Optimizing Bounds on Security Prices in Incomplete Markets. Does Stochastic Volatility Specification Matter?

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Abstract

We extend and generalize some results on bounding security prices under two stochastic volatility models that provide closed-form expressions for option prices. In detail, we compute analytical expressions for benchmark and standard good-deal bounds. For both models, our findings show that our benchmark results generate much tighter bounds. A deep analysis of the properties of option prices and bounds involving a sensitivity analysis and analytical derivation of Greeks for both option prices and bounds is also presented. These results provide strong practical applications taking into account the relevance of pricing and hedging strategies for traders, financial institutions, and risk managers.

Key words: Incomplete Markets, Stochastic Volatility Model, CIR Process, Ornstein-Uhlenbeck Process, Good-deal Bounds.

JEL classification: C61, G12, G13

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1 Introduction

Under the assumption of complete markets, every contingent claim can be replicated by a portfolio formed of the underlying basic assets of the market. In this case, the equivalent martingale measure and the market price of risk are unique and, then, the price of any security is uniquely determined by this martingale measure. Nevertheless, in the real world this situation barely happens and there can be (infinitely) many equivalent martingale measures.

When there are sources of risk that are not directly traded (such as stochastic volatility, jumps or weather) the assumption of complete market fails. Staum (2008) surveys many approaches to pricing and hedging derivative securities under incomplete markets. As there will not exist a unique martingale measure, there will exist infinitely many arbitrage-free price processes for a certain financial security. Then, it can be interesting to derive no-arbitrage bounds on asset prices and obtain a no-arbitrage interval where the price of the asset should lie.

Several papers have computed bounds on option prices. For instance, Basso and Pianca (2001) consider a state-preference approach and provide lower and upper bounds for European option prices by solving a non-linear optimization problem. No-arbitrage bounds can also be computed using information about prices of other options on the same underlying asset, see Bertsimas and Popescu (2002) or d'Aspremont and El Ghaoui (2006), among others. Working in discrete-time, Reynaerts *et al.* (2006) focus on the Cox, Ross and Rubinstein (1979) model with daily time step and derive bounds on prices for arithmetic Asian options with discrete sampling. These bounds can also be obtained assuming an incomplete knowledge of the underlying price distribution. For example, Zuluaga *et al.* (2009) derive closed-form semi-parametric bounds for the payoff of a European call option, given up to third-order statistical moments for the underlying asset distribution at maturity.

Considering incomplete markets, Bernardo and Ledoit (2000) and Cochrane and Saá-Requejo (2000) try to find no-arbitrage bounds on prices as tight as possible using the stochastic discount factor (SDF) as starting point.[1] Then, both papers restrict the pricing kernel to derive these bounds. Extracting a paragraph from Franke *et al.* (2007, p. 215), "Cochrane and Saá-Requejo (2000) show that the option price can be bounded by limiting the variance of the pricing kernel. In similar vein, Bernardo and Ledoit (2000) show that the option price can be bounded by limiting the convexity of the pricing kernel." The intuition behind these two papers is that investors will choose a trading asset price according to some optimality criterion. As mentioned in Pinar *et al.* (2010, p. 771), "in Cochrane and Saá-Requejo (2000), the absence of arbitrage is replaced by the concept of a good deal, defined as an investment with a high Sharpe ratio. While they do not use the term "good-deal", Bernardo and Ledoit (2000) replace the high Sharpe ratio by the gain-loss ratio."

[1]The SDF was introduced in Hansen and Jaganathan (1991) who demonstrated that a bound on the maximum available Sharpe ratio is equivalent to a bound on the volatility of the admissible SDF.

In more detail, Bernardo and Ledoit (2000) analyze different investment opportunities using the *gain-loss* ratio as a performance measure using a benchmark or reference asset pricing model. They demonstrate that a high *gain-loss* ratio is related to SDF's that are "specially" far from the benchmark SDF. In this way, an appropriate benchmark SDF can tighten the no-arbitrage bounds and the corresponding no-arbitrage interval. Inspired by this paper, Pinar *et al.* (2010) apply linear programming to price and hedge contingent claims in a multi-period setting and propose an optimality criterion, the "λ gain-loss ratio", that treats asymmetrically gains and losses. The pricing bounds obtained are tighter than the no-arbitrage ones and, as expected, converge to the no-arbitrage ones as the gain-loss preference parameter tends to infinity. These authors also show that a unique claim price may be found for a limiting case of the risk aversion parameter.

Alternatively, Cochrane and Saá-Requejo (2000) also measure the attractiveness of an investment but using the Sharpe ratio and suggested to rule out usual arbitrage opportunities with too high Sharpe ratios. Thus, they obtain tighter price bounds that are named *benchmark good-deal* (BGD) bounds. Several authors have dealt with this approach and proposed different methodologies to compute this type of bounds. For instance, Cerny and Hodges (2002) present the theory of good-deal pricing in financial markets and shows that "any such technique can be seen as a generalization of no-arbitrage pricing and that, with a little bit of care, it will contain the no-arbitrage and the representative agent equilibrium as the two opposite ends of a spectrum of possible no-good-deal equilibrium restrictions." In a related paper, Cerny (2003) replaces the Sharpe ratio (connected to quadratic utility) with a generalized Sharpe ratio based on an arbitrary increasing smooth utility function and shows that "for Itô processes the Cochrane and Saá-Requejo (2000) bounds are invariant to the choice of the utility function, and that in the limit they tend to a unique price determined by the minimal martingale measure."

Björk and Slinko (2006) extend the setting in Cochrane and Saá-Requejo (2000) by studying arbitrage-free good-deal pricing bounds for derivative assets and presented results for the Merton-jump diffusion model. Additionally, they derive extended Hansen-Jagannathan bounds for the Sharpe Ratio process in the point process setting. Albanese and Tompaidis (2008) consider the good-deal pricing literature and perform a dynamic risk-reward analysis for a type of time-based hedging strategies in the presence of transaction costs. Pinar (2008) uses an arbitrage-adjusted Sharpe-ratio criterion and convex optimization and provides bounds on contingent claim prices that are tighter than the no-arbitrage ones. Finally, Bondarenko and Longarela (2009) present asset price bounds as the result of an optimization problem over a set of admissible SDF's. They consider the option pricing model presented in Heston (1993) and assume certain limits for the volatility risk premium. They derive closed-form solutions for the BGD bounds and for a particular case, standard good-deal (GD) bounds, showing that the former are much tighter than the latter.

Continuing with this research area, our paper focuses on computing and analyzing BGD and GD bounds for different asset prices under two stochastic volatility option pricing models, that introduced in Heston (1993) and an extension of that posited in Schöbel and Zhu (1999). In this way, we can get an insight into the effects of different specifications for the stock volatility process on the aforementioned bounds.

Heston (1993) generalizes the classical model for stock prices presented in Black and Scholes (1973) allowing the stock volatility to follow a "square-root" (CIR-type) stochastic process as presented in Cox, Ingersoll and Ross (1985). Additionally, Schöbel and Zhu (1999) extended the stochastic volatility model of Stein and Stein (1991) where the stock volatility follows an Ornstein-Uhlenbeck process. They allow correlation to exist between the underlying stock returns and the instantaneous volatility and found a closed-form expression for option prices.

As these two models deal with stochastic volatility, markets are incomplete. However, both provide unique closed-form expressions for the prices of certain securities assuming a certain functional form of the market price(s) of risk of the corresponding factor(s). In fact, each functional form is associated to a martingale measure and, thus, to a price for the security.

This paper contributes to the existing literature in three ways: firstly, we analyze deeply Bondarenko and Longarela (2009) and fix different errors in their numerical analysis. One of our main results is that, now, the difference between GD and BGD bounds is stronger than that previously reported by these authors. Secondly, we extend the Schöbel and Zhu (1999) model allowing the market price of risk of the volatility to be different from zero. In this extended model, with no new mathematical ideas, we also obtain analytical expressions for option prices and their bounds. Numerical illustrations are shown for all the bounds obtained.

Our final contribution is that, for both models, extensive sensitivity analysis are carried out studying how changes in the models' parameters affect prices and bounds. Additionally, we also implement a hedging analysis by computing several Greeks for prices and bounds. Computation of these Greeks is relevant because, as shown in Carr (2001), these amounts can be interpreted as the values of certain quantoed contingent claims. Besides, as this author states, "this interpretation allows one to transfer intuitions regarding values to these Greeks and to apply any valuation methodology to determine them".

The structure of the paper is as follows. Section 2 describes the theoretical framework that is needed to find the bounds on option prices. Stochastic volatility models are presented in Section 3. Section 4 derives analytical expressions for option prices and their bounds under these models. A deep analysis of the properties of option prices and bounds involving a sensitivity analysis and derivation of Greeks for both option prices and bounds is included in Section 5. Finally, Section 6 summarizes the main findings and conclusions.

2 Theoretical Framework

We present now our theoretical framework. Consider a probability space (Ω, F, P) with the corresponding filtration $\{F_t\}_{t\geq 0}$. Assume that we have a bond that pays the risk-free rate r_t, a risky asset S_t (stock), and one (non-tradable) state variable V_t. Let (W_t^s, W_t^v) be two standard and independent Brownian motions and let $h_t = (h_t^s, h_t^v)$ be an adapted two dimensional process, which satisfies the Novikov condition. Departing from the probability measure P, we define the measure Q via the Radon-Nikodim derivative, that is,

$$\frac{dQ}{dP} = \xi_T$$

where, for all t,

$$\xi_t = \exp\left[-\int_0^t h_u dW_u - \frac{1}{2}\int_0^t \|h_u\|^2 du\right]$$

and ξ is a P-martingale with expected value equal to one. The SDF process is defined as

$$\Lambda_t = B_t \xi_t$$

where $B_t = \exp\left(-\int_0^t r_u du\right)$. Applying Itô's lemma, we get that

$$\frac{d\Lambda_t}{\Lambda_t} = -r_t dt - h_t' dW_t$$

We can define the benchmark model in terms of the vector process $h_t^* = (h_t^{*s}, h_t^{*v})$ with the corresponding martingale measure, Q^*, and the SDF process, Λ^*. Now, we will mention some important statements and restrictions that are necessary for our purposes.

By definition, Λ prices the risk-free bond and also the risky asset if it satisfies

$$E_t^P\left[\frac{d(\Lambda_t S_t)}{\Lambda_t S_t}\right] = -\delta_t dt$$

where δ_t is the continuous dividend yield. The market price of risk of the stock is given by

$$\lambda_t^s = \frac{\mu_t - r_t}{\sigma_t}$$

where μ_t and σ_t denote the expected instantaneous rate of return and the instantaneous volatility of the stock, respectively. We now establish the set of admissible SDF's as stated in this Lemma.[2]

Lemma 1 *The SDF process Λ prices the stock if and only if it has an associated process satisfying $h^s = \lambda^s$.* ■

[2] This result corresponds to Lemma 1 in Bondarenko and Longarela (2009).

