DECOMPOSITION OF THE CONDITIONAL ASSET RETURN DISTRIBUTION

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We estimate the conditional asset return distribution by modelling a finite number of quantiles. The motivation for this is to jointly incorporate time - varying dynamics of shape and scale of the asset return distribution in a robust manner. Additionally, we want to address any violations of the quantile orderings when estimating such models.

PRELIMINARY AND INCOMPLETE: PLEASE DO NOT QUOTE

1. INTRODUCTION

In this paper we jointly model selected quantiles of the asset return distribution, standardised by the inter - quartile range (IQR). By this, we are able to model both the scale and the shape of the conditional return distribution in a robust way. Our aim is to decompose the conditional distribution by using quantile regression, in particular a CAViaR model, see Engle and Manganelli (2004), for each of a set of quantiles.

Estimation of the conditional distribution of asset return y_t is of great importance. In a parametric dynamic framework where the return is modelled as $y_t = \mu_t + \sigma_t \epsilon_t$, we first estimate either the conditional mean $\mu_t = E(y_t|F_{t-1}(x))$ or the conditional variance $\sigma_t^2 = E((y_t - \mu_t)^2|F_{t-1}(x))$, and then recover the conditional distribution. The shock ϵ_t is drawn from a specified distribution.

This methodology assumes that the standardised error $\hat{\epsilon}_t = \frac{y_t - \mu_t}{\sigma_t}$ is independent from the information $F_{t-1}(x)$ set in the past, Engle (1982). The validity of this assumption is criticised by many authors in literature. This is mainly because the distribution of returns is characterised by skewness and kurtosis. Therefore, assuming that the only features of the conditional distribution which depend upon the conditioning information are the mean and variance is not justified, Hansen (1994). Thus, in many empirical studies other moments of the distribution are considered, such as the conditional skewness and kurtosis.

The accurate estimation of the conditional distribution is essential in many cases where the aim is to predict the risk in financial returns i.e. VaR forecasting, asset pricing etc. Thus, in order to obtain an accurate estimation of the conditional asset return distribution we need to account for richer dynamics.

Some examples in the literature consider parametric approaches (Gallant et al. (1991)) and non parametric approaches (Engle and Gonzalez-Rivera (1991)) for the estimation of the conditional asset return distribution. Although the approaches differ in terms of the underlying assumptions, they all consider a constant distribution for the error term. This implies that the shape of the distribution is not allowed to vary through time. Hansen (1994) accounts for both time - varying shape and skewness in the conditional distribution. Harvey and Siddique (1999) study the conditional skewness of asset returns, by explicitly modelling the conditional second and third moments jointly in a parametric framework.

The approach of modelling the quantiles of the distribution directly has been shown to be a robust approach in cases where non - normality holds, as in the case of asset returns or in cases where the aim is to fit the tails of the distribution. Engle and Manganelli (2004), Chernozhukov and Umantsev (2001), Chen et al. (2011) for example use the CAViaR formulation of the regression quantile criterion (Koenker and Bassett (1978)), and find that this is successful in estimating the Value at Risk (VaR).

The joint estimation of multiple quantiles is a natural way to represent the distribution, and comes with the added advantage that the problem of incorrectly ordered quantiles can be dealt with. By modelling the quantiles jointly constraints are imposed by construction and the crossing problem is directly addressed. By using a single quantile model one might end up with an estimate for the 1% that is higher than the 5% quantile for example. This violates the correct ordering of the quantiles, see Chernozhukov et al. (2008), and Chernozhukov et al. (2010) for some improvements on the single quantile functions so that they do not cross.

By extending the single quantile and combining quantile estimates at different probability levels we are able to use valuable and different information provided from the different sides of the distribution. White et al. (2008) provided results for the Multi - Quantile (MQ) CAViaR model, whereby several quantiles are jointly estimated to obtain time - varying indicators of skewness and kurtosis.

However, the estimation of the MQ CAViaR is a difficult procedure, possibly more so because each quantile being modelled is assumed to depend linearly on all other quantiles. Here, we attempt an alternative link between the individual quantiles, that is the scale of the distribution given by the time - varying IQR at time t.

IQR is the difference between the 75% and the 25% quartile. It is a robust measure of scale and therefore useful for modelling asset returns which are found to be skewed with heavy tails relative to the Normal distribution. Similar to decomposing the return y_t into $\sigma_t \epsilon_t$, where σ_t is the conditional standard deviation and ϵ_t a shock drawn from a specified distribution, our approach involves standardising the quantiles by the estimated time - varying IQR. By this, fat - tails should be reduced and the dynamics of the shape are separated from the scale dynamics.

Combining quantile estimates is not new in literature. Granger et al. (1989) and Granger and Ramanathan (1984) modelled the time varying dynamics of the shape by combining quantile forecasts.

There is also an extensive literature on combining shape and scale dynamics of the distribution. Xiao and Koenker (2009) for example estimate GARCH models by using quantile regression in a two - step approach. In particular, in the first step they employ a quantile autoregression sieve approximation for the GARCH model by combining information over different quantiles. In the second step a GARCH model is obtained based on the first stage minimum distance estimation of the scale process of the time series.

In a later study Jeon and Taylor (2013) use quantile models with implied volatility for VaR estimation in order to use information provided by both quantile models and information supplied by the market's expectation of risk. Chen and Gerlach (2014) use the intra - day sources of data to capture the dynamic volatility (scale) and tail risk (shape) of the conditional distribution. In particular, they implicitly model the Expected Shortfall (ES) by an autoregressive expectile class of model. Cai and Xiao (2012) study quantile regression estimation for dynamic models in a three - step semi - parametric procedure. They assume that some coefficients are functions of informative covariates and thus they are partially varying coefficients.

Our contribution is to use information provided by quantile forecasts at different probability levels jointly with the dynamics of IQR. By standardising by the estimated IQR simultaneously we are able to separate and study the time - varying dynamics of the shape and scale. In terms of estimation, we estimate the models using the CAViaR formulation of the regression quantile criterion (Koenker and Bassett (1978), Engle and Manganelli (2004), Chernozhukov and Umantsev (2001)). We choose to use the CAViaR formulation because it offers a general framework for modelling various forms of non - i.i.d. (independent and identically distributed) error distributions where both error densities and volatilities are non - constant, Engle and Manganelli (2004). This is our starting point as we seek a robust way to model both the scale and the shape of the conditional asset return distribution.

Various specifications of the joint quantile model with IQR are given. These are compared with a simple model that jointly estimates multiple quantiles but not IQR. The comparisons are made in terms of in - sample fitting and out - of - sample forecasting. In particular, comparisons are conducted graphically, in terms of the regression quantile criterion (RQ), and back - testing criteria. The proposed models are also able to obtain accurate forecasts at 1% VaR.

The paper is structured as follows. Section 2 introduces the joint quantile models. Section 3 reviews the literature on regression quantile estimation, consistency and asymptotic normality of the estimator. Section 4 discusses the estimation method, and section 5 presents an empirical application to real data. Section 6 concludes the paper.

2. THE MODEL

We decompose the asset return distribution by separating the dynamics of the shape (quantiles) and scale (IQR) in a semi - parametric framework. By this, we are able to model the asset return distribution, and identify the type of departures during different periods such as those of high / low volatility, and skewness.

This setting is superior to traditional approaches for single quantile in terms of forecasting and better explaining the evolution of the tails of the distribution. We base quantile estimation on a finite sample of quantiles of the left and the right side of the distribution that are estimated jointly with the time - varying IQR and standardised by IQR. Let the θ- quantile at time t, $q_{\theta,t}$ be modelled as

$$q_{\theta,t} = IQR_t \left(u_\theta + \sum_{i=1}^{p} \beta_{\theta,i} \frac{q_{\theta,t-i}}{IQR_{t-i}} + \sum_{i=1}^{p} \frac{\ell(\mathbb{F}_{t-1}; \gamma_{\theta,i}, \dots, \gamma_{\theta,p})}{IQR_{t-i}} \right) \tag{1}$$

for $\theta = 0.99, 0.95, 0.25, 0.05, 0.01$, where F_{t-1} is the information set up to and including time $t-1$, and $\ell(.)$ is a possibly non - linear function. u_θ is the intercept of the quantile, $\beta_{\theta,i}$ is the autoregressive parameter, and $\gamma_{\theta,i}$ is the parameter on the lagged values of returns, quantiles etc. Let the time t- IQR be modelled as

$$IQR_t = u + \sum_{i=1}^{p} \beta_i IQR_{t-i} + \sum_{i=1}^{p} \ell(\mathbb{F}_{t-1}; \gamma_i, \dots, \gamma_p), \tag{2}$$

and

$$q_{0.75,t} = IQR_t + q_{0.25,t}. \tag{3}$$

2.1. JOINT QUANTILE SPECIFICATIONS

Next, we discuss some examples of joint quantile processes with IQR that we estimate. As a first example we model the dynamics of one of the quartiles (25%) and IQR, inferring the counterpart quartile (75%) from the two quantities by using a Symmetric Absolute Value (SAV) process as in Engle and Manganelli (2004) to obtain the quantiles and IQR. Thus, the Joint SAV IQR model (J-SAV-IQR) is given by

- J-SAV-IQR

$$IQR_t = u + \sum_{i=1}^{p} \beta_i IQR_{t-i} + \sum_{i=1}^{p} \gamma_i |y_{t-i}|$$

  and

$$q_{0.75,t} = IQR_t + q_{0.25,t}$$

  and

$$q_{\theta,t} = IQR_t \left(u_\theta + \sum_{i=1}^{p} \beta_{\theta,i} \frac{q_{\theta,t-i}}{IQR_{t-i}} + \sum_{i=1}^{p} \gamma_{\theta,i} \frac{|y_{t-i}|}{IQR_{t-i}} \right),$$

  for $\theta = 0.99, 0.95, 0.25, 0.05, 0.01$.

The strong asymmetries detected in empirical studies suggest that negative returns are more likely to cause higher increases in market risk than positive ones. A natural way of considering this is to use an Asymmetric Slope (AS) process process as in Engle and Manganelli (2004) for modelling the quantiles and the IQR. Thus, the Joint AS IQR (J-AS-IQR) model is given by

- J-AS-IQR

$$IQR_t = u + \sum_{i=1}^{p} \beta_i IQR_{t-i} + \sum_{i=1}^{p} \gamma_i y^{+}_{t-i} + \sum_{i=1}^{p} \delta_i y^{-}_{t-i},$$

$$q_{0.75,t} = IQR_t + q_{0.25,t}$$

  and

$$q_{\theta,t} = IQR_t \left(u_\theta + \sum_{i=1}^{p} \beta_{\theta,i} \frac{q_{\theta,t-i}}{IQR_{t-i}} + \sum_{i=1}^{p} \gamma_{\theta,i} \frac{y^{+}_{t-i}}{IQR_{t-i}} + \sum_{i=1}^{p} \delta_{\theta,i} \frac{y^{-}_{t-i}}{IQR_{t-i}} \right),$$

  for $\theta = 0.99, 0.95, 0.25, 0.05, 0.01$. Here $y^+ = \max(y, 0)$ and $y^- = -\min(y, 0)$.

We also consider estimating the dynamics of all the quantiles and imputing IQR as the difference between the first and the third quartile. The 25% and 75% quartiles may follow a SAV process. Thus, the Joint SAV difference (J-SAV-diff) model is given by

- J-SAV-diff

$$q_{0.75,t} = u_{0.75} + \sum_{i=1}^{p} \beta_{0.75,i} q_{0.75,t-i} + \sum_{i=1}^{p} \gamma_{0.75,i} |y_{t-i}|,$$

$$q_{0.25,t} = u_{0.25} + \sum_{i=1}^{p} \beta_{0.25,i} q_{0.25,t-i} + \sum_{i=1}^{p} \gamma_{0.25,i} |y_{t-i}|,$$

$$IQR_t = q_{0.75,t} - q_{0.25,t}.$$

The remaining quantiles are given by

$$q_{\theta,t} = IQR_t \left(u_\theta + \sum_{i=1}^{p} \beta_{\theta,i} \frac{q_{\theta,t-i}}{IQR_{t-i}} + \sum_{i=1}^{p} \gamma_{\theta,i} \frac{|y_{t-i}|}{IQR_{t-i}} \right),$$

for $\theta = 0.99, 0.95, 0.05, 0.01$.

We also propose the Joint component AS IQR model (J-C-AS-IQR). This specification uses a two component process to model IQR in order to account for a slow moving component. Thus, we replace u with a time - varying process that induces a long memory property to the IQR and allows for smooth adjustments to the level of the IQR under different market conditions. The deviation $IQR_{t-1} - u_{t-1}$ is the component that represents an adjusted distance from the unconditional mean. The dynamics of u_t capture the dependence in $\overline{IQR}$ (unconditional mean of IQR), albeit with an adjusted mean level. Overall, we introduce a long memory feature in the IQR_t process similar to that in the component GARCH models, see Engle and Lee (1999). Thus,

- J-C-AS-IQR

$$IQR_t = u_t + \sum_{i=1}^{p} \beta_i (IQR_{t-i} - u_{t-i}) + \sum_{i=1}^{p} \gamma_i y_{t-i}^{+} + \sum_{i=1}^{p} \delta_i y_{t-i}^{-},$$

$$u_t = \alpha + \sum_{j=1}^{r} \beta_j u_{t-j} + \sum_{j=1}^{r} \gamma_j y_{t-j}.$$

The quantiles are given by

$$q_{\theta,t} = IQR_t \left(u_\theta + \sum_{i=1}^{p} \beta_{\theta,i} \frac{q_{\theta,t-i}}{IQR_{t-i}} + \sum_{i=1}^{p} \gamma_{\theta,i} \frac{|y_{t-i}|}{IQR_{t-i}} \right),$$

for $\theta = 0.99, 0.95, 0.25, 0.05, 0.01$, and $q_{0.75,t} = IQR_t + q_{0.25,t}$.

3. PARAMETER ESTIMATION OF THE JOINT QUANTILE MODEL

The parameters of the model in equations 1, 2, 3 are obtained by Quasi - Maximum Likelihood Estimator (QMLE). The estimator is the solution of

$$- \min \frac{1}{T} \sum_{t=1}^{T} \sum_{\theta} [\theta - I(y_t < q_{\theta,t})][y_t < q_{\theta,t}], \tag{4}$$

where T is the sample size, and I is an indicator function. The objective function allocates different weights at different parts of the distribution according to whether or not the inequality $y_t < q_{\theta,t}$ holds.

The log - likelihood

$$- \sum_{\theta} [\theta - I(y_t < q_{\theta,t})][y_t < q_{\theta,t}]$$

is the log - likelihood of a vector of θ independent asymmetric double exponential random variables, see Komunjer (2005). There is no need to impose any distributional assumption in order to solve the above minimization problem. The methodology acquired here is semi - parametric and uses the Quasi - Maximum Likelihood (QML). Thus, we do not need to assume that $y_t - q_{\theta,t}$ follows an asymmetric double exponential distribution. White et al. (2008) establish the consistency of the estimator following Powell (1984) along with the asymptotic normality, using a method as in Huber (1967) and Weiss (1991). A problem that remains unsolved in the quantile setting is that the estimator is not asymptotically efficient. Komunjer and Vuong (2006), and Komunjer and Vuong (2010) used a tick exponential family in order to restore the efficiency asymptotically.

4. ESTIMATION

In this section we discuss the estimation method that we used in order to estimate the joint quantile model with IQR. Our model is similar to the model proposed by White et al. (2008) in the sense that they both combine several quantiles which are estimated jointly but it differs in the links used for building the dependencies between quantiles. The MQ - CAViaR model assumes that the quantiles are linearly dependent on all other quantiles, whereas we attempt an alternative link between the individual quantiles using the IQR.

We estimate the parameters of the joint quantile model with IQR by using a different algorithm than that in White et al. (2008). White et al. (2008) conduct the computations in a step - wise fashion, where they first estimate the MQ - CAViaR model containing only the 2.5% and 25% quantiles. The starting values for the optimization are obtained by the single CAViaR estimates, and the remaining parameters are initialised at zero. However, this procedure might propose initial values that allow for quantile crossings from the beginning of the algorithm, because single CAViaR models are found to produce crossings. This procedure is repeated for the 75% and 97.5% quantiles, as well.

