ON THE EXERCISE OF AMERICAN QUANTO OPTIONS

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Abstract

We provide a comprehensive description of the optimal exercise policies associated with American quanto options in a parsimonious diffusive currency market. We also show that a non-standard exercise

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policy may be optimal in the presence of non–positive domestic interest rates.

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1 Introduction

Quanto options are derivatives written on a foreign security. In the most common call and put options on foreign stocks or indexes the exchange rate is fixed and settled at the beginning of the contract. The exchange rate can be fixed at the forward level (typically with the same maturity of the option), or the level of the initial spot exchange rate. The foreign risky security has a domestic risk neutral drift that is different from the domestic riskless interest rate (diminished by dividend yield, if any). This makes quanto options sensibly different from plain vanilla options (see Chapter 17 in Hull [8]). Moreover, quanto options come very frequently in the American variety, so that they can be exercised during their whole life.

In a parsimonious diffusive model (see Bjork [4]) we provide an exhaustive characterization of the optimal exercise policies of American quanto put options, that depend on the payoff structure as well as on the interplay between the domestic and the foreign riskless interest rate. In particular, we show that in the presence of a domestic non-positive short-term interest rate (as for instance the Euro denominated or the Yen denominated markets) and of a foreign positive short-term interest rate (as for instance the US Dollar denominated market) these options may exhibit unusual optimal exercise policies. We show by means of a numerical example that as the sign of the domestic interest rate r_d changes form positive to negative a non-standard double continuation can appear. Battauz, De Donno and Sbuelz (2015) determine sufficient conditions for the emergence of a double continuation region for American perpetual put options. It follows that the same conditions are sufficient also for the finite maturity case. Since such conditions are quite restrictive in the proximity of the expiration date, Battauz, De Donno and Sbuelz (2015) work out a necessary condition for the finite maturity case. We contribute by providing examples of quanto options that can be reduced to put options exhibiting a non standard double continuation region in the finite maturity case, without meeting the sufficient perpetual condition.

2 American quanto options in a lognormal currency market

We consider a frictionless continuous-time market, modeled through a stochastic basis $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\leq t\leq T}, P)$ satisfying the usual assumptions (in the sense of Definitions I.1.2 and I.1.3 in [9]). Let $B_d(t) = e^{r_d t}$ be the domestic riskless bond price, where r_d is the constant *domestic* riskless interest rate. Denote with $B_f(t) = e^{r_f\, t}$ the foreign riskless bond price, where r_f is the constant *foreign* riskless interest rate and with $S_f = \{S_f(t)\}_{t\in[0,T]}$ the foreign risky security price described by

$$\frac{dS_f(t)}{S_f(t)} = \mu_f dt + \sigma_f\ dW^P(t)$$

where W^P is the $\Re^2$-Brownian motion under the historical probability measure P with respect to the filtration $\mathcal{F}$ and σ_f is the $\Re^2-$vector of volatilities of the foreign security. Let G_f be the cumulative gain process obtained by buying 1 unit of the the foreign risky security at the initial date $t = 0$. If the foreign security pays a continuous dividend yield q_f and the dividend $q_f S_f(t)dt$ is continuously reinvested in the security S_f, the value of the cumulative gain process at t is

$$G_f(t) = e^{q_f\ t}\ S_f(t)$$

and its differential

$$dG_f(t) = e^{q_f\ t} dS_f(t) + q_f S_f(t)dt = G_f(t)\left((\mu_f + q_f)\,dt + \sigma_f\ dW^P(t)\right). \tag{2.1}$$

The two markets are connected via the *foreign to domestic exchange rate* X. If, for instance, we pick the Euro market as the domestic one and the US market as the foreign one, X is the *dollar to euros* exchange rate. Assume that X is lognormal and driven by

$$\frac{dX(t)}{X(t)} = \mu_X dt + \sigma_X\ dW^P(t)$$

under the historical probability measure P (σ_X is an $\Re^2-$vector). For sake of notation we will denote the scalar product of two vectors $\sigma_1, \sigma_2 \in \Re^2$ with $\sigma_1\sigma_2$. The following proposition describes the domestic risk neutral distribution of the assets in the market and the forward rate. The proof extends the arguments of Chapter 17 in Bjork [4], to the case of a dividend-paying foreign asset.

