected welfare of the economy with a limit on the short-term borrowing and without any limit for different α values and different objective probabilities $p \in [0, 1]$. Figure 6 shows the difference between the expected welfare of the economy when investors can pledge only α of their asset to borrow minus the expected welfare of the economy without a borrowing limit. The difference of the expected welfare is calculated for all objective probabilities and α values. In Figure 6, it is seen that for low objective probabilities, limiting borrowing is welfare improving and it is welfare decreasing for high objective probabilities. This is expected since the optimal α is increasing as the economy is doing better.

Figure 7 shows only the nonnegative difference to see the objective probabilities where the limit on borrowing has a positive effect on welfare. The blue line shown in the figure is the belief of the marginal buyer that determines the price of the asset when there is no borrowing limit and the green line is the belief of the marginal buyer when there is limited borrowing.

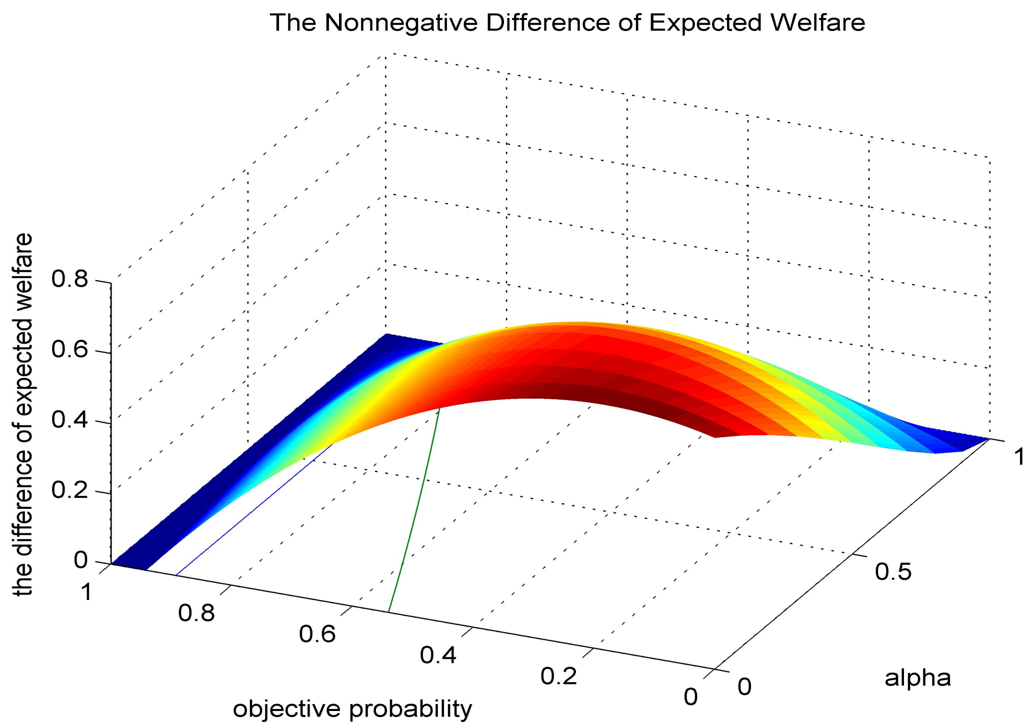

Figure 7: The nonnegative difference of the welfare of the economy when there is full borrowing and when there is limited borrowing with changing α values. The blue line is the belief of the marginal buyer when there is full borrowing and the green line is the belief of the marginal buyer when there is limited borrowing.

This figure clearly shows that when the investors hold so lower beliefs compared to the objective probability, then limiting borrowing is decreasing the welfare of the economy. In other words, limiting borrowing is welfare decreasing during recessions when the investors are pessimistic about future. Moreover, a limit on collateral short-term borrowing is welfare improving even in the case that the investors hold the correct beliefs with the objective probability. This increase in the welfare is increasing as they hold higher beliefs compared to the objective probability meaning as they get more optimistic like in an expansion. Therefore, we can conclude that a counter-cyclical regulatory limit on the amount of borrowing is optimal.

2.2 Equilibrium when the investors are risk-averse

The utility function for each investor is $U(C^h)$ where $U'(C^h) > 0$ and $U''(C^h) \leq 0$ since the investors are assumed to be risk-averse in this section. The date 0 utility of each investor is $U(C_0^h) + hU(C_U^h) + (1-h)U(C_D^h)$ where C_0^h is the consumption at date 0, C_U^h and C_D^h are the consumptions in the good state and in the bad state at date 1, respectively.

At date 0, each investor chooses her consumption (C_0), borrowing (φ_0), the amount that she warehouses (ω_0), and the amount of the asset that she buys (y_0) by solving the following optimization problem

$$\max_{\{C_0, \varphi_0, \omega_0, y_0\}} \quad \{U(C_0^h) + hU(C_U^h) + (1-h)U(C_D^h)\}$$

$$\begin{aligned}
C_U^h &= \omega_0 + y_0 - \varphi_0 \\
C_D^h &= \omega_0 + P_D y_0 - \varphi_0 \\
C_0^h + \omega_0 + P_0 y_0 &= P_0 + 1 + \varphi_0 \\
\varphi_0 &\leq \alpha P_D y_0 \\
\omega_0 &\geq 0 \\
y_0 &\geq 0.
\end{aligned}$$

In equilibrium, the investor holding the belief (marginal buyer)

$$h^*(P) = \frac{U'(C_D^h)(P_0 - P_D)}{U'(C_D^h)(P_0 - P_D) + U'(C_U^h)(1 - P_0)}$$

is indifferent between buying and selling the asset where P is the vector of the asset prices. The belief of the marginal buyer determines the price of the asset. The investors who are holding higher beliefs (optimistic investors) $h \geq h^*(P)$ are the buyers of the asset and they are the borrowers. The ones holding lower beliefs (pessimistic investors) $h \leq h^*(P)$ than the marginal belief are the sellers of the asset and they are the lenders.

The Lagrange multiplier for the borrowing constraint is $\lambda_2 = U'(C_0^h) - hU'(C_U^h) + (1 - h)U'(C_D^h)$. When the borrowing constraint binds,

$$U'(C_0^h) > hU'(C_U^h) + (1-h)U'(C_D^h)$$

the sellers (pessimistic investors) prefer not to lend money since the marginal utility of consuming now is higher than the marginal utility of consuming in the next period. They prefer consuming now rather than lending now and consuming tomorrow. The only condition that the sellers are convinced to lend money is only if they are indifferent between consuming now or the next period where

$$U'(C_0^h) = hU'(C_U^h) + (1-h)U'(C_D^h)$$

and this is the condition where the borrowing constraint is not binding. For this case, they borrow $\varphi_0^S = -\dfrac{P_0+1}{2}$ (meaning that they lend the positive amount) which equates their date 0 consumption to their date 1 consumption which is the same for the good and the bad state.

For the buyers of the asset (optimistic investors), the borrowing constraint binds for the ones that hold the beliefs $h \geq h'(P)$ where

$$h'(P) = \frac{U'(C_D^h) - U'(C_0^h)}{U'(C_D^h) - U'(C_U^h)}$$

so the optimistic investors with beliefs $h \geq h'(P)$ borrow at the maximum level $\varphi_0 = \alpha P_D y_0$. The less optimistic investors ($h \leq h'(P)$) borrow $\varphi_0 < \alpha P_D y_0$.

The belief $h^{'}(P)$ is increasing in α so as the regulatory limit is increasing less buyers can have binding borrowing constraint. Moreover, the belief of the marginal buyer $h^*(P)$ is also increasing in α which means as the regulatory limit is increasing, more optimistic investors become buyers and the belief of the economy is increasing. One can easily guess that this also leads to an increase in the price of the asset since the buyers become more optimistic and borrow more to buy the asset.

For the case where $h'(P) \geq h^*(P)$, there are two different borrowers with binding and non-binding borrowing constraint. The amount of the asset that is bought by the more optimistic investors (the ones with binding borrowing constraint) is denoted by y_0^B and satisfies the following condition

$$y_0^B \geq \frac{P_0+1}{P_0+1-2P_D\alpha}.$$

The amount of the asset that the less optimistic investors (with non-binding borrowing constraint) buy is denoted as y_0^N and satisfies the following inequalities

$$\frac{P_0+1+2\varphi_0^N}{P_0+1} \leq y_0^N \leq \frac{P_0+1+2\varphi_0^N}{P_0+P_D}$$

where φ_0^N is the amount that each optimistic investor with non-binding borrowing constraint borrows.

The market clears for the asset if

$$\int\limits_{h^*(P)}^{h^{'}(P)} y_0^N \, \mathrm{d}h + \int\limits_{h^{'}(P)}^{1} y_0^B \, \mathrm{d}h = 1.$$

The borrowing market clears if

$$\int_{h^*(P)}^{h'(P)} \varphi_0^N \, \mathrm{d}h + \int_{h'(P)}^{1} \alpha P_D y_0^B \, \mathrm{d}h = \int_{0}^{h^*(P)} \frac{P_0 + 1}{2} \, \mathrm{d}h$$

where the demand of the borrowers is equal to the supply of the lenders.

For the case where $h'(P) \leq h^*(P)$, all the borrowers have binding borrowing constraint. The market clearing conditions change for this case,

$$\int_{h^*(P)}^{1} y_0^B \, \mathrm{d}h = 1$$

and

$$\int_{h^*(P)}^{1} \alpha P_D y_0^B \, \mathrm{d}h = \int_{0}^{h^*(P)} \frac{P_0 + 1}{2} \, \mathrm{d}h.$$

Proposition 2.7 *In equilibrium at date* 0*, the investors holding beliefs (optimistic investors) higher than* $h^*(P)$ *are the borrowers and buyers of the asset while the investors holding beliefs (pessimistic investors) lower than* $h^*(P)$ *are the lenders and the sellers of the asset where*

$$h^*(P) = \frac{U'(C_D^h)(P_0 - P_D)}{U'(C_D^h)(P_0 - P_D) + U'(C_U^h)(1 - P_0)}$$

is the belief of the marginal buyer who is indifferent between buying and selling the asset. The more optimistic investors ($h \geq h'(P)$) *have binding borrowing constraint and borrow* $\varphi_0^B = \alpha P_D y_0^B$ *and the less optimistic investors* ($h \leq h'(P)$) *have non-binding borrowing constraint* $\varphi_0^N < \alpha P_D y_0^N$ *where*

$$h'(P) = \frac{U'(C_D^h) - U'(C_0^h)}{U'(C_D^h) - U'(C_U^h)}.$$

When they come to date 1, if the good state occurs, all of the investors continue to hold their optimal choices till date 2. However, if the bad state occurs, the more optimistic investors ($h \geq h'(P)$) go bankrupt for $\alpha = 1$ where there is no regulatory limit on borrowing and they are wiped out of the market. If there exist a regulatory limit $\alpha < 1$ and they end up in state D, all of the optimistic investors pay back their debt and borrow again by using the asset they keep as collateral. Remember that the more optimistic ones borrow at the maximum level $\varphi_0^B = \alpha P_D y_0^B$ so they continue holding $(1 - \alpha) y_0^B$ of the asset after they pay their debt back. The less optimistic investors ($h \leq h'(P)$) continue to hold less than the more optimistic ones.