In a second step, the parameters obtained at the first step are used as starting values for the optimization of the MQ - CAViaR model containing two more probability levels, the 75% and 97.5% quantiles, initializing the remaining parameters at zero. In the last step, they use the estimates from the second step as starting values for the full MQ - CAViaR model optimization containing all the quantiles of interest, again setting to zero the remaining parameters. Such a procedure might be computationally expensive.

In addition, the authors notice that the choice of initial conditions is crucial as the optimization procedure is sensitive to them resulting in a quite flat likelihood function around the optimum. Thus, they find that when choosing different combinations of quantile couples in the first step of the estimation procedure tends to produce different parameter estimates for the model.

In order to avoid complex roots in the joint quantile model parametrization we set some reasonable starting values. By this we mean that the autoregressive roots (AR) should lie in the unit root, in order to allow for stationarity. Also, the coefficient for the intercept should be negative for the left - tail quantiles, and positive for the right - tail quantiles, accordingly the parameters on the returns should be negative or positive depending on the side of the distribution they belong.

Thus, initial values for the parameters are chosen according to the properties of the model each time. We allow for a range of sensible initial values for the parameters of the models. The starting values for the quantiles and the IQR are chosen to be the empirical quantiles and IQR for the first 300 observations, respectively. Given these, we target the empirical (unconditional) quantile of the return series. In the case of J-AS-IQR model the intercept becomes

$$u_\theta = \sum_{i=1}^{p}(1-\beta_{\theta,i})\overline{q}_\theta - \left(\sum_{i=1}^{p}\gamma_{\theta,i}\overline{y}^{+} + \sum_{i=1}^{p}\delta_{\theta,i}\overline{y}^{-}\right),$$

where $\overline{q}_\theta = E[\frac{q_{\theta,t-i}}{IQR_{t-i}}]$ is the standardised quantile at θ probability level, and

$\overline{y}^{+} = E[\frac{y^{+}_{t-i}}{IQR_{t-i}}]$, $\overline{y}^{-} = E[\frac{y^{-}_{t-i}}{IQR_{t-i}}]$ is the expectation of the positive and negative standardised returns respectively. In particular, we assume that $y_t = \sigma_t\epsilon_t$ for $\epsilon_t \sim N(0,1)$, where $\overline{y}^{+} = E[\frac{\sigma|\epsilon^{+}|}{1.349\sigma}]$ and $\overline{y}^{-} = E[\frac{\sigma|\epsilon^{-}|}{1.349\sigma}]$.

We choose the standardised quantile to be equal to $N^{-1}(\theta,0,1)$, where N^{-1} is the inverse distribution of the standard Normal distribution. Accordingly we choose the empirical (unconditional) quantile of the return series for the remaining models that we introduced. The targeted empirical quantiles are used only as an input to the optimizer.

We also consider a model according to which the quantiles are not standardised by the IQR. We call this model Joint SAV (J-SAV) because it uses a SAV process for estimating the quantiles,

$$q_{\theta,t} = u_\theta + \sum_{i=1}^{p}\beta_{\theta,i}q_{\theta,t-i} + \sum_{i=1}^{p}\gamma_{\theta,i}|y_{t-i}|,$$

for $\theta = 0.99, 0.95, 0.75, 0.25, 0.05, 0.01$.

The starting values of the quantiles for this model are chosen as the empirical quantile of the first 300 observations at θ probability level. The intercept of the quantile is given by

$$u_\theta = \sum_{i=1}^{p}(1-\beta_{\theta,i})\overline{q}_\theta - \sum_{i=1}^{p}\gamma_{\theta,i}\overline{|y|},$$

where $\overline{q}_\theta$ is the unconditional quantile and $\overline{|y|}$ is the unconditional mean of the absolute returns.

By this procedure we are able to obtain appropriate initial values for the optimiser and improve chances of finding a global minimum. We do not reduce one parameter as is done in GARCH models for variance targeting.

Given a small range of initial values and starting points, we conduct a grid search and we choose the parameters that minimize (fminsearch) equation (4).

5. EMPIRICAL ANALYSIS AND RESULTS

We estimate the models on a set of stocks and indices using a long time horizon of 12 years. We analyse FTSE trading on the London Stock Exchange, NASDAQ trading on the NASDAQ Stock Market, Standard and Poor's 500 (SP500), International Business Machines (IBM), Walt Disney company (DIS), Caterpillar Inc. (CAT), Dow Chemical company (DOW), and Boeing company (Boeing) trading on the New York Stock Exchange.

The data set ranges from 01 January 2002 to 14 November 2014. We divided the data into two samples where the first one is used for the estimation of the model and the second sample, consisting of 500 observations which correspond to 2 financial years, for the out - of - sample testing. Zero returns were removed and the sample sizes differ across the assets. The starting dates, the in - sample dates, and the sample sizes for the different assets are presented in table 1 along with some summary statistics of the data. All assets have positive return median, and mean close to 0. All series have negative skewness except for Boeing and DIS. All series have also fat tails.

In figures 1 through 16, and in tables 2 through 16 we present the results. In figures 1 through 16, we present the estimated conditional quantiles $q_{\theta,t}$ and the corresponding standardised quantiles by IQR $\hat{q}_{\theta,t} = \frac{q_{\theta,t}}{IQR_t}$ for all the assets under study. In the figures we use various colours in order to depict the quantiles at different probability levels. In particular, we choose magenta to depict the quantiles at 99%, for all the plots that follow. Yellow is chosen to illustrate the quantile at 95%. Green and black are used for the quartiles at 75% and 25%, respectively. We use red colour to depict the 5% quantile, and light blue for the 1% quantile.

In table 2 we give the RQ criterion that corresponds to the whole sample (both in - sample and out - of - sample) for all the assets and models under study. In tables 3 through 10 we provide the ratio of violations to the length of the testing period for the joint quantile models both in - sample and out - of - sample. Finally, in tables 11 - 16 we give the test statistics for the models. In particular, we test the models for the out - of - sample period by the Likelihood Ratio (LR) test by Christoffersen (1998) and the Dynamic Quantile (DQ) test by Engle and Manganelli (2004).

Time - varying shape and scale

In Figures 1- 16, we show that by modelling IQR jointly with the quantiles (standardised by IQR) we are able to capture the time - varying scale, leaving few outliers. In addition, the tails are clearly more volatile than the inner quantiles which correspond to the main body of the distribution and the 5% and 95%. This suggests that the shape of the tail at the extreme levels is influencing the time variation in the shape of the distribution the most.

In addition, it is clear that the models fit the data well apart from a few cases (CAT for J-AS-IQR, NASDAQ for J-AS-IQR, and DIS for J-SAV-IQR) and after standardising the quantiles by IQR the time - varying scale is removed leaving the shape evolving through time.By this we are able to study the evolution of the shape. This was one of our initial motivations for modelling the quantiles jointly with IQR. In particular we wanted to study how the shape evolves after removing the scale.

It worth mentioning that in figure 16, and in particular the lower panel (J-AS-IQR), the evolution of the quartiles is somewhat different from that of the other models / assets. More specifically, it seems that both the quartiles are driven by the IQR_t process as the spike at the beginning of the in - sample period is driven downwards and its magnitude is high although it represents the main body of the distribution. This can be explained by a highly persistent IQR, which drives both the processes in the case of DIS.

Figures 1- 16 also show that the proposed models do not produce crossings for both the in - sample fit and out - of - sample forecasting period. We show that we were able to address the crossing problem, as there are no crossings and the quantiles are correctly ordered across all the assets and models. This result was mainly due to the initial values that we chose to use for feeding the optimisation algorithm, and due to the fact that we model jointly the quantiles in an one - step procedure, where by construction the correct ordering is taken into account, as the link between the quantiles is the time - varying scale.

Another important finding is that the parameters of the models are reasonable in the sense that they are complying with the theoretical concept of the quantiles. In figures 1- 16 we show that the intercept of the quantiles is negative when the left side of the distribution is estimated, and positive in the case of the right tail. The autoregressive parameters ensure stationarity for all the quantiles, IQR is highly persistent while others are not. Parameters on the returns of positive quantiles are positive, increasing the quantile each time while parameters on the returns of negative quantiles are negative, decreasing it.

In and out - of - sample performance

Table 2 gives the value of the minimisation problem for all the models that estimate quantiles jointly with IQR, and J-SAV which is a simpler model that is chosen for comparison reasons. In particular, we want to point out why using a model which accounts for both the scale and the shape of the distribution is more beneficial for estimating and forecasting purposes. In almost all cases (with the exceptions of Boeing and DOW for J-SAV-diff, and NASDAQ for J-SAV-IQR) J-SAV produces the highest value compared to the RQ obtained by joint quantile models with IQR across different assets, implying that joint quantile models with IQR were able to succeed a lower RQ criterion.

Back - testing results of the joint quantile models with the IQR indicate that the in - sample exceedances obtained by all specifications are very close to the critical value at all the probability levels. This finding suggests that the models fit well the data.

In terms of out - of - sample forecasting, results differ among data. In the case of IBM, J-SAV seems to work very well with the exception of the 95% where it provides with estimates that imply that the model overestimates. J-SAV-diff and J-SAV-IQR do a really good job for IBM, but at 1% there are some biases. For the models that build on an AS process the picture is different for IBM: they tend to overestimate / underestimate the body, still obtaining the desired IQR, and they predict accurately the tails of the distribution, resulting in exactly the targeted value in the case of J-AS-IQR.

In predicting quantiles the J-SAV model for SP500 has a tendency to overestimate / under - estimate quantiles, leading to out - of - sample coverages below / above the targeted probability level depending on the side of the distribution. As for the remaining models for SP500, J-SAV-

IQR, J-AS-IQR, and J-C-AS-IQR forecast well, although the quartiles are not close to the value that is targeted each time. In particular, the difference of the quartiles is always close to the desired value making these models appropriate for forecasting.

The picture for the other assets is similar to that of SP500. Coverages that correspond to the body and the tails of the distribution are quite accurate with some trade off between the quantiles, as the link that we use to model them is dynamic. Overall, the models that seem to be superior in terms of forecasting are the models that first estimate the dynamics of the 25% quartile and IQR, inferring the 75% quartile from the two quantities. Also the model that uses a two component process to model IQR in order to account for a slow moving component, seems to do a good job in forecasting the violations out - of - sample. This model is richer in the sense that not only accounts for the shape and scale dynamics, but also addresses the term structure of the IQR. CAT is the only asset for which all the models fail as they tend to underestimate the 25%, 5%, and 1% quantiles.

It worth mentioning that J-SAV is a simple model in the sense that does not account for the time - varying scale of the distribution, almost in all cases produces biases out - of - sample at all probability levels, even those that belong to the right tail of the distribution and they are supposed to be easier to estimate. With the exception of IBM, this model over - estimates the right tail of the distribution and under - estimates the left tail. This finding is in accordance with our initial intuition that the use of a richer model, still parsimonious, that accounts for both the scale and the shape of the distribution should be able to describe the evolution of the distribution more accurately.

The model performs quite poorly for all the probability levels because it overestimates the right side of the distribution, and underestimates the left side of the distribution. This holds for all the assets under study. There are some exceptions when this model predicts well the 1% (IBM, CAT, FTSE, NASDAQ, DIS) and at 5% (IBM, DOW, NASDAQ), but these are rather random and do not prove the model's adequacy in forecasting the conditional quantiles.

Sometimes the entire distribution is not the quantity of interest, but a specific quantile of the distribution. For instance, we may be interested in estimating and forecasting the VaR, which is the 1% quantile of the conditional asset return distribution. The results for the 1% VaR show that joint quantile models estimated and standardised by IQR do a good job describing the evolution of the left tail. The results are very accurate for J-SAV-IQR, J-AS-IQR and J-C-AS-IQR for most of the assets. In many cases the models produce exactly the targeted hits, which is very important as we know that it is very hard to predict the left tail of the conditional asset return distribution because of its statistical properties.

Summing up, the out - of - sample quantile prediction results produced by the joint quantile models are satisfactory. The simple model is able to fit the data but out - of - sample does poorly. The joint quantile models with IQR, on the other hand, perform very well both - in - sample and out - of - sample. This makes them a robust alternative in estimating and forecasting the quantiles of the conditional distribution.

Tests

In tables 11- 16 we present the test statistics for the models. We use $-$ in order to show that a test statistic was not possible to be computed because of the sample size. We choose to use the LR test by Christoffersen (1998). For this test it is well known that it requires some out - of - sample exceedances in order to be defined, and not lead to multicollinearity issues. However, the LR test statistic is not always possible to be computed because of the fact that there might not be exceedances within the forecasting period. Thus, in order to be able to compare the different models in cases where there no exceedances, we also employ the DQ test by Engle and Manganelli (2004).

Another reason for using these tests is because it is easy to check the criterion for the out - of - sample forecasts using one of them, as none of them is making assumptions about the underlying data generating process. More specifically, in order to construct the above tests one needs the information set in the past that consists of the hit sequence $[I(y_{t-1} < q_{\theta,t-1}), \ldots, I(y_1 < q_{\theta,1})]$.

LR test checks both the general criterion of goodness for an out - of - sample forecast of quantile series, and model misspecification. For the test of conditional coverage a LR testing framework is required in order to obtain the LR-uc unconditional coverage test, the LR-i independence coverage test, and the LR-cc conditional coverage test.

Back - testing of the models is also carried out using the DQ out - of - sample test to test and compare the performance of the joint quantile models with IQR. This particular test is one of the standard tests to compare CAViaR models. The null hypothesis of this model is given by

$$E[Hit_t|F_{t-1}] = 0,$$

where the conditional expectation of the hits $Hit_t = \theta - I(y_t < q_{\theta,t})$ should be zero so that the null hypothesis is not likely to be rejected. Thus, the quantiles are estimated correctly, if independently for each day of the forecasting period the probability of exceeding it equals θ and the sequence of Hit_t is uncorrelated with its own lagged values. Under the null hypothesis, the test statistic is asymptotically X^2 distributed with N degrees of freedom.

In - sample and out - of - sample coverages should be close to the underlying probability level each time. If the p-value of the DQ test is larger, this indicates that the null hypothesis of independent quantile exceedances is more likely not to be rejected, suggesting that a model is more appropriate. On the other hand if the p-value is smaller (compared to 1% significance level), then this implies that the null hypothesis is more likely to be rejected, suggesting that the quantile exceedances are not independent and the model is less adequate.

Let's turn to the results for the DQ test. According to the results, IBM passes the test for all the models, therefore we can not reject the null hypothesis at 1%. The same findings hold for the LR test, whenever this was possible to be computed. These two models agree in the case of IBM.

If we turn now to SP500 results differ. The two tests only agree for J-SAV-IQR. For the remaining models the results vary among the tests. For CAT the two tests agree for J-SAV, but not for the other models that we study. DOW passes the LR test at all probability levels for all the models, but does not pass the DQ test at 1% for J-SAV, J-SAV-diff, and J-SAV-IQR.

In the case of NASDAQ the two models agree and reject J-SAV-IQR, and J-AS-IQR at 25%, but LR rejects also the remaining models at 25%, and J-C-AS-IQR at 95% . In the case of DIS, although we can not reject the models with the DQ test at 1% confidence interval given the p-values provided, we can reject the 25% quartile by J-SAV-diff, J-SAV-IQR, J-AS-IQR, and J-C-AS-IQR. DQ test results suggest that the models are appropriate in adequately estimating the exceedances, as the DQ test p-value is in most cases higher than 1%, indicating that the null hypothesis of the DQ test cannot be rejected. In contrast to the findings about the DQ test, our analysis suggests that some of the models are rejected by LR tests.

6. CONCLUSION

This paper presents a joint quantile model estimated and standardised by IQR. Joint quantile models estimated with IQR provide evidence for being able to capture dynamics that are consistent with the concept of time - varying risk.

Extreme quantiles at the tails are more volatile than the quantiles of the main body, as expected. In line with prior evidence asymmetry parameters suggest that negative returns are more likely to cause higher increases in the left tail of the distribution than positive returns, this is stronger for more extreme quantiles.

We are able to analyse the dynamics of both the scale and the shape which would be difficult to capture using traditional models, and address the crossing problem.

