Equilibrium Currency Hedging under Equity-Currency Contagion

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Abstract

27 years after Black (1990, JF), the universal hedging formula (that is, in equilibrium, every investor would have the same hedging ratio towards any foreign currency regardless of the investor's home currency) remains the prevalent opinion on currency hedging both in the industry and in academia, despite the sophistication of the equity and foreign exchange market. In this paper, we propose a mutually exciting jump-diffusion model that explicitly accounts for equity-currency contagion. We characterize the "safe haven" currencies by a small equity-currency excitor, indicating that a price plunge in the equity market is not likely to trigger a depreciation of that currency. The "investment" currency, on the other hand, is characterized by a large equity-currency excitor, indicating that a price plunge in the equity market is very much likely to trigger a substantial depreciation of that currency. We first solve the portfolio optimization problem involving equity assets and risky currencies in closed form in a partial equilibrium framework, taking equity and exchange rate dynamics as given. Second, we impose security market clearing conditions and derive the equilibrium currency hedging strategies. We find that all else equal, investors hedge less safe haven currency risk than investment currency risk, a result that challenges the classic Black (1990) universal hedging formula.

Keywords: Portfolio choice; Currency contagion; Currency hedging; Market clear; Safe haven currency; Investment currency; Mutually exciting jumps.

JEL classification: G11, G12, G15, F31.

1 Introduction

The last two decades witnessed several episodes of financial and currency crises, most notably the 1994 Mexico peso crisis, the 1997 Asian crisis, the 1998 Russian crisis, the more recent 2008 global financial crisis and the subsequent 2009 European debt crisis. A common feature in these currency crises is that, among other things, the currency devaluation in a crisis is usually accompanied by dramatic capital market drops. During the Asian crisis, for instance, initiated by sharp currency devaluations in Southeast Asia, the Dow Jones Industrial Average plummets 554 points for its biggest point loss by then. Shortly after, the Korean won hits a record low in December, followed by Indonesian rupiah's free fall in January 1998 and the collapse of Russia's financial system in mid 1998.

The equity-currency contagion is a well documented phenomenon in the literature. For example, Caramazza, Ricci, and Salgado (2004) conclude that financial linkages are significant causes of currency crises after controlling for the role of domestic and external fundamentals, trade spillovers, and financial weaknesses in the affected countries. A strong financial linkage to the crisis country of origin not only raises the probability of contagion substantially, but also helps explain the observed regional concentration of currency crises. Pesenti and Tille (2000) study the Asian currency crisis and find that while weak or unsustainable economic policies provide a partial explanation of the currency crisis, they cannot account for the severity of the crises. One also need to take into account the volatile capital markets.[1] Fratzscher (2003) finds that the Latin American crisis in 1994-95 and the Asian crisis of 1997 spread across emerging markets are not primarily due to the weakness of those countries' fundamentals but rather to a high degree of financial interdependence among affected economies. Brunnermeier, Nagel, and Pedersen (2008) link the crash risk of carry trade strategies to funding constraints of speculators, with funding constraints measured by the implied volatility of the S&P 500 stock index. Ferreira Filipe and Suominen (2014) investigate how the financial market conditions in a major carry trade funding country, Japan, affect the global currency markets and currency trading and find that funding risks in Japan (measured by the stock options implied volatility and crash risk in the stock market in Japan) affect the global currency market. Consistent with these findings, De Bock and de Carvalho Filho (2015) find that during the risk-off episodes, currency markets exhibit recurrent patterns, as the Japanese yen, Swiss franc, and U.S. dollar appreciate against other G-10 and emerging market currencies. Lettau, Maggiori, and Weber (2014) study the cross section of currency returns using the downside risk capital asset pricing model. They find that high yield currencies earn higher excess returns than low-yield currencies because their co-movement with aggregate market returns is stronger conditional on bad market returns than it is

[1] This view is also shared by practitioners. For example, Bluford Putnam, the managing director and chief economist from CME Group claims that the emerging market currency contagion in 2013-2014 was driven by asset allocation shifts from emerging markets to US equities and other mature industrial markets.

conditional on good market returns. Francis, Hasan, and Hunter (2006) and Chernov, Graveline, and Zviadadze (2012) find that spillover from equity to currency market exists not only in mean but also in volatility.

The interdependence between equities and currencies leads to the concept of “safe haven” currencies and “investment currencies”. Ranaldo and Söderlind (2010) define an asset to be “safe haven” if it offers hedging benefits on average or in times of stress. They find that safe haven currencies tend to have low yield but immune to market downturns. “Investment currencies”, on the other hand, are like the mirror image of safe haven currencies – high yield and high exposure to systemic risks. When the global market is in stress, investors tend to move into “safe haven” currencies (Cenedese 2012). They identify safe haven currencies by regressing currency returns on current or lagged risk factors such as stock returns and bond returns. They conclude that the Swiss franc, the Japanese yen, and the British pound display safe currency characteristics. Nevertheless, covariance between currency returns and equity returns can be time varying, and can even change signs over time. Cenedese (2012) finds that during periods of bear, volatile world equity markets, currencies provide different hedging benefits than in bull markets. The 2008 financial crisis emerged as an important case study where safe haven effects went against typical patterns partially in contrast with the results of Ranaldo and Söderlind (2010). During the crisis, a large number of currencies that were not at the centre of the turmoil depreciated, even those which were regarded as safe haven currencies preceding the crisis (Kohler 2010). Habib and Stracca (2012) study what makes a safe haven currency in a systematic way and find that only a few factors are robust associated to a safe haven status.

The interplay between the equity market and the currency market poses challenges on optimal currency hedging. So far, there is no consensus on how much currency risks to hedge and even whether to hedge currency risks at all. Empirical work has been carried out to answer this question. On one hand, many studies have found that hedging currency risks reduces portfolio risks. For example, Glen and Jorion (1993) investigate the benefits from currency hedging with forward contracts and find that currency hedging significantly improves the performance of portfolios. Campbell et al. (2010) consider an investor with an exogenous portfolio of equities or bonds and ask how the investor can use foreign currencies to manage the risk of the portfolio. They find that the correlations between exchange rates and equity returns vary a lot across different currency pairs. On the other hand, papers like De Roon, Eiling, Gerard, and Hillion (2012) find that currency hedging reduces the volatility of portfolio returns at a cost of lower expected return distribution and fatter tails of international equity returns. On the extreme, Froot (1993) claims that currency exposure should be left unhedged for long term investors based on the assumption that purchasing power parity holds in the long run and exchange rates display mean reversion.

The complication of the equity and currency returns calls for models that account for the dependence structure of the equity market and the currency market. Indeed, as pointed out in Backus, Foresi, and Telmer (2001), the gross return of a foreign currency is exactly the ratio of the foreign stochastic discount factor return over the domestic one. As long as risk factors are compensated differently in the two economies, priced risk factors that drive the equity returns should in principle drive exchange rates. Therefore the equity market and the foreign exchange rate are interconnected in theory. Modeling equity and exchange rate jointly is not only empirically interesting but also of theoretical relevance.

Modeling equity and exchange rate dynamics is developed into separate strands of literature. On the equity side, to name a few, Pan (2002) estimates the S&P 500 index to a model in which both volatility and price may jump; Aït-Sahalia, Cacho-Diaz, and Laeven (2015) introduce the mutually exciting jump diffusion processes to model equity prices; Boswijk, Laeven, and Lalu (2015) add, on top of Aït-Sahalia et al. (2015), stochastic volatility and estimate the model using option prices.

On the currency side, an active research area in the exchange rate literature is to explain the forward premium puzzle and the carry trade returns. Examples are the factor models proposed by Backus, Foresi, and Telmer (2001), later extended by Lustig, Roussanov, and Verdelhan (2011) to account for the cross section of carry trade returns. Bates (1996) is one of the pioneers that include jumps in stochastic volatility models to capture exchange rate dynamics. Since then, FX models with jumps to capture crash risk in currency returns can be found in Chernov, Graveline, and Zviadadze (2012), Farhi and Gabaix (2008), Farhi, Fraiberger, Gabaix, Ranciere, and Verdelhan (2009), Carr and Wu (2007), Jurek (2014). Jumps in exchange rates have also been documented and studied using high frequency data by Lahaye, Laurent, and Neely (2011), Chatrath, Miao, Ramchander, and Villupuram (2014) and Lee and Wang (2014), etc.

Theoretical studies that account for the interdependence between the capital market and the exchange rate market are relatively scarce. One example is Bakshi, Carr, and Wu (2008), who decompose the stochastic discount factor (hence exchange rates) into interest rate risk, equity risk and an orthogonal component. A similar factor structure model can be found in Brusa, Ramadorai, and Verdelhan (2016), who include an equity factor, a carry factor and a Dollar factor in modeling exchange rate dynamics. Another attempt is Lettau, Maggiori, and Weber (2014), who propose to explain the currency return in a downside risk capital asset pricing model by including a downside equity beta.

The interdependent structure between equity and currency has important implications on international portfolio choice and optimal currency hedging strategies. The study on the theoretical multi-currency hedging in an equilibrium framework starts with Solnik (1974), which is expanded by Sercu (1980), Stulz (1981), Adler and Dumas (1983), etc. While the literature on modeling equity and exchange rate dynamics has grown fast in the past decade, relatively little is known on the international portfolio choice with currency risks in more realistic scenarios.[2] One of the first attempts on the equilibrium currency hedging is made by Black (1990). Using the geometric Brownian motion model for equity and currency returns, the paper derives a striking result: in equilibrium, every investor hedges the same amount of any risky currency regardless of the investor's home currency. This universal currency hedging ratio depends only on the average risk tolerance and on total wealth and total assets held by investors in each country. Surprisingly, Black's universal hedging ratio remains the prevalent opinion on currency hedging both in the industry and in academia even 27 years after the paper was published. Among existing literature that studies international portfolio choice problem with currency risks, (conditional) covariance between exchange rates and equity (bond) risks is used exclusively as the measure of interdependence between currencies and other asset classes, despite how sophisticated and interdependent the equity market and the foreign exchange market have become.

We contribute to the equity-currency literature by bridging this gap. We revisit the Black's equilibrium currency hedging problem under the context of equity-currency contagion. We propose a realistic model that generates the equity-currency contagion, which enables a theoretical characterization of the "safe haven" properties of a risky currency. We derive the equilibrium currency hedging strategies under this context.

To focus on the impact of equity-currency contagion, we strike a balance between model parsimony and consistency with the extant literature. In particular, we propose a mutually exciting jump diffusion model to describe equity and exchange rate processes jointly. In this model, an equity price jump today increases the probability of experiencing further price jumps in the equity market in the future as well as the probability of experiencing price jumps in the exchange rate market, and similar for the exchange rate jumps. The model therefore produces a rich dependence structure between the equities and foreign exchange rates – the normal dependence is captured by instantaneous covariance and the dependence during market turmoil is generated by jump excitation. Jump excitation in this case is a better candidate than time-varying covariance for two reasons. First, although there appears to be excess dependence between the equity market and the exchange rates, the comovement is neither exactly simultaneous nor certain. By mutually exciting jumps, a crash in the equity market only increase the *probability* of *future* currency jumps. Second, investment currencies, which are more prone to capital market turmoil, not necessarily have an equally strong impact on equities as equities on them, especially during recessions. Dependence generated by covariance is symmetric in nature, in the sense that if a currency were of the

[2]Some exceptions include Brown, Dark, and Zhang (2012), who study the optimal currency hedging problem in the context of stochastic volatility, and Torres (2012), who explore the optimal portfolio choice problem in a Poissonian jump diffusion model. Empirical papers include Glen and Jorion (1993) who investigate the benefits from currency hedging with forward contracts and find that currency hedging significantly improves the performance of portfolios. Campbell et al. (2010) consider an investor with an exogenous portfolio of equities or bonds and ask how the investor can use foreign currency to manage the risk of the portfolio. By looking at higher moments of hedged portfolio returns, De Roon, Eiling, Gerard, and Hillion (2012) find that while hedging lowers the volatility of international equity and bond portfolios, it results in unfavorable Sharpe ratios, skewness and kurtosis.

"investment" type measured by covariance, then its movement should also stir the equity market equally well. By capturing tail dependence using jump excitation, we allow for asymmetric excitation structure, in which case a currency that barely influences the equity market may acutely respond to equity market downturns.

While deviating from the log normal stochastic discount factors in the carry trade literature, the model complies with the foreign exchange literature findings that both global and country-specific risk factors are essential ingredients to generate the observed carry trade return patterns. In particular, our model is consistent with Brusa, Ramadorai, and Verdelhan (2016), in which three types factors are driving the stochastic discount factors: an equity factor that only drives the equity, a currency factor which only appears in the currency returns, and an equity-currency factor, which drives both the equity market and the foreign exchange rate. Our model can be regarded as an extension and variation of Farhi, Fraiberger, Gabaix, Ranciere, and Verdelhan (2009), where we allow the equity market and the exchange rate to be mutually exciting, while maintaining the factor structure that prevails the exchange rate literature.

We first solve the portfolio optimization problem with country-specific stocks and currencies in closed form in a partial equilibrium framework, taking equity and exchange rate dynamics as given. We show that the optimal net weight on a risky currency can be decomposed into four components: (1) the risk premium demand that earns the expected excess currency return by taking currency risks, (2) the risk management demand that exploits the diversification benefits through the instantaneous covariance matrix with other assets in the portfolio, (3) the myopic buy-and-hold demand which is induced by the discontinuities (jumps) in the currency returns, and (4) the intertemporal hedging demand that hedges the state variable uncertainty – the stochastic jump intensities in our case. The myopic buy-and-hold demand and the intertemporal hedging demand distinguish our prediction of the optimal currency holdings from, say, that of Solnik (1974) and Black (1990). The intertemporal hedging demand, in particular, is a result of the mutually exciting nature of the jump components.

To see the implication on the equilibrium currency hedging strategies under the equity-currency contagion context, we impose security market clearing conditions. Our equilibrium currency hedging differs from that of Black (1990) in the following aspects. Investors with different home currencies will in general have different hedging ratios towards a risky currency. More importantly, all else equal, investors have a larger hedging ratio for investment currencies, those that are prone to equity market turmoil than that for the safe haven currencies, those that are immune to equity market downturns. The preference for the safe haven currencies cannot be readily replicated by symmetric dependence measures, such as correlation.

This paper is organized as follows: Section 2 proposes the equity and exchange rate dynamics. We show that our model is able to generate equity-currency contagion while at the same time comply with the extant literature. Section 3 solves the optimal asset allocation problem in a partial equilibrium framework. Section 4 studies the property of the optimal net currency weights. Section 5 imposes the security market clearing conditions and derives the equilibrium currency hedging strategies. Section 6 illustrates the safe haven bias: all else equal, investors will have a larger hedging ratio towards investment currencies than the safe haven currencies, a result that cannot be directly replicated using linear correlation in classic models. Section 7 concludes.

2 A parsimonious model that allows for equity-currency contagion

In this section, we propose a model of equity and exchange rates that generates equity-currency contagion. This is achieved by including in the equity and currency returns both cross sectionally and serially dependent jump components, namely, mutually exciting jumps. We specify country-specific stocks, pricing kernels and exchange rates in Section 2.1. For market completeness, we also introduce country-specific stock derivatives. In Section 2.2, we give the pricing formula for call options in this context. Section 2.3 discusses how our model is related to the extant literature.

2.1 Set up

2.1.1 The equity market

In this section, we propose an equity-currency model that generates tail risk contagion between equity risk and currency risk. Let there be $n+1$ countries. Each country has its own currency. Let there be a risk free money market, a country-specific stock index and a derivative written on the stock index in each country, denominated in the domestic currency. We use superscript to denote in which currency the quantities are measured and subscript to denote the referred object. Variables without a superscript are prices denominated in the domestic currency. For instance, the domestic money market account of country i by is denoted by $B_i(t)$. We adopt the convention of denoting vectors and matrices using boldface characters to distinguish them from scalars. It holds that

$$\mathrm{d}B_i(t) = B_i(t) r_i(t)\,\mathrm{d}t, \tag{1}$$

where $r_i(t)$ is the continuously compounded risk free rate of country i.

Suppose the country stocks are exposed to a country-specific Brownian risk $W_i, i = 0, \dots, n$ and a global equity crash risk, which is modeled by a counting process N_m with intensity $\{\lambda_m(t)\}$. Denote the domestic stock of of country i by S_i^i, which follows the dynamics

$$\frac{\mathrm{d}S_i}{S_i} = r_i\,\mathrm{d}t + \mu_{s_i}\lambda_m\,\mathrm{d}t + \sigma_{s_i}\sqrt{\lambda_m}\,\mathrm{d}W_i + j_{s_i}(\,\mathrm{d}N_m - \lambda_m\,\mathrm{d}t), \tag{2}$$

where μ_{s_i} is the expected excess return; W_i is a standard Brownian motion; and j_{s_i} is the jump amplitude of country i, assumed to be a negative constant. The Brownian risk $W_i, i = 0, \ldots, n$, are independent of the jump risk N_m. The Brownian motions that drive the stocks of different countries are allowed to be correlated. Denote the correlation between W_i and W_j by ρ_{ij}.

In addition to the country stock, we introduce a stock option $O_i^i(t)$ in each country. If the market is free of arbitrage opportunities, there exists a risk neutral measure Q_i,[3] under which

$$O_i(t) = \mathbb{E}_t^{Q_i}[g(S_i(\tau), \lambda_m(\tau))],$$

for any $t \leq \tau$, where τ is the time to expiration.

The stock option provides exposure to the same risk factors to which the stock return is exposed to. As we will see later, the introduction of stock options completes the equity market in the sense that (1) the risk premiums of the equity Brownian motions and the equity jump component can be uniquely pinned down; (2) a portfolio that belongs to the $\mathcal{H}^2$ space with any exposure to the equity Brownian motion and equity jump component can be replicated using the stock and the stock option. To illustrate the latter point, as Liu and Pan (2003) explain, one can start with the stock and add out-of-the-money put options to the portfolio which provide more exposure to jump risk, in order to separate exposure to jump risk from that to the diffusive price shock.

Let $E_i^j(t)$ be the exchange rate between currency j and currency i, understood as the currency j price per unit of currency i. We choose currency $i = 0$ as the base currency. A superscript of 0 indicates base currency denominated variables.

If the exchange rate is stochastic, the money market of country i is a risky investment for investor from country $j, j \neq i$. The money market of country i denominated in currency j has price $B_i^j(t) = B_i^i(t)E_i^j(t)$, it holds that

$$\frac{\mathrm{d}B_i^j(t)}{B_i^j(t)} = r_j(t) + \frac{\mathrm{d}E_i^j(t)}{E_i^j(t)}.$$

Similarly, equity i denominated in currency j has price $S_i^j(t) = S_i^i(t)E_i^j(t)$ at time t.

Define the currency-hedged stock $\hat{S}_i^j(t)$ as

$$\frac{\mathrm{d}\hat{S}_i^j(t)}{\hat{S}_i^j(t)} = \frac{\mathrm{d}S_i^j(t)}{S_i^j(t)} - \Big(\frac{\mathrm{d}B_i^j(t)}{B_i^j(t)} - \frac{\mathrm{d}B_j(t)}{B_j(t)}\Big).$$

[3] More on the risk neutral measure and derivative prices in the next section.

The base-currency-hedged stock i for investor j can be constructed by a continuous-rebalanced portfolio that invests 100% in the unhedged stock i, borrowing 100% from country i and lending domestically. Effectively, borrowing abroad and lending domestically mimics the payoff of a currency forward contract (see Campbell, Serfaty-de Medeiros, and Viceira (2010)).

Define the currency-hedged global equity index as the weighted average of country stocks. Denoted in the base currency, it holds that

$$\hat{M}^0 = \sum_{i=0}^{n} h_i \hat{S}_i^0.$$

where h_i is country i's market capital as a proportion to the global capital. We have $\sum_{i=0}^{n} h_i = 1$. Note that h_i is a currency-independent variable.

The return dynamic of the global equity index is given by

$$\begin{aligned}\frac{\mathrm{d}\hat{M}^0(t)}{\hat{M}^0(t^-)} &= \sum_{i=0}^{n} \left(h_i \hat{\mu}_{s_i}^0 \lambda_m dt + \sigma_{s_i} \sqrt{\lambda_m} dW_i + \hat{j}_{s_i}^0 (dN_m - \lambda_m dt) \right) \\ &=: \mu_m^0 \lambda_m \,\mathrm{d}t + \sigma_m \sqrt{\lambda_m}\,\mathrm{d}W_m + j_m^0 (\,\mathrm{d}N_m - \lambda_m \,\mathrm{d}t),\end{aligned} \tag{3}$$

where

$$\sigma_m \sqrt{\lambda_m} dW_m(t) = \sum_{i=0}^{n} h_i \sigma_{s_i} \sqrt{\lambda_m} dW_i(t), \quad j_m^0 = \sum_{i=1}^{n} h_i \hat{j}_{s_i}^0,$$

Define $\boldsymbol{\sigma}_s$ as a diagonal matrix containing $\sigma_{s_i}, i = 0, \ldots, n$ on the diagonal, $\boldsymbol{h}$ as a vector containing $h_i, i = 0, \ldots, n$, and $\boldsymbol{W}(t)$ as a vector containing $W_i(t), i = 0, \ldots, n$. We can see that $W_m(t) = \frac{1}{\sigma_m} \boldsymbol{h}' \boldsymbol{\sigma}_s \boldsymbol{W}(t)$, $\boldsymbol{\sigma}_m = \sqrt{\boldsymbol{h}' \boldsymbol{\Sigma} \boldsymbol{h}}$. Here, $\boldsymbol{\Sigma}$ is the covariance matrix of countries' equities. Define $\boldsymbol{L}$ as the is a correlation matrix with ones on the diagonal and correlation coefficients of the currency-hedged equities off-diagonal.

$$\boldsymbol{L}\boldsymbol{L}' = \begin{pmatrix} 1 & \rho_{01} & \ldots & \rho_{0n} \\ \rho_{01} & 1 & \ldots & \rho_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ \rho_{0n} & \rho_{1n} & \ldots & 1 \end{pmatrix}.$$

It holds that $\boldsymbol{\Sigma} = \boldsymbol{\sigma}_s \boldsymbol{L} \boldsymbol{L}' \boldsymbol{\sigma}_s'$.

