On the Time-varying Stock-Bond Correlation: Deciphering Heterogeneous Expectations

Jintao DU

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Abstract

Heterogeneous expectations are difficult to pin down and most of the predictions are confirmed only by calibrated simulations. This paper presents both a full characterization of equilibrium and an estimation of a dynamic general-equilibrium model in which two groups of agents hold heterogeneous expectations about future economic growth and inflation. Heterogeneous expectations and endogenous consumption fluctuations change the stock-bond correlation without the need to change agents' risk preferences or the dynamics between economic growth and inflation. Earnings and inflation processes are jointly estimated by maximizing the likelihood of their time series. The time series of heterogeneous expectations are uncovered using historical prices by means of the unscented Kalman Filter. The conditional correlation implied by the model can predict the conditional correlation in the data. Our model works for both Treasury bonds and Treasury Inflation Protected Securities. A time-varying term-structure of stock-bond correlation is identified and explained by the implied-disagreements. Our estimates of disagreements are correlated with dispersions of corresponding forecasts in the Survey of Professional Forecasters. Unconditional correlations and impulse responses are obtained from simulations.

1 Introduction

The correlation between stock market and Treasury bonds is crucial for asset allocation in portfolio management, but what decides its dynamics? The existing literature resorts to changes either in the correlation between economic growth and inflation or in agents' risk preferences, both of which do not change very frequently. This paper provides a novel explanation of the *time-varying* stock-bond correlation: a dynamic general-equilibrium model in which two groups of agents hold heterogeneous expectations about the future economic growth and inflation. The model shows that combinations of a small amount of heterogeneous expectations on both economic growth and inflation, and the endogenous consumption fluctuation resulting from trading can generate enough variation in the stock-bond correlation without the need to change risk preferences or correlations between fundamentals.

Despite ample evidence in surveys and analyst forecasts supporting the existence of heterogeneous expectations among investors, heterogeneous expectation models are like double-edged swords: on the one hand, they generate a pricing kernel that implies time-varying volatilities of equilibrium asset prices and risk premiums; on the other hand, the non-linearity harms the tractability of prices, and dispersions in expectations are difficult to pin down. Therefore, most of the predictions are confirmed by calibrations.

This paper attacks the heterogeneous-expectations problem head-on. The parameters of the model are estimated, rather than calibrated, using the Expectation-Maximization approach, an algorithm designed for missing data and applied to state-space models. In contrast to the literature that uses survey data as proxies of disagreements, the time-series of dispersion in expectations are uncovered from historical prices, together with the model using the unscented Kalman Filter, a machine-learning technique that deals with non-linearity. This paper directly regresses the historical conditional stock-bond correlation on the one implied by the model, and finds the latter predicts the former. Filtered dispersions in expectations are also compared with survey data. Regressions of asset prices that are not used in the filtering serve as out-of-sample tests for the model.

The most intriguing fact about stock-bond correlation is the switch of its sign in recent years. In the late 90s, researchers documented positive correlation between stock and bond prices (Shiller and Beltratti 1992; Campbell and Ammer 1993), while the market observed persistent negative stock-bond correlation after 2000 (Ilmanen, 2003; Li, 2002; Jones and Wilson, 2004; Connolly et al, 2005). Also, negative correlation is a common feature in many crises episodes, and is not merely a country-specific event but occurs simultaneously across countries (Baur and Lucey, 2009). The benchmark of stock-bond correlation has switched from positive to negative.

Is negative stock-bond correlation really shocking? Let us look at the data. Figure 1 and 2 present the time-series of conditional correlations between the $S\&P$ 500 index and US Treasury bonds with maturity of 5/10/30 year, both nominal and inflation-protected. The correlations are calculated with daily returns in each year. The correlation series exhibits great time-variation. Moreover, negative stock-bond correlations are not novel to the market since we observe a lot of years with negative stock-bond correlation in the last century.

To understand the stock-bond correlation, we should notice the difference between *conditional* correlation and *unconditional* correlation. At different points of time with its specific information set, the correlation looking forward is the conditional one. The unconditional correlation, however, is the overall correlation in the entire probability space. Therefore, economists need to answer two questions to explain the stock-bond correlation: first, why the conditional one switches sign under different conditions; second, why the unconditional correlation is positive. This paper provides a coherent answer: investors' heterogeneous expectations and the endogenous consumption share fluctuation resulting from trading. When investors have homogeneous expectations, the stock-bond correlation is negative because real stocks hedge inflation risks in nominal bonds and nominal bonds hedge risks of economic growth in stocks. In the presence of disagreements on inflation growth, the hedging links are weakened: both stock and bond prices are depressed by the additional risk in inflation, and they are expected either to recover together in the future with decreased disagreements on inflation expectation or to be depressed further if investors disagree even more, thus the stock-bond correlation is positive. In general, investors are more likely to disagree on inflation, so the unconditional stock-bond correlation is positive and the result is confirmed by simulations. In recent years, when inflation is persistently low and monetary policies are stable, investors have minor disagreements on expected inflation, and thus the stock-bond correlation is negative.

This paper also incorporates heterogeneous expectations on economic growth for three reasons: first, it is a basic form of heterogeneous expectation that we cannot neglect; second, it also changes the stock-bond correlation and thus helps to explain the full dynamics; third, it helps to construct stationarity in simulations. Models with heterogeneous expectations may suffer from the issue that one group of agents will end up taking the whole consumption and turn the model into a representative-agent model. To bring about stationarity without degenerating the model, a certain form of symmetric learning seems necessary. Researchers propose several tricks to model the source of heterogeneous expectations: investors may interpret a Markov chain differently (David 2008, Buss et al. 2016), disagree on volatility (Ehling et al. 2016), disagree on mean-reverting speed (Whelan 2014), have contrary interpretation of signals (Xiong and Yan 2010) or investors have the same learning but with different signals (Dumas, Lewis and Osambela 2016). This paper proposes another way to promote symmetry: one group of investors has a relative advantage in predicting future inflation and the other group has a relative advantage in predicting future economic growth. With balanced combinations of parameters, the asymmetric learning will end up as symmetric relative advantages through learning about economic growth and inflation. From simulations we observe that the model generates non-degenerated distributions of consumption shares for more than a thousand years, therefore, we can view the model as quasi-stationary.

Many attempts have been made to explain the intuition of the seemingly "unexpected pattern" of negative stock-bond correlation. The first story is "flight to quality" (Chicago Federal Reserve Bank News Letter, #4, December 1987): *"When investors are scared, they look for safety. They adjust their portfolios to include more safe assets and fewer risky assets."* Flight to quality asserts that in busts, investors' trading decreases stock prices and increases bond prices and thus stock and bond prices are negatively correlated. However, following the same logic, the other side of the story is "flight to risky": in booms investors include more risky assets and fewer safe assets, which also generates negative stock-bond correlation. Thus, stock-bond correlation should be negative all the time.

The negative stock-bond correlation reflects the hedging role between stocks and bonds: bonds as safe assets may hedge the economic risk in stocks, while stocks as real assets hedge the inflation risk in nominal bonds. The two hedging roles amplify each other. When the economic growth and inflation are positively correlated, the hedging role becomes even more significant: if inflation signals good economic growth, a positive inflation shock will increase stocks and depress bonds not only because the inflation rises, but the economic growth will rise as well. When the economic growth and inflation are negatively correlated, the hedging roles offset each other: if inflation signals bad economic growth, a positive inflation shock may lead to negative stock-bond correlations when the depression on economic growth is strong to offset the rise of inflation. This is exactly the intuition from models featuring regime-shifts (David and Veronesi 2013; Campbell, Pflueger and Viceira 2014; Song 2017), asymmetric monetary policies near zero lower bound (Evans et al. 2016; Howard 2016), or business cycles (Koijen, Lustig and Van Nieuwerburgh, 2010). These representative-agent models all need postulated pricing kernels, and some ways to embed inflation shocks either in the utility, like money illusion, or in the process of economic growth.

Different risk preferences also impact the stock-bond correlation. Barsky

(1989) shows that an elasticity of inter-temporal substitution (EIS) greater than unity implies a positive correlation while EIS less than unity implies negative correlation. Also, risk aversion helps to determine the magnitude. Kozak (2013) shows that the stock-bond correlation is negative if risk aversion is high keeping the same EIS. These results yield the intuition that an unexpected increase in economic growth will increase both the discount rate and the dividend payout in the future, so that bond prices will unambiguously decrease while the change in stock prices depends on whether the increase in dividend payout can offset that in the discount rate. According to Vissing-Jorgensen (2002), estimates of EIS are around 0.3 to 0.4 for stockholders, so the stock-bond correlation should be negative when there is no heterogeneous expectation.

The model in this paper is an extension to the sentiment-risk model proposed by Dumas, Kurshev and Uppal (2009, henceforth DKU), where two groups of investors are intellectually but not financially segmented and there is no inflation. In their model, one group of investors is fully rational, and the other group is over-confident about some pure-noise signals in the market. The change of measure between the two groups' beliefs is named "sentiment". In this paper, an inflation process is introduced, and it is correlated with the economic growth in both outputs and growth rates. There are two pure-noise signals in the market: one is perceived (falsely) as correlated with the growth rate of the economy; the other is perceived (falsely) as correlated with the rate of inflation. The more correlated an investor believes the signal is with the true growth rate, the more likely he will have biased expectations. Alternatively, we could assume the signals are true signals. Both models will have decreased investors' perceived variance of asset prices and excess volatility from the econometrician's viewpoint. Also, the solution and the equilibrium characterization are the same. The technical difference lies in the estimation task: if signals are pure noise, an econometrician could form a better estimate of the model using data that are publicly available; however, if signals are informative, she will not be able to justify her estimation unless she observes all the signals .

The rest of the paper is organized as follows. Section 2 describes the fundamental processes, signals, investors' beliefs and learning. In Section 3, the investors' optimization problem is introduced and solved; the equilibrium pricing kernel is obtained by market clearing. Equilibrium asset prices are determined from the pricing kernel. Section 4 estimates the fundamental processes and decides the investor-related parameters using grid search. Section 5 discusses the stock-bond correlation based on impulse responses and simulations. Section 6 uncovers the time-series of disagreements, compares them with survey data, predicts the conditional stock-bond correlation, and discusses its fit to the data. Section 7 concludes. All the proofs, tables, and figures are provided in the appendix.

2 Beliefs and Information Structure

2.1 The Fundamental Processes

This paper develops an information model in which two groups of investors (group A and group B) populate an endowment economy. In this economy, there are two fundamental variables: the aggregate dividend output δ_t, and the nominal price index P_t (CPI). We start by specifying their law of motion:

Assumption 1. *δ_t and P_t jointly follow geometric Brownian motions*

$$\frac{d\delta_t}{\delta_t} = f_t d_t + \sigma_{\delta,\delta} dZ_t^{\delta} + \sigma_{\delta,P} dZ_t^{P}, \tag{1}$$

$$\frac{dP_t}{P_t} = \pi_t d_t + \sigma_{P,\delta} dZ_t^{\delta} + \sigma_{P,P} dZ_t^{P}, \tag{2}$$

with their growth rates f_t and π_t following the joint mean-reverting process:

$$df_t = -\zeta_f(f_t - \bar{f})dt + \sigma_{f,f} dZ_t^{f} + \sigma_{f,\pi} dZ_t^{\pi}; \zeta_f > 0, \tag{3}$$

$$d\pi_t = -\zeta_\pi(\pi_t - \bar{\pi})dt + \sigma_{\pi,f} dZ_t^{f} + \sigma_{\pi,\pi} dZ_t^{\pi}; \zeta_\pi > 0, \tag{4}$$

where $dZ_t^{\delta}, dZ_t^{P}, dZ_t^{f}, dZ_t^{\pi}$ are one-dimensional processes following independent Brownian motions.

ζ_f and ζ_π are the mean-reverting parameters, $\bar{f}$ and $\bar{\pi}$ are the long-run mean of aggregate dividend growth rate and inflation rate. If $\sigma_{\delta,P}, \sigma_{P,\delta}$ are not both 0, the aggregate dividend growth is correlated with inflation. Z^i is the main driver of innovation in each process of i ($i \in \{\delta, P, f, \pi\}$). If $\sigma_{f,\pi}, \sigma_{\pi,f}$ are not both 0, the aggregate expected dividend-growth rate and inflation rate are correlated.

We assume that investors observe δ_t and P_t, and that they do not observe the conditional expected growth rates of dividends and inflation, f_t and π_t. Therefore, investors need to estimate the growth rates.

2.2 The Bogus Signals and Investors' Models

Assumption 2. *All investors filter out the current values of f and π using the observation of the current dividend, δ_t, the current price index, P_t, and two public bogus signals, $s_{f,t}$ and $s_{\pi,t}$, which have the following processes:*

$$ds_{f,t} = \sigma_{s_{f,t}} dZ_t^{s_f}, \tag{5}$$

$$ds_{\pi,t} = \sigma_{s_{\pi,t}} dZ_t^{s_\pi}, \tag{6}$$

where $dZ_t^{s_f}$ and $Z_t^{s_\pi}$ are one-dimensional processes following independent Brownian motions.

The signals $s_{f,t}$ and $s_{\pi,t}$ are pure noise. Investors in group i (for $i = A, B$) believe that the signal s_f is correlated with f with correlations ϕ_f^i, and that the signal s_π is correlated with π with correlation ϕ_π^i:

$$ds_{f,t} = \sigma_{s_f} \phi_f^i dZ_t^f + \sigma_{s_f} \sqrt{1 - {\phi_f^i}^2} dZ_t^{s_f}; |\phi_f^i| \leq 1, \tag{7}$$

$$ds_{\pi,t} = \sigma_{s_\pi} \phi_f^i dZ_t^\pi + \sigma_{s_\pi} \sqrt{1 - {\phi_\pi^i}^2} dZ_t^{s_\pi}; |\phi_\pi^i| \leq 1. \tag{8}$$

Equation (7) and (8) states the source of heterogeneous expectations: the investors have different "models" in their mind. The more they believe in the pure-noise signal, the more they are biased in general. The more they believe in the pure-noise signal than others do, the more likely they are to lose. One can have different assumptions on the relationship of ϕ between groups. We assume that $\phi_f^A > \phi_f^B$ and $\phi_\pi^A < \phi_\pi^B$. Thus group A is better in predicting inflation and are worse in predicting the aggregate dividend growth compared with group B. We can interpret the model as a reflection of the real world where news floods the market and only skilled investors can tease out true information from the bogus. Investors are not omniscient: if they are experts in stocks and specialize in predicting the aggregate dividend growth, they are not so good at predicting inflation, and vice versa. This assumption also implies the efficient market hypothesis because every investor has the same information set. For the same reason, no one needs to cooperate with anyone else.

2.3 Beliefs and Their Dynamics

Investors update their beliefs of the expected growth rates f and π with the aggregate dividend output and price level δ, P, and the two pure noise signals s_f and s_π. Lemma 1 presents the outcome of filtering for investors in group i (for $i = A, B$):

Lemma 1. *Optimal Filtering. The investors in group i have the following beliefs:*

$$\frac{d\delta_t}{\delta_t} = \hat{f}_t^i d_t + \sigma_{\delta\delta} dW_{\delta,t}^i + \sigma_{\delta P} dW_{P,t}^i, \tag{9}$$

$$\frac{dP_t}{P_t} = \hat{\pi}_t^i d_t + \sigma_{P\delta} dW_{\delta,t}^i + \sigma_{PP} dW_{P,t}^i, \tag{10}$$

with their growth rates f_t and π_t following the joint mean-reverting process:

$$\begin{aligned} d\hat{f}_t^i &= -\zeta_f(\hat{f}_t^i - \bar{f})dt \\ &+ \frac{\gamma_f^i \sigma_{PP} - \gamma_{f\pi}^i \sigma_{\delta P}}{\sigma_{\delta\delta}\sigma_{PP} - \sigma_{\delta P}\sigma_{P\delta}} dW_{\delta,t}^i + \frac{\gamma_{f\pi}^i \sigma_{\delta\delta} - \gamma_f^i \sigma_{P\delta}}{\sigma_{\delta\delta}\sigma_{PP} - \sigma_{\delta P}\sigma_{P\delta}} dW_{P,t}^i \\ &+ \phi_f^i \sigma_{ff} dW_{s_f,t}^i + \phi_\pi^i \sigma_{f\pi} dW_{s_\pi,t}^i, \end{aligned} \tag{11}$$

$$\begin{aligned} d\hat{\pi}_t^i &= -\zeta_\pi(\hat{\pi}_t^i - \bar{\pi})dt \\ &+ \frac{\gamma_{f\pi}^i \sigma_{PP} - \gamma_\pi^i \sigma_{\delta P}}{\sigma_{\delta\delta}\sigma_{PP} - \sigma_{\delta P}\sigma_{P\delta}} dW_{\delta,t}^i + \frac{\gamma_\pi^i \sigma_{\delta\delta} - \gamma_{f\pi}^i \sigma_{P\delta}}{\sigma_{\delta\delta}\sigma_{PP} - \sigma_{\delta P}\sigma_{P\delta}} dW_{P,t}^i \\ &+ \phi_f^i \sigma_{\pi f} dW_{s_f,t}^i + \phi_\pi^i \sigma_{\pi\pi} dW_{s_\pi,t}^i, \end{aligned} \tag{12}$$

their stationary estimates of variance and covariance of state variables are

$$\begin{aligned} \gamma_f^i &= h^i \sigma_{\delta\delta} sin(\alpha) - \zeta_f \sigma_{\delta\delta}^2 - h^i \sigma_{\delta P} cos(\alpha) - \zeta_f \sigma_{\delta P}^2 \\ \gamma_{f\pi}^i &= j^i \sigma_{\delta\delta} sin(\beta) - \zeta_\pi \sigma_{\delta\delta}\sigma_{P\delta} - j^i \sigma_{\delta P} cos(\beta) - \zeta_\pi \sigma_{\delta P}\sigma_{PP} \\ \gamma_\pi^i &= j^i \sigma_{P\delta} sin(\beta) - \zeta_\pi \sigma_{P\delta}^2 - j^i \sigma_{PP} cos(\beta) - \zeta_\pi \sigma_{PP}^2 \\ where& \\ h^i &= \sqrt{\sigma_{ff}^2(1 - (\phi_f^i)^2) + \sigma_{f\pi}^2(1 - (\phi_\pi^i)^2) + \zeta_f^2 \sigma_{\delta\delta}^2 + \zeta_f^2 \sigma_{\delta P}^2} \\ j^i &= \sqrt{\sigma_{\pi f}^2(1 - (\phi_f^i)^2) + \sigma_{\pi\pi}^2(1 - (\phi_\pi^i)^2) + \zeta_\pi^2 \sigma_{\delta\delta}^2 + \zeta_\pi^2 \sigma_{\delta P}^2} \end{aligned} \tag{13}$$

and α β are constants specified in the appendices.