In state D at date 1, the investors solve the following optimization problem

$$\max_{\{C_D^h, \varphi_1, \omega_1, y_1\}} \quad \{U(C_D^h) + hU(C_{DU}^h) + (1 - h)U(C_{DD}^h)\}$$

$$
\begin{aligned}
C^h_{DU} &= \omega_1 + y_1 - \varphi_1 \\
C^h_{DD} &= \omega_1 + dy_1 - \varphi_1 \\
C^h_D + \omega_1 + P_D y_1 &= \omega_0 + P_D y^i_0 - \varphi^i_0 + \varphi_1 \\
\varphi_1 &\leq d(\alpha y_1) \\
\omega_1 &\geq 0 \\
y_1 &\geq 0
\end{aligned}
$$

where C^h_{DU} is the consumption at date 2 when the asset goes up and C^h_{DD} is the consumption in the worst state at date 2. In the optimization problem, $i = S, N, B$ for the sellers, the buyers with non-binding borrowing constraint and the binding buyers, respectively.

At date 1, the belief of the marginal buyer decreases to

$$h^{**}(P) = \frac{U'(C^h_{DD})(P_D - d)}{U'(C^h_{DD})(P_D - d) + U'(C^h_{DU})(1 - P_D)}$$

where $h^*(P) \leq h^{**}(P)$ as they end up in the state D. The investors holding beliefs $h \geq h^{**}(P)$ become the buyers of the asset and the ones holding lower beliefs are the sellers of the asset. Moreover, the belief that determines the borrowing constraint of the investors also decreases to

$$h^{''}(P) = \frac{U'(C^h_{DD}) - U'(C^h_D)}{U'(C^h_{DD}) - U'(C^h_{DU})}$$

where $h^{''}(P) \leq h^{'}(P)$. Therefore, similar to date 0, some of the buyers have binding and some have non-binding borrowing constraint.

Proposition 2.8 *In equilibrium at date* 1 *in the state* D*, the investors holding beliefs (optimistic investors) higher than* $h^{**}(P)$ *are the borrowers and buyers of the asset while the investors holding beliefs (pessimistic investors) lower than* $h^{**}(P)$ *are the lenders and the sellers of the asset where*

$$h^{**}(P) = \frac{U'(C^h_{DD})(P_D - d)}{U'(C^h_{DD})(P_D - d) + U'(C^h_{DU})(1 - P_D)}$$

is the belief of the marginal buyer who is indifferent between buying and selling the asset. The more optimistic investors ($h \geq h^{''}(P)$) *have binding borrowing constraint and borrow* $\varphi^B_0 = \alpha P_D y^B_0$ *and the less optimistic investors* ($h \leq h^{''}(P)$) *have non-binding borrowing constraint* $\varphi^N_0 < \alpha P_D y^N_0$ *where*

$$h^{''}(P) = \frac{U'(C^h_{DD}) - U'(C^h_D)}{U'(C^h_{DD}) - U'(C^h_{DU})}.$$

For the case $h^*(P) \leq h^{''}(P)$, the more optimistic investors with beliefs $h > h^{'}(P)$ continue to be the buyers of the asset at date 1 and they continue to borrow the maximum amount $\varphi_1^{BB} = \alpha d y_1^{BB}$. The asset that these investors buy satisfies the following inequality

$$y_1^{BB} \geq \frac{P_D(1-\alpha)y_0^B}{1+P_D-2d\alpha}.$$

The investors with beliefs $h^{'}(P) \geq h \geq h^{''}(P)$ were non-binding at date 0 and they become binding at date 1 so they borrow $\varphi_1^{NB} = \alpha d y_1^{NB}$ where the asset they buy satisfies

$$y_1^{NB} \geq \frac{P_D y_0^N - \varphi_0^N}{1+P_D-2d\alpha}.$$

The less optimistic investors with beliefs $h^{''}(P) \geq h \geq h^*(P)$ continue to be non-binding and buy y_1^{NN} asset which satisfies the following inequality

$$\frac{P_D y_0^N - \varphi_0^N + 2\varphi_1^N}{P_D+1} \leq y_1^{NN} \leq \frac{P_D y_0^N - \varphi_0^N + 2\varphi_1^N}{P_D+d}.$$

The investors with beliefs $h^{**}(P) \leq h \leq h^*(P)$ were the sellers at date 0, they become buyers with non-binding borrowing constraint and they buy y_1^N of the asset and this satisfies

$$\frac{\frac{P_0+1}{2}+2\varphi_1^N}{P_D+1} \leq y_1^N \leq \frac{\frac{P_0+1}{2}+2\varphi_1^N}{P_D+d}.$$

The pessimistic investors with beliefs $h \leq h^{**}(P)$ are the sellers of the asset and they lend $\varphi_1^h = \dfrac{P_0+1}{4}$. The market clearing condition for the asset is as following

$$\int_{h^{**}(P)}^{h^*(P)} y_1^N \, \mathrm{d}h + \int_{h^*(P)}^{h^{''}(P)} y_1^{NN} \, \mathrm{d}h + \int_{h^{''}(P)}^{h^{'}(P)} y_1^{NB} \, \mathrm{d}h + \int_{h^{'}(P)}^{1} y_1^{BB} \, \mathrm{d}h = 1.$$

For the borrowing market, the market clearing condition is as below

$$\int_{h^{**}(P)}^{h^*(P)} \varphi_1^N \, \mathrm{d}h + \int_{h^*(P)}^{h^{''}(P)} \varphi_1^{NN} \, \mathrm{d}h + \int_{h^{''}(P)}^{h^{'}(P)} d\alpha y_1^{NB} \, \mathrm{d}h + \int_{h^{'}(P)}^{1} d\alpha y_1^{BB} \, \mathrm{d}h = \int_{0}^{h^{**}(P)} \frac{P_0+1}{4} \, \mathrm{d}h.$$

For the case $h^{''}(P) \leq h^*(P) \leq h^{'}(P)$, the more optimistic investors with beliefs $h > h^{'}(P)$ continue to be the buyers of the asset at date 1 and they continue to borrow the maximum amount $\varphi_1^{BB} = \alpha d y_1^{BB}$. The asset that these investors buy satisfies the following inequality

$$y_1^{BB} \geq \frac{P_D(1-\alpha)y_0^B}{1+P_D-2d\alpha}.$$

The investors with beliefs $h^{'}(P) \geq h \geq h^*(P)$ were non-binding at date 0 and they become binding at date 1 so they borrow $\varphi_1^{NB} = \alpha d y_1^{NB}$ where the asset they buy satisfies

$$y_1^{NB} \geq \frac{P_D y_0^N - \varphi_0^N}{1+P_D-2d\alpha}.$$

The less optimistic investors with beliefs $h^*(P) \geq h \geq h^{''}(P)$ were the sellers at date 0, they become buyers with binding borrowing constraint and they buy y_1^B of the asset and this satisfies

$$\frac{\frac{P_0+1}{2}}{P_D+1-2d\alpha} \leq y_1^B \leq \frac{\frac{P_0+1}{2}}{P_D+d-2d\alpha}.$$

The investors with beliefs $h^{**}(P) \leq h \leq h^{''}(P)$ were the sellers at date 0, they become buyers with non-binding borrowing constraint and they buy y_1^N of the asset and this satisfies

$$\frac{\frac{P_0+1}{2}+2\varphi_1^N}{P_D+1} \leq y_1^N \leq \frac{\frac{P_0+1}{2}+2\varphi_1^N}{P_D+d}.$$

The pessimistic investors with beliefs $h \leq h^{**}(P)$ are the sellers of the asset and they lend $\varphi_1^h = \dfrac{P_0+1}{4}$.

For the case $h^*(P) \leq h^{'}(P)$ and $h^{''}(P) \leq h^{**}(P)$, the more optimistic investors with beliefs $h > h^{'}(P)$ continue to be the buyers of the asset at date 1 and they continue to borrow the maximum amount $\varphi_1^{BB} = \alpha d y_1^{BB}$. The asset that these investors buy satisfies the following inequality

$$y_1^{BB} \geq \frac{P_D(1-\alpha)y_0^B}{1+P_D-2d\alpha}.$$

The investors with beliefs $h^{'}(P) \geq h \geq h^*(P)$ were non-binding at date 0 and they become binding at date 1 so they borrow $\varphi_1^{NB} = \alpha d y_1^{NB}$ where the asset they buy satisfies

$$y_1^{NB} \geq \frac{P_D y_0^N - \varphi_0^N}{1+P_D-2d\alpha}.$$

The less optimistic investors with beliefs $h^*(P) \geq h \geq h^{**}(P)$ were the sellers at date 0, they become buyers with binding borrowing constraint and they buy y_1^B of the asset and this satisfies

$$\frac{\frac{P_0+1}{2}}{P_D+1-2d\alpha} \leq y_1^B \leq \frac{\frac{P_0+1}{2}}{P_D+d-2d\alpha}.$$

The pessimistic investors with beliefs $h \leq h^{**}(P)$ are the sellers of the asset and they lend $\varphi_1^h = \dfrac{P_0+1}{4}$.

For the case $h^{'}(P) \leq h^*(P)$ and $h^{''}(P) \geq h^{**}(P)$, the optimistic investors with beliefs $h > h^*(P)$ continue to be the buyers of the asset at date 1 and they continue to borrow the maximum amount $\varphi_1^{BB} = \alpha d y_1^{BB}$. The asset that these investors buy satisfies the following inequality

$$y_1^{BB} \geq \frac{P_D(1-\alpha)y_0^B}{1+P_D-2d\alpha}.$$

The less optimistic investors with beliefs $h^*(P) \geq h \geq h^{''}(P)$ were the sellers at date 0, they become buyers with binding borrowing constraint and they buy y_1^B of the asset and this satisfies

$$\frac{\frac{P_0+1}{2}}{P_D+1-2d\alpha} \leq y_1^B \leq \frac{\frac{P_0+1}{2}}{P_D+d-2d\alpha}.$$

The investors with beliefs $h^{**}(P) \leq h \leq h^{''}(P)$ were the sellers at date 0, they become buyers with non-binding borrowing constraint and they buy y_1^N of the asset and this satisfies

$$\frac{\frac{P_0+1}{2}+2\varphi_1^N}{P_D+1} \leq y_1^N \leq \frac{\frac{P_0+1}{2}+2\varphi_1^N}{P_D+d}.$$

The pessimistic investors with beliefs $h \leq h^{**}(P)$ are the sellers of the asset and they lend $\varphi_1^h = \frac{P_0+1}{4}$.