2.1.2 Pricing kernel processes and exchange rates

The change in exchange rate is effectively the ratio of the change in pricing kernel processes of the two countries (Backus, Foresi, and Telmer 2001). We first specify the pricing kernel process of each country and then derive the consistent exchange rate process thereafter.

Following Pan (2002), we specify the pricing kernel process of country $i, i = 1, \ldots, n$, to be of the following parametric form

$$\frac{\mathrm{d}\pi_i(t)}{\pi_i(t^-)} = -\left(r_i(t)\,\mathrm{d}t + \eta_i\sqrt{\lambda_m(t)}\,\mathrm{d}W_m + v_i\sqrt{\lambda_i(t)}\,\mathrm{d}Z_i\right) + \kappa_i(\,\mathrm{d}N_m(t) - \lambda_m(t)\,\mathrm{d}t)$$
$$+ \left(y_i\,\mathrm{d}N_i(t) - \mathbb{E}[y_i]\lambda_i(t)\,\mathrm{d}t\right). \tag{4}$$

Here, the pricing kernel also prices risks that do not drive equity returns. We introduce two new priced risk factors, a country-specific Brownian motion Z_i, assumed to be independent of other risk factors and also of $Z_j, j \neq i$, and a jump component N_i with jump intensity $\lambda_i(t)$ at time t. In Equation (4), η_i, κ_i are equity Brownian and jump risk premium in country i; v_i is a constant that represents the risk premium of Brownian motion Z_i; y_i is allowed to be a random variable.

The literature has shown that there are risk dimensions that influence currency returns in international economies but are absent in a single-economy equity market.[4] Our pricing kernel specification (4) is consistent with Bakshi, Carr, and Wu (2008) and Brusa, Ramadorai, and Verdelhan (2016), in that apart from domestic equity risk factors, the pricing kernel process of a country is also driven by foreign equity risk factors as well as risk factors which are not spanned in the international equity market.

Notice that the pricing kernel is driven by the global equity Brownian risk W_m, the global equity jump risk N_m and currency-specific risks Z_i, N_i. This is intuitive, the pricing kernel process π_i determines the risk premium of risky investment for investors in country i. On the equity side, consistent with international CAPM (see Solnik (1974)), only systematic risks are compensated. In our case, the systematic equity risk factors are the Brownian risk W_m and the jump risk N_m, as those that drive the global equity index.

The equity jump has deterministic jump sizes and is compensated only with the jump timing risk. Under the risk neutral measure, the jump component N_m has intensity $(1 + \kappa_i)\lambda_m$ under measure Q_i. With respect to the currency jump component, if we restrict that the jump risks are compensated for jump size risk but not for jump timing risk, as in Pan (2002), then only the jump size distribution changes under Q_i (determined by the distribution of y_i) and the jump intensity remains the same after the measure change.

As a normalization, we assume that the base currency is stable and is free of currency-specific risks. Following Pan (2002), the pricing kernel process of the base country is given by

$$\frac{\mathrm{d}\pi_0(t)}{\pi_0(t^-)} = -\,\eta_0\sqrt{\lambda_m(t)}\,\mathrm{d}W_m + \kappa_0(\,\mathrm{d}N_m(t) - \lambda_m(t)\,\mathrm{d}t), \tag{5}$$

[4] See, for example, Brandt, Cochrane, and Santa-Clara (2006).

where η_0, κ_0 represent equity Brownian and jump risk premium in the base country. Note that the base currency can be regarded as a reserve currency. The reserve currency feature of the base currency is produced by the fact that only systematic equity risks W_m, N_m are priced in the base country.

According to Backus, Foresi, and Telmer (2001), if the markets are integrated, exchange rates reflect the differences in pricing kernels in the associated markets.

$$E_i^0(t) = \pi_i(t)/\pi_0(t), \tag{6}$$

or, in SDE representation, x where

$$\mu_{e_i}^0 = \eta_0(\eta_0 - \eta_i) - \kappa_i j_i, \quad \sigma_{e_i} = \eta_0 - \eta_i, \quad j_{e_i}^0 = \frac{\kappa_i - \kappa_0}{1+\kappa_0}. \tag{7}$$

Notice that we may kill the equity jump in the exchange rate process by setting $\kappa_i = \kappa_0, \forall i$. In other words, currencies are free of equity jump risks only if the equity jump component is compensated the same way in every market.

We see that in the base country, since only systematic equity risk factors are priced, the country-specific currency risks do not lead to any risk premia. Investors from the base country are compensated for taking currency risk only through the common risk factors that drive both the equity and the exchange rate.

It is clear from Equation (4) and (5) that $\pi_i, i = 0, \ldots, n$, are local martingales under the real world measure. If π_i are actually martingales, one can verify according to the Lenglart-Girsanov Theorem that the pricing kernels are the Radon-Nikodym derivatives that changes the measure P to a risk neutral measure Q_i, under which the global equity index and country equities denominated in currency j follow

$$\begin{aligned}
\frac{\mathrm{d}\hat{M}^j(t)}{\hat{M}^j(t^-)} =& \big(\hat{\mu}_m^j - \eta_j\sigma_m + \hat{j}_m^j\kappa_j\big)\lambda_m(t)\,\mathrm{d}t + \sigma_m\sqrt{\lambda_m(t)}\,\mathrm{d}W_m^{Q_j}(t) + \hat{j}_m^j(\,\mathrm{d}N_{m,t}^{Q_j} - (1+\kappa_j)\lambda_m(t)\,\mathrm{d}t), \\
\frac{\mathrm{d}\hat{S}_i^j(t)}{\hat{S}_i^j(t^-)} =& \big(\hat{\mu}_{s_i}^j - \frac{\eta_j}{\sigma_m}\sum_{l=0}^{n} h_l\rho_{il}\sigma_{s_i}\sigma_{s_l} + \hat{j}_{s_i}^j\kappa_j\big)\lambda_m(t)\,\mathrm{d}t + \sigma_i\sqrt{\lambda_m(t)}\,\mathrm{d}W_i^{Q_j}(t) \\
&+ \hat{j}_{s_i}^j(\,\mathrm{d}N_{m,t}^{Q_j} - (1+\kappa_j)\lambda_m(t)\,\mathrm{d}t),
\end{aligned}$$

where $W_i^{Q_j}$ is a standard Brownian motion under the risk neutral measure of country j, with $W_i^{Q_j}(t) = W_i(t) - \eta_j\int_0^t\sqrt{\lambda_m(s)}\,\mathrm{d}s$. The jump process $N_m^{Q_j}(t)$ has intensity $(1+\kappa_j)\lambda_m(t)$ under the martingale measure Q_j. In order that $\hat{M}(t), \hat{S}_i^j(t)$ are local martingales under the risk neutral measure of country j, it should hold that

$$\begin{cases} \hat{\mu}^j_m = \sigma_m \eta_j - \kappa_j \hat{j}^j_m, \\ \hat{\mu}^j_{s_i} = \frac{\eta_j}{\sigma_m} \sum_{l=0}^n h_l \rho_{il} \sigma_{s_i} \sigma_{s_l} - \kappa_j \hat{j}^j_{s_i}. \end{cases} \tag{8}$$

Note that the expected excess returns of a country's stock consists of the risk premium of (I) the country-specific Brownian risk, and (II) the global equity crash risk. In particular, $\sum_{l=0}^n h_l \rho_{il} \sigma_{s_i} \sigma_{s_l}$ is the instantaneous covariance between the market equity return and stock i. Then the premium for country-specific Brownian risk is the premium for the market equity times the covariance between the market equity and stock i divided by the instantaneous variance of the market equity. Similarly, the premium for the jump risk of equity i is the ratio of the jump amplitude of equity i and that of the market equity. i.e.,

$$\hat{\mu}^j_{s_i}(I) := \frac{\eta_j}{\sigma_m} \sum_{l=0}^n h_l \rho_{il} \sigma_{s_i} \sigma_{s_l} = \frac{\mathrm{Cov}(\hat{R}^j_i(c), \hat{R}^j_m(c))}{\mathrm{Var}(\hat{R}^j_m(c))} \hat{\mu}^j_m(I), \quad \hat{\mu}^j_{s_i}(II) := -\kappa_j \hat{j}^j_{s_i} = \frac{\hat{j}^j_{s_i}}{\hat{j}^j_m} \hat{\mu}^j_m(II).$$

Here, $\hat{R}^j_i(c), \hat{R}^j_m(c)$ denote the continuous part of the return of stock i and the market equity, respectively. $\hat{\mu}^j_{s_i}(I)$ is the country-specific volatility risk premium that exhibits a CAPM structure, and $\hat{\mu}^j_{s_i}(II)$ is the jump premium.

In addition, free of arbitrage opportunities implies that similar structure applies to the countries' derivative prices

$$\hat{\mu}^j_{o_i} = \frac{\eta_j}{\sigma_m} \sum_{l=0}^n h_l \rho_{il} \sigma_{o_i} \sigma_{o_l} - \kappa_j \hat{j}^j_{o_i}.$$

2.1.3 Equity-currency contagion

We allow for jump propagation between equity and currencies by letting N_m, N_i to be mutually exciting with intensities $\lambda_m(t), \lambda_i(t)$ that follow

$$\begin{aligned} \mathrm{d}\lambda_m(t) &= \alpha_m(\lambda_{m,\infty} - \lambda_m(t))\,\mathrm{d}t + \beta_{m,m}\,\mathrm{d}N_m(t) + \sum_{l=1}^n \beta_{l,m}\,\mathrm{d}N_l(t), \\ \mathrm{d}\lambda_i(t) &= \alpha_i(\lambda_{i,\infty} - \lambda_i(t))\,\mathrm{d}t + \beta_{m,i}\,\mathrm{d}N_m(t) + \sum_{l=1}^n \beta_{l,i}\,\mathrm{d}N_l(t), \end{aligned}$$

where $\alpha_m, \alpha_i, \lambda_{m,\infty}, \lambda_{i,\infty}, \beta_{i,m}, \beta_{m,i}, \beta_{i,j}, \beta_{m,m} \geq 0, \ \forall i, j$.

The occurrence of a jump in the equity market at time t, i.e., $\mathrm{d}N_m(t) = 1$, not only raises the intensity of the equity jump component, $\lambda_m(t)$, by a non-negative amount $\beta_{m,m}$, but also increases the intensity of the currency jump component, $\lambda_i(t)$, by a non-negative amount $\beta_{m,i}$. After being excited, the intensity of both the equity jump component $\lambda_m(t)$ and the currency jump component $\lambda_i(t)$ mean revert to their respective steady state, $\lambda_{m,\infty}\lambda_{i,\infty}$, at exponential decaying rates α_m, α_i, until they get excited by a next jump occurrence.

In the remainder, we call β, defined as

$$\boldsymbol{\beta} := (\beta_m, \beta_1, \ldots, \beta_n) = \begin{pmatrix} \beta_{m,m} & \beta_{1,m} & \cdots & \beta_{n,m} \\ \beta_{m,1} & \beta_{1,1} & \cdots & \beta_{n,1} \\ \vdots & \vdots & \ddots & \vdots \\ \beta_{m,n} & \beta_{1,n} & \cdots & \beta_{n,n} \end{pmatrix},$$

the *excitation matrix* between equity and currency i; $\beta_{m,m}$ is called the *equity self excitor*; $\beta_{i,i}$ is called the *currency self excitor* of currency i; $\beta_{m,i}$ is called the *equity-currency excitor* of currency i, which measures the excitation from the equity jump component to the jump component of currency i; $\beta_{i,m}$ is called the *currency-equity excitor* of currency i, which measures the excitation from the jump component of currency i to the equity jump component.

Let $\boldsymbol{\lambda}(t) = (\lambda_m(t), \lambda_1(t), \ldots, \lambda_n(t))'$. The unconditional expectation of the jump intensity is given by

$$\mathbb{E}[\boldsymbol{\lambda}(t)] = (\boldsymbol{I}_n - \boldsymbol{\beta}./(\boldsymbol{\alpha}\boldsymbol{\iota}'))^{-1}\boldsymbol{\lambda}_\infty,$$

where $\boldsymbol{I}_n$ is an n by n identity matrix; $\boldsymbol{\alpha}, \boldsymbol{\lambda}_\infty$ are vectors of $\alpha_i, \lambda_{i,\infty}, i = m, 1, \ldots, n$, respectively; $\boldsymbol{\iota}$ is a column vector of all ones. The intensity processes can be made stationary by imposing

$$(\boldsymbol{I}_n - \boldsymbol{\beta}./(\boldsymbol{\alpha}\boldsymbol{\iota}'))^{-1} > 0.$$

This is a general yet parsimonious model which generates contagion between the equity market and the foreign exchange market. The equity and currency model given in (2) and (**??**) is a natural extension of the geometric Brownian motion models prevailing the equity-currency portfolio literature (see Solnik (1974), Black (1990), Campbell, Serfaty-de Medeiros, and Viceira (2010)). The model also generates stochastic volatility driven by jump intensity processes.

The mutually exciting jump components in Equation (2) and (**??**) are able to produce important stylized facts of equity-currency behavior. For example, the stock returns exhibit jump clustering as a result of the time series excitation, and equity-currency contagion as a result of the cross section excitation between these two asset classes. The market equity and currencies have an instantaneous covariance of $\sigma_m \sigma_{e_i} \lambda_m(t)$, which is stochastic, and increases when the equity market is in turmoil. This is consistent with the empirical findings of stochastic covariance between the equity market and the foreign exchange market. In addition, during market downturns, equity market turbulence can lead to currency turmoil and vice versa, creating non-linear excess dependence between equity market and the foreign exchange market.

The dependence generated by mutually exciting jumps have two distinctive features. The first is that the dependence between equity and currency is not simultaneous. Under contagious equity-currency risks, the exchange rate is likely to experience a jump in succession of an equity market plunge once the currency intensity builds up as a result of equity jumps. The dependence of the extreme movements in these two markets is neither simultaneous nor certain. Therefore it is not replicable by common risk factors. The second property is that the model allows for asymmetric excitation. Even if two currencies have the same influence on the equity market (i.e., same currency-equity excitors), they are allowed to have different exposure to equity market turmoil (i.e., different equity-currency excitors). It allows for separate analysis on the two way equity-currency contagion. A currency whose value remains relatively stable during equity market turbulence (i.e., low equity-currency excitor) has the property of a safe haven currency.

2.2 Option pricing

Suppose the market is free of arbitrage, then the price of an option $O_j(t)$ written on stock $S_j(t)$, with payoff function $f(S_j(\tau))$ is given by

$$O_j(t) = e^{-r_j(\tau-t)}\mathbb{E}_t^{Q_j}[f(S_j(\tau))]. \tag{9}$$

In this section, we consider the price of a standard call option on the domestic equity of each country. Appendix D also gives the pricing formula for put options and straddles. The payoff function for the call option is given by

$$C_j(\tau) = f(S_j(\tau)) = (S_j(\tau) - K)^+,$$

where K is the strike price.

The following proposition gives the call option price formula $C_j(t), t \leq \tau$, as a function of the stock price and the equity jump intensity at time t.

Proposition 1. *The call option price $C_j(t), t \leq \tau$ is given by*

$$C_j(t) = G_{1,-1}(-\log K) - KG_{0,-1}(-\log K), \tag{10}$$

where

$$G_{a,b}(w) = \frac{1}{2}\psi_t(a) - \frac{1}{\pi}\int_0^\infty \frac{Im[e^{-iuw}\psi_t(a+iub)]}{u}du, \tag{11}$$

with

$$\psi_t(u) = S_j(t)^u \exp(\mathcal{P} + \mathcal{Q}\lambda_m(t)). \tag{12}$$

Here, $\mathcal{P} = \mathcal{P}(t), \mathcal{Q} = \mathcal{Q}(t)$,

$$\begin{aligned}
\frac{d}{dt}\mathcal{Q}(t) =& \Big(\frac{1}{2}\sigma_{e_j}^2 + j_{e_j}(1+\kappa_j)\Big)u + \alpha_m \mathcal{Q}(t) - \frac{1}{2}u^2\sigma_{e_j}^2 - (1+\kappa_j)\Big((1+j_{e_j})^u e^{\beta_{m,m}\mathcal{Q}(t)} - 1\Big), \;\; \mathcal{Q}(\tau) = 0 \\
\frac{d}{dt}\mathcal{P}(t) =& -\alpha_m \lambda_{m,\infty}\mathcal{Q}(t), \;\; \mathcal{P}(\tau) = 0.
\end{aligned}$$

Therefore the dynamics of the options prices are given by

$$\begin{aligned}
\frac{\mathrm{d}O_j(t)}{O_j(t^-)} =& r_j(t)\,\mathrm{d}t + \mu_o(S_j(t), \lambda_m(t))\lambda_m(t)\,\mathrm{d}t + \sigma_o(S_j(t), \lambda_m(t))\sqrt{\lambda_m(t)}\,\mathrm{d}W_m(t) \\
&+ j_o(S_j(t), \lambda_m(t))(\,\mathrm{d}N_m(t) - \lambda_m(t)\,\mathrm{d}t),
\end{aligned}$$

where

$$\begin{aligned}
\sigma_{o_j}(S_j(t), \lambda_m(t)) &= \frac{\sigma_m S_j(t)}{O(t^-)} \frac{\partial f(S_j(t), \lambda_m(t))}{\partial S_j(t)}\bigg|_{(S_j(t), \lambda_m(t))} \\
j_{o_j}(S_j(t), \lambda_m(t)) &= \frac{1}{O_j(t^-)}\Big(f\big((1+j_{s_j})S_j(t), \lambda_m(t) + \beta_{m,m}\big) - f\big(S_j(t), \lambda_m(t)\big)\Big).
\end{aligned}$$

The price dynamics of the index option can be expressed as

$$\begin{aligned}
\frac{\mathrm{d}O_j(t)}{O_j(t^-)} =& r_j(t)\,\mathrm{d}t + \mu_{o_j}(S_j(t), \lambda_m(t))\lambda_m(t)\,\mathrm{d}t + \sigma_{o_j}(S_j(t), \lambda_m(t))\sqrt{\lambda_m(t)}\,\mathrm{d}W_j(t) \\
&+ j_{o_j}(S_j(t), \lambda_m(t))(\,\mathrm{d}N_m(t) - \lambda_m(t)\,\mathrm{d}t),
\end{aligned}$$

where

$$\begin{aligned}
\sigma_{o_j}(S_j(t), \lambda_m(t)) &= \frac{\sigma_{s_j} S_j(t)}{O_j(t^-)} \frac{\partial g(S_j(t), \lambda_m(t))}{\partial S_j(t)}\bigg|_{(S_j(t), \lambda_m(t))} \\
j_{o_i}(S_j(t), \lambda_m(t)) &= \frac{1}{O_j(t^-)}\Big(g\big((1+j_{s_j}S_j(t), \lambda_m(t) + \beta_{m,m}\big) - g\big(S_j(t), \lambda_m(t)\big)\Big).
\end{aligned}$$

We will drop the arguments of $\mu_{o_j}(S_j(t), \lambda_m(t)), \sigma_{o_j}(S_j(t), \lambda_m(t)), j_{o_j}(S_j(t), \lambda - M(t))$ and simply denote them by $\mu_{o_j}(t), \sigma_{o_j}(t)$ for notation simplicity, with the time t argument indicating state dependence.