Notice that the true information δ and P serve the same role as the bogus signal s_f and s_π do, which cause the update in growth rates. Also, there are only four independent Brownian motions in each group's mind. We define

$$\hat{g}_{f,t} = \hat{f}_t^B - \hat{f}_t^A, \tag{14}$$

$$\hat{g}_{\pi,t} = \hat{\pi}_t^B - \hat{\pi}_t^A, \tag{15}$$

so $\hat{g}_{f,t}$ is the disagreement in aggregate dividend growth rate, and $\hat{g}_{\pi,t}$ is the disagreement in inflation rate.

Now we have developed three sets of probability measures concerning the aggregate dividend and inflation processes: the true probability measure, and the probability measures under the probability beliefs of investors in group A and those in group B. Group A's probability beliefs at time t are represented by a change of measure η, and $\{\eta_t\}$ is a strictly positive martingale process. For any event e_u belonging to the σ-*algebra* of time u, we have

$$E_t^A[\mathbb{1}_{e_u}] = E_t^B[\frac{\eta_u}{\eta_t}\mathbb{1}_{e_u}] \tag{16}$$

η_t is named the "sentiment" variable in DKU as it captures the way in which group A's probability beliefs differ from group B's under the probability measure of group B. In Lemma 2, we show the dynamics of the change of measure from group A's probability beliefs to that from group B's.

Lemma 2. *Girsanov's Theorem. The change of measure, η_t, follows*

$$\frac{d\eta_t}{\eta_t} = -\frac{\sigma_{PP}\hat{g}_{f,t} - \sigma_{\delta P}\hat{g}_{\pi,t}}{\sigma_{\delta\delta}\sigma_{PP} - \sigma_{\delta P}\sigma_{P\delta}} dW_{\delta,t}^B - \frac{\sigma_{\delta\delta}\hat{g}_{\pi,t} - \sigma_{P\delta}\hat{g}_{f,t}}{\sigma_{\delta\delta}\sigma_{PP} - \sigma_{\delta P}\sigma_{P\delta}} dW_{P,t}^B \tag{17}$$

Observe that the disagreements are both incorporated into the volatility applied to the Brownian motions because the aggregate dividend growth and inflation are correlated. If $\sigma_{\delta P} = 0$, which means that the aggregate dividend growth has no exposure to the inflation shock, the information about inflation will not help the learning on aggregate dividend growth shock (but still helps the learning on the growth rate), and thus the volatility on $dW_{\delta,t}^B$ will be reduced to $-\frac{\hat{g}_{f,t}}{\sigma_{\delta\delta}}$, which is the representation of η_t in DKU. The same intuition works for the volatility applied to the inflation Brownian. Under group B's probability measure, the disagreements between two groups follow the dynamics below:

Lemma 3. *Dynamics of Disagreements.*

$$\begin{aligned} d\hat{g}_{f,t} = &- \psi_{ff}\hat{g}_{f,t}dt - \psi_{f\pi}\hat{g}_{\pi,t}dt \\ &+ \sigma_{\hat{g}_f,\delta}dW_{\delta,t}^B + \sigma_{\hat{g}_f,P}dW_{P,t}^B + \sigma_{\hat{g}_f,s}dW_{s_f,t}^B + \sigma_{\hat{g}_f,s'}dW_{s_\pi,t}^B \end{aligned} \tag{18}$$

$$\begin{aligned} d\hat{g}_{\pi,t} = &- \psi_{\pi f}\hat{g}_{f,t}dt - \psi_{\pi\pi}\hat{g}_{\pi,t}dt \\ &+ \sigma_{\hat{g}_\pi,\delta}dW_{\delta,t}^B + \sigma_{\hat{g}_\pi,P}dW_{P,t}^B + \sigma_{\hat{g}_\pi,s}dW_{s_f,t}^B + \sigma_{\hat{g}_\pi,s'}dW_{s_\pi,t}^B \end{aligned} \tag{19}$$

by comparing parameters, we can find that

$$
\begin{aligned}
\psi_{ff} &= \zeta_f + \frac{\gamma_f^A(\sigma_{P\delta}^2 + \sigma_{PP}^2) - \gamma_{f\pi}^A(\sigma_{\delta\delta}\sigma_{P\delta} + \sigma_{\delta P}\sigma_{PP})}{(\sigma_{\delta\delta}\sigma_{PP} - \sigma_{\delta P}\sigma_{P\delta})^2}, \\
\psi_{f\pi} &= \frac{\gamma_{f\pi}^A(\sigma_{\delta\delta}^2 + \sigma_{\delta P}^2) - \gamma_f^A(\sigma_{\delta\delta}\sigma_{P\delta} + \sigma_{\delta P}\sigma_{PP})}{(\sigma_{\delta\delta}\sigma_{PP} - \sigma_{\delta P}\sigma_{P\delta})^2}, \\
\sigma_{\hat{g}_f,\delta} &= \frac{(\gamma_f^B - \gamma_f^A)\sigma_{PP} - (\gamma_{f\pi}^B - \gamma_{f\pi}^A)\sigma_{\delta P}}{\sigma_{\delta\delta}\sigma_{PP} - \sigma_{\delta P}\sigma_{P\delta}}, \\
\sigma_{\hat{g}_f,P} &= \frac{(\gamma_{f\pi}^B - \gamma_{f\pi}^A)\sigma_{\delta\delta} - (\gamma_f^B - \gamma_f^A)\sigma_{P\delta}}{\sigma_{\delta\delta}\sigma_{PP} - \sigma_{\delta P}\sigma_{P\delta}}, \\
\sigma_{\hat{g}_f,s} &= (\phi_f^B - \phi_f^A)\sigma_{ff}, \\
\sigma_{\hat{g}_f,s'} &= (\phi_\pi^B - \phi_\pi^A)\sigma_{f\pi},
\end{aligned}
\tag{20}
$$

$$
\begin{aligned}
\psi_{\pi f} &= \frac{\gamma_{f\pi}^A(\sigma_{P\delta}^2 + \sigma_{PP}^2) - \gamma_\pi^A(\sigma_{\delta\delta}\sigma_{P\delta} + \sigma_{\delta P}\sigma_{PP})}{(\sigma_{\delta\delta}\sigma_{PP} - \sigma_{\delta P}\sigma_{P\delta})^2}, \\
\psi_{\pi\pi} &= \zeta_\pi + \frac{\gamma_\pi^A(\sigma_{\delta\delta}^2 + \sigma_{\delta P}^2) - \gamma_{f\pi}^A(\sigma_{\delta\delta}\sigma_{P\delta} + \sigma_{\delta P}\sigma_{PP})}{(\sigma_{\delta\delta}\sigma_{PP} - \sigma_{\delta P}\sigma_{P\delta})^2}, \\
\sigma_{\hat{g}_\pi,\delta} &= \frac{(\gamma_{f\pi}^B - \gamma_{f\pi}^A)\sigma_{PP} - (\gamma_\pi^B - \gamma_\pi^A)\sigma_{\delta P}}{\sigma_{\delta\delta}\sigma_{PP} - \sigma_{\delta P}\sigma_{P\delta}}, \\
\sigma_{\hat{g}_\pi,P} &= \frac{(\gamma_\pi^B - \gamma_\pi^A)\sigma_{\delta\delta} - (\gamma_{f\pi}^B - \gamma_{f\pi}^A)\sigma_{P\delta}}{\sigma_{\delta\delta}\sigma_{PP} - \sigma_{\delta P}\sigma_{P\delta}}, \\
\sigma_{\hat{g}_\pi,s} &= (\phi_f^B - \phi_f^A)\sigma_{\pi f}, \\
\sigma_{\hat{g}_\pi,s'} &= (\phi_\pi^B - \phi_\pi^A)\sigma_{\pi\pi}.
\end{aligned}
\tag{21}
$$

First, we can show that $\psi_{ff} > \psi_{f\pi}, \psi_{\pi f} < \psi_{\pi\pi}$ and $\psi_{ff} > 0, \psi_{\pi\pi} > 0$, therefore, the disagreements $\hat{g}_{f,t}$ and $\hat{g}_{\pi,t}$ are jointly mean-reverting. Second, the long-run mean of both disagreements is zero, because the bogus signals have no drift-term. If investors are biased in the exactly same way (not necessarily unbiased), the disagreements will have no diffusion and investors will converge in their opinions even if they disagree at the beginning.

3 The Equilibrium Allocation of Consumption and Asset Prices

3.1 The Individual's Optimization Problem

Assumption 3. *The financial market is complete, and both groups have the same CRRA utility with the relative risk aversion $1-\alpha$ and the rate of impatience ρ. Investors maximize their expected utility from lifetime consumption.*

Assumption 3 allows us to use the static martingale formulation (as in Cox and Huang, 1989). Investors in group B solve the below lifetime optimization problem:

$$\max_c E^B \int_0^\infty e^{-\rho t}\frac{1}{\alpha}(c_t^B)^\alpha dt;\ \ \alpha < 1, \tag{22}$$

subject to the lifetime budget constraint

$$E^B \int_0^\infty \xi_t^B P_t c_t^B dt = \bar{\theta}^B E^B \int_0^\infty \xi_t^B P_t \delta_t dt, \tag{23}$$

ξ^B is the change of measure from group B's probability measure to the risk-neutral measure, which is the *nominal* pricing kernel. $\bar{\theta}^B$ is the share of equity with which B is initially endowed. The first-order condition for consumption is

$$e^{-\rho t}(c_t^B)^{\alpha-1} = \lambda^B \xi_t^B P_t. \tag{24}$$

where λ^B is the Lagrange multiplier of the budget constraint (23). Group A holds an initial share $\bar{\theta}^A = 1-\bar{\theta}^B$ of equity and faces an analogous optimization problem. The only difference is that group A uses a probability measure that is different from that of group B. Under B's probability measure, the problem of A can be stated as follows:

$$\max_c E^B \int_0^\infty \eta_t e^{-\rho t}\frac{1}{\alpha}(c_t^A)^\alpha dt; \alpha < 1 \tag{25}$$

subject to the lifetime budget constraint

$$E^B \int_0^\infty \xi_t^B P_t c_t^A dt = \bar{\theta}^A E^B \int_0^\infty \xi_t^B P_t \delta_t dt \tag{26}$$

The first-order condition for consumption in this case is

$$\eta_t e^{-\rho t}(c_t^A)^{\alpha-1} = \lambda^A \xi_t^B P_t \tag{27}$$

where λ^A is the Lagrange multiplier of the budget constraint (27).

3.2 The Equilibrium Pricing Measure

Equilibrium is defined as a price system and a pair of consumption-portfolio policies such that (i) investors maximize their expected utility from lifetime consumption based on their own beliefs; (ii) the asset prices are the same under anyone's probability measure; (iii) the market clears. In equilibrium, investors agree to disagree on the expectations. They take speculative positions against each other, and thus their relative consumption fluctuates endogenously.

The aggregate resource constraint, from (24) and (27), is

$$(\frac{\lambda^A \xi_t^B P_t e^{\rho t}}{\eta_t})^{\frac{1}{\alpha-1}} + (\lambda^B \xi_t^B P_t e^{\rho t})^{\frac{1}{\alpha-1}} = \delta_t \tag{28}$$

Solving (28), we obtain

$$\xi_t^B(\delta_t, \eta_t) = e^{-\rho t}\big[(\frac{\eta_t}{\lambda^A P_t})^{\frac{1}{1-\alpha}} + (\frac{1}{\lambda^B P_t})^{\frac{1}{1-\alpha}}\big]^{1-\alpha} \delta_t^{\alpha-1} \tag{29}$$

and, therefore

$$c_t^A = \omega(\eta_t)\delta_t \tag{30}$$

$$c_t^B = (1-\omega(\eta_t))\delta_t \tag{31}$$

where

$$\omega(\eta_t) = \frac{(\frac{\eta_t}{\lambda^A})^{\frac{1}{1-\alpha}}}{(\frac{\eta_t}{\lambda^A})^{\frac{1}{1-\alpha}} + (\frac{1}{\lambda^B})^{\frac{1}{1-\alpha}}} \tag{32}$$

A linear consumption-sharing rule manifests itself in Equation (30) and (31), but the relative consumption, $\omega(\eta_t)$, is stochastic because the sentiment variable η_t is stochastic. Also, the price level does not enter the consumption allocation equations directly, but the price level process does influence the consumption allocation through η_t.

3.3 Asset Prices

Five securities that are linearly independent are required to complete financial markets and implement the equilibrium, because there are four Brownian motions that investors can observe: $W_{\delta,t}^B$, $W_{P,t}^B$, $W_{s_f,t}^B$, and $W_{s_\pi,t}^B$. The choice of securities is arbitrary. We are interested in the stock-bond correlation, so we assume that these assets are a nominal bond that pays one dollar at its maturity, an inflation-linked bond that pays one unit of consumption at its maturity, and a stock that pays the aggregate dividend δ perpetually. The other two securities can be modeled as one that has the diffusion of $W^B_{s_f,t}$, and the other that has the diffusion of $W^B_{s_\pi,t}$. These can be swaps on the two bogus signals.

There are seven state variables in the economy: two output variables, δ_t and P_t, two estimates of growth rates, $\hat{f}^B$ and $\hat{\pi}^B$, two disagreements, $\hat{g}_f$ and $\hat{g}_\pi$, and the sentiment variable, η_t. η_t is one-to-one related to group A's consumption share ω^A.

The equilibrium price of the nominal bond with maturity T, which I denote by BN, can be obtained directly from the pricing measure:

$$BN(\hat{f}^B, \hat{\pi}^B, \hat{g}_f, \hat{g}_\pi, \eta, t, T) = E^B_{\hat{f}^B, \hat{\pi}^B, \hat{g}_f, \hat{g}_\pi, \eta}\Big[\frac{\xi^B_T}{\xi^B_t}\Big] \tag{33}$$

Similarly, the equilibrium price of an inflation-linked bond with maturity T, which I denote by BI:

$$BI(P, \hat{f}^B, \hat{\pi}^B, \hat{g}_f, \hat{g}_\pi, \eta, t, T) = P_t E^B_{P, \hat{f}^B, \hat{\pi}^B, \hat{g}_f, \hat{g}_\pi, \eta}\Big[\frac{\xi^B_T}{\xi^B_t}\frac{P_T}{P_t}\Big] \tag{34}$$

The equilibrium price of the stock, which I denote by F (to be consistent with the notation in DKU), is the sum of all the future aggregate dividends:

$$F(P, \delta, \hat{f}^B, \hat{\pi}^B, \hat{g}_f, \hat{g}_\pi, \eta, t) = \delta_t P_t \int_t^\infty E^B_{P, \delta, \hat{f}^B, \hat{\pi}^B, \hat{g}_f, \hat{g}_\pi, \eta}\Big[\frac{\xi^B_u}{\xi^B_t}\frac{\delta_u}{\delta_t}\frac{P_u}{P_t}\Big] du \tag{35}$$

We also denote the single-payoff version of stock by F^T, which pays the aggregate dividend at time T:

$$F^T(P, \delta, \hat{f}^B, \hat{\pi}^B, \hat{g}_f, \hat{g}_\pi, \eta, t) = \delta_t P_t E^B_{P, \delta, \hat{f}^B, \hat{\pi}^B, \hat{g}_f, \hat{g}_\pi, \eta}\Big[\frac{\xi^B_T}{\xi^B_t}\frac{\delta_T}{\delta_t}\frac{P_T}{P_t}\Big] \tag{36}$$

Assuming that α is an integer, we can expand ξ^B_t from equation (29):

$$\xi^B_t = \frac{1}{\lambda^B}\sum_{j=0}^{1-\alpha}\Big[\frac{(1-\alpha)!}{(1-\alpha-j)!j!}\Big(\frac{\eta_t \lambda^B}{\lambda^A}\Big)^j\Big] \tag{37}$$

Therefore, to find out the asset prices, we need to calculate the moment-generating function of the joint distribution for $\{ln\delta_u, lnP_u, ln\eta_u\}$.

3.4 The Moment-generating Function

The moment-generating function of the joint distribution for $\{ln(\delta_u), ln(P_u), ln(\eta_u)\}$ can be obtained in closed form.

Proposition 1. *The moment-generating function for* $\{ln\delta_u, lnP_u, ln\eta_u\}$ *is*

$$\begin{aligned} E^B_{\hat{f}^B,\hat{\pi}^B,\hat{g}_f,\hat{g}_\pi}[(\delta_u)^\epsilon (P_u)^\kappa (\eta_u)^\chi] = \delta^\epsilon \eta^\chi P^\kappa \times H_{f\pi}(\hat{f}^B, \hat{\pi}^B, u, t; \epsilon, \kappa) \\ \times H_g(\hat{g}, \hat{g}_\pi, u, t; \epsilon, \kappa, \chi) \end{aligned} \tag{38}$$

where

$$\begin{aligned} H_{f\pi}(\hat{f}^B, \hat{\pi}^B, u, t; \epsilon, \kappa) = exp\Big\{\epsilon[\bar{f}(u-t) + \frac{1}{\zeta_f}(\hat{f}^B - \bar{f})(1 - e^{-\zeta_f(u-t)})] \\ + \kappa[\bar{\pi}(u-t) + \frac{1}{\zeta_\pi}(\hat{\pi}^B - \bar{\pi})(1 - e^{-\zeta_\pi(u-t)})] + Af\pi(\epsilon, \kappa; u-t)\Big\} \end{aligned} \tag{39}$$

$$\begin{aligned} H_g(\hat{g}_f, \hat{g}_\pi, u, t; \epsilon, \kappa, \chi) = exp\Big\{(A1_{f\pi}(\chi; u-t) \\ + \epsilon^2 A2_f(\chi; u-t) + \kappa^2 A2_\pi(\chi; u-t) + \epsilon\kappa A2_{f\pi}(\chi; u-t) \\ + \epsilon\hat{g}_f B_f(\chi; u-t) + \kappa\hat{g}_\pi B_\pi(\chi; u-t) \\ + \hat{g}_f^2 C_f(\chi; u-t) + \hat{g}_\pi^2 C_\pi(\chi; u-t) + \hat{g}_f\hat{g}_\pi C_{f\pi}(\chi; u-t)\Big\} \end{aligned} \tag{40}$$

and the functions $Af\pi$*,* $A1_{f\pi}$*,* $A2_f$*,* $A2_\pi$*,* $A2_{f\pi}$*,* B_f*,* B_π*,* $C_f(\chi; u-t)$*,* $C_\pi(\chi; u-t)$*, and* $C_{f\pi}$ *are given in the proof.*

This moment-generating function belongs to the linear-quadratic jump-diffusion model class, whose solution is generalized in Cheng and Scaillet (2007). The solution boils down to solving a system of *non-symmetric matrix Riccati differential equations* (RDE), using Radon's lemma. The asset prices are weighted averages of exponential linear-quadratic functions of state variables with stochastically time-varying weights.