Proposition 2.1 *The domestic risk neutral dynamics of the foreign risky security price is*

$$\frac{dS_f(t)}{S_f(t)} = \mu_f^Q dt + \sigma_f\ dW^Q(t)$$

where W^Q *is a* $\Re^2-$*Brownian motion,* Q *denotes the domestic risk neutral measure and* $\mu_f^Q = r_f - q_f - \sigma_f\sigma_X$*. The exchange rate is*

$$\frac{dX(t)}{X(t)} = \mu_X^Q dt + \sigma_X\ dW^Q(t)$$

with $\mu_X^Q = r_d - r_f$*. The correlation between the foreign security* S_f *and the exchange rate* X *is*

$$\rho = \frac{\sigma_f \cdot \sigma_X}{\|\sigma_f\|\,\|\sigma_X\|}.$$

The domestic-denominated foreign security price $S_f^*(t) = S_f(t)X(t)$ *is driven by*

$$dS_f^*(t) = S_f^*(t)\left[(r_d - q_f)\ dt + \left(\sigma_X\ + \sigma_f\right) dW^Q(t)\right].$$

The forward exchange rate F_0, *i.e. the number of units of domestic currency settled at* $t = 0$ *to receive one unit of the foreign currency at the maturity* T *is*

$$F_0 = E^Q\left[X(T)\right] = X(0)\ e^{\mu_X^Q T} = X_0 e^{\left(r_d - r_f\right)T}.$$

Proof. Let $B_f^*(t) = B_f(t)X(t)$. B_f^* and S_f^* are both *risky domestic securities.* Then, exploiting the integration by parts' formula and observing that the covariation between B_f and X is 0, we have

$$\begin{aligned} dB_f^*(t) &= d\left(B_f(t)X(t)\right) = X(t)dB_f(t) + B_f(t)dX(t) = \\ &= B_f^*(t)\left[\left(r_f + \mu_X^Q\right)dt + \sigma_X\ dW^Q(t)\right] \end{aligned}$$

and no-arbitrage implies

$$r_f + \mu_X^Q = r_d$$

delivering

$$\mu_X^Q = r_d - r_f.$$

Consider the cumulative gain process denominated in domestic currency

$$G_f^*(t) = G_f(t)X(t).$$

The differential of G_f in Equation (2.1) can be rewritten with respect to the domestic risk neutral measure Q as

$$dG_f(t) = dG_f(t) = e^{q_f\ t}dS_f(t) + q_f S_f(t)dt = G_f(t)\left(\left(\mu_f^Q + q_f\right)dt + \sigma_f\ dW^Q(t)\right).$$

Ito formula implies

$$\begin{aligned} dG_f^*(t) &= d(G_f(t)X(t)) = X(t)dG_f(t) + G_f(t)dX(t) + \sigma_f G_f(t)\sigma_X X(t)dt = \\ &= G_f^*(t)\left[\left(\mu_f^Q + q_f + \mu_X^Q + \sigma_f\sigma_X\right)dt + \left(\sigma_X\ + \sigma_f\right)dW^Q(t)\right]. \end{aligned}$$

No-arbitrage implies that G_f^* discounted at the rate r_d is a $Q-$martingale, i.e. the domestic-risk neutral drift of G_f^* equals r_d. This delivers the equation

$$\mu_f^Q + q_f + \mu_X^Q + \sigma_f \sigma_X = r_d,$$

which implies

$$\mu_f^Q = r_f - q_f - \sigma_f \sigma_X.$$

The equation for S_f^* becomes

$$\begin{aligned} dS_f^*(t) &= S_f^*(t) \left[\left(\mu_f^Q + \mu_X^Q + \sigma_f \sigma_X \right) dt + \left(\sigma_X \ + \sigma_f \right) dW^Q(t) \right] \\ &= S_f^*(t) \left[\left(r_f - q_f - \sigma_f \sigma_X + r_d - r_f + \sigma_f \sigma_X \right) dt + \left(\sigma_X \ + \sigma_f \right) dW^Q(t) \right] \\ &= S_f^*(t) \left[\left(r_d - q_f \right) \ dt + \left(\sigma_X \ + \sigma_f \right) dW^Q(t) \right] \end{aligned}$$