2.3 Relation to the literature

2.3.1 Relation to a Poisson jump-diffusion economy

The pricing kernel specification of the base country can also be regarded as a normalization on the domestic pricing kernel. Suppose the jump components N_m, N_i were Poisson jumps with intensity λ_m and λ_i. Then the assumption on the base currency can be regarded as a normalization without loss of generality. To see this, observe the exchange rate of currency i against currency j:

$$\begin{aligned}\frac{\mathrm{d}E^j_{i,t}}{E^j_{i,t^-}} =&\Big(r_j(t) - r_i(t) + (\mu_{e_i} - \mu_{e_j} - \sigma_{e_i}\sigma_{e_j} + \sigma^2_{e_j})\lambda_m + (v_j^2 - \mathbb{E}\Big[\frac{y_j^2}{1+y_j}\Big])\lambda_j\Big)\,\mathrm{d}t \\ &+ (\sigma_i - \sigma_j)\sqrt{\lambda_m}\,\mathrm{d}W_m(t) - v_i\sqrt{\lambda_i}\,\mathrm{d}Z_i(t) + v_j\sqrt{\lambda_j}\,\mathrm{d}Z_j(t) + \frac{j_{e_i} - j_{e_j}}{1+j_{e_j}}(\,\mathrm{d}N_m(t) - \lambda_m(t)\,\mathrm{d}t) \\ &+ (y_i\,\mathrm{d}N_i(t) - \mathbb{E}[y_i]\lambda_i(t)\,\mathrm{d}t) - \Big(\frac{y_j}{1+y_j}\,\mathrm{d}N_j(t) - \mathbb{E}\Big[\frac{y_j}{1+y_j}\Big]\lambda_j(t)\,\mathrm{d}t\Big) \\ =&\Big(r_j(t) - r_i(t) + \bar{\mu}_{e_i}\bar{\lambda}_m\Big)\,\mathrm{d}t + \bar{\sigma}_{e_i}\sqrt{\bar{\lambda}_m}\,\mathrm{d}\bar{W}_m(t) + \bar{v}_i\sqrt{\bar{\lambda}_i}\,\mathrm{d}\bar{Z}_i(t) + (\bar{y}_i\,\mathrm{d}\bar{N}_i(t) - \mathbb{E}[\bar{y}_i]\bar{\lambda}_i(t)\,\mathrm{d}t).\end{aligned}$$

And the global equity denominated in currency j is given by

$$\begin{aligned}\frac{\mathrm{d}(\hat{M}^j/E^0_j)(t)}{(\hat{M}^j/E^0_j)(t^-)} =&r_j(t)\,\mathrm{d}t + \Big(\mu_m - \sigma_m\sigma_j - \mu_j + \sigma_j^2\Big)\lambda_m(t)\,\mathrm{d}t + (v_j^2 - \mathbb{E}\Big[\frac{y_j^2}{1+y_j}\Big])\lambda_j\,\mathrm{d}t \\ &+ (\sigma_m - \sigma_j)\sqrt{\lambda_m(t)}\,\mathrm{d}W_m(t) - v_j\sqrt{\lambda_j}\,\mathrm{d}Z_j + \frac{j_m - j_j}{1+j_j}(\,\mathrm{d}N_m(t) - \lambda_m(t)\,\mathrm{d}t) \\ &- \Big(\frac{y_j}{1+y_j}\,\mathrm{d}N_j - \mathbb{E}\Big[\frac{y_j}{1+y_j}\Big]\lambda_j\,\mathrm{d}t\Big) \\ &=: r_j(t)\,\mathrm{d}t + \bar{\mu}_m\bar{\lambda}_m\,\mathrm{d}t + \bar{\sigma}_m\sqrt{\bar{\lambda}_m}\,\mathrm{d}\bar{W}_m + \bar{j}_m(\,\mathrm{d}\bar{N}_m - \lambda_m\mathrm{d}t).\end{aligned}$$

Here, $\bar{W}_m(t), \bar{Z}_i(t)$ are independent and standard Brownian motions. $\bar{N}_m, \bar{N}_i$ are Poissonian jumps with intensities $\bar{\lambda}_m, \bar{\lambda}_i$ given by

$$\bar{\lambda}_m = \lambda_m + \lambda_j, \quad \bar{\lambda}_i = \lambda_i + \lambda_j, \tag{13}$$

In addition,

$$
\begin{aligned}
\bar{\mu}_m &= \frac{(\mu_m - \sigma_m\sigma_j + \sigma_j^2)\lambda_m + \left(v_j^2 - \mathbb{E}\left[\frac{y_j^2}{1+y_j}\right]\right)\lambda_j}{\lambda_m + \lambda_j},\\
\bar{\sigma}_i^2 &= \frac{(\sigma_i - \sigma_j)^2\lambda_m + v_j^2\lambda_j}{\lambda_i + \lambda_j},\\
\bar{v}_i &= v_i^2\frac{\lambda_i}{\lambda_i + \lambda_j},\\
\bar{\sigma}_m^2 &= \frac{(\sigma_m - \sigma_j)^2\lambda_m + v_j^2\lambda_j}{\lambda_m + \lambda_j}.
\end{aligned}
$$

From which we see that currency j can be regarded as the base currency and the model can be rewritten as the parametric form of Equation (5) by redefining the parameters.

The model proposed in the last section, therefore, can be regarded as a natural extension of the standard Poissonian jump diffusion model, in the sense that we let the Poissonian jumps to be mutually exciting in order to generate equity-currency contagion.

2.3.2 Relation to factor models

Should the jump factors $N_m, N_i, i = 1, \ldots, n$, be Poissonian, the pricing kernel of country i could be decomposed orthogonally into an equity component π_i^m and a currency component π_i^s as in Bakshi, Carr, and Wu (2008),

$$
\begin{cases}
\frac{\mathrm{d}\pi_i^{(m)}(t)}{\pi_i^{(m)}(t^-)} = -\eta_i\sqrt{\lambda_m}\,\mathrm{d}W_m + \kappa_i(\,\mathrm{d}N_m(t) - \lambda_m\,\mathrm{d}t),\\
\frac{\mathrm{d}\pi_i^{(s)}(t)}{\pi_i^{(s)}(t^-)} = -v_i\sqrt{\lambda_i}\,\mathrm{d}Z_i + (y_i\,\mathrm{d}N_i(t) - \mathbb{E}[y_i]\lambda_i(t)\,\mathrm{d}t),\\
\pi_{i,t} = \exp(-\int_0^t r_i(s)\,\mathrm{d}s)\pi_i^{(m)}(t)\pi_i^{(s)}(t).
\end{cases}
\tag{14}
$$

The pricing kernel processes given in Equation (4) exhibit a factor structure. Notice that not all risk factors are priced in the pricing kernel of country j. The consequence of this is that exchange rates of different currencies are exposed to different risk factors. Similar assumptions regarding the pricing kernel (that the pricing kernels are driven by both global factors and country-specific factors) can be found in Lustig, Roussanov, and Verdelhan (2011), Bakshi, Carr, and Wu (2008), and Farhi, Fraiberger, Gabaix, Ranciere, and Verdelhan (2009); consistent with Bates (1996), Carr and Wu (2007), the model has Gaussian and non-Gaussian factors; it is also consistent with Lustig, Roussanov, and Verdelhan (2014) who find countercyclical risk premium. We allow for the possibility that risk factors which are not priced in the equity to be priced in the pricing kernels, as in Bakshi, Carr, and Wu (2008).

Our model is consistent with Brusa, Ramadorai, and Verdelhan (2016), in which case there are three global factors that drive the stochastic discount factors. The first is a global equity factor N_m which can be priced exactly the same way in every country by setting $\kappa_i = \kappa_j,\ \forall i, j$. In this case, this factor will not appear in the exchange rate process but drives the world equity return. The second is a country-specific currency factor N_i. This factor only drives the exchange rate but not the equity returns, potentially capturing the crash risk in the carry trade. The third is an equity-currency factor W_m, which drives both the equity market and the foreign exchange rate, mimicking the "dollar factor" in Brusa, Ramadorai, and Verdelhan (2016).

3 Optimal asset allocation

Let there be a representative investor from each country. In this section we define and solve the optimal asset allocation problem for every investor with different home currencies. Instead of raw assets, which are foreign assets quoted in local-currency-denominated prices, we will look at currency-hedged asset prices, which has a one-to-one correspondence to the raw prices but easier to work with. Section 3.1 derives the dynamics of currency-hedged asset returns. Section 3.2 solves the asset allocation problem with the countries' stocks, stock options and bonds as the asset universe. Section 3.3 presents the Separation Theorem which states that the asset universe can be collapsed into a global equity, a global derivative portfolio and countries' bonds without any utility cost.

3.1 Returns on the currency-hedged-assets

When investing in a foreign stock, the investor is faced with not only the equity risk but also the currency risk. We will formulate the optimal asset allocation problem in terms of currency-hedged assets instead of the original assets. There is a one-to-one correspondence between the allocation strategy on the currency-hedged assets and that on the original assets. In the extreme case, an unhedged position in foreign stock j corresponds to a long position in currency j equal to the holding of stock j, whereas a fully hedged stock position corresponds to a net zero position in that foreign currency.

For an investor whose domestic currency is the base currency, a currency-hedged equity $j, j = 0, 1, \ldots, n$, has return dynamics

$$\begin{aligned}
\frac{\mathrm{d}\hat{S}_j^0(t))}{\hat{S}_j^0(t)} =& \frac{\mathrm{d}(S_j(t)E_j^0(t))}{S_j(t)E_j^0(t)} - \Big(\frac{\mathrm{d}B_j^0(t)}{B_j^0(t)} - \frac{\mathrm{d}B_0(t)}{B_0(t)}\Big) \\
=& \Big(\mu_{s_j} + \frac{\sigma_{e_i}\sigma_{s_i}}{\sigma_m}\sum_{l=1}^{n}\rho_{jl}h_l\sigma_{e_l} + \frac{j_{e_j}(j_{e_j} - j_{s_i})}{1 + j_{e_j}}\Big)\lambda_m(t)\,\mathrm{d}t + \Big(\mathbb{E}\Big[\frac{y_j^2}{1 + y_j}\Big] + v_j^2\Big)\lambda_j(t)\,\mathrm{d}t \\
&+ \sigma_{s_j}\sqrt{\lambda_m(t)}\,\mathrm{d}W_j(t) + j_{s_j}(1 + j_{e_j})(\,\mathrm{d}N_m(t) - \lambda_m(t)\,\mathrm{d}t) \\
=:& \hat{\mu}_{s_j}^0\lambda_m\,\mathrm{d}t + \sigma_{s_j}\sqrt{\lambda_m}\,\mathrm{d}W_j(t) + \hat{j}_{s_j}^0(\,\mathrm{d}N_m - \lambda_m\,\mathrm{d}t),
\end{aligned}$$

where $\hat{\mu}_{s_j}^0 = \mu_{s_j} + \frac{\sigma_{e_i}\sigma_{s_i}}{\sigma_m}\sum_{l=1}^{n}\rho_{jl}h_l\sigma_{e_l} + \frac{j_{e_j}(j_{e_j} - j_{s_i})}{1+j_{e_j}}$ is the expected return on stock j hedged against currency j risk for an investor from the base country; $\hat{j}^0_{s_j} = j_{s_j}(1 + j_{e_j})$ is the jump amplitude of the currency hedged stock.

One can construct a currency-hedged derivative j in the same way

$$\begin{aligned}\frac{\mathrm{d}\hat{O}^0_j(t)}{\hat{O}^0_j(t)} &= \frac{\mathrm{d}(O_j(t)E^0_j(t))}{O_j(t)E^0_j(t)} - \Big(\frac{\mathrm{d}B^0_j(t)}{B^0_j(t)} - \frac{\mathrm{d}B_0(t)}{B_0(t)}\Big) \\ &=: \hat{\mu}^0_{o_j}\lambda_m \,\mathrm{d}t + \sigma_{o_j}\sqrt{\lambda_m}\,\mathrm{d}W_j(t) + \hat{j}^0_{o_j}(\,\mathrm{d}N_m - \lambda_m \,\mathrm{d}t),\end{aligned}$$

with $\hat{\mu}^0_{o_j} = \mu_{o_j} + \frac{\sigma_{e_j}\sigma_{o_j}}{\sigma_m}\sum_{l=1}^n \rho_{jl}h_l\sigma_{e_l} + \frac{j_{e_j}(j_{e_j}-j_{o_i})}{1+j_{e_j}}$, $\hat{j}^0_{o_j} = j_{o_j}(1 + j_{e_j})$.

Similarly, for investor $i, i = 1, \ldots, n$, the return on currency hedged stock $j, j = 0, 1, \ldots, n$, and currency hedged derivative j are given by

$$\begin{aligned}\frac{\mathrm{d}\hat{S}^i_j(t)}{\hat{S}^i_j(t)} &= \frac{\mathrm{d}(S_j(t)E^0_j(t)/E^0_i(t))}{S_j(t)E^0_j(t)/E^0_i(t)} - \Big(\frac{\mathrm{d}B^i_j(t)}{B^i_j(t)} - \frac{\mathrm{d}B_i(t)}{B_i(t)}\Big) \\ &=: \hat{\mu}^i_{s_j}\lambda_m dt + \sigma_{s_j}\sqrt{\lambda_m}\,\mathrm{d}W_j(t) + \hat{j}^i_{s_j}(\,\mathrm{d}N_m - \lambda_m \,\mathrm{d}t), \\ \frac{\mathrm{d}\hat{O}^i_j(t)}{\hat{O}^i_j(t)} &= \frac{\mathrm{d}(O_j(t)E^0_j(t)/E^0_i(t))}{O_j(t)E^0_j(t)/E^0_i(t)} - \Big(\frac{\mathrm{d}B^i_j(t)}{B^i_j(t)} - \frac{\mathrm{d}B_i(t)}{B_i(t)}\Big) \\ &=: \hat{\mu}^i_{o_j}\lambda_m dt + \sigma_{o_j}\sqrt{\lambda_m} dW_j(t) + \hat{j}^i_{o_j}(dN_m - \lambda_m dt).\end{aligned}$$

Here, $\hat{\mu}^i_{s_j} = \mu_{s_j} + \frac{(\sigma_{e_i}-\sigma_{e_j})\sigma_{s_j}}{\sigma_m}\sum_{l=1}^n \rho_{jl}h_l\sigma_{sl}$, $\hat{j}^i_{s_j} = j_{s_j}(1 + j_{e_i})$, $\hat{\mu}^i_{o_j} = \mu_{o_j} + \frac{(\sigma_{e_i}-\sigma_{e_j})\sigma_{o_j}}{\sigma_m}\sum_{l=1}^n \rho_{jl}h_l\sigma_{e_l}$, $\hat{j}^i_{o_j} = j_{o_j}(1 + j_{e_i})$.

3.2 Solving for the optimal asset allocation problem

Define the portfolio weights vector $\hat{\boldsymbol{w}}^j(t) = (\hat{w}^j_{s_0}(t), \ldots, \hat{w}^j_{s_n}(t), \hat{w}^j_{o_0}(t), \ldots, \hat{w}^j_{o_n}(t), \hat{w}^j_{e_0}(t), \ldots, \hat{w}^j_{e_n}(t))'$ be a $3(n+1) \times 1$ vectored process, which are adapted, cáglád, and bounded in $\mathcal{L}^2$.[5] Problem 1 defines the asset allocation problem.

Problem 1. Let there be a representative investor from each country j with initial wealth x, who has expected power utility with risk aversion $u(x^j_j) = \frac{1}{1-\gamma_j}x^{1-\gamma_j}$, $\gamma_j > 0, \forall j$. Each investor is allowed to invest in foreign as well as domestic risk-free and (currency-hedged) risky assets. Investors neither consume nor receive any intermediate income. Assume that investors can rebalance their portfolios in continuous-time without incurring any transaction costs. The objective is to maximize the expected utility over terminal wealth $X_j(T)$ through optimal continuous time trading.

$$\sup_{\hat{\boldsymbol{w}}^j} \mathbb{E}_0\Big[\frac{X_j(T)^{1-\gamma_j}}{1-\gamma_j}\Big], \tag{15}$$

[5]Since portfolio weights cannot anticipate jumps, they are $\mathcal{F}_{t-}$ measurable and left continuous (cf. Aït-Sahalia and Hurd (2012)).

subject to the budget constraint:

$$\frac{\mathrm{d}X_j(t)}{X_j(t^-)} = r_j(t)\,\mathrm{d}t + \sum_{i=0}^{n} \hat{w}^j_{s_i} \frac{\mathrm{d}\hat{S}^j_i(t)}{\hat{S}^j_i(t^-)} + \sum_{i=0}^{n} \hat{w}^j_{o_i} \frac{\mathrm{d}\hat{O}^j_i(t)}{\hat{O}^j_i(t^-)} + \sum_{i=1}^{n} \hat{w}^j_{e_i} \frac{\mathrm{d}B^j_i(t)}{B^j_i(t^-)}. \tag{16}$$

Define the indirect utility function J for investor j at time $t = 0$ as

$$J(t, x, \boldsymbol{\lambda}) = \sup_{\hat{\boldsymbol{w}}^j} \mathbb{E}\Big[\frac{X_j(T)^{1-\gamma_j}}{1-\gamma_j}\Big],$$

where $x = X_j(t), \boldsymbol{\lambda} = (\lambda_m(t), \lambda_1(t), \ldots, \lambda_n(t))'$ are current values of the wealth and jump intensities. Define $\theta^j_{w_i}, \theta^j_n$ as the portfolio exposure to equity risk factors W_i, N_m,

$$\begin{cases} \theta^j_{w_i} = \hat{w}^j_{s_i}\sigma_{s_i} + \hat{w}^j_{o_i}\sigma_{o_i} + \sum_{l=1}^{n} \hat{w}^j_{s_l}\sigma_{s_l} h_l \sigma_{e_i}/\sigma_m, \\ \theta^j_n = \sum_{i=0}^{n}(\hat{w}^j_{s_i}\hat{j}^j_{s_i} + \hat{w}^j_{o_i}\hat{j}^j_{o_i}) + \sum_{i=1}^{n} \hat{w}^j_{e_i}\hat{j}^j_{e_i}. \end{cases} \tag{17}$$

Notice that $\sigma_{e_i} dW_m = \sum_{l=0}^{n} \sigma_{e_i} h_l \sigma_{s_l}/\sigma_m dW_l$. Therefore $\sigma_{e_i} h_l \sigma_{s_l}/\sigma_m$ is the exposure to the Brownian motion W_l in exchange rate E^0_i. Write in matrix notation $\boldsymbol{\theta}^j_w = (\theta^j_{w_0}, \ldots, \theta^j_{w_n})'$.

We are going to solve for the optimal $\boldsymbol{\theta}^j_w, \theta^j_n, \hat{w}^j_{e_i}, i = 1, \ldots, n$, by first conjecturing (which we later verify) that the indirect utility function is of the form

$$J(t, x, \boldsymbol{\lambda}) = \frac{\big(x_j\big)^{1-\gamma_j}}{1-\gamma_j} \exp(P(t) + Q(t)'\lambda) \tag{18}$$

where $P(t)$ and $Q(t)$ are functions of time but not of the state variables x and $\boldsymbol{\lambda}$.

The following proposition provides an analytical solution to the optimal portfolio strategy.

Proposition 2. *There exists a solution* $\hat{\boldsymbol{w}}^j(t) = (\hat{w}^j_{s_0}(t), \ldots, \hat{w}^j_{s_n}(t), \hat{w}^j_{o_0}(t), \ldots, \hat{w}^j_{o_n}(t), \hat{w}^j_{e_0}(t), \ldots, \hat{w}^j_{e_n}(t))'$ *for Problem 1. The optimal portfolio weight is given by solving the following equations for the elements* *of* $\hat{w}^j$*,*

$$\begin{cases} -\mathbb{E}[y_i] - \gamma_j \hat{w}^j_{e_i} v_i^2 + e^{Q'\beta_i}\mathbb{E}[(1+\hat{w}^j_{e_i} y_i)^{-\gamma_j} y_i] = 0, \quad i = 1,\dots,n, \\ \begin{pmatrix} \hat{w}^j_{s_0} \\ \vdots \\ \hat{w}^j_{s_n} \\ \hat{w}^j_{o_0} \\ \vdots \\ \hat{w}^j_{o_n} \end{pmatrix} = \begin{pmatrix} \sigma_{s_0} & 0 & \dots & 0 & \sigma_{o_0} & 0 & \dots & 0 \\ 0 & \sigma_{s_1} & \dots & 0 & 0 & \sigma_{o_1} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \sigma_{s_n} & 0 & 0 & \dots & \sigma_{o_n} \\ \hat{j}^j_{s_0} & \hat{j}^j_{s_1} & \dots & \hat{j}^j_{s_n} & \hat{j}^j_{o_0} & \hat{j}^j_{o_1} & \dots & \hat{j}^j_{o_n} \end{pmatrix}^{-1} \begin{pmatrix} \theta^j_{w_0} - \sum_{l=1}^n \hat{w}^j_{e_l}\sigma_{e_l} h_0 \sigma_{s_0}/\sigma_m \\ \vdots \\ \theta^j_{w_n} - \sum_{l=1}^n \hat{w}^j_{e_l}\sigma_{e_l} h_n \sigma_{s_n}/\sigma_m \\ \theta^j_n - \sum_{l=1}^n \hat{w}_{e_l}\hat{j}_{e_l} \end{pmatrix}, \\ \hat{w}^j_{e_0} = 1 - \sum_{l=0}^n (\hat{w}^j_{s_l} + \hat{w}^j_{o_l}) - \sum_{l=1}^n \hat{w}_{e_l}\hat{j}_{e_l}. \end{cases} \tag{19}$$

where

$$\begin{cases} \boldsymbol{\theta}^j_w = \frac{\eta_j}{\gamma_j \sigma_m}\boldsymbol{\sigma}'_s \boldsymbol{h}, \\ \theta^j_n = (1+\kappa_j)^{-\frac{1}{\gamma_j}} \exp(\frac{1}{\gamma_j}\boldsymbol{Q'\beta}_m) - 1. \end{cases} \tag{20}$$

Here, $\hat{w}^j_{e_i} = \hat{w}^j_{e_i}(t=0)$*,* $\boldsymbol{Q} = \boldsymbol{Q}(t=0)$*, in which* $\boldsymbol{Q}(t) = (Q_0, Q_1, \dots, Q_n)$ *is a deterministic process defined by the ordinary differential equation, for* $i = 1,\dots,n$*,*

$$\begin{cases} \dot{Q}_m(t) = \alpha_m Q_m(t) + \frac{\gamma_j - 1}{2\gamma_j}\eta_j^2 + (\gamma_j - 1)\kappa_j - \gamma_j (1+\kappa_j)^{\frac{\gamma_j-1}{\gamma_j}} \exp\left(\frac{1}{\gamma_j}\boldsymbol{Q}(t)'\boldsymbol{\beta}_m\right) + \gamma_j \\ \dot{Q}_i(t) = \alpha_i Q_i(t) + (1-\gamma_j)\hat{w}^j_{e_i}(t)\mathbb{E}[y_i] + \frac{1}{2}\gamma_j(1-\gamma_j)\hat{w}^j_{e_i}(t)^2 v_i^2 - \mathbb{E}[(1+\hat{w}^j_{e_i}(t) y_i)^{1-\gamma_j}] \exp(\boldsymbol{Q}(t)'\boldsymbol{\beta}_i) + 1, \\ \dot{P}(t) = -(1-\gamma_j) r_j - \alpha_m \lambda_{m,\infty} Q_0(t) - \sum_{i=1}^n \alpha_i \lambda_{i,\infty} Q_i(t). \end{cases} \tag{21}$$

with $P(T) = 0,\ \boldsymbol{Q}(T) = 0$*.*

The pricing kernel of the base country shows that only the two equity risk factors are priced in the base country. The stocks and stock derivatives are sufficient to complete the equity market in the sense that they are able to provide any (admissible) exposure to the two risk factors. It seems that the optimal portfolio should consist of the stocks and the stock options exclusively and any foreign currency should be redundant. It would be true if the equity jump component N_m and the currency jump component N_i are independent or have linear dependency. Due to the nonlinear dependence structure generated by the mutually exciting jumps, the investor would demand foreign currency exposure for their hedging potentials even if the currency-specific risks are not rewarded.