For a fixed benchmark SDF process, Λ^*, the volatility constraint in continuous time is given by

$$E_t^{Q^*}\left[\frac{d\left(\Lambda_t/\Lambda_t^*\right)}{\Lambda_t/\Lambda_t^*}\right]^2 \leq A_t^2 dt \tag{1}$$

where A_t is some adapted process. Using that the change of measure does not affect the volatility process, the left-hand side of (1) can be simplified as

$$E_t^{Q^*}\left[\frac{d\left(\Lambda_t/\Lambda_t^*\right)}{\Lambda_t/\Lambda_t^*}\right]^2 = E_t^P\left[\frac{d\left(\Lambda_t/\Lambda_t^*\right)}{\Lambda_t/\Lambda_t^*}\right]^2 = E_t^P\left[\frac{d\Lambda_t}{\Lambda_t} - \frac{d\Lambda_t^*}{\Lambda_t^*}\right]^2 = \|h_t - h_t^*\|^2 dt \tag{2}$$

Lemma 1 implies that, for an admissible SDF process, it holds that

$$\|h_t - h_t^*\|^2 = (\lambda_t^s - h_t^{s*})^2 + (h_t^v - h_t^{v*})^2 \tag{3}$$

Using (1)-(3), the volatility constraint becomes

$$(h_t^v - h_t^{v*})^2 \leq A_t^2 - (\lambda_t^s - h_t^{s*})^2 \tag{4}$$

In most applications, the benchmark is usually an admissible SDF. In this case, the volatility constraint discards those SDF processes for which $|h_t^v - h_t^{v*}| \geq A_t$. $A_t \equiv A$ is sometimes called a ceiling process.

We can now define the BGD bounds for asset prices. Consider an asset that pays a stream of dividends given by an adapted process X and a terminal payoff X_T. Its price C_t under a candidate SDF process Λ is given by

$$C_t(h) = E_t^P\left(\int_t^T \frac{\Lambda_u}{\Lambda_t} X_u du\right) + E_t^P\left(\frac{\Lambda_T}{\Lambda_t} X_T\right)$$

Let H^v be the set of processes h^v which satisfy (4) on the interval [0,T]. The lower bound is defined by the solution of a certain minimization problem, namely,

$$\underline{C_0}(h^*) = \min_{h^v} C_0(h^v, h^*, A)$$

The equivalent maximization problem provides the corresponding upper bound. Formally, we can state the following Proposition.[3]

Proposition 1 *The BGD bounds are given by*

$$\underline{C_0}(h^*) = C_0(\underline{h^v}), \quad \overline{C_0}(h^*) = C_0(\overline{h^v})$$

where

$$\underline{h_t^v} = h_t^{v*} + \sqrt{A_t^2 - (\lambda_t^s - h_t^{s*})^2}, \quad \overline{h_t^v} = h_t^{v*} - \sqrt{A_t^2 - (\lambda_t^s - h_t^{s*})^2}$$

■

Remark 1 *The GD bounds are derived as a particular case of the BGD bounds with* $h_t^* = (0, 0)$. ■

[3] This result corresponds to Proposition 5 in Bondarenko and Longarela (2009).

3 Stochastic Volatility Models

3.1 The Heston (1993) Model

Heston (1993) proposes to model the stock price evolution in time in a more general form than the standard classical model proposed in Black and Scholes (1973). In more detail, this author assumes a Geometric Brownian motion for the stock price evolution with the addition of a second state variable, namely, the variance of the stock return. Then, under the true probability measure P, the processes for the stock price S_t and for the variance of the stock return V_t are given by

$$\frac{dS_t}{S_t} = (r + sV_t)dt + \sqrt{V_t}dW_t^S \tag{5}$$

$$dV_t = (\alpha - \beta V_t)dt + \sigma\sqrt{V_t}\left(\rho dW_t^S + \sqrt{1-\rho^2}dW_t^V\right) \tag{6}$$

where r, s, α, β, σ, and ρ are constants and W_t^S and W_t^V are two uncorrelated standard Brownian motions. Then, the stock variance V_t follows the square-root mean-reverting process posited in Cox *et al.* (1985). In more detail, V_t converges to a long-run mean α/β with a certain speed of adjustment β. Additionally, the diffusion of the process is proportional to the variance level. The restriction $\sigma^2 \leq 2\alpha$ guarantees the positiveness of V_t. By construction, both processes are correlated with $\mathrm{corr}(dS_t/S_t, dV_t) = \rho$.[4]

Standard arbitrage arguments show that the price at time t of any derivative asset on the stock, $U(S,V,t)$, must satisfy the following partial differential equation (PDE)

$$\frac{1}{2}VS^2U_{SS} + \rho\sigma VSU_{SV} + \frac{1}{2}\sigma^2VU_{VV} + rSU_S + \left[\alpha - \beta V_t - \lambda(S,V,t)\right]U_V - rU + U_t = 0$$

where subscripts indicate the corresponding partial derivative and with $\lambda(S,V,t)$ denoting the market price of risk related to the stock volatility.[5] Moreover, similarly to Cox *et al.* (1985), we will assume that the risk premium of the variance is proportional to the variance level, that is,

$$\lambda(S,V,t) = \lambda V_t \tag{7}$$

This implies that, under the risk-neutral probability measure Q, the variance follows a mean-reverting process with long-run mean $\alpha/(\beta+\lambda)$ and speed of adjustment $(\beta+\lambda)$.

Under these assumptions, Heston (1993) derived a closed-form expression for the price of a European call stock option via Fourier inversion of the conditional characteristic functions. The Heston theoretical price at time $t = 0$ of this option will be denoted as

$$C_0^H(\lambda) = C^H(K, T, S_0, V_0, t = 0, \theta, \lambda)$$

[4] Broadie and Kaya (2006) suggest a method based on Fourier inversion techniques and conditioning arguments for the exact simulation of equations (5)-(6). See also Andersen (2008) that proposes new algorithms for time-discretization and Monte Carlo simulation of this model.

[5] Lamoureux and Lastrapes (1993) present empirical evidence on the significativeness of this term for equity options.

where $\theta = (\alpha, \beta, \sigma, \rho)$ is the set of parameters included in the process for the variance of the stock return, V_t. As shown in Heston (1993), the expression for this price is stated as follows.

Proposition 2 *Under the Heston (1993) model, the closed-form expression for the price at time t of a European call stock option is given as*

$$C^H(K, T, S_t, V_t, t, \theta, \lambda) = S_t P_1 - e^{-r(T-t)} K P_2$$

where, for $j = 1, 2$, we get

$$\begin{aligned}
P_j(S_t, V_t, \tau, K) &= \frac{1}{2} + \frac{1}{\pi} \int_0^\infty Re\left(\frac{e^{-i\phi \ln(K)} F_j(S_t, V_t, \tau, \phi)}{i\phi}\right) d\phi \\
F_j(S_t, V_t, \tau, K) &= \exp\left\{C_j(\tau, \phi) + D_j(\tau, \phi)V_t + i\phi \ln(S_t)\right\} \\
C_j(\tau, \phi) &= r\tau\phi i + \frac{\alpha}{\sigma^2}\left((\beta_j - \rho\sigma\phi i + h_j)\tau - 2\ln\left(\frac{1 - g_j e^{h_j \tau}}{1 - g_j}\right)\right) \\
D_j(\tau, \phi) &= \frac{\beta_j - \rho\sigma\phi i + h_j}{\sigma^2} \frac{1 - e^{h_j \tau}}{1 - g_j e^{h_j \tau}} \\
g_j &= \frac{\beta_j - \rho\sigma\phi i + h_j}{\beta_j - \rho\sigma\phi i - h_j} \\
h_j &= \sqrt{(\rho\sigma\phi i)^2 - \sigma^2(2u_j \phi i - \phi^2)}
\end{aligned}$$

where $\tau = T - t$, $u_1 = 0.5$, $u_2 = -0.5$, $\beta_1 = \beta + \lambda - \rho\sigma$, $\beta_2 = \beta + \lambda$. ■

3.2 Extended Schöbel and Zhu (1999) Model

Now we will study the stochastic volatility model presented in Schöbel and Zhu (1999) in which the volatility follows an Ornstein-Uhlenbeck (O-U) process. Under the risk-neutral measure Q, the processes for the logarithm of the stock price and the volatility for the stock return are given as

$$\begin{aligned}
dx_t &= \left(r - \frac{1}{2}v_t^2\right) dt + v_t d\widetilde{W}_t^s && (8) \\
dv_t &= \kappa(\theta - v_t)dt + \sigma\left(\rho d\widetilde{W}_t^s + \sqrt{1 - \rho^2} d\widetilde{W}_t^v\right) && (9)
\end{aligned}$$

where $\widetilde{W}_t^s$ and $\widetilde{W}_t^V$ denote independent standard Brownian motions under the measure Q. Hence, both processes are correlated with $corr(dx_t, dv_t) = \rho$. The stock volatility tends to a long-term value θ with speed κ. As the volatility is a non-traded asset, the measure Q is not unique and depends on the market price of volatility, λ_t, which is implicity determined by the market participants. Schöbel and Zhu (1999) assumed $\lambda_t = 0$. Considering now that $\lambda_t \neq 0$ (further on we will assume that (7) holds), we can rewrite the processes (8)-(9) in terms of the stock price and variance to get

$$\begin{aligned}
\frac{dS_t}{S_t} &= r dt + \sqrt{V_t} d\widetilde{W}_t^s && (10) \\
dV_t &= \left(2\kappa\sqrt{V_t}(\theta - \sqrt{V_t}) + \sigma^2 - \lambda_t\right)dt + 2\sigma\sqrt{V_t}\left(\rho d\widetilde{W}_t^s + \sqrt{1 - \rho^2} d\widetilde{W}_t^V\right) && (11)
\end{aligned}$$

Thus, we have generalized the model of Schöbel and Zhu (1999). In this general model, we can obtain analytically the option price as stated in the following Proposition.