For the case $h'(P) \leq h^*(P)$ and $h^{''}(P) \leq h^{**}(P)$, the optimistic investors with beliefs $h > h^*(P)$ continue to be the buyers of the asset at date 1 and they continue to borrow the maximum amount $\varphi_1^{BB} = \alpha d y_1^{BB}$. The asset that these investors buy satisfies the following inequality

$$y_1^{BB} \geq \frac{P_D(1-\alpha)y_0^B}{1+P_D-2d\alpha}.$$

The less optimistic investors with beliefs $h^{**}(P) \geq h \geq h^*(P)$ were the sellers at date 0, they become buyers with binding borrowing constraint and they buy y_1^B of the asset and this satisfies

$$\frac{\frac{P_0+1}{2}}{P_D+1-2d\alpha} \leq y_1^B \leq \frac{\frac{P_0+1}{2}}{P_D+d-2d\alpha}.$$

The pessimistic investors with beliefs $h \leq h^{**}(P)$ are the sellers of the asset and they lend $\varphi_1^h = \frac{P_0+1}{4}$.

2.2.1 Comparative Statics

In this section, I will show the comparative statics for the equilibrium of the risk-averse agents. Let's define the utility function $U = (C)^\gamma$ where $\gamma \in (0,1)$. For ease of calculation, I concentrate on the equilibrium where $C_0^h = C_D^h$ for the buyers which leads to $h^{'}(P) = 0$. This means the borrowing constraint binds for the borrowers. Figure **??** shows the belief of the economy at date 0 and date 1. When the investors are risk averse, the belief of the economy is lower than the belief in the risk-neutral investors equilibrium. As the investors get risk-averse, they become less optimistic.

The price of the asset is increasing in α as shown in Figure **??**. The utility of the investors are increasing in the best state. The utility of the seller is increasing in the worst state whereas the utility of the buyers are decreasing in the worst state.

The main question I focus on is whether the welfare results for the risk-neutral investors come from the distribution of the buyers and the sellers which is captured by the CES aggregator. In this section, I define the welfare as the sum of the utilities of the investors to answer this question.

Let us denote the date 2 welfare of the economy $W_i(P)$ where $i = 1, 2, 3, 4$, from the best to the worst state respectively. Then the welfare for the four different states (for the case $h^*(P) \leq h^{''}(P)$) is calculated as

$$W_1(P) = \int_0^{h^*(P)} U\left(\frac{1+\varphi_0}{h^*(P)}\right) \mathrm{d}h + \int_{h^*(P)}^{h^{'}(P)} U\left(\frac{y_0 - (\varphi_0 - \varphi_0^B)}{h^{'}(P) - h^*(P)}\right) \mathrm{d}h + \int_{h^{'}(P)}^{1} U\left(\frac{1 - y_0 - \varphi_0^B}{1 - h^{'}(P)}\right) \mathrm{d}h$$

$$W_2(P) = \int_0^{h^*(P)} U\left(\frac{1+\varphi_0}{h^*(P)}\right) \mathrm{d}h + \int_{h^*(P)}^{h^{'}(P)} U\left(\frac{y_0 - (\varphi_0 - \varphi_0^B)}{h^{'}(P) - h^*(P)}\right) \mathrm{d}h + \int_{h^{'}(P)}^{1} U\left(\frac{1 - y_0 - \varphi_0^B}{1 - h^{'}(P)}\right) \mathrm{d}h$$

$$W_3(P) = \int_0^{h^{**}(P)} U\left(\frac{1+\varphi}{h^{**}(P)}\right) \mathrm{d}h + \int_{h^{**}(P)}^{h^*(P)} U\left(\frac{(1 - (y_1 + y_2 + y_3)) - (\varphi - (\varphi_1 + \varphi_2 + \varphi_3))}{h^*(P) - h^{**}(P)})\right) \mathrm{d}h$$

$$+ \int_{h^*(P)}^{h^{''}(P)} U\left(\frac{y_3 - \varphi_3}{h^*(P) - h^{**}(P)}\right) \mathrm{d}h + \int_{h^{''}(P)}^{h^{'}(P)} U\left(\frac{y_2 - \varphi_2}{h^{'}(P) - h^{''}(P)}\right) \mathrm{d}h + \int_{h^{'}(P)}^{1} U\left(\frac{y_1 - \varphi_1}{1 - h^{'}(P)}\right) \mathrm{d}h$$

$$W_4(P) = \int_0^{h^{**}(P)} U\left(\frac{1+\varphi}{h^{**}(P)}\right) \mathrm{d}h + \int_{h^{**}(P)}^{h^*(P)} U\left(\frac{(1 - (y_1 + y_2 + y_3))d - (\varphi - (\varphi_1 + \varphi_2 + \varphi_3))}{h^*(P) - h^{**}(P)})\right) \mathrm{d}h$$

$$+ \int_{h^*(P)}^{h^{''}(P)} U\left(\frac{y_3 d - \varphi_3}{h^*(P) - h^{**}(P)}\right) \mathrm{d}h + \int_{h^{''}(P)}^{h^{'}(P)} U\left(\frac{y_2 d - \varphi_2}{h^{'}(P) - h^{''}(P)}\right) \mathrm{d}h + \int_{h^{'}(P)}^{1} U\left(\frac{y_1 d - \varphi_1}{1 - h^{'}(P)}\right) \mathrm{d}h$$

where φ_0 is the aggregate amount that the borrowers borrow, φ_0^B is the aggregate amount that the borrowers with binding borrowing constraint borrow, y_0 is the aggregate amount of asset that the buyers with non-binding borrowing constraint buy at date 0, y_1 (φ_1) is the aggregate amount of asset that the buyers with beliefs $h \geq h^{'}(P)$ (the ones that have binding borrowing constraints at both dates) buy (borrow), y_2 (φ_2) is the aggregate amount of asset that the buyers with beliefs $h^{'}(P) \geq h \geq h^{''}(P)$ (the ones that become binding at date 1) buy (borrow), y_3 (φ_3) is the aggregate amount of asset that the buyers with beliefs $h^{''}(P) \geq h \geq h^*(P)$ (the ones that have non-binding borrowing constraints at both dates) buy (borrow) at date 1. Let the objective probability of the good state, denote as p, be the same for two dates and independent. The expected welfare of the economy at date 0 is

$$E(W_0(P)) = p^2W_1(P) + p(1-p)W_2(P) + (1-p)pW_3(P) + (1-p)^2W_4(P)$$

and we can see that the welfare for the two good states is the same. So the expected welfare of the economy becomes

$$E(W_0(P)) = pW_1(P) + (1-p)pW_3(P) + (1-p)^2W_4(P).$$

This calculation is the same with the risk-neutral case.

Now I want to talk about the effects of an increase in α on the expected welfare of the economy. We can follow the same order as in the risk-neutral case and start with the discussion if the probability of the good state is 1. This means they end up in the good state and the welfare of the economy is W_1. In this case, an increase in α increases the price of the asset and this increases the amount that the banks can borrow by using this asset is collateral. The lenders can always pay back their debt without any problem. In this case, an increase in α will increase the welfare of the economy.

Proposition 2.9 *The date* 2 *welfare of the economy when the asset goes up at date* 1 *(in the best state) is increasing in* α

$$\frac{\partial W_1}{\partial \alpha} \geq 0.$$

This implies that $\alpha = 1$ is the optimal α if $p = 1$. If we know that the probability of going up is one then no regulatory limit on borrowing is the optimal and we can let the investors borrow and lend as much as they want.

Secondly, we can discuss the effect of an increase in α if the probability of the good state is zero. This means they will end up in the worst state at date 2. In this case, an increase in α will increase the borrowing capacity of the buyers and this will increase their loss when they come to the worst state. As α increases, their loss will increase. So, an increase in α leads to a decrease in the welfare of the economy if the probability of the good state is zero.

Proposition 2.10 *The date* 2 *welfare of the economy, when the probability of the good state is zero* ($p = 0$)*, is decreasing in* α

$$\frac{\partial W_4}{\partial \alpha} \leq 0.$$

This proposition implies that the optimal regulatory limit $\alpha = 0$ if we have the probability of the good state $p = 0$. The expected welfare $E(W_0(P))$ is increasing in α when $p = 1$ and it is decreasing in α when $p = 0$ which is the same with the risk-neutral case. This means the optimal α is zero $\alpha^* = 0$ at $p = 0$ and it is one $\alpha^* = 1$ at $p = 1$. We can find the optimal alpha for different p values in between by taking the derivative of the expected welfare with respect to α

$$\frac{\partial E(W_0(P))}{\partial \alpha} = 0$$

where $\frac{\partial^2 E(W_0(P))}{\partial \alpha^2} \leq 0$. Now let's denote this partial derivative as

$$F(\alpha^*, p) = \frac{\partial E(W_O)}{\partial \alpha} = 0.$$

The derivative of the optimal α^* with respect to p can be calculated as

$$\frac{d\alpha^*}{dp} = \frac{-\frac{\partial F}{\partial P}}{\frac{\partial F}{\partial \alpha}}$$

and this derivative is positive since $\frac{\partial W_1}{\partial \alpha} \geq 0$ and $\frac{\partial W_4}{\partial \alpha} \leq 0$.

When we calculate the difference of the expected welfare when there exist a regulatory limit and when there is no regulation. The difference is shown in Figure **??**. This difference is calculated as the welfare of the economy when the investors have limited borrowing for all α values minus the welfare of the economy when the investors have no borrowing limit (when $\alpha = 1$). Figure **??** shows that limiting borrowing is welfare decreasing for high objective probabilities and it is welfare increasing for low objective probabilities. For a comparison of the objective probabilities with the marginal belief in the market, Figure **??** shows the non-negative difference of the expected welfare and the blue line shows the belief of the market. Limiting borrowing is welfare improving when the marginal belief is higher than the objective probability and it is welfare decreasing when the marginal belief is lower than the objective probability. In other words, a liquidity regulation improves welfare during expansion when the investors are very optimistic compared to the objective probability and it decreases welfare during recession when the investors are very pessimistic.

