Underestimation of the solvency capital and risk measurements

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Abstract. This paper examines why a financial entity's solvency capital estimation might be underestimated if the total amount required is obtained directly from a risk measurement. Using Monte Carlo simulation we show that, in some instances, a common risk measure such as Value-at-Risk is not subadditive when certain dependence structures are considered. Higher risk evaluations are obtained for independence between random variables than those obtained in the case of comonotonicity. The paper stresses, therefore, the relationship between dependence structures and capital estimation.

Keywords: Solvency II, Solvency Capital Requirement, Value-at-Risk, Tail Value-at-Risk, Monte Carlo, Copulas

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1 Introduction

Recent years have seen the developement of regulatory frameworks designed to guarantee the financial stability of banking and insurance entities around the world. Europe developed Basel II and, more recently, Basel III for its banking market, and parallel to these accords drew up the Solvency II directive in 2010 for its insurance market and the Swiss Solvency Test in Switzerland. The regulatory frameworks seek to stablish what might be considered a reasonably amount of capital (referred to as *Solvency Capital* in Basel II and III and as *Solvency Capital Requirement* in Solvency II and the Swiss Solvency Test) to put aside to ensure financial stability in the case of adverse fluctuations on losses. This quantity must reflect the entity's specific risk profile and, under the aforementioned frameworks, it can be arrived at by applying either the *Standard Model* proposed by the regulator or an *Internal Model* proposed by the entity itself. In the later case, a number of requirements must first be satisfied before the model can be used for the purposes of capital estimation. In the European frameworks is regulated by the calibration of a risk measurement given a confidence level over a given time horizon.
In this paper we focus our attention on the European insurance market, and more specifically in non-life underwriting risk, in relation to the Solvency II and Swiss

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Solvency Test regulations. Under both frameworks, the total capital estimation is obtained by aggregating individual capitals requirements arising from a company's various sources of risk, based on the correlation between them as defined by the *Standard Model*. As a means of aggregating risks, we proposed a simulation of a multivariate random variable where each marginal distribution function represents the claims of a given line of business. We simulate a sample of this multivariate random variable taking into account the correlation between lines of business and, then, we aggregate the results of each simulated claim by line of business in order to obtain the distribution of the total claims. Finally, we estimate the capital requirements by applying a risk measure over the total claims distribution.
By representing an example of a multivariate random variable simulation we show that under certain assumptions there are risk measures that fail to satisfy the subadditive property. This is tipically the case where there is a very heavy tailed or skewed distribution on the margin and/or in which a special dependence structure is assumed for its joint distribution. Such circumstances can lead to an underestimation of the solvency capital if we incorrectly assimilate a risk measurement to the capital requirements, i.e., the appropiate distribution is not fitted to the marginals or the joint behavior of the marginal distributions is unknown.
The paper also emphasizes typical misunderstanding in the meaning of risk measures and the relationship between these risk measures and the underlying dependence structures of the variables, which represents the sources of risk.

2 Misunderstanding in the meaning of Value-at-Risk measure

Risk measures are tipically employed to determine the amount of capital that should be aside to cover unexpected losses. Artzner *et al.* (1999) proposed a number of desirable properties that a risk measure should satisfy in order to be considered a *coherent* risk measure. One such property, that of *subadditivity*, captures the idea of diversification across random variables since in the words of Artzner *et al.* (1999): *"...the merger [of two risks] does not create extra risk"*. Suppose we have n random variables, $X_i, i = 1, \dots, n$ and its sum, $S = \sum_{i=1}^{n} X_i$; then we say that risk measure ρ has the *subadditivity* property if and only if for all X_i

$$\rho(S) \leq \sum_{i=1}^{n} \rho(X_i) \quad . \tag{1}$$

Although several risk measures are available, here we focus on two that are on loss distribution, namely *Value-at-Risk* and *Tail Value-at-Risk*. These risk measures seek to describe how risky a portfolio is. Of the two, the most frequently adopted is *the Value-at-Risk* measure given that it is employed under Basel III and Solvency II as a tool for calibrating the solvency capital requirements. *Value-at-Risk* and its properties has been widely discussed (see Jorion, 2007). The *Value-at-Risk* measure is simply a quantile of a distribution function. However, it is not a *coherent* risk measure since it does not satisfy condition (1) for all X_i, although it does in the case in which X_i are normally distributed and, more generally, in the case in which elliptical distributions are considered for X_i as is shown in Fang *et al.* (1990) and Embrechts *et al.* (2002). Formally, given a confidence level $\alpha \in [0,1]$, *Value-at-Risk* is defined as the infimum value of the distribution of a random variable X such that the probability that a value x

exceeds a certain threshold x is no greater than $1-\alpha$. Usually this probability is taken to be 0.05 or less.