As one may expect, in a Poisson jump-diffusion economy where both equity and currency dynamics follow (correlated) Poisson jump-diffusion processes, the base investor would never choose to invest in any foreign currency (provided that he has access to stocks and stock options) if the currency-specific risks are not compensated in his country. This property is summarized in Remark 1.

From Proposition 2, the optimal portfolio weights for any investor can be calculated by solving simultaneously a simple pair of equations, Equation (19) and (21). Each pair of equations are easily solved numerically using standard finite difference method. Specifically, for $j = 0, \ldots, n$, one starts with the terminal condition $Q_i(T) = 0$ to derive the optimal weights at the terminal time T, $\hat{w}^j_{e_i}(T)$. Then one goes back a small time interval Δ, and calculate $Q^j_i(T - \Delta)$ using $\hat{w}^j_{e_i}(T - \Delta)$. Continue with the recursive algorithm until one reaches time zero. A step size as small as a quarter of a day is enough to generate the desired accuracy. The computation burden in Proposition 2 is almost negligible compared to numerically solving the multi-dimensional HJB equation (40).

The semi-closed form of the portfolio weights crucially depends on the state-independent feature of $\boldsymbol{\theta}^j_w, \theta^j_n$. Note that although $\boldsymbol{\theta}^j_w, \theta^j_n$ are independent of the state variables, the weights on the stocks and stock options are not. This is due to the fact that the price dynamics of the stock options, in particular, $\sigma_{o_i}(t), j_{o_i}(t)$, are driven by state variables $S^i_i(t), \lambda_m(t)$. Therefore there is market timing in the optimal portfolio weights on the stocks and stock options.

Remark 1. *In the context of equity-currency contagion, even if the idiosyncratic currency risk is not rewarded in the base country, the investor would not hedge 100 percent of the currency risk, as long as the currency risk is able to spillover to the equity market. The nonlinearity of dependence between the equity market and the foreign exchange market as a result of contagion leads to non-zero currency exposure for hedging purposes. If the dependence structure were linear, the investor will opt for a total hedge of currency risks in the absence of currency risk premia.*

Remark 2. *One important distinction between the optimal currency hedging strategy predicted by our model and that of Campbell, Serfaty-de Medeiros, and Viceira (2010) is that the optimal currency demand in our model is home currency dependent. In Campbell, Serfaty-de Medeiros, and Viceira (2010), for example, the residents of both the United States and Germany will have the same optimal demands for Australian Dollar corresponding to a given equity portfolio. In our model, however, since the currency demand is generated through nonlinear dependence between currencies and the equity market, investors from different home currencies would have different demand for a foreign currency per unit of world equity invested.*

3.3 The Separation Theorem

Solnik (1974) proves a three-fund separation theorem in case of a geometric Brownian motion model. In particular, he shows that investors are indifferent between the country stocks and a market equity index. Observe that in Equation (16), while there are $n+2$ equity risk factors ($n+1$ country-specific Brownian risk factors and one global equity jump factor), there are $2(n+1)$ equity assets, one stock and one stock option from each country. One can also see from Equation (19) that the matrix to be inverted has full row rank but not full column rank, indicating redundant equity assets.

The next theorem presents the $n+1+2$ fund separation result of our model.

Theorem 1. *Every investor is indifferent between choosing portfolios from the original* $3(n+1)$ *assets or from* $(n+1)+2$ *funds. From the perspective of investor* j*, a possible choice for those funds is*

- *the market equity index (hedged against currency risk)* $\hat{M}^j$*, as defined in Equation* (**??**).
- *a portfolio of stock derivatives (hedged against currency risk)* $\hat{D}^j$*, defined as*

$$\hat{D}^j = \sum_{i=0}^{n} k_i \hat{O}_i^j,$$

with

$$(k_0, \ldots, k_n)' = \frac{\boldsymbol{\sigma}_o^{-1}\boldsymbol{\sigma}_e'\boldsymbol{h}}{\boldsymbol{\iota}'(\boldsymbol{\sigma}_o^{-1}\boldsymbol{\sigma}_e'\boldsymbol{h})}, \tag{22}$$

- *the* $n+1$ *bonds of each country*

In light of Theorem 1, the investable asset universe for every investor is the $n+1$ bonds of each country (the domestic bond is regarded as the risk-free asset), a currency-hedged global equity index

$$\frac{\mathrm{d}\hat{M}^j(t)}{\hat{M}^j(t)} = \mu_m^j \lambda_m(t)\,\mathrm{d}t + \sigma_m \sqrt{\lambda_m(t)}\,\mathrm{d}W_m(t) + j_m^j(\,\mathrm{d}N_m(t) - \lambda_m(t)\,\mathrm{d}t),$$

and a currency-hedged portfolio of derivatives

$$\frac{\mathrm{d}\hat{D}^j(t)}{\hat{D}^j(t)} = \mu_d^j \lambda_m(t)\,\mathrm{d}t + \sigma_d \sqrt{\lambda_m(t)}\,\mathrm{d}W_m(t) + j_d^j(\,\mathrm{d}N_m(t) - \lambda_m(t)\,\mathrm{d}t),$$

with

$$\sigma_d = \frac{\sigma_m}{\boldsymbol{\iota}'(\boldsymbol{\sigma}_o^{-1}\boldsymbol{\sigma}_e'\boldsymbol{h})}, \quad j_d^j = \frac{\boldsymbol{h}'\boldsymbol{\sigma}_e\boldsymbol{\sigma}_o^{-1}}{\boldsymbol{\iota}'(\boldsymbol{\sigma}_o^{-1}\boldsymbol{\sigma}_e'\boldsymbol{h})}\boldsymbol{j}_o^j.$$

We may redefine the optimal asset allocation problem in terms of the market equity, market derivative portfolio and the bonds.

Problem 2. Let there be a representative investor from each country j, who has expected power utility with risk aversion γ_j and aims to maximize his expected utility at time $t = 0$ through optimally investing:

$$\sup_{\hat{\boldsymbol{w}}^j} \mathbb{E}_0\Big[\frac{X_j(T)^{1-\gamma_j}}{1-\gamma_j}\Big], \tag{23}$$

subject to the budget constraint:

$$\frac{\mathrm{d}X_j(t)}{X_j(t^-)} = r_j\,\mathrm{d}t + \hat{w}_m^j \frac{\mathrm{d}\hat{M}^j(t)}{\hat{M}^j(t^-)} + \hat{w}_d^j \frac{\mathrm{d}\hat{D}^j(t)}{\hat{D}^j(t^-)} + \sum_{i=1}^n \hat{w}_{e_i}^j \frac{\mathrm{d}\hat{B}_i^j(t)}{\hat{B}_i^j(t^-)}. \tag{24}$$

The following proposition solves the above portfolio choice problem.

Proposition 3. *The asset allocation problem in Problem 2 has a solution $\hat{w}^j = (\hat{w}_m^j, \hat{w}_d^j, \hat{w}_{e_0}^j, \dots, \hat{w}_{e_n}^j)$. The optimal portfolio weight is given by solving the following nonlinear equation for $\hat{w}^j$,*

$$\begin{cases} -\mathbb{E}[y_i] - \gamma_j \hat{w}_{e_i}^j v_i^2 + e^{Q'\beta_i}\mathbb{E}[(1+\hat{w}_{e_i}^j y_i)^{-\gamma_j} y_i] = 0, \; i = 1,\dots,n \\ \begin{pmatrix} \hat{w}_m^j \\ \hat{w}_d^j \end{pmatrix} = \begin{pmatrix} \sigma_m & \sigma_o \\ \hat{j}_m^j & \hat{j}_d^j \end{pmatrix}^{-1} \begin{pmatrix} \theta_m^j - \sum_{l=1}^n \hat{w}_{e_l}^j \sigma_{e_l} h_0 \sigma_{s_0}/\sigma_m \\ (1+\hat{j}_{e_j}^j)\theta_n^j - \sum_{l=1}^n \hat{w}_{e_l}^j j_{e_l} \end{pmatrix} \end{cases}, \tag{25}$$

where

$$\begin{cases} \theta_m^j = \frac{1}{\gamma_j}\eta_j, \\ \theta_n^j = (1+\kappa_j)^{-\frac{1}{\gamma_j}} \exp(\frac{1}{\gamma_j}\boldsymbol{Q}'\boldsymbol{\beta}_m) - 1. \end{cases} \tag{26}$$

Here, $\hat{w}_{e_i}^j = \hat{w}_{e_i}^j(t=0)$, $\boldsymbol{Q} = \boldsymbol{Q}(t=0)$, in which $\boldsymbol{Q}(t)$ is a deterministic vectored process given by Equation (21)*.*

4 Properties

In Section 3.3 we show that the asset allocation problem boils down to optimally investing in the global equity index, the global derivative portfolio and currencies. In this section, we focus on the optimal weights on this simplified universe of assets (instead of the country-specific stocks and derivatives), especially the optimal weights on currencies. In Section 4.1, we decompose the optimal net currency weight into four components, among which the intertemporal hedging component is of particular interests. In Section 4.2, we conduct comparative statics analysis of the intertemporal hedging demand with respect to jump risk parameters.

4.1 Decompose the currency weight

Now that we have solved the asset allocation problem for investors from each country, we study the property of the optimal net currency weights in their portfolios in this section. Note that the solutions given by proposition 3 are general results where exchange rates are exposed to both the equity jump component and the currency-specific jump component. In this section, we make the simplified assumption that $j_i = 0, \forall i = 1, \ldots, n$.

We can write the HJB equation in terms of portfolio weights on the global equity index, the derivative portfolio and the risky currencies.

$$\begin{aligned}
0 = \sup_{\hat{\boldsymbol{w}}^j} \Big\{ & J_t + \Big(r_j + \hat{w}_m^j(\hat{\mu}_m^j - \hat{j}_m^j)\lambda_m + \hat{w}_d^j(\hat{\mu}_d^j - \hat{j}_d^j)\lambda_m + \sum_{i=1}^{n} \hat{w}_{e_i}^j(\hat{\mu}_{e_i}^j\lambda_m - \mathbb{E}[y_i]\lambda_i)\Big) J_x x \\
& + \alpha_m(\lambda_{m,\infty} - \lambda_m)J_{\lambda_m} + \sum_{i=1}^{n} \alpha_i(\lambda_{i,\infty} - \lambda_i)J_{\lambda_i} + \frac{1}{2}\Big(\big(\hat{w}_m^j\sigma_m\big)^2\lambda_m + \big(\hat{w}_d^j\sigma_d\big)^2\lambda_m \\
& + \sum_{i=1}^{n} \big(\hat{w}_{e_i}^j\big)^2(\sigma_{e_i}^2\lambda_m + v_i^2\lambda_i) + 2\hat{w}_m^j\hat{w}_d^j\sigma_m\sigma_d\lambda_m + 2\sum_{i=1}^{n} \hat{w}_m^j\hat{w}_{e_i}^j\sigma_m\sigma_{e_i}\lambda_m \\
& + \sum_{i=1}^{n}\sum_{l=1}^{n\backslash i} \hat{w}_{e_i}^j\hat{w}_{e_l}^j\sigma_{e_i}\sigma_{e_l}\lambda_m + 2\sum_{i=1}^{n} \hat{w}_d^j\hat{w}_{e_i}^j\sigma_o\sigma_{e_i}\lambda_m\Big) J_{xx}x^2 \\
& + \lambda_m\Big(J\big(x(1 + \hat{w}_m^j\hat{j}_m^j + \hat{w}_d^j\hat{j}_d^j), \boldsymbol{\lambda} + \boldsymbol{\beta}_m\big) - J\Big) \\
& + \sum_{i=1}^{n} \lambda_i\mathbb{E}\Big[\Big(J\big(x(1 + \hat{w}_{e_i}^j y_i), \boldsymbol{\lambda} + \boldsymbol{\beta}_i\big) - J\Big)\Big]\Big\}.
\end{aligned}$$

If $\hat{w}_m^j, \hat{w}_d^j, \hat{w}_{e_i}^j$ given by Proposition 3 are optimal, then by substituting J for its functional form (18), $\hat{w}_{e_i}^j$ must satisfy the following first order conditions

$$\begin{aligned}
0 = & \hat{\mu}_{e_i}^j\lambda_m - \mathbb{E}[y_i]\lambda_i - \gamma_j\Big(\hat{w}_d^j\sigma_d\sigma_{e_i}\lambda_m + \hat{w}_m^j\sigma_m\sigma_{e_i}\lambda_m + \sum_{l=1}^{n} \sigma_{e_l}\sigma_{e_i}\lambda_m + \hat{w}_{e_i}^j(\sigma_{e_i}^2\lambda_m + v_i^2\lambda_i)\Big) \\
& + \lambda_i e^{\boldsymbol{Q}'\boldsymbol{\beta}_i}\mathbb{E}[(1 + \hat{w}_{e_i}^j y_i)^{-\gamma_j} y_i], \qquad (27) \\
=: & a_i^j - \gamma_j\Big(\hat{w}_d^j\sigma_{id} + \hat{w}_m^j\sigma_{im} + \sum_{l=1}^{n} \hat{w}_{e_l}^j\sigma_{jl}\Big) + \lambda_i(Y_i - \mathbb{E}[y_i]) + \lambda_i(e^{\boldsymbol{Q}'\boldsymbol{\beta}_i} - 1)Y_i,
\end{aligned}$$

where $a_i^j = \hat{\mu}_i^j\lambda_m$ is the expected excess return of currency i; $\sigma_{id} = \sigma_d\sigma_{e_i}\lambda_m$ is the covariance between the index option and currency i for the investor from the base country; $\sigma_{im} = \sigma_m\sigma_{e_i}\lambda_m$ is the covariance between the market equity portfolio and currency i; $\sigma_{il} = \sigma_{e_i}\sigma_{e_l}\lambda_m$ is the covariance between currency j and currency l; $b_i = \sigma_{e_i}^2\lambda_m + v_i^2\lambda_i$ is the instantaneous variance of currency i; and $Y_i = \mathbb{E}[(1 + \hat{w}_{e_i}y_i)^{-\gamma_j}y_i]$ is the marginal utility increase induced by jump component N_i from investing in one unit of the foreign currency i for the investor from the base country.

Rearrange Equation (27) and get

$$\hat{w}^j_{e_i} = \frac{1}{\gamma_j b_i}\Big\{ \underbrace{a^j_i}_{I} \underbrace{-\gamma_j(\hat{w}^j_d \sigma_{id} + \hat{w}^j_m \sigma_{im} + \sum_{l}^{n\setminus i} \hat{w}^j_{e_l}\sigma_{e_i})}_{II} + \underbrace{\lambda_i Y_i}_{III} + \underbrace{\lambda_i (e^{\boldsymbol{Q}'\boldsymbol{\beta}_i} - 1) Y_i}_{IV} \Big\}. \tag{28}$$

The optimal portfolio weights consist of a risk premium demand (I), a risk management demand (II), a myopic buy-and-hold demand (III), and an intertemporal hedging demand (IV).

The risk premium demand (I) is determined by the expected excess return on investing in the foreign currency i. It is a return-driven demand. A larger expected excess return indicates larger appreciation of the currency in expectation with respect to the domestic currency. The risk management demand (II) exploits the diversification benefit of investing in the risky currency contained in the covariance between the market equity, equity derivative portfolio and other risky currencies. This is the demand that has been extensively studied in the international empirical finance literature. For example, Glen and Jorion (1993), Campbell et al. (2010), De Roon et al. (2012) all base their currency hedging strategies solely on the risk management demand (covariance with the market equity portfolio).

The myopic buy-and-hold demand (III) arises because foreign exchange rates have jumps. As explained by Liu et al. (2003), unlike continuous fluctuations, jumps may occur before the investor has the opportunity to adjust the portfolio. Jump risks, therefore, are similar to "illiquidity risk": the investor has to hold the asset until the jump has occurred. Observe that

$$Y^j_i \propto \nabla_{\hat{w}^j_{e_i}} \mathbb{E}[u(X_j(t)) - u(X_j(t^-)|N_i(t) - N_i(t^-) = 1].$$

$\mathbb{E}[u(X_j(t)) - u(X_j(t^-)|N_i(t) - N_i(t^-) = 1]$ is the expected utility gain at time t conditional on an occurrence in jump component l at time t . Therefore (III) is the expected marginal utility increase induced by jump component i from investing in one unit of risky assets at time t. The buy-and-hold demand is "myopic" in the sense that it does not take into account the uncertainties of future jump intensities. Note that in case of $\gamma_j = 0$, meaning that the investor is risk neutral, this term is zero.

The last term (IV) is tailored to account for the fact that the jumps are mutually exciting. Since the asset prices $B^j_i(t)$ and the state variables $\boldsymbol{\lambda}(t)$ are both driven by jumps $N_i(t)$, foreign bond i can be used to hedge future realizations of the state variables to hedge changes in $\boldsymbol{\lambda}(t)$. Intuitively, the mean-variance demand and the myopic buy-and-hold demand exploit the risk-return trade-off of the risky assets, whereas the intertemporal hedging demand is only concerned with state variable uncertainties.

All four components of the portfolio weights can be time-varying, but for different reasons. The risk premium demand (I), risk management demand (II) and myopic buy-and-hold demand (III) depend on the state variables $M^j(t), \lambda_m(t), \lambda_i(t)$. Hence they change with the state variables instantaneously. The intertemporal hedging demand (IV), on the other hand, depends not only on the spot values of the state variables, but also on how the returns and the state variables evolve within the investment horizon. The information of future outcomes is contained in $\boldsymbol{Q}(t)$, which is horizon dependent.

As one may expect, net currency weights predicted by special cases of our model are combinations of the decomposed terms. In particular, if the currency returns are independent of equity returns, as assumed by Solnik (1974), the risk management demand (II) of the currency is zero. If the economy is free of jumps as in Sercu (1980), Adler and Dumas (1983), Black (1990), both the myopic buy-and-hold demand (III) and the intertemporal hedging demand (IV) are zero. If the jumps are Poissonian with constant jump intensities as in Torres (2012), the intertemporal hedging demand (IV) is zero.

4.2 Comparative statics

The risk premium demand, risk management demand and the buy-and-hold demand components of the currency hedging strategy can be interpreted in a straightforward way by observing Equation (28). One can immediately tell that the risk premium demand (I) increases when the investor can earn a higher expected excess return from investing in the risky currency; the risk management demand (II) is negative when there is positive correlation between other assets and the currency and positive otherwise; the buy-and-hold demand (III) is negative when the currency jumps downward and positive if it jumps upward.

The intertemporal hedging demand (IV), however, depends on the excitation structure between the equity jump component and the currency jump component. To see how (IV) is determined by the jump excitation parameters, we conduct comparative statics analysis in Figure 1 and 2.

We consider three countries: a base country with the base currency, Country I, and Country II. We make the simplified assumption that these three countries represent the global financial market. We denote the currencies from these countries by the base currency, Currency I, and Currency II. Similarly, we call the representative investors from these countries the base investor, Investor I and Investor II, respectively.

Here, we only study the comparative statics of the optimal net weight on Currency I from the perspective of the base investor. The behaviour of the optimal net weight on Currency I for investor II has a qualitatively similar pattern. To keep the analysis clean, we adopt a deterministic currency jump size $y_1,\ y_2 < 0$ in this section.