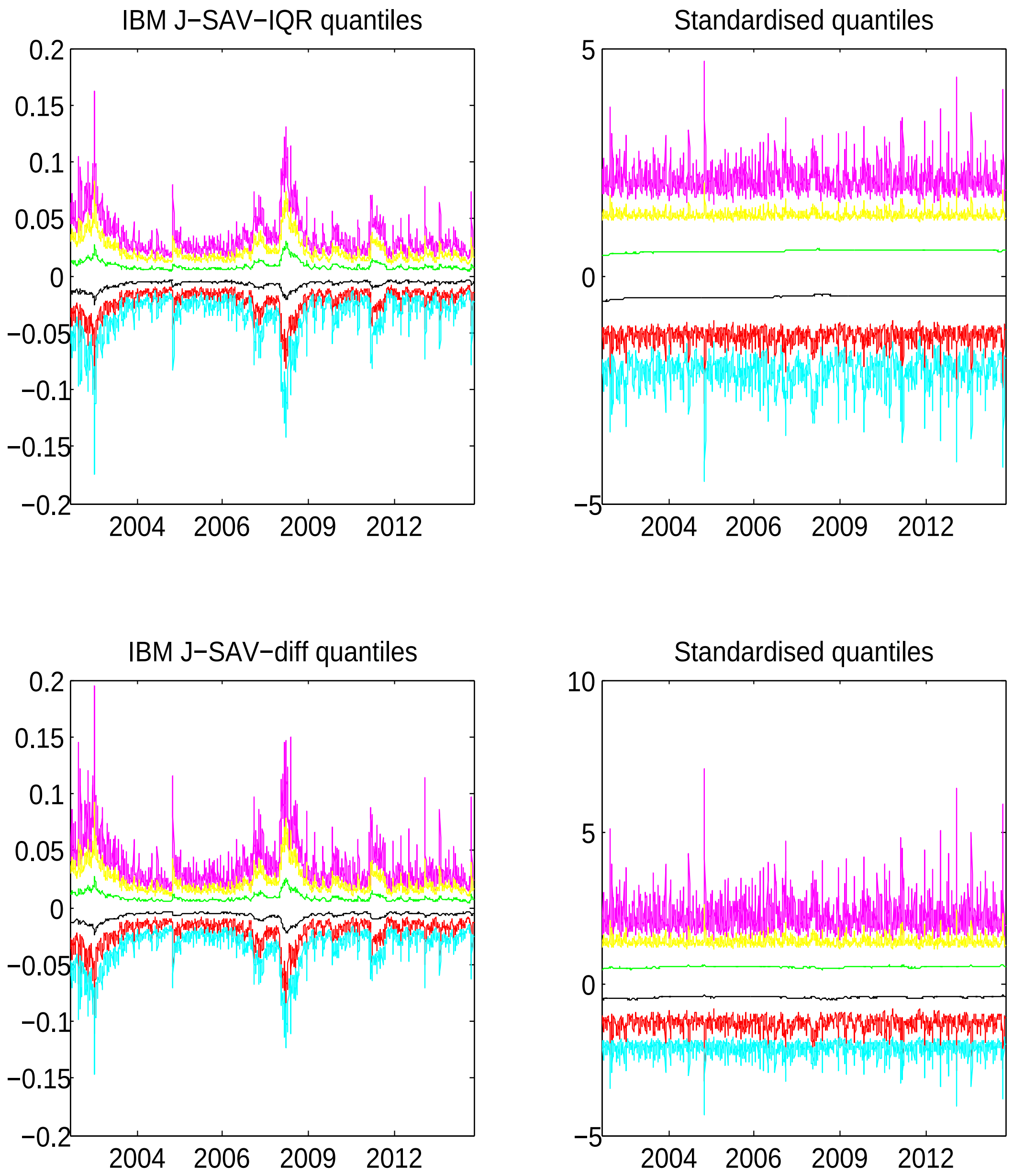

Figure 1: Estimated $q_{t,\theta}$ and standardised $\hat{q}_{t,\theta}$ by IQR_t: J-SAV-IQR quantiles (upper panel) and J-SAV-diff quantiles (lower panel) for IBM.

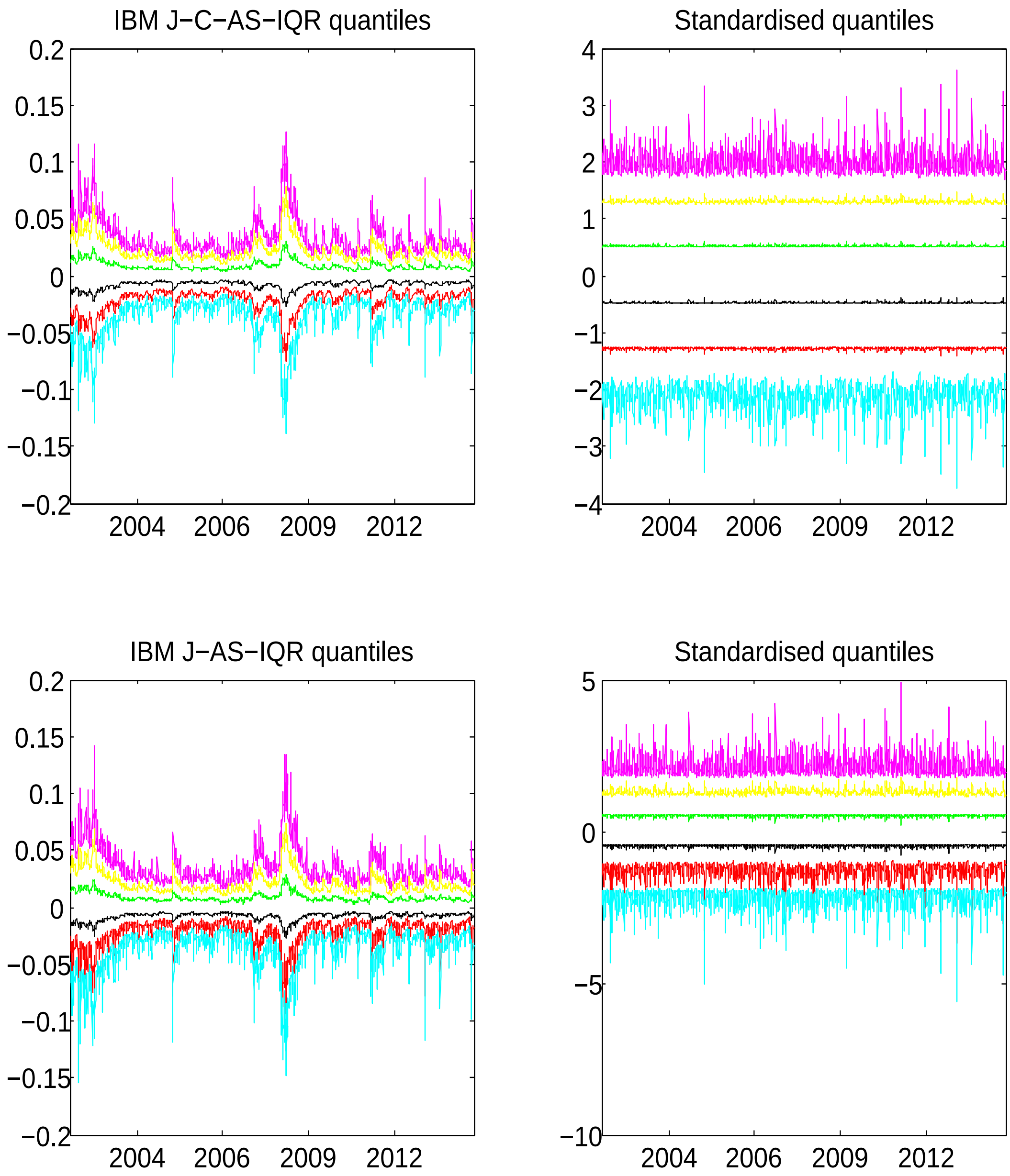

Figure 2: Estimated $q_{t,\theta}$ and standardised $\hat{q}_{t,\theta}$ by IQR_t: J-C-AS-IQR quantiles (upper panel) and J-AS-IQR quantiles (lower panel) for IBM.

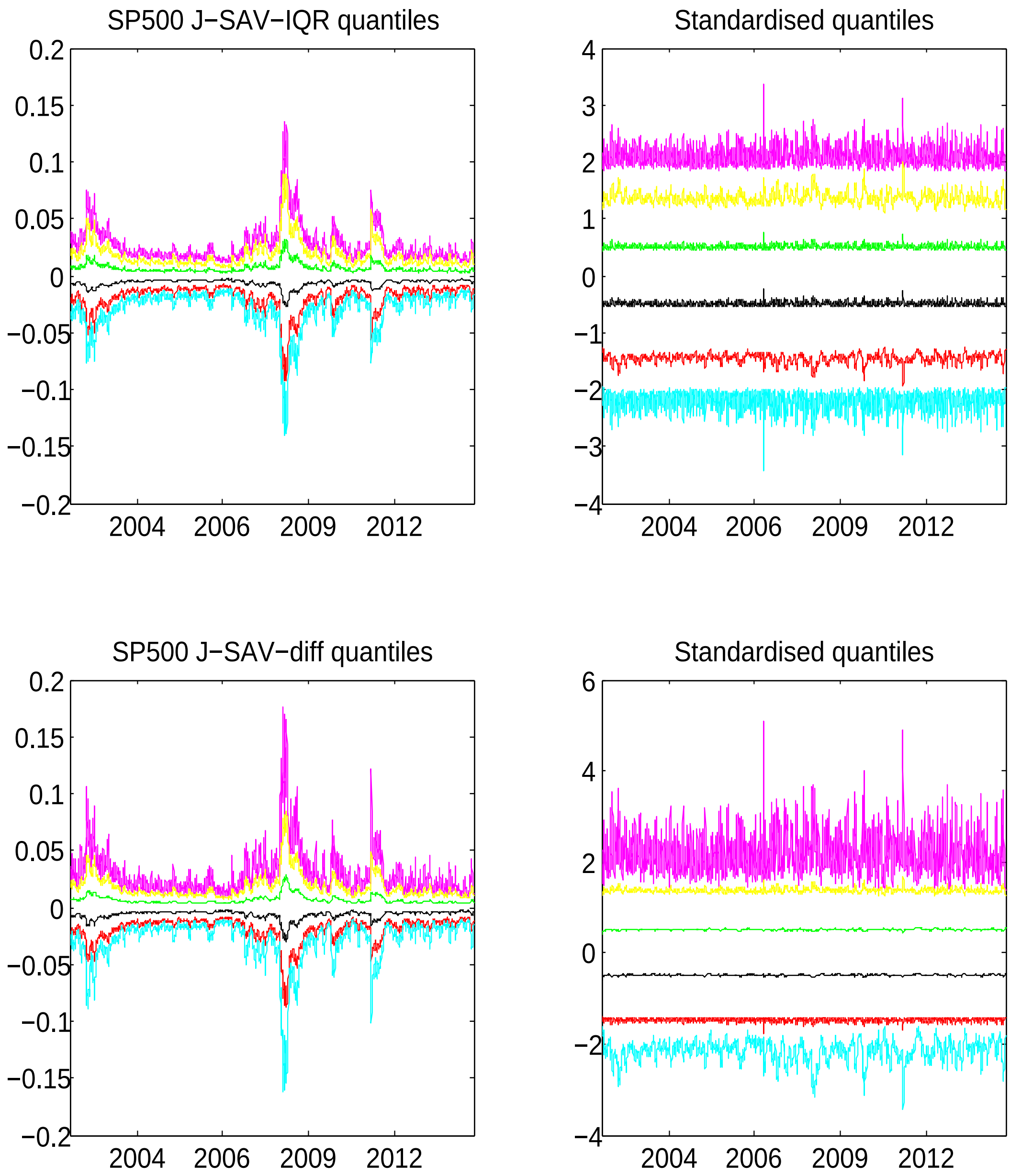

Figure 3: Estimated $q_{t,\theta}$ and standardised $\hat{q}_{t,\theta}$ by IQR_t: J-SAV-IQR quantiles (upper panel) and J-SAV-diff quantiles (lower panel) for SP500.

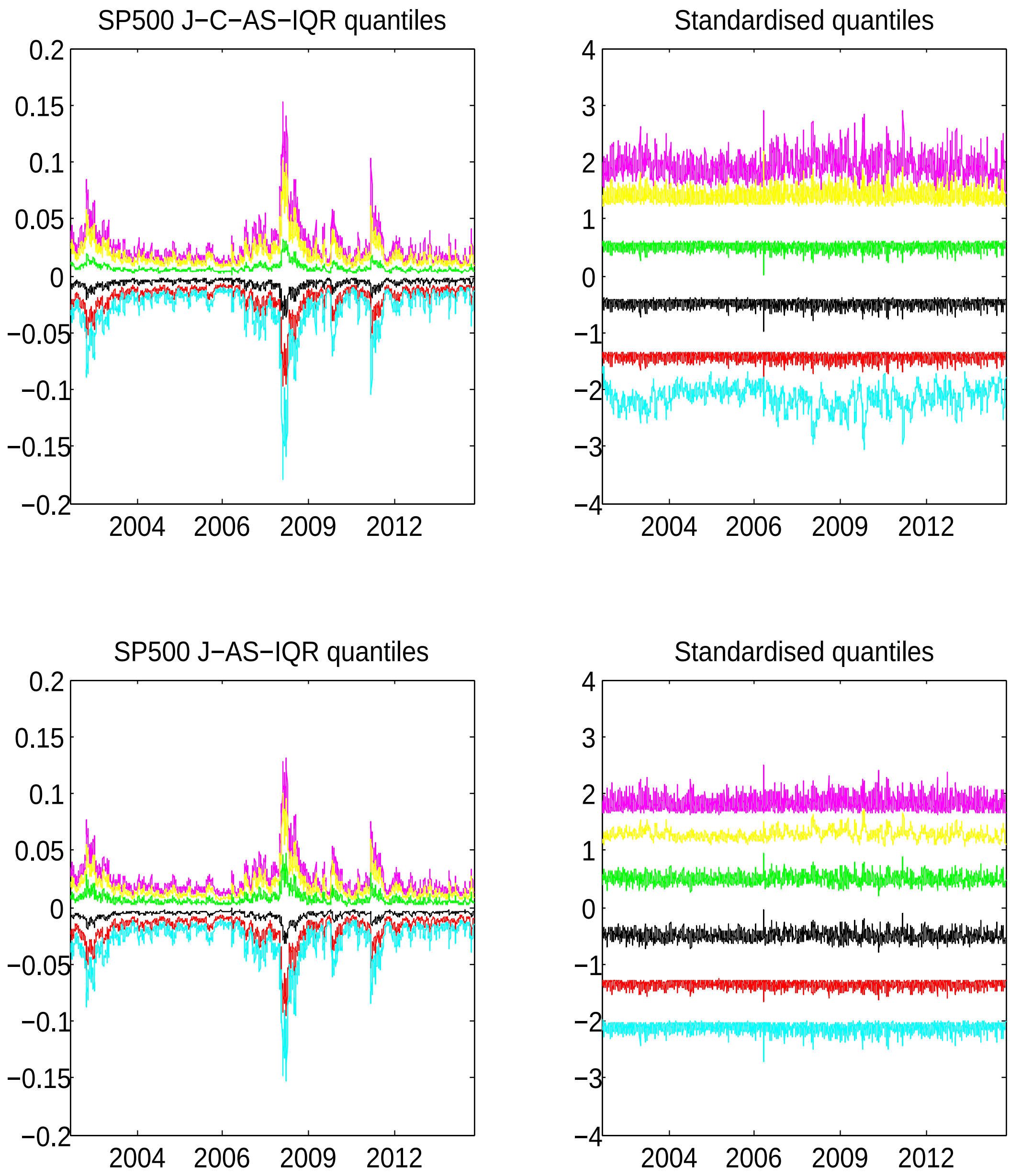

Figure 4: Estimated $q_{t,\theta}$ and standardised $\hat{q}_{t,\theta}$ by IQR_t: J-C-AS-IQR quantiles (upper panel) and J-AS-IQR quantiles (lower panel) for SP500.