Proposition 3 *Under the extended Schöbel and Zhu (1999) model, the closed-form expression for the price at time t of a European call stock option is given as*

$$C^{OU}(K,T,S_0,V_0,t,\theta,\lambda) = S_t P_1 - e^{-r(T-t)} K P_2$$

where the probabilities $P_j,\ j=1,2$ are given by

$$P_j = \frac{1}{2} + \frac{1}{\pi}\int_0^\infty Re\left(\frac{e^{-i\phi\ln(K)} f_j(\phi)}{i\phi}\right) d\phi, \quad j=1,2$$

where

$$\begin{aligned}
f_1(\phi) &= E^Q[\exp(-r(T-t) - x(t) + (1+i\phi)x(T))] \\
&= \exp\left\{ i\phi(r(T-t)+x(t)) - \frac{1}{2}(1+i\phi)\rho\left[\sigma^{-1}v^2(t) + \sigma(T-t)\right]\right\} \\
&\times \exp\left\{\frac{1}{2}D(t,T,s_1,s_3)v^2(t) + B(t,T,s_1,s_2,s_3)v(t) + C(t,T,s_1,s_2,s_3)\right\}
\end{aligned}$$

with

$$\begin{aligned}
s_1 &= -\frac{1}{2}(1+i\phi)^2(1-\rho^2) + \frac{1}{2}(1+i\phi)(1-2(\kappa+\lambda')\rho\sigma^{-1}) \\
s_2 &= (1+i\phi)\kappa\theta\rho\sigma^{-1} \\
s_3 &= \frac{1}{2}(1+i\phi)\rho\sigma^{-1} \\
\lambda' &= \frac{\lambda}{2}
\end{aligned}$$

and

$$\begin{aligned}
f_2(\phi) &= E^Q[\exp\{i\phi x(T)\}] \\
&= \exp\left\{ i\phi(r(T-t)+x(t)) - \frac{1}{2}i\phi\rho\left[\sigma^{-1}v^2(t) + \sigma(T-t)\right]\right\} \\
&\times \exp\left\{\frac{1}{2}D(t,T,\widehat{s}_1,\widehat{s}_3)v^2(t) + B(t,T,\widehat{s}_1,\widehat{s}_2,\widehat{s}_3)v(t) + C(t,T,\widehat{s}_1,\widehat{s}_2,\widehat{s}_3)\right\}
\end{aligned}$$

with

$$\begin{aligned}
\widehat{s}_1 &= \frac{1}{2}\phi^2(1-\rho^2) + \frac{1}{2}i\phi(1-2(\kappa+\lambda')\rho\sigma^{-1}) \\
\widehat{s}_2 &= i\phi\kappa\theta\rho\sigma^{-1} \\
\widehat{s}_3 &= \frac{1}{2}i\phi\rho\sigma^{-1}
\end{aligned}$$

Finally,[6]

$$
\begin{aligned}
D(t,T) &= \frac{1}{\sigma^2}\left(\kappa+\lambda'-\gamma_1\frac{\sinh\{\gamma_1(T-t)\}+\gamma_2\cosh\{\gamma_1(T-t)\}}{\cosh\{\gamma_1(T-t)\}+\gamma_2\sinh\{\gamma_1(T-t)\}}\right)\\
B(t,T) &= \frac{1}{\sigma^2\gamma_1}\left(\frac{\kappa\theta\gamma_1-\gamma_2\gamma_3+\gamma_3\left(\sinh\{\gamma_1(T-t)\}+\gamma_2\cosh\{\gamma_1(T-t)\}\right)}{\cosh\{\gamma_1(T-t)\}+\gamma_2\sinh\{\gamma_1(T-t)\}}-\kappa\theta\gamma_1\right)\\
C(t,T) &= -\frac{1}{2}\ln\left(\cosh\{\gamma_1(T-t)\}+\gamma_2\sinh\{\gamma_1(T-t)\}\right)+\frac{1}{2}(\kappa+\lambda')(T-t)\\
&+ \frac{\kappa^2\theta^2\gamma_1^2-\gamma_3^2}{2\sigma^2\gamma_1^3}\left(\frac{\sinh\{\gamma_1(T-t)\}}{\cosh\{\gamma_1(T-t)\}+\gamma_2\sinh\{\gamma_1(T-t)\}}-\gamma_1(T-t)\right)\\
&+ \frac{(\kappa\theta\gamma_1-\gamma_2\gamma_3)\gamma_3}{\sigma^2\gamma_1^3}\left(\frac{\cosh\{\gamma_1(T-t)\}-1}{\cosh\{\gamma_1(T-t)\}+\gamma_2\sinh\{\gamma_1(T-t)\}}\right)
\end{aligned}
$$

with[7]

$$
\gamma_1=\sqrt{2\sigma^2 s_1+(\kappa+\lambda')^2},\quad \gamma_2=\frac{1}{\gamma_1}\left[\kappa+\lambda'-2\sigma^2 s_3\right],\quad \gamma_3=(\kappa+\lambda')\kappa\theta-s_2\sigma^2
$$

■

In this Proposition, for getting the probabilities P_1 and P_2, we derive their corresponding characteristic functions and follow the same schedule as Schöbel and Zhu (1999) until reaching the previous expressions in which the market price of risk of the volatility appears now explicitly.

Firstly, for deriving the characteristic functions, we need the volatility process. Applying the Itô's lemma to the variance process (11), we get

$$
dv_t=\kappa\left[\theta-\left(1+\frac{\lambda}{2\kappa}\right)v_t\right]dt+\sigma\left(\rho d\widetilde{W}_t^s+\sqrt{1-\rho^2}d\widetilde{W}_t^v\right)
$$

Secondly, the obtention of the characteristic functions is similar to that shown in Schöbel and Zhu (1999) and involves solving the following system of ordinary differential equations

$$
\begin{aligned}
D_t &= -\sigma^2D^2+2(\kappa+\lambda')D+2s_1\\
B_t &= \left[\kappa+\lambda'-\sigma^2D\right]B-\kappa\theta D+s_2\\
C_t &= -\frac{1}{2}\sigma^2B^2-\kappa\theta B-\frac{1}{2}\sigma^2 D
\end{aligned}
$$

subject to the terminal conditions $D(T,T)=2s_3,\ B(T,T)=C(T,T)=0$.

Comparing the results obtained in Schöbel and Zhu (1999) with our Proposition, we can see that s_1 and $\widehat{s_1}$ have changed, thus the functions $D(t,T)$, $B(t,T)$, and $C(t,T)$ have changed. Additionally, note that γ_1, γ_2, and γ_3 have also changed.

[6]Note that we are using interchangeably the notation for the functions $B(\cdot), C(\cdot)$ and $D(\cdot)$ for f_1 and f_2 with more arguments than right now. This is just to emphasize the variables that affect these functions.

[7]In a similar way, we can define $\widehat{\gamma}_i,\ i=1,2,3$ for the function f_2.

4 Computation of Bounds for Option Prices

In this section we will apply our general theoretical framework to obtain analytically bounds on call option prices in the previous stochastic volatility models. We start with the Heston (1993) model.

4.1 Bounds for the Heston (1993) Model

Consider an adapted process $h_t = (h_t^s, h_t^v)$, the associated martingale measure $Q = Q(h)$, and the SDF process Λ, where

$$\frac{d\Lambda_t}{\Lambda_t} = -rdt - h_t^s dW_t^s - h_t^v dW_t^v$$

Under the risk-neutral measure Q, the stock price and variance processes (see (5)-(6)) are given as

$$\begin{aligned} \frac{dS_t}{S_t} &= (r + sV_t - \lambda_t^s \sqrt{V_t})dt + \sqrt{V_t} d\widetilde{W}_t^S & (12) \\ dV_t &= (\alpha - \beta V_t - \lambda_t)dt + \sigma\sqrt{V_t}\left(\rho d\widetilde{W}_t^S + \sqrt{1-\rho^2} d\widetilde{W}_t^V\right) & (13) \end{aligned}$$

The market price of risk of the variance is assumed to be

$$\lambda_t = \sigma\sqrt{V_t}(\rho h_t^s + \sqrt{1-\rho^2} h_t^v) \tag{14}$$

for certain processes h_t^s and h_t^v to be obtained. By Lemma 1, it is known that $h_t^s = \lambda_t^s$ and, for making the process (12) risk-neutral, λ_t^s has to satisfy $\lambda_t^s = s\sqrt{V_t}$. Hence,

$$h_t^s = s\sqrt{V_t} \tag{15}$$

Following equation (15), we get an expression of the form in (7) if

$$h_t^v = v\sqrt{V_t} \tag{16}$$

for some constant v. Replacing (15)-(16) in (14), we get that $\lambda = \sigma(s\rho + v\sqrt{1-\rho^2})$.

The problem for the lower bound for a European call option with strike K and maturity T is

$$\underline{C_0} = \min C_0(h) = \min E_0^P \left[\frac{\Lambda_T}{\Lambda_0} \max\{0, S_T - K\}\right]$$

for some $h \in H^v$ where H^v is the set of processes $h_t = (s\sqrt{V_t}, h_t^v)$ for which the volatility constraint in (4) is satisfied. Then, our benchmark is assumed to be of the form $h_1^* = (s\sqrt{V_t}, v^*\sqrt{V_t})$ for some constant v^*, which will be chosen afterwards. This benchmark is admissible and it satisfies (7), so Heston's formula applies. For the special case of standard GD bounds, we set $h_2^* = (0, 0)$ and apply this to all the previous expressions wherever it is necessary.

Cochrane and Saá-Requejo (2000) consider a case where, in the volatility constraint, A is a positive constant. We assume the ceiling process is proportional to V_t, that is,

$$A_t = \overline{A}\sqrt{V_t} \tag{17}$$

where $\overline{A}$ is a positive constant. This specification allows us to derive the analytical expression for the BGD bounds via the Heston (1993) formula.

According to the benchmark, as stated in Proposition 1, to compute the upper and lower bounds, we need two processes in which $\overline{h_t^v} = \overline{v}\sqrt{V_t}$, $\underline{h_t^v} = \underline{v}\sqrt{V_t}$, with $\overline{v}$ and $\underline{v}$ certain constants.

For the BGD bounds, where $h_1^* = (s\sqrt{V_t}, v^*\sqrt{V_t})$, from Proposition 1 we can derive $\underline{v} = v^* + \overline{A}$ and $\overline{v} = v^* - \overline{A}$. Similarly, for the GD bounds, where $h_2^* = (0,0)$, we get $\underline{v} = \sqrt{\overline{A}^2 - s^2}$, $\overline{v} = -\sqrt{\overline{A}^2 - s^2}$. In both cases, the proper values for $\overline{A}$ will be chosen later. With this result and Proposition 1, we are able to enunciate the following Proposition.[8]

Proposition 4 *The lower and upper bounds for the price of a European call option are written as*

$$\underline{C_0}(h^*) = C_0^H(\underline{\lambda}), \quad \overline{C_0}(h^*) = C_0^H(\overline{\lambda})$$

where $\underline{\lambda} = \sigma(s\rho + \underline{v}\sqrt{1-\rho^2})$ *and* $\overline{\lambda} = \sigma(s\rho + \overline{v}\sqrt{1-\rho^2})$. ∎

Consider a trader who wants to compute the call option price and the bounds for this price. However, she is concerned about the potential misspecification of the unobservable variance risk premium λ_t in (14). In this case, she can allow some uncertainty assuming that the true variance risk premium is bounded as follows:

$$\lambda_l V_t \leq \lambda V_t \leq \lambda_h V_t \tag{18}$$

where λ_l and λ_h are constants given by $\lambda_j = \sigma(s\rho + \sqrt{1-\rho^2}v_j),\ j = l, h$ and

$$v_l = v - 0.5\Delta, \quad v_h = v + 0.5\Delta \tag{19}$$

with $\Delta > 0$ indicating the distance between v_h and v_l. Obviously, setting $\lambda_l = \lambda_h$, we get the Heston (1993) theoretical call price. The inequalities in (18) define the whole set of "plausible" candidate processes h, which produce a range of candidate call prices. As it is natural, we would like to have the price bounds as tight as possible.

It is easy to verify analytically that the optimal choices for $\overline{A}$ for BGD bounds and GD bounds are, respectively, $\overline{A} = \max\{|v_l - v^*|, |v_h - v^*|\}$ and $\overline{A} = \sqrt{s^2 + \max\{v_l^2, v_h^2\}}$.