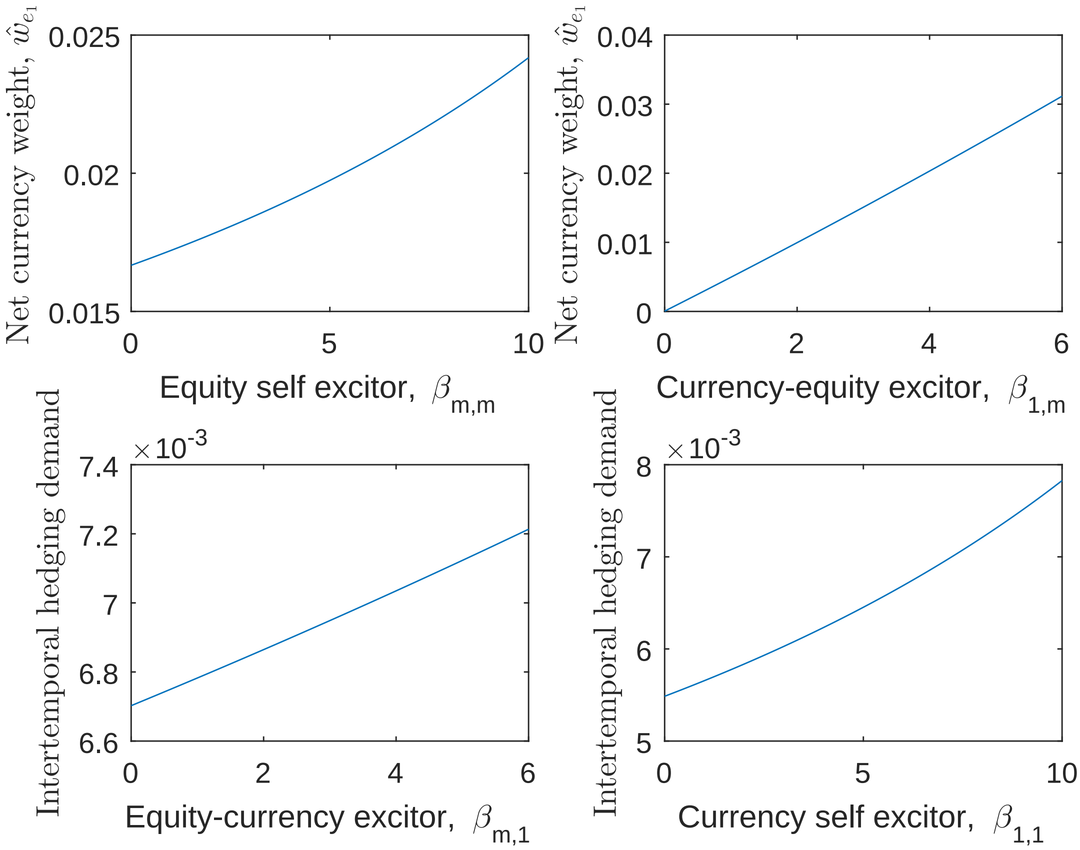

Figure 1: The intertemporal hedging demand (IV) of Currency I for the base investor as functions of elements in the excitation matrix $\boldsymbol{\beta}$. The base case parameters are $\eta_0 = 0.1,\ \sigma_m = 0.2,\ \sigma_d = 0.1,\ \sigma_{e_1} = \sigma_{e_2} = -0.1,\ v_1 = v_2 = 0.05,\ j_m = -0.03,\ j_d = 0.1,\ \alpha_m = \alpha_1 = \alpha_2 = 35,\ \boldsymbol{\beta} = (15, 6, 6; 6, 8, 0; 6, 0, 8),\ T = 1,\ \kappa_1 = \kappa_2 = 0.02,\ y_1 = y_2 = -2\%,\ \gamma_0 = \gamma_1 = \gamma_2 = 3,\ \lambda_m = \lambda_1 = 2.$

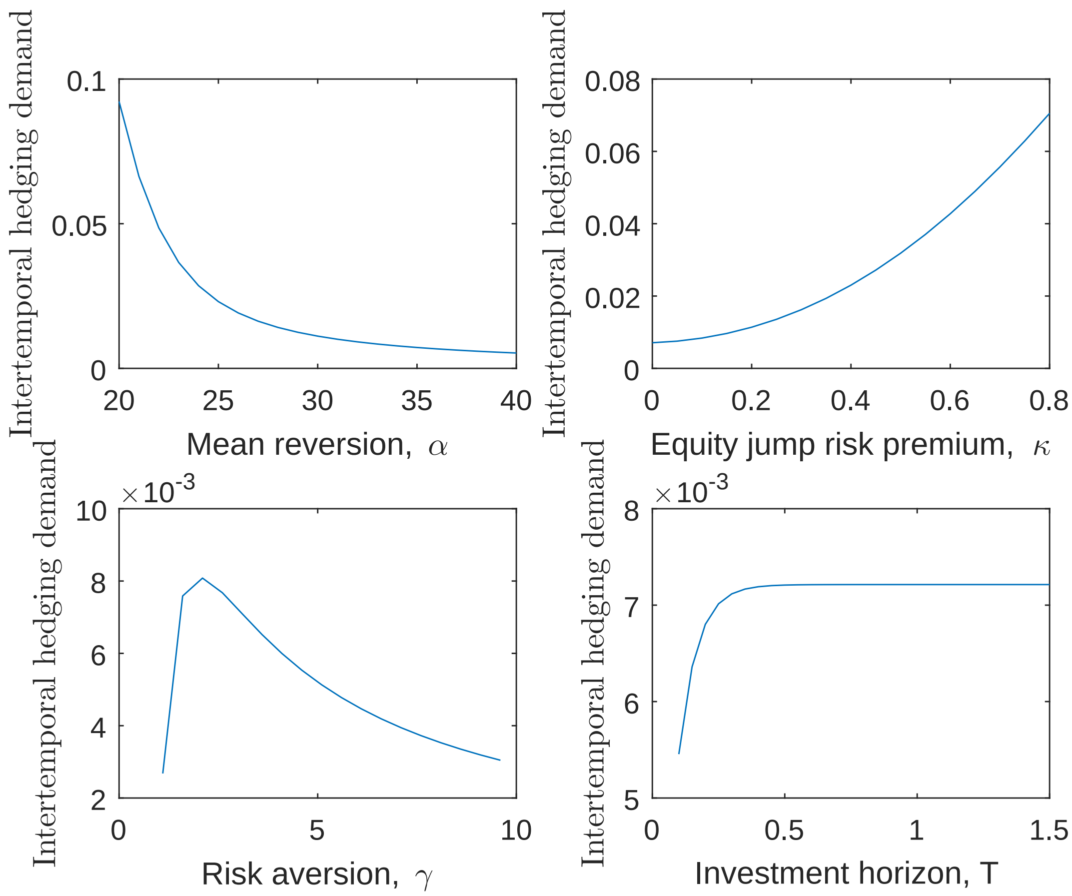

Figure 2: The intertemporal hedging demand (IV) of Currency I for the investor from the base country as functions of the mean reversion rate α_m (top left), equity jump risk premium κ_0 (top right), risk aversion γ_0 (bottom left) and investment horizon T (bottom right). The base case parameters are $\eta_0 = 0.1,\ \sigma_m = 0.2,\ \sigma_d = 0.1,\ \sigma_{e_1} = \sigma_{e_2} = -0.1,\ v_1 = v_2 = 0.05,\ j_m = -0.03,\ j_d = 0.1,\ \alpha_m = \alpha_1 = \alpha_2 = 35,\ \boldsymbol{\beta} = (15, 6, 6; 6, 8, 0; 6, 0, 8),\ T = 1,\ \kappa_1 = \kappa_2 = 0.02,\ y_1 = y_2 = -2\%,\ \gamma_0 = \gamma_1 = \gamma_2 = 3,\ \lambda_m = \lambda_1 = 2.$

We plot the intertemporal hedging demand (IV) of Currency I for the base investor as functions of elements in the excitation matrix $\boldsymbol{\beta}$ in Figure 1. The figure shows that increasing any element of the excitation matrix $\boldsymbol{\beta}$ leads to increasing hedging demand (IV) of Currency I in the base investor's portfolio, whether it be the self excitor of the market equity, $\beta_{m,m}$ (top left), the equity-currency excitor, $\beta_{m,1}$ (bottom left), the currency-equtiy excitor, $\beta_{1,m}$ (top right), or the self excitor of Currency I, $\beta_{1,1}$ (bottom right).

Figure 2 plots the intertemporal hedging demand (IV) of Currency I for the base investor as functions of the mean reversion rate α_m (top left), equity jump risk premium κ_0 (top right),[6] domestic risk aversion γ_0 (bottom left) and the investment horizon T (bottom right). Larger jump risk premium and longer investment horizon result in increasing hedging demand for Currency I. On the contrary, faster mean

[6] For every κ_0, we maintain that $\kappa_0 = \kappa_1 = \kappa_2$.

reversion rate decreases the hedging demand for Currency I for the base investor. Interestingly, increasing the risk aversion first increases then decreases the base investor's hedging demand.

When $y_1 < 0$, from the perspective of the base investor, the foreign currency jumps downward, opposite to the jumps in equity and currency jump intensities. Currency I, therefore, can be used as a static hedge against the state variables. As a result, the base investor has a positive hedging demand for Currency I. The larger the hedging potential the risky currency is against the state variables, the larger hedging demand investors have.

Larger excitation and slower mean reversion imply that the currency jump intensity process $\lambda_1(t)$ is more volatile. As one may expect, the more uncertainty there is in the state variable, the larger hedging incentive investors have. Loosely speaking, as the equity jump risk premium increases, more weight is assigned to the market equity, which leads to more jump risks to be hedged. Similarly, longer investment horizon leads to increased sensitivity of indirect utility to state variables. In short, hedging demand rises when there are increasing uncertainties in investor's indirect utility.

The effect of increasing the risk aversion, however, is not clear. On one hand, increasing the risk aversion decreases the demand for Currency I in general, implying a smaller amount to be hedged, thereby decreasing the hedging demand. On the other hand, a more risk averse investor is more inclined to hedge the changes in the state variable, and may therefore have a larger hedging demand. The final result depends on which effect is larger. Figure 2 shows that the effect of increasing risk aversion is not monotone: it first increases the jump risk demand and then reduces it.

An interesting phenomenon is that while the currency weight $\hat{w}^j_{e_i}$ does not display market timing, its components do. Figure 3, 4 plot the volatility-scaled four components of the weight on Currency I in the base investor's portfolio as functions of the equity jump intensity λ_m and the currency jump intensity λ_1, respectively.

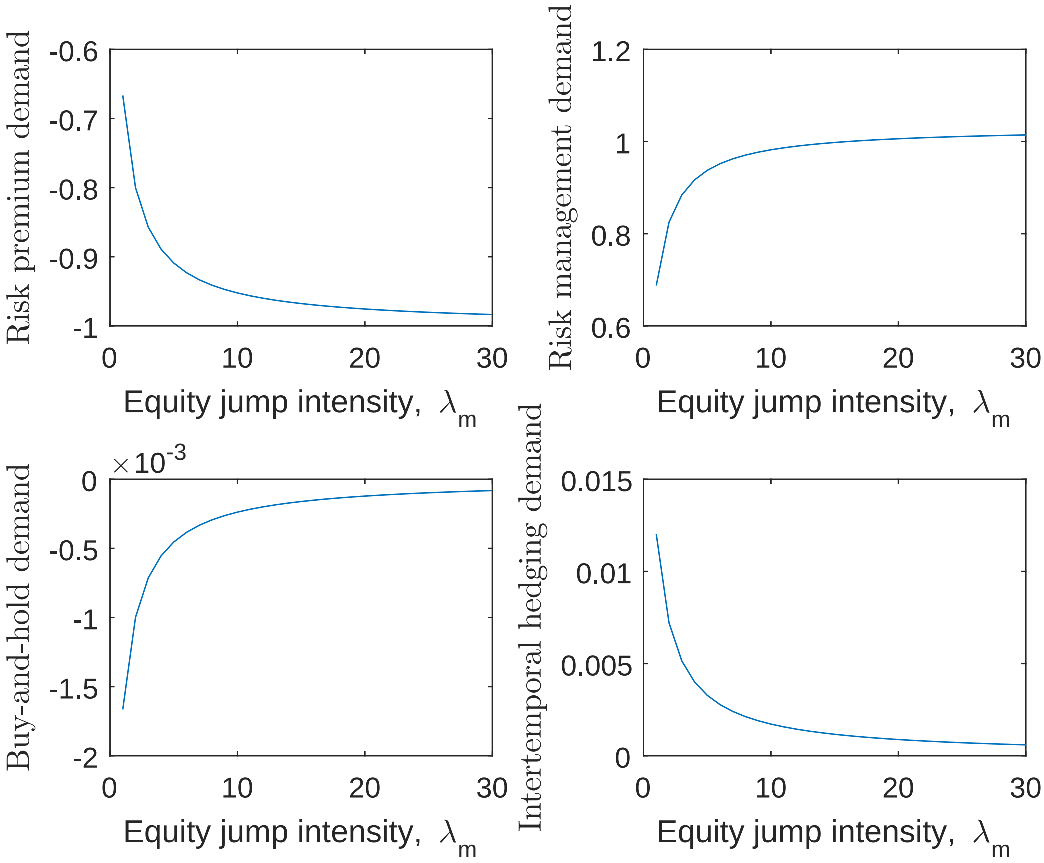

Figure 3: Comparative statics of the components (I, II, III, IV) of Currency I for the base investor as functions of the current equity jump intensity λ_m. The upper left panel plots the risk premium demand I, the upper right panel plots the risk management demand II, the bottom left panel plots the buy-and-hold demand III, and bottom right panel plots the hedging demand IV. The base case parameters are $\eta_0 = 0.2,\ \sigma_m = 0.2,\ \sigma_d = 0.1,\ \sigma_{e_1} = \sigma_{e_2} = -0.1,\ v_1 = v_2 = 0.05,\ j_m = -0.03,\ j_d = 0.02,\ \alpha_m = \alpha_1 = \alpha_2 = 35,\ \boldsymbol{\beta} = (15, 6, 6; 6, 8, 0; 6, 0, 8),\ T = 1,\ \kappa_0 = \kappa_1 = \kappa_2 = 0.02,\ y_1 = y_2 = -2\%,\ \gamma = 3,\ \lambda_m = \lambda_1 = 2$.

Figure 3 plots the components of the optimal weight on Currency I for an investor from the base country as functions of the current equity jump intensity λ_m. The figure shows that the risk premium demand I (upper left panel) increases with the equity jump intensity. Since the expected excess return is proportional to the jump intensity, larger jump intensity increases the compensation for the base investor. It is not surprising that given the negative covariance with other assets, the risk management component II (upper right panel) decreases with the equity jump intensity. Larger equity intensity increases the covariance, resulting in more negative risk management demand. Both the myopic buy-and-hold demand III (bottom left) and the intertemporal hedging demand IV (bottom right) approach zero as the equity jump intensity increases. Increasing the equity jump intensity increases the currency volatility b_i but leaves the myopic demand and intertemporal hedging demand unchanged. Therefore after volatility scaling, both demand components approach zero as volatility increases.

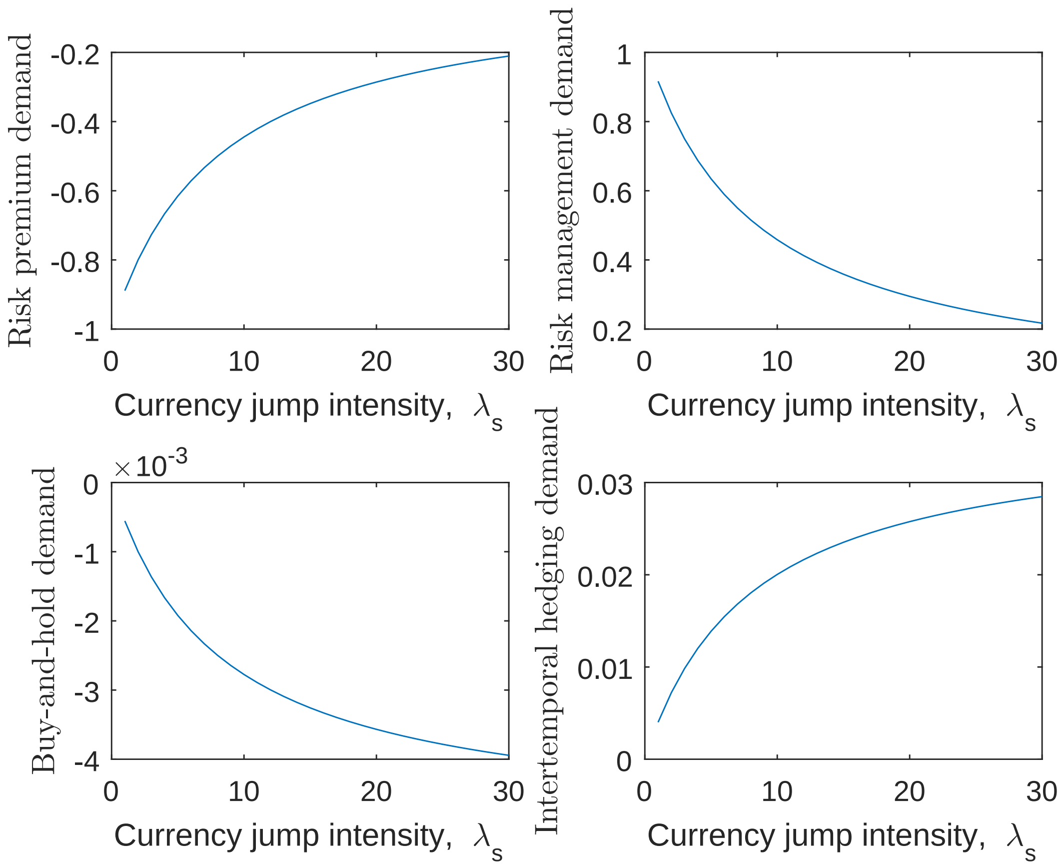

Figure 4: Comparative statics of the components (I, II, III, IV) of the foreign currency demand for the domestic investor as functions of the current currency jump intensity λ_s. The upper left panel plots the speculation demand I, the upper right panel plots the risk management demand II, the bottom left panel plots buy-and-hold demand III, and the bottom right panel plots the hedging demand IV. The base case parameters are $\eta_0 = 0.2$, $\sigma_m = 0.2$, $\sigma_d = 0.1$, $\sigma_{e_1} = \sigma_{e_2} = -0.1$, $v_1 = v_2 = 0.05$, $j_m = -0.03$, $j_d = 0.02$, $\alpha_m = \alpha_1 = \alpha_2 = 35$, $\boldsymbol{\beta} = (15, 6, 6; 6, 8, 0; 6, 0, 8)$, $T = 1$, $\kappa_0 = \kappa_1 = \kappa_2 = 0.02$, $y_1 = y_2 = -2\%$, $\gamma = 3$, $\lambda_m = \lambda_1 = 2$.

Figure 4 plots the components of the optimal weight on Currency I for the base investor as functions of the current currency jump intensity λ_1. We see opposite patterns to Figure 3. Both the risk premium demand I (upper left) and the risk management demand II (upper right) converge to zero as currency jump intensity increases, because larger currency jump intensity leads to larger currency volatility but does not affect the expected excess return or covariance with other assets. Both buy-and-hold demand III (bottom left) and intertemporal hedging demand IV (bottom right) increase in absolute value as currency jump intensity rises. This is because increasing currency jump intensity magnifies the currency jump effect on portfolio weight.

5 Market equilibrium

This section derives the equilibrium currency hedging strategies. Previous to this section, we work with the net currency weight $\hat{w}^j_{e_i}$. The raw currency weights are given by the net weights plus the implicit currency investment in the currency-hedged assets. In order to calculate the equilibrium currency hedging strategy, one needs to get to the raw currency weights. In this section, we impose market clearing conditions in order to see how much currency risks investors are exposed to by investing in the global equity and the global derivative portfolio.

5.1 Equilibrium condition

Denote country i's wealth as a proportion to the world wealth as $f_i = X^j_i/M^j$, where j can be any currency. Following Wang (1996) and Bongaerts, De Jong, and Driessen (2011), Definition 1 defines the market equilibrium as the condition that the security markets clear.

Definition 1 (Market Equilibrium)**.** Market equilibrium consists of the asset price processes $(M^i(t), D^i(t))$ and the trading strategies $(\boldsymbol{w}^i)$ for $i = 0, \ldots, n$, such that the investors' expected utilities are maximized

$$\boldsymbol{w}^i = \arg\sup \mathbb{E}_0\Big[\frac{X_i(T)^{1-\gamma_i}}{1-\gamma_i}\Big],$$

subject to their respective wealth dynamics:

$$\frac{\mathrm{d}X_i(t)}{X_i(t^-)} = w^i_m \frac{\mathrm{d}M^i(t)}{M^i(t^-)} + w^i_d \frac{\mathrm{d}D^i(t)}{D^i(t^-)} + \sum_{l=0}^{n} w^i_{e_l} \frac{\mathrm{d}B^i_l(t)}{B^i_l(t^-)}, \quad i = 0, \ldots, n. \tag{29}$$

and the security markets clear

$$\begin{cases} \sum_{i=0}^n f_i = 1, \\ \sum_{i=0}^n h_i = 1, \\ \sum_{i=0}^n f_i w^i_d = 0, \\ \sum_{i=0}^n f_i w^i_{e_l} = 0, \quad l = 0, \ldots, n. \end{cases} \tag{30}$$

The first equation implies that the sum of the market capitalization of each country equals the total market capitalization. The second condition says that the total capital in the market comes from the wealth of nations. The third and fourth equations impose that the net supply of the equity derivatives and bonds should be zero, meaning that the gross lending in the equity derivatives as well as each currency should be equal to the gross borrowing.

The security market clearing conditions imply that the wealth distribution (f_i), country's market capitalization (h_i) and each country's share in the derivative portfolio (k_i) need to be consistent with the return dynamics of equities and exchange rates.

In Section 3, we have derived the optimal asset allocation on currency-hedged assets. The following lemma shows how to compute the weights on the raw assets from the weights on the hedged assets.

Lemma 1. *The portfolio weights on the currency-unhedged assets are given by*

$$w_m^j = \hat{w}_m^j, \quad \hat{w}_o^j = \hat{w}_o^j, \quad j = 0, \dots, n \tag{31}$$

$$w_{e_i}^j = \hat{w}_{e_i}^j - h_i \hat{w}_m^j - k_i \hat{w}_d^j, \quad i, j = 1, \dots, n, \ i \neq j, \tag{32}$$

$$w_{e_j}^j = 1 + \hat{w}_{e_j}^j - h_j \hat{w}_m^j - k_j \hat{w}_d^j, \quad i, j = 1, \dots, n, \tag{33}$$

$$w_{e_0}^j = -\Big(\sum_{i=1}^{n} \hat{w}_{e_i}^j + h_0 \hat{w}_m^j + k_0 \hat{w}_d^j\Big), \quad j = 1, \dots, n. \tag{34}$$

Therefore we can equivalently construct the market clearing conditions using the weights on the currency-hedged assets.