In this section, we show that when we calculate the welfare as the sum of the utilities then the change in the welfare is the same as in the previous section when we calculate the welfare with CES aggregator. We can conclude that this result does not come from the distribution of the sellers and the buyers. Therefore, a regulatory limit affects the utility of the investors and this changes the welfare of the economy. According to this study, I find that a regulatory limit on the amount of short-term debt is welfare decreasing when the investors are pessimistic about the future compared to the objective probability and it is welfare improving when the investors are optimistic about the future. As a result, a counter-cyclical regulatory limit on the amount of borrowing is the efficient regulation that improves the welfare of the economy.

3 Conclusion

This paper develops a general equilibrium model that shows the interaction between the borrowing capacity of the investors and the asset price and how this amplifies the effects of credit booms and busts. The main objective of the paper is to analyze the efficiency of a limit on the borrowing capacity of the investors to regulate the liquidity in the market and mitigate the effects of the credit cycles. As a limit on the borrowing capacity of the investors, I introduce a fraction α that determines the borrowing capacity of the collateral. This limit α can be seen as the fraction of the asset that can be pledged to borrow against or α is the fraction that the investors can borrow after they pay the tax on their debt (in this case, investors are paying $1 - \alpha$ of the asset price as tax and borrow only the α fraction of the asset price).

The results of the paper show that when the investors are holding higher beliefs than the objective probability (more optimistic about the future) of the economy, limiting borrowing is welfare improving and when the investors are holding lower beliefs compared to the objective probability (more pessimistic about the future) of the economy, limiting borrowing is welfare decreasing. In other words, limiting short-term collateral borrowing is welfare improving during the expansions when investors are overoptimistic about the future of the asset and it is welfare decreasing during the recessions when investors are overpessimistic about the future of the economy. Therefore, a counter-cyclical quantity restriction on the short-term borrowing (lower during expansions and higher during recessions) is the optimal regulation according to the model.

The results show that limiting the amount of short-term debt in the economy can be used as an efficient tool for liquidity regulation of the market. The welfare effects of this limit provide a rationale for macro-prudential regulation. This paper concludes that liquidity regulation can mitigate the extreme effects of a credit cycle.

A Appendix

Proof of Proposition 2.1

The Lagrangian of the optimization problem is

$$\mathcal{L} = C_0^h + hC_U^h + (1-h)C_D^h + \lambda_1(-C_0^h - P_0 y_0 - \omega_0 + P_0 + 1 + \varphi_0) + \lambda_2(P_D y_0 \alpha - \varphi_0).$$

The investors choose the optimal C_O^h, φ_0, ω_0 and y_0 so the first order conditions are

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial C_0^h} &= 1 - \lambda_1 = 0 \Longrightarrow \lambda_1 = 1 \\
\frac{\partial \mathcal{L}}{\partial \varphi_0} &= -h - (1-h) + \lambda_1 - \lambda_2 = 0 \Longrightarrow \lambda_2 = 0 \\
\frac{\partial \mathcal{L}}{\partial y_0} &= h + (1-h)P_D - \lambda_1 P_0 + \lambda_2 P_D \alpha \leq 0 \\
\frac{\partial \mathcal{L}}{\partial \omega_0} &= h + (1-h) - \lambda_1 \leq 0
\end{aligned}$$

The Kuhn-Tucker conditions are $\lambda_1(-C_0^h - \omega_0 - P_0 y_0 + P_0 + 1 + \varphi_0) = 0$, $\lambda_2(P_D y_0 \alpha - \varphi_0) = 0$, $y_0(h + (1-h)P_D - \lambda_1 P_0 + \lambda_2 P_D \alpha) = 0$ and $\omega_0(h + (1-h) - \lambda_1) = 0$. From the first order conditions, we have $\lambda_1 > 0$ and this implies the budget constraint is binding $C_0^h + \omega_0 + P_0 y_0 = P_0 + 1 + \varphi_0$.

If we have $y_0 = 0$ then $h \leq \dfrac{P_0 - P_D}{1 - P_D}$.These investors are the sellers and the lenders. They are indifferent between consuming and warehousing.

If $y_0 > 0$ and $\varphi_0 = P_D \alpha y_0$ so these investors are the buyers and the borrowers. The budget constraint becomes $P_0 y_0 = P_0 + 1 + P_D \alpha y_0$ and this implies $y_0 = \dfrac{P_0 + 1}{P_0 - P_D \alpha}$. They borrow at the maximum level $\varphi_0 = P_D \alpha \dfrac{P_0 + 1}{P_0 - P_D \alpha}$.

The market for the asset clears if $\int_{h^*(P)}^{1} y_0 \, \mathrm{d}h = 1$. This leads to $(1 - h^*(P))y_0 = 1 \Longrightarrow (1 - \dfrac{P_0 - P_D}{1 - P_D})y_0 = 1$ and if we plug in y_0, we get $(1 - \dfrac{2P_D \alpha}{P_0 + 1})\dfrac{P_0 + 1}{P_0 - P_D \alpha} = 1$.

Proof of Proposition 2.2

The only difference between date 1 optimization problem and date 0 optimization problem is the budget constraints. The optimistic investors with beliefs $h \geq h^*(P)$ have $\dfrac{P_D(1-\alpha)}{1 - h^*(P)}$ amount of consumption good at date 1 whereas the pessimistic investors with beliefs $h \leq h^*(P)$ have $\dfrac{P_D \alpha + 1}{h^*(P)}$ amount of consumption good. The Lagrangian of the optimization problem for the optimistic investors is

$$\mathcal{L} = C_D^h + hC_{DU}^h + (1-h)C_{DD}^h + \lambda_1(-C_D^h - P_D y_1 - \omega_1 + P_D \frac{1-\alpha}{1 - h^*(P)} + \varphi_1) + \lambda_2(d y_1 \alpha - \varphi_1).$$

The solution of the optimization problem is the same. The investors with the beliefs $h \geq h^*(P)$ continue to be the buyers and they buy $y_1^O = \dfrac{P_D(1-\alpha)}{(1 - h^*(P))(P_D - d\alpha)}$ of the asset. I call these investors the old buyers. They are also the borrowers and they borrow at the maximum level $\varphi_1 = y_1^O d\alpha$. The amount of their consumption at date 2 is $C_{DU}^h = y_1^O(1 - d\alpha)$ if the asset goes up and $C_{DD} = y_1^O d(1-\alpha)$ at the worst state.

The Lagrangian of the optimization problem for the pessimistic investors with beliefs $h \leq h^*(P)$ is

$$\mathcal{L} = C_D^h + hC_{DU}^h + (1-h)C_{DD}^h + \lambda_1(-C_D^h - P_D y_1 - \omega_1 + \frac{P_D\alpha + 1}{h^*(P)} + \varphi_1) + \lambda_2(dy_1\alpha - \varphi_1).$$

The solution is similar to the above problem. The investors holding beliefs $h \geq h^{**}(P)$ where $h^{**}(P) = \dfrac{P_D - d}{1-d}$ are the buyers of the asset and also the borrowers. They buy $y_1^N = \dfrac{P_D\alpha + 1}{h^*(P)(P_D - d\alpha)}$ of the asset and they borrow at the maximum level $\varphi_1 = dy_1^N\alpha$. I call these investors the new buyers since they become buyers of the asset at date 1 although they were the sellers at date 0.

Proof of Proposition 2.3

At date 0, $\int\limits_{h^*(P)}^{1} y_0 \, \mathrm{d}h = 1$ implies

$$\left(1 - \frac{P_0 - P_D}{1 - P_D}\right)\left(\frac{P_0 + 1}{P_0 - P_D\alpha}\right) = 1$$

and this is simplified to

$$(1 - P_0)(1 + P_0) - (1 - P_D)(P_0 - P_D\alpha) = 0.$$

At date 1,

$$\int\limits_{h^{**}(P)}^{h^*(P)} y_1^N \, \mathrm{d}h + \int\limits_{h^*(P)}^{1} y_1^O \, \mathrm{d}h = 1$$

implies

$$\left(\frac{P_0 - P_D}{1 - P_D} - \frac{P_D - d}{1 - d}\right)\left(\frac{(P_D\alpha + 1)(1 - P_D)}{(P_0 - P_D)(P_D - \alpha d)}\right) + \left(1 - \frac{P_0 - P_D}{1 - P_D}\right)\left(\frac{P_D(1-\alpha)(1 - P_D)}{(1 - P_0)(P_D - \alpha d)}\right) = 1.$$

These two equations are simplified to

$$P_0 + 1 = \frac{(1 + P_D\alpha)(1 - P_D)}{P_0 - P_D} = \frac{(1 + d\alpha)(1 - d)}{P_D - d}.$$

We can define two functions

$$F_1(P_0, P_D, \alpha) = (P_0 + 1)(P_0 - P_D) - (1 - P_D)(P_D\alpha + 1) = 0$$
$$F_2(P_0, P_D, \alpha) = (P_0 + 1)(P_D - d) - (1 - d)(d\alpha + 1) = 0.$$

The derivatives are

$$\frac{\partial F_1}{\partial P_0} = (P_0 - P_D) + (P_0 + 1) > 0, \; \frac{\partial F_2}{\partial P_D} = P_0 + 1 > 0$$
$$\frac{\partial F_1}{\partial P_D} = -\alpha(1 - P_D) - (P_0 - P_D\alpha) < 0, \; \frac{\partial F_2}{\partial P_0} = P_D - d > 0$$
$$\frac{\partial F_1}{\partial \alpha} = -(1 - P_D)P_D < 0, \; \frac{\partial F_2}{\partial \alpha} = -(1 - d)d < 0.$$

By Cramer's rule, the derivative of P_0 with respect to α can be written as

$$\frac{dP_0}{d\alpha} = \frac{-\frac{\partial F_1}{\partial \alpha}\frac{\partial F_2}{\partial P_D} + \frac{\partial F_1}{\partial P_D}\frac{\partial F_2}{\partial \alpha}}{\frac{\partial F_1}{\partial P_0}\frac{\partial F_2}{\partial P_D} - \frac{\partial F_1}{\partial P_D}\frac{\partial F_2}{\partial P_0}}$$

and we plug in the above derivatives, we get

$$\frac{dP_0}{d\alpha} = \frac{(1-P_D)P_D(P_0+1) + (\alpha(1-P_D) + (P_0 - P_D\alpha))(1-d)d}{((P_0-P_D)+(P_0+1))(P_0+1) + (\alpha(1-P_D) + (P_0-P_D\alpha))(P_D-d)} > 0.$$

Remember $P_0 = h^{**} + (1-h^{**})P_D$ and $h^{**} = \frac{P_D - d}{1-d}$ so $\frac{dP_0}{d\alpha} = \frac{dh^{**}}{d\alpha}(1-P_D) + (1-h^{**})\frac{dP_D}{d\alpha}$ and $\frac{dh^{**}}{d\alpha} = \frac{1}{1-d}\frac{dP_D}{d\alpha}$. After some algebra, $\frac{dP_0}{d\alpha} = \frac{2(1-P_D)}{1-d}\frac{dP_D}{d\alpha}$ which implies that $\frac{dP_D}{d\alpha} > 0$.