The *forward exchange rate* F_0, i.e. the number of units of domestic currency settled at $t = 0$ to receive one unit of the foreign currency at the maturity T, is determined by

$$\mathbf{E}^Q \left[e^{-r_d T} \left(1 \cdot X(T) - F_0 \right) \right] = 0$$

that implies the following relation between the forward and the spot exchange rate in terms of the domestic and foreign interest rates:

$$F_0 = \mathbf{E}^Q \left[X(T) \right] = X(0) \, e^{\mu_X^Q T} = X_0 e^{\left(r_d - r_f \right) T}. \qquad \square$$

Quanto put options have different payoffs, depending on contract specifications. In Equation (2.2), the time t instantaneous payoff has a foreign-denominated strike K_f and is converted in domestic currency at the floating exchange rate at exercise:

$$V(t) = \sup_{t \le \tau \le T} \mathbf{E}^Q \left[e^{-r_d (T-t)} \, X(\tau) \left(K_f - S_f(\tau) \right)^+ \middle| \, \mathcal{F}_t \right] \tag{2.2}$$

In this case the American quanto option coincides with the foreign American option, converted in domestic currency at the current floating exchange rate.

Thus early exercise is optimal for the American quanto put option if the foreign underlying risky security enters the optimal early exercise region of the American put option (see Proposition 2.3).

But American quanto options are more appealing to investors, if the currency risk is reduced. This goal is achieved by settling a domestic denominated strike price K_d as in (2.3) , where the foreign security is converted in domestic currency at the floating exchange rate at exercise,

$$V(t) = \sup_{t \leq \tau \leq T} \mathbf{E}^Q \left[e^{-r_d(T-t)} \left(K_d - X(\tau) S_f(\tau) \right)^+ \middle| \mathcal{F}_t \right] \tag{2.3}$$

or, as in Equation (2.4), where the strike price is denominated in domestic currency and the exchange rate for conversion at any exercise date is fixed at its initial spot level

$$V(t) = \sup_{t \leq \tau \leq T} \mathbf{E}^Q \left[e^{-r_d(T-t)} \left(K_d - X_0 S_f(\tau) \right)^+ \middle| \mathcal{F}_t \right]. \tag{2.4}$$

A similar payoff structure in Equation (2.5) maintains the domestic-denominated strike price and fixes the exchange rate at the initial forward level:

$$V(t) = \sup_{t \leq \tau \leq T} \mathbf{E}^Q \left[e^{-r_d(T-t)} \left(K_d - F_0 S_f(\tau) \right)^+ \middle| \mathcal{F}_t \right]. \tag{2.5}$$

In another popular version of the quanto option, the strike price is denominated in the foreign currency and the payoff converted at the initial spot exchange rate

$$V(t) = \sup_{t \leq \tau \leq T} \mathbf{E}^Q \left[e^{-r_d(T-t)} X_0 \cdot \left(K_f - S_f(\tau) \right)^+ \middle| \mathcal{F}_t \right] \tag{2.6}$$

We introduce now a template to classify and characterize the optimal exercise policy of American quanto put options in terms of American put options for various combination in the level, sign and hierarchy of the key paramteres r_d, r_f and the volatility vectors σ_f and σ_X. Let B^Q be a one-dimensional $Q-$Brownian motion and denote with

$$v(t,s;\mu,\sigma,\delta,K) = \sup_{0\leq\Theta\leq T-t} \mathbf{E}^Q \left[e^{-\delta\Theta} \left(K - s\cdot\exp\left(\left(\mu - \frac{\sigma^2}{2}\right)\Theta + \sigma B^Q(\Theta) \right) \right)^+ \right] \tag{2.7}$$

the time$-t$ value of an American put option on a lognormal security with drift μ, volatility σ, interest rate δ, strike price K and maturity T. The drift can be expressed as drift $\mu = \delta - q$, where q is the dividend yield. Throughout our analysis we assume $\sigma > 0$.