The equilibrium price of the nominal bond BN is:

$$\begin{aligned} E^B_{\hat{f}^B,\hat{\pi}^B,\hat{g}_f,\hat{g}_\pi}[\frac{\xi^B_T}{\xi^B_t}] = e^{\rho(T-t)}(1-\omega(\eta))^{1-\alpha} H_{f\pi}(\hat{f}^B, \hat{\pi}^B, T, t; \alpha-1, -1) \\ \times \sum_{j=0}^{1-\alpha}\Big\{\frac{(1-\alpha)!}{(1-\alpha-j)!j!}\Big(\frac{\omega(\eta)}{1-\omega(\eta)}\Big)^j H_g(\hat{g}_f, \hat{g}_\pi, T, t; \alpha-1, -1, \frac{j}{1-\alpha})\Big\} \end{aligned} \tag{41}$$

The equilibrium price of the inflation-linked bond BI is:

$$\begin{aligned}
P_t E^B_{P,\hat{f}^B,\hat{\pi}^B,\hat{g}_f,\hat{g}_\pi}[\frac{\xi^B_u}{\xi^B}_t \frac{P_T}{P_t}] &= P_t e^{\rho(T-t)}(1-\omega(\eta))^{1-\alpha} \\
&\times H_{f\pi}(\hat{f}^B, \hat{\pi}^B, T, t; \alpha-1, 0) \\
&\times \sum_{j=0}^{1-\alpha} \Big\{ \frac{(1-\alpha)!}{(1-\alpha-j)!j!} \Big(\frac{\omega(\eta)}{1-\omega(\eta)} \Big)^j H_g(\hat{g}_f, \hat{g}_\pi, T, t; \alpha-1, 0, \frac{j}{1-\alpha}) \Big\}
\end{aligned} \tag{42}$$

The equilibrium price of single-payoff stock F^T is:

$$\begin{aligned}
\delta_t P_t E^B_{P,\delta,\hat{f}^B,\hat{\pi}^B,\hat{g}_f,\hat{g}_\pi}[\frac{\xi^B_u}{\xi^B}_t \frac{\delta_T}{\delta_t} \frac{P_T}{P_t}] &= \delta_t P_t e^{\rho(T-t)}(1-\omega(\eta))^{1-\alpha} \\
&\times H_{f\pi}(\hat{f}^B, \hat{\pi}^B, T, t; \alpha, 0) \\
&\times \sum_{j=0}^{1-\alpha} \Big\{ \frac{(1-\alpha)!}{(1-\alpha-j)!j!} \Big(\frac{\omega(\eta)}{1-\omega(\eta)} \Big)^j H_g(\hat{g}_f, \hat{g}_\pi, T, t; \alpha, 0, \frac{j}{1-\alpha}) \Big\}
\end{aligned} \tag{43}$$

The equilibrium price of stock F is the integral of F^T:

$$\delta_t P_t \int_{T=t}^{\infty} E^B_{P,\delta,\hat{f}^B,\hat{\pi}^B,\hat{g}_f,\hat{g}_\pi}[\frac{\xi^B_T}{\xi^B}_t \frac{\delta_T}{\delta_t} \frac{P_T}{P_t}] dT = \int_{T=t}^{\infty} F^T dT \tag{44}$$

Following Detemple and Murthy (1994), the pricing formulas are weighted averages of exponential-quadratic functions, which can be seen as the weighted averages of prices in several homogeneous-agent economies, each populated with group B investors with risk aversion $1-j$. The pricing formulas tell us the risks that each security bears. Particularly, the real assets, inflation-linked bond BN and stock F, are immune to the inflation rate π_t. However, they are subject to the disagreement on inflation rate, and the influence is the same irrespective of the sign. When investors agree on the inflation rate, the price index serves only as a scaling factor from the real to the nominal; while if investors disagree, they will take speculative positions against each other, and the disagreement on inflation serves as a measure of riskiness.

3.5 The Expected Conditional Stock-Bond Correlation

The second moments of prices are stochastic because their volatilities are stochastic. We can study their expected second moments with the product of their diffusion vectors, which under group B's probability measure are the loadings on $dW^B_{\delta,t}$ $dW^B_{P,t}$ and $dW^B_{s_f,t}$ $dW^B_{s_\pi,t}$. The diffusion vectors can be obtained from the gradient of the price function post-multiplied by the diffusion matrix of state variables. Each security's price exposure has seven components corresponding to the seven elements of the gradient vector. All these derivatives are known in closed form when $1-\alpha \in \mathbb{N}$. The variance-covariance matrix of security prices is simply

$$\begin{bmatrix} diffF \\ diffBI \\ diffBN \end{bmatrix} \begin{bmatrix} diffF \\ diffBI \\ diffBN \end{bmatrix}'$$

The conditional correlation between securities' prices and the conditional return correlation are the same, because the return is the expected price divided by the previous price, while the previous price is already available in the information set and will be canceled out from both the numerator and the denominator.

Proposition 2. *The correlation between two single-payment securities is immune to both the level and the current growth rates of aggregate dividend and price index.*

The proof is straightforward: because the output and growth rates are functions multiplied to the weighted average of H_g, and thus canceled out from their variance in the denominator and covariance in the numerator. This proposition states that current growth rates do have an impact on the variance of asset prices but not the correlations if assets do not have lasting effects on investors' consumption. This is the reason why some models have constant correlations between asset prices.

4 Empirical Features of the Model

In this section, we will estimate the parameters of the aggregate dividend growth and inflation processes, and calibrate the parameters that decide investors' risk and time preference. We apply grid search to pin down the parameters that generate heterogeneous expectations.

4.1 Data

This paper uses daily prices of $S\&P$ 500 and Treasury bonds from Bloomberg. The bond prices are implied by their yields. Data on annual earnings of $S\&P$ 500 are from Aswath Damodaran's website, and they are used as proxies for nominal dividends in that the actual dividends are scheduled and manipulated. Annual CPI data are from CRSP. These data are not smoothed, which keeps their original volatility. I also use the dispersion data in Survey of Professional Forecasters from the Federal Reserve Bank of Philadelphia.

4.2 Estimating the Aggregate Dividend Growth and Inflation Processes with the E-M Approach

The model has both parameters and state variables. The plan is that we estimate the parameters of the fundamental processes, and then use the estimates to uncover the state variables. The objective is that the model can predict stock-bond correlation using the estimates of parameters and state variables. This plan is ambitious because we need to take the model in a rigorous manner. Most of models with heterogeneous expectations use calibrated parameter values, which seem to match some moments of the data. I argue that this is not the correct way because the *sample moments* are hardly the *unconditional moments*. The most salient feature of models with heterogeneous expectations is the time-varying volatility, while by assuming the equivalence between sample and unconditional moments we deny it.

How to take the model seriously? In this paper, as we have specified the law of motion of aggregate dividend growth and inflation, we just need to maximize the likelihood of time series of earnings growth and inflation. First, as data are discrete observations, we need to transform the processes from the continuous-time form to the discrete-time form using exact discretization under the true probability measure. Define the log aggregate dividend growth ratio and inflation:

$$Y_t \equiv \begin{bmatrix} ln(\frac{\delta_t}{\delta_{t-1}}) \\ ln(\frac{P_t}{P_{t-1}}) \end{bmatrix} = \Delta \begin{bmatrix} f_{t-1} \\ \pi_{t-1} \end{bmatrix} - \frac{1}{2}\Delta \begin{bmatrix} \sigma_{\delta\delta}^2 + \sigma_{\delta P}^2 \\ \sigma_{P\delta}^2 + \sigma_{PP}^2 \end{bmatrix} + \sqrt{\Delta} \begin{bmatrix} \sigma_{\delta\delta} & \sigma_{\delta P} \\ \sigma_{P\delta} & \sigma_{PP} \end{bmatrix} \begin{bmatrix} Z_t^\delta \\ Z_t^P \end{bmatrix} \tag{45}$$

$$\begin{aligned} X_t \equiv \begin{bmatrix} f_t \\ \pi_t \end{bmatrix} &= \begin{bmatrix} f_{t-1}e^{-\zeta_f\Delta} \\ \pi_{t-1}e^{-\zeta_\pi\Delta} \end{bmatrix} + \begin{bmatrix} \bar{f}(1-e^{-\zeta_f\Delta}) \\ \bar{\pi}(1-e^{-\zeta_\pi\Delta}) \end{bmatrix} \\ &+ \begin{bmatrix} \sqrt{\frac{1-e^{-2\zeta_f\Delta}}{2\zeta_f}}\sigma_{ff} & \sqrt{\frac{1-e^{-2\zeta_f\Delta}}{2\zeta_f}}\sigma_{f\pi} \\ \sqrt{\frac{1-e^{-2\zeta_\pi\Delta}}{2\zeta_\pi}}\sigma_{\pi f} & \sqrt{\frac{1-e^{-2\zeta_\pi\Delta}}{2\zeta_\pi}}\sigma_{\pi\pi} \end{bmatrix} \begin{bmatrix} Z_t^f \\ Z_t^\pi \end{bmatrix} \end{aligned} \tag{46}$$

where Δ is the time interval between observations and it is one if the model are in the same frequency with the data.

Note that if the aggregate dividend growth is independent from inflation, the discretization of δ and P can be done separately. This is the dividend growth process that prevails in the literature. If we take its expectation and variance, we can easily find that

$$Var[ln(\frac{\delta_t}{\delta_{t-1}})] = \Delta^2 \frac{\sigma_f^2}{2\zeta_f} + \sigma_\delta^2 \Delta$$

$$E[ln(\frac{\delta_t}{\delta_{t-1}})] = \Delta(\bar{f} - \frac{1}{2}\sigma_\delta^2)$$

If we replace $Var(Y_t)$ with the sample variance $V\hat{ar}(Y_t)$ and replace $E(Y_t)$ with the sample mean $E(\hat{Y_t})$, we have two equations and four parameters, which are σ_f^2, ζ_f, σ_δ^2 and $\bar{f}$. This is why we have some degrees of freedom. Particularly, $V\hat{ar}(Y_t) \neq \sigma_\delta^2$ and $E(\hat{Y_t}) \neq \bar{f}$: the sample variance is not equal to the output volatility and the sample mean is not equal to the long-term mean of growth rate, which some previous studies take them as equal.

In total, there are twelve parameters in the fundamental processes: $\sigma_{\delta\delta}$, $\sigma_{\delta P}$, $\sigma_{P\delta}$, σ_{PP}, ζ_f, ζ_π, $\bar{f}$, $\bar{\pi}$, σ_{ff}, $\sigma_{f\pi}$, $\sigma_{f\pi}$, $\sigma_{\pi\pi}$. We also need to estimate the state variables: the dividend growth rate f_t and the inflation rate π_t. This is a typical dual-estimation problem in which statisticians need to estimate the parameters and state variables at the same time. We can use the Expectation-maximization Approach, which is an iterative algorithm designed for missing data and applied to state-space models (see, for example, Chapter 6, Haykin 2001). The idea is as follows: first, we choose the initial values of the parameters (which I take the values from Brenna and Xia, 2006, and Xiong and Yan, 2010); second, we calculate the distribution of state variable vector $\{X_t\}$ with observations $\{Y_t\}$ and parameters using the Kalman filter and backward pass; third, we estimate the parameters by maximizing the log likelihood of $\{X_t\}$ and $\{Y_t\}$ together. We iterate the procedure until all the parameters converge. Mathematically, the E-M algorithm guarantees the convergence (Dempster et al. 1977). In practice, I use the gradient descent for convergence: the algorithm stops when all the parameters' gradients are smaller than a certain percentage (we choose 2×10^{-6}) of the log likelihood.

Figure 3-5 plots the estimates of $\bar{f}$, $Cov(f, \pi)$, and $\sigma_{\delta\delta}$ against the number of iterations as illustrations of the convergence. The choices of illustration are arbitrary. Clearly, they converge. We have clean solutions to the first-order conditions of the log likelihood with respect to the parameters in the growth rate systems and the convergence is nice. We do not have analytical solution to the first-order conditions concerning the output volatilities, and use numerical method to maximize the log likelihood, which results in the perturbations. The convergence is quite fast as it only takes 165 iterations.

Table 1 reports the estimates of all the parameters and their variances from the inverse of the Fisher information matrix. The volatility of the earnings is 0.0658, which is close to the overall sample standard deviation of earnings growth (0.0664). The volatility of inflation is close to 0.011, which is the overall sample standard deviation of inflation (0.012). After correction for the volatility in the growth rates, the variance implied by the estimates are even closer to the deviation from the overall sample variance. The mean-reverting parameter of the earnings growth rate is 0.12, which means the it takes $\ln(2)/\zeta_f$=6 years for the difference between the earnings growth rate and its long-term mean to die out by half; similarly, the mean-reverting parameter of inflation is 0.03 and implies that the half-life of deviation from long-term inflation rate is about 23 years.

On a technical note, the numerical method also causes the variance from the Inverse fisher information matrix to be negative. This means that, before other parameters converge, the parameters identified by the numerical method have already reached their optimum and have started to decrease the likelihood. This is somehow inevitable with numerical methods in the sense that it is difficult and of a bit of luck to find out the path along which all the parameters are jointly reaching the optimum.

Notice that we do not have the estimates of σ_{ff}, $\sigma_{\pi f}$, $\sigma_{f\pi}$, and $\sigma_{\pi\pi}$ individually, but rather we have estimated the variance-covariance matrix of the growth rates. This is because given the model structure, it is impossible to pin them down from the likelihood. There is one degree of freedom as there are 4 unknowns and 3 equations. We can rewrite them as below:

$$\begin{aligned}
\sigma_{ff} =& Sin(\pi - ArcSin[\frac{Cov(f,\pi)}{\sqrt{Var(f)Var(\pi)}}])\sqrt{Var(f)},\\
\sigma_{f\pi} =& Cos[\pi - ArcSin[\frac{Cov(f,\pi)}{\sqrt{Var(f)Var(\pi)}}]]\sqrt{Var(f)},\\
\sigma_{\pi f} =& Cos(\beta)\sqrt{Var(\pi)}\\
\sigma_{\pi\pi} =& Sin(\beta)\sqrt{Var(\pi)}
\end{aligned} \tag{47}$$

where $\frac{\pi}{2} < \beta < \frac{3\pi}{4} - ArcSin(\frac{Cov(f,\pi)}{\sqrt{Var(f)Var(\pi)}})$. If we impose the restriction that $\sigma_{f\pi} = \sigma_{\pi f}$, we will find $\beta = 1.13$.

4.3 Parameters about Investors and Grid Search

First, we need to choose the risk-aversion parameter $1-\alpha$. Barsky (1989) shows that the correlation between the riskless interest rate and the stock market depends on the aversion to inter-temporal substitution. As elasticity of inter-temporal substitution (EIS) is the inverse of relative risk aversion, we should choose α so that its implied EIS is matched with the data. According to Vissing-Jorgensen (2002) that estimates of EIS are around 0.3 to 0.4 for stock holders, we can pin down that the relative risk aversion in integer should be 3 and α should be -2. The rate of impatience ρ is chosen at 0.05 to match the level of prices.

To choose the parameters characterizing the source of heterogeneous expectations, ϕ_f^A, ϕ_π^A, ϕ_f^B, and ϕ_π^B, I implement a grid search. The purpose is to find out the combination of $\{\phi\}$ that generates the most stable consumption distribution. We search all four $\{\phi\}$ from 0.01 to 0.99 with increments being 0.01, applying the constrains that $\phi_f^A > \phi_f^B$ and $\phi_\pi^A < \phi_\pi^B$. We antithetically simulate 5000 paths for 1000-year data, and record the 10th, 50th, and 90th quantile of the final consumption distribution. From all the combinations that generates a consumption distribution whose 50th quantile is close 0.5, we pick the $\{\phi\}$ that has the lowest dispersion between the 10th and 90th quantile, and they are $\phi_f^A = 0.89$, $\phi_\pi^A = 0.15$, $\phi_f^B = 0.1$, and $\phi_\pi^B = 0.7$. The final consumption distribution of group A can be found in Figure 6, which shows that indeed the consumption distribution does not degenerate to the tails.

5 The Stock-Bond Correlation

This section examines both unconditional and conditional correlation implied by the model. We start from the unconditional correlation.

5.1 The Unconditional Stock-Bond Correlation

When there is no sentiment risk, the correlation between bond and stock become constant, so the conditional correlation is also the unconditional one, and the model is reduced to a learning-based Vasicek model (1977). From Figure 7 and Figure 8, we can see that the stock-bond correlation increases with risk aversion. This is consistent with Kozak (2013) who says that, when the discount rate is high, the stock-bond correlation is negative. With CRRA utility, the discount rate decreases with risk aversion, so the stock-bond correlation increases with risk aversion. The maturity of bonds also increases the correlation on nominal bond and single-payment stock, but the impact of maturity of bonds is weaker on the correlation between nominal bond and perpetual normal stock because the impact diminishes with time and is netted off by the future dividends. Note that, around risk aversion being unity, the correlation starts to decrease with risk aversion. This corresponds to the effect of EIS on the stock-bond correlation. EIS greater than unity switches the correlation from negative to positive in Barsky (1989), while, in a learning model with CRRA utility, risk aversion and EIS are inverse, and the pricing kernel is a smooth function of risk aversion. As a result, EIS increases stock-bond correlation when it is smaller than unity rather than brings about a sudden change of sign.