An investor can not be made better off while making the others not worse off

Remember the Lagrangian of the optimization problem at date 0

$$\mathcal{L} = C_0^h + hC_U^h + (1-h)C_D^h + \lambda_1(-C_0^h - P_0 y_0 + P_0 + 1 + \varphi_0) + \lambda_2(P_D y_0 \alpha - \varphi_0).$$

By Roy's Identity, we have $\frac{\partial \nu(P)}{\partial \alpha} = \frac{\partial \mathcal{L}}{\partial P_0}\frac{dP_0}{d\alpha} + \frac{\partial \mathcal{L}}{\partial P_D}\frac{dP_D}{d\alpha}$ where $\nu(P)$ is the indirect utility. It is the consumption of the investors in our model since the investors are risk-neutral. The consumption of the seller $\frac{\partial\left(\frac{1+P_D\alpha}{h^*(P)}\right)}{\partial \alpha} = \lambda_1 \frac{dP_0}{d\alpha} \geq 0$ and the consumption of the buyer $\frac{\partial\left(\frac{1-P_D\alpha}{1-h^*(P)}\right)}{\partial \alpha} = \lambda_1(1-y_0)\frac{dP_0}{d\alpha} \leq 0$ since $y_0 = \frac{P_0+1}{P_0 - P_D\alpha} \geq 1$.

Now remember the Lagrangian of the optimization problem for the old buyers at date 1 $\mathcal{L} = C_D^h + hC_{DU}^h + (1-h)C_{DD}^h + \lambda_1(-C_D^h - P_D y_1 + P_D\frac{1-\alpha}{1-h^*(P)} + \varphi_1) + \lambda_2(dy_1\alpha - \varphi_1)$. In state DD, the consumption of the seller is again increasing since the consumption of the seller is always the same in any state. The consumption of the old buyer is decreasing in α

$$\frac{\partial C(P)}{\partial \alpha} = -\lambda_1(y_1^O - \frac{(1-\alpha)}{1-h^*(P)})\frac{dP_D}{d\alpha} = -\lambda_1 \frac{\alpha d(1-\alpha)}{(1-h^*(P))(P_D - \alpha d)}\frac{dP_D}{d\alpha} \leq 0$$

since $y_1^O = \frac{P_D(1-\alpha)}{(1-h^*)(P_D - \alpha d)}$.

Now remember the Lagrangian of the optimization problem for the new buyers at date 1 $\mathcal{L} = C_D^h + hC_{DU}^h + (1-h)C_{DD}^h + \lambda_1(-C_D^h - P_D y_1 + \frac{P_D\alpha + 1}{h^*(P)} + \varphi_1) + \lambda_2(dy_1\alpha - \varphi_1)$. The consumption of the new buyer is also decreasing in α

$$\frac{\partial B(P)}{\partial \alpha} = -\lambda_1(y_1^N - \frac{(\alpha)}{h^*(P)})\frac{dP_D}{d\alpha} = -\lambda_1 \frac{1+\alpha^2 d}{(h^*(P))(P_D - \alpha d)}\frac{dP_D}{d\alpha} \leq 0$$

since $y_1^N = \frac{P_D\alpha + 1}{(h^*(P))(P_D - \alpha d)}$.

Proof of Proposition 2.4

The welfare of the economy in the best state is $W_1 = ((A)^\rho h^*(P) + (B)^\rho(1-h^*(P)))^{\frac{1}{\rho}}$ where $A = \left(\frac{1+\varphi}{h^*}\right)$ and $B = \left(\frac{1-\varphi}{1-h^*}\right)$ and $\varphi = P_D\alpha$ to ease the notation. When we take the derivative with respect to α, we have

$$\frac{\partial W_1}{\partial \alpha} = \frac{1}{\rho}\left(A^\rho h^* + B^\rho(1-h^*)\right)^{\frac{1}{\rho}-1}\left(\rho A^{\rho-1}\frac{\partial A}{\partial \alpha}h^* + \rho B^{\rho-1}\frac{\partial B}{\partial \alpha}(1-h^*) + (A^\rho - B^\rho)\frac{\partial h^*}{\partial \alpha}\right).$$

The derivative of the consumptions with respect to α are $\frac{\partial A}{\partial \alpha} = \left(\frac{\partial \varphi}{\partial \alpha} - \frac{\partial h^*}{\partial \alpha}A\right)\frac{1}{h^*} \geq 0$ and $\frac{\partial B}{\partial \alpha} = \left(-\frac{\partial \varphi}{\partial \alpha} + \frac{\partial h^*}{\partial \alpha}B\right)\frac{1}{1-h^*} \leq 0$. When we plug these derivatives in the above derivative, we have $\frac{\partial W_1}{\partial \alpha} = \frac{1}{\rho}\left(A^\rho h^* + B^\rho(1-h^*)\right)^{\frac{1}{\rho}-1} G(P)$ where

$$G(P) = \left(\rho A^{\rho-1}\left(\frac{\partial \varphi}{\partial \alpha} - \frac{\partial h^*}{\partial \alpha}A\right) - \rho B^{\rho-1}\left(\frac{\partial \varphi}{\partial \alpha} - \frac{\partial h^*}{\partial \alpha}B\right) + (A^\rho - B^\rho)\frac{\partial h^*}{\partial \alpha}\right).$$

Define a function $f : \mathbb{R}_+ \to \mathbb{R}$ such that $f(x) = x^\rho \frac{\partial h^*}{\partial \alpha} - \rho x^{\rho-1}\left(\frac{\partial h^*(P)}{\partial \alpha}x - \frac{\partial (P_D\alpha)}{\partial \alpha}\right)$. Now we can write the derivative in terms of the new function f as $G(P) = (f(A) - f(B))$. First, I want to show that $\frac{1+P_D\alpha}{h^*(P)} \leq \frac{1-P_D\alpha}{1-h^*}$ or $A \leq B$. Assume the reverse relation then we have $1 + P_D\alpha - h^*(P) - P_D\alpha h^* > h^* - P_D\alpha h^*$ and this implies $\frac{1+P_D\alpha}{h^*} > 2$. Remember that $\frac{1+P_D\alpha}{h^*} = P_0 + 1$ since $C_0^h = C_D^h$ and $1 + P_0 > 2$ gives contradiction since $P_0 \leq 1$. This implies $\frac{1+P_D\alpha}{h^*} \leq \frac{1-P_D\alpha}{1-h^*}$ so $A \leq B$. The consumption of the seller is smaller than the consumption of the buyer when they end up in the best state.

Now let's write the derivative of the function $f(x)$ as $f'(x) = -\rho(\rho-1)x^{\rho-2}\left(\frac{\partial h^*}{\partial \alpha}x - \frac{\partial (P_D\alpha)}{\partial \alpha}\right)$ so $f'(x) \geq 0$ at both A and B since $\frac{\partial A}{\partial \alpha} = \left(\frac{\partial \varphi}{\partial \alpha} - \frac{\partial h^*}{\partial \alpha}A\right)\frac{1}{h^*} \geq 0$ and $A \leq B$. This implies $f(A) - f(B) \leq 0$ which concludes that $\frac{\partial W_1}{\partial \alpha} \geq 0$ for $\rho \leq 0$.

Proof of Proposition 2.5

The welfare of the economy in the worst state is

$$W_4 = \left(A^\rho h^{**}(P) + B^\rho(h^*(P) - h^{**}(P)) + C^\rho(1-h^*(P))\right)^{\frac{1}{\rho}}$$

where $A = \frac{1+d\alpha}{h^{**}(P)}$, $B = y_1^N(d - \alpha d)$ and $C = y_1^O(d - \alpha d)$.

First, I want to show that $C \leq B \leq A$. The consumption of the new buyer is $B = y_1^N(d - \alpha d) = \frac{P_D\alpha + 1}{h^*(P)(P_D - \alpha d)}(d - \alpha d)$ and the consumption of the old buyer is $C = y_1^O(d - \alpha d) = \frac{P_D(1-\alpha)}{(1-h^*(P))(P_D - \alpha d)}(d - \alpha d)$. Let's assume $C > B$ then $P_D(1-\alpha)h^*(P) > (1-h^*(P))(P_D\alpha+$

1) and this implies $P_D + 1 > \dfrac{P_D\alpha + 1}{h^*(P)} = P_0 + 1$ which contradicts. So, we have $B \leq C$. And $C = \dfrac{P_D\alpha + 1}{h^*(P)(P_D - \alpha d)}(d - \alpha d) \leq A = \dfrac{d\alpha + 1}{h^{**}(P)} = \dfrac{P_D\alpha + 1}{h^*(P)}$ since $\dfrac{d - d\alpha}{P_D - d\alpha} \leq 1$.