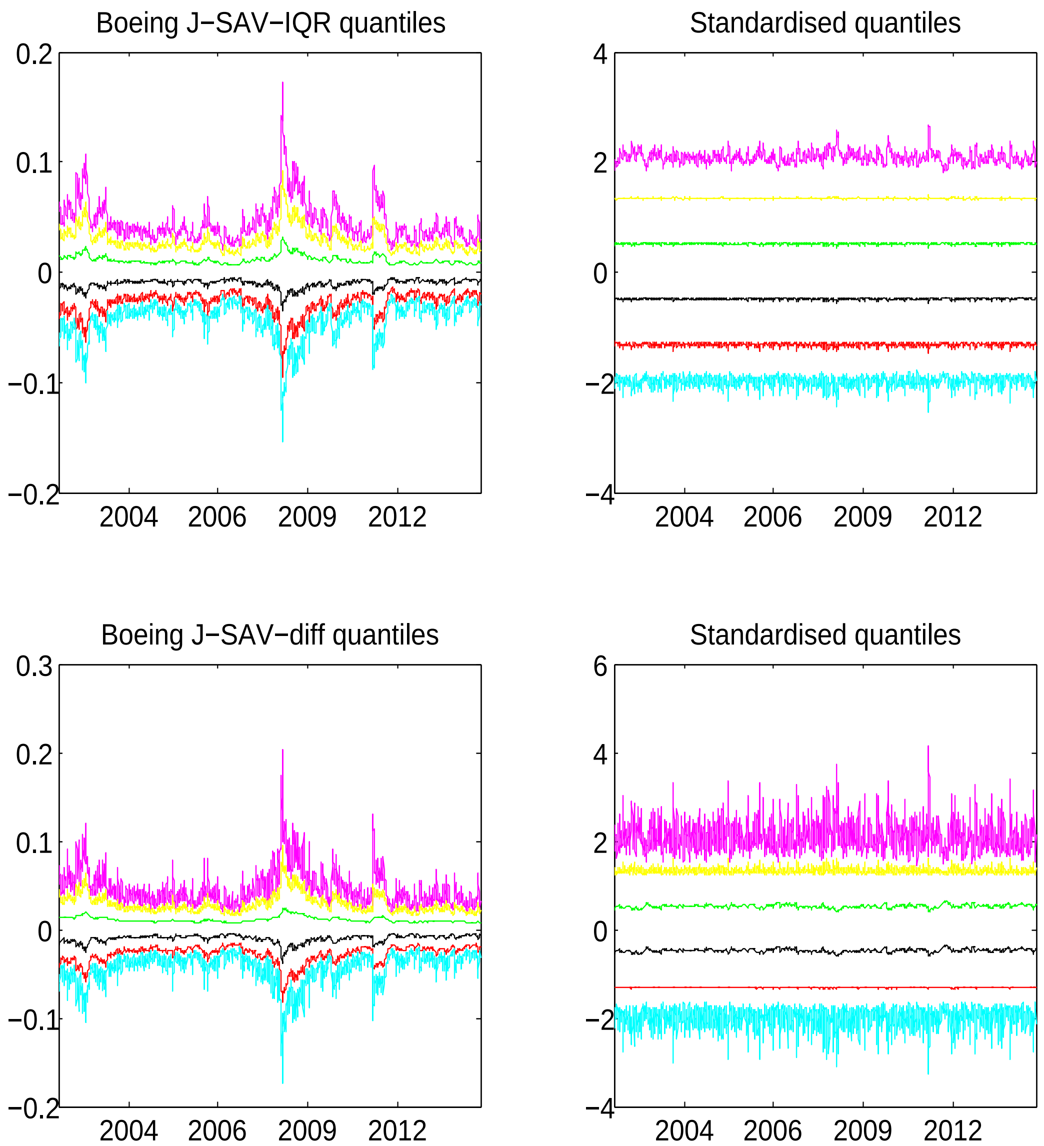

Figure 5: Estimated $q_{t,\theta}$ and standardised $\hat{q}_{t,\theta}$ by IQR_t: J-SAV-IQR quantiles (upper panel) and J-SAV-diff quantiles (lower panel) for Boeing.

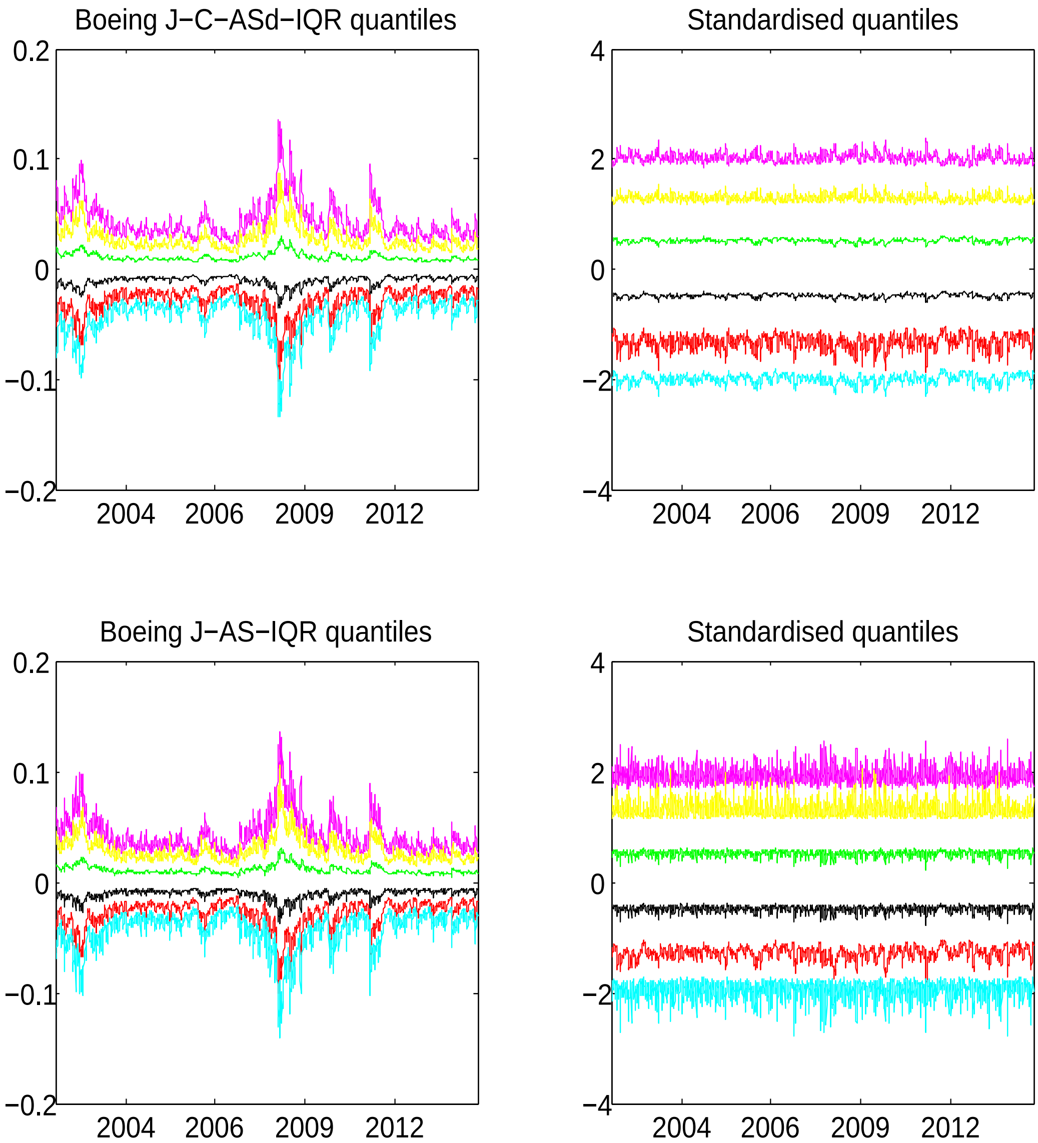

Figure 6: Estimated $q_{t,\theta}$ and standardised $\hat{q}_{t,\theta}$ by IQR_t: J-C-AS-IQR quantiles (upper panel) and J-AS-IQR quantiles (lower panel) for Boeing.

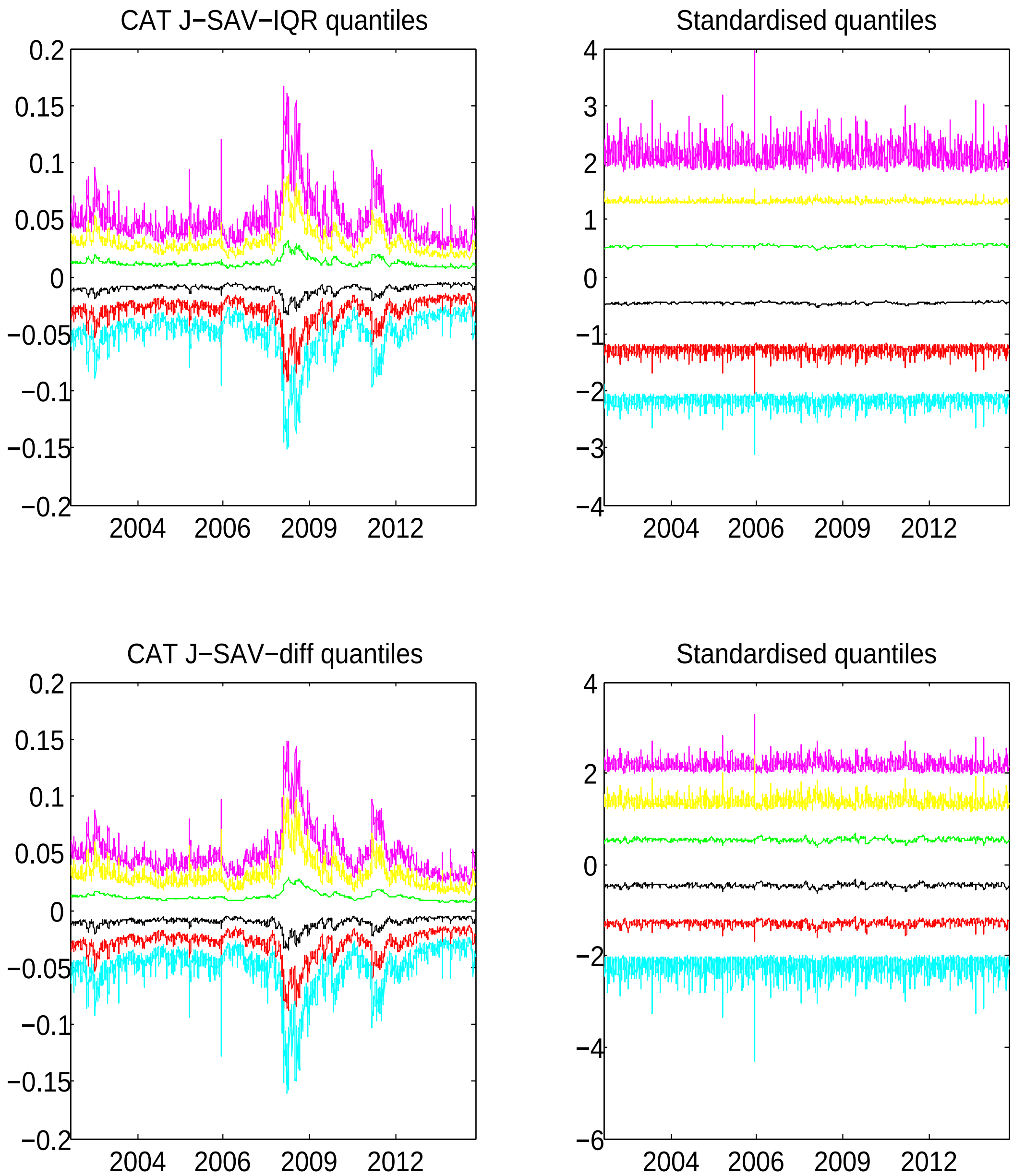

Figure 7: Estimated $q_{t,\theta}$ and standardised $\hat{q}_{t,\theta}$ by IQR_t: J-SAV-IQR quantiles (upper panel) and J-SAV-diff quantiles (lower panel) for CAT.

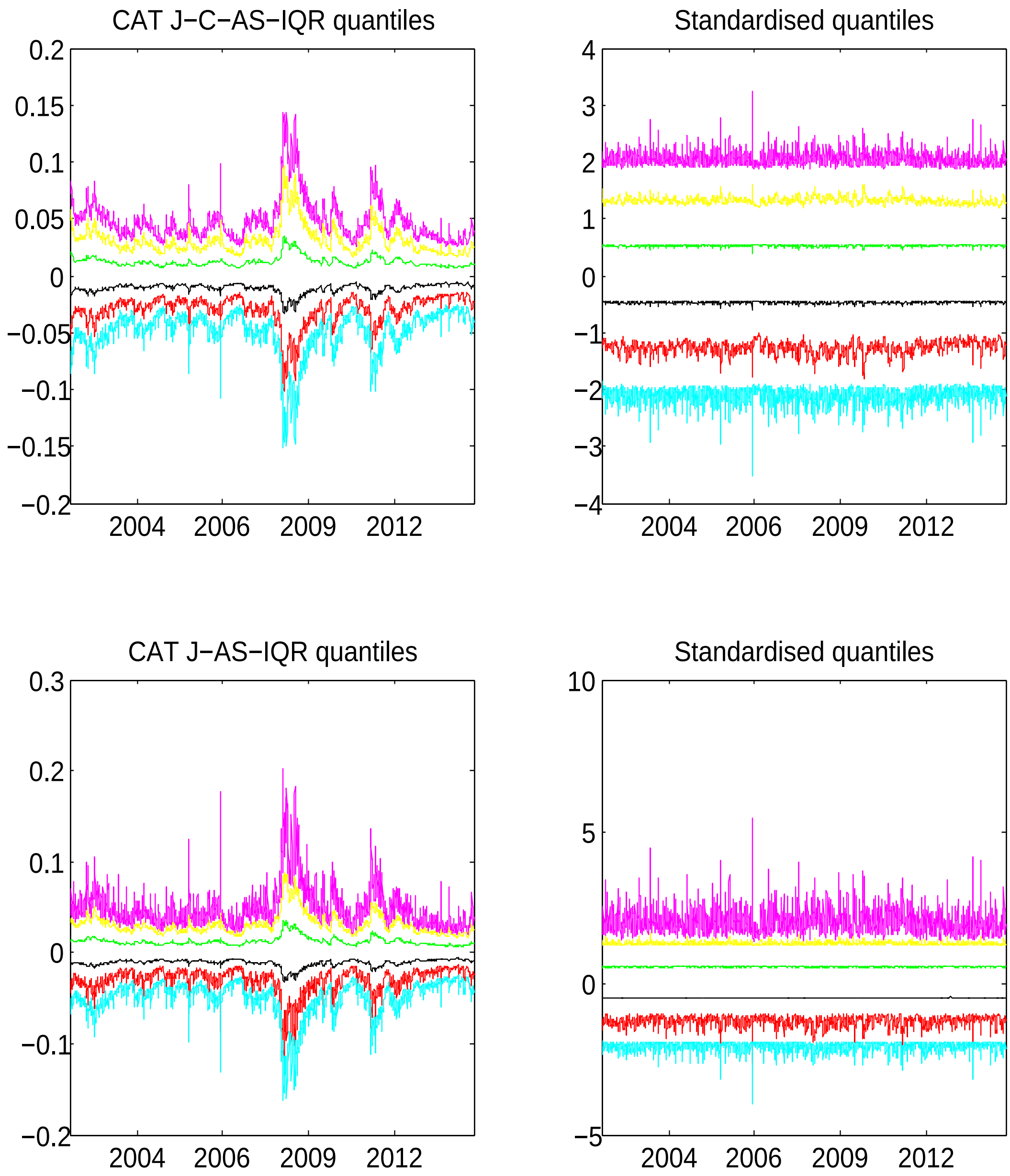

Figure 8: Estimated $q_{t,\theta}$ and standardised $\hat{q}_{t,\theta}$ by IQR_t: J-C-AS-IQR quantiles (upper panel) and J-AS-IQR quantiles (lower panel) for CAT.