Although the conditional stock-bond correlation varies a lot, the overall correlation between $S\&P$ 500 and US 5/10/30 year Treasury bond is 0.36, 0.51 and 0.65. We simulate 500 antithetic paths for 200 years and compute stock-bond correlations in the last 100 year. The distribution of simulated correlations are in Figure 9 to 11. The simulated correlations are nicely distributed and centered around 0. The average correlation between stock and 5/10/30 year bond are 0.019, 0.027, and 0.037. The magnitude is only one-tenth of the data. This may due to the difference between the average of paths and one specific path.

5.2 The Impulse Responses to Shocks

Figure 12 plots the impulse responses to a shock in the aggregate dividend of each group's deviation from the true aggregate dividend growth rate. Although there is no other shock, investors' estimates of aggregate dividend growth immediately deviates from the truth for two reasons: first, both groups misinterpret the signal; second, even if they update correctly, they will still deviate from the true signal, because they do not observe the shock in expected dividend growth rate. In fact, as there is no shock from the expected dividend growth rate and no shock from bogus signal, investors in group B, whose beliefs are less distorted, end up deviating further from the true growth rate than those who are biased do, because they attribute more weight to the actual dividend growth. We see their deviations from the truth slowly converges, but the impact will last because it results in a transfer of the consumption share from group B to group A. The changes in their disagreements and consumption share will change the conditional stock-bond correlation.

5.3 The Conditional Stock-Bond Correlation

The conditional stock-bond correlation is very complicated because it is susceptible to five state variables. In this subsection, we try to illustrate the main mechanisms. We start again by plotting the correlation in response to maturity and relative risk aversion in Figure 13. With sentiment risks, the term-structure of stock-bond correlation becomes obvious: the stock-bond correlation increases with the bond maturity. Also, it increases with relative risk aversion. The term-structure of stock-bond correlation sheds light upon the fact that in recent years the overall correlation between 5-year Treasury bond and $S\&P$ 500 has decreased from positive to negative, while the correlation between 10/30-year Treasury bond and $S\&P$ 500 is still positive. The impact of risk-aversion on stock-bond correlation explains that the conditional stock-bond correlation increases with group A's consumption as in Figure 15. Recall that asset prices are the weighted averages of prices in several homogeneous-agent economies populated with group B investors with risk aversion $1 - j$. The weight is decided by the relative consumption share of group A to group B, and thus the higher consumption share group A has, the more weight will be put to the pricing of equilibrium in which agents have higher risk-aversion. As risk-aversion increases stock-bond correlation, the consumption share of group A will also increase it.

Then we move on to the disagreements. Figure 14 plots the conditional stock-bond correlation against both disagreements on aggregate dividend growth rate and inflation rate. The conditional correlation varies from -1 to 1 smoothly with disagreements. Note that the edge of this plot where we observe correlations continuously close to 1 or -1 are extreme cases where one Brownian shock dominates the correlation.

6 Uncovering Disagreements and Predicing Conditional Stock-Bond Correlations

In this section, we first estimate the time series of disagreements, and compare them with the dispersion of forecasts in the Survery of Professional Forecasters. Then we use the estimates of disagreements together with other state variables to calculate the conditional correlation implied by the model. In the end, we run a predictive regression of the historical stock-bond correlation on the model implied correlation.

6.1 Uncovering Disagreements from Prices

The problem of estimating disagreements is again to filter a state-space model. In this model, we have four observables: the aggregate dividend growth, inflation, stock price, and bond price. One may also use several bond prices. Under the true probability measure, we have seven state variables: the true current growth rate and two estimates by investors in group A and group B; the true inflation rate and two estimates by investors in group A and group B; the consumption share of group A. We can apply the Kalman Filter, assuming that there are some pricing errors. The difficulty lies in the non-linearity of the prices that prevent us from obtaining the variance-covariance matrix between the observable and state variables. To deal with non-linearity, we choose the unscented transform (see, for example, Chapter 10, Durbin and Koopman 2012).

The unscented transform is implemented for an approximation of the probability distribution of the variables. It can capture higher moments and perform better than the Extended Kalman Filter. We start from zero disagreements and equal consumption share, and then let the unscented Kalman Filter tell us what the time series of disagreements should be. There is no backward smoothing in this excercise. From the filter, we obtain the conditional mean and variance of each state variable. We take the mean as the estimates of state variables. Figure 17 and 18 shows the model-implied (means of conditional) disagreements. The disagreements are quite smooth over time, and they are very small.

We first compare our estimates of disagreements with the time series of the dispersion in forecasts at the 4th quarter each year in the Survey of Professional Forecasters (SPF), which has been used as a standard proxy for disagreements in the literature. The horizon of forecast is one year. We use the dispersion in real GDP growth forecast as the proxy for the disagreement in aggregate dividend growth because it is the most comparable variable in the survey. We use the dispersion in CPI forecast as the proxy for the disagreement in inflation. As the CPI forecast is on the level rather than on the growth, we also use the ratio of dispersion in CPI forecast to the contemporaneous CPI as the second proxy for disagreement in inflation. As the dispersion is measured by the 75th percentile estimate minus the 25th percentile estimate, the dispersion in SPF is always positive. Therefore, we should compare dispersions with the *absolute* values of disagreements.

Table 5 shows shows the regression result of dispersions in forecasts on the model-implied disagreements. The regression of the dispersion of real GDP gorwth forecasts on the model-implied disagreement in aggregate dividend growth shows an R^2 of 0.2 and a p-value of 0.0009. The regression of dispersion of CPI level forecasts on the model-implied disagreement of inflation shows an R^2 of 0.44 and a p-value of 8×10^{-6}. The regression of the ratio of dispersion of CPI level forecasts to the contemporaneous CPI level on the model-implied disagreement of inflation shows an R^2 of 0.67 and a p-value of 6.7×10^{-10}. We also conduct cross checks by regressing the disagreement of inflation on real GDP forecast dispersion and by regressing the disagreement of aggregate dividend growth on CPI forecast dispersion and the ratio of that to the contemporaneous CPI, and the results have high p-value and low R^2. Therefore, we can conclude that our model-implied disagreements are correlated with the corresponding dispersions in the survey data. We can infer the disagreements on aggregate dividend growth and inflation from publicly available prices, and they are correlated with the dispersions in survey data collected from individual forecasters. This piece of evidence shows that my estimates of disagreements are sensible and credible.

6.2 Predicting Conditional Stock-Bond Correlation

In year t, we can use the state variable in the same year to calculate the conditional stock-bond correlation implied by the model. By doing that, we only use information up to year t. As we use the mean as the estimate of disagreement, the model-implied bond-stock correlation is actually the "expected value" of the model-implied bond-stock correlation. There are two kinds of correlations: one is the correlation of prices, and the other is the correlation of returns. In the model, the two correlations are the same, because the price at the previous point is canceled out. However, they are different when we calculate the sample correlation, because the price at the previous point is different at each day. We first discuss the correlation of prices.

Figure 16 shows the fit of the model-implied bond-stock correlation. Indeed, the model-implied bond-stock correlation looks like the "expected value" of the historical correlation. Also, in contrast to the regime-shift stories where the regime shifted after 2000, the model-implied bond-stock correlation starts to decline in early 1990s. This decline is due to the gradual change of disagreements and the endogenous decrease of group A's consumption share.

We regress the historical time-series of the price correlation between Treasury bonds and $S\&P$ 500 index in year $t+1$ on the model implied correlation between nominal bonds and the stock at the end of year t. We apply no other controls. We drop the first five data points as burn-in period. Table 2 shows the regression results. Ideally, the coefficient should be one, p-value should be 0, and R^2 should be 100%. Unfortunately, the result is not perfect. However, in general, the model-implied correlations predict the historical correlations in the coming year, except the 5 year Treasury inflation-protected security. For the nominal Treasury bonds, the coefficient is positive, which means the model tracks and predicts the trend of correlations in the data. The p-values of coefficients are mostly significant at least at the 0.05 level, and R^2 ranges from 0.11 to 0.24. For the 10-year and 30-year TIPS, the coefficient is negative. This is because the pricing kernel is pushing for a decreasing stock-bond correlation, while the correlation between TIPS and $S\&P$ 500 index in the data is positive and strong. The correlations in the data are not predicted by their lags. This addresses two things, on the one hand, the predictive power of the model-implied correlations does not come from the autocorrelation of correlations in the data (as there is no autocorrelation); on the other hand, the autocorrelation of the smoothed correlation (by dynamic conditional correlation or exponential smoothers, etc.) is mechanical. The above reasoning suggests that, compared with smoothing methods, our estimation practice seems to be a better way to calculate and predict the conditional stock-bond correlation.

We then regress the historical time-series of the return correlation between Treasury bonds and $S\&P$ 500 index in year $t+1$ on the model implied correlation between nominal bonds and the stock at the end of year t. There is significant autocorrelation in the stock-bond return correlation, which is due to the significant autocorrelation in the bond returns. The benchmark regression applies no controls, and we also run another two regressions including controls of one lagged return correlation and two lags of return correlation, respectively. Table three shows that our model-implied conditional correlation of stock and bond returns predicts that in the data. Our model works well with nominal bonds, but fails with TIPS. There is significant auto-correlation in the stock-bond return correlation, but our model-implied correlations are still significant after including the lags. From BIC and AICc information criterion, the regression with one lag has the best quality. Considering the fact that the stock-bond correlation implied by our model is the "expected" value, which is much more smooth than the realized correlations, we also run a robustness check by standardizing both correlations, and applying the same regressions. From Table 4, we see the results do not change.

The model-implied bond-stock correlation for all the nominal bonds are in Figure 20, from which we can see the change in term-structure of the stock-bond correlation. Before 1980s, the stock-bond correlation increase in bond maturity; from 1980s to late 1990s, the stock-bond correlation decreases in bond maturity; in recent 20 years, the stock-bond correlation reverts to increasing in bond maturity. In the real data, we observe changes in the slope of the correlation term-structure in many year, but the changes are not explained by the model because the pricing kernel appears less variant than that in the data. The disagreements can explain the gap between the model-implied stock-bond correlation with different bond maturities. We run the regression of the difference between model-implied stock-10-year-bond correlation and stock-2-year-bond correlation on each disagreements, the R^2 is 0.7 with disagreement on inflation and 0.64 with disagreement on aggregate dividend growth. The regression of the difference between model-implied stock-10-year-bond correlation and stock-2-year-bond correlation on group A's consumption share shows small R^2 and high p-value. Therefore, the fluctuations in consumption share do not explain the term-structure of stock-bond correlation.

The consumption share of group A is plotted in Figure 19. Over the 56 years group A gradually lost 10 percent of the total consumption to group B. The stock-bond correlation decreases with group A's consumption share, thus the stock-bond correlation decreases from positive to negative. The consumption share does not explain stock-bond correlations from regression tests, but it is important to explain the sign of correlation. The fit of filtered PE ratios and bond prices to real data are plotted in Figure 21 and 22. Again, we find this heterogeneous model is not as "wild" as one may suspect in the beginning, in the sense that from a serious estimation, we will not get outrageously big or fluctuating disagreements. Disagreements persist, which results in less variation in the pricing kernel, and thus the model-implied prices are less variant than the data looks.

7 Conclusion

There are four facts about the stock-bond correlation. First, it varies with time. Second, it can be positive or negative conditionally. Third, it is overall positive. Last, we have identified a time-varying term-structure of the stock-bond correlation. A model with heterogeneous expectations and endogenous consumption fluctuation can explain the above facts.

We take the estimation of the model seriously. We jointly estimate the earnings growth and inflation processes by maximizing their likelihood using the E-M approach. We use grid search to pin down the sentiment parameters. We use the unscented Kalman Filter to uncover the time series of disagreements, and calculate the time series of conditional stock-bond correlation implied by the model. The model-implied conditional stock-bond correlation predicts the conditional stock-bond correlation in the data. The predictive power comes from the auto-regressive structure of the fundamental processes. Disagreements on aggregate dividend-growth rate and inflation rate can explain the difference in conditional stock-bond correlations with different bond maturities, but it is the endogenous consumption fluctuation that changes the sign of the conditional stock-bond correlation.

In our model, the changes in conditional stock-bond correlation changes are driven by risks from heterogeneous expectations rather than agents' risk preferences or the dynamics between economic growth and inflation. With small and persistent disagreements, we can explain 10-20% of the conditional stock-bond correlation in the data, across almost all the maturities of Treasury bonds except the TIPS-5. Our estimate of disagreements are correlated with the dispersion of corresponding forecasts in SPF.

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A Proofs

A.1 Proof for Lemma 2

From optimal filtering theory, (see Theorem 12.7 p.36, Lipster and Shiryaev 2001), we can find out the growth rates satisfying the stochastic differential equations in Lemma 2. Their steady-state variances γ_f^i $\gamma_{f\pi}^i$ γ_π^i satisfy the quadratic equation system:

$$
\begin{aligned}
&\frac{\sigma_{PP}^2+\sigma_{P\delta}^2}{(\sigma_{\delta\delta}\sigma_{PP}-\sigma_{P\delta}\sigma_{\delta P})^2}(\gamma_f^i)^2-2\frac{\sigma_{PP}\sigma_{\delta P}+\sigma_{P\delta}\sigma_{\delta\delta}}{(\sigma_{\delta\delta}\sigma_{PP}-\sigma_{P\delta}\sigma_{\delta P})^2}\gamma_f^i\gamma_{f\pi}^i\\
&+\frac{\sigma_{\delta P}^2+\sigma_{\delta\delta}^2}{(\sigma_{\delta\delta}\sigma_{PP}-\sigma_{P\delta}\sigma_{\delta P})^2}(\gamma_\pi^i)^2=\sigma_{ff}^2(1-(\phi_f^i)^2)+\sigma_{f\pi}^2(1-(\phi_\pi^i)^2)-2\zeta_f\gamma_f^i,\\
&\frac{\sigma_{PP}^2+\sigma_{P\delta}^2}{(\sigma_{\delta\delta}\sigma_{PP}-\sigma_{P\delta}\sigma_{\delta P})^2}\gamma_f^i\gamma_{f\pi}^i-\frac{\sigma_{PP}\sigma_{\delta P}+\sigma_{P\delta}\sigma_{\delta\delta}}{(\sigma_{\delta\delta}\sigma_{PP}-\sigma_{P\delta}\sigma_{\delta P})^2}[(\gamma_{f\pi}^i)^2+\gamma_f^i\gamma_\pi^i]\\
&+\frac{\sigma_{\delta P}^2+\sigma_{\delta\delta}^2}{(\sigma_{\delta\delta}\sigma_{PP}-\sigma_{P\delta}\sigma_{\delta P})^2}\gamma_{f\pi}^i\gamma_\pi^i=\sqrt{(\sigma_{ff}^2(1-(\phi_f^i)^2))(\sigma_{\pi f}^2(1-(\phi_f^i)^2))}\\
&+\sqrt{(\sigma_{f\pi}^2(1-(\phi_\pi^i)^2))(\sigma_{\pi\pi}^2(1-(\phi_\pi^i)^2))}-(\zeta_f+\zeta_\pi)\gamma_{f\pi}^i,\\
&\frac{\sigma_{PP}^2+\sigma_{P\delta}^2}{(\sigma_{\delta\delta}\sigma_{PP}-\sigma_{P\delta}\sigma_{\delta P})^2}(\gamma_{f\pi}^i)^2-2\frac{\sigma_{PP}\sigma_{\delta P}+\sigma_{P\delta}\sigma_{\delta\delta}}{(\sigma_{\delta\delta}\sigma_{PP}-\sigma_{P\delta}\sigma_{\delta P})^2}\gamma_{f\pi}^i\gamma_\pi^i\\
&+\frac{\sigma_{\delta P}^2+\sigma_{\delta\delta}^2}{(\sigma_{\delta\delta}\sigma_{PP}-\sigma_{P\delta}\sigma_{\delta P})^2}(\gamma_\pi^i)^2=\sigma_{\pi f}^2(1-(\phi_f^i)^2)+\sigma_{\pi\pi}^2(1-(\phi_\pi^i)^2)-2\zeta_\pi\gamma_\pi^i.
\end{aligned}
\tag{48}
$$

Substitute γ_f^i $\gamma_{f\pi}^i$ γ_π^i from Lemma 1, and we can find the below system of trigonometric function after some algebraic operations:

$$
\begin{aligned}
&cos(\alpha-\beta)=\frac{\sqrt{(\sigma_{ff}^2(1-(\phi_f^i)^2))(\sigma_{\pi f}^2(1-(\phi_f^i)^2))}}{hj}\\
&+\frac{\sqrt{(\sigma_{f\pi}^2(1-(\phi_\pi^i)^2))(\sigma_{\pi\pi}^2(1-(\phi_\pi^i)^2))}}{hj}+\frac{2\zeta_f\zeta_\pi(\sigma_{\delta\delta}\sigma_{P\delta}+\sigma_{\delta P}\sigma_{PP})}{hj},\\
&cos(\beta-\mu)=(\zeta_f-\zeta_\pi)(\sigma_{\delta\delta}\sigma_{P\delta}+\sigma_{\delta P}\sigma_{PP}),\\
&sin(\mu)\\
&=(h\sigma_{P\delta}cos(\alpha-\beta)+h\sigma_{PP}sin(\alpha-\beta)-j\sigma_{\delta\delta})/\\
&\Big((h\sigma_{P\delta}cos(\alpha-\beta)+h\sigma_{PP}sin(\alpha-\beta)-j\sigma_{\delta\delta})^2\\
&+(h\sigma_{P\delta}sin(\alpha-\beta)-h\sigma_{PP}cos(\alpha-\beta)-j\sigma_{\delta P})^2\Big),\\
&cos(\mu)\\
&=h\sigma_{P\delta}sin(\alpha-\beta)-h\sigma_{PP}cos(\alpha-\beta)-j\sigma_{\delta P})/\\
&\Big((h\sigma_{P\delta}cos(\alpha-\beta)+h\sigma_{PP}sin(\alpha-\beta)-j\sigma_{\delta\delta})^2\\
&+(h\sigma_{P\delta}sin(\alpha-\beta)-h\sigma_{PP}cos(\alpha-\beta)-j\sigma_{\delta P})^2\Big).
\end{aligned}
\tag{49}
$$

As γ_f^i and γ_π^i must be positive, one can find a unique pair of $\{\alpha,\beta\}$ within the range of $[0,2\pi]$.