$$VaR^{\alpha}(X) = \ infimum\,\{x/P(X > x) = 1-\alpha\} = F_X^{\leftarrow}(\alpha)\ , \qquad (2)$$

where $F_X(x)$ denotes the distribution function of X and $F_X^{\leftarrow}(\alpha)$ denotes the inverse distribution function of $F_X(x)$.
It is common to choose $1-\alpha$ based on a very large period. For example, under the Solvency II directive, this choice is made on the basis of an occurrence of an event every two hundred years, i.e., a probability of 0.5%. For this reason, this risk measure is known as a frequency measure. It describes the distribution up to the α -th percentile, but it provides no information as to how the distribution behaves at higher percentiles.
Unlike the *Value-at-Risk* measure, the *Tail Value-at-Risk* describes the behavior of the tail of the distribution. This risk measure is defined, therefore, as the expected value of all percentiles higher than the *Value-at-Risk*. Formally,

$$TVaR^{\alpha}(X) = E[X|x \geq VaR^{\alpha}(X)]\ , \qquad (3)$$

where $E[\cdot\mid\cdot]$ denotes the conditional expectation operator.
Thus, while two different distributions might have the same *Value-at-Risk* for a given a confidence level, their *Tail Value-at-Risk* may differ due to a different heaviness of the tail of the distribution. The fact of having to consider not just the choice of $1-\alpha$ but also the values of the distribution higher than the α-th percentile means the *Tail Value-at-Risk* is known as a severity risk measure. Likewise, the *Tail Value-at-Risk* measure is a subadditive and *coherent* since it satisfies property (1), as is shown in Embrechts *et al* (2005).
Despite the mathematical principles underpinning the *Value-at-Risk* and *Tail Value-at-Risk* measures, a misunderstanding arises when seeking to apply them (specially, in the case of the former) to capital requirements. It is common to interpret *Value-at-Risk* as the value that will not be exceeded with a probability α. If, as is usual, the random variable is considerd as the loss of a portfolio, this definition is equivalent to the loss that will not be exceeded with a probability of α, or the maximum loss given α as pointed out by Jorion (2007). Yet, this interpretation is not stictly correct since maximum loss is not generally given by *Value-at-Risk* for a given confidence level as we shall see bellow.
Inequality (1) shows an upper bound for $\rho(S)$ which is $\sum_{i=1}^{n}\rho(X_i)$. A typical misunderstanding arises from this bound given that only in cases of diversifiable risks can such a bound ocurr. In instances of non-diversifiable risks this bound fails. As Embrechts *et al.* (2003) showed, if we consider *Value-at-Risk* as a risk measure and a sequence of comonotonic random variables $X_i, i = 1, \dots, n$ then $VaR(S) = VaR(\sum_{i=1}^{n} X_i) = \sum_{i=1}^{n} VaR(X_i)$, which is known as the comonotonic bound. But this is not the worst possible case since comonotonicity does not necessarily result in the worst loss a company might suffer. Dependence structures between random variables X_i might be found so that the *Value-at-Risk* of the sum of them exceeds the comonotonic bound. Translating this to the definition of *Value-at-Risk* we can conclude that it is possible to have greater losses than those arising from the comonotonic case given α. Then $\sum_{i=1}^{n} VaR(X_i)$ is not the maximum loss we could have given α.

Generally, *Value-at-Risk* fails to be a sub-additive risk measure in those cases where we have very heavy-tailed random variables, which is the case of those variables representing catastrophic or operational risks. It also fails in the case of skewed random variables and in some instances in which special dependence structures are imposed on the joint behavior of margins. Embrechts *et al.* (2005) showed that when Pareto random variables are considered, *Value-at-Risk* fails to be subadditive. Moreover, when the infinite mean case is considered, i.e., a *tail-parameter* of Pareto distribution is equal to one for all random variables considered, *Value-at-Risk* fails to be sub-additive at all confidence levels, otherwise, there's a confidence level up to which *Value-at-Risk* is sub-additive and beyond which it is not.

In short, the estimation of capital requirements using a risk measure should be approached with caution since the resulting valuation might underestimate the capital needs depending on the joint distribution of the random variables representing the implicit risks within a company as well as on its individual statistical distributions. Therefore, risk measurements derived from non *coherent* risk measures could lead the company to financial and solvency instability.