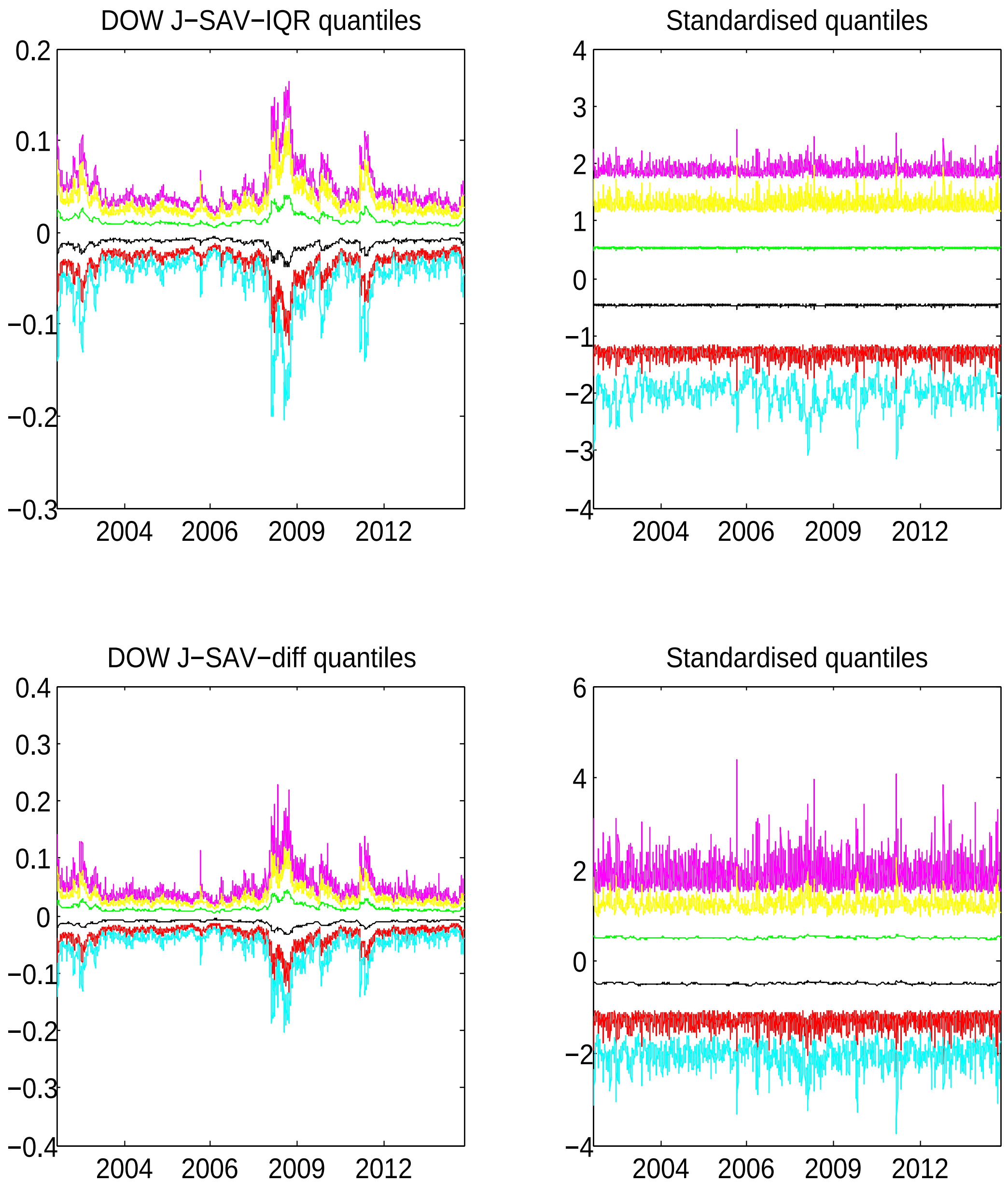

Figure 9: Estimated $q_{t,\theta}$ and standardised $\hat{q}_{t,\theta}$ by IQR_t: J-SAV-IQR quantiles (upper panel) and J-SAV-diff quantiles (lower panel) for DOW.

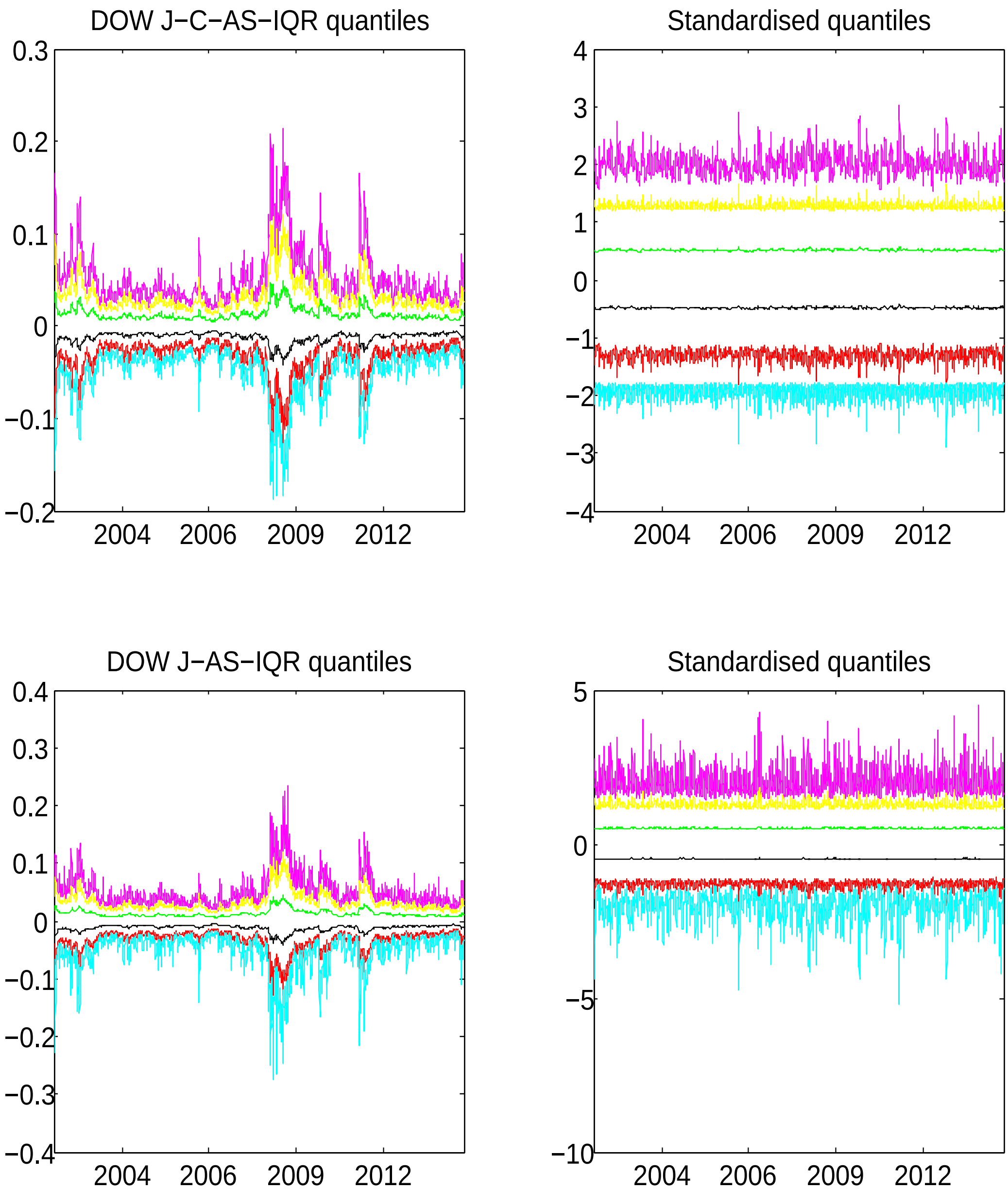

Figure 10: Estimated $q_{t,\theta}$ and standardised $\hat{q}_{t,\theta}$ by IQR_t: J-C-AS-IQR quantiles (upper panel) and J-AS-IQR quantiles (lower panel) for DOW.

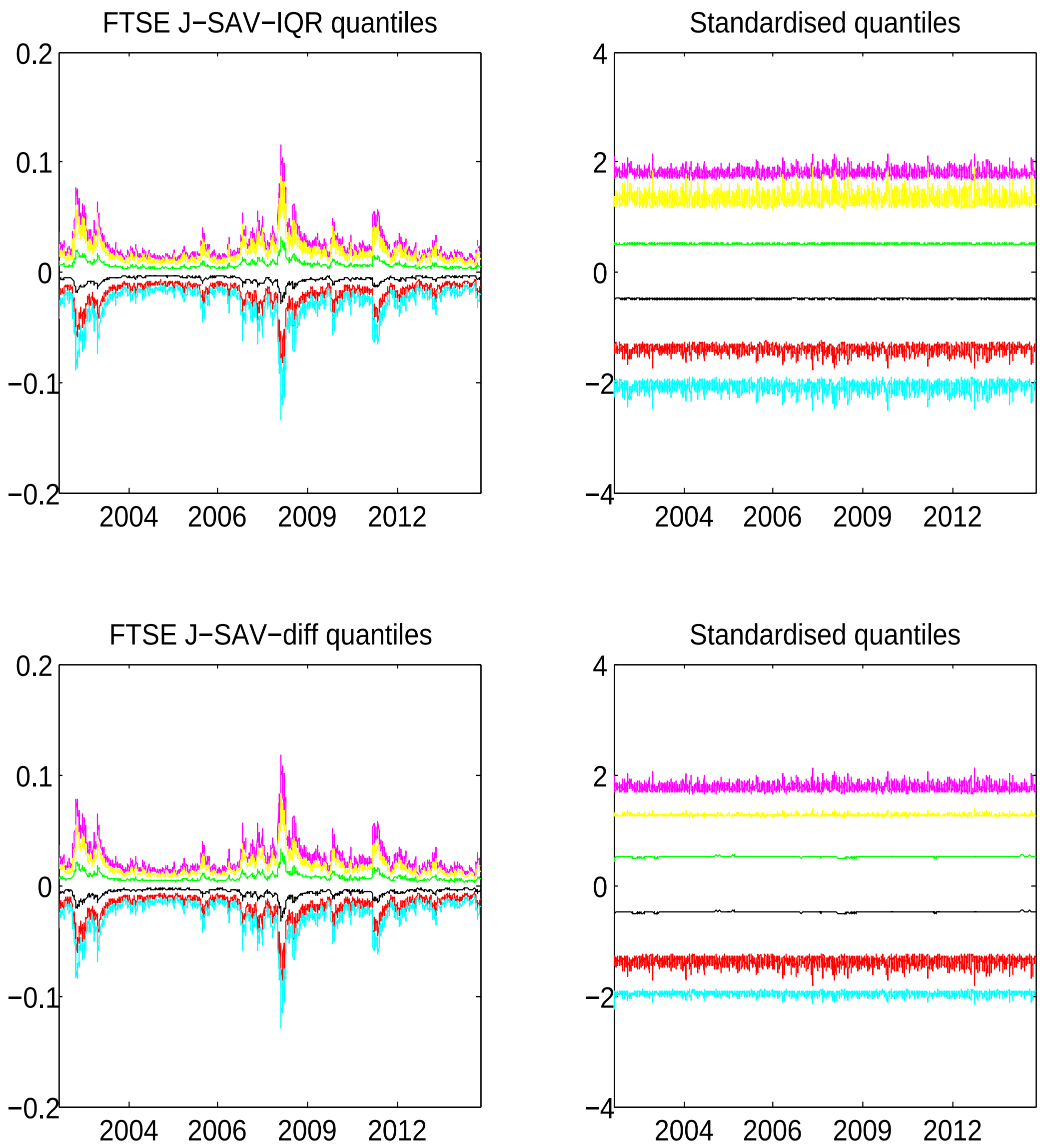

Figure 11: Estimated $q_{t,\theta}$ and standardised $\hat{q}_{t,\theta}$ by IQR_t: J-SAV-IQR quantiles (upper panel) and J-SAV-diff quantiles (lower panel) for FTSE.

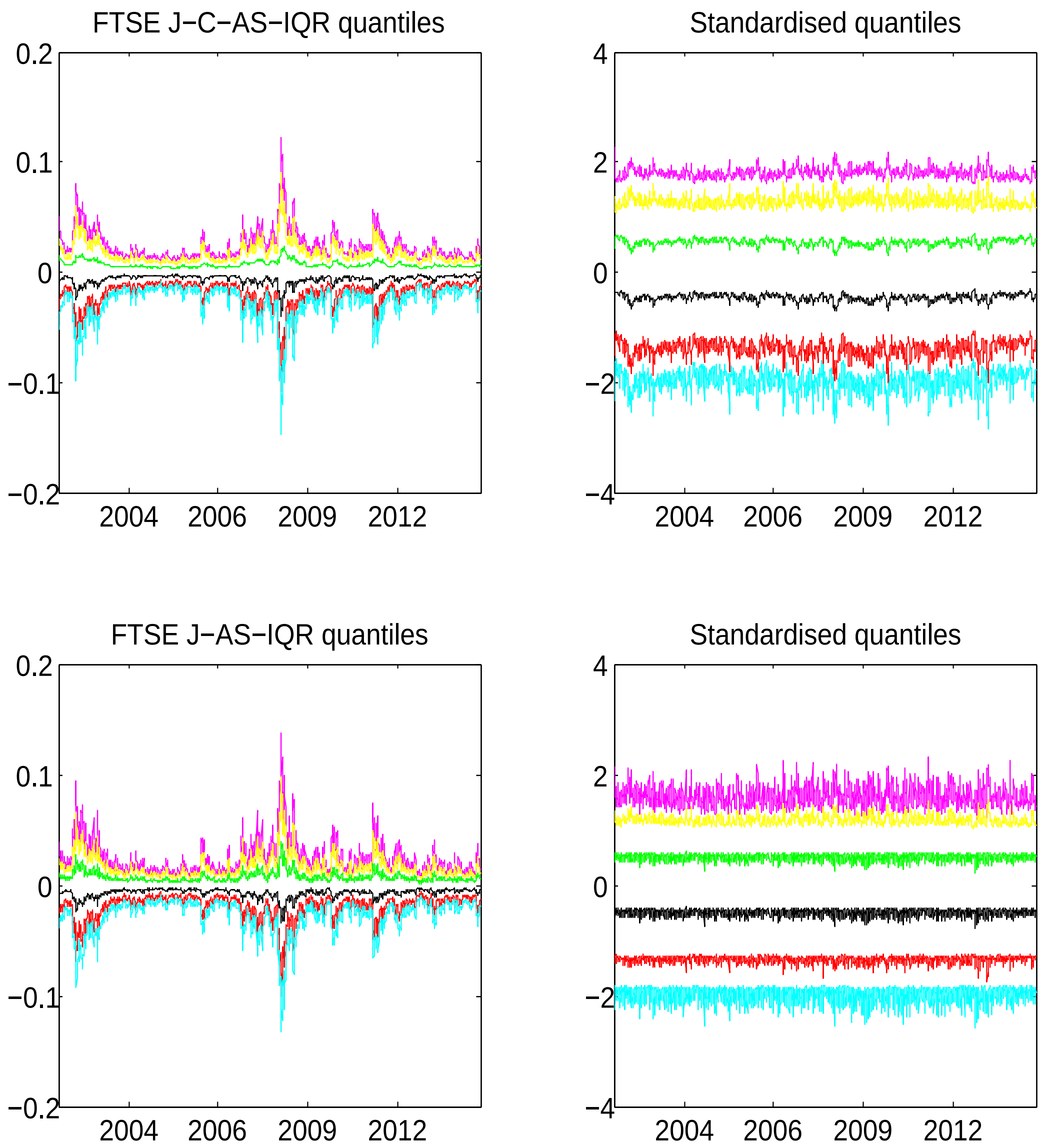

Figure 12: Estimated $q_{t,\theta}$ and standardised $\hat{q}_{t,\theta}$ by IQR_t: J-C-AS-IQR quantiles (upper panel) and J-AS-IQR quantiles (lower panel) for FTSE.