Denote by

$$v_\infty(s;\mu,\sigma,\delta,K) = \sup_{0\leq\Theta\leq\infty} \mathbf{E}^Q \left[e^{-\delta\Theta} \left(K - s\cdot\exp\left(\left(\mu - \frac{\sigma^2}{2}\right)\Theta + \sigma B^Q(\Theta) \right) \right)^+ \right] \tag{2.8}$$

the value of the perpetual American put option. Obviously,

$$(K-s)^+ \leq v(t,s;\mu,\sigma,\delta,K) \leq v_\infty(s;\mu,\sigma,\delta,K), \text{ for all } t\in[0,T],$$

no matter of the parameters' values. Hence, if there exists an optimal early exercise opportunity for the perpetual put option, this is also the case for the finite-maturity one.

In Theorem 2.2 we provide a comprehensive description of the optimal early exercise region for American put options on a lognormal underlying asset in case of positive, zero, and negative interest rates. The resulting asymptotic behavior of the critical price at maturity depends also on the interplay with the underlying risk neutral drift.

In particular, Theorem 2.2, Point 1, focuses on the standard case of a positive interest rate δ. When $\delta > 0$, it is well known that there exists a constant critical price that triggers optimal early exercise for the perpetual option and the American put option value is finite (see also Remark 2.1). On the contrary, when the interest rate δ is negative, the perpetual American put option may have an infinite value. Assumption (2.13) in Theorem 2.2, Point 4, ensures that the perpetual put option has a finite value and displays optimal early exercise opportunities above (resp. below) an upper (resp. lower) constant critical price (see also Proposition 2.2 in De Donno and Sbuelz [3]). As a consequence, the finite maturity put option does also have optimal early exercise opportunities above (resp. below) an upper (resp. lower) critical price. However, Assumption (2.13) is not satisfied in many practical examples, that display optimal early exercise opportunities in the finite-maturity case only (see Section 4). Therefore, in Theorem 2.2 Point 5, we extend the results of Theorem 2.4 in Battauz, De Donno and Sbuelz [3] to describe the asymptotics of the upper and lower critical prices at maturity under the milder condition of the existence of *some* optimal early exercise opportunity.

Theorem 2.2 *1. If $\delta > 0$, early exercise is optimal for the perpetual American put option when the underlying price $S(t) \leq S^c_\infty$, where S^c_∞ is the (constant) critical price of the perpetual American put option,*

$$S^c_\infty = -\frac{\alpha}{1-\alpha}K < K, \qquad A = \frac{(S^c_\infty)^{1-\alpha}}{-\alpha} > 0, \tag{2.9}$$

and

$$\alpha = \frac{-\left(\mu - \frac{\sigma^2}{2}\right) - \sqrt{\left(\mu - \frac{\sigma^2}{2}\right)^2 + 2\delta\sigma^2}}{\sigma^2} < 0. \tag{2.10}$$

The perpetual put value is

$$v_\infty(s; \mu, \sigma, \delta, K) = \begin{cases} As^\alpha & \text{for } s > S^c_\infty \\ K - s & \text{for } s \leq S^c_\infty. \end{cases} \tag{2.11}$$

Moreover, if $0 \leq \mu = \delta - q \leq \delta$ the finite-maturity critical price is such that

$$\lim_{t \to T} S^c(t) = K$$

with

$$\lim_{t\to T}\frac{K-S^c(t)}{\sigma K\sqrt{(T-t)\ln\frac{\gamma}{(T-t)}}}=1 \text{ if } \mu>0,$$

where $\gamma=\frac{\sigma^2}{8\pi\mu^2}$, *and*

$$\lim_{t\to T}\frac{K-S^c(t)}{\sigma K\sqrt{(T-t)\ln\frac{1}{(T-t)}}}=\sqrt{2} \text{ if } \mu=0.$$