We will use Proposition 4 to compute analytically the bounds on call option prices, when $\Delta = 0,\ 2,\ 4$. Similarly to Bondarenko and Longarela (2009), we analyze the effect of correlation on the price bounds considering $\rho = -0.1,\ -0.53,\ -0.9$. We use their parameters $r = 0.05,\ S_0 = 100,\ V_0 = \alpha/\beta = 0.0137,\ T = 0.25$, and $v^* = -22.6$ to compute the BGD bounds for the process h_1^*. The remaining parameters ($\alpha = 0.097,\ \beta = 7.1,\ \sigma = 0.32,\ s = 8.6,\ v = -22.6$) are based on Pan (2002), who fits the Heston (1993) model to a sample of S&P 500 index options over the period 1989-1996.[9]

[8] This result corresponds to Proposition 7 in Bondarenko and Longarela (2009).

[9] The parameter v for the variance risk premium is the hardest one to estimate as the variance is a non-traded asset. However, the remaining parameters can be easily estimated as the stock price process S_t can be observed.

Table 1 shows the price bounds for call options with strikes from 85 to 115 and for the three correlations aforementioned. We notice analytically that if $\Delta = 0$, the BGD bounds are equal to the theoretical call price. Moreover, it is also easy to prove that the upper GD bounds coincide with the upper BGD bounds. Looking at the Heston prices, C_0, we see that correlation affects ITM and OTM options in a different way. As ρ increases in absolute value, OTM option prices decrease, while ITM option prices first increase and then decrease. The same happens for BGD bounds and thus for upper GD bounds. The behavior of lower GD bounds is different: as $|\rho|$ gets higher, bounds on ITM option prices increase while, for OTM options, these bounds first decrease and then increase.

[INSERT TABLE 1 AROUND HERE]

Note that the lower GD bounds, $\underline{C_0}^{GD}$, included in this Table are much smaller than those reported in Table 1 in Bondarenko and Longarela (2009). For example, if $\rho = -0.10, K = 100$, we obtain the values 2.5238, 2.5092, 2.4949 for $\Delta = 0,\ 2,\ 4$, respectively. On the contrary, the corresponding values reported by these authors are 2.977, 2.951, and 2.916. The same happens for all the correlations, strikes, and values of Δ we are considering. Thus our case of BGD versus GD bounds is now stronger, that is, the benchmark good-deal bounds are even tighter than the standard good-deal ones than previously illustrated by these authors.[10]

For the three cases of correlation and three values of Δ, Figure 1 presents the difference between the bounds, $\underline{C_0}$ and $\overline{C_0}$, and the Black-Scholes price C_0^{BS}, in which the volatility is set to $\sqrt{V_0}$.

[INSERT FIGURE 1 AROUND HERE]

Figure 2 shows the size of bounds, that is, the difference between the bounds $\overline{C_0}$ and $\underline{C_0}$. As the graph shows, the largest size of the bounds corresponds to near at-the-money options.

[INSERT FIGURE 2 AROUND HERE]

It can be clearly seen that the bounds for h_1^* are considerably tighter than those for h_2^*. This is particulary pronounced when Δ approaches zero, as the size of bounds is zero for h_1^* and strictly positive for h_2^*. As expected, as Δ increases, the size of bounds increases as we are separating bounds further from the theoretical price.

As the correlation increases in absolute value, the bounds shrink in both GD and BGD bounds. It could be said that the larger $|\rho|$ is, the tighter the sizes of bounds are. For complete markets ($|\rho| = 1$), it can be proved analytically that the size of bounds turns zero. Intuitively, although the

[10] The reason for these differences is due to some mistakes that, after code comparison, were identified in the programming codes of Bondarenko and Longarela (2009). We thank both authors for helpful interaction with us. We also thank an anonymous referee for highlighting us this issue.

volatility is a "virtual" product, we know its behavior as changes in stock prices would transmit to changes in the volatility. To sum up, stochastic volatility involves incomplete market when $|\rho| < 1$.

Economically, we could deduce some results. The BGD bounds determine the range of possible prices for which the admissible SDFs lie in the "neighborhood" of a given benchmark. The "radius" of this neighborhood is determined by the ceiling $\overline{A}$. The ceiling must contain all the SDFs for which the variance risk premium is bounded as in (18). For the GD bounds, the neighborhood is symmetric with respect to the shock dW_t^V. Hence, even for $\Delta = 0$, their neighborhood must include some SDFs that are economically implausible. Thus, the GD bounds are wider even when there is no uncertainty about the variance risk premium.

4.2 Bounds for the Extended Schöbel and Zhu (1999) Model

Now, we follow the same procedure as in the Heston (1993) model to compute the option prices and their bounds in the extended Schöbel and Zhu (1999) model. Under this extended model, the theoretical price of a European call option at time $t = 0$ is denoted as $C_0^{OU}(\lambda)$. We deduce from (11) that the market price of volatility will have the form $\lambda_t = 2\sigma\sqrt{V_t}(\rho h_t^s + \sqrt{1-\rho^2}h_t^v)$ for some processes h_t^s and h_t^v to be obtained. As the spot price process is already risk-neutral we get $\lambda_t^s = 0 = h_t^s$ where the second equality comes from Lemma 1. As we want to obtain the form of the market price of volatility as in (7), we assume again (16). Thus, it is easy to verify that $\lambda = 2\sigma\sqrt{1-\rho^2}v$.

Remember the form of the ceiling process in (17) and that λ_t lies as in (18) so as to contain all the plausible candidate processes h that generate the candidate call prices, where (19) holds.

Following Proposition 1, we will substitute $\underline{h_t^v} = \underline{v}\sqrt{V_t}$ and $\overline{h_t^v} = \overline{v}\sqrt{V_t}$ for computing the bounds. Moreover, for the case of BGD bounds, $h_1^* = (0, v^*\sqrt{V_t})$, thus we get $\overline{v} = v^* - \overline{A}$ and $\underline{v} = v^* + \overline{A}$. In a similar way, for GD bounds, $h_2^* = (0, 0)$ and we obtain $\overline{v} = -\overline{A}$ and $\underline{v} = \overline{A}$.

Then, we can state the next Proposition that arises directly from Proposition 1.

Proposition 5 *The lower and upper bounds for the price of a European call option are written as*

$$\underline{C_0}(h^*) = C_0^{OU}(\underline{\lambda}), \qquad \overline{C_0}(h^*) = C_0^{OU}(\overline{\lambda})$$

where $\underline{\lambda} = 2\sigma\sqrt{1-\rho^2}\underline{v}$ *and* $\overline{\lambda} = 2\sigma\sqrt{1-\rho^2}\overline{v}$. ■

As λ_t lies as in (18), it is easy to prove that the optimal choice for the ceiling process for the BGD and GD bounds are, respectively, $\overline{A} = \max\{|v_h - v^*|, |v_l - v^*|\}$ and $\overline{A} = \max\{|v_h|, |v_l|\}$.

We will compute the prices and bounds for European call options. Following Schöbel and Zhu (1999), we consider the parameters $t = 0$, $T = 0.5$, $r = 0.093$, $\kappa = 4$, $\theta = 0.2$, $\sigma = 0.1$ while, for computing the bounds, we choose $v = v^* = -22.6$ in line with Bondarenko and Longarela (2009). Additionally, we analyze a range of strikes K from 90 to 120 and correlations $\rho = -0.25, -0.5, -0.75$.

Table 2 show prices and bounds for these correlations. As $|\rho|$ increases, prices and bounds decrease, except for the lower GD bounds that increase except for the very deep OTM options.

[INSERT TABLE 2 AROUND HERE]

Figure 3 graphs the difference between bounds and Black-Scholes prices for these correlations. The Black-Scholes prices will be denoted AVBS as they are based on the expected average variance AV that, as stated in Schöbel and Zhu (1999), is given by

$$AV = \frac{\sigma^2}{2\kappa} + \theta^2 + \frac{1 - e^{-\kappa(T-t)}}{T-t} \left(\frac{2\theta}{\kappa} (v_t - \theta) - \frac{\sigma^2 - 2\kappa(v_t - \theta)^2}{4\kappa^2} \left(1 + e^{-\kappa(T-t)} \right) \right)$$

This Figure shows that, as $|\rho|$ increases, the size of bounds minus the AVBS prices shrinks. Moreover, the bounds for the BGD case are much tighter than for the GD one.

[INSERT FIGURE 3 AROUND HERE]

Figure 4 shows the size of bounds. We see that as $|\rho|$ increases, the size of the bounds shrinks as we are closer to a complete market. If $|\rho| = 1$, we would be in the case of perfect correlation, and the market is complete as we explained for the Heston (1993) model.

[INSERT FIGURE 4 AROUND HERE]

If we compare Figures 1 and 3 and Figures 2 and 4, the size of bounds decreases as $|\rho|$ gets closer to one. Moreover, BGD bounds are always tighter than GD ones.

After having shown numerically and graphically these results, we will perform a more detailed analysis of the properties of option prices and their bounds.

5 Properties of Option Prices and Bounds

This Section includes the computation of several Greeks for option prices and a detailed sensitivity analysis with respect to some parameters for the previous stochastic volatility models.[11] In addition, we also provide closed-form expressions and discussion of Greeks for bounds on option prices. Concretely, we focus on delta, gamma, and vega as traders and institutions use to design hedging schemes that involve these Greeks. The delta (vega) of an option indicates the rate of change in the option price with respect to the underlying price (volatility). The option's gamma measures the rate of change in the option delta with respect to the underlying price.

[11] For the sake of brevity, Tables and Figures illustrating numerically and graphically the analytical expressions are not reported here and are available in Marroquín-Martínez and Moreno (2010).

5.1 Analysis for the Heston (1993) Model

5.1.1 Greeks for the Heston (1993) Model

As shown in Proposition 2, this model provides a closed-form expression for the call option price that includes explicitly two probabilities P_1 and P_2. Let C denote the call option price at time t. Then, we can compute analytically several Greeks as stated in the next Proposition.