3 A non-*subadditivity* example of the Value-at-Risk risk measure

In this section we present an example in which we demonstrate that while the *Value-at-Risk* measure fails to be subadditive, the *Tail Value-at-Risk* measure satisfies the property of *subadditivity*. We use a historical non-life insurance market data set for Spain corresponding to the period 2000 to 2010. The data were obtained from public information published at the website of the Dirección General de Seguros (DGS)[2]. The data represents the yearly annual aggregate claims and three lines of business are considered (see Table 1 for their descriptive statistics[3]). A complete description of the risks included in each line of business can be found in the QIS-5 Technical Specifications, CEIOPS (2010).

We assume that each line of business behaves statistically as a generalized Pareto random variable. The parameters resulting from fitting the data to the Pareto distribution are shown in Table 2.

Table 1. Descriptive statistics * of the yearly annual aggregate ** claims by line of business

	Min.	1st Qu.	Median	Mean	Sd	3rd Qu.	Max.
Motor, third party liability	4.286	4.904	5.182	5.085	0.332	5.318	5.439
Fire and other property damage	2.337	3.006	3.740	3.539	0.723	4.067	4.544
Third party liability	0.552	0.684	0.819	0.842	0.195	1.011	1.084

Source: Own source from DGS / *thousand millions euros / ** deflated

[2]www.dgsfp.meh.es

[3]Minimum (Min.), Quartile (Qu.), Standard deviation (Sd), Maximum (Max.)

After fitting the data of each line of business to its corresponding distribution, we performed a multivariate Monte Carlo simulation and we compute the *Value-at-Risk* and the *Tail Value-at-Risk* for several confidence levels. For the joint behavior of risks we imposed several copulas, which in fact entails considering several different dependence structures. An introduction to multivariate models and dependence concepts and their properties can be found in Nelsen (2006) and Joe (1997).

Table 2. Generalized Pareto shape and scale * parameters

	Shape	**Scale**
Motor, third party liability	0.93	0.30
Fire and other property damage	0.95	0.23
Third party liability	0.75	0.19

*thousand millions euros

First, we established two extreme cases of dependence: comonotonicity and independence. We used the upper Fréchet copula, which reflects the case of comonotonicity between margins, and then we used the independence copula, which reflects the case of independence between margins. Finally, we employed two further copulas, the Clayton copula, a very right- skewed distribution, and the Frank copula, a symmetric but very heavy tailed distribution, with two dependence parameters (θ) each one.

Table 3 shows the results of *Value-at-Risk* and *Tail Value-at-Risk* for several confidence levels. Under the comonotonic assumption, which leads to the comonotonic bound, *Value-at-Risk* underestimates the risk (i.e., fails to be subadditive) compared to the independence, the Clayton and Frank copula cases at all condidence level up to some point between 0.90 and 0.99, and beyond which it *Value-at-Risk* satisty the *subadditivity* property. The values in bold indicate where the *subadditivity* property fails to be satisfied compared with the case of comonotonicity.

Table 3. VaR and TVaR * from a simulation ** of a tridimensional multivariate distribution for several dependence structures and Pareto margins

Value at Risk						
c.l.	**independence copula**	**Clayton copula**		**Frank copula**		**comonotonic copula**
		$\boldsymbol{\theta = 1}$	$\boldsymbol{\theta = 2}$	$\boldsymbol{\theta = 1}$	$\boldsymbol{\theta = 2}$	
0.8	**3.41**	**3.53**	**3.57**	**3.45**	**3.48**	**2.67**
0.9	**6.36**	**6.78**	**6.99**	**6.59**	**6.85**	**5.78**
0.99	47.69	49.6	50.7	50.02	50.17	53.33
0.999	398.05	396.45	421.82	371.26	401.18	435.69
Tail Value at Risk						
c.l.	**independence copula**	**Clayton copula**		**Frank copula**		**comonotonic copula**
		$\theta = 1$	$\theta = 2$	$\theta = 1$	$\theta = 2$	
0.8	23.14	25.09	26.76	24.78	31.5	32.66
0.9	41.7	45.34	48.57	44.84	58.35	61.49
0.99	289.28	318.72	347.22	315.22	446.51	488.76
0.999	1868.51	2151.37	2419.51	2163.13	3409.88	3777.41

Source: Own source / *thousand millions euros / c.l.: confidence level / **100.000 simulations

Since, *Tail Value-at-Risk* is a subadditive measure of risk, all values under independence case, as well as those for the Clayton and Frank copula assumptions are, as expected, lower than those in the comonotonic case at all confidence level. In the *Tail Value-at-Risk* case, the more dependence assumption implies higher values of risk measurements at all confidence levels.