If $\mu=\delta-q<0<\delta$ *the finite-maturity critical price is such that*

$$S^c(T^-)=\lim_{t\to T}S^c(t)=\frac{\delta}{q}K<K$$

with

$$\lim_{t\to T}\frac{S^c(T^-)-S^c(t)}{S^c(T^-)\sigma\sqrt{(T-t)}}=y^*,$$

where $y^*\approx-0.638$ *is the number such that*

$$\phi(y)=\sup_{0\le\Theta\le1}\mathbb{E}\left[\int_0^\Theta(y+B(s))\,ds\right]=0 \qquad (2.12)$$

for all $y\le y^*$ *and* $\phi(y)>0$ *for all* $y>y^*$.

2. *If* $\delta\le0$, *and* $\mu\le0$ *i.e.* $q\ge0$, *then early exercise is never optimal, and the value of the American put option coincides with the European one.*

3. *If* $\delta=0$, *and* $\mu-\frac{\sigma^2}{2}>0$, *then* α *in Equation* (2.10) *becomes* $\alpha=\frac{-2\left(\mu-\frac{\sigma^2}{2}\right)}{\sigma^2}<0$. *There exists a unique critical price* $S^c(t)\le S^c_\infty$, *with* S^c_∞ *defined in Equation* (2.9), *and*

$$S^c(t)-K\sim-K\sigma\sqrt{(T-t)\ln\frac{\sigma^2}{8\pi(T-t)\mu^2}} \text{ as } t\to T$$

4. *If* $\delta<0$,

$$\mu-\frac{\sigma^2}{2}>0, \text{ and } \left(\mu-\frac{\sigma^2}{2}\right)^2+2\delta\sigma^2>0, \qquad (2.13)$$

then the perpetual American put option value v_∞ *is*

$$v_\infty(x) = \begin{cases} A_l \cdot x^{\xi_l} & \text{for } x \in (0; l_\infty) \\ K - x & \text{for } x \in [l_\infty; u_\infty] \\ A_u \cdot x^{\xi_u} & \text{for } x \in (u_\infty; +\infty) \end{cases} \tag{2.14}$$

where $\xi_u < \xi_l$ *are the negative solutions of the equation*

$$\frac{1}{2}\sigma^2\xi^2 + \left(\mu - \frac{\sigma^2}{2}\right)\xi - \delta = 0, \tag{2.15}$$

The critical prices are

$$l_\infty, u_\infty = K\frac{\xi_i}{\xi_i - 1} \quad \text{for } i = l, u \tag{2.16}$$

and the constant A_l *and* A_u *are given by*

$$A_l = -\frac{(l_\infty)^{1-\xi_l}}{\xi_l} \text{ and } A_u = -\frac{(u_\infty)^{1-\xi_u}}{\xi_u}. \tag{2.17}$$

There exist a lower critical price $l(t)$ *and an upper critical price* $u(t)$ *such that*

$$\frac{\delta K}{\delta - \mu} \leq l(t) < u(t) \leq K \tag{2.18}$$

such that the finite-maturity American put option is optimally exercised at t *if* $S(t) \in [l(t), u(t)]$ *and optimally continued if* $S(t) < l(t)$ *or* $S(t) > u(t)$. *Moreover,*

$$u(t) - K \sim -K\sigma\sqrt{(T-t)\ln\frac{\sigma^2}{8\pi(T-t)\mu^2}} \quad \text{for } t \to T \tag{2.19}$$

The lower free boundary satisfies

$$l(t) - \frac{\delta K}{\delta - \mu} \sim \frac{\delta K}{\delta - \mu}\left(-y^*\sigma\sqrt{(T-t)}\right) \quad \text{for } t \to T \tag{2.20}$$

where $y^* \approx -0.638$ *is the number defined in Equation* (2.12).