A.2 Proof for Proposition 1

The moment-generating function H satisfies the linear PDE

$$0 = \mathcal{L}H(\delta, \hat{f}^B, \hat{g}_f, P, \hat{\pi}, \hat{g}_\pi, \eta, t, u; \epsilon, \kappa, \xi) + \frac{\partial H}{\partial t}(\delta, \hat{f}^B, \hat{g}_f, P, \hat{\pi}, \hat{g}_\pi, \eta, t, u; \epsilon, \kappa, \xi) \tag{50}$$

with the initial conditional $H(\delta, \hat{f}^B, \hat{g}_f, P, \hat{\pi}, \hat{g}_\pi, \eta, t, u; \epsilon, \kappa, \xi) = \delta^\epsilon P^\kappa \eta^\xi$, and $\mathcal{L}$ is the differential generator of $(\delta, \hat{f}^B, \hat{g}_f, P, \hat{\pi}, \hat{g}_\pi, \eta)$ under the probability measure of group B. Now assume the H is exponential affine-quadratic:

$$\begin{aligned} &H = \delta^\epsilon P^\kappa \eta^\chi exp\{(Af(t) + Bf(t)\hat{f}^B) + (A\pi(t) + B\pi(t)\hat{\pi}^B) \\ &+ (A1_f(t) + \epsilon^2 A2_f(t) + \epsilon \hat{g}_f B_f(t) + \hat{g}_f^2 C_f(t))\} \\ &+ (A1_\pi(t) + \kappa^2 A2_\pi(t) + \kappa \hat{g}_\pi B_\pi(t) + \hat{g}_\pi^2 C_\pi(t)) \\ &+ \hat{g}_f \hat{g}_\pi D(t) + \epsilon\kappa E(t)\} \end{aligned} \tag{51}$$

Here we prove the independent version of the model. Note that $\frac{\partial H}{\partial t} = H \times (\frac{\partial H_f}{\partial t} + \frac{\partial H_\pi}{\partial t} + \frac{\partial H_g}{\partial t} + \frac{\partial H_{g\pi}}{\partial t})$.

Define

$$H_1 = \delta^\epsilon \eta^{chi} \times H_f(\hat{f}^B, u, t; \epsilon) \times H_g(\hat{g}_f, u, t; \epsilon, \chi) \tag{52}$$

$$H_2 = P^\kappa \eta^{ki} \times H_\pi(\hat{\pi}^B, u, t; \kappa) \times H_{g^\pi}(\hat{g}_\pi, u, t; \kappa, \chi) \tag{53}$$

Following Dumas, Kurshev and Uppal (2009), H_1 is the solution to the following PDE:

$$\begin{aligned} 0 = &\frac{\partial H_1}{\partial \delta}\delta \hat{f}^B - \frac{\partial H_1}{\partial \hat{f}^B}\zeta_f(\hat{f}^B - \bar{f}) - \frac{\partial H_1}{\partial \hat{g}^f}\hat{g}^f(\zeta_f + \frac{\gamma_\delta^A}{\sigma_\delta^2}) \\ &+ \frac{1}{2}\frac{\partial^2 H_1}{\partial \delta^2}(\delta\sigma_\delta)^2 + \frac{1}{2}\frac{\partial^2 H_1}{\partial (\hat{g}^f)^2}[(\frac{\gamma_\delta^B - \gamma_\delta^A}{\sigma_\delta})^2 + (\phi_f \sigma_f)^2] + \frac{1}{2}\frac{\partial^2 H_1}{\partial (\hat{f}^B)^2}(\frac{\gamma_\delta^B}{\sigma_\delta})^2 \\ &- \frac{\partial^2 H_1}{\partial \delta \partial \eta}\delta\eta\hat{g}^f + \frac{\partial^2 H_1}{\partial \delta \partial \hat{g}^f}\delta(\gamma_\delta^B - \gamma_\delta^A) + \frac{\partial^2 H_1}{\partial \delta \partial \hat{f}^B}\delta\gamma_\delta^B - \frac{\partial^2 H_1}{\partial \eta \partial \hat{g}^f}\eta\hat{g}^f(\frac{\gamma_\delta^B - \gamma_\delta^A}{\sigma_\delta^2}) \\ &- \frac{\partial^2 H_1}{\partial \eta \partial \hat{f}^B}\eta\hat{g}^f\frac{\gamma_\delta^B}{\sigma_\delta^2} + \frac{\partial^2 H_1}{\partial \hat{g}^f \partial \hat{f}^B}(\frac{\gamma_\delta^B - \gamma_\delta^A}{\sigma_\delta^2})\gamma_\delta^B + \frac{1}{2}\frac{\partial^2 H}{\partial \eta^2}\eta^2(\frac{\hat{g}_f}{\sigma_\delta})^2 + \frac{\partial H_1}{\partial t} \\ &\quad where\ \frac{\partial H_1}{\partial t} = H_1 \times (\frac{\partial H_f}{\partial t} + \frac{\partial H_g}{\partial t}). \end{aligned} \tag{54}$$

Similarly, H_2 is the solution to the following PDE:

$$
\begin{aligned}
0 = & \frac{\partial H_2}{\partial P} P \hat{\pi}^B - \frac{\partial H_2}{\partial \hat{\pi}^B} \zeta_\pi (\hat{\pi}^B - \bar{\pi}) - \frac{\partial H_2}{\partial \hat{g}_\pi} \hat{g}_\pi (\zeta_\pi + \frac{\gamma_P^A}{\sigma_P^2}) \\
& + \frac{1}{2} \frac{\partial^2 H_2}{\partial P^2} (P \sigma_P)^2 + \frac{1}{2} \frac{\partial^2 H_2}{\partial (\hat{g}_\pi)^2} [(\frac{\gamma_P^B - \gamma_P^A}{\sigma_P})^2 + (\phi_\pi \sigma_\pi)^2] + \frac{1}{2} \frac{\partial^2 H_2}{\partial (\hat{\pi}^B)^2} (\frac{\gamma_P^B}{\sigma_P})^2 \\
& - \frac{\partial^2 H_2}{\partial P \partial \eta} P \eta \hat{g}_\pi + \frac{\partial^2 H_2}{\partial P \partial \hat{g}_\pi} P (\gamma_P^B - \gamma_P^A) + \frac{\partial^2 H_2}{\partial P \partial \hat{\pi}^B} P \gamma_P^B - \frac{\partial^2 H_2}{\partial \eta \partial \hat{g}_\pi} \eta \hat{g}_\pi (\frac{\gamma_P^B - \gamma_P^A}{\sigma_P^2}) \\
& - \frac{\partial^2 H_2}{\partial \eta \partial \hat{\pi}^B} \eta \hat{g}_\pi \frac{\gamma_P^B}{\sigma_P^2} + \frac{\partial^2 H_2}{\partial \hat{g}_\pi \partial \hat{\pi}^B} (\frac{\gamma_P^B - \gamma_P^A}{\sigma_P^2}) \gamma_P^B + \frac{1}{2} \frac{\partial^2 H}{\partial \eta^2} \eta^2 (\frac{\hat{g}_\pi}{\sigma_P})^2 + \frac{\partial H_2}{\partial t}
\end{aligned} \tag{55}
$$

$$
where \ \frac{\partial H_2}{\partial t} = H \times (\frac{\partial H_\pi}{\partial t} + + \frac{\partial H_{g^\pi}}{\partial t}).
$$

Note that

$$
\begin{aligned}
&\frac{1}{H_1}\times[\frac{\partial H_1}{\partial\delta}\delta\hat{f}^B-\frac{\partial H_1}{\partial\hat{f}^B}\zeta_f(\hat{f}^B-\bar{f})-\frac{\partial H_1}{\partial\hat{g}^f}\hat{g}^f(\zeta_f+\frac{\gamma_\delta^A}{\sigma_\delta^2})\\
&+\frac{1}{2}\frac{\partial^2 H_1}{\partial\delta^2}(\delta\sigma_\delta)^2+\frac{1}{2}\frac{\partial^2 H_1}{\partial(\hat{g}^f)^2}[(\frac{\gamma_\delta^B-\gamma_\delta^A}{\sigma_\delta})^2+(\phi_f\sigma_f)^2]+\frac{1}{2}\frac{\partial^2 H_1}{\partial(\hat{f}^B)^2}(\frac{\gamma_\delta^B}{\sigma_\delta})^2\\
&-\frac{\partial^2 H_1}{\partial\delta\partial\eta}\delta\eta\hat{g}^f+\frac{\partial^2 H_1}{\partial\delta\partial\hat{g}^f}\delta(\gamma_\delta^B-\gamma_\delta^A)+\frac{\partial^2 H_1}{\partial\delta\partial\hat{f}^B}\delta\gamma_\delta^B-\frac{\partial^2 H_1}{\partial\eta\partial\hat{g}^f}\eta\hat{g}^f(\frac{\gamma_\delta^B-\gamma_\delta^A}{\sigma_\delta^2})\\
&-\frac{\partial^2 H_1}{\partial\eta\partial\hat{f}^B}\eta\hat{g}^f\frac{\gamma_\delta^B}{\sigma_\delta^2}+\frac{\partial^2 H_1}{\partial\hat{g}^f\partial\hat{f}^B}(\frac{\gamma_\delta^B-\gamma_\delta^A}{\sigma_\delta^2})\gamma_\delta^B+\frac{1}{2}\frac{\partial^2 H}{\partial\eta^2}\eta^2(\frac{\hat{g}_f}{\sigma_\delta})^2+\frac{\partial H_1}{\partial t}]\\
&+\frac{1}{H_2}\times[\frac{\partial H_2}{\partial P}P\hat{\pi}^B-\frac{\partial H_2}{\partial\hat{\pi}^B}\zeta_\pi(\hat{\pi}^B-\bar{\pi})-\frac{\partial H_2}{\partial\hat{g}_\pi}\hat{g}_\pi(\zeta_\pi+\frac{\gamma_P^A}{\sigma_P^2})\\
&+\frac{1}{2}\frac{\partial^2 H_2}{\partial P^2}(P\sigma_P)^2+\frac{1}{2}\frac{\partial^2 H_2}{\partial(\hat{g}_\pi)^2}[(\frac{\gamma_P^B-\gamma_P^A}{\sigma_P})^2+(\phi_\pi\sigma_\pi)^2]+\frac{1}{2}\frac{\partial^2 H_2}{\partial(\hat{\pi}^B)^2}(\frac{\gamma_P^B}{\sigma_P})^2\\
&-\frac{\partial^2 H_2}{\partial P\partial\eta}P\eta\hat{g}_\pi+\frac{\partial^2 H_2}{\partial P\partial\hat{g}_\pi}P(\gamma_P^B-\gamma_P^A)+\frac{\partial^2 H_2}{\partial P\partial\hat{\pi}^B}P\gamma_P^B-\frac{\partial^2 H_2}{\partial\eta\partial\hat{g}_\pi}\eta\hat{g}_\pi(\frac{\gamma_P^B-\gamma_P^A}{\sigma_P^2})\\
&-\frac{\partial^2 H_2}{\partial\eta\partial\hat{\pi}^B}\eta\hat{g}_\pi\frac{\gamma_P^B}{\sigma_P^2}+\frac{\partial^2 H_2}{\partial\hat{g}_\pi\partial\hat{\pi}^B}(\frac{\gamma_P^B-\gamma_P^A}{\sigma_P^2})\gamma_P^B+\frac{1}{2}\frac{\partial^2 H}{\partial\eta^2}\eta^2(\frac{\hat{g}_\pi}{\sigma_P})^2+\frac{\partial H_2}{\partial t}]\\
&=\frac{1}{H}\times[\frac{\partial H}{\partial\delta}\delta\hat{f}^B-\frac{\partial H}{\partial\hat{f}^B}\zeta_f(\hat{f}^B-\bar{f})-\frac{\partial H}{\partial\hat{g}_f}\hat{g}_f(\zeta_f+\frac{\gamma_\delta^A}{\sigma_\delta^2})\\
&+\frac{1}{2}\frac{\partial^2 H}{\partial\delta^2}(\delta\sigma_\delta)^2+\frac{1}{2}\frac{\partial^2 H}{\partial(\hat{g}_f)^2}[(\frac{\gamma_\delta^B-\gamma_\delta^A}{\sigma_\delta})^2+(\phi_f\sigma_f)^2]+\frac{1}{2}\frac{\partial^2 H}{\partial(\hat{f}^B)^2}(\frac{\gamma_\delta^B}{\sigma_\delta})^2\\
&-\frac{\partial^2 H}{\partial\delta\partial\eta}\delta\eta\hat{g}_f+\frac{\partial^2 H}{\partial\delta\partial\hat{g}_f}\delta(\gamma_\delta^B-\gamma_\delta^A)+\frac{\partial^2 H}{\partial\delta\partial\hat{f}^B}\delta\gamma_\delta^B-\frac{\partial^2 H}{\partial\eta\partial\hat{g}_f}\eta\hat{g}_f(\frac{\gamma_\delta^B-\gamma_\delta^A}{\sigma_\delta^2})\\
&-\frac{\partial^2 H}{\partial\eta\partial\hat{f}^B}\eta\hat{g}_f\frac{\gamma_\delta^B}{\sigma_\delta^2}+\frac{\partial^2 H}{\partial\hat{g}_f\partial\hat{f}^B}(\frac{\gamma_\delta^B-\gamma_\delta^A}{\sigma_\delta^2})\gamma_\delta^B\\
&+\frac{\partial H}{\partial P}P\hat{\pi}^B-\frac{\partial H}{\partial\hat{\pi}^B}\zeta_\pi(\hat{\pi}^B-\bar{\pi})-\frac{\partial H}{\partial\hat{g}_\pi}\hat{g}_\pi(\zeta_\pi+\frac{\gamma_P^A}{\sigma_P^2})\\
&+\frac{1}{2}\frac{\partial^2 H}{\partial P^2}(P\sigma_P)^2+\frac{1}{2}\frac{\partial^2 H}{\partial(\hat{g}_\pi)^2}[(\frac{\gamma_P^B-\gamma_P^A}{\sigma_P})^2+(\phi_\pi\sigma_\pi)^2]+\frac{1}{2}\frac{\partial^2 H}{\partial(\hat{\pi}^B)^2}(\frac{\gamma_P^B}{\sigma_P})^2\\
&-\frac{\partial^2 H}{\partial P\partial\eta}P\eta\hat{g}_\pi+\frac{\partial^2 H}{\partial P\partial\hat{g}_\pi}P(\gamma_P^B-\gamma_P^A)+\frac{\partial^2 H}{\partial P\partial\hat{\pi}^B}P\gamma_P^B-\frac{\partial^2 H}{\partial\eta\partial\hat{g}_\pi}\eta\hat{g}_\pi(\frac{\gamma_P^B-\gamma_P^A}{\sigma_P^2})\\
&-\frac{\partial^2 H}{\partial\eta\partial\hat{\pi}^B}\eta\hat{g}_\pi\frac{\gamma_P^B}{\sigma_P^2}+\frac{\partial^2 H}{\partial\hat{g}_\pi\partial\hat{\pi}^B}(\frac{\gamma_P^B-\gamma_P^A}{\sigma_P^2})\gamma_P^B\\
&+\frac{1}{2}\frac{\partial^2 H}{\partial\eta^2}\eta^2[(\frac{\hat{g}_f}{\sigma_\delta})^2+(\frac{\hat{g}_\pi}{\sigma_P})^2]+\frac{\partial H}{\partial t}]
\end{aligned}\tag{56}
$$

The above equality holds for each and every derivative in the partial differential equation.

Therefore, $\delta^{\epsilon}\eta^{\chi}P^{\kappa} \times H_f(\hat{f}^B, u, t; \epsilon) \times H_g(\hat{g}_f, u, t; \epsilon, \chi) \times H_{\pi}(\hat{\pi}^B, u, t; \kappa) \times H_{g^{\pi}}(\hat{g}_{\pi}, u, t; \kappa, \chi)$ is the solution to the whole PDE. The definitions of the functions inside $\delta^{\epsilon}\eta^{\chi}P^{\kappa} \times H_f(\hat{f}^B, u, t; \epsilon) \times H_{g^f}(\hat{g}_f, u, t; \epsilon, \chi) \times H_{\pi}(\hat{\pi}^B, u, t; \kappa) \times H_{g^{\pi}}(\hat{g}_{\pi}, u, t; \kappa, \chi)$ are similar to those in Dumas Kurshev and Uppal (2009).

A.3 The E-M Approach Applied to Estimate the Fundamental Processes

Here I illustrate the case for the aggregate dividend growth.

First is the expectation part. With the discrete-time processes (57) and (58), given parameters of σ_f^2, ζ_f, σ_δ^2 and $\bar{f}$, applying the Kalman filter we have:

$$\hat{f}_{t+1|t} = e^{-\zeta_f \Delta}\hat{f}_{t|t} + \bar{f}(1 - e^{-\zeta_f \Delta})$$

$$P_{t+1|t} = e^{-2\zeta_f \Delta}P_{t|t} + \frac{1 - e^{-2\zeta_f \Delta}}{2\zeta_f}\sigma_f^2$$

$$K_{t+1} = \frac{P_{t+1|t}}{\Delta P_{t+1|t} + \sigma_\delta^2}$$

$$\hat{f}_{t+1|t+1} = \hat{f}_{t+1|t} + K_{t+1}(Y_{t+1} - \Delta\hat{f}_{t+1|t} + 0.5\Delta\sigma_\delta^2)$$

$$P_{t+1|t+1} = P_{t+1|t} - K_{t+1}\Delta P_{t+1|t}$$

Then applying backward pass we have:

$$L_t = \frac{P_{t|t}e^{-\zeta_f \Delta}}{P_{t+1|t}}$$

$$\hat{f}_{t|T} = \hat{f}_{t|t} + L_t(\hat{f}_{t+1|T} - \hat{f}_{t+1|t})$$

$$P_{t|T} = P_{t|t} + L_t(P_{t+1|T} - P_{t+1|t})L_t$$

Second, we need to calculate the likelihood function

$$\begin{aligned} L =& constant + \sum_1^T [-\frac{1}{2}ln(\Delta\sigma_\delta^2) - \frac{1}{2\Delta\sigma_\delta^2}(Y_i - (\hat{f}_{i-1} - \frac{1}{2}\sigma_\delta^2)\Delta)^2 \\ & - \frac{1}{2}ln(\frac{1 - e^{-2\zeta_f \Delta}}{\zeta_f}\sigma_f^2) - \frac{\zeta_f}{(1 - e^{-2\zeta_f \Delta})\sigma_f^2}(\hat{f}_i - (\hat{f}_{i-1}e^{-\zeta_f \Delta} + \bar{f}(1 - e^{-\zeta_f \Delta}))^2] \end{aligned}$$

We maximize the likelihood by taking derivatives over σ_f^2, ζ_f, σ_δ^2 and $\bar{f}$, respectively, and equate the derivatives to 0. Solving these first order conditions we find that σ_f^2, σ_δ^2 and $\bar{f}$ can be rewritten as a function of ζ_f. Substitute σ_f^2, σ_δ^2 and $\bar{f}$ into the first order condition with respect to ζ_f, we choose the smallest positive solution as the optimizing ζ_f, and then we can obtain the optimizing values of σ_f^2, σ_δ^2 and $\bar{f}$. Note that $\hat{f}_i$ and $\hat{f}_{i-1}$ are not independent with each other, so the concerning calculations are

$$E(\hat{f}_i|Y) = \hat{f}_{t|T}$$

$$E(\hat{f}_i\hat{f}_i|Y) = P_{t|T} + \hat{f}_{t|T}\hat{f}_{t|T}$$

$$E(\hat{f}_i\hat{f}_{i+1}|Y) = \hat{f}_{t|t}\hat{f}_{t+1|T} + L_t(P_{t+1|T}(\hat{f}_{t+1|T} - \hat{f}_{t+1|t})\hat{f}_{t+1|T})$$

Obtaining the new values of σ_f^2, ζ_f, σ_δ^2 and $\bar{f}$, we can redo the above procedures until σ_f^2, ζ_f, σ_δ^2 and $\bar{f}$ converge. When σ_f^2, ζ_f, σ_δ^2 and $\bar{f}$ converge, we not only get the values of σ_f^2, ζ_f, σ_δ^2 and $\bar{f}$, but also the estimated distributions of $\{\hat{f}_{t|Y}\}$.