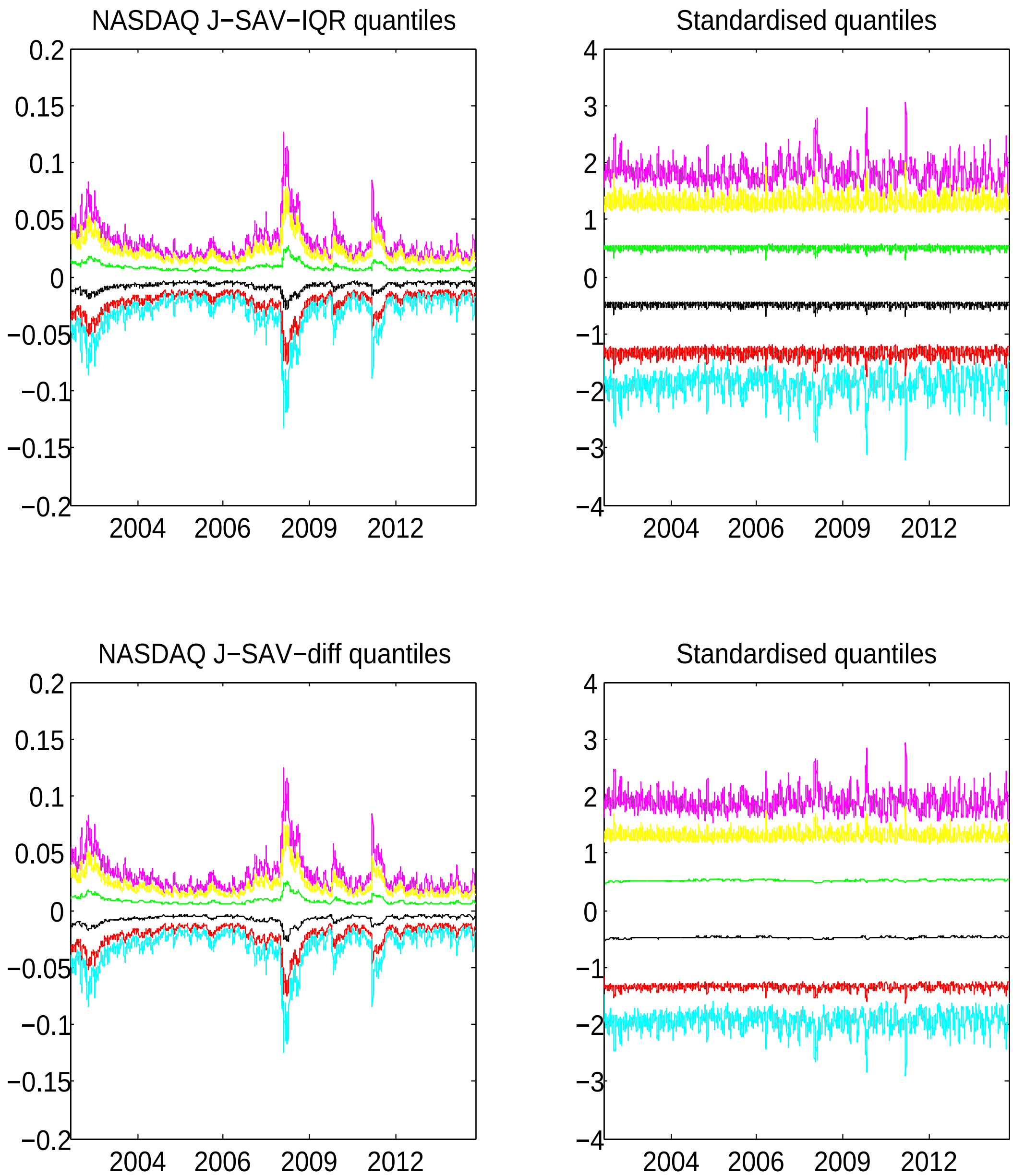

Figure 13: Estimated $q_{t,\theta}$ and standardised $\hat{q}_{t,\theta}$ by IQR_t: J-SAV-IQR quantiles (upper panel) and J-SAV-diff quantiles (lower panel) for NASDAQ.

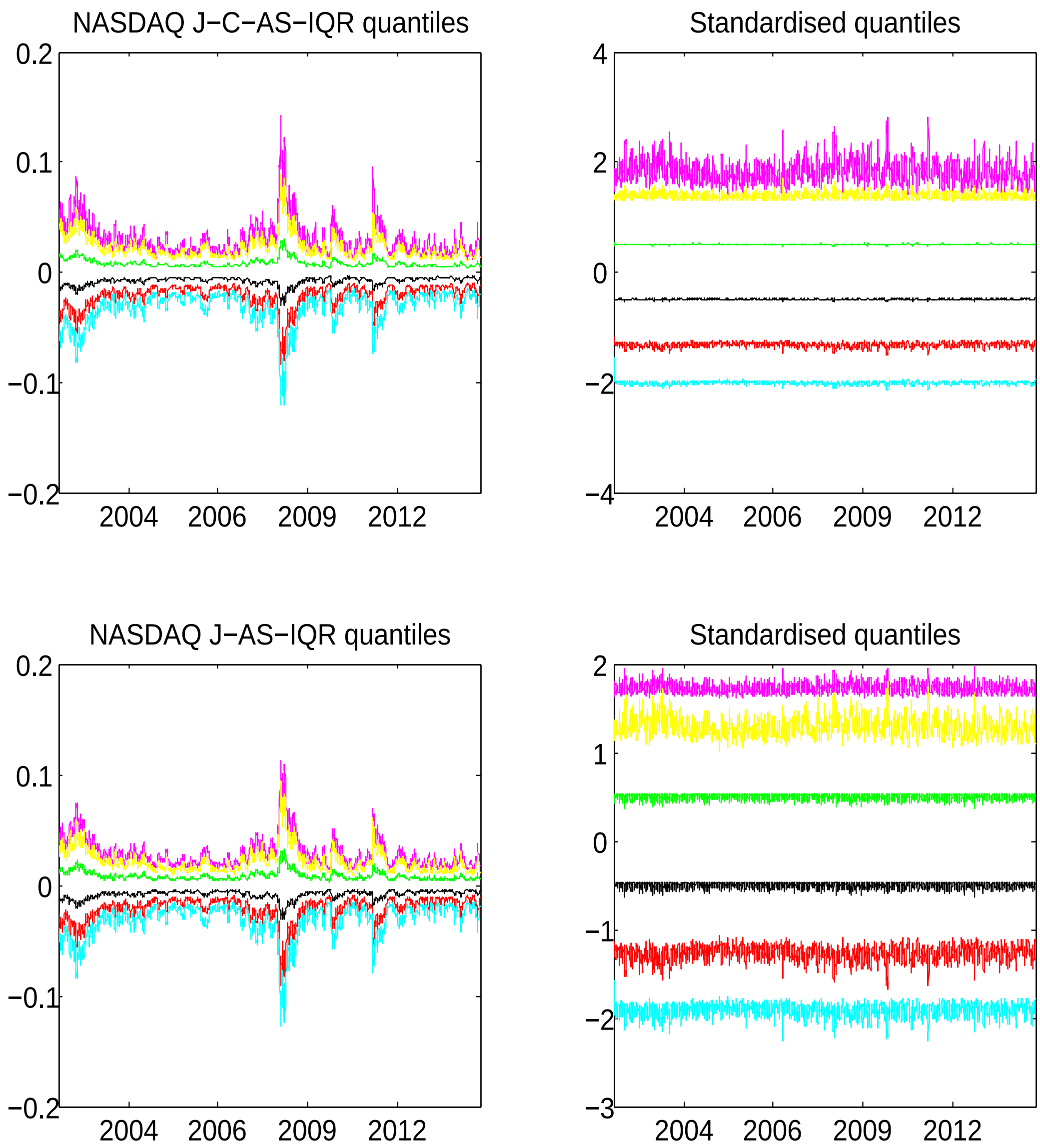

Figure 14: Estimated $q_{t,\theta}$ and standardised $\hat{q}_{t,\theta}$ by IQR_t: J-C-AS-IQR quantiles (upper panel) and J-AS-IQR quantiles (lower panel) for NASDAQ.

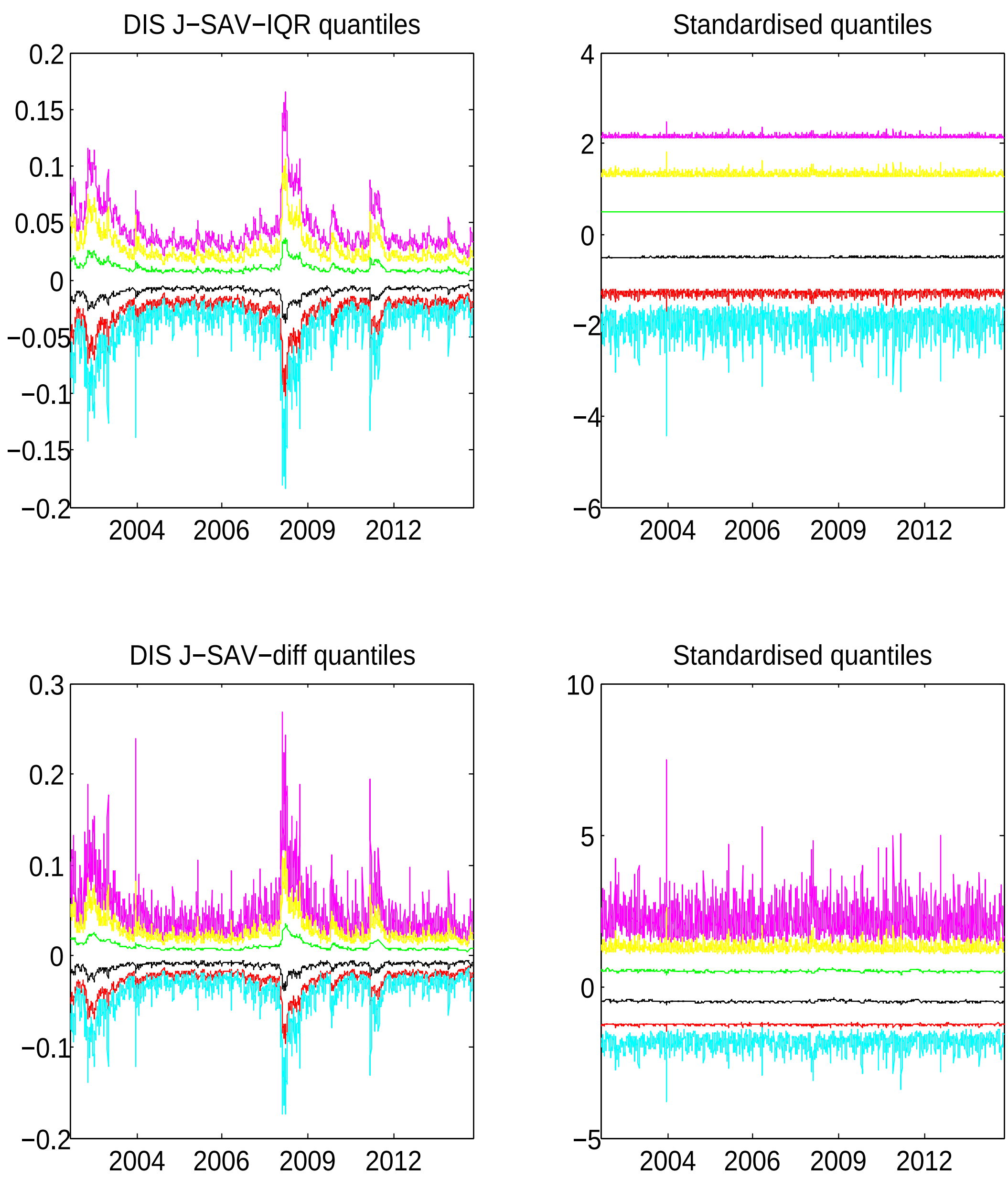

Figure 15: Estimated $q_{t,\theta}$ and standardised $\hat{q}_{t,\theta}$ by IQR_t: J-SAV-IQR quantiles (upper panel) and J-SAV-diff quantiles (lower panel) for DIS.

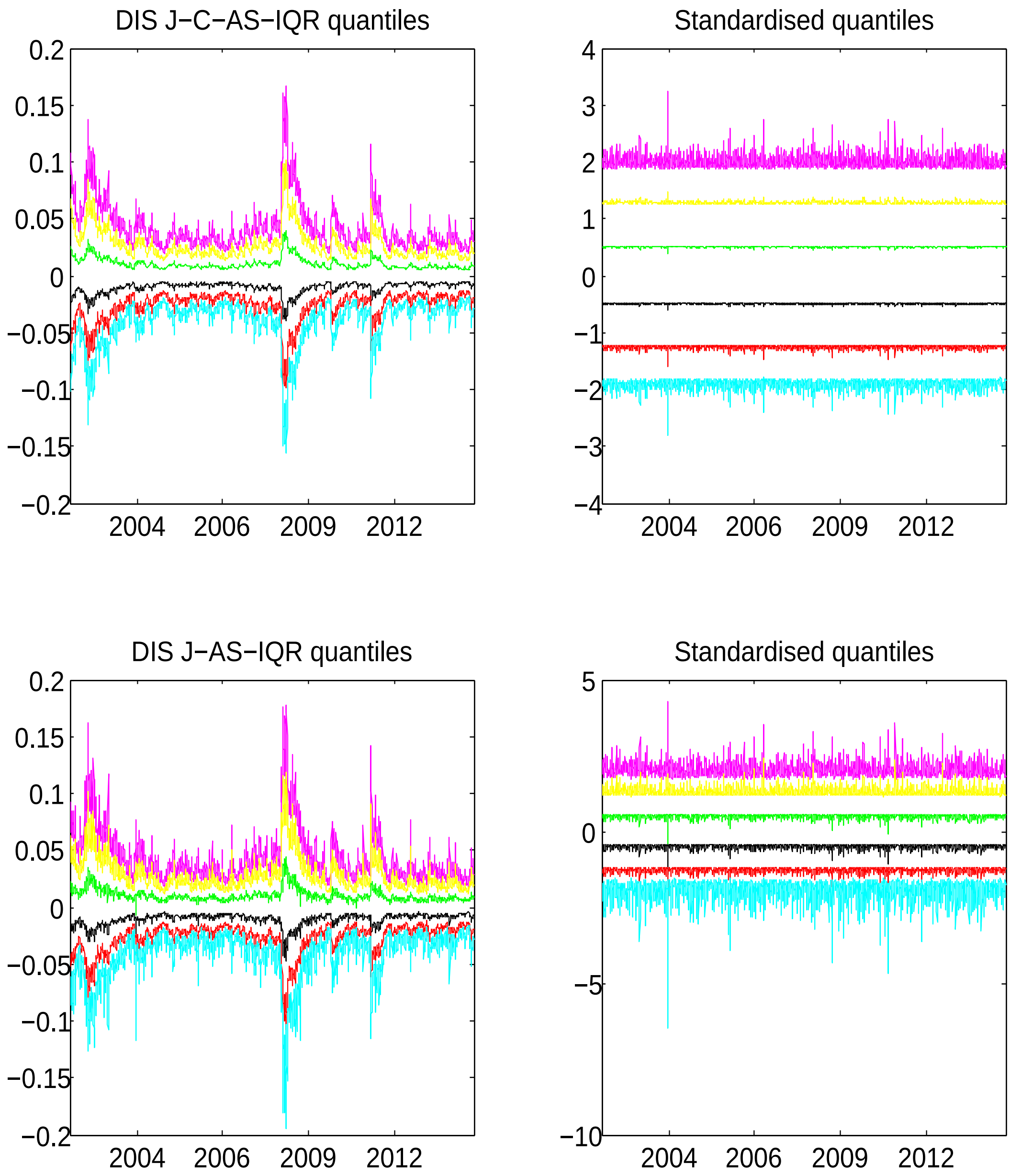

Figure 16: Estimated $q_{t,\theta}$ and standardised $\hat{q}_{t,\theta}$ by IQR_t: J-C-AS-IQR quantiles (upper panel) and J-AS-IQR quantiles (lower panel) for DIS.

	Start Date	End in - sample period	Sample size	Mean	Median	0.5% quantile	99.5% quantile	Skewness	Kurtosis
IBM	02/01/02	21/11/12	3229	0.0002	0.0002	-0.0568	0.0481	-0.0253	9.7280
SP500	01/01/02	21/11/12	3241	0.0002	0.0007	-0.0483	0.0427	-0.2189	12.3505
Boeing	02/01/02	21/11/12	3227	0.0005	0.0007	-0.0649	0.0611	0.0560	6.7509
CAT	02/01/02	21/11/12	3222	0.0005	0.0007	-0.0671	0.0681	-0.1059	8.2583
DOW	02/01/02	21/11/12	3205	0.0003	0.0007	-0.0902	0.0754	-0.4159	10.3794
FTSE	01/01/02	10/12/12	3248	0.0001	0.0005	-0.0488	0.0426	-0.1350	10.1849
NASDAQ	02/01/02	21/11/12	3241	0.0003	0.0009	-0.0467	0.0483	-0.0923	8.2130
DIS	02/01/02	21/11/12	3202	0.0005	0.0008	-0.0625	0.0662	0.2631	9.1386

Table 1: Data summary for the time series used in the estimation.