4 Implications for Risk Management

New regulatory frameworks in Europe establish the capital requirements that financial companies need to maintain in order to ensure acceptable levels of solvency. These requirements are calibrated by applying a risk measure based on the distribution of a random variable that represents the total losses a company would suffer.

Typically, in the field of risk management, managers are aware of the individual risks for which they have responsibility, but are unaware of the relationship between the sources of risk that they manage and those arising from other departments or other areas within the same company. This means that the joint behavior of the random variables representing the different sources of risk remains unknown. As such, the process of aggregating risk measurements to obtain an global capital requirement is complex and difficult to address.

As consequence of this, it's usual for risk managers to obtain the risks measurements of each variable that represent a single source of risk considered and after that add them up to obtain a single amount and assimilate it to the regulatory capital.

In so doing, if companies choose a non *coherent* risk measure (as defined by Artzner *et al.*, 1999) such as *Value-at-Risk*, (thereby adhering to Solvency II or Basel III criteria), they may overlook a number of crucial points. First, they may overlook the tail behavior of each random variable considerd individually, which means, they fail to incorporate the possibility of severe losses that could greatly undermine the company's solvency. Second, by summing up risk measurements, such as those resulting from the *Value-at-Risk*, a company implicitly assumes, perhaps inadvertenly, that its random variables are comonotonic, which is in conflict with the idea of their ignoring the joint behavior of variables. Even in those instance in which risk managers believe that the joint behavior is comonotonic, it is possible to suffer worse losses than those that are consequence of summing the *Value at Risk* measurements derived from comonotonic random variables, as shown in the example in the previous section (see Table 3). In this example, even when the random variables are independent, the *Value at Risk* of the sum of random variables is greater than it is in the case of comonotonicity at certain confidence levels.

This finding highlights not only the importance of choosing of a *coherent* risk measure, but also the need, first, to select the right fit and distribution function so as to reflect correctly the joint behavior of sources of risk and their dependence structures; and, second, to select the right fit and distribution functions so as to reflect the margin behavior of each random variable representing a single sources of risk.

In example presented in Table 3, we used Pareto distributions for margins. When considering this distribution, it is possible to find cases in which we might have infinite expected values or variance, which are extreme cases that cause non-*subadditivity* for the *Value-at-Risk* measure, but they are not the only cases in which this might occur.

There are many risks, generally catastrophic risks, for which the Pareto distribution fits well, and for which the non-*subadditivity* property could fail when using non-*coherent* risk measures. Examples of the Pareto distribution being fitted to operational and catastrophic risks can be found in Guillén *et al.* (2011) and Sarabia *et al.* (2009), respectively.

A number of lessons can be drawn by those involved in risk management from the preceding analysis. First, knowledge of the joint statistic behavior of risks is essential when considering a company's overall level of risk. This does not simply mean the need to estimate the correlation coefficients but also to consider underlying dependence structures. Second, although using a *coherent* risk measure such as the *Tail Value-at-Risk* leads to higher capital estimations than in the case when using the *Value-at-Risk* measure, it serves to ensure that capital requirements are not underestimated due to the property of subadditivity should the hypothesis regarding the joint behavior of sources of risk prove to be incorrect. However, errors in the model estimation could well lead to the real capital needs being underestimated, even though a *coherent* risk measure such as *Tail Value-at-Risk* has been used. Third, comonotonicity does not represent the worst possible scenario when estimating capital requirements using loss and risk measurements. This might appear counterintuitive, but managers of risks need to bear in mind that risks may well be superadditive as opposed to subadditive, and as such, no diversification effect occurs when several random variables are merged. Fourth, while it may appear obvious, differences in capital estimation derived from the application of different risk measures do not mean the portfolio has varying degrees of risk; rather what we see is simply different ways of measuring what might happen beyond a given threshold, i.e. the losses that would exceed the threshold in the case of the *Value-at-Risk* measure, and the severity of these losses beyond the threshold in the case of *Tail Value-at-Risk* measure.

Appendix

A multivariate simulation of the random variable was conducted using the R-Project software, version 2.13.1 and the *copula* package implemented therein. Bellow we describe the simulation performance.

A random variable X has a Generalized Pareto distribution (GPD) if its distribution function is

$$G_{X,\xi,\beta}(x) = \begin{cases} 1 - \left(1 + \frac{\xi \cdot x}{\beta}\right)^{\frac{-1}{\xi}}; if\ \xi \neq 0 \\ 1 - exp\left(\frac{-x}{\beta}\right); if\ \xi = 0 \end{cases}$$

where ξ the shape parameter and $\beta > 0$ the scale parameter. The expexted value $E[X]$ is

$$E[X] = \frac{\beta}{1 - \xi}$$

and the standard deviation is the scale parameter β.