Proposition 6 *Under the Heston (1993) model, the deltas and gammas for the call option are*

$$\begin{aligned} delta_S(S_t, V_t, t, T) &= \frac{\partial C}{\partial S_t} = P_1 && (20) \\ delta_V(S_t, V_t, t, T) &= \frac{\partial C}{\partial V_t} = S_t \frac{\partial P_1}{\partial V_t} - e^{-r(T-t)} K \frac{\partial P_2}{\partial V_t} && (21) \\ \Gamma_S(S_t, V_t, t, T) &= \frac{\partial^2 C}{\partial S_t^2} = \frac{\partial P_1}{\partial S_t} && (22) \\ \Gamma_V(S_t, V_t, t, T) &= \frac{\partial^2 C}{\partial V_t^2} = S_t \frac{\partial^2 P_1}{\partial V_t^2} - e^{-r(T-t)} K \frac{\partial^2 P_2}{\partial V_t^2} && (23) \end{aligned}$$

where for $h = S_t, V_t,\ j, n = 1, 2$

$$\frac{\partial^n P_j}{\partial h^n} = \frac{1}{\pi} \int_0^\infty Re \left(\frac{e^{-i\phi \ln(K)}}{i\phi} \frac{\partial^n F_j}{\partial h^n} \right) d\phi$$

and

$$\begin{aligned} \frac{\partial F_j}{\partial S_t} &= F_j \left(i\phi \frac{1}{S_t} \right) \\ \frac{\partial^2 F_j}{\partial S_t^2} &= F_j \frac{\phi}{S_t^2} (-\phi - i) \\ \frac{\partial^n F_j}{\partial V_t^n} &= F_j D_j^n \end{aligned}$$

In addition, applying the chain rule, the vega of the call option, ν*, is given as*

$$\nu(S_t, V_t, t, T) = \frac{\partial C}{\partial v_t} = 2v_t \frac{\partial C}{\partial V_t} = 2v_t\ delta_V(S_t, V_t, t, T)$$

■

We have illustrated graphically how these Greeks work. We check that as K decreases, $delta_S$ approaches faster to one confirming that, as expected, the option price is more sensitive to the underlying one for deeper ITM options. In addition, we also find that the highest values of both gamma and vega take place when the option is near ATM, indicating that (a) delta is highly sensitive to small changes in the underlying price and (b) the investor is concerned about moving to the ITM or OTM area, respectively. As in Black-Scholes (1973), the underlying price that maximizes both Greeks is always lower than the strike.

5.1.2 Sensitivity Analysis for the Heston (1993) Model

Now we compute the sensitivities of the call option with respect to the parameters α, β, σ, ρ, and V_0. The most cumbersome expressions are those for the sensitivities with respect to ρ and σ. The analytical formulas in a general form are stated in the following Proposition.

Proposition 7 *The sensitivities of the call option with respect to the parameters α, β, σ, ρ, and V_0 are given by*

$$\frac{\partial C}{\partial h} = S_t \frac{\partial P_1}{\partial h} - e^{-r(T-t)} K \frac{\partial P_2}{\partial h}$$

for $h = \alpha, \beta, \sigma, \rho, V_0$, where for $j = 1, 2$, we get

$$\frac{\partial P_j}{\partial h} = \frac{1}{\pi} \int_0^{\infty} Re\left(\frac{e^{-i\phi \ln(K)}}{i\phi} \frac{\partial F_j}{\partial h} \right) d\phi$$

■

For the particular case of the sensitivity with respect to ρ, we get

$$\frac{\partial F_j}{\partial \rho} = F_j \left(\frac{\partial C_j}{\partial \rho} + \frac{\partial D_j}{\partial \rho} V_t \right)$$

and

$$\begin{aligned}
\frac{\partial C_j}{\partial \rho} &= \frac{\alpha}{\sigma^2}\left[\left(\beta_j'(\rho) - \sigma\phi i + h_j'(\rho)\right) T - 2\left(\frac{e^{h_jT}(-g_j'(\rho) - g_j h_j'(\rho) T)(1-g_j) + g_j'(\rho)(1-g_j e^{h_jT})}{(1-g_j)(1-g_j e^{h_jT})} \right) \right] \\
\frac{\partial D_j}{\partial \rho} &= \frac{1}{\sigma^2}\left[\left(\beta_j'(\rho) - \sigma\phi i + h_j'(\rho)\right) \left(\frac{1-e^{h_jT}}{1-g_j e^{h_jT}} \right) + (\beta_j - \rho\sigma\phi i + h_j) \right] \\
&\times \left[\left(\frac{-h_j'(\rho) T e^{h_jT}(1-g_j e^{h_jT}) + (g_j'(\rho) e^{h_jT} + g_j h_j'(\rho) T e^{h_jT})(1-e^{h_jT})}{(1-g_j e^{h_jT})^2} \right) \right] \\
h_j'(\rho) &= \frac{(\rho\sigma\phi i - \beta_j)(\sigma\phi i - \beta_j'(\rho))}{\sqrt{(\rho\sigma\phi i - \beta_j)^2 - \sigma^2(2u_j\phi i - \phi^2)}} \\
g_j'(\rho) &= \frac{(\beta_j'(\rho) - \sigma\phi i + h_j'(\rho))(\beta_j - \rho\sigma\phi i - h_j) - (\beta_j'(\rho) - \sigma\phi i - h_j'(\rho))(\beta_j - \rho\sigma\phi i + h_j)}{(\beta_j - \rho\sigma\phi i - h_j)^2}
\end{aligned}$$

taking in account that

$$\beta_1'(\rho) = \lambda'(\rho) - \sigma, \quad \beta_2'(\rho) = \lambda'(\rho)$$

It can be shown analytically that the call price is always increasing in σ, V_0, α and decreasing in β. However, the correlation affects options in a different way depending on their moneyness degree. Our graphs (not included here) show that, when $\rho > 0$, the correlation affects less when the option is deep OTM. As ρ tends to one, the call price decreases. But when $\rho < 0$, the correlation affects in a different way depending on how deep OTM the options are. For negative enough values of the correlation, the highest increase in call prices correspond to the smallest strike (K=105).

Taking the parameter values of the previous sections as point of reference, we have built several Tables and Figures that illustrate changes in theoretical call prices and bounds when varying a model input parameter, remaining all the other constant. A qualitative discussion of the results follows:

- **Parameter** α: increasing α, the mean reversion level of volatility increases and also do prices and bounds. This is logical as, if this mean reversion level is higher, the volatility of the model will become higher at some point in the future. This introduces more uncertainty to the model and then, prices and bounds increase. In fact, the higher α is, the higher the distance between the theoretical call price and the lower GD bounds is. Thus, for higher values of α, the lower GD bound become less informative. If we decrease α, then the opposite happens.

- **Speed of adjustment of the variance,** β: starting from our initial speed (7.1), a decrease in β leads to higher theoretical call prices and bounds. This happens because the variance process will be slower in reaching the mean reversion level, adding more uncertainty to the model. We also find that, for small values of β, the lower GD bound is less informative, hence less useful. On the contrary, if we increase β, the opposite happens.

- **Initial variance,** V_0: if this value increases, theoretical prices, bounds, and the length of the interval, $\overline{C}(GD)-\underline{C}(GD)$, increase. Although V_0 changes, the mean reversion level of volatility keeps constant. Then, starting from a higher V_0, the model incorporates more volatility as the variance process will not experiment big changes. Thus, we will have higher option prices and bounds. If V_0 decreases, the opposite happens. In both cases, the length of the interval for the BGD bounds does not change remarkably. Finally, for higher values of V_0, the lower GD bound is less informative.

- **Volatility of the volatility,** σ: if we change σ, then λ, $\overline{\lambda}$, and $\underline{\lambda}$ also change and, consequently, theoretical prices and bounds. In more detail, if we increase σ, λ's become more negative and theoretical prices and bounds increase. But if σ decreases, λ's turn higher and the opposite happens. We can mention that a higher value of σ implies higher volatility with two opposite effects: (a) it directly increases the call price and (b) as we are considering negative values of ρ, a higher volatility implies an smaller underlying price, and then an smaller call price. In our case, the first effect weights more than the second one.

5.2 Analysis for the Extended Schöbel and Zhu (1999) Model

As shown in Proposition 3, this model provides closed-form expressions for call option prices. Then, we can compute analytically the Greeks and different sensitivities with respect to the model parameters. Following the formulas for the Greeks expressed for the Heston (1993) model, we need to calculate the partial derivatives for the characteristic functions f_1 and f_2 in this case. Then, we obtain the following.

Proposition 8 *Under the extended Schöbel and Zhu (1999) model, the deltas and gammas for the call option are given by equations (20)-(23) where, for $h = S_t, V_t,\ j, n = 1, 2$,*

$$\frac{\partial^n P_j}{\partial h^n} = \frac{1}{\pi}\int_0^\infty Re\left(\frac{e^{-i\phi\ln(K)}}{i\phi}\frac{\partial^n f_j(\phi)}{\partial h^n}\right)d\phi$$

and

$$\begin{aligned}
\frac{\partial f_j}{\partial S_t} &= f_j i\phi\frac{1}{S_t}, \quad j = 1, 2 \\
\frac{\partial^2 f_j}{\partial S_t^2} &= f_j\frac{\phi}{S_t^2}(-\phi - i), \quad j = 1, 2 \\
\frac{\partial f_1}{\partial V_t} &= f_1\left(-(1+i\phi)\rho\sigma^{-1}V_t + D(t,T)V_t + B(t,T)\right) \\
\frac{\partial^2 f_1}{\partial V_t^2} &= f_1\left(\left[-(1+i\phi)\rho\sigma^{-1}V_t + D(t,T)V_t + B(t,T)\right]^2 - (1+i\phi)\rho\sigma^{-1} + D(t,T)\right) \\
\frac{\partial f_2}{\partial V_t} &= f_2\left(-i\phi\rho\sigma^{-1}V_t + D(t,T)V_t + B(t,T)\right) \\
\frac{\partial^2 f_2}{\partial V_t^2} &= f_2\left(\left[-i\phi\rho\sigma^{-1}V_t + D(t,T)V_t + B(t,T)\right]^2 - i\phi\rho\sigma^{-1} + D(t,T)\right)
\end{aligned}$$

Similarly to the Heston (1993) model, the vega of the call option is given as

$$\nu(S_t, V_t, t, T) = 2v_t\ delta_V(S_t, V_t, t, T)$$

■

As we did for the Heston (1993) model, we have illustrated numerically and graphically these Greeks jointly with additional sensitivity measures. The main conclusion is that results do not change qualitatively with respect to those for the Heston (1993) model.

5.3 Greeks for Bounds

We will compute several Greeks for the bounds and graph some of the results. For computing these Greeks we apply the same expressions as those for the theoretical ones, but replacing the input variable λ by $\overline{\lambda}$ and $\underline{\lambda}$ as stated in the following Proposition. We report just the expressions for the upper bound as those for the lower bound are completely similar.

Proposition 9 *Under the Heston(1993) and extended Schöbel and Zhu (1999) models, the Greeks for the upper bound of the call option are given by*

$$\begin{aligned}
\overline{delta_S}(S_t, V_t, t, T) &= \frac{\partial \overline{C}}{\partial S_t} = P_1(\overline{\lambda}) \\
\overline{\Gamma_S}(S_t, V_t, t, T) &= \frac{\partial^2 \overline{C}}{\partial S_t^2} = \frac{\partial P_1(\overline{\lambda})}{\partial S_t} \\
\overline{delta_V}(S_t, V_t, t, T) &= \frac{\partial \overline{C}}{\partial V_t} = S_t \frac{\partial P_1(\overline{\lambda})}{\partial V_t} - e^{-r(T-t)} K \frac{\partial P_2(\overline{\lambda})}{\partial V_t}
\end{aligned}$$

and, for computing the vega, the chain rule is needed:

$$\overline{\nu}(S_t, V_t, t, T) = \frac{\partial \overline{C}}{\partial v_t} = 2v_t \frac{\partial \overline{C}}{\partial V_t} = 2v_t\, \overline{delta_V}(S_t, V_t, t, T)$$

where $\overline{C}$ denotes the upper bound obtained for each model. Obviously, this bound presents a different closed-form expression in each model. ∎

The results will be reported in detail for the Heston (1993) model as we have obtained similar conclusions for the extended Schöbel and Zhu (1999) model.[12]

Figure 5 presents, for different correlations, deltas for BGD and GD bounds as a function of the underlying price. Although it is hard to see the lines, going deeply into the graphs, we can see that $\overline{delta_S}$ is above $\underline{delta_S}$ (delta for the lower bound) just in the OTM area. Besides, it is clear that the theoretical $delta_S$ is always between $\overline{delta_S}$ and $\underline{delta_S}$. We have also checked that increasing the strike moves the graph to the right, keeping the same form and characteristics.