Now let's write the derivative of the welfare with respect to α in terms of A, B and C

$$\frac{\partial W_4}{\partial \alpha} = \frac{1}{\rho}\left(A^\rho h^{**}(P) + B^\rho(h^*(P) - h^{**}(P)) + C^\rho(1 - h^*(P))\right)^{\frac{1}{\rho}-1} F(P)$$

where $F(P) = \rho A^{\rho-1}\dfrac{\partial A}{\partial \alpha}h^{**} + \rho B^{\rho-1}\dfrac{\partial B}{\partial \alpha}(h^* - h^{**}) + \rho C^{\rho-1}\dfrac{\partial C}{\partial \alpha}(1 - h^*) + \dfrac{\partial h^{**}}{\partial \alpha}(A^\rho - B^\rho) + \dfrac{\partial h^*}{\partial \alpha}(B^\rho - C^\rho)$. For ease of notation, I want to define the consumption as following $A = \dfrac{1 + \varphi}{h^{**}}$, $B = \dfrac{y_0 d - (\varphi - \varphi^B)}{h^* - h^{**}}$ and $C = \dfrac{(1 - y_0)d - \varphi^B}{1 - h^*}$ where φ is the aggregate amount that the sellers lend, φ^B is the total amount that the old sellers borrowed and y_0 is the aggregate amount of the asset that the new buyers buy. The derivative of the consumption are $\dfrac{\partial A}{\partial \alpha} = \left(\dfrac{\partial \varphi}{\partial \alpha} - \dfrac{\partial h^{**}}{\partial \alpha}A\right)\dfrac{1}{h*} \geq 0$, $\dfrac{\partial B}{\partial \alpha} = \left(\dfrac{\partial y_0}{\partial \alpha}d - \dfrac{\partial \varphi}{\partial \alpha} + \dfrac{\partial \varphi^B}{\partial \alpha} - \dfrac{\partial h^*}{\partial \alpha}B + \dfrac{\partial h^{**}}{\partial \alpha}B\right)\dfrac{1}{h* - h^{**}} \leq 0$ and $\dfrac{\partial C}{\partial \alpha} = \left(-\dfrac{\partial y_0}{\partial \alpha}d - \dfrac{\partial \varphi^B}{\partial \alpha} + \dfrac{\partial h^*}{\partial \alpha}C\right)\dfrac{1}{1 - h*} \leq 0$. If we plug these in $F(P)$, we have

$$\begin{aligned}
F(P) = \rho A^{\rho-1}\left(\frac{\partial \varphi}{\partial \alpha} - \frac{\partial h^{**}}{\partial \alpha}A\right) + \rho B^{\rho-1}\left(\frac{\partial y_0}{\partial \alpha}d - \frac{\partial \varphi}{\partial \alpha} + \frac{\partial \varphi^B}{\partial \alpha} - \frac{\partial h^*}{\partial \alpha}B + \frac{\partial h^{**}}{\partial \alpha}B\right) + \\
\rho C^{\rho-1}\left(-\frac{\partial y_0}{\partial \alpha}d - \frac{\partial \varphi^B}{\partial \alpha} + \frac{\partial h^*}{\partial \alpha}C\right) + \frac{\partial h^{**}}{\partial \alpha}(A^\rho - B^\rho) + \frac{\partial h^*}{\partial \alpha}(B^\rho - C^\rho) = \\
\rho A^{\rho-1}\left(\frac{\partial \varphi}{\partial \alpha} - \frac{\partial h^{**}}{\partial \alpha}A\right) - \rho B^{\rho-1}\left(\frac{\partial \varphi}{\partial \alpha} - \frac{\partial h^{**}}{\partial \alpha}B\right) + \frac{\partial h^{**}}{\partial \alpha}(A^\rho - B^\rho) + \\
\rho B^{\rho-1}\left(\frac{\partial y_0}{\partial \alpha}d + \frac{\partial \varphi^B}{\partial \alpha} - \frac{\partial h^*}{\partial \alpha}B\right) - \rho C^{\rho-1}\left(\frac{\partial y_0}{\partial \alpha}d + \frac{\partial \varphi^B}{\partial \alpha} - \frac{\partial h^*}{\partial \alpha}C\right) + \frac{\partial h^*}{\partial \alpha}(B^\rho - C^\rho).
\end{aligned}$$

Define two functions $f : \mathbb{R}_+ \to \mathbb{R}$ and $g : \mathbb{R}_+ \to \mathbb{R}$ such that

$$\begin{aligned}
f(X) = \rho X^{\rho-1}\left(\frac{\partial \varphi}{\partial \alpha} - \frac{\partial h^{**}}{\partial \alpha}X\right) + \frac{\partial h^{**}}{\partial \alpha}X^\rho \text{ and} \\
g(X) = \rho X^{\rho-1}\left(\frac{\partial y_0}{\partial \alpha}d + \frac{\partial \varphi^B}{\partial \alpha} - \frac{\partial h^*}{\partial \alpha}X\right) + \frac{\partial h^*}{\partial \alpha}X^\rho.
\end{aligned}$$

Then, we have $F(P) = f(A) - f(B) + g(B) - g(C)$. The derivative of the functions are $f'(X) = \rho(\rho - 1)X^{(\rho-2)}\left(\dfrac{\partial \varphi}{\partial \alpha} - \dfrac{\partial h^{**}}{\partial \alpha}X\right) \geq 0$ at A and B since $\dfrac{\partial A}{\partial \alpha} \geq 0$ and $A \geq B$ which implies $f(A) - f(B) \geq 0$, $g'(X) = \rho(\rho - 1)X^{(\rho-2)}\left(\dfrac{\partial y_0}{\partial \alpha}d + \dfrac{\partial \varphi^B}{\partial \alpha} - \dfrac{\partial h^*}{\partial \alpha}X\right) \geq 0$ at B and C since $\dfrac{\partial C}{\partial \alpha} \leq 0$ and $B \geq C$ which implies $g(B) - g(C) \geq 0$. Therefore, $F(P) \geq 0$ and this implies that $\dfrac{\partial W_4}{\partial \alpha} \leq 0$ for $\rho \leq 0$.

Proof of Proposition 2.6

The derivative of optimal α^* with respect to the probability is $\frac{d\alpha^*}{dp} = \frac{-\frac{\partial F}{\partial p}}{\frac{\partial F}{\partial \alpha}}$. First, let's start from the denominator, we have $\frac{\partial F}{\partial \alpha} = \frac{\partial^2 E(W)}{\partial \alpha^2} \leq 0$ since we know that the optimal α^* is maximizing the expected welfare $E(W)$. Now, let's find the sign of the numerator, we have $\frac{\partial E(W)}{\partial \alpha} = p\frac{\partial W_1}{\partial \alpha} + (1-p)p\frac{\partial W_3}{\partial \alpha} + (1-p)^2\frac{\partial W_4}{\partial \alpha} = 0$ which implies $\frac{\partial W_3}{\partial \alpha} = -\frac{1-p}{p}\frac{\partial W_4}{\partial \alpha} - \frac{1}{1-p}\frac{\partial W_1}{\partial \alpha}$. The derivative in the numerator is

$$\frac{\partial F}{\partial p} = \frac{\partial\left(\frac{\partial E(W)}{\partial \alpha}\right)}{\partial p} = \frac{\partial W_1}{\partial \alpha} + (1-2p)\frac{\partial W_3}{\partial \alpha} - 2(1-p)\frac{\partial W_4}{\partial \alpha}$$

and if we plug in $\frac{\partial W_3}{\partial \alpha}$, we have $\frac{\partial F}{\partial p} = \frac{p}{1-p}\frac{\partial W_1}{\partial \alpha} - \frac{1-p}{p}\frac{\partial W_4}{\partial \alpha} \geq 0$ since we know that $\frac{\partial W_1}{\partial \alpha} \geq 0$ and $\frac{\partial W_4}{\partial \alpha} \leq 0$. This implies that $\frac{d\alpha^*}{dp} \geq 0$.

Proof of Proposition 2.7

The Lagrangian of the optimization problem is

$$\mathcal{L} = U(C_0^h) + hU(C_U^h) + (1-h)U(C_D^h) + \lambda_1(-C_0^h - P_0 y_0 - \omega_0 + P_0 + 1 + \varphi_0) + \lambda_2(P_D y_0 \alpha - \varphi_0).$$

The investors choose the optimal C_O^h, φ_0 and y_0 so the first order conditions are

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial C_0^h} &= U'(C_0^h) - \lambda_1 = 0 \Longrightarrow \lambda_1 = U'(C_0^h) > 0 \\
\frac{\partial \mathcal{L}}{\partial \varphi_0} &= -hU'(C_U^h) - (1-h)U'(C_D^h) + \lambda_1 - \lambda_2 = 0 \Longrightarrow \lambda_2 = U'(C_0^h) - [hU'(C_U^h) + (1-h)U'(C_D^h)] \\
\frac{\partial \mathcal{L}}{\partial y_0} &= hU'(C_U^h) + (1-h)U'(C_D^h)P_D - \lambda_1 P_0 + \lambda_2 P_D \alpha \leq 0 \\
\frac{\partial \mathcal{L}}{\partial \omega_0} &= hU'(C_U^h) + (1-h)U'(C_D^h) - \lambda_1 \leq 0
\end{aligned}$$

The Kuhn-Tucker conditions are $\lambda_1(-C_0^h - \omega_0 - P_0 y_0 + P_0 + 1 + \varphi_0) = 0$, $\lambda_2(P_D y_0 \alpha - \varphi_0) = 0$, $y_0(hU'(C_U^h) + (1-h)U'(C_D^h)P_D - \lambda_1 P_0 + \lambda_2 P_D \alpha) = 0$ and $\omega_0(hU'(C_U^h) + (1-h)U'(C_D^h) - \lambda_1) = 0$. From the first order conditions, we have $\lambda_1 > 0$ and this implies the budget constraint is binding $C_0^h + \omega_0 + P_0 y_0 = P_0 + 1 + \varphi_0$. And for the investors who have binding borrowing constraint as $\lambda_2 > 0$, we have $\omega_0 = 0$.

If $\lambda_2 > 0$, we have $U'(C_0^h) > [hU'(C_U^h) + (1-h)U'(C_D^h)]$ and $\varphi_0 = P_D y_0 \alpha$. If $y_0 = 0$, then $\varphi_0 = 0$, $C_U^h = C_D^h = 0$ and $C_0^h = P_0 + 1$. The condition $U'(C_0^h) > [hU'(C_U^h) + (1-h)U'(C_D^h)]$ implies that $U'(C_0^h) > U'(C_U^h)$ which also implies $C_0^h = P_0 + 1 < C_U^h = 0$. This is contradiction. So, the sellers never have binding borrowing constraint. If $y_0 > 0$ and $\varphi_0 = P_D \alpha y_0$ then

$U'(C_0^h) > [hU'(C_U^h)+(1-h)U'(C_D^h)]$ implies $h > \dfrac{U'(C_D^h) - U'(C_0^h)}{U'(C_D^h) - U'(C_U^h)}$. This condition also implies that $U'(C_U^h) \leq U'(C_0^h)$ and this leads to $C_U^h = y_0^B(1 - P_D\alpha) \geq C_0^h = P_0 + 1 + y_0^B(P_D\alpha - P_0)$. Thus, the asset that the very optimistic investors buy satisfies $y_0^B \geq \dfrac{P_0+1}{P_0+1-2P_D\alpha}$.