The shape parameter can be estimated using maximum likelihood estimation or the method of moments. Using the moments estimation procedure, the shape parameter

results in $E[X]=\frac{\beta}{1-\xi}\Rightarrow\xi=\frac{-\beta}{E[X]}+1=-CoVa[X]+1$; being $CoVa[X]$ the coefficient of variation of X.
Having estimated the sample expected value and the standard deviation for each line of business considered we obtain the coefficient of variation and construct a 2x3 matrix which containing the coefficients of variation and standard deviation in each of the rows. The columns represent each one of the three line of business.

```
data<-read.table('CoV.csv', header=TRUE, sep=";")
data<-as.matrix(data)
```

The estimation of the shape and scale parameters becomes

```
shape1<- -data[1,1]+1 # the shape parameter for the first margin.
shape2<- -data[1,2]+1 # the shape parameter for the second margin.
shape3<- -data[1,3]+1 # the shape parameter for the third margin.
scale1<-data[2,1] # the scale parameter for the first margin.
scale2<-data[2,2] # the scale parameter for the second margin.
scale3<-data[2,3] # the scale parameter for the third margin.
```

The comonotonic bound are derived from the sum of the $\alpha-th$ quantiles of each marginal distribution. We compute the quantiles for each margin after simulating one thousand observations of the GPD for each margin given its corresponding parameters.

```
n=100000 # number of simulations.
l<-c(0.80,0.90,0.99,0.999) # vector of given confidence level.
```

The simulation for the first, second and third margin are

```
pareto1<-rGPD(n,shape1, beta=scale1) # for the first margin.
pareto2<-rGPD(n,shape2, beta=scale2) # for the second margin.
pareto3<-rGPD(n,shape3, beta=scale3) # for the third margin.

VaR1<-quantile(pareto1,l) # Value-at-Risk for the first margin.
VaR2<-quantile(pareto2,l) # Value-at-Risk for the second margin.
VaR3<-quantile(pareto3,l) # Value-at-Risk for the third margin.
VaR_com<-(VaR1+VaR2+VaR3)# comonotonic bound vector.
```

When considering independence between the random variables, we obtain *the Value-at-Risk* and the *Tail Value-at-Risk* by simulating the three-dimensional Gaussian copula with Pareto margins given shape and scale parameters and dependence parameters (linear correlations) for the copula equal to zero.

```
corr<-c(0,0,0) # vector of linear correlations for the Gaussian copula.
copulagaussiana<-mvdc(normalCopula(corr,dim=3,
dispstr="un"),c(rep("GPD",3)),list(param1,param2,
param3))
```

```
gaussian.sample<-rmvdc(copulagaussiana,1000000)
```
simulation of the copula.

```
gauss.sample.aggrega<-rowSums(gaussian.sample)# distribution of total losses.
VaR_ind<-quantile(gauss.sample.aggrega,l)
TVaR_ind<-rep(0,4)
for(i in 1:4){
ES<-mean(gauss.sample.aggrega[gauss.sample.
aggrega>VaR_ind[i]])
TVaR_ind[i]<-ES}
```

`VaR_ind` # *Value-at-Risk* vector under independence assumption.
`TVaR_ind` # *Tail Value-at-Risk* vector under independence assumption.

The *Tail Value-at-Risk* under the comonotonicity assumption between random variables is obtained by simply setting a new correlation vector for the Gaussian copula, i.e, `corr<-c (0.9999,0.9999,0.9999)` as the comonotonic copula can be obtained when the dependence parameters of the Gaussian copula tend to one.
The *Value-at-Risk* and *Tail Value-at-Risk* for the Clayton and Frank copulas are obtained in a similar way but changing the kind of copula used and the corresponding dependence parameter. We show the Clayton copula case for a dependence parameter $\theta = 1$.

`theta=1` # dependence parameter for the Clayton copula.

```
Copulaclayton<-mvdc(claytonCopula(theta,dim=3),
c(rep("GPD",3)),list(param1,param2,param3))
clayton.sample<-rmvdc(copulaclayton,1000000)
clayton.sample.aggrega<-rowSums(clayton.sample)
VaR_cl<-quantile(clayton.sample.aggrega,l)
TVaR_cl<-rep(0,4)
for(i in 1:4){
ES<-mean(clayton.sample.aggrega[clayton.sample.
aggrega>VaR_cl[i]])
TVaR_cl[i]<-ES}
```

`VaR_cl` # *Value-at-Risk vector* under Clayton copula assumption.
`TVaR_cl` #*Tail Value-at-Risk vector* under Clayton copula assumption.

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