[INSERT FIGURE 5 AROUND HERE]

Figure 6 shows gammas for bounds as a function of the underlying price. When the option is deep OTM or deep ITM, $\overline{\Gamma_S} > \Gamma_S > \underline{\Gamma_S}$ (gamma for the lower bound). However, when the call option is near ATM, the opposite happens. In the case of GD bounds, the lower ones for $\Delta = 2, 4$ take similar values and then, it is not possible to distinguish them visually. This Figure also shows that the maximum Γ_S is achieved in the OTM area. If we increased the strike, the graph moves to the right, maintaining the same shape and features except for the value that maximizes Γ_S. As K increases, we have checked that the maximum Γ_S decreases, similarly to the Black-Scholes model.

[INSERT FIGURE 6 AROUND HERE]

Finally, Figure 7 presents vegas for BGD and GD bounds for different correlations. These Greeks respect the expected order, this is, $\overline{\nu}$ is always above $\underline{\nu}$ (vega for the lower bound) and the theoretical vega ν is always located between the vegas for bounds.

[12] Results are available upon request.

[INSERT FIGURE 7 AROUND HERE]

Graphing these vegas for bounds on options with different strikes, we have also checked that higher values of K provide results that are qualitatively similar to those obtained for gammas for bounds except that the maximum ν increases with K.

Summarizing, we could say that Greeks for bounds do not behave exactly as bounds for prices. Sometimes what we have called "Greek for the upper bound" takes the place of the "Greek for the lower bound" and viceversa. Anyway, as expected, the theoretical Greek is always located between both. However, the two main conclusions observed for the bounds for prices keep for the Greeks for bounds. First of all, BGD bounds are much tighter than GD ones. Secondly, as $|\rho|$ increases, the size of the Greeks for bounds decreases and bounds get tighter. Thus, as $|\rho|$ gets closer to one, we are closer to a complete market, hence, both size of bounds and Greeks for bounds tend to zero.

6 Conclusions

As stated in the previous literature, it is possible to find an arbitrage-free interval for bounds of option prices under incomplete markets. In this framework, we have analyzed the stochastic volatility models proposed in Heston (1993) and an extension of that introduced in Schöbel and Zhu (1999). Both models provide unique closed-form expressions for the prices of certain securities assuming a certain functional form of the market price of risk of the corresponding factors. In fact, each functional form is associated to a martingale measure and, thus, to a price for the security. These analytical expressions are relevant from both practical and theoretical perspectives as they allow us to compute easily the bounds on option prices, to perform a sensitivity analysis over the parameters involved in these bounds, and to design the corresponding hedging strategy.

Following the research area initiated in Cochrane and Saá-Requejo (2000), for both models, we have obtained analytical expressions for BGD and GD bounds and have performed a detailed sensitivity and hedging analysis for prices and bounds. In both cases, the main qualitative conclusion is that BGD bounds are much tighter than GD ones. This also happens for Greeks for bounds.

Therefore, the answer for the question in the paper's title is that the specification of the stochastic volatility does not seem to be relevant in qualitative terms when computing bounds on option prices and their Greeks. Moreover, as expected, approaching to the perfect correlation case (that is, complete markets), the size of bounds becomes smaller.

Comparing the results obtained in both models for our chosen parameters, we notice that the correlation affects option prices in a different way depending on the degree of moneyness for the Heston (1993) model. Nevertheless, for the extended Schöbel and Zhu (1999) model, the correlation treats option prices in a similar way regardless the moneyness degree.

The main contributions of the paper are as follows: compared to Bondarenko and Longarela (2009), we have enlarged the analysis of bounds for option prices in the Heston (1993) model and fixed some errors in their numerical analysis finding now higher differences between GD and BGD bounds. Afterwards, we have extended the Schöbel and Zhu (1999) model and obtained analytical expressions for option prices and bounds. Our final contribution is that, for both models, we have performed extensive sensitivity and hedging analysis showing, respectively, the effect of changes in the model parameters on these bounds and the risk management properties of these bounds.

More concretely, the sensitivity analysis for the Heston (1993) model has illustrated that, as the mean reversion level of the volatility, the volatility of the volatility, or the initial variance increase (and the speed of adjustment of the variance decreases), all the bounds increase and the distance between upper and lower GD bounds increases. The intuition behind these results is that, in these cases, we have a wider no-arbitrage interval for option prices. Hence, we have less information about the unique price we would find in complete markets. This analysis has also shown that BGD bounds are tighter than GD ones. The analysis for the extended Schöbel and Zhu (1999) model provided similar qualitative conclusions.

A possible extension of this paper involves computing bounds on option prices using another pricing models that also provide closed-form expressions for option prices. For instance, some possible pricing techniques and candidate models are the following: a) the stochastic volatility model presented in Jizba *et al.* (2009) that applies a Mellin transform to obtain a generalized Black-Scholes formula assuming that the log-returns are given by a superposition of Gaussian distributions and the variance follows a Gamma distribution, b) Chiu *et al.* (2010) that applies asymptotic techniques to price European options whose underlying asset follows a mean-reverting log-normal process with (a two-factor) stochastic volatility, c) the fuzzy approach employed in Nowak and Romaniuk (2010) assuming that the underlying asset follows a Levy process with jumps, and d) Wong and Lo (2010) applied the Fourier transform to value analytically European options whose underlying asset follows a mean-reverting log-normal process with stochastic volatility. A final alternative is Pillay and O' Hara (2011) that considers a process for the underlying asset similar to that in Wong and Lo (2010) but enlarged with jumps. This issue is left as avenue for further research.

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Appendix of Tables

Table 1: BGD and GD bounds for the Heston (1993) model.

K	C_0	$\underline{C_0}^{BGD}$		$\overline{C_0}^{BGD}$		$\underline{C_0}^{GD}$			$\overline{C_0}^{GD}$		
		$\Delta = 2$	$\Delta = 4$	$\Delta = 2$	$\Delta = 4$	$\Delta = 0$	$\Delta = 2$	$\Delta = 4$	$\Delta = 0$	$\Delta = 2$	$\Delta = 4$
$\rho =$ -0.10											
85	16.1421	16.1348	16.1281	16.1500	16.1586	16.0571	16.0570	16.0569	16.1421	16.1500	16.1586
90	11.4383	11.4193	11.4014	11.4584	11.4798	11.1376	11.1364	11.1354	11.4383	11.4584	11.4798
95	7.1796	7.14254	7.1068	7.2184	7.2587	6.3876	6.3812	6.3750	7.1796	7.2184	7.2587
100	3.8190	3.7677	3.7181	3.8720	3.9267	2.5238	2.5092	2.4949	3.8190	3.8720	3.9267
105	1.6997	1.6520	1.6059	1.7492	1.8004	0.5714	0.5601	0.5492	1.6997	1.7492	1.8004
110	0.6601	0.6286	0.5986	0.6932	0.7280	0.0780	0.0746	0.0713	0.6601	0.6932	0.7280
115	0.2372	0.2207	0.2052	0.2549	0.2739	0.0080	0.0074	0.0069	0.2372	0.2549	0.2739
$\rho =$ -0.53											
85	16.2056	16.1966	16.1881	16.215	16.2252	16.0641	16.0636	16.0631	16.2056	16.2151	16.2252
90	11.5578	11.5386	11.5202	11.5779	11.5989	11.1769	11.1744	11.1720	11.5578	11.5779	11.5989
95	7.3249	7.2915	7.2592	7.3594	7.3950	6.5131	6.5050	6.4970	7.3249	7.3594	7.3950
100	3.8715	3.8271	3.7839	3.9171	3.9640	2.6652	2.6504	2.6358	3.8715	3.9171	3.9640
105	1.5757	1.5347	1.4950	1.6180	1.6616	0.5587	0.5480	0.5376	1.5757	1.6180	1.6616
110	0.4749	0.4509	0.4280	0.5001	0.5267	0.0454	0.0432	0.0411	0.4749	0.5001	0.5267
115	0.1133	0.1040	0.0954	0.1234	0.1343	0.0017	0.0015	0.0014	0.1133	0.1234	0.1343
$\rho=$ -0.9											
85	16.1933	16.1894	16.1857	16.1973	16.2013	16.1030	16.1023	16.0999	16.1933	16.1973	16.2013
90	11.5131	11.5049	11.4970	11.5214	11.5299	11.2749	11.2753	11.2755	11.5131	11.5214	11.5299
95	7.1997	7.1855	7.1716	7.2141	7.2287	6.7268	6.7198	6.7136	7.1997	7.2141	7.2287
100	3.6020	3.5829	3.5640	3.6213	3.6410	2.9465	2.9338	2.9211	3.6020	3.6213	3.6410
105	1.1679	1.1509	1.1343	1.1851	1.2027	0.6411	0.6346	0.6281	1.1679	1.1851	1.2027
110	0.1439	0.1379	0.1321	0.1503	0.1568	0.0094	0.0080	0.0066	0.1439	0.1503	0.1568
115	0.0029	0.0026	0.0023	0.0033	0.0037	0.0061	0.0068	0.0075	0.0029	0.0033	0.0037

This Table reports BGD and GD bounds for the Heston (1993) model. Strikes K range from 85 to 115. $\rho = -0.1,\ -0.53,\ -0.9$. When $\Delta = 0$, lower and upper BGD bounds are equal to the theoretical call price C_0, that is why are not reported. $\alpha = 0.097,\ \beta = 7.1,\ \sigma = 0.32,\ s = 8.6,\ v = v^* = -22.6$. The remaining parameters are set as $r = 0.05,\ S_0 = 100,\ V_0 = \alpha/\beta = 0.0137,\ T = 0.25$.

Table 2: BGD and GD bounds for the extended Schöbel and Zhu (1999) model.