	IBM	SP500	Boeing	CAT	DOW	FTSE	NASDAQ	DIS
J-SAV	34.9097	29.0782	44.6173	49.9866	52.4068	28.5122	34.0265	43.6203
J-SAV-diff	34.8765	28.9526	44.6361	49.8810	52.4554	28.4229	34.0029	43.6006
J-SAV-IQR	34.8502	28.8707	44.5441	49.8775	52.3501	28.4332	34.0963	43.4475
J-AS-IQR	34.6341	28.3496	44.3225	49.6785	52.1826	27.8413	33.6923	43.1711
J-C-AS-IQR	34.6116	28.7366	44.2637	49.6304	52.2807	28.0768	33.7138	43.1813

Table 2: Value of the RQ criterion for the proposed joint quantile models.

	θ	J-SAV	J-SAV-diff	J-SAV-IQR	J-AS-IQR	J-C-AS-IQR
In-Sample	0.9900	0.9916	0.9894	0.9897	0.9912	0.9894
Out-Of-Sample	0.9900	0.9920	0.9920	0.9920	0.9920	0.9920
In-Sample	0.9500	0.9502	0.9498	0.9483	0.9469	0.9480
Out-Of-Sample	0.9500	0.9680	0.9660	0.9620	0.9640	0.9700
In-Sample	0.7500	0.7545	0.7549	0.7534	0.7556	0.7523
Out-Of-Sample	0.7500	0.7560	0.7620	0.7660	0.7820	0.7740
In-Sample	0.2500	0.2506	0.2528	0.2473	0.2433	0.2440
Out-Of-Sample	0.2500	0.2540	0.2660	0.2620	0.2280	0.2420
In-Sample	0.0500	0.0487	0.0509	0.0498	0.0517	0.0495
Out-Of-Sample	0.0500	0.0520	0.0500	0.0480	0.0460	0.0440
In-Sample	0.0100	0.0095	0.0110	0.0117	0.0117	0.0110
Out-Of-Sample	0.0100	0.0100	0.0140	0.0140	0.0100	0.0120

Table 3: The ratio of violations to the length of the testing period, both in - sample and out - of - sample, for joint quantile models for IBM. The sample ranges from January 01, 2002, to November 14, 2014.

	θ	**J-SAV**	**J-SAV-diff**	**J-SAV-IQR**	**J-AS-IQR**	**J-C-AS-IQR**
In-Sample	0.9900	0.9934	0.9909	0.9920	0.9872	0.9887
Out-Of-Sample	0.9900	0.9980	0.9940	0.9940	0.9920	0.9940
In-Sample	0.9500	0.9518	0.9489	0.9467	0.9507	0.9555
Out-Of-Sample	0.9500	0.9760	0.9680	0.9680	0.9660	0.9740
In-Sample	0.7500	0.7464	0.7512	0.7505	0.7494	0.7497
Out-Of-Sample	0.7500	0.7060	0.7180	0.6940	0.6980	0.7220
In-Sample	0.2500	0.2547	0.2444	0.2422	0.2382	0.2481
Out-Of-Sample	0.2500	0.2300	0.2080	0.1980	0.1940	0.2140
In-Sample	0.0500	0.0452	0.0500	0.0522	0.0529	0.0503
Out-Of-Sample	0.0500	0.0360	0.0420	0.0460	0.0480	0.0420
In-Sample	0.0100	0.0080	0.0131	0.0120	0.0106	0.0117
Out-Of-Sample	0.0100	0.0040	0.0140	0.0080	0.0080	0.0120

Table 4: The ratio of violations to the length of the testing period, both in - sample and out - of - sample, for joint quantile models for SP500. The sample ranges from January 01, 2002, to November 14, 2014.

	θ	**J-SAV**	**J-SAV-diff**	**J-SAV-IQR**	**J-AS-IQR**	**J-C-ASd-IQR**
In-Sample	0.9900	0.9905	0.9879	0.9897	0.9883	0.9894
Out-Of-Sample	0.9900	0.9940	0.9920	0.9920	0.9920	0.9940
In-Sample	0.9500	0.9509	0.9494	0.9505	0.9501	0.9505
Out-Of-Sample	0.9500	0.9640	0.9580	0.9640	0.9560	0.9540
In-Sample	0.7500	0.7484	0.7492	0.7473	0.7539	0.7528
Out-Of-Sample	0.7500	0.7260	0.7280	0.7260	0.7400	0.7360
In-Sample	0.2500	0.2516	0.2501	0.2490	0.2439	0.2483
Out-Of-Sample	0.2500	0.2220	0.2160	0.2100	0.2080	0.2100
In-Sample	0.0500	0.0491	0.0473	0.0484	0.0510	0.0484
Out-Of-Sample	0.0500	0.0420	0.0440	0.0440	0.0480	0.0440
In-Sample	0.0100	0.0099	0.0103	0.0106	0.0110	0.0106
Out-Of-Sample	0.0100	0.0040	0.0060	0.0080	0.0100	0.0100

Table 5: The ratio of violations to the length of the testing period, both in - sample and out - of - sample, for joint quantile models for Boeing. The sample ranges from January 01, 2002, to November 14, 2014.

	θ	**J-SAV**	**J-SAV-diff**	**J-SAV-IQR**	**J-AS-IQR**	**J-C-AS-IQR**
In-Sample	0.9900	0.9923	0.9904	0.9897	0.9875	0.9901
Out-Of-Sample	0.9900	0.9980	0.9920	0.9900	0.9920	0.9920
In-Sample	0.9500	0.9486	0.9504	0.9489	0.9511	0.9511
Out-Of-Sample	0.9500	0.9720	0.9600	0.9660	0.9680	0.9680
In-Sample	0.7500	0.7509	0.7494	0.7502	0.7535	0.7461
Out-Of-Sample	0.7500	0.8320	0.8000	0.8100	0.8100	0.8100
In-Sample	0.2500	0.2509	0.2494	0.2476	0.2458	0.2465
Out-Of-Sample	0.2500	0.2340	0.2260	0.2140	0.1880	0.1900
In-Sample	0.0500	0.0507	0.0507	0.0529	0.0518	0.0503
Out-Of-Sample	0.0500	0.0340	0.0340	0.0300	0.0280	0.0280
In-Sample	0.0100	0.0107	0.0103	0.0099	0.0099	0.0107
Out-Of-Sample	0.0100	0.0120	0.0100	0.0080	0.0080	0.0060

Table 6: The ratio of violations to the length of the testing period, both in - sample and out - of - sample, for joint quantile models for CAT. The sample ranges from January 01, 2002, to November 14, 2014.

	θ	**J-SAV**	**J-SAV-diff**	**J-SAV-IQR**	**J-AS-IQR**	**J-C-AS-IQR**
In-Sample	0.9900	0.9911	0.9893	0.9882	0.9878	0.9882
Out-Of-Sample	0.9900	0.9920	0.9840	0.9880	0.9900	0.9900
In-Sample	0.9500	0.9519	0.9486	0.9508	0.9490	0.9482
Out-Of-Sample	0.9500	0.9560	0.9420	0.9460	0.9380	0.9440
In-Sample	0.7500	0.7431	0.7542	0.7542	0.7538	0.7516
Out-Of-Sample	0.7500	0.7060	0.7400	0.7500	0.7380	0.7260
In-Sample	0.2500	0.2514	0.2466	0.2499	0.2503	0.2477
Out-Of-Sample	0.2500	0.2100	0.2080	0.2100	0.2140	0.2080
In-Sample	0.0500	0.0506	0.0514	0.0506	0.0488	0.0518
Out-Of-Sample	0.0500	0.0480	0.0480	0.0500	0.0520	0.0540
In-Sample	0.0100	0.0104	0.0089	0.0107	0.0107	0.0096
Out-Of-Sample	0.0100	0.0180	0.0120	0.0180	0.0180	0.0120

Table 7: The ratio of violations to the length of the testing period, both in - sample and out - of - sample, for joint quantile models for DOW. The sample ranges from January 01, 2002, to November 14, 2014.

	θ	J-SAV	J-SAV-diff	J-SAV-IQR	J-AS-IQR	J-C-ASd-IQR
In-Sample	0.9900	0.9916	0.9905	0.9913	0.9876	0.9916
Out-Of-Sample	0.9900	0.9920	0.9880	0.9900	0.9880	0.9900
In-Sample	0.9500	0.9487	0.9509	0.9531	0.9491	0.9534
Out-Of-Sample	0.9500	0.9620	0.9600	0.9600	0.9620	0.9680
In-Sample	0.7500	0.7544	0.7460	0.7445	0.7507	0.7529
Out-Of-Sample	0.7500	0.7880	0.7660	0.7660	0.7740	0.7860
In-Sample	0.2500	0.2529	0.2467	0.2485	0.2453	0.2595
Out-Of-Sample	0.2500	0.2400	0.2300	0.2280	0.1900	0.2440
In-Sample	0.0500	0.0426	0.0517	0.0517	0.0473	0.0513
Out-Of-Sample	0.0500	0.0300	0.0500	0.0440	0.0420	0.0480
In-Sample	0.0100	0.0080	0.0113	0.0095	0.0095	0.0106
Out-Of-Sample	0.0100	0.0120	0.0160	0.0140	0.0160	0.0160

Table 8: The ratio of violations to the length of the testing period, both in - sample and out - of - sample, for joint quantile models for FTSE. The sample ranges from January 01, 2002, to November 14, 2014.

	θ	J-SAV	J-SAV-diff	J-SAV-IQR	J-AS-IQR	J-C-ASd-IQR
In-Sample	0.9900	0.9901	0.9898	0.9887	0.9901	0.9898
Out-Of-Sample	0.9900	0.9980	0.9980	0.9960	0.9980	0.9980
In-Sample	0.9500	0.9515	0.9493	0.9493	0.9497	0.9591
Out-Of-Sample	0.9500	0.9700	0.9620	0.9620	0.9680	0.9780
In-Sample	0.7500	0.7541	0.7552	0.7508	0.7512	0.7486
Out-Of-Sample	0.7500	0.7400	0.7440	0.7380	0.7440	0.7380
In-Sample	0.2500	0.2437	0.2433	0.2422	0.2382	0.2444
Out-Of-Sample	0.2500	0.1840	0.1860	0.1780	0.1740	0.1840
In-Sample	0.0500	0.0500	0.0503	0.0514	0.0547	0.0536
Out-Of-Sample	0.0500	0.0500	0.0480	0.0480	0.0440	0.0460
In-Sample	0.0100	0.0099	0.0120	0.0117	0.0117	0.0106
Out-Of-Sample	0.0100	0.0080	0.0140	0.0140	0.0120	0.0100

Table 9: The ratio of violations to the length of the testing period, both in - sample and out - of - sample, for joint quantile models for NASDAQ. The sample ranges from January 01, 2002, to November 14, 2014.

	θ	**J-SAV**	**J-SAV-diff**	**J-SAV-IQR**	**J-AS-IQR**	**J-C-AS-IQR**
In-Sample	0.9900	0.9904	0.9893	0.9896	0.9896	0.9893
Out-Of-Sample	0.9900	1.0000	0.9940	0.9980	0.9960	0.9960
In-Sample	0.9500	0.9523	0.9474	0.9489	0.9486	0.9508
Out-Of-Sample	0.9500	0.9680	0.9440	0.9400	0.9440	0.9440
In-Sample	0.7500	0.7476	0.7531	0.7483	0.7498	0.7543
Out-Of-Sample	0.7500	0.7160	0.7280	0.7320	0.7140	0.7380
In-Sample	0.2500	0.2513	0.2402	0.2446	0.2472	0.2454
Out-Of-Sample	0.2500	0.2060	0.1820	0.1840	0.1960	0.1980
In-Sample	0.0500	0.0507	0.0474	0.0492	0.0485	0.0492
Out-Of-Sample	0.0500	0.0360	0.0440	0.0440	0.0420	0.0440
In-Sample	0.0100	0.0115	0.0130	0.0118	0.0107	0.0107
Out-Of-Sample	0.0100	0.0080	0.0120	0.0120	0.0180	0.0120

Table 10: The ratio of violations to the length of the testing period, both in - sample and out - of - sample, for joint quantile models for DIS. The sample ranges from January 01, 2002, to November 14, 2014.

	IBM	**J-SAV**	**J-SAV-diff**	**J-SAV-IQR**	**J-AS-IQR**	**J-C-AS-IQR**
LR-uc (p-values)	0.9900	0.6445	0.6445	0.6445	0.6445	0.6445
LR-i (p-values)	0.9900	-	-	-	-	-
LR-cc (p-values)	0.9900	-	-	-	-	-
LR-uc (p-values)	0.9500	0.0497	0.0839	0.2028	0.1338	0.0278
LR-i (p-values)	0.9500	-	-	-	-	-
LR-cc (p-values)	0.9500	-	-	-	-	-
LR-uc (p-values)	0.7500	0.7756	0.5501	0.4197	0.0984	0.2194
LR-i (p-values)	0.7500	0.0322	0.0386	0.0649	0.6328	0.1608
LR-cc (p-values)	0.7500	0.0968	0.0986	0.1314	0.2277	0.1759
LR-uc (p-values)	0.2500	0.8164	0.3971	0.5205	0.2616	0.6973
LR-i (p-values)	0.2500	0.6933	0.7232	0.8881	0.6214	0.6872
LR-cc (p-values)	0.2500	0.9006	0.6562	0.8054	0.4715	0.8549
LR-uc (p-values)	0.0500	0.8303	0.9918	0.8444	0.6850	0.5367
LR-i (p-values)	0.0500	0.7371	-	-	-	0.9746
LR-cc (p-values)	0.0500	0.9238	-	-	-	0.8258
LR-uc (p-values)	0.0100	0.9964	0.3939	0.3939	0.9964	0.6596
LR-i (p-values)	0.0100	-	-	-	-	-
LR-cc (p-values)	0.0100	-	-	-	-	-
	SP500	**J-SAV**	**J-SAV-diff**	**J-SAV-IQR**	**J-AS-IQR**	**J-C-AS-IQR**
LR-uc (p-values)	0.9900	0.0285	0.3336	0.3336	0.6445	0.3336
LR-i (p-values)	0.9900	-	-	-	-	-
LR-cc (p-values)	0.9900	-	-	-	-	-
LR-uc (p-values)	0.9500	**0.0032**	0.0497	0.0497	0.0839	**0.0071**
LR-i (p-values)	0.9500	0.2831	0.5311	0.0962	0.1234	0.3391
LR-cc (p-values)	0.9500	**0.0073**	0.1198	0.0365	0.0685	0.0169
LR-uc (p-values)	0.7500	0.0239	0.0975	**0.0043**	**0.0079**	0.1455
LR-i (p-values)	0.7500	**0.0003**	0.0473	**0.0028**	0.2237	0.0147
LR-cc (p-values)	0.7500	**0.0001**	0.0354	**0.0002**	0.0140	0.0177
LR-uc (p-values)	0.2500	0.3091	0.0286	**0.0063**	**0.0032**	0.0620
LR-i (p-values)	0.2500	0.5315	0.7207	0.7039	0.3758	0.6026
LR-cc (p-values)	0.2500	0.4902	0.0855	0.0224	**0.0087**	0.1530
LR-uc (p-values)	0.0500	0.1338	0.4048	0.6850	0.8444	0.4048
LR-i (p-values)	0.0500	-	-	-	-	-
LR-cc (p-values)	0.0500	-	-	-	-	-
LR-uc (p-values)	0.0100	0.1260	0.3939	0.6445	0.6445	0.6596
LR-i (p-values)	0.0100	-	-	-	-	-
LR-cc (p-values)	0.0100	-	-	-	-	-

Table 11: Back - testing results for joint quantile models: The p-values for the LR test by Christoffersen (1998) are provided for IBM and SP500. Models that are rejected by the LR test are in bold for rejection at 1% significance level. $-$ denotes that the p-value was not produced.