5. *If $\delta < 0$, and there exists $\overline{x} > 0$ such that the finite-maturity American put option is optimally exercised at $\bar{t} \in (0, T)$ if $S\left(\bar{t}\right) = \overline{x}$, then the segment with extremes*

$$l(t) = \inf\left\{s \geq 0 : v(t, s; \mu, \sigma, \delta, K) = (K - s)^{+}\right\} \tag{2.21}$$

$$u(t) = \sup\left\{s \geq 0 : v(t, s) = (K - s)^{+}\right\} \wedge K \tag{2.22}$$

is non-empty for any $t \in \left[\bar{t}, T\right]$. The option is optimally exercised at any $t \geq \bar{t}$ whenever $S(t) \in [l(t), u(t)]$. The lower and the upper free boundary branches satisfy the inequality (2.18) *for any $t \geq \bar{t}$ as well as the asymptotics* (2.20) *and* (2.19)*.*

Proof. When $\delta > 0$ and $0 < \mu = \delta - q < \delta$ or $\mu < 0 < \delta$, the asymptotics of $S^c(t)$ at maturity are determined by Evans, Kuske and Keller [7], and further improved by De Marco and Henry-Labordè [5]. For the case $\delta > 0$ and $0 = \mu$ the asymptotics are provided in Theorem 3 in Lamberton and Villeneuve [10]. Hence Point 1 and 3 are proved.

If $\delta \leq 0$, and $q = \delta - \mu \geq 0$, then Jensen inequality implies that for any $0 < \Theta \leq T - t$

$$\begin{aligned}
\mathbf{E}^Q \left[e^{-\delta\Theta} \left(K - s \cdot \exp\left(\left(\mu - \frac{\sigma^2}{2} \right) \Theta + \sigma B^Q(\Theta) \right) \right)^{+} \right] \geq \\
\geq e^{-\delta\Theta} \left(K - s \cdot e^{\mu\Theta} \right)^{+} \\
= \left(K e^{-\delta\Theta} - s \cdot e^{-q\Theta} \right)^{+} \\
> \left(K - s \cdot e^{-q\Theta} \right)^{+} \text{ since } e^{-\delta\Theta} > 1 \\
\geq (K - s)^{+} \text{ since } e^{-q\Theta} \leq 1.
\end{aligned}$$

Since $(K - s)^{+}$, the immediate payoff at t, is dominated by the continuation value at any future $0 < \Theta \leq T - t$, exercise is optimal at T only. In the limiting case $\delta = 0$, if $\mu \leq 0$ i.e. $q \geq 0$, then early exercise is never optimal, and the value of the American put option coincides with the European one. Indeed, if $\delta = 0$ and $\mu \leq 0$ Equation (2.15) does not admit any negative solution. Hence Point 2 follows.

If $\delta < 0$, and $q = \delta - \mu < 0$, then early exercise may be optimal even in the perpetual case. The proof follows by Proposition 2.2, and the geometry and the asymptotics for the finite-maturity critical prices can be retrieved by Theorems 2.3 and 2.4 in Battauz, De Donno and Sbuelz [3]. Point 4 follows.

Consider now Point 5. If there exists $\overline{x} > 0$ such that the finite-maturity American put option is optimally exercised at $\overline{t} \in (0, T)$ if $S\left(\overline{t}\right) = \overline{x}$, then early exercise is optimal for all $t \geq \overline{t}$ and $S(t) = \overline{x}$ since

$$(K - \overline{x})^+ \leq v(t, \overline{x}; \mu, \sigma, \delta, K) \leq v(\overline{t}, \overline{x}; \mu, \sigma, \delta, K) = (K - \overline{x})^+ ,$$

where the first inequality follows by the payoff value dominance of the American option, and the second inequality as the American option value is decreasing with respect to time t. Since $v(t, 0; \mu, \sigma, \delta, K) = Ke^{-\delta(T-t)} > K = (K-0)^+$, as $\delta < 0$, it follows that $l(t) > 0$ for all $t \in \left(\overline{t}, T\right)$. The remaining part of Point 5 follows by the proof of Theorems 2.3 and 2.4 in Battauz, De Donno and Sbuelz [3] restricted to $t \in \left[\overline{t}, T\right]$. □