K	C_0	$\underline{C_0}^{BGD}$		$\overline{C_0}^{BGD}$		$\underline{C_0}^{GD}$			$\overline{C_0}^{GD}$		
		$\Delta = 2$	$\Delta = 4$	$\Delta = 2$	$\Delta = 4$	$\Delta = 0$	$\Delta = 2$	$\Delta = 4$	$\Delta = 0$	$\Delta = 2$	$\Delta = 4$
$\rho =$ -0.25											
90	16.7217	16.6236	16.5292	16.8239	16.9301	14.6130	14.5966	14.5809	16.7217	16.8239	16.9301
95	13.3648	13.2459	13.1309	13.4877	13.6150	10.5046	10.4761	10.4484	13.3648	13.4877	13.6150
100	10.4519	10.3179	10.1880	10.5900	10.7325	6.9961	6.9565	6.9177	10.4519	10.5900	10.7325
105	7.9990	7.8576	7.7203	8.1447	8.2948	4.2683	4.2236	4.1798	7.9990	8.1447	8.2948
110	5.9943	5.8534	5.7168	6.1396	6.2895	2.3688	2.3274	2.2870	5.9943	6.1396	6.2895
115	4.4028	4.2696	4.1408	4.5405	4.6830	1.1936	1.1617	1.1307	4.4028	4.5405	4.6830
120	3.1736	3.0534	2.9377	3.2984	3.4281	0.5478	0.5268	0.5067	3.1736	3.2984	3.4281
$\rho =$ -0.5											
90	16.5714	16.4911	16.4136	16.6545	16.7406	14.7034	14.6864	14.6701	16.5714	16.6545	16.7406
95	13.1468	13.0489	12.9538	13.2477	13.3517	10.6303	10.6025	10.5754	13.1468	13.2477	13.3517
100	10.1650	10.0539	9.9458	10.2791	10.3964	7.1294	7.0918	7.0550	10.1650	10.2791	10.3964
105	7.6519	7.5342	7.4195	7.7728	7.8969	4.3715	4.3295	4.2882	7.6519	7.7728	7.8969
110	5.6049	5.4877	5.3737	5.7252	5.8490	2.4193	2.3802	2.3421	5.6049	5.7252	5.8490
115	3.9942	3.8842	3.7776	4.1074	4.2242	1.1965	1.1665	1.1372	3.9942	4.1074	4.2242
120	2.7700	2.6724	2.5780	2.8711	2.9758	0.5259	0.5065	0.4879	2.7700	2.8711	2.9758
$\rho =$ -0.75											
90	16.2462	16.1948	16.1446	16.2990	16.3532	14.8551	14.8390	14.8232	16.2462	16.2990	16.3532
95	12.7146	12.6506	12.5881	12.7800	12.8469	10.8450	10.8203	10.7960	12.7146	12.7800	12.8469
100	9.6333	9.5598	9.4877	9.7084	9.7849	7.3777	7.3454	7.3137	9.6333	9.7084	9.7849
105	7.0438	6.9654	6.8885	7.1238	7.2053	4.6027	4.5670	4.5318	7.0438	7.1238	7.2053
110	4.9556	4.8779	4.8018	5.0349	5.1158	2.5880	2.5545	2.5216	4.9556	5.0349	5.1158
115	3.3452	3.2735	3.2034	3.4185	3.4936	1.2874	1.2612	1.2355	3.3452	3.4185	3.4936
120	2.1609	2.0992	2.0393	2.2243	2.2893	0.5559	0.5388	0.5223	2.1609	2.2243	2.2893

This Table reports BGD and GD bounds for the extended Schöbel and Zhu (1999) model. Strikes K range from 90 to 120. $\rho = -0.25,\ -0.5,\ -0.75$. When $\Delta = 0$, lower and upper BGD bounds are equal to the theoretical call price C_0, that is why are not reported. The parameters are $t = 0,\ T = 0.5,\ r = 0.093,\ \kappa = 4,\ \theta = 0.2,\ \sigma = 0.1,\ v = v^* = -22.6$.

Appendix of Figures

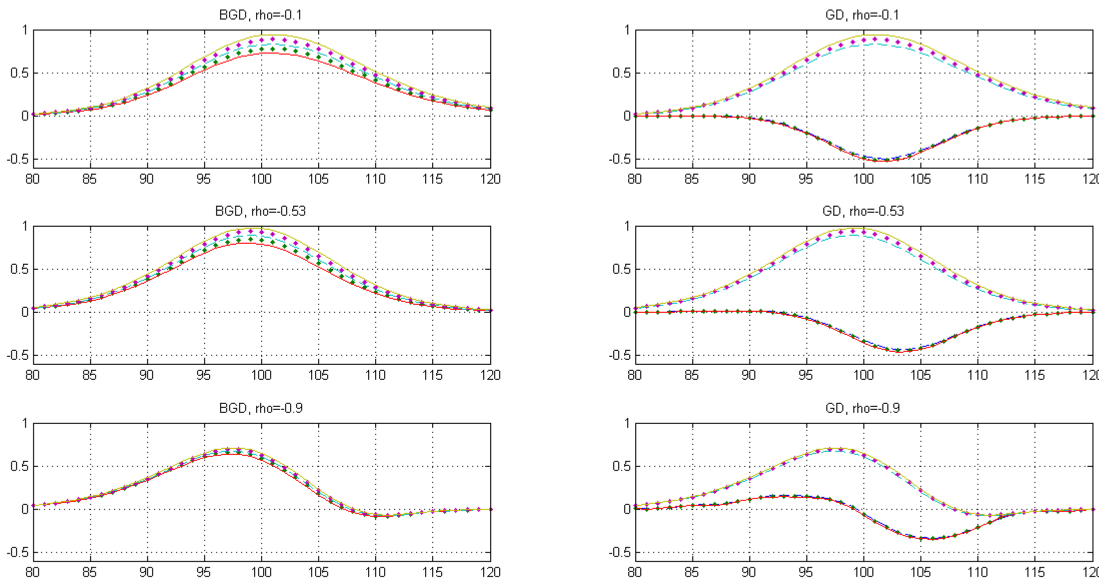

Figure 1: Difference between option price bounds and Black-Scholes prices versus strike for the Heston (1993) model. Strikes K range from 80 to 120. We consider correlations $\rho = -0.1, -0.53, -0.9$. $\Delta = 0$ (dashed lines), $\Delta = 2$ (dotted lines), and $\Delta = 4$ (solid lines). The left and right panels are for h_1^* and h_2^*, respectively. $\alpha = 0.097,\ \beta = 7.1,\ \sigma = 0.32,\ s = 8.6,\ v = v^* = -22.6$. The remaining parameters are set as $r = 0.05,\ S_0 = 100,\ V_0 = \alpha/\beta = 0.0137,\ T = 0.25$.

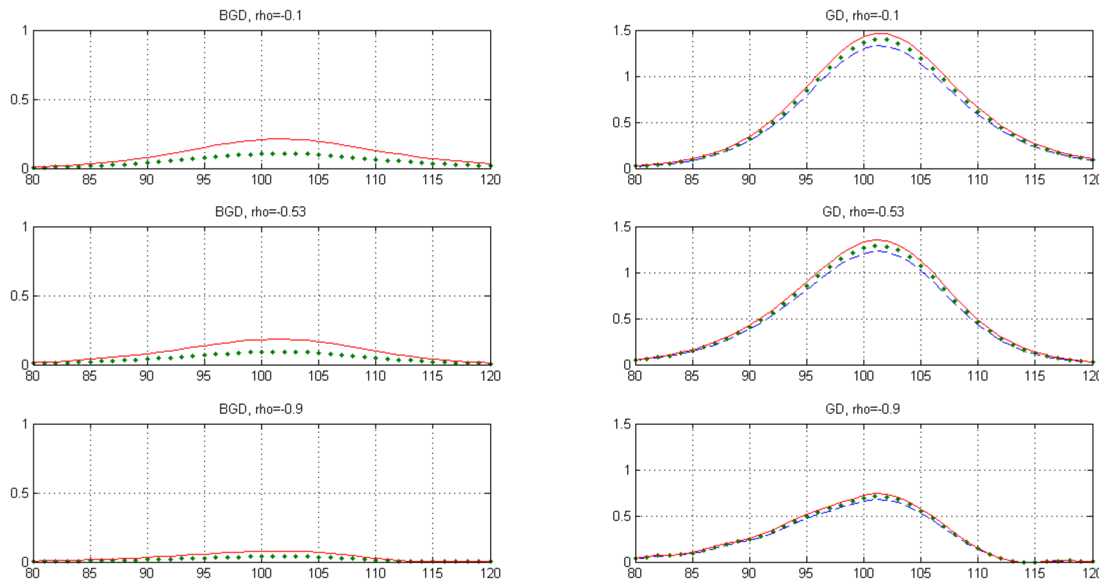

Figure 2: Size of bounds $\overline{C_0} - \underline{C_0}$ versus strike K. Heston (1993) model. Strikes K range from 80 to 120. $\rho = -0.1, -0.53, -0.9$. $\Delta = 0$ (dashed lines), $\Delta = 2$ (dotted lines), and $\Delta = 4$ (solid lines). The left and right panels are for h_1^* and h_2^*, respectively. $\alpha = 0.097,\ \beta = 7.1,\ \sigma = 0.32,\ s = 8.6,\ v = v^* = -22.6$. The remaining parameters are set as $r = 0.05,\ S_0 = 100,\ V_0 = \alpha/\beta = 0.0137,\ T = 0.25$.

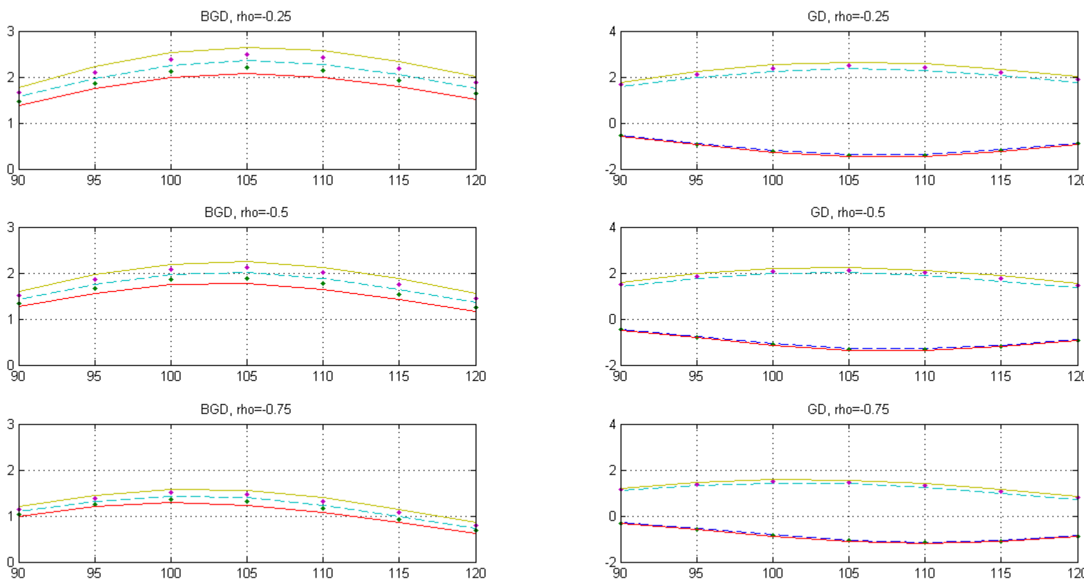

Figure 3: The difference between the option price bounds and the AVBS price versus strike K. Extended Schöbel and Zhu (1999) model. Strikes K range from 90 to 120. $\rho = -0.25, -0.5, -0.75$. $\Delta = 0$ (dashed lines), $\Delta = 2$ (dotted lines), and $\Delta = 4$ (solid lines). The left and right panels are for h_1^* and h_2^*, respectively. $t = 0,\ T = 0.5,\ r = 0.093,\ \kappa = 4,\ \theta = 0.2,\ \sigma = 0.1,\ v = v^* = -22.6$.