If $\lambda_2 = 0$, we have $U'(C_0^h) = [hU'(C_U^h) + (1-h)U'(C_D^h)]$ and $\varphi_0 \leq P_D y_0 \alpha$. This implies $C_D^h \leq C_0^h \leq C_U^h$. If $y_0 = 0$, then $\varphi_0 \leq 0$, $C_D^h = C_U^h = -\varphi_0$ and $C_0^h = P_0 + 1$. The condition $U'(C_0^h) = [hU'(C_U^h)+(1-h)U'(C_D^h)]$ implies that $C_D^h = C_U^h = C_0^h$ which leads to $\varphi_0 = -\dfrac{P_0+1}{2}$. So, the sellers are the lenders. If $y_0 > 0$, we have $C_0^h = P_0 + 1 + \varphi_0^N - P_0 y_0^N$, $C_U^h = y_0^N - \varphi_0^N$ and $C_D^h = P_D y_0^N - \varphi_0^N$. The condition $C_D^h \leq C_0^h \leq C_U^h$ implies that $\dfrac{P_0+1+2\varphi_0}{P_0+1} \leq y_0^N \leq \dfrac{P_0+1+2\varphi_0}{P_0+P_D}$.

Proof of Proposition 2.8

The Lagrangian of the optimization problem for the more optimistic investors ($h \geq h^{'}(P)$) is

$$\mathcal{L} = C_D^h + hC_{DU}^h + (1-h)C_{DD}^h + \lambda_1(-C_D^h - P_D y_1 - \omega_1 + P_D y_0^B(1-\alpha) + \varphi_1) + \lambda_2(dy_1\alpha - \varphi_1).$$

These investors continue to be the buyers with binding borrowing constraint and they buy $y_1^B B \geq \dfrac{P_D(1-\alpha)y_0^B}{1+P_D-2d\alpha}$ of the asset.

The Lagrangian of the optimization problem for the less optimistic investors ($h^{'}(P) \geq h \geq h^*(P)$) is

$$\mathcal{L} = C_D^h + hC_{DU}^h + (1-h)C_{DD}^h + \lambda_1(-C_D^h - P_D y_1 - \omega_1 + P_D y_0^N - \varphi_0^N + \varphi_1) + \lambda_2(dy_1\alpha - \varphi_1).$$

The investors holding the belief $h^{'}(P) \geq h \geq h^{''}(P)$ continue to be the buyers with binding borrowing constraint and they buy $y_1^N B \geq \dfrac{P_D y_0^N - \varphi_0^N}{1+P_D-2d\alpha}$ of the asset where $h^{''}(P) = dfrac U'(C_{DD}^h) - U'(C_D^h)U'(C_{DD}^h) - U'(C_{DU}^h)$. The investors holding the belief $h^{''}(P) \geq h \geq h^*(P)$ continue to be buyers with non-binding borrowing constraint and they buy $\dfrac{P_D y_0^N - \varphi_0^N + 2\varphi_1^N}{1+P_D} \leq y_1^N \leq \dfrac{P_D y_0^N - \varphi_0^N + 2\varphi_1^N}{d+P_D}$ amount of assets.

The lagrangian of the optimization problem for pessimistic investors with beliefs $h \leq h^*(P)$ is

$$\mathcal{L} = C_D^h + hC_{DU}^h + (1-h)C_{DD}^h + \lambda_1(-C_D^h - P_D y_1 - \omega_1 + \frac{P_0+1}{2} + \varphi_1) + \lambda_2(dy_1\alpha - \varphi_1).$$

The solution is similar to the above problem. The investors holding beliefs $h \geq h^{**}(P)$ where $h^{**}(P) = \dfrac{P_D - d}{1-d}$ are the buyers of the asset and also the non-bonding borrowers. They buy $\dfrac{\frac{P_0+1}{2} + 2\varphi_1^N}{1+P_D} \leq y_1^N \leq \dfrac{\frac{P_0+1}{2} + 2\varphi_1^N}{d+P_D}$ of the asset. The investors holding beliefs $h \leq h^{**}(P)$ continue to be sellers of the asset.

This proof is for the case where I assume $h^{''}(P) \geq h^{*}(P)$. If the inequality is reversed, then the only difference will be that all the buyers at date 0 will become a binding borrower and the new buyers become either binding or non-binding instead of only non-binding here.

The proof that $h^{''}(P) \leq h^{'}(P)$ is as following. Assume that $h^{''}(P) > h^{'}(P)$ then $\dfrac{U'(C^h_{DD}) - U'(C^h_D)}{U'(C^h_{DD}) - U'(C^h_{DU})} > \dfrac{U'(C^h_D) - U'(C^h_0)}{U'(C^h_D) - U'(C^h_U)}$. This implies that $U'(C^h_D)(U'(C^h_U) - U'(C^h_D)) > U'(C^h_{DD})(U'(C^h_U) - U'(C^h_0)) + U'(C^h_{DU})(U'(C^h_0) - U'(C^h_D))$ and we know that $U'(C^h_{DD}) \geq U'(C^h_{DU})$ since $C^h_{DD} \leq C^h_{DU}$. This leads to $U'(C^h_D)(U'(C^h_U) - U'(C^h_D)) > U'(C^h_{DU})(U'(C^h_U) - U'(C^h_D))$ and $U'(C^h_U) < U'(C^h_D)$ implies that $U'(C^h_D) < U'(C^h_{DU})$ and this gives $C^h_D > C^h_{DU}$ which is contradiction.

Moreover, $h^{'}(P) = \dfrac{U'(C^h_D) - U'(C^h_0)}{U'(C^h_D) - U'(C^h_U)}$ is increasing in α since the consumption $C^h_0 = P_0 + 1 + y_0(P_D\alpha - P_0)$ is increasing in α and the consumption $C^h_U = y_0(1 - P_D\alpha)$ is decreasing in α.

Proof of Proposition 2.9

The welfare of the economy in the best state is $W_1 = \Big(U(A)h^*(P) + U(B)(h^{'}(P) - h^*(P)) + U(C)(1 - h^{'}(P))\Big)$ where $A = \left(\dfrac{1+\varphi}{h^*}\right)$, $B = \left(\dfrac{y_0 - (\varphi - \varphi^B)}{h' - h^*}\right)$ and $C = \left(\dfrac{1 - y_0 - \varphi^B}{1 - h^*}\right)$ where φ_0 is the aggregate amount that the borrowers borrow, φ^B_0 is the aggregate amount that the binding borrowers borrow and y_0 is the aggregate amount of asset that the non-binding borrowers buy to ease the notation. When we take the derivative with respect to α, we have $\dfrac{\partial W_1}{\partial \alpha} = h^*U'(A)\dfrac{\partial A}{\partial \alpha} + (h^{'} - h^*)U'(B)\dfrac{\partial B}{\alpha} + (1 - h^{'})U'(C)\dfrac{\partial C}{\partial \alpha} + \dfrac{\partial h^*}{\partial \alpha}(U(A) - U(B)) + \dfrac{\partial h^{'}}{\partial \alpha}(U(B) - U(C))$. The derivative of the consumptions are $\dfrac{\partial A}{\partial \alpha} = \left(\dfrac{\partial \varphi}{\partial \alpha} - \dfrac{\partial h^*}{\partial \alpha}A\right)\dfrac{1}{h^*} \geq 0$, $\dfrac{\partial B}{\partial \alpha} = \left(\dfrac{\partial y_0}{\partial \alpha} - \dfrac{\partial \varphi}{\partial \alpha} + \dfrac{\partial \varphi^B}{\partial \alpha} - \dfrac{\partial h^{'}}{\partial \alpha}B + \dfrac{\partial h^*}{\partial \alpha}B\right)\dfrac{1}{h' - h^*} \leq 0$ and $\dfrac{\partial C}{\partial \alpha} = \left(-\dfrac{\partial y_0}{\partial \alpha} - \dfrac{\partial \varphi^B}{\partial \alpha} + \dfrac{\partial h^{'}}{\partial \alpha}C\right)\dfrac{1}{1 - h'} \leq 0$. When we plug these derivatives in the above derivative, we have $\dfrac{\partial W_1}{\partial \alpha} = U'(A)\left(\dfrac{\partial \varphi}{\partial \alpha} - \dfrac{\partial h^*}{\partial \alpha}A\right) - U'(B)\left(\dfrac{\partial \varphi}{\partial \alpha} - \dfrac{\partial h^*}{\partial \alpha}B\right) + \dfrac{\partial h^*}{\partial \alpha}(U(A) - U(B)) + U'(B)\left(\dfrac{\partial y_0}{\partial \alpha} + \dfrac{\partial \varphi^B}{\partial \alpha} - \dfrac{\partial h^{'}}{\partial \alpha}B\right) - U'(C)\left(\dfrac{\partial y_0}{\partial \alpha} + \dfrac{\partial \varphi^B}{\partial \alpha} - \dfrac{\partial h^{'}}{\partial \alpha}C\right) + \dfrac{\partial h^{'}}{\partial \alpha}(U(B) - U(C))$. Define functions $f : \mathbb{R}_+ \to \mathbb{R}$ and $g : \mathbb{R}_+ \to \mathbb{R}$ such that $f(X) = U'(X)\left(\dfrac{\partial \varphi}{\partial \alpha} - \dfrac{\partial h^*}{\partial \alpha}X\right) + \dfrac{\partial h^*}{\partial \alpha}U(X)$ and

$g(X) = U'(X)\left(\dfrac{\partial y_0}{\partial \alpha} + \dfrac{\partial \varphi^B}{\partial \alpha} - \dfrac{\partial h^{'}}{\partial \alpha}X\right) + \dfrac{\partial h^{'}}{\partial \alpha}U(X)$ then $\dfrac{\partial W_1}{\partial \alpha} = f(A) - f(B) + g(B) - g(C)$. The derivative of f is $f'(X) = U''(X)\left(\dfrac{\partial \varphi}{\partial \alpha} - \dfrac{\partial h^*}{\partial \alpha}X\right)$, $f'(A) \leq 0$ since $U'' \leq 0$, $\dfrac{\partial A}{\partial \alpha} \geq 0$

and $f'(B) \leq 0$ since $B \leq C$ implies that $\left(\frac{\partial y_0}{\partial \alpha} + \frac{\partial \varphi^B}{\partial \alpha} - \frac{\partial h^{'}}{\partial \alpha}B\right) \geq 0$ and this implies $\left(\frac{\partial \varphi}{\partial \alpha} - \frac{\partial h^*}{\partial \alpha}B\right) \leq 0$. This leads to $f(A) - f(B) \geq 0$ since $A \leq B$. The derivative of g is $g'(X) = U''(X)\left(\frac{\partial y_0}{\partial \alpha} + \frac{\partial \varphi^B}{\partial \alpha} - \frac{\partial h^{'}}{\partial \alpha}X\right)$. We have $g'(B) \leq 0$ and $g'(C) \leq 0$ from the same reasons above. This leads to $g(B) - g(C) \geq 0$ since $B \leq C$. So, we have $\frac{\partial W_1}{\partial \alpha} = f(A) - f(B) + g(B) - g(C) \geq 0$.