Remark 2.1 *We comment here Assumption* (2.13) . *Intuitively, a sufficient condition for the existence of optimal early exercise of the perpetual put with value* v_∞ *is*

$$\left(\mu - \frac{\sigma^2}{2}\right)^2 + 2\delta\sigma^2 > 0. \tag{2.23}$$

Condition (2.23) *ensures that the function* $v_\infty(x) = As^\alpha$ *has at least one tangency point with the immediate put payoff in the extreme(s) of the early exercise region. This condition is always satisfied when* $\delta \geq 0$. *The tangency* *equation has always two real opposite solutions*

$$\alpha = \frac{-\left(\mu - \frac{\sigma^2}{2}\right) - \sqrt{\left(\mu - \frac{\sigma^2}{2}\right)^2 + 2\delta\sigma^2}}{\sigma^2} < 0$$

$$\alpha_+ = \frac{-\left(\mu - \frac{\sigma^2}{2}\right) + \sqrt{\left(\mu - \frac{\sigma^2}{2}\right)^2 + 2\delta\sigma^2}}{\sigma^2} > 0,$$

no matter of the sign of μ*. Therefore the perpetual American put option admits the representation of Equation* (2.11)*, and it is optimally exercised when the underlying is below the unique critical price* S_∞^c*. If* $\delta < 0$*, Condition* (2.23) *is not always true. Battauz, De Donno and Sbuelz [3] show in Proposition 2.2 (see also [2]) that a sufficient condition for the existence of optimal early exercise of the perpetual put* v_∞ *is* (2.13)*. This conditions ensures the existence of the two real negative roots of the tangency equation* (2.15)*.*

The sufficient condition (2.13) may not be true, even if binomial approximations show that the finite-maturity American put option displays optimal early exercise opportunities. These cases satisfy a necessary condition for early exercise established in Proposition 2.5 in Battauz, De Donno and Sbuelz [3]. For the ease of the reader, we state the necessary condition (2.24) herefollows.

Condition 2.1 ***(necessary condition for early exercise, negative interest rate).*** *If* $\delta < 0$ *and* $\mu > 0$ *a necessary condition for the optimal exercise of the finite-maturity American put option at* $t \in [0; T)$ *is*

$$\mathcal{N}^{-1}\left(e^{\delta(T-t)}\right) - \mathcal{N}^{-1}\left(e^{(\delta-\mu)(T-t)}\right) \geq \sigma\sqrt{T-t}, \tag{2.24}$$

where $\mathcal{N}^{-1}(\cdot)$ *denotes the inverse of the standard normal cumulative distribution function.*

Assumption 2.1 *Condition 2.2 (necessary condition for early*

Conditions (2.24) and (2.13) point in the same direction, requiring the growth rate of the underlying μ to be relatively high compared to the (negative) interest rate δ. However, condition (2.24) is definitely milder than (2.13), as we one can see also in the examples of our numerical section for American quanto put options. In particular, condition (2.24) forces the European put option v_e

$$v_e(t,x;\mu,\sigma,\delta,K) = Ke^{-\delta(T-t)}\mathcal{N}(\overline{z}) - xe^{(\mu-\delta)(T-t)}\mathcal{N}\left(\overline{z} - \sigma\sqrt{(T-t)}\right), \tag{2.25}$$

with $\mathcal{N}(y)$ denoting the distribution function of a standard normal random variable, and $\overline{z} = \left(\ln\frac{K}{x} - \left(\mu - \frac{\sigma^2}{2}\right)(T-t)\right)\frac{1}{\sigma\sqrt{T-t}}$, to fall below the immediate payoff at t for some values of the underlying. In fact, if this is not the case and the European option always dominates the immediate payoff at t for all values of the underlying x, then there is no optimal exercise for the American option at t.