A.4 The Discretization for the Unscented Kalman Filter

First, we need to discretize the disagreements process under the effective measure. Define $\hat{g}^i_{f,t} = \hat{f}^i_t - f_t, \hat{g}^i_{\pi,t} = \hat{\pi}^i_t - \pi_t, i = \{A, B\}$, their dynamics jointly follow a mean-reverting system:

$$\begin{aligned} d\hat{g}^i_{f,t} = &- \psi^i_{ff}\hat{g}_{f,t}dt - \psi^i_{f\pi}\hat{g}_{\pi,t}dt + \sigma_{\hat{g}^i_f,\delta}dZ^\delta_t + \sigma_{\hat{g}^i_f,P}dZ^P_t \\ &+ \sigma_{\hat{g}^i_f,s_f}dZ^{s_f}_t + \sigma_{\hat{g}^i_f,s_\pi}dZ^{s_\pi}_t - \sigma_{ff}dZ^f_t - \sigma_{f\pi}dZ^\pi_t \end{aligned} \tag{57}$$

$$\begin{aligned} d\hat{g}^i_{\pi,t} = &- \psi^i_{\pi f}\hat{g}_{f,t}dt - \psi^i_{\pi\pi}\hat{g}_{\pi,t}dt + \sigma_{\hat{g}^i_\pi,\delta}dZ^\delta_t + \sigma_{\hat{g}^i_\pi,P}dZ^P_t \\ &+ \sigma_{\hat{g}^i_\pi,s_f}dZ^{s_f}_t + \sigma_{\hat{g}^i_\pi,s_\pi}dZ^{s_\pi}_t - \sigma_{\pi f}dZ^f_t - \sigma_{\pi\pi}dZ^\pi_t \end{aligned} \tag{58}$$

where the coefficients are

$$\begin{aligned}
\psi^i_{ff} &= \zeta_f + \frac{\gamma^i_f(\sigma^2_{P\delta} + \sigma^2_{PP}) - \gamma^i_{f\pi}(\sigma_{\delta\delta}\sigma_{P\delta} + \sigma_{\delta P}\sigma_{PP})}{(\sigma_{\delta\delta}\sigma_{PP} - \sigma_{\delta P}\sigma_{P\delta})^2}, \\
\psi^i_{f\pi} &= \frac{\gamma^i_{f\pi}(\sigma^2_{\delta\delta} + \sigma^2_{\delta P}) - \gamma^i_f(\sigma_{\delta\delta}\sigma_{P\delta} + \sigma_{\delta P}\sigma_{PP})}{(\sigma_{\delta\delta}\sigma_{PP} - \sigma_{\delta P}\sigma_{P\delta})^2}, \\
\sigma^i_{\hat{g}_f,\delta} &= \frac{\gamma^i_f\sigma_{PP} - \gamma^i_{f\pi}\sigma_{\delta P}}{\sigma_{\delta\delta}\sigma_{PP} - \sigma_{\delta P}\sigma_{P\delta}}, \\
\sigma^i_{\hat{g}_f,P} &= \frac{\gamma^i_{f\pi}\sigma_{\delta\delta} - \gamma^i_f\sigma_{P\delta}}{\sigma_{\delta\delta}\sigma_{PP} - \sigma_{\delta P}\sigma_{P\delta}}, \\
\sigma^i_{\hat{g}_f,s_f} &= \phi^i_f\sigma_{ff}, \\
\sigma^i_{\hat{g}_f,s_\pi} &= \phi^i_\pi\sigma_{f\pi},
\end{aligned} \tag{59}$$

$$\begin{aligned}
\psi^i_{\pi f} &= \frac{\gamma^i_{f\pi}(\sigma^2_{P\delta} + \sigma^2_{PP}) - \gamma^i_\pi(\sigma_{\delta\delta}\sigma_{P\delta} + \sigma_{\delta P}\sigma_{PP})}{(\sigma_{\delta\delta}\sigma_{PP} - \sigma_{\delta P}\sigma_{P\delta})^2}, \\
\psi^i_{\pi\pi} &= \zeta_\pi + \frac{\gamma^i_\pi(\sigma^2_{\delta\delta} + \sigma^2_{\delta P}) - \gamma^i_{f\pi}(\sigma_{\delta\delta}\sigma_{P\delta} + \sigma_{\delta P}\sigma_{PP})}{(\sigma_{\delta\delta}\sigma_{PP} - \sigma_{\delta P}\sigma_{P\delta})^2}, \\
\sigma^i_{\hat{g}_\pi,\delta} &= \frac{\gamma^i_{f\pi}\sigma_{PP} - \gamma^i_\pi\sigma_{\delta P}}{\sigma_{\delta\delta}\sigma_{PP} - \sigma_{\delta P}\sigma_{P\delta}}, \\
\sigma^i_{\hat{g}_\pi,P} &= \frac{\gamma^i_\pi\sigma_{\delta\delta} - \gamma^i_{f\pi}\sigma_{P\delta}}{\sigma_{\delta\delta}\sigma_{PP} - \sigma_{\delta P}\sigma_{P\delta}}, \\
\sigma^i_{\hat{g}_\pi,s_f} &= \phi^i_f\sigma_{\pi f}, \\
\sigma^i_{\hat{g}_\pi,s_\pi} &= \phi^i_\pi\sigma_{\pi\pi}.
\end{aligned} \tag{60}$$

The solution to this system is as below:

$$\begin{aligned}
\hat{g}^i_{f,t} =& \hat{g}^i_{f,0}\frac{\frac{1}{\theta_2(t)} + \frac{1}{\theta_2 p(t)}}{\frac{\theta_1(t)}{\theta_2(t)} + \frac{\theta_1 p(t)}{\theta_2 p(t)}} + \hat{g}^i_{\pi,0}\frac{\frac{1}{\theta_2(t)} - \frac{1}{\theta_2 p(t)}}{\frac{\theta_1(t)}{\theta_2(t)} + \frac{\theta_1 p(t)}{\theta_2 p(t)}} \\
&+ \frac{1}{\frac{\theta_1(t)}{\theta_2(t)} + \frac{\theta_1 p(t)}{\theta_2 p(t)}}\int_0^t [(\frac{\theta_1(s)}{\theta_2(t)} + \frac{\theta_1 p(s)}{\theta_2 p(t)})\sigma_{\hat{g}^i_f} + (\frac{\theta_2(s)}{\theta_2(t)} - \frac{\theta_2 p(s)}{\theta_2 p(t)})\sigma_{\hat{g}^i_\pi}]dW_s
\end{aligned} \tag{61}$$

$$\hat{g}^i_{\pi,t} = \hat{g}^i_{f,0}\frac{\frac{1}{\theta_1(t)} - \frac{1}{\theta_1 p(t)}}{\frac{\theta_2(t)}{\theta_1(t)} + \frac{\theta_2 p(t)}{\theta_1 p(t)}} + \hat{g}^i_{\pi,0}\frac{\frac{1}{\theta_1(t)} + \frac{1}{\theta_1 p(t)}}{\frac{\theta_2(t)}{\theta_1(t)} + \frac{\theta_2 p(t)}{\theta_1 p(t)}}$$
$$+ \frac{1}{\frac{\theta_2(t)}{\theta_1(t)} + \frac{\theta_2 p(t)}{\theta_1 p(t)}} \int_0^t [(\frac{\theta_1(s)}{\theta_1(t)} - \frac{\theta_1 p(s)}{\theta_1 p(t)})\sigma_{\hat{g}^i_f} + (\frac{\theta_2(s)}{\theta_1(t)} + \frac{\theta_2 p(s)}{\theta_1 p(t)})\sigma_{\hat{g}^i_\pi}]dW_s \tag{62}$$

where

$$\begin{aligned}
\sigma_{\hat{g}^i_f} &= \{\sigma^i_{\hat{g}_f,\delta}, \sigma_{\hat{g}^i_f,P}, \sigma_{\hat{g}^i_f,s_f}, \sigma_{\hat{g}^i_f,s_\pi}, -\sigma_{ff}, -\sigma_{f\pi}\} \\
\sigma_{\hat{g}^i_\pi} &= \{\sigma_{\hat{g}^i_\pi,\delta}, \sigma_{\hat{g}^i_\pi,P}, \sigma_{\hat{g}^i_\pi,s_f}, \sigma_{\hat{g}^i_\pi,s_\pi}, -\sigma_{\pi f}, -\sigma_{\pi\pi}\} \\
dW_s &= \{dZ^\delta_s, dZ^P_s, dZ^{s_f}_s, dZ^{s_\pi}_s, dZ^f_s, dZ^\pi_s\}'
\end{aligned} \tag{63}$$

and $\theta_1(t), \theta_2(t), \theta_1 p(t), \theta_2 p(t)$ are solutions to two systems of matrix defferential equations:

$$\begin{bmatrix}\theta'_1(t)\\ \theta'_2(t)\end{bmatrix} = \begin{bmatrix}\psi^i_{ff} & \psi^i_{f\pi}\\ \psi^i_{\pi\pi} & \psi^i_{\pi f}\end{bmatrix}\begin{bmatrix}\theta_1(t)\\ \theta_2(t)\end{bmatrix} \tag{64}$$

$$\begin{bmatrix}\theta_1 p'(t)\\ \theta_2 p'(t)\end{bmatrix} = \begin{bmatrix}\psi^i_{ff} & -\psi^i_{f\pi}\\ \psi^i_{\pi\pi} & -\psi^i_{\pi f}\end{bmatrix}\begin{bmatrix}\theta_1 p(t)\\ \theta_2 p(t)\end{bmatrix} \tag{65}$$

The solution to each θ function is a linear combination of two exponential functions if the corresponding coefficient matrix is of full rank, or just one exponential function if the corresponding coefficient matrix has rank of one. For neat notations, I omit the "i" in all the θ functions.

$\hat{g}^i_{f,t}$ and $\hat{g}^i_{\pi,t}$ are jointly normally distributed. After obtaining their means and variances, the exact discretization is as below:

$$
\begin{aligned}
\hat{g}^i_{f,t+\Delta} =\ & \hat{g}^i_{f,t}\frac{\frac{1}{\theta_2(\Delta)}+\frac{1}{\theta_2 p(\Delta)}}{\frac{\theta_1(\Delta)}{\theta_2(\Delta)}+\frac{\theta_1 p(\Delta)}{\theta_2 p(\Delta)}} + \hat{g}^i_{\pi,t}\frac{\frac{1}{\theta_2(\Delta)}-\frac{1}{\theta_2 p(\Delta)}}{\frac{\theta_1(\Delta)}{\theta_2(\Delta)}+\frac{\theta_1 p(\Delta)}{\theta_2 p(\Delta)}} \\
& + (\frac{\mathcal{I}(\sqrt{\int_0^\Delta((\frac{\theta_1(s)}{\theta_2(\Delta)}+\frac{\theta_1 p(s)}{\theta_2 p(\Delta)})\sigma_{\hat{g}^i_f,\delta}+(\frac{\theta_2(s)}{\theta_2(\Delta)}-\frac{\theta_2 p(s)}{\theta_2 p(\Delta)})\sigma_{\hat{g}^i_\pi,\delta})^2 ds})}{\frac{\theta_1(\Delta)}{\theta_2(\Delta)}+\frac{\theta_1 p(\Delta)}{\theta_2 p(\Delta)}} W_{\delta,\Delta} \\
& + \frac{\mathcal{I}(\sqrt{\int_0^\Delta((\frac{\theta_1(s)}{\theta_2(\Delta)}+\frac{\theta_1 p(s)}{\theta_2 p(\Delta)})\sigma_{\hat{g}^i_f,P}+(\frac{\theta_2(s)}{\theta_2(\Delta)}-\frac{\theta_2 p(s)}{\theta_2 p(\Delta)})\sigma_{\hat{g}^i_\pi,P})^2 ds})}{\frac{\theta_1(\Delta)}{\theta_2(\Delta)}+\frac{\theta_1 p(\Delta)}{\theta_2 p(\Delta)}} W_{P,\Delta} \\
& + \frac{\mathcal{I}(\sqrt{\int_0^\Delta((\frac{\theta_1(s)}{\theta_2(\Delta)}+\frac{\theta_1 p(s)}{\theta_2 p(\Delta)})\sigma_{\hat{g}^i_f,s_f}+(\frac{\theta_2(s)}{\theta_2(\Delta)}-\frac{\theta_2 p(s)}{\theta_2 p(\Delta)})\sigma_{\hat{g}^i_\pi,s_f})^2 ds})}{\frac{\theta_1(\Delta)}{\theta_2(\Delta)}+\frac{\theta_1 p(\Delta)}{\theta_2 p(\Delta)}} W_{s_f,\Delta} \\
& + \frac{\mathcal{I}(\sqrt{\int_0^\Delta((\frac{\theta_1(s)}{\theta_2(\Delta)}+\frac{\theta_1 p(s)}{\theta_2 p(\Delta)})\sigma_{\hat{g}^i_f,s_\pi}+(\frac{\theta_2(s)}{\theta_2(\Delta)}-\frac{\theta_2 p(s)}{\theta_2 p(\Delta)})\sigma_{\hat{g}^i_\pi,s_\pi})^2 ds})}{\frac{\theta_1(\Delta)}{\theta_2(\Delta)}+\frac{\theta_1 p(\Delta)}{\theta_2 p(\Delta)}} W_{s_\pi,\Delta} \\
& - \frac{\mathcal{I}(\sqrt{\int_0^\Delta((\frac{\theta_1(s)}{\theta_2(\Delta)}+\frac{\theta_1 p(s)}{\theta_2 p(\Delta)})\sigma_{ff}+(\frac{\theta_2(s)}{\theta_2(\Delta)}-\frac{\theta_2 p(s)}{\theta_2 p(\Delta)})\sigma_{\pi f})^2 ds})}{\frac{\theta_1(\Delta)}{\theta_2(\Delta)}+\frac{\theta_1 p(\Delta)}{\theta_2 p(\Delta)}} W_{f,\Delta} \\
& - \frac{\mathcal{I}(\sqrt{\int_0^\Delta((\frac{\theta_1(s)}{\theta_2(\Delta)}+\frac{\theta_1 p(s)}{\theta_2 p(\Delta)})\sigma_{f\pi}+(\frac{\theta_2(s)}{\theta_2(\Delta)}-\frac{\theta_2 p(s)}{\theta_2 p(\Delta)})\sigma_{\pi\pi})^2 ds})}{\frac{\theta_1(\Delta)}{\theta_2(\Delta)}+\frac{\theta_1 p(\Delta)}{\theta_2 p(\Delta)}} W_{\pi,\Delta}
\end{aligned} \tag{66}
$$