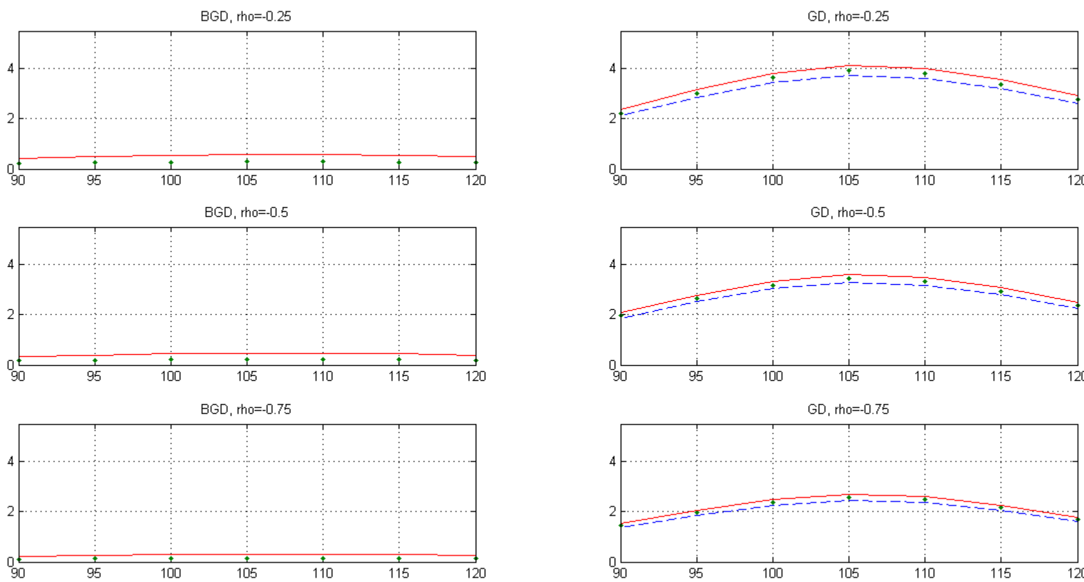

Figure 4: Size of bounds $\overline{C_0} - \underline{C_0}$ versus strike K. Extended Schöbel and Zhu (1999) model. Strikes K range from 90 to 120. $\rho = -0.25, -0.5, -0.75$. $\Delta = 0$ (dashed lines), $\Delta = 2$ (dotted lines), and $\Delta = 4$ (solid lines). The left and right panels are for h_1^* and h_2^*, respectively. $t = 0$, $T = 0.5$, $r = 0.093$, $\kappa = 4$, $\theta = 0.2$, $\sigma = 0.1$, $v = v^* = -22.6$.

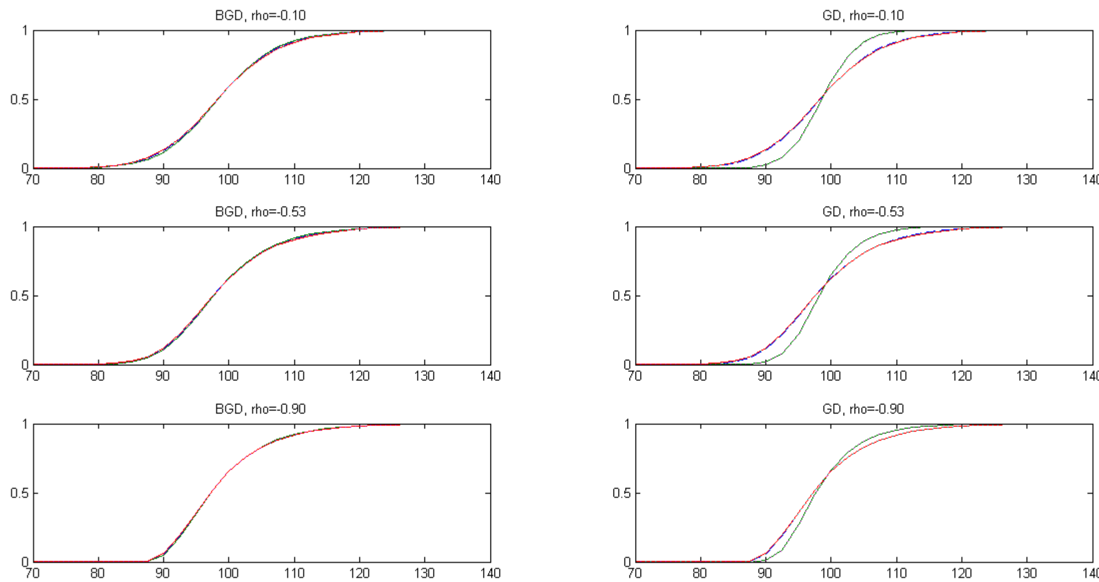

Figure 5: Deltas for bounds versus S_0. Heston (1993) model. $K = 100$. $\rho = -0.1, -0.53, -0.9$. $\Delta = 0$ (dashed line), $\Delta = 2$ (dotted lines), and $\Delta = 4$ (solid lines). The left and right panels are for h_1^* and h_2^*, respectively. $\alpha = 0.097,\ \beta = 7.1,\ \sigma = 0.32,\ s = 8.6,\ v = -22.6, v^* = -22.6$. The remaining parameters are set as $r = 0.05,\ V_0 = \alpha/\beta = 0.0137,\ T = 0.25$.

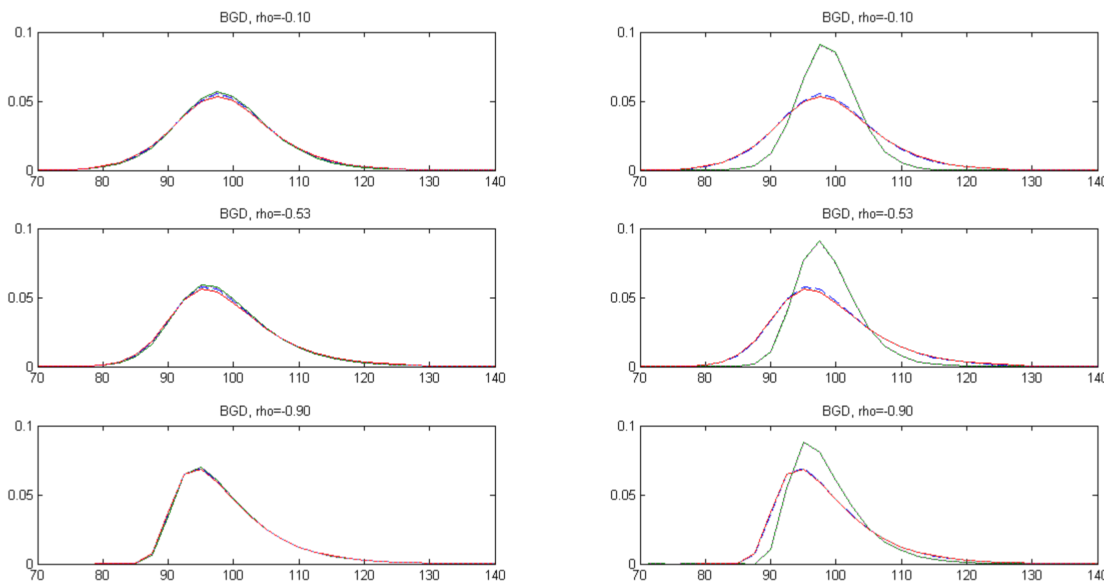

Figure 6: Gammas for bounds versus S_0. Heston (1993) model. $K = 100$. $\rho = -0.1, -0.53, -0.9$. $\Delta = 0$ (dashed line), $\Delta = 2$ (dotted lines), and $\Delta = 4$ (solid lines). The left and right panels are for h_1^* and h_2^*, respectively. $\alpha = 0.097,\ \beta = 7.1,\ \sigma = 0.32,\ s = 8.6,\ v = v* = -22.6$. The remaining parameters are set as $r = 0.05,\ V_0 = \alpha/\beta = 0.0137,\ T = 0.25$.

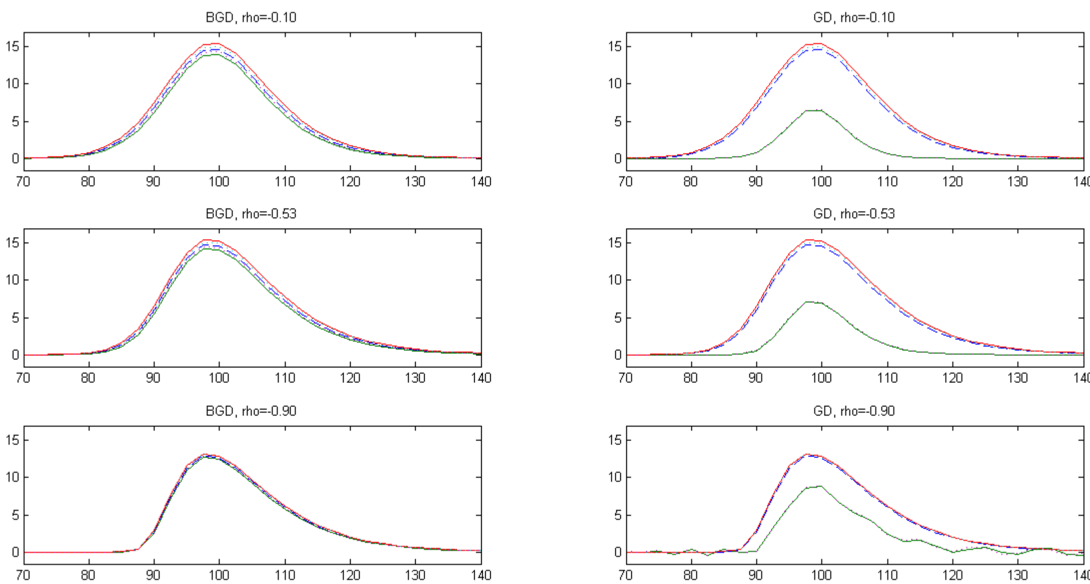

Figure 7: Vegas for bounds versus S_0. Heston (1993) model. K=100. $\rho = -0.1, -0.53, -0.9$. $\Delta = 0$ (dashed line), $\Delta = 2$ (dotted lines), and $\Delta = 4$ (solid lines). The left and right panels are for h_1^* and h_2^*, respectively. $\alpha = 0.097,\ \beta = 7.1,\ \sigma = 0.32,\ s = 8.6,\ v = -22.6, v^* = -22.6$. The remaining parameters are set as $r = 0.05,\ V_0 = \alpha/\beta = 0.0137,\ T = 0.25$.