Proof of Proposition 2.10

The welfare of the economy in the worst state is

$$W_4 = U(A)h^{**} + U(B)(h^* - h^{**}) + U(C)(h^{''} - h^*) + U(D)(h^{'} - h^{''}) + U(E)(1 - h^{'})$$

where $A = \frac{1+\varphi}{h^{**}}$, $B = \frac{(1-(y_1+y_2+y_3))d - (\varphi - (\varphi_1+\varphi_2+\varphi_3))}{h^* - h^{**}}$, $C = \frac{y_3 d - \varphi_3}{h'' - h^*}$, $D = \frac{y_2 d - \varphi_2}{h' - h''}$ and $E = \frac{y_1 d - \varphi_1}{1 - h^{'}}$. The derivative is $\frac{\partial W_4}{\partial \alpha} = U'(A)\frac{\partial A}{\partial \alpha}h^{**} + U'(B)\frac{\partial B}{\partial \alpha}(h^* - h^{**}) + U'(C)\frac{\partial C}{\partial \alpha}(h^{''} - h^*) + U'(D)\frac{\partial D}{\partial \alpha}(h^{'} - h^{''}) + U'(E)\frac{\partial E}{\partial \alpha}(1 - h^{'}) + (U(A) - U(B))\frac{\partial h^{**}}{\partial \alpha} + (U(B) - U(C))\frac{\partial h^*}{\partial \alpha} + (U(C) - U(D))\frac{\partial h^{''}}{\partial \alpha} + (U(D) - U(E))\frac{\partial h^{'}}{\partial \alpha}$.

The derivatives of the consumption of the investors are $\frac{\partial A}{\partial \alpha} = \left(\frac{\partial \varphi}{\partial \alpha} - \frac{\partial h^{**}}{\partial \alpha}A\right)\frac{1}{h^{**}} \geq 0$,
$\frac{\partial B}{\partial \alpha} = \left(-\frac{\partial y_1}{\partial \alpha}d - \frac{\partial y_2}{\partial \alpha}d - \frac{\partial y_3}{\partial \alpha}d - \frac{\partial \varphi}{\partial \alpha} + \frac{\partial \varphi_1}{\partial \alpha} + \frac{\partial \varphi_2}{\partial \alpha} + \frac{\partial \varphi_3}{\partial \alpha} - \frac{\partial h^*}{\partial \alpha}B + \frac{\partial h^{**}}{\partial \alpha}B\right)\frac{1}{h^* - h^{**}} \leq 0$,
$\frac{\partial C}{\partial \alpha} = \left(\frac{\partial y_3}{\partial \alpha}d - \frac{\partial \varphi_3}{\partial \alpha} - \frac{\partial h^{''}}{\partial \alpha}C + \frac{\partial h^*}{\partial \alpha}C\right)\frac{1}{h'' - h^*} \leq 0$,
$\frac{\partial D}{\partial \alpha} = \left(\frac{\partial y_2}{\partial \alpha}d - \frac{\partial \varphi_2}{\partial \alpha} - \frac{\partial h^{'}}{\partial \alpha}D + \frac{\partial h^{''}}{\partial \alpha}D\right)\frac{1}{h' - h''} \leq 0$ and
$\frac{\partial E}{\partial \alpha} = \left(\frac{\partial y_1}{\partial \alpha}d - \frac{\partial \varphi_1}{\partial \alpha} + \frac{\partial h^{'}}{\partial \alpha}E\right)\frac{1}{1 - h^{'}} \leq 0$. When we plug in these derivatives, we get

$$\frac{\partial W_4}{\partial \alpha} = U'(A)\left(\frac{\partial \varphi}{\partial \alpha} - \frac{\partial h^{**}}{\partial \alpha}A\right) - U'(B)\left(\frac{\partial \varphi}{\partial \alpha} - \frac{\partial h^{**}}{\partial \alpha}B\right) + (U(A) - U(B))\frac{\partial h^{**}}{\partial \alpha} +$$
$$U'(C)\left(\frac{\partial y_3}{\partial \alpha}d - \frac{\partial \varphi_3}{\partial \alpha} - \frac{\partial h^{''}}{\partial \alpha}C + \frac{\partial h^*}{\partial \alpha}C\right) - U'(B))\left(\frac{\partial y_3}{\partial \alpha}d - \frac{\partial \varphi_3}{\partial \alpha} - \frac{\partial h^{''}}{\partial \alpha}B + \frac{\partial h^*}{\partial \alpha}B\right) +$$
$$(U(C) - U(B))\left(\frac{\partial h^{''}}{\partial \alpha} - \frac{\partial h^*}{\partial \alpha}\right) + U'(D)\left(\frac{\partial y_2}{\partial \alpha}d - \frac{\partial \varphi_2}{\partial \alpha} - \frac{\partial h^{'}}{\partial \alpha}D + \frac{\partial h^{''}}{\partial \alpha}D\right) -$$
$$U'(B)\left(\frac{\partial y_2}{\partial \alpha}d - \frac{\partial \varphi_2}{\partial \alpha} - \frac{\partial h^{'}}{\partial \alpha}B + \frac{\partial h^{''}}{\partial \alpha}B\right) + (U(D) - U(B))\left(\frac{\partial h^{'}}{\partial \alpha} - \frac{\partial h^{''}}{\partial \alpha}\right) +$$
$$U'(E)\left(\frac{\partial y_1}{\partial \alpha}d - \frac{\partial \varphi_1}{\partial \alpha} + \frac{\partial h^{'}}{\partial \alpha}E\right) - U'(B)\left(\frac{\partial y_1}{\partial \alpha}d - \frac{\partial \varphi_1}{\partial \alpha} + \frac{\partial h^{'}}{\partial \alpha}B\right) - (U(E) - U(B))\frac{\partial h^{'}}{\partial \alpha} =$$
$$f_1(A) - f_1(B) + f_2(C) - f_2(B) + f_3(D) - f_3(B) + f_4(E) - f_4(B)$$

where I define $f_i : \mathbb{R}_+ \to \mathbb{R}$ for $i = 1, 2, 3, 4$ such that $f_1(X) = U'(X)\left(\frac{\partial \varphi}{\partial \alpha} - \frac{\partial h^{**}}{\partial \alpha}X\right) + U(X)\frac{\partial h^{**}}{\partial \alpha}$,
$f_2(X) = U'(X)\left(\frac{\partial y_3}{\partial \alpha}d - \frac{\partial \varphi_3}{\partial \alpha} - \frac{\partial h^{''}}{\partial \alpha}X + \frac{\partial h^{*}}{\partial \alpha}X\right) + U(X)\left(\frac{\partial h^{''}}{\partial \alpha} - \frac{\partial h^{*}}{\partial \alpha}\right)$,
$f_3(X) = U'(X)\left(\frac{\partial y_2}{\partial \alpha}d - \frac{\partial \varphi_2}{\partial \alpha} - \frac{\partial h^{'}}{\partial \alpha}D + \frac{\partial h^{''}}{\partial \alpha}X\right) + U(X)\left(\frac{\partial h^{'}}{\partial \alpha} - \frac{\partial h^{''}}{\partial \alpha}\right)$ and
$f_4(X) = U'(X)\left(\frac{\partial y_1}{\partial \alpha}d - \frac{\partial \varphi_1}{\partial \alpha} + \frac{\partial h^{'}}{\partial \alpha}X\right) - U(X)\frac{\partial h^{'}}{\partial \alpha}$. We have $f_1'(X) = U''(X)\left(\frac{\partial \varphi}{\partial \alpha} - \frac{\partial h^{**}}{\partial \alpha}X\right) \leq 0$ at A and B since $\frac{\partial A}{\partial \alpha} \geq 0$ and $A \geq B$ so $f_1(A) - f_1(B) \leq 0$. For the second function, $f_2'(X) = U''(X)\left(\frac{\partial y_3}{\partial \alpha}d - \frac{\partial \varphi_3}{\partial \alpha} - \frac{\partial h^{''}}{\partial \alpha}X + \frac{\partial h^{*}}{\partial \alpha}X\right) \geq 0$ at B and C since $\frac{\partial C}{\partial \alpha} \leq 0$ and $B \geq C$ so $f_2(C) - f_2(B) \leq 0$. The same logic applies to f_3 and f_4. Therefore, $\frac{\partial W_4}{\partial \alpha} \leq 0$.

References

V. Acharya and S. Viswanathan. Leverage, moral hazard, and liquidity. *The Journal of Finance*, 66(1):99–138, 2011.

V. Acharya, D. Gale, and T. Yorulmazer. Rollover risk and market freezes, 2010.

J. Bengui. Systemic risk and inefficient debt maturity. university of maryland. Technical report, mimeo, 2010.

M. Brunnermeier and M. Oehmke. The maturity rat race. Technical report, National Bureau of Economic Research, 2010.

C. Calomiris and C. Kahn. The role of demandable debt in structuring optimal banking arrangements. *The American Economic Review*, pages 497–513, 1991.

D. Diamond and P. Dybvig. Bank runs, deposit insurance, and liquidity. *The Journal of Political Economy*, pages 401–419, 1983.

D. Diamond and R. Rajan. Liquidity risk, liquidity creation and financial fragility: A theory of banking, 1999.

D. Diamond and R. Rajan. Fear of fire sales and the credit freeze, 2009.

J. Geanakoplos. *The leverage cycle*. Citeseer, 2009.

G. Gorton. Slapped in the face by the invisible hand: Banking and the panic of 2007. *Yale School of Management mimeo*, 2009.

O. Jeanne and A. Korinek. Managing credit booms and busts: A pigouvian taxation approach. Technical report, National Bureau of Economic Research, 2010.

N. Kiyotaki and J. Moore. Credit cycles. *Journal of Political Economy*, 105(2):211–246.

E. Perotti and J. Suarez. A Pigovian Approach to Liquidity Regulation. *Tinbergen Institute Discussion Papers*, 2011.

H. Shin. Reflections on Northern Rock: the bank run that heralded the global financial crisis. *Journal of Economic Perspectives*, 23(1):101–119, 2009.

H. S. Shin. Macroprudential Policies Beyond Basel III. *International Centre for Financial Regulation*, 2011.