	Boeing	J-SAV	J-SAV-diff	J-SAV-IQR	J-AS-IQR	J-C-ASd-IQR
LR-uc (p-values)	0.9900	0.3336	0.6445	0.6445	0.6445	0.3336
LR-i (p-values)	0.9900	-	-	-	-	-
LR-cc	0.9900	-	-	-	-	-
LR-uc (p-values)	0.9500	0.1338	0.4048	0.1338	0.5367	0.6850
LR-i (p-values)	0.9500	-	-	-	-	-
LR-cc (p-values)	0.9500	-	-	-	-	-
LR-uc (p-values)	0.7500	0.2101	0.2493	0.2101	0.5890	0.4564
LR-i (p-values)	0.7500	0.1322	0.3549	0.1322	0.3647	0.6578
LR-cc (p-values)	0.7500	0.1468	0.3357	0.1468	0.5730	0.6869
LR-uc (p-values)	0.2500	0.1499	0.0785	0.0374	0.0286	0.0374
LR-i (p-values)	0.2500	0.1765	0.1450	0.1941	0.2483	0.1192
LR-cc (p-values)	0.2500	0.1422	0.0735	0.0494	0.0468	0.0341
LR-uc (p-values)	0.0500	0.4048	0.5367	0.5367	0.8444	0.5367
LR-i (p-values)	0.0500	0.2808	0.0769	0.3324	0.4494	0.3324
LR-cc (p-values)	0.0500	0.3951	0.1729	0.5165	0.7369	0.5165
LR-uc (p-values)	0.0100	0.1260	0.3336	0.6445	0.9964	0.9964
LR-i (p-values)	0.0100	-	-	-	-	-
LR-cc (p-values)	0.0100	-	-	-	-	-
	CAT	**J-SAV**	**J-SAV-diff**	**J-SAV-IQR**	**J-AS-IQR**	**J-C-AS-IQR**
LR-uc (p-values)	0.9900	0.0285	0.6445	0.9964	0.6445	0.6445
LR-i (p-values)	0.9900	-	-	0.0343	-	-
LR-cc (p-values)	0.9900	-	-	0.1065	-	-
LR-uc (p-values)	0.9500	0.0145	0.2930	0.0839	0.0497	0.0497
LR-i (p-values)	0.9500	0.3993	0.8236	0.6015	0.5311	0.5311
LR-cc (p-values)	0.9500	0.0354	0.5611	0.1958	0.1198	0.1198
LR-uc (p-values)	0.7500	**0.0000**	**0.0063**	**0.0010**	**0.0010**	**0.0010**
LR-i (p-values)	0.7500	0.7530	0.6026	0.5762	0.7929	0.7929
LR-cc (p-values)	0.7500	**0.0000**	0.0210	**0.0040**	**0.0045**	**0.0045**
LR-uc (p-values)	0.2500	0.4197	0.2194	0.0620	**0.0010**	**0.0015**
LR-i (p-values)	0.2500	0.3783	0.3883	0.4218	0.5072	0.5820
LR-cc (p-values)	0.2500	0.4898	0.3242	0.1269	**0.0037**	**0.0057**
LR-uc (p-values)	0.0500	0.0839	0.0839	0.0278	0.0145	0.0145
LR-i (p-values)	0.0500	0.6015	-	-	-	-
LR-cc (p-values)	0.0500	0.1958	-	-	-	-
LR-uc (p-values)	0.0100	0.6596	0.9964	0.6445	0.6445	0.3336
LR-i (p-values)	0.0100	-	-	-	-	-
LR-cc (p-values)	0.0100	-	-	-	-	-

Table 12: Back - testing results for joint quantile models: The p-values for the LR test by Christoffersen (1998) are provided for Boeing, and CAT. Models that are rejected by the LR test are in bold for rejection at 1% significance level. − denotes that the p-value was not produced.

	DOW	J-SAV	J-SAV-diff	J-SAV-IQR	J-AS-IQR	J-C-AS-IQR
LR-uc (p-values)	0.9900	0.6445	0.2131	0.6596	0.9964	0.9964
LR-i (p-values)	0.9900	-	-	-	-	-
LR-cc (p-values)	0.9900	-	-	-	-	-
LR-uc (p-values)	0.9500	0.5367	0.4169	0.6776	0.2304	0.5386
LR-i (p-values)	0.9500	-	-	-	-	-
LR-cc (p-values)	0.9500	-	-	-	-	-
LR-uc (p-values)	0.7500	0.0239	0.5890	0.9794	0.5205	0.2101
LR-i (p-values)	0.7500	0.3840	0.3969	0.3264	0.6202	0.7624
LR-cc (p-values)	0.7500	0.0534	0.6036	0.6176	0.7194	0.4356
LR-uc (p-values)	0.2500	0.0286	0.0216	0.0286	0.0484	0.0216
LR-i (p-values)	0.2500	0.5704	0.4947	0.5704	0.5482	0.3427
LR-cc (p-values)	0.2500	0.0776	0.0567	0.0776	0.1191	0.0456
LR-uc (p-values)	0.0500	0.8444	0.8444	0.9918	0.8303	0.6776
LR-i (p-values)	0.0500	0.1250	0.4494	0.5140	0.1909	0.2312
LR-cc (p-values)	0.0500	0.3024	0.7369	0.8082	0.4156	0.4478
LR-uc (p-values)	0.0100	0.1049	0.6596	0.1049	0.1049	0.6596
LR-i (p-values)	0.0100	-	-	-	-	-
LR-cc (p-values)	0.0100	-	-	-	-	-
	FTSE	**J-SAV**	**J-SAV-diff**	**J-SAV-IQR**	**J-AS-IQR**	**J-C-ASd-IQR**
LR-uc (p-values)	0.9900	0.6445	0.6596	0.9964	0.6596	0.9964
LR-i (p-values)	0.9900	-	-	-	-	-
LR-cc (p-values)	0.9900	-	-	-	-	-
LR-uc (p-values)	0.9500	0.2028	0.2930	0.2930	0.2028	0.0497
LR-i (p-values)	0.9500	-	-	-	-	-
LR-cc (p-values)	0.9500	-	-	-	-	-
LR-uc (p-values)	0.7500	0.0374	0.3618	0.3618	0.1822	0.0484
LR-i (p-values)	0.7500	0.0422	0.0347	0.0649	0.9258	0.0642
LR-cc (p-values)	0.7500	0.0146	0.0710	0.1200	0.4090	0.0257
LR-uc (p-values)	0.2500	0.6219	0.3091	0.2616	**0.0015**	0.7756
LR-i (p-values)	0.2500	0.1380	0.1717	0.4568	0.1633	0.4455
LR-cc (p-values)	0.2500	0.2947	0.2343	0.4038	**0.0025**	0.7177
LR-uc (p-values)	0.0500	0.0278	0.9918	0.5367	0.4048	0.8444
LR-i (p-values)	0.0500	0.4635	0.1556	0.3324	0.2808	0.1250
LR-cc (p-values)	0.0500	0.0679	0.3649	0.5165	0.3951	0.3024
LR-uc (p-values)	0.0100	0.6596	0.2131	0.3939	0.2131	0.2131
LR-i (p-values)	0.0100	-	-	-	-	-
LR-cc (p-values)	0.0100	-	-	-	-	-

Table 13: Back - testing results for joint quantile models: The p-values for the LR test by Christoffersen (1998) are provided for DOW and FTSE. Models that are rejected by the LR test are in bold for rejection at 1% significance level. $-$ denotes that the p-value was not produced.

	NASDAQ	J-SAV	J-SAV-diff	J-SAV-IQR	J-AS-IQR	J-C-AS-IQR
LR-uc	0.9900	0.0285	0.0285	0.1260	0.0285	0.0285
LR-i	0.9900	-	-	-	-	-
LR-cc	0.9900	-	-	-	-	-
LR-uc	0.9500	0.0278	0.2028	0.2028	0.0497	0.0013
LR-i	0.9500	0.4635	0.7483	0.7483	0.5311	0.2319
LR-cc	0.9500	0.0679	0.4222	0.4222	0.1198	0.0029
LR-uc	0.7500	0.5890	0.7376	0.5205	0.7376	0.5205
LR-i	0.7500	0.0624	0.0607	0.4297	0.1648	0.0821
LR-cc	0.7500	0.1523	0.1628	0.5955	0.3603	0.1795
LR-uc	0.2500	0.0005	0.0007	0.0001	0.0000	0.0005
LR-i	0.2500	0.5486	0.8444	0.5214	0.3870	0.7587
LR-cc	0.2500	0.0018	0.0031	0.0005	0.0002	0.0021
LR-uc	0.0500	0.9918	0.8444	0.8444	0.5367	0.6850
LR-i	0.0500	0.8062	0.8775	0.8775	-	0.9508
LR-cc	0.0500	0.9703	0.9693	0.9693	-	0.9193
LR-uc	0.0100	0.6445	0.3939	0.3939	0.6596	0.9964
LR-i	0.0100	-	-	-	-	-
LR-cc	0.0100	-	-	-	-	-
	DIS	**J-SAV**	**J-SAV-diff**	**J-SAV-IQR**	**J-AS-IQR**	**J-C-AS-IQR**
LR-uc (p-values)	0.9900	-	0.3336	0.0285	0.1260	0.1260
LR-i (p-values)	0.9900	-	-	-	-	-
LR-cc (p-values)	0.9900	-	-	-	-	-
LR-uc (p-values)	0.9500	0.0497	0.5386	0.3141	0.5386	0.5386
LR-i (p-values)	0.9500	-	-	0.8782	0.6074	0.2763
LR-cc (p-values)	0.9500	-	-	0.5954	0.7254	0.4577
LR-uc (p-values)	0.7500	0.0788	0.2493	0.3428	0.0631	0.5205
LR-i (p-values)	0.7500	0.9283	0.8333	0.8222	0.0508	0.9279
LR-cc (p-values)	0.7500	0.2125	0.5038	0.6217	0.0264	0.8101
LR-uc (p-values)	0.2500	0.0216	**0.0003**	**0.0005**	**0.0045**	**0.0063**
LR-i (p-values)	0.2500	0.7290	0.6285	0.5544	0.2175	0.2961
LR-cc (p-values)	0.2500	0.0674	**0.0013**	**0.0018**	**0.0083**	0.0139
LR-uc (p-values)	0.0500	0.1338	0.5367	0.5367	0.4048	0.5367
LR-i (p-values)	0.0500	0.6741	0.9746	0.9746	0.8992	0.9746
LR-cc (p-values)	0.0500	0.2976	0.8258	0.8258	0.7011	0.8258
LR-uc (p-values)	0.0100	0.6445	0.6596	0.6596	0.1049	0.6596
LR-i (p-values)	0.0100	-	-	-	-	-
LR-cc (p-values)	0.0100	-	-	-	-	-

Table 14: Back - testing results for joint quantile models: The p-values for the LR test by Christoffersen (1998) are provided for NASDAQ, and DIS. Models that are rejected by the LR test are in bold for rejection at 1% significance level. $-$ denotes that the p-value was not produced.

	θ	**J-SAV**	**J-SAV-diff**	**J-SAV-IQR**	**J-AS-IQR**	**J-C-AS-IQR**
IBM	0.9900	0.9963	0.9936	0.9860	0.9966	0.9916
	0.9500	0.4089	0.5494	0.4754	0.3220	0.1173
	0.7500	0.5154	0.6216	0.6901	0.4459	0.5098
	0.2500	0.6400	0.7784	0.8560	0.5378	0.2454
	0.0500	0.6802	0.3873	0.3994	0.5537	0.8204
	0.0100	0.8728	0.9606	0.9630	0.9973	0.9921
	θ	**J-SAV**	**J-SAV-diff**	**J-SAV-IQR**	**J-AS-IQR**	**J-C-AS-IQR**
SP500	0.9900	0.7202	0.9548	0.9863	0.9979	0.9904
	0.9500	0.1899	0.5659	0.2653	0.3387	0.2806
	0.7500	**0.0001**	0.1563	**0.0004**	0.0428	**0.0063**
	0.2500	0.0358	**0.0094**	**0.0022**	**0.0023**	0.0694
	0.0500	0.7460	0.1656	0.1675	0.4725	0.9395
	0.0100	0.9098	0.3424	0.9946	0.9950	0.7176
	θ	**J-SAV**	**J-SAV-diff**	**J-SAV-IQR**	**J-AS-IQR**	**J-C-ASd-IQR**
Boeing	0.9900	0.9904	0.9975	0.9978	0.9947	0.9888
	0.9500	0.7054	0.9276	0.6946	0.7969	0.0538
	0.7500	0.4485	0.6671	0.5365	0.6829	0.7370
	0.2500	0.2530	0.1688	0.2347	0.2344	0.0737
	0.0500	0.2796	**0.0010**	0.0124	**0.0096**	0.1777
	0.0100	0.9304	0.9652	**0.0005**	0.9992	**0.0035**
	θ	**J-SAV**	**J-SAV-diff**	**J-SAV-IQR**	**J-AS-IQR**	**J-C-AS-IQR**
CAT	0.9900	0.6871	0.7883	**0.0030**	0.9355	0.8808
	0.9500	0.3999	0.7807	0.6467	0.5014	0.5368
	0.7500	**0.0025**	0.0869	0.0645	0.0486	0.0596
	0.2500	0.0614	0.0883	0.1522	0.0589	0.0753
	0.0500	0.1871	0.3482	0.2794	0.3588	0.3086
	0.0100	0.8393	0.9650	0.9490	0.8762	0.8664

Table 15: Back - testing results for joint quantile models: The p-values for the DQ test by Engle and Manganelli (2004) are provided for each probability level for IBM, SP500, Boeing, CAT. Models which are rejected by the DQ test are in bold for rejection at 1% significance level.

	θ	**J-SAV**	**J-SAV-diff**	**J-SAV-IQR**	**J-AS-IQR**	**J-C-AS-IQR**
DOW	0.9900	0.9406	0.6090	0.9710	0.9925	0.8649
	0.9500	0.7277	0.2471	0.4425	0.3152	0.6320
	0.7500	0.1143	0.5940	0.3453	0.2857	0.3439
	0.2500	0.1659	0.0876	0.1985	0.1493	0.0531
	0.0500	0.1919	0.2979	0.4578	0.3823	0.0582
	0.0100	**0.0035**	**0.0072**	**0.0030**	0.0955	0.9334
	θ	**J-SAV**	**J-SAV-diff**	**J-SAV-IQR**	**J-AS-IQR**	**J-C-ASd-IQR**
FTSE	0.9900	0.9995	0.9943	0.9986	**0.0000**	0.9995
	0.9500	0.4372	0.6804	0.6607	0.0757	0.2894
	0.7500	0.0181	0.0640	0.0481	0.4391	**0.0041**
	0.2500	0.2941	0.2644	0.5166	0.0140	0.4355
	0.0500	0.4785	0.1290	0.5291	0.0743	0.3263
	0.0100	0.9354	0.6319	0.8369	0.5988	0.6657
	θ	**J-SAV**	**J-SAV-diff**	**J-SAV-IQR**	**J-AS-IQR**	**J-C-ASd-IQR**
NASDAQ	0.9900	0.7682	0.7601	0.9358	0.7652	0.7653
	0.9500	0.3653	0.7607	0.7607	0.5617	0.1447
	0.7500	0.0833	0.2689	0.8538	0.1441	0.0212
	0.2500	0.0143	0.0123	0.0096	0.0064	0.0349
	0.0500	0.3881	0.4942	0.5246	0.5972	0.9717
	0.0100	0.8570	0.5253	0.5529	0.9321	0.8880
	θ	**J-SAV**	**J-SAV-diff**	**J-SAV-IQR**	**J-AS-IQR**	**J-C-AS-IQR**
DIS	0.9900	-	0.9900	0.6231	0.9180	0.9045
	0.9500	0.5718	0.6922	0.7208	0.0903	0.2054
	0.7500	0.5360	0.6969	0.8228	0.2241	0.9031
	0.2500	0.1190	0.0175	0.0234	0.0697	0.0889
	0.0500	0.6707	0.7536	0.1722	0.7410	0.8193
	0.0100	0.8849	0.9770	0.7765	0.1633	0.6645

Table 16: Back - testing results for joint quantile models: The p-values for the DQ test by Engle and Manganelli (2004) are provided for each probability level for DOW, FTSE, NASDAQ, DIS. Models which are rejected by the DQ test are in bold for rejection at 1% significance level.

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