Theorem 2. *For $f_i, h_i \in [0,1]$, Equation (30) is equivalent to*

$$\begin{cases} \sum_{i=0}^n f_i = 1, \\ \sum_{i=0}^n h_i = 1, \\ \sum_{i=0}^n f_i \hat{w}_m^i = 1, \\ \sum_{i=0}^n f_i \hat{w}_d^i = 0, \\ \sum_{i=0}^n \hat{w}_{e_j}^i f_i - h_j + f_j = 0, \quad \forall j = 1, \dots, n. \end{cases} \tag{35}$$

5.2 Equilibrium currency hedging

In Black (1990), the equilibrium hedging strategy of currency i for investor j is defined as the negative of the investment on currency i per unit of the global equity index invested,

$$H_i^j := -\frac{w_{e_i}^j}{w_m^j}. \tag{36}$$

In terms of weights on the currency-hedged assets, Equation (36) can be expressed as

$$H_i^j = -\frac{\hat{w}_{e_i}^j - (h_i \hat{w}_m^j + k_i \hat{w}_d^j)}{\hat{w}_m^j}. \tag{37}$$

Proposition 4 (Black (1990))**.** *If all prices follow geometric Brownian motion processes and all investors* *have the same risk aversion coefficient γ, then the equilibrium hedging strategy of currency i for any investor $j, j \neq i$ is given by*

$$H_i^j\text{-}Black = f_i(1 - 1/\gamma), \quad \forall j \neq i. \tag{38}$$

Equation (56) is the well-known universal hedging formula derived by Black (1990). The two key implications are: (1) In equilibrium, every investor hedges the same amount of any risky currency i regardless of their home currencies j; (2) The universal currency hedging ratio of currency i only depends on two variables: the coefficient of relative risk aversion and the total wealth held by investors in country i. This means that the currency's expected excess return, volatility or correlation with the equity market do not have a direct impact on the hedging ratio of the currency, as long as the wealth holdings and risk attitude are fixed.

6 Safe haven vs. investment currencies

In Ranaldo and Söderlind (2010), a safe haven currency is a currency that offers hedging benefits on average. For instance, Campbell et al. (2010) show that Swiss franc and Euro are negatively related to equity. However, the correlations between currencies and the equity market are very unstable and may switch between positive and negative values periodically. Even worse, during the 2007-2009 financial crisis, as Kohler (2010) notes, "a large number of currencies that were not at the center of the turmoil depreciated, even those which were regarded as safe haven currencies preceding the crisis". For example, the 2008 financial crisis emerged as an important case study where safe haven effects went against typical patterns partially in contrast with the results of Ranaldo and Söderlind (2010).

Therefore we focus on the alternative definition of safe haven currencies in Ranaldo and Söderlind (2010). A currency is considered a safe haven if it gives hedging benefits in times of stress.

We can intuitively distinguish a safe haven currency and an investment currency in our framework. A safe haven currency provides a safe haven to investors during a recession. Therefore a safe haven currency should be relatively immune to capital market turmoil. In our model, the excitor $\beta_{m,i}$ measures how large a jump occurrence in the equity market N_m raises the intensity of the currency jump component λ_i. A safe haven currency, therefore, should have a relatively smaller $\beta_{m,i}$. An investment currency, on the contrary, is like the mirror image of the safe haven currencies, and is characterized by a relatively larger $\beta_{m,i}$. As a consequence, a safe haven currency is not as prone to the equity market downturns as investment currencies.

Whether a currency is of the "safe haven" type or "investment" type has important implication in determining the optimal currency exposure. Observe that $\beta_{m,i}$ plays a different role from $\beta_{i,m}$ in determining the currency demand. Recall that the intertemporal hedging demand (IV) is a function of

$\boldsymbol{\beta}_i$, in which $\beta_{i,m}$ and $\beta_{m,i}$ are not weighted symmetrically. Therefore imagine a safe haven currency and an investment currency with identical risk profile (including expected return, covariance, jump size, jump intensity, etc.) except that the safe haven currency has smaller equity-currency excitor $\beta_{m,i}$, the demand for these two risky currencies for a foreign investor is in general different.

6.1 Equilibrium net currency weight

In this section, we are going to illustrate investors' preferences towards safe haven currency numerically. Similar to the numerical studies in Section 4.2, we consider a three-currency scenario including a base currency.

Figure 5 and Figure 6 plot the equilibrium net Currency I weight for the base investor, $\hat{w}^0_{e_1}$ when the contagion structure between the equity and Currency I changes, using the first Equation of (25). We fix the first and third row of the excitation matrix, in order that the equity-currency contagion structure for Currency II does not vary. In Figure 5, we let the equity-currency excitor $\beta_{m,1}$ increase while finding the corresponding currency self excitor $\beta_{1,1}$ that delivers the same expected jump intensity $\mathbb{E}[\lambda_1]$. Conversely in Figure 6, we let the currency-equity excitor $\beta_{1,m}$ increase while finding the corresponding currency self excitor $\beta_{m,m}$ that delivers the same expected equity jump intensity $\mathbb{E}[\lambda_m]$. The equilibrium net weight on Currency I is plotted in the solid curves and that on Currency II is depicted in dotted curves.

Note that in both Figure 5 and 6, the two non-base currencies have the same risk profile (volatility, covariance with the equity, jump amplitude, expected jump intensity) except the excitation structure. Figure 5 shows what happens when Currency I moves from a safe haven currency to an investment currency. When $\beta_{m,1}$ is small, the currency has the safe haven characteristic and is stable during market downturns. As $\beta_{m,1}$ increases, the currency becomes more liable to depreciate during capital market turmoil. We observe from the figure that the base investor demands more currency exposure when the currency is of the safe haven type. In Figure 6, even though the dependence between equity and Currency I increases all the same (just like Figure5), the optimal net currency weight displays an opposite pattern to Figure 5. The figure shows that when it comes to portfolio choices, the direction of excitation matters. In particular, a currency is only safe haven when the *equity-currency* excitor is small.

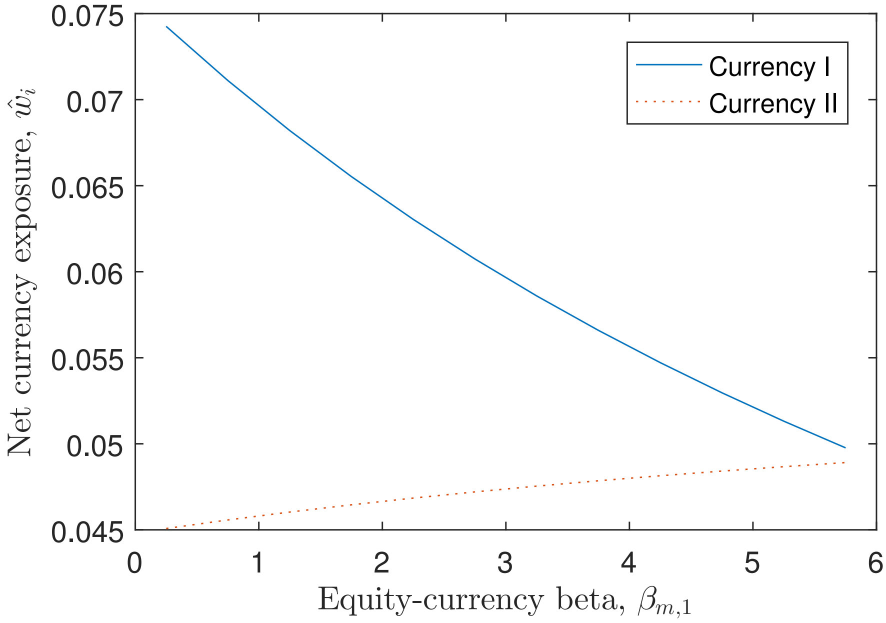

Figure 5: Equilibrium net weight on Currency I (solid line) and II (dotted line) of the base investor as a function of the equity-currency excitor $\beta_{m,1}$. The equilibrium net currency weight for the base investor is computed using Equation (25). The excitation matrix is $\boldsymbol{\beta} = (15, 6, 6; \beta_{m,1}, \beta_{1,1}, 0; 6, 0, 8)$. We let $\beta_{m,1}$ increase and find the corresponding $\beta_{1,1}$ such that the expected equity and currency jump intensities do not vary with the excitation matrix. All the other parameters are kept constant with $\eta_0 = 0.3,\ \sigma_{e_1} = \sigma_{e_2} = -0.1,\ v_1 = v_2 = 0.05,\ \alpha_m = \alpha_1 = \alpha_2 = 35,\ T = 1,\ \kappa_0 = \kappa_1 = \kappa_2 = 0.02,\ y_1 = y_2 = -2\%,\ \gamma_0 = \gamma_1 = \gamma_2 = 3.$

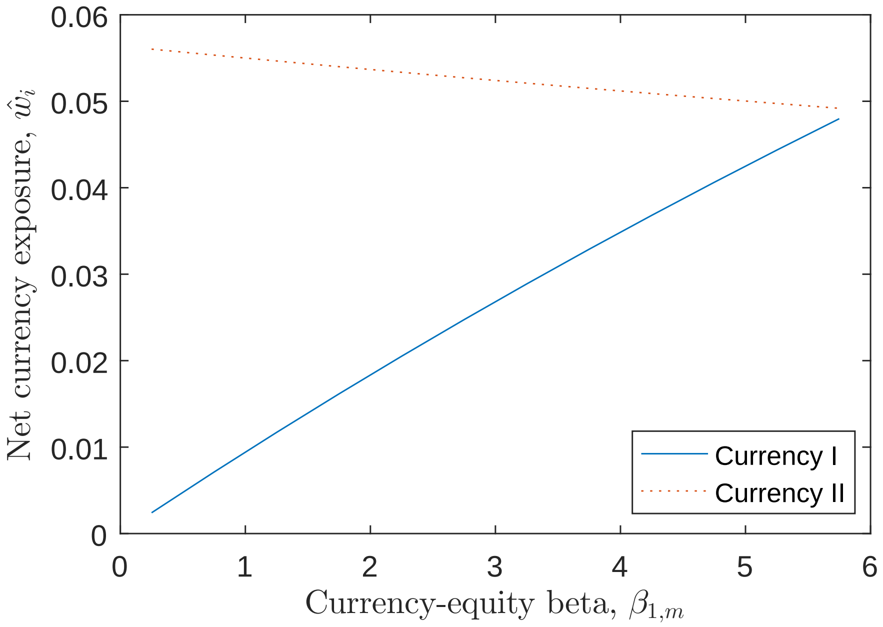

Figure 6: Equilibrium net weight on Currency I (solid line) and II (dotted line) of the base investor as a function of the currency-equity excitor $\beta_{1,m}$. The equilibrium net currency weight for the base investor is computed using Equation (25). The excitation matrix is $\boldsymbol{\beta} = (\beta_{m,m}, \beta_{1,m}, 6; 6, 8, 0; 6, 0, 8)$. We let $\beta_{1,m}$ increase and find the corresponding $\beta_{1,1}$ such that the expected equity and currency jump intensities do not vary with the excitation matrix. All the other parameters are kept constant with $\eta_0 = 0.3,\ \sigma_{e_1} = \sigma_{e_2} = -0.1,\ v_1 = v_2 = 0.05,\ \alpha_m = \alpha_1 = \alpha_2 = 35,\ T = 1,\ \kappa_0 = \kappa_1 = \kappa_2 = 0.02,\ y_1 = y_2 = -2\%,\ \gamma_0 = \gamma_1 = \gamma_2 = 3$.

Jump excitation structure determines the intertemporal hedging demand for currencies. The x-axis in Figure 5 starts with 0, indicating that an occurrence in the equity jump component does not increase the probability of a depreciation of Currency I. In comparison, at the end point of the x-axis, an occurrence in the equity jump component raises the jump intensity $\lambda_1(t)$ by 6. As $\beta_{m,1}$ increases, the impact of a price plunge in the equity market on the value of Currency I increases, making Currency I less safe haven. When Currency I moves away from a safe haven currency and towards an investment currency, the base investor decreases the net weight on Currency I and slightly increases that on the other risky currency, Currency II.

6.2 Equilibrium currency hedging strategy

In this section we study what happens to the equilibrium currency hedging strategy given in Equation (37) as the equity-currency excitor increases.

For a cleaner illustration of the distinction between the equilibrium currency hedging prediction of our model and that of Black (1990), every time we increase the equity-currency excitor, we keep the risk aversion parameter γ and the wealth distribution vector (f_i) fixed, such that the Black (1990)'s prediction does not vary with the equity-currency excitor. To restore equilibrium, however, the global equity index and derivative portfolio are allowed to be endogenous. The detailed algorithm of finding and restoring the market equilibrium can be found in Appendix B.

Figure 7 compares the equilibrium currency hedging ratio of our model to the universal hedging ratio (Equation (56)) of Black (1990). The left panel plots the model prediction of the hedging ratio of currency I in equilibrium when Currency I moves from a safe haven currency to an investment currency. The hedging ratio of Currency I for the base investor is plotted in the solid curve, while that for Investor II is plotted in the dotted curve. The figure is produced in the same way as Figure 5 but with the dependent variable being the hedging ratio of Currency I. We see that as Currency I becomes more prone to equity market downturns, both investors from the base country and country II hedge a larger proportion of the risk of Currency I.

The right panel plots the currency hedging prediction calculated using the Black hedging formula (56). We see that the equilibrium currency hedging in the Black (1990) model does not change when Currency I is no longer safe haven. Notice that in the right panel, one curve is visible because the base investor and Investor II have the same hedging ratio, namely, the universal hedging ratio.

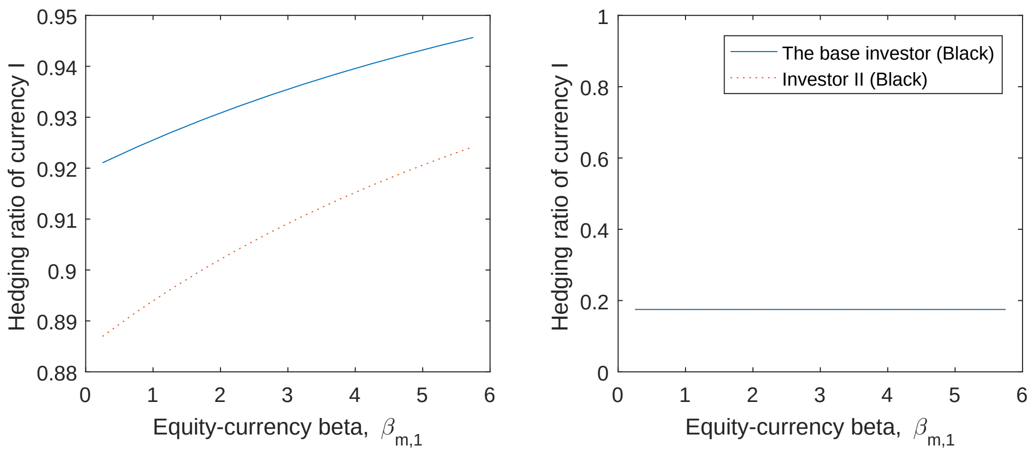

Figure 7: Equilibrium hedging ratio of Currency I when Currency I moves from a safe haven currency to an investment currency. The left panel plots the hedging ratio calculated by Equation (37). The hedging ratio for the base investor is plotted in the solid curve, while that for Investor II is plotted in the dotted curve. The right panel plots Black's universal hedging ratio for Currency I (see Equation (56)). Here, one curve is visible because the base investor and Investor II have the same hedging ratio. The excitation matrix is $\boldsymbol{\beta} = (15, 6, 6; \beta_{m,1}, \beta_{1,1}, 0; 6, 0, 8)$. We let $\beta_{m,1}$ increase and find the corresponding $\beta_{1,1}$ such that the expected equity and currency jump intensities do not vary with the excitation matrix. The following parameters are kept constant with $\eta_0 = 0.3$, $\sigma_{e_1} = \sigma_{e_2} = -0.1$, $v_1 = v_2 = 0.05$, $\alpha_m = \alpha_1 = \alpha_2 = 35$, $T = 1$, $\kappa_0 = \kappa_1 = \kappa_2 = 0.02$, $y_1 = y_2 = -2\%$, $\gamma_0 = \gamma_1 = \gamma_2 = 3$.

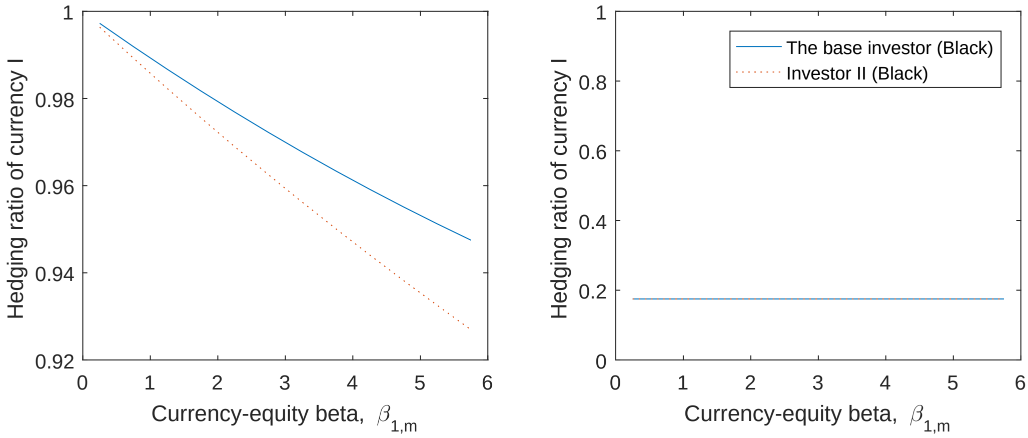

Figure 8: Equilibrium hedging ratio of Currency I as a function of the currency-equity excitor $\beta_{1,m}$. The left panel plots the hedging ratio calculated by Equation (37). The hedging ratio for the base investor is plotted in the solid curve, while that for Investor II is plotted in the dotted curve. The right panel plots Black's universal hedging ratio for Currency I (see Equation (56)). Here, one curve is visible because the base investor and Investor II have the same hedging ratio. The excitation matrix is $\boldsymbol{\beta} = (15, 6, 6; \beta_{m,1}, \beta_{1,1}, 0; 6, 0, 8)$. We let $\beta_{m,1}$ increase and find the corresponding $\beta_{1,1}$ such that the expected equity and currency jump intensities do not vary with the excitation matrix. The following parameters are kept constant with $\eta_0 = 0.3,\ \sigma_{e_1} = \sigma_{e_2} = -0.1,\ v_1 = v_2 = 0.05,\ \alpha_m = \alpha_1 = \alpha_2 = 35,\ T = 1,\ \kappa_0 = \kappa_1 = \kappa_2 = 0.02,\ y_1 = y_2 = -2\%,\ \gamma_0 = \gamma_1 = \gamma_2 = 3$.

Increasing the currency-equity excitor, while also increases the equity currency dependence, leads to opposite patterns, as shown in Figure 6 and 8. Still, we see that increasing the currency-equity beta does not have an impact on the Black's hedging strategy.

One may argue that in Figure 7 and 8, the Black's equilibrium currency hedging ratio plotted in the right panels are not optimal should investors fit the asset returns in a geometric Brownian motion model. The question arises that if in an economy of Black (1990), where the asset returns, investors' preferences, and wealth, capital distributions form an equilibrium market, will increasing the linear correlation coefficient between the market equity and currencies, deliver the same safe haven currency effect. We therefore redo the analysis in a Black model.

Indeed, the phenomenon that investors hedge more investment currency risks than safe haven currency risks cannot be replicated by linear correlation in a geometric Brownian motion model. Figure 9 plots the hedging ratio of Currency I for every investor as a function of the linear correlation between Currency I and the currency-hedged market equity, predicted by the equilibrium model of Black (1990). Similar to Figure 7, we let the risk aversion to be the same across all investors, in which case each investor invests 100% in the market equity, however the correlation between currency and equity changes. As a result, all investors hedge the same amount of currency risks, regardless of the risk profile of the home currency or the risky currencies.

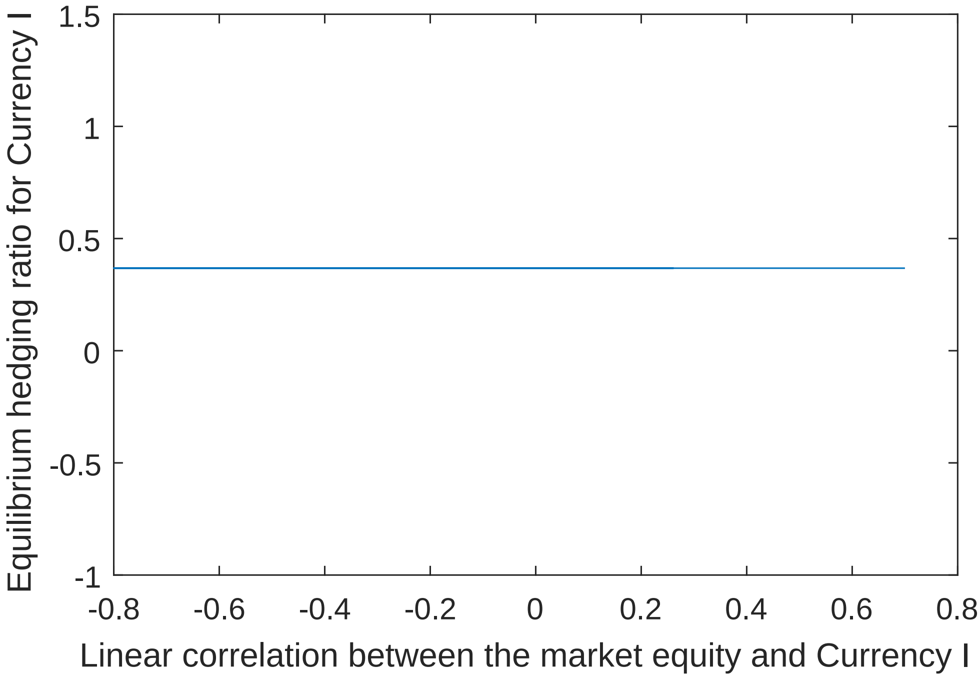

Figure 9: Equilibrium hedging ratio of Currency I as a function of the linear correlation coefficient between the market equity and Currency I. All investors have the same hedging ratio of Currency I, hence one curve. We start with an initial equilibrium, then change the correlation and restore the equilibrium by finding a consistent market equity. The construction of the Black model is given in Appendix C.