$$
\begin{aligned}
\hat{g}^i_{\pi,t+\Delta} =& \hat{g}^i_{f,0}\frac{\frac{1}{\theta_1(\Delta)}-\frac{1}{\theta_1 p(\Delta)}}{\frac{\theta_2(\Delta)}{\theta_1(\Delta)}+\frac{\theta_2 p(\Delta)}{\theta_1 p(\Delta)}} + \hat{g}^i_{\pi,0}\frac{\frac{1}{\theta_1(\Delta)}+\frac{1}{\theta_1 p(\Delta)}}{\frac{\theta_2(\Delta)}{\theta_1(\Delta)}+\frac{\theta_2 p(\Delta)}{\theta_1 p(\Delta)}} \\
&+ \frac{\mathcal{I}(\sqrt{\int_0^\Delta((\frac{\theta_1(s)}{\theta_1(\Delta)}-\frac{\theta_1 p(s)}{\theta_1 p(\Delta)})\sigma_{\hat{g}^i_f,\delta}+(\frac{\theta_2(s)}{\theta_1(\Delta)}+\frac{\theta_2 p(s)}{\theta_1 p(\Delta)})\sigma_{\hat{g}^i_\pi,\delta})^2 ds})}{\frac{\theta_2(\Delta)}{\theta_1(\Delta)}+\frac{\theta_2 p(\Delta)}{\theta_1 p(\Delta)}} W_{\delta,\Delta} \\
&+ \frac{\mathcal{I}(\sqrt{\int_0^\Delta((\frac{\theta_1(s)}{\theta_1(\Delta)}-\frac{\theta_1 p(s)}{\theta_1 p(\Delta)})\sigma_{\hat{g}^i_f,P}+(\frac{\theta_2(s)}{\theta_1(\Delta)}+\frac{\theta_2 p(s)}{\theta_1 p(\Delta)})\sigma_{\hat{g}^i_\pi,P})^2 ds})}{\frac{\theta_2(\Delta)}{\theta_1(\Delta)}+\frac{\theta_2 p(\Delta)}{\theta_1 p(\Delta)}} W_{P,\Delta} \\
&+ \frac{\mathcal{I}(\sqrt{\int_0^\Delta((\frac{\theta_1(s)}{\theta_1(\Delta)}-\frac{\theta_1 p(s)}{\theta_1 p(\Delta)})\sigma_{\hat{g}^i_f,s_f}+(\frac{\theta_2(s)}{\theta_1(\Delta)}+\frac{\theta_2 p(s)}{\theta_1 p(\Delta)})\sigma_{\hat{g}^i_\pi,s_f})^2 ds})}{\frac{\theta_2(\Delta)}{\theta_1(\Delta)}+\frac{\theta_2 p(\Delta)}{\theta_1 p(\Delta)}} W_{s_f,\Delta} \quad (67) \\
&+ \frac{\mathcal{I}(\sqrt{\int_0^\Delta((\frac{\theta_1(s)}{\theta_1(\Delta)}-\frac{\theta_1 p(s)}{\theta_1 p(\Delta)})\sigma_{\hat{g}^i_f,s_\pi}+(\frac{\theta_2(s)}{\theta_1(\Delta)}+\frac{\theta_2 p(s)}{\theta_1 p(\Delta)})\sigma_{\hat{g}^i_\pi,s_\pi})^2 ds})}{\frac{\theta_2(\Delta)}{\theta_1(\Delta)}+\frac{\theta_2 p(\Delta)}{\theta_1 p(\Delta)}} W_{s_\pi,\Delta} \\
&- \frac{\mathcal{I}(\sqrt{\int_0^\Delta((\frac{\theta_1(s)}{\theta_1(\Delta)}-\frac{\theta_1 p(s)}{\theta_1 p(\Delta)})\sigma_{ff}+(\frac{\theta_2(s)}{\theta_1(\Delta)}+\frac{\theta_2 p(s)}{\theta_1 p(\Delta)})\sigma_{\pi f})^2 ds})}{\frac{\theta_2(\Delta)}{\theta_1(\Delta)}+\frac{\theta_2 p(\Delta)}{\theta_1 p(\Delta)}} W_{f,\Delta} \\
&- \frac{\mathcal{I}(\sqrt{\int_0^\Delta((\frac{\theta_1(s)}{\theta_1(\Delta)}-\frac{\theta_1 p(s)}{\theta_1 p(\Delta)})\sigma_{f\pi}+(\frac{\theta_2(s)}{\theta_1(\Delta)}+\frac{\theta_2 p(s)}{\theta_1 p(\Delta)})\sigma_{\pi\pi})^2 ds})}{\frac{\theta_2(\Delta)}{\theta_1(\Delta)}+\frac{\theta_2 p(\Delta)}{\theta_1 p(\Delta)}} W_{\pi,\Delta}
\end{aligned}
$$

where $\mathcal{I}(\sqrt{\int_0^\Delta x(s)^2 ds}) = \begin{cases} \sqrt{\int_0^\Delta x(s)^2 ds} & if\ x(s) > 0 \\ -\sqrt{\int_0^\Delta x(s)^2 ds} & if\ x(s) < 0 \end{cases}$ $(s < 0 < \Delta)$, so $\mathcal{I}$ is a sign indicator function to keep the sign of Brownnian parts consistent with that of their volatility across all $\hat{g}^i$.

B Tables and Figures

Table 1: The Converged Estimates of Fundamental Processes

This table lists all the estimates of aggregate dividend growth and inflation from the E-M Approach. The aggregate dividend is proxied by annual reported earnings of $S\&P$ 500. The inflation is proxied by the CPI index in CRSP. The variances are obtained from the inverse of Fisher Information matrix.

Parameter	Symbol	Value	Variance
Long-term average growth rate of aggregate dividend	$\bar{f}$	.01	4.6×10^{-5}
Volatility of aggregate dividend on dividend shock	$\sigma_{\delta\delta}$	.066	-2.2×10^{-4}
Volatility of aggregate dividend on inflation shock	$\sigma_{\delta P}$	.003	-5.2×10^{-4}
Mean-reverting parameter of aggregate dividend growth	ζ_f	.12	3.7×10^{-3}
Long-term average growth rate of inflation	$\bar{\pi}$	.018	3.4×10^{-5}
Volatility of inflation on dividend shock	$\sigma_{P\delta}$	.00047	-1.5×10^{-5}
Volatility of inflation on inflation shock	σ_{PP}	.011	-9.1×10^{-5}
Mean-reverting parameter of inflation	ζ_π	.03	4.8×10^{-4}
Variance of aggregate dividend growth rate	$\sigma_{ff}^2+\sigma_{f\pi}^2$	3.5×10^{-5}	4.03×10^{-11}
Variance of inflation rate	$\sigma_{\pi f}^2+\sigma_{\pi\pi}^2$	4.4×10^{-6}	2.56×10^{-13}
Covariance of aggregate dividend growth rate and inflation rate	$\sigma_{ff}\sigma_{\pi f}+\sigma_{f\pi}\sigma_{\pi\pi}$	2.1×10^{-6}	2.02×10^{-12}

Table 2: The predictive regression of stock-bond price correlation

This table shows the regression of the historical time-series of the correlation between Treasurys bond prices and $S\&P$ 500 index levels on the model implied time-series of the stock-bond correlation. The regressions are specified with the maturity of the bond. It also shows the regression of historical correlation between 10-year Treasury bond and $S\&P$ 500 index on itself up to two lags.

Independent Variable	coefficient	P-value	R^2
T 2	3.18	.0002	.24
T 5	3.14	.027	.18
T 10	2.85	.015	.11
T 30	3.85	.027	.12
TIPS 5	-1.76	.4	.03
TIPS 10	-4.8	.05	.18
TIPS 30	-5.1	.1	.14
T 10_{t-1}	.12	.38	.015
T 20_{t-2}	.17	.22	.028

Table 3: The predictive regression of stock-bond return correlation

This table shows the regression of the historical time-series of the correlation between Treasurys bond returns and $S\&P$ 500 index returns on the model implied time-series of the stock-bond correlation. The regressions are specified with the maturity of the bond. The regression specification is as below.

$$Corr_t = constant + \beta ImpliedCorr_t$$

$$Corr_t = constant + \beta ImpliedCorr_t + \beta_1 Corr_{t-1}$$

$$Corr_t = constant + \beta ImpliedCorr_t + \beta_1 Corr_{t-1} + \beta_2 Corr_{t-2}$$

Corr	β	β_1	β_2	$Adjusted R^2$
T 2	2.6(1.4×10^{-8})			.55
T 2	2.0(8.7×10^{-4})	.23(.16)		.56
T 2	1.8(8.6×10^{-3})	.21(.23)	.094(.58)	.54
T 5	3.4(1.3×10^{-11})			.54
T 5	2.2(1.7×10^{-4})	.29(.04)		.62
T 5	2.2(1.7×10^{-3})	.26(.07)	.07(.58)	.62
T 10	4.2(2.8×10^{-12})			.53
T 10	2.7(2.6×10^{-4})	.37(.007)		.67
T 10	2.5(1.4×10^{-3})	.34(.02)	.058(.68)	.66
T 30	5.8(1.0×10^{-10})			.65
T 30	3.9(1.0×10^{-3})	.34(.035)		.69
T 30	3.5(.014)	.34(.056)	.07(.68)	.68
TIPS 5	-.65(.38)			-.011
TIPS 5	-.91(.3)	-.021(.93)		-.04
TIPS 5	1.2(.21)	.033(.89)	-.47(.06)	.12
TIPS 10	1.4(.14)			.062
TIPS 10	.58(.58)	.044(.84)		-.08
TIPS 10	1.2(.36)	.022(.93)	-.3(.23)	-.074
TIPS 30	1.6(.15)			.06
TIPS 30	.99(.47)	.16(.52)		-.02
TIPS 30	1.7(.34)	.17(.51)	-.31(.3)	-.06

Table 4: Robustness:The predictive regression of stock-bond return correlation

This table shows the regression of the historical time-series of the correlation between Treasurys bond returns and $S\&P$ 500 index returns on the model implied time-series of the stock-bond correlation. The regressions are specified with the maturity of the bond. The regression specification is the same as that in Table 3, but the variables are standardized.

Corr	β	β_1	β_2	R^2
T 2	.75(1.4$\times 10^{-8}$)			.55
T 2	.58(8.7$\times 10^{-4}$)	.23(.16)		.56
T 2	.16(.0086)	.064(.23)	.029(.58)	.54
T 5	.78(1.3$\times 10^{-11}$)			.6
T 5	.57(1.4$\times 10^{-4}$)	.28(.054)		.63
T 5	.53(.0022)	.26(.087)	.069(.64)	.62
T 10	.79(2.8$\times 10^{-12}$)			.62
T 10	.54(1.8$\times 10^{-4}$)	.33(.014)		.67
T 10	.52(.0017)	.33(.034)	.04(.78)	.66
T 30	.81(1.0$\times 10^{-10}$)			.65
T 30	.56(.001)	.34(.035)		.69
T 30	.49(.014)	.34(.056)	.07(.68)	.68
TIPS 5	-.2(.38)			-.011
TIPS 5	-.28(.3)	-.021(.93)		-.048
TIPS 5	-.36(.21)	-.033(.89)	-.47(.056)	.12
TIPS 10	.33(.144)			.062
TIPS 10	.13(.58)	.044(.84)		-.086
TIPS 10	.28(.36)	.022(.93)	-.3(.23)	-.074
TIPS 30	.33(.15)			.06
TIPS 30	.2(.47)	.16(.52)		-.024
TIPS 30	.34(.34)	-.31(.3)	.17(.051)	-.06

Table 5: The regressions of dispersions in Survey of Professional Forecasters

This table shows the regression of dispersions of forecasts of real GDP growth and inflation on the model-implied disagreements. The dependent variables list the tickers of dispersions, and the independent variables list the tickers of disagreements. RGDP is short for real GDP growth. CPI refers to the dispersion in CPI level forecast. CPI ratio is the ratio of dispersion in CPI

Dependent Variable	Independent Variable	coefficient	P-value	R^2
RGDP	dividend growth	-1.6	.00099	.2
CPI	inflation	61.6	8.1×10^{-6}	.44
CPI ratio	inflation	15081	6.7×10^{-10}	.67
RGDP	inflation	7.35	.82	.0011
RGDP	inflation ratio	7.4	.8	.0011
CPI	dividend growth	.17	.5	.013
CPI ratio	dividend growth	-40.99	.42	.018

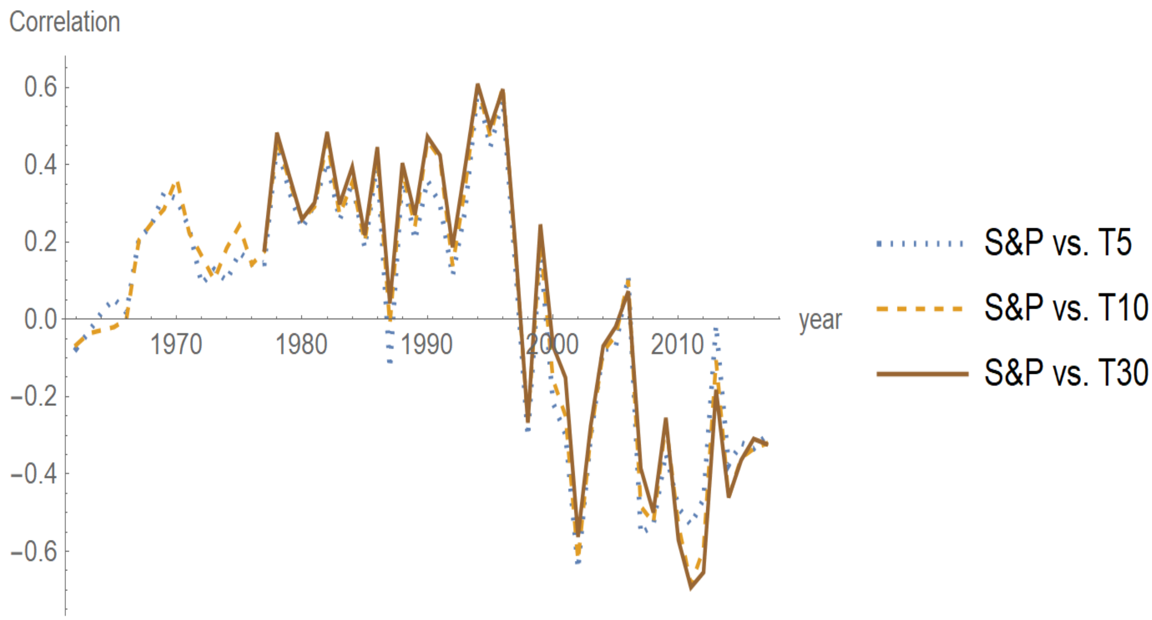

Figure 1: The annual correlation between $S\&P$ 500 and 5/10/30 year US Treasury bond based on daily returns

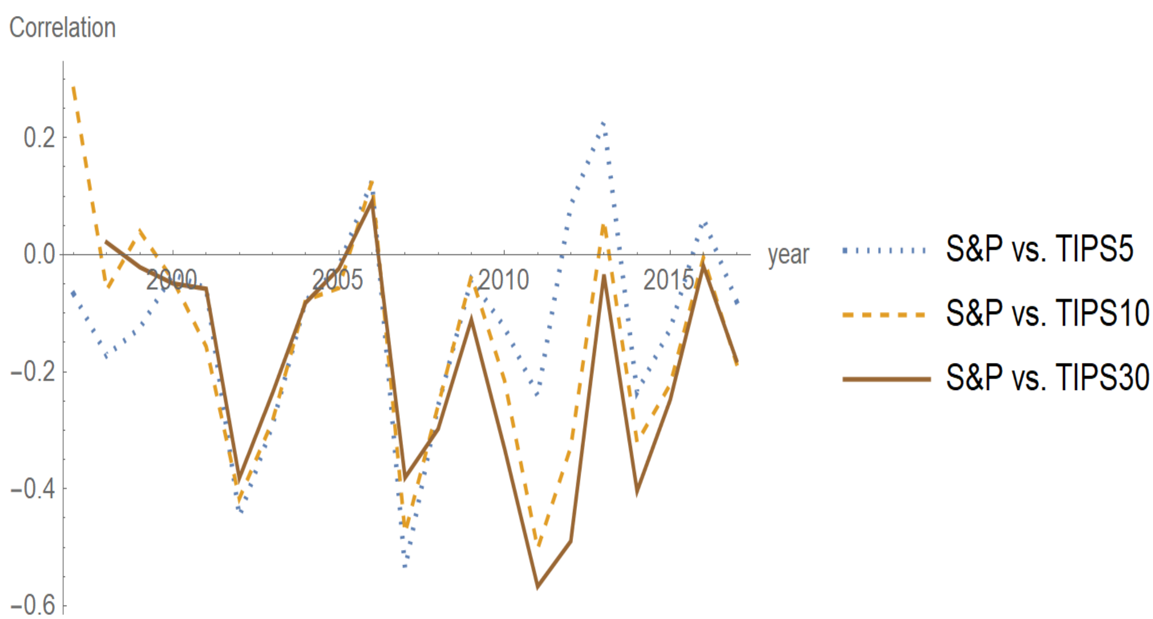

Figure 2: The annual correlation between $S\&P$ 500 and 5/10/30 year US TIPS based on daily returns