Note that Figure 9 is produced differently from the right panels of Figure 7 and 8. In Figure 7 and 8, the equilibrium currency hedging ratio is computed using the Black's hedging formula, $f_i(1-1/\gamma)$, where the wealth distributor (f_i) are found such that the equity-currency contagious market is in equilibrium. It is not equal to the definition of the currency hedging, $-\frac{w^j_{e_i}}{w^j_m}$, which is plotted in the left panels of Figure 7 and8. In Figure 9, by contrast, the hedging ratio is calculated using the definition $H^j_i = -\frac{w^j_{e_i}}{w^j_m}$. In Black's model, when the market is in equilibrium, the currency hedging ratio can be shown to be equal to $f_i(1-1/\gamma)$. Here, the wealth distributor $\boldsymbol{f}$ is consistent with the equilibrium portfolio weights, $w^j_{e_i}, w^j_m$, in the sense that $\boldsymbol{f}, w^j_{e_i}, w^j_m$ satisfy the equilibrium condition (57). The details on the construction of the Black model and how Figure 9 is produced can be found in Appendix C.

In Figure 7 and 8, the asset universe consists of country-specific stocks, call options and currencies. In Appendix D, we show that if we use put options and straddles instead of call options, the patterns are qualitatively similar.

All else equal, investors prefer safe haven currencies to investment currencies regardless of their home currencies. The latter is more likely to go through a substantial depreciation once the equity market experiences a price plunge. Investors with exposure to investment currencies have to risk the possibly that the currency investment will go down during financial crises. Exposure to safe haven currencies, however, can act as a shield to the equity investment: when the equity is in turmoil, the value of the currency investment typically remains stable.

7 Conclusion

Inspired by the empirical findings that there exists risk spillover from the equity market to the currency market, we revisit the classic equilibrium currency hedging problem established by Solnik (1974) and Black (1990) under the context of equity-currency contagion. We postulate a mutually exciting jump diffusion model to jointly model the equity returns and currency returns. Our model is consistent with the extant literature in that (1) the currency returns are subjected to country-specific risk factors as well as global risk factors, (2) currency returns are subjected, but not limited, to equity risks. On top of these features, we further allow for cross excitation between the equity jump component and the currency jump components.

We assume that the global market is integrated and free of arbitrage opportunities, in which case the currency returns is equal to the difference in the returns of the pricing kernels of the two countries. We first solve analytically the asset allocation problem for every representative investor in terms of the currency-hedged assets. We show that the optimal net currency weights can be decomposed into four components: (1) the risk premium demand that earns the expected excess returns by taking currency risks; (2) the risk management demand that exploits the diversification benefits embedded in the instantaneous covariance structure with other assets in the portfolio; (3) the myopic buy-and-hold demand which is induced by the discontinuities (jumps) in the returns; and (4) the intertemporal hedging demand that hedges the state variable risks. The intertemporal hedging demand is a result of the mutually exciting nature of the jump components. Loosely speaking, the intertemporal hedging demand for currencies increases when there is more uncertainty in the state variables and when there is more jump risk to hedge.

Next we impose security market clearing conditions to derive the equilibrium currency hedging strategy, defined as the negative of the investment on a risky currency per unit of global equity index invested. Compared with the classic equilibrium currency hedging ratio of Black (1990), our prediction has two distinctive features: (1) The universal hedging ratio no longer holds: investors with different domestic currencies will in general have different currency hedging ratios; (2) The dependence structure between the equity market and the currency market does matter: Everything else equal, investors hedge more investment currency risk than the safe haven currency risk, whereas investors can be indifferent in Black (1990).

Appendix

A Proofs

Proof for Proposition 1. Define

$$X_j(t) = \log S_j(t).$$

$$\psi(u) = e^{-r_j(\tau-t)}\mathbb{E}^{Q_j}[\exp(uX_j(\tau))|\mathcal{F}_t].$$

Under the risk neutral measure Q_j of country j, the dynamics of $X_j(t)$ follows

$$\mathrm{d}X_j(t) = \Big(r_j - \frac{1}{2}\sigma_{s_j}^2\lambda_m(t)\Big)\,\mathrm{d}t + \sigma_{s_j}\,\mathrm{d}W_j^{Q_j}(t) + \log(1+j_{s_j})(\,\mathrm{d}N_m^{Q_j}(t) - (1+\kappa_j)\lambda_m(t)\,\mathrm{d}t).$$

The jump process $N_m^{Q_j}(t)$ has intensity $(1+\kappa_j)\lambda_m(t)$ under the risk neutral measure of country j. Duffie, Pan, and Singleton (2000) show that the price of a call option $C_j(t)$ is given by

$$C_j(t) = G_{1,-1}(-\log K_j) - K_j G_{0,-1}(-\log K_j)$$

where $G_{a,b}(y)$ denotes the price of a security that pays $e^{aX_j(T)}$ at time T in case of $bX_j(t) \leq y$. The Fourier transform $\mathcal{G}_{a,b}(\cdot)$ is defined as

$$\begin{aligned}\mathcal{G}_{a,b}(u) :=& \int_{-\infty}^{+\infty} e^{izy} dG_{a,b}(y)\\ =&\mathbb{E}_t^{Q_j}[\exp((a+iub)X_j(T)]\\ =&\psi_t(a+iub)\end{aligned}$$

Employ the Duffie et al. (2000) transform analysis, define

$$K_0 = \begin{pmatrix} 0 \\ \alpha_m\lambda_\infty \end{pmatrix},\; K_1 = \begin{pmatrix} 0 & -\frac{1}{2}\sigma_{s_j}^2 - j_{s_j}(1+\kappa_j) \\ 0 & -\alpha_m \end{pmatrix},$$

$$(H_1)_{11} = (0, \sigma_{s_j}^2)', \quad H_0 = 0,$$

$$l_0 = 0,\; l_1 = (0, 1+\kappa_j)',\; \theta(c) = \exp\Big(j_{s_j}c_1 + c_2\beta_{m,m}\Big).$$

It holds that

$$\psi_t(u) = S_j(t)^u \exp(\mathcal{P} + \mathcal{Q}\lambda_m(t)) \tag{39}$$

where $\mathcal{P} = \mathcal{P}(t), \mathcal{Q} = \mathcal{Q}(t)$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{Q}(t) =\Big(\frac{1}{2}\sigma_{s_j}^2 + j_{s_i}(1+\kappa_j)\Big)u + \alpha_m \mathcal{Q}(t) - \frac{1}{2}u^2\sigma_{s_j}^2 - (1+\kappa_j)\Big((1+j_{s_j})^u e^{\beta_{m,m}\mathcal{Q}(t)} - 1\Big),\ \mathcal{Q}(\tau) = 0$$
$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{P}(t) =-\alpha_m\lambda_{m,\infty}\mathcal{Q}(t),\ \mathcal{P}(\tau) = 0.$$

$G_{a,b}(y)$ can be recovered by applying the inverse Fourier transform formula

$$G_{a,b}(y) = \frac{1}{2}\psi_t(a) - \frac{1}{\pi}\int_0^\infty \frac{\mathrm{Im}[e^{-iuy}\psi_t(a+iub)]}{u}du,$$

□

Proof of Proposition 2. Since the market is incomplete, we employ the stochastic control method to solve the portfolio optimization problem. We can rewrite the budget constraint (16), replacing the portfolio weight on stocks and stock options by the portfolio exposure to equity risk factors, while keeping the portfolio weight on the foreign currency:

$$\frac{\mathrm{d}X_j(t)}{X_j(t^-)} =r_j(t)\,\mathrm{d}t + \boldsymbol{\theta}_w^{j\prime}\Big(\sqrt{\lambda_m}\,\mathrm{d}\boldsymbol{W} + \frac{1}{\sigma_m}\eta_j\boldsymbol{h}'\boldsymbol{\sigma}_s(\boldsymbol{L}\boldsymbol{L}')\boldsymbol{\theta}_w^j\lambda_m\,\mathrm{d}t\Big) + \theta_n^j(\,\mathrm{d}N_m - (1+\kappa_j)\lambda_m\,\mathrm{d}t)$$
$$+\sum_{i=1}^n \hat{w}_{e_i}^j(\lambda_i\,\mathrm{d}t - v_i\sqrt{\lambda_i}\,\mathrm{d}Z_i + y_i\,\mathrm{d}N_i - \mathbb{E}[y_i]\lambda_i\,\mathrm{d}t).$$

Bellman's optimality principle implies that

$$0 = \sup_{\hat{w}^j}\mathcal{A}J,$$

where $\mathcal{A}$ denotes the infinitesimal generator operator. The Hamilton-Jacobi-Bellman (HJB) equation reads

$$0 = \sup_{\theta_w^j,\theta_n^j,\hat{w}_{e_i}^j}\Big\{J_t + \Big(r_j + \eta_j\boldsymbol{h}'\boldsymbol{\sigma}_s(\boldsymbol{L}\boldsymbol{L}')\boldsymbol{\theta}_w^j\lambda_m/\sigma_m - \theta_n^j(1+\kappa_j)\lambda_m - \sum_{i=1}^n\hat{w}_{e_i}^j\mathbb{E}[y_i]\lambda_i\Big)J_x x + \alpha_m(\lambda_{m,\infty}-\lambda_m)J_{\lambda_m}$$
$$+\sum_{i=1}^n\alpha_i(\lambda_{i,\infty}-\lambda_i)J_{\lambda_i} + \frac{1}{2}\Big(\boldsymbol{\theta}_w^{j\prime}\boldsymbol{L}\boldsymbol{L}'\boldsymbol{\theta}_w^j\lambda_m + \sum_{i=1}^n(\hat{w}_{e_i}^j v_i)^2\lambda_i\Big)J_{xx}x^2$$
$$+\lambda_m\Big(J\big(x(1+\theta_n^j),\boldsymbol{\lambda}+\boldsymbol{\beta}_m\big) - J\Big) + \sum_{i=1}^n\lambda_i\mathbb{E}\Big[J\big(x_j(1+\hat{w}_{e_i}y_i),\boldsymbol{\lambda}+\boldsymbol{\beta}_i\big) - J\Big]\Big\}, \tag{40}$$

We use $J_t, J_x, J_{\lambda_m}, J_{\lambda_i}$ to denote the partial derivatives of J with respect to $t, x, \lambda_m, \lambda_i$ and similarly for the higher order derivatives.

We take derivatives of $J(t,x,\lambda)$ with respect to its arguments, substitute into the HJB equation in Equation (40), and differentiate with respect to the portfolio risk exposure θ_w^j, θ_n^j, and the currency weights $\hat{w}^j_{e_i}, i = 1, \dots, n$, to obtain the following first-order conditions:

$$0 = \frac{1}{\sigma_m}\eta_j \boldsymbol{h}'\boldsymbol{\sigma}_s(\boldsymbol{LL}')\lambda_m - \gamma_j(\boldsymbol{LL}')\boldsymbol{\theta}^j_w\lambda_m, \quad (41)$$

$$0 = -(1+\kappa_j)\lambda_m + \lambda_m e^{\boldsymbol{Q}'\boldsymbol{\beta}_m}(1+\theta^j_n)^{-\gamma_j}, \quad (42)$$

$$0 = -\mathbb{E}[y_i]\lambda_i - \gamma_j \hat{w}^j_{e_i} v_i^2 \lambda_i + \lambda_i e^{\boldsymbol{Q}'\boldsymbol{\beta}_i}\mathbb{E}[(1+\hat{w}^j_{e_i}y_i)^{-\gamma_j}y_i]. \quad (43)$$

which results in Equation (19).

It should be noted that $\boldsymbol{\theta}^j_w, \theta^j_n, \hat{w}^j_{e_i}$ are independent of $X_t, \lambda(t)$ and are functions of $\boldsymbol{Q}$. We now proceed to derive the ordinary differential equations for the time-varying coefficients $P(t)$ and $\boldsymbol{Q}(t)$, under which the conjectured form (18) for the indirect utility function J indeed satisfies the HJB equation (40). For this, we substitute (18), (41) and (42) into the HJB equation and obtain,

$$\begin{aligned}0 =& \dot{P} + \dot{\boldsymbol{Q}}'\lambda + (1-\gamma_j)\Big(r_j(t) + \eta_j \boldsymbol{h}'\boldsymbol{\sigma}_s(\boldsymbol{LL}')\boldsymbol{\theta}^j_w\lambda_m/\sigma_m - (1+\kappa_j)\theta^j_n\lambda_m - \sum_{i=1}^n \hat{w}^j_{e_i}\mathbb{E}[y_i]\lambda_i\Big) + \alpha_m(\lambda_{m,\infty} - \lambda_m)Q_m \\ &+ \sum_{i=1}^n \alpha_i(\lambda_{i,\infty} - \lambda_i)Q_i - \frac{1}{2}\gamma_j(1-\gamma_j)(\boldsymbol{\theta}^{j\prime}_w \boldsymbol{LL}'\boldsymbol{\theta}^j_w\lambda_m + \sum_{i=1}^n (\hat{w}^j_{e_i}v_i)^2\lambda_i) \\ &+ \lambda_m\Big((1+\theta_n)^{1-\gamma}\exp(\boldsymbol{Q}'\boldsymbol{\beta}_m) - 1\Big) + \sum_{i=1}^n \lambda_i\Big(\mathbb{E}[(1+\hat{w}_{e_i}y_i)^{1-\gamma_j}]\exp(\boldsymbol{Q}'\boldsymbol{\beta}_i) - 1\Big),\end{aligned}$$

where $\dot{P}, \dot{Q}$ denote the derivative of $P(t), Q(t)$ with respect to time t. The left-hand side of this expression is an affine function in λ_m, λ_i. For this expression to hold for all λ_m, λ_i, the constant term and the linear coefficient of λ_m, λ_i on the left-hand side must be set equal to zero separately, which leads to the ordinary differential equation for $Q(t)$ given in (21). □

Proof of Theorem 1. By replacing the country-specific equities with a global market equity, Equation (19) can be written as

$$\begin{pmatrix} \hat{w}^j_m \\ \hat{w}^j_{o_0} \\ \vdots \\ \hat{w}^j_{o_n} \end{pmatrix} = \begin{pmatrix} h_0\sigma_{s_0} & \sigma_{o_0} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ h_n\sigma_{s_n} & 0 & \dots & \sigma_{o_n} \\ j_m & j_{o_0} & \dots & j_{o_n} \end{pmatrix}^{-1} \begin{pmatrix} \theta^j_{w_0} - \sum_{i=1}^n \hat{w}^j_{e_i}\sigma_{e_i} \\ \vdots \\ \theta^j_{w_n} - \sum_{i=1}^n \hat{w}^j_{e_i}\sigma_{e_i} \\ \theta^j_n - \sum_{i=1}^n \hat{w}^j_{e_i}\hat{j}^j_{e_i} \end{pmatrix}. \quad (44)$$

Notice that the matrix to be inverted is of full column rank. Therefore the equity weights vector $(w^j_m, w^j_{o0}, \dots, w^j_{on})$ exists and is unique.

Next we show that all investors, regardless of their home currencies, will invest in the same global derivative portfolio. Denote investor j's position on the currency-hedged global equity index by $\hat{w}^j_m$. By multiplying each country equity's weight in the market equity index, $h_i\hat{w}^j_m$ therefore gives the weight on the country equities in investor j's portfolio, i.e.,

$$h_i \hat{w}^j_m = \hat{w}^j_{s_i}.$$

Define $\boldsymbol{\sigma}_o$ as an $(n+1)\times(n+1)$ diagonal matrix with σ_{o_i} on the diagonal. Further define σ_e as an $n\times(n+1)$ matrix with with the $[i,j]^{\text{th}}$ element containing currency i's exposure to country j's equity Brownian motion. The first equation of (17) implies that

$$\boldsymbol{\sigma}'_s \boldsymbol{h} \hat{w}^j_m + \boldsymbol{\sigma}_o \hat{\boldsymbol{w}}^j_o = \boldsymbol{\theta}^j_w - \boldsymbol{\sigma}'_e \hat{\boldsymbol{w}}^j_e. \tag{45}$$

Note that $\boldsymbol{\sigma}_e$ can be written as

$$\boldsymbol{\sigma}'_e = \begin{pmatrix} \frac{\eta_0-\eta_1}{\sigma_m}\boldsymbol{\sigma}'_s\boldsymbol{h} & \ldots & \frac{\eta_0-\eta_n}{\sigma_m}\boldsymbol{\sigma}'_e\boldsymbol{h} \end{pmatrix}.$$

In addition, Proposition 2 shows that

$$\boldsymbol{\theta}^j_w = \frac{\eta_j}{\sigma_m\gamma_j}\boldsymbol{\sigma}'_s\boldsymbol{h}.$$

Therefore we have

$$\begin{aligned}\boldsymbol{\sigma}'_e\hat{\boldsymbol{w}}^j_e &= \boldsymbol{\sigma}'_s\boldsymbol{h}\begin{pmatrix} \frac{\eta_0-\eta_1}{\sigma_m} & \ldots & \frac{\eta_0-\eta_n}{\sigma_m}\end{pmatrix}\hat{\boldsymbol{w}}^j_e \\ &:= (\boldsymbol{c}\hat{\boldsymbol{w}}^j_e)\boldsymbol{\sigma}'_s\boldsymbol{h},\end{aligned}$$

where $\boldsymbol{c}$ denote the $1\times n$ vector $\begin{pmatrix} \frac{\eta_0-\eta_1}{\sigma_m} & \ldots & \frac{\eta_0-\eta_n}{\sigma_m}\end{pmatrix}$. Equation (45) becomes

$$\left(\hat{w}^j_m - \frac{\eta_i}{\sigma_m\gamma_j} + \boldsymbol{c}\hat{\boldsymbol{w}}^j_e\right)\boldsymbol{\sigma}'_s\boldsymbol{h} + \boldsymbol{\sigma}_o\hat{\boldsymbol{w}}^j_o = 0,$$

from which we get

$$\hat{\boldsymbol{w}}^j_o = \left(\frac{\eta_i}{\sigma_m\gamma_j} - \hat{w}^j_m - \boldsymbol{c}\hat{\boldsymbol{w}}^j_e\right)\boldsymbol{\sigma}_o^{-1}\boldsymbol{\sigma}'_s\boldsymbol{h}.$$

Notice that $\left(\frac{\eta_i}{\sigma_m\gamma_j} - w^j_m - \boldsymbol{c}\hat{\boldsymbol{w}}^j_e\right)$ is a single number, and $\boldsymbol{\sigma}_o^{-1}\boldsymbol{\sigma}'_s\boldsymbol{h}$ is an $(n+1)\times 1$ vector, and is independent of the investor identity j. Therefore all investors will invest in the same global derivative portfolio. □

Proof for Lemma 1. The weights on the unhedged equity and equity derivatives should be equal to the weights on the hedged ones,

$$w^j_m = \hat{w}^j_m, \quad w^j_d = \hat{w}^j_d.$$

The raw weight on currency i, $\hat{w}^j_{e_i}$ should be the net currency weight $\hat{w}^j_{e_j}$ plus the currency position embedded in the hedged assets,

$$w_{e_i}^j = \hat{w}_{e_i}^j - h_i \hat{w}_m^j - k_i \hat{w}_d^j, \quad j = 1, \ldots, n.$$

The budget constraint (24) can be written as

$$\begin{aligned}
\frac{\mathrm{d}X_j(t)}{X_j(t^-)} =& \hat{w}_m^j \frac{\mathrm{d}\hat{M}^j(t)}{\hat{M}_{t^-}^j} + \hat{w}_d^j \frac{\mathrm{d}\hat{D}^j(t)}{\hat{D}_{t^-}^j} + \sum_{i=1}^{n} \hat{w}_{e_i}^j \frac{\mathrm{d}\hat{B}_i^j(t)}{\hat{B}_i^j(t^-)} + \Big(1 - \hat{w}_m^j - \hat{w}_d^j - \sum_{i=1}^{n} \hat{w}_{e_i}^j\Big) \frac{\mathrm{d}\hat{B}_0^j(t)}{\hat{B}_0^j} \\
=& \hat{w}_m^j \frac{\mathrm{d}M^j(t)}{M_{t^-}^j} + \hat{w}_d^j \frac{\mathrm{d}D^j(t)}{D_{t^-}^j} + \sum_{i=1}^{n\backslash j} \Big(\hat{w}_{e_i}^j - h_i \hat{w}_m^j - k_i \hat{w}_d^j\Big) \frac{\mathrm{d}B_i^j(t)}{B_i^j(t^-)} - \Big(h_0 \hat{w}_m^j + k_0 \hat{w}_d^j + \sum_{i=1}^{n} \hat{w}_{e_i}^j\Big) \frac{\mathrm{d}B^j(t)}{B^j} \\
&+ \Big(1 + \hat{w}_{e_j}^j - h_j \hat{w}_m^j - k_j \hat{w}_d^j\Big) \frac{\mathrm{d}B_j(t)}{B_j(t)}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
w_m^j &= \hat{w}_m^j, \quad \hat{w}_d^j = \hat{w}_d^j, \quad j = 0, \ldots, n, \\
w_{e_i}^j &= \hat{w}_i^j - h_i \hat{w}_m^j - k_i \hat{w}_o^j, \quad w_{e_j}^j = 1 + \hat{w}_{e_j}^j - h_j \hat{w}_m^j - k_j \hat{w}_d^j, \quad i, j = 1, \ldots, n, \ i \neq j, \\
w_{e_0}^j &= -\Big(\sum_{i=1}^{n} \hat{w}_{e_i}^j + h_0 \hat{w}_m^j + k_0 \hat{w}_d^j\Big), \quad j = 1, \ldots, n.
\end{aligned}$$

□

Proof for Theorem 2. The third equation is obtained by multiplying the third equation in (30) by h_j on both sides. And since $w_m^i h_j = w_{s_j}^i$, we get $\sum_{i=0}^n f_i \hat{w}_m^i = 1$. The other equations can be easily verified by replacing the weights on the unhedged assets by the hedged counterparts using Lemma 1. □

Proof for Proposition 4. See Black (1990). □

B Numerical equilibrium calculation

In this section, we explain the numerical algorithms we use for the equilibrium calculations. In particular, the procedure to restore equilibrium when the equity-currency excitor changes is different from that of finding an equilibrium for exogenous asset return generating process. To produce Figure 7 and Figure 8, we use the latter to find an initial equilibrium, taking the equity and exchange rate dynamics, investors' preferences as given and looking for the equilibrium wealth distributor f and market capitalization ratio h. When we change the equity-currency excitor or the currency -equity excitor, we use the former to restore equilibrium, taking the wealth distributor f and exchange rate dynamics as given and looking for the equilibrium market equity process.