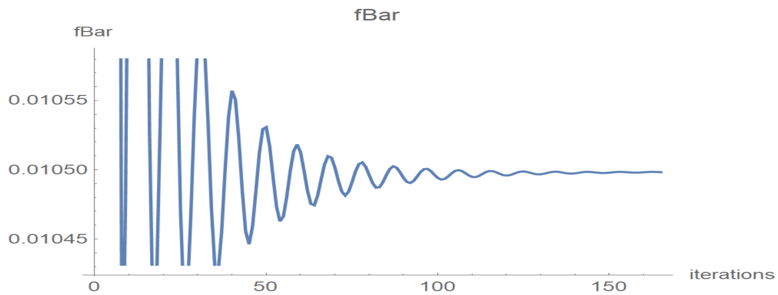

Figure 3: The convergence of $\bar{f}$

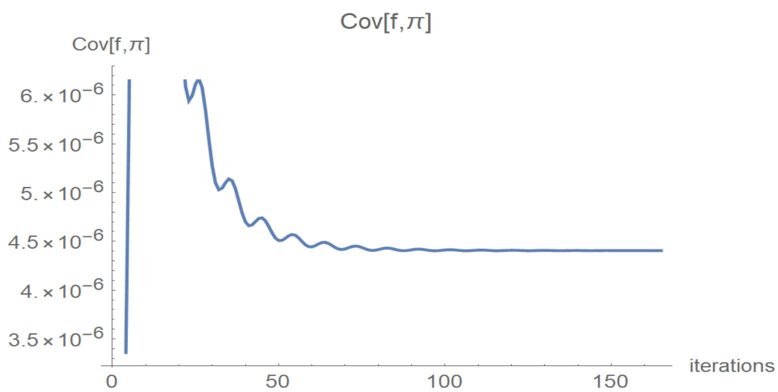

Figure 4: The convergence of $Cov(f, \pi)$

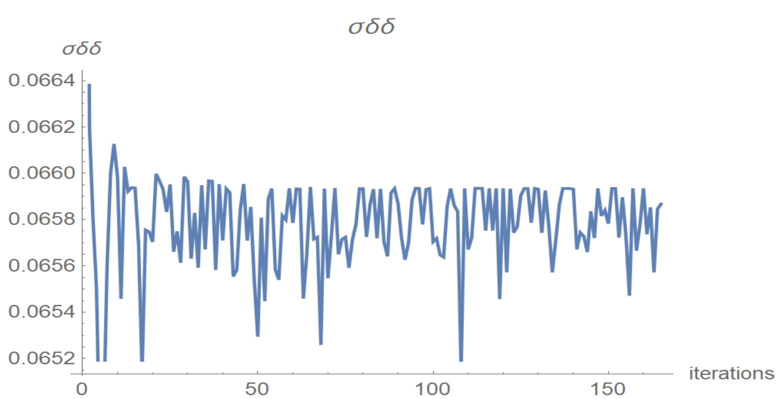

Figure 5: The convergence of $\sigma_{\delta\delta}$

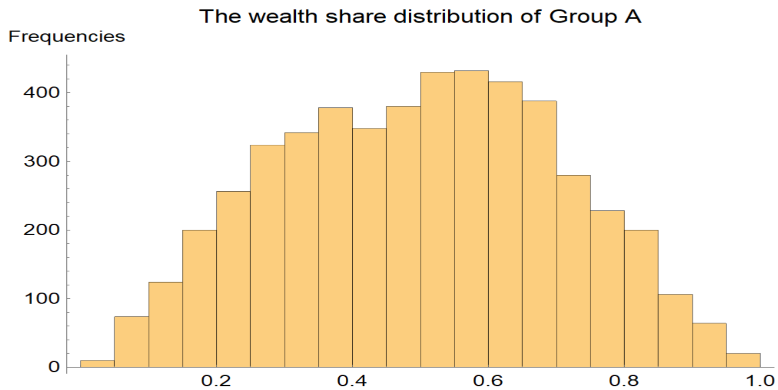

Figure 6: The final consumption share distribution of group A

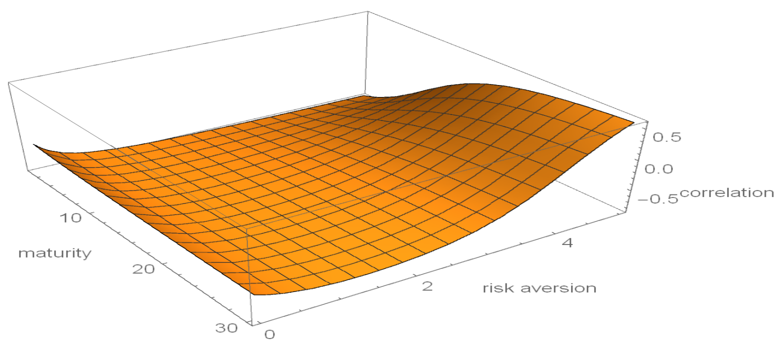

Figure 7: The correlation between nominal bond and single-payment stock

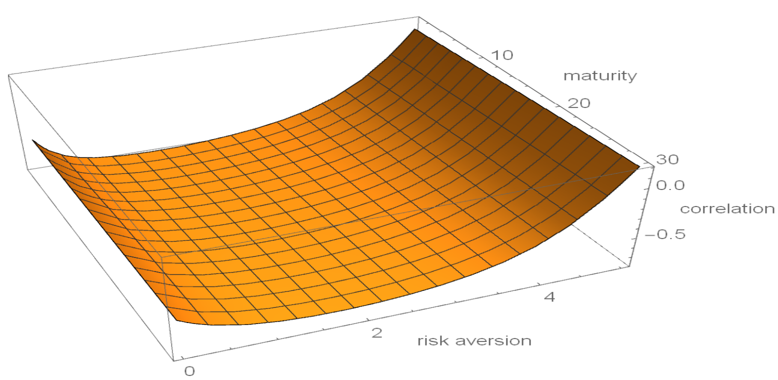

Figure 8: The correlation between nominal bond and stock

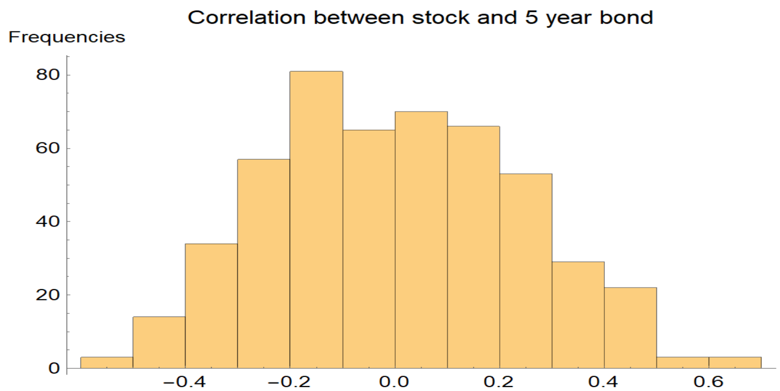

Figure 9: The distribution of simulated correlation between 5 year nominal bond and stock

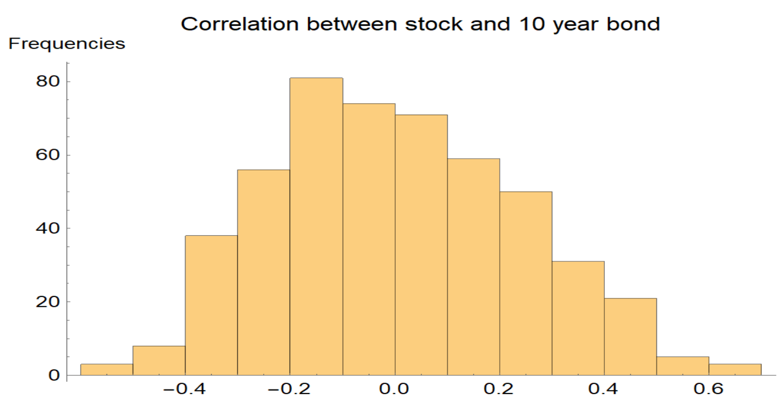

Figure 10: The distribution of simulated correlation between 10-year nominal bond and stock

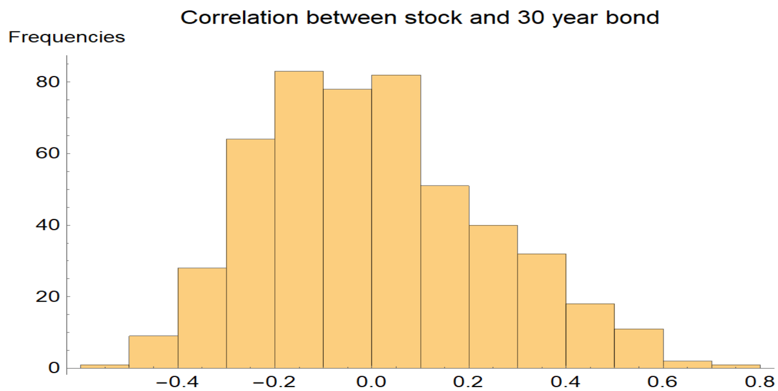

Figure 11: The distribution of simulated correlation between 30 year nominal bond and stock

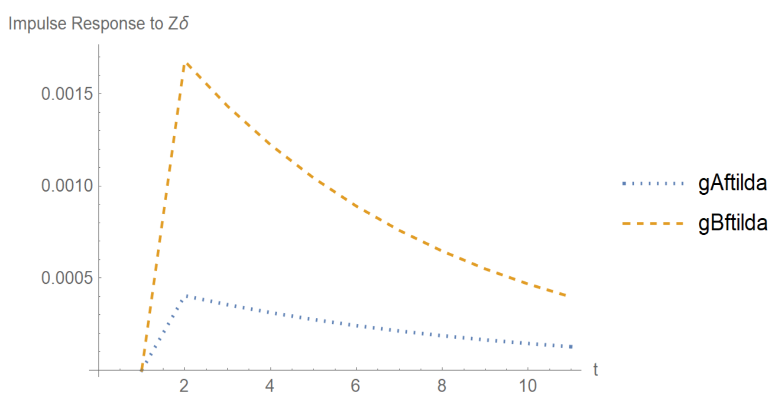

Figure 12: Impulse response of $\tilde{g}_f^A$ and $\tilde{g}_f^B$ to Z_δ

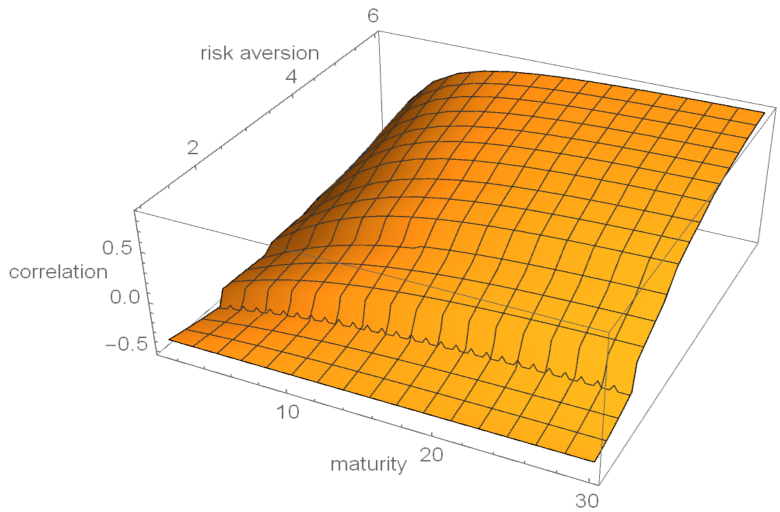

Figure 13: The conditional correlation between nominal bond and stock against risk aversion and bond maturity

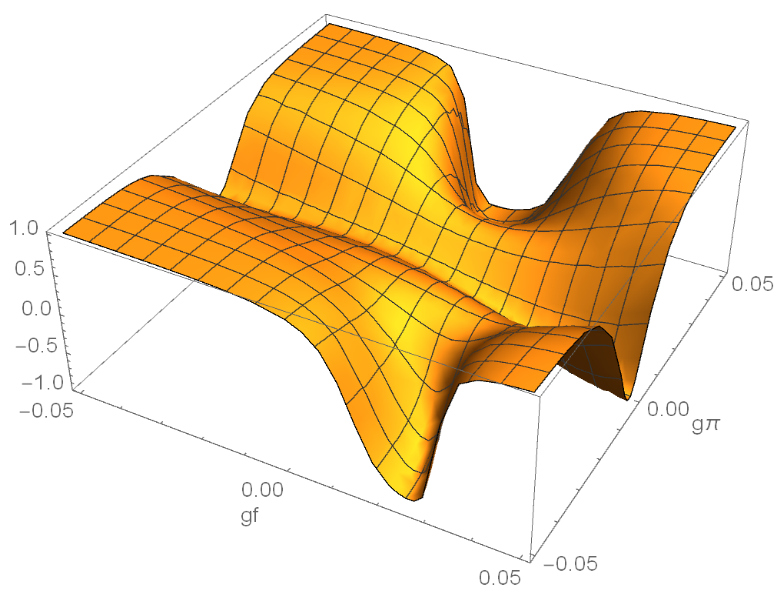

Figure 14: The conditional correlation between nominal bond and stock against disagreements

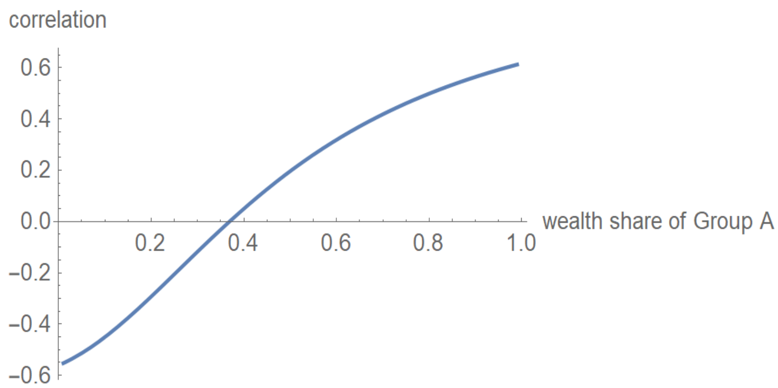

Figure 15: The conditional correlation between nominal bond and stock against the consumption share of group A

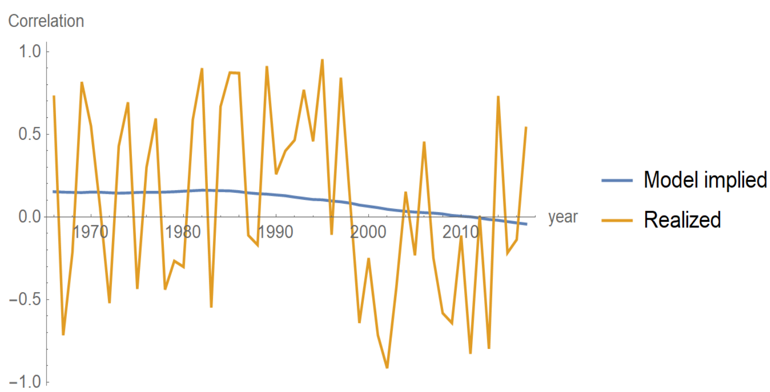

Figure 16: The fit of model-implied bond-stock correlation and the historical stock-bond correlation series

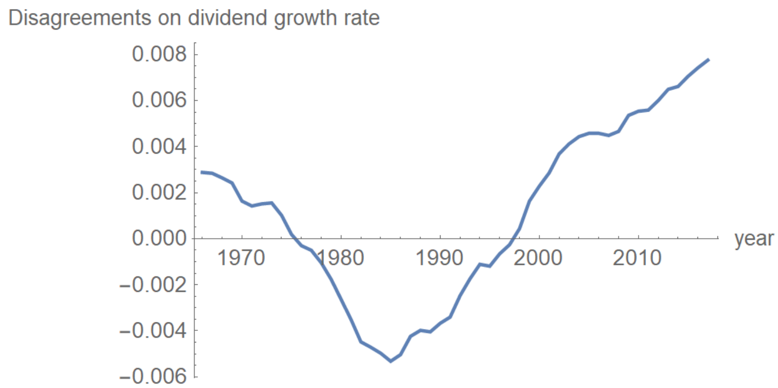

Figure 17: The time series of implied disagreements on aggregate dividend growth rate

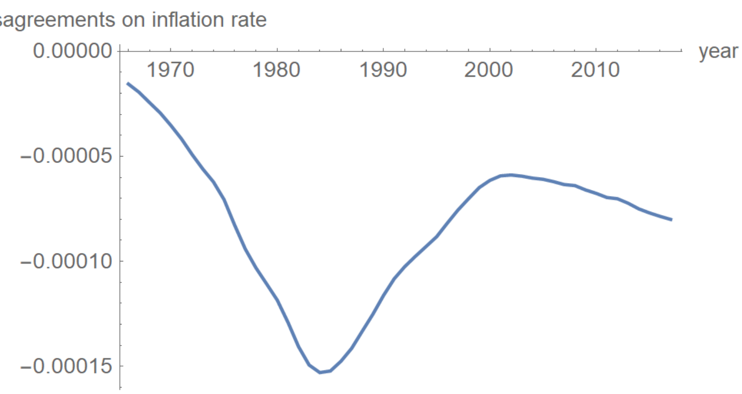

Figure 18: The time series of implied disagreements on inflation

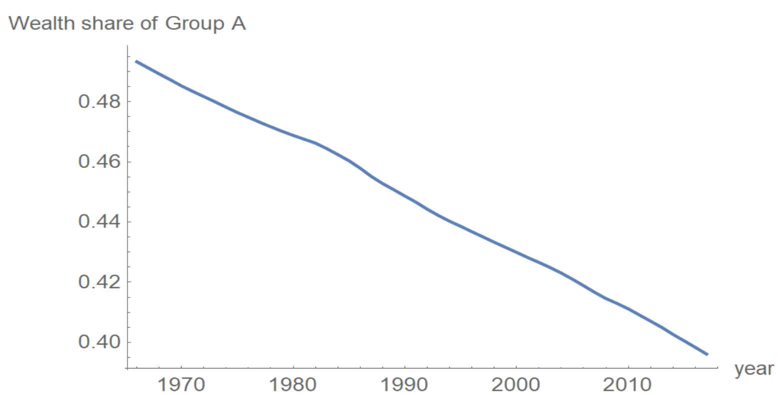

Figure 19: The time series of group A's consumption share

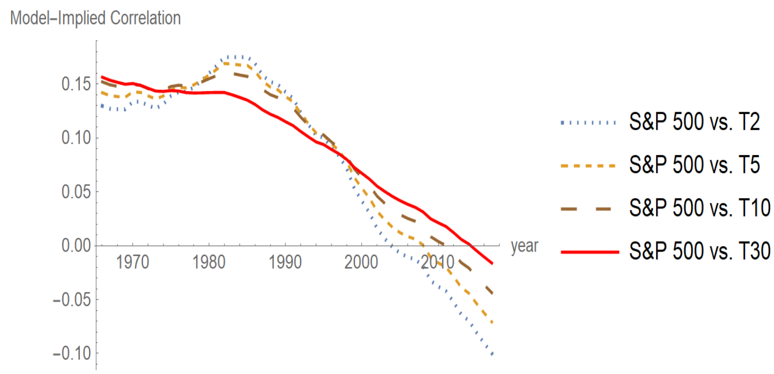

Figure 20: The conditional correlation between nominal bonds and stock implied by the model

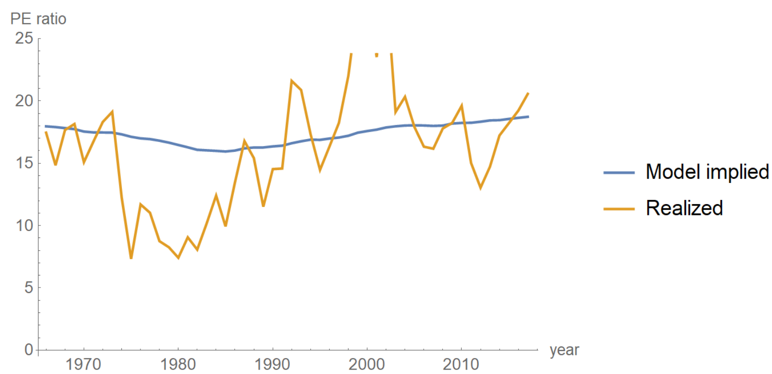

Figure 21: The fit of model-implied PE ratio and the historical PE ratio series

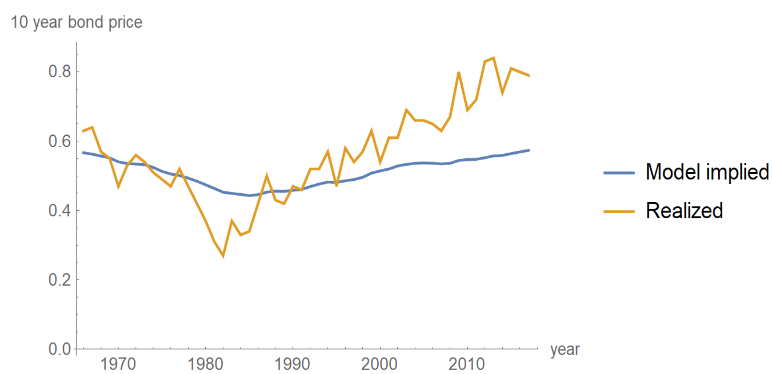

Figure 22: The fit of model-implied 10-year nominal bond price and the historical T10 price series