B.1 Algorithm to find an initial equilibrium

Let the equity returns and exchange rate dynamics be given. Now we are going to find a wealth distributor f, which tells how much wealth each country is holding, and a market capital distributor h, which tells how much assets each country is holding. We use the following algorithm to find an initial equilibrium, where Figure 7 and Figure 8 start with.

1. Solve for the optimal net currency holding $\hat{w}_{e_i}^j$ for each investor $j = 0, \ldots, n$, and for each currency $i = 1, \ldots, n$, using Proposition 2.

2. According to the security market clearing conditions given by (35), the clearing of the bonds market implies that

$$\boldsymbol{h}^- = \hat{\boldsymbol{w}}_e^0 \boldsymbol{f} + \boldsymbol{f}^-, \tag{46}$$

   where $\boldsymbol{h}^-$ is a vector containing $h_1, \ldots, h_n$; $\boldsymbol{f}^-$ is a vector containing $f_1, \ldots, f_n$; $\hat{\boldsymbol{w}}_e$ is an $n \times (n+1)$ matrix defined as

$$\hat{\boldsymbol{w}}_e = \begin{pmatrix} \hat{w}_{e_1} & \ldots & \hat{w}_{e_1}^n \\ \vdots & \vdots & \vdots \\ \hat{w}_{e_n} & \ldots & \hat{w}_{e_n}^n \end{pmatrix}.$$

   Here, since $\boldsymbol{h} = (1 - \boldsymbol{\iota}'\boldsymbol{h}^-, \boldsymbol{h}^-)'$, $\boldsymbol{f} = (1 - \boldsymbol{\iota}'\boldsymbol{f}^-, \boldsymbol{f}^-)'$, we see that the global equity index composition vector $\boldsymbol{h}$ can be expressed as a function of the wealth distribution vector $\boldsymbol{f}$ and the net currency holdings.

3. According to the third and forth equations of (35), the equity and derivative market clearing condition implies that

$$\begin{cases} \sigma_m = \sum_{j=0}^n f_j \theta_m^j + \sum_{i=1}^n \sigma_{e_i}(f_i - h_i), \\ j_m = \sum_{j=0}^n (1 + j_{e_j}) f_j \theta_n^j + \sum_{i=1}^n j_{e_i}(f_i - h_i). \end{cases} \tag{47}$$

4. But the global equity index found in the last step need to be a weighted average of countries' equities with the weights h, therefore

$$\sigma_m^2 = \boldsymbol{h}'\boldsymbol{\Sigma}\boldsymbol{h}, \quad j_m^j = \sum_{i=0}^n h_i j_{s_i}(1 + j_{e_j}), \tag{48}$$

5. Substitute σ_m and j_m on the LHS of Equation (48) by Equation (47) in Step 3 and get

$$\Big(\sum_{j=0}^{n} f_j \theta_m^j + \sum_{i=1}^{n} \sigma_{e_i}(f_i - h_i)\Big)^2 = \boldsymbol{h}'\boldsymbol{\Sigma}\boldsymbol{h}, \tag{49}$$

$$\sum_{j=0}^{n}(1+j_{e_j}) f_j \theta_n^j + \sum_{i=1}^{n} j_{e_i}(f_i - h_i) = \sum_{i=0}^{n} h_i j_{s_i}(1+j_{e_j}), \tag{50}$$

with $\boldsymbol{h}$ a function of $\boldsymbol{f}$ given by Equation (46). We hence arrive at two equations of the vector $\boldsymbol{f}$ (which has $n-1$ unknown elements). With carefully specified exogenous parameters, one can easily find an $n-$dimensional simplex $\boldsymbol{f}$ such that the above equations hold. Note that in case of $n \geq 2$, the solution is not necessarily unique.

6. Once a solution $\boldsymbol{f}$ to Equation (49) and (50) is found, one can calculate the corresponding $\boldsymbol{h}$ using Equation (46).

B.2 Algorithm to restore equilibrium

Let the international market be in equilibrium. Now we independently change the equity-currency excitor $\beta_{m,i}$. The new equilibrium is found as follows (variables that vary with $\beta_{m,i}$ are denoted by a bar, $\bar{\cdot}$)

1. Solve for the new optimal net currency holdings $\bar{w}_{e_i}^j$ for each investor $j = 0, \ldots, n$, and for each currency $i = 1, \ldots, n$, using Proposition 3.

2. For fixed $\boldsymbol{f}$, the bonds market clearing condition implies that

$$\bar{\boldsymbol{h}}^- = \bar{\boldsymbol{w}}_e \boldsymbol{f} + \boldsymbol{f}^-, \tag{51}$$

where $\boldsymbol{f}^-$ denotes the vector $(f_1, \ldots, f_n)$, and similar for $\bar{\boldsymbol{h}}^-$.

3. From the perspective of the base investor, the equity and derivative market clearing condition implies that

$$\begin{cases} \bar{\sigma}_m = \sum_{j=0}^{n} f_j \theta_m^j + \sum_{i=1}^{n} \sigma_{e_i}(f_i - \bar{h}_i), \\ \bar{j}_m^j = \sum_{j=0}^{n}(1+j_{e_j}) f_j \theta_n^j + \sum_{i=1}^{n} j_{e_i}(f_i - \bar{h}_i). \end{cases} \tag{52}$$

4. But the global equity index found in the last step need to be a weighted average of countries' equities with the weights h,

$$\bar{\sigma}_m^2 = \bar{\boldsymbol{h}}'\bar{\boldsymbol{\Sigma}}\bar{\boldsymbol{h}}, \quad \bar{j}_m^j = \sum_{i=0}^{n} \bar{h}_i \bar{j}_{s_i}(1+j_{e_j}), \tag{53}$$

which leads to renewed countries' stock volatility $\bar{\boldsymbol{\Sigma}}$ and jump amplitude $\bar{\boldsymbol{j}}_s$, that are compatible with Equation (52).

5. For the same type of the derivative contract, the new countries' stock imply new countries' derivatives, denoted by $\bar{\boldsymbol{\sigma}}_o, \bar{\boldsymbol{j}}_o$, which eventually leads to new global derivative portfolio

$$\bar{\sigma}_d = \bar{\boldsymbol{k}}'\bar{\boldsymbol{\Sigma}}_o\bar{\boldsymbol{k}}, \quad \bar{j}_d = \sum_{i=0}^{n} \bar{h}_i \bar{j}_{o_i}(1 + j_{e_i}),$$

with

$$\bar{\boldsymbol{k}} = \frac{\bar{\boldsymbol{\sigma}}_o^{-1}\bar{\boldsymbol{\sigma}}_e'\bar{\boldsymbol{h}}}{\boldsymbol{\iota}'(\bar{\boldsymbol{\sigma}}_o^{-1}\bar{\boldsymbol{\sigma}}_e'\bar{\boldsymbol{h}})}.$$

6. Calculate the new portfolio weights on the global equity index and global derivative portfolio using

$$\begin{pmatrix} \bar{w}_m^j \\ \bar{w}_d^j \end{pmatrix} = \begin{pmatrix} \bar{\sigma}_m & \bar{\sigma}_d \\ \bar{j}_m & \bar{j}_d \end{pmatrix}^{-1} \begin{pmatrix} \theta_m^j - \sum_{i=1}^n \bar{w}_{e_i}^j \sigma_{e_i} \\ (1 + j_{e_j})\theta_n^j - \sum_{i=1}^n \bar{\hat{w}}_{e_i}^j j_{e_i} \end{pmatrix}.$$

7. Calculate the new hedging strategy

$$\bar{H}_i^j := -\frac{\bar{w}_i^j}{\bar{w}_m^j} = -\frac{\bar{\hat{w}}_i^j - \bar{h}_i \bar{w}_m^j - \bar{k}_i \bar{w}_d^j}{\bar{w}_m^j}.$$

C A recap of the geometric Brownian motion model

The Black (1990) model is constructed as the following. From the perspective of the base investor, let the equity in country i, denominated in the base currency be

$$\frac{\mathrm{d}S_i^0(t)}{S_i^0(t)} = r_0 + \mu_{s_i}^0 \,\mathrm{d}t + \sigma_{s_i} \,\mathrm{d}W_i(t).$$

And the return on currency i for the base investor follows the dynamics

$$\frac{\mathrm{d}E_i^0(t)}{E_i^0(t)} = (r_0 - r_i)\,\mathrm{d}t + \mu_{e_i}^0 \,\mathrm{d}t + \sigma_{e_i} \,\mathrm{d}B_i(t).$$

The price of equity i quoted in currency j is therefore S_i^0/E_j^0.

Define $\boldsymbol{\Sigma}$ as the covariance matrix among the country-specific stocks quoted in the base currency and exchange rates between currencies and the base currency,

$$\boldsymbol{\Sigma} = \begin{pmatrix} \sigma^2_{s_0} & \sigma_{s_0,s_1} & \dots & \sigma_{s_0,s_n} & \sigma_{s_0,e_1} & \dots & \sigma_{s_0,e_n} \\ \sigma_{s_1,s_0} & \sigma^2_{s_1} & \dots & \sigma_{s_1,s_n} & \sigma_{s_1,e_1} & \dots & \sigma_{s_1,e_n} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \sigma_{s_n,s_0} & \sigma_{s_n,s_1} & \dots & \sigma^2_{s_n} & \sigma_{s_n,e_1} & \dots & \sigma_{s_n,e_n} \\ \sigma_{e_1,s_0} & \sigma_{e_1,s_1} & \dots & \sigma_{e_1,s_n} & \sigma^2_{e_1} & \dots & \sigma_{e_1,e_n} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \sigma_{e_n,s_0} & \sigma_{e_n,s_1} & \dots & \sigma_{e_n,s_n} & \sigma_{e_n,e_1} & \dots & \sigma^2_{e_n} \end{pmatrix},$$

where $\sigma_{a,b}$ denotes the covariance between asset a and asset b.

One of the benefits of working with currency-hedged assets is that if denominated in a different currency, the covariance matrix of currency-hedged assets does not change. Only expected excess returns are dependent on the home currency. The currency-hedged expected excess return of any risky asset in currency j is equal to the expected excess return denominated in the base currency, minus the covariance between the equity and the exchange rate between currency j and the base currency. Define the currency-hedged equity return as

$$\frac{\mathrm{d}\hat{S}^j_i(t)}{\hat{S}^j_i(t)} = \frac{\mathrm{d}S^j_i(t)}{S^j_i(t)} - \Big(\frac{\mathrm{d}B^j_i(t)}{B^j_i(t)} - \frac{\mathrm{d}B_j(t)}{B_j(t)}\Big).$$

Suppose that the expected return vector from the perspective of the base investor is given by $\hat{\mu}^0$, then it holds that

$$\hat{\boldsymbol{\mu}}^j = \hat{\boldsymbol{\mu}}^0 - \boldsymbol{\Sigma}[:,j],$$

where $\boldsymbol{\Sigma}[:,j]$ is the j^{th} column of the covariance matrix. Denote the holdings of equity i by investor from country j by $w^j_{s_i}$. It holds that

$$\hat{\boldsymbol{w}}^j = (\hat{w}^j_{s_0}, \dots, \hat{w}^j_{s_n} \hat{w}^j_{e_1}, \dots, \hat{w}^j_{e_n})' = \frac{1}{\gamma_j}\boldsymbol{\Sigma}^{-1}\hat{\boldsymbol{\mu}}^j.$$

As shown by Solnik (1974), an important property of asset holdings in a geometric Brownian motion market is that every investor would hold the same equity portfolio regardless of his home currency. That is,

$$\frac{w^i_{s_j}}{w^l_{s_j}} = \frac{\gamma_l}{\gamma_i}. \tag{54}$$

Denote the weight on equity S_i in the market equity portfolio by h_i, with $\sum_{i=0}^n h_i = 1$. Since every investor is indifferent between holding the market equity and country-specific equities, we can replace the country-specific stocks by a market equity index. Now we are in a market of Black (1990) with a market equity and country-specific currencies. Now use $\bar{\hat{\boldsymbol{w}}}^j$ to denote the vector containing the portfolio weight on the market equity index and currencies, i.e., $\bar{\hat{\boldsymbol{w}}}^j = (\hat{w}^j_m, \hat{w}^j_{e_1}, \ldots, \hat{w}^j_{e_n})'$. Define

$$\boldsymbol{\omega} = \begin{pmatrix} \boldsymbol{h}' & 0 \\ 0 & \boldsymbol{I}_n \end{pmatrix}.$$

The weights on the market equity index and currencies are therefore given by

$$\bar{\hat{\boldsymbol{w}}}^j = \frac{1}{\gamma_j}(\boldsymbol{\omega}\boldsymbol{\Sigma}\boldsymbol{\omega}')^{-1}(\boldsymbol{\omega}\hat{\boldsymbol{\mu}}^j).$$

One can easily verify that

$$w^j_m = \hat{w}^j_m = \sum_{i=0}^{n} \hat{w}^j_{s_i} = \sum_{i=0}^{n} w^j_{s_i}.$$

In the absence of an equity derivative, we use the following conditions to find equilibrium

$$\sum_{i=0}^{n} f_i = 1, \quad \sum_{i=0}^{n} \hat{w}^i_m f_i = 1, \quad \sum_{i=0}^{n} \hat{w}^i_{e_j} f_i - h_j + f_j = 0, \quad \forall j = 1, \ldots, n.$$

The equilibrium condition $\sum_{i=0}^{n} \hat{w}^i_m f_i = 1$ implies that

$$\sum_{j=0}^{n} f_j \Big(\sum_{i=0}^{n} \hat{w}^j_{s_i} \Big) = 1.$$

According to Equation (54), we have

$$\Big(\sum_{l=0}^{n} \frac{\gamma_j}{\gamma_l} f_l \Big) \Big(\sum_{i=0}^{n} w^j_{s_i} \Big) = 1.$$

In the special case that every investor has the same risk preference γ, we have

$$\sum_{i=0}^{n} w^j_{s_i} = 1, \quad \forall j. \tag{55}$$

Equation (55) imposes a condition on country-specific equity returns for the existence of the equilibrium when the risk aversion coefficients are the same across countries.

The equilibrium currency hedging, defined as the negative of the currency investment per unit investment in the market equity index, is given by

$$H^i_j := -\frac{w^i_{e_j}}{w^i_m} = f_j \Big(1 - w^j_m / \gamma_m \Big). \tag{56}$$

The second equality is proved by Black (1990), where γ_m is the weighted average of investors' risk attitude, given by

$$1/\gamma_m = \sum_{i=0}^{n} f_i/\gamma_i.$$

To make it comparable to the analysis in Section 6, we assume that all investors share the same risk aversion parameter γ. We employ the following procedure to find the market equilibrium in the model of Black (1990):

1. Specify the the country-specific equity and currency dynamics $\boldsymbol{\mu}, \boldsymbol{\Sigma}$, making sure that Equation (55) is satisfied.

2. Solve the optimal portfolio weights on the country-specific equities and currencies.

3. Calculate the weight on each country's equity in the market equity index $\boldsymbol{h}$.

4. Calculate the joint dynamics of the asset universe. In the geometric Brownian motion case in particular, compute the expected excess return vector and the covariance matrix, using

$$\bar{\hat{\boldsymbol{\mu}}}^j = \boldsymbol{\omega}\hat{\boldsymbol{\mu}}^j, \quad \bar{\boldsymbol{\Sigma}} = \boldsymbol{\omega}\boldsymbol{\Sigma}\boldsymbol{\omega}'.$$

5. Compute the optimal portfolio weights $\bar{\hat{\boldsymbol{w}}}^j = \frac{1}{\gamma}\bar{\boldsymbol{\Sigma}}^{-1}\bar{\hat{\boldsymbol{\mu}}}^j$.

6. Find the wealth distribution vector f such that

$$\sum_{i=0}^{n} f_i = 1, \quad \sum_{i=0}^{n} \hat{w}^i_{e_j} f_i - h_j + f_j = 0, \quad \forall j = 1, \ldots, n. \tag{57}$$

We plot the equilibrium currency hedging ratio in the model of Black (1990) as a function of the correlation between the market equity and Currency I in a three-country world in Figure 9. Similar to Figure 7 and 8, we start with an initial equilibrium. We keep f fixed and change the linear correlation coefficient between the market equity and Currency I and restore the equilibrium by finding the renewed market equity. The details of restoring the equilibrium with fixed wealth distribution f is explained below.

1. Find an initial equilibrium with correlation coefficient ρ.

2. For a new correlation coefficient, find a constant c, such that the first entry of $1/\gamma(\boldsymbol{\Sigma})^{-1}(c\bar{\hat{\boldsymbol{\mu}}}^0)$ is 1, thereby satisfying Equation (55).

3. Find the new market equity composition vector h, such that

$$f_0\hat{w}^0_{e_1} + f_1\hat{w}^1_{e_1} + f_2\hat{w}^2_{e_1} = h_1 - f_1,$$
$$f_0\hat{w}^0_{e_2} + f_1\hat{w}^1_{e_2} + f_2\hat{w}^2_{e_2} = h_2 - f_2.$$

4. Calculate the new equilibrium currency hedging ratio $-\frac{w^i_{e_i}}{w^i_m}$.

D Robustness check

The safe-haven preference seen in the equilibrium currency hedging strategies of investors is free of particular derivative contract chosen. Section 6 uses call options to complete the equity market and presents the safe-haven preference. In this section, we use different derivative contracts and show that investors' preference for safe-haven currencies in equilibrium is not affected qualitatively.

Similar to the call option pricing, the put option price $P_j(t)$ with maturity τ and strike price K is given by

$$P_j(t) = KG_{0,1}(\log K) - G_{1,1}(\log K), \tag{58}$$

where $G_{a,b}(w)$ can be calculated according to Equation (11) in Proposition 1.

Having priced the call and put options, we also consider a straddle. Inspired by Liu, Longstaff, and Pan (2003), we consider the following "delta-neutral" straddle:

$$\text{Straddle}_j(t) = C_j(S_j(t), \lambda_m(t); K, \tau) + P_j(S_j(t), \lambda_m(t); K, \tau),$$

where C and P are pricing formulas for call and put options with the same strike price K and time to expiration τ. As Liu, Longstaff, and Pan (2003) put it, the "delta-neutral" is made of call and put options that are typically very close to the money, which can be used to intentionally avoid deep out-of-the-money options in the quantitative examples due to liquidity issues.

Table 1 reports the equilibrium hedging ratio of Currency I and Currency II for the base investor when different derivative contracts are used for given equity-currency excitation structure. The rows correspond to different derivative contracts. The first row is the hedging ratios when call options are used. The second row corresponds to put options and the third row straddles. The three main columns are hedging ratios of Currency I and Currency II in different equity-currency excitation scenarios. In case of "Large excitation", Currency I and Currency II have the same risk profile, including the excitation structure with the equity market. "Medium excitation" refers to the case where the equity-currency excitor of Currency I is smaller than that in the "Large excitation" scenario, while the excitation structure involving Currency

II remains unchanged from the "Large excitation" scenario. The equity-currency excitor of Currency II is smallest in the "Small excitation" case. Across all three scenarios, the excitation structure between the equity and Currency II does not vary. Also invariant are the expected jump intensities of the equity jump component, Currency I jump component and Currency II jump component.

The hedging ratio of Currency I is always the largest in case of "Large excitation" and smallest in case of "Small excitation", regardless of which derivative contracts are used. The investor has a preference for the safe-haven currency, in the sense that the more immune the currency is to the equity turmoil, the less currency risk the investor hedges away in equilibrium. This conclusion is robust with regard to the derivative contracts that are used to complete the equity market.

	Large excitation		Medium excitation		Small excitation	
	Currency I	Currency II	Currency I	Currency II	Currency I	Currency II
call	0.937	0.937	0.928	0.938	0.920	0.940
put	0.936	0.936	0.927	0.937	0.918	0.939
straddle	0.927	0.927	0.917	0.929	0.907	0.931

Table 1: This table reports the equilibrium hedging ratio of Currency I and Currency II for the base investor when different derivative contracts are used for given equity-currency excitation structures. The rows correspond to different derivative contracts: call options in the first row; put options in the second and straddles in the third. "Large", "medium" and "small" excitation refer to the different scenario of equity-currency excitor of Currency I. The equity-currency excitor of Currency I is the largest in "Large excitation", and smallest in "Small excitation". The equity-currency excitation structure of Currency II is the same as that of Currency I in the "Large excitation" scenario and remains the same through all scenarios. The expected jump intensities of the equity jump component, Currency I jump component and Currency II jump component are kept constant in all scenarios. The excitation matrix is set to be $\boldsymbol{\beta} = (15, 6, 6; 6, 8, 0; 6, 0, 8)$ in case of "Large excitation", $\boldsymbol{\beta} = (15, 6, 6; 4, 11.8, 0; 6, 0, 8)$ in case of "Medium excitation", and $\boldsymbol{\beta} = (15, 6, 6; 2, 15, 0; 6, 0, 8)$ in case of "Small excitation". The expected jump intensities are intentionally kept constant with $\mathbb{E}[\lambda_m] = 1.28$, $\mathbb{E}[\lambda_1] = \mathbb{E}[\lambda_2] = 0.67$ in all three scenarios.

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