In the next propositions we rewrite the American quanto put options in terms of American put options on a lognormal (onedimensional) security. We start from the option defined in Equation (2.2), whose behavior is unaffected by r_d.

Proposition 2.3 *Consider the American quanto put option defined in Equation* (2.2)*. Then*

$$V(t) = \sup_{t\le\tau\le T}\mathbf{E}^Q\left[e^{-r_d(T-t)}\,X(\tau)\left(K_f - S_f(\tau)\right)^+\Big|\,\mathcal{F}_t\right] =$$
$$= X(t)\cdot v(t,S_f(t);r_f - q_f,\|\sigma_f\|,r_f,K_f),$$

i.e. the American quanto put option price coincides with the foreign American put option price converted at the current spot exchange rate. Therefore, *early exercise is optimal at t if $S_f(t)$ is in the early exercise region of the foreign American put option $v(t, S_f(t); r_f - q_f, \|\sigma_f\|, r_f, K_f)$ as described in Theorem 2.2 with $\delta = r_f$, $\mu = r_f - q_f$, $\sigma = \|\sigma_f\|$, and $K = K_f$.*

Proof. Let $\widetilde{N}(t) = X(t)e^{(r_f - r_d)t}$ the numeraire (see Battauz [1]) associated to the equivalent probability measure Q^N, whose density is

$$\frac{dQ^N}{dQ} = \frac{\widetilde{N}(T)}{\widetilde{N}(0)}.$$

Bayes' theorem implies that

$$\begin{aligned} V(t) &= \sup_{t \le \tau \le T} \mathbf{E}^Q \left[e^{-r_d(\tau - t)} X(\tau) (K_f - S_f(\tau))^+ \middle| \mathcal{F}_t \right] \\ &= \sup_{t \le \tau \le T} \frac{\mathbf{E}^{Q^N} \left[\frac{dQ}{dQ^N} e^{-r_d(\tau - t)} X(\tau) (K_f - S_f(\tau))^+ \middle| \mathcal{F}_t \right]}{\mathbf{E}^{Q^N} \left[\frac{dQ}{dQ^N} \middle| \mathcal{F}_t \right]} \\ &= \sup_{t \le \tau \le T} \frac{\mathbf{E}^{Q^N} \left[\frac{X(0)}{X(\tau)e^{(r_f - r_d)\tau}} e^{-r_d(\tau - t)} X(\tau) (K_f - S_f(\tau))^+ \middle| \mathcal{F}_t \right]}{\frac{X(0)}{X(t)e^{(r_f - r_d)t}}} \\ &= X(t) \sup_{t \le \tau \le T} \mathbf{E}^{Q^N} \left[e^{-r_f(\tau - t)} (K_f - S_f(\tau))^+ \middle| \mathcal{F}_t \right]. \end{aligned}$$

The factor

$$\sup_{t \le \tau \le T} \mathbf{E}^{Q^N} \left[e^{-r_f(\tau - t)} (K_f - S_f(\tau))^+ \middle| \mathcal{F}_t \right]$$

is the foreign price of the American put option on S_f if the Q^N-drift of S_f coincides with the foreign riskless interest rate r_f. Indeed, Girsanov theorem implies that the process

$$dW^N(t) = -\sigma_f \; dt \; + \; dW^Q(t)$$

is a $2-$*dimensional* Q^N *Brownian motion.* Therefore

$$\begin{aligned} \frac{dS_f(t)}{S_f(t)} &= \mu_f^Q dt + \sigma_f \; dW^Q(t) \\ &= \left(r_f - q_f - \sigma_f \sigma_X\right) dt + \sigma_f \; \left(dW^N(t) + \sigma_f \; dt\right) \\ &= (r_f - q_f) \, dt + \sigma_f \; dW^N(t), \end{aligned}$$

and thus the Q^N-distribution of S_f coincides with its foreign risk-neutral distribution. The American quanto option coincides with

$$V(t) = X(t) \cdot v(t, S_f(t); r_f - q_f, \|\sigma_f\|, r_f, K_f),$$

and the rest of the proposition follows by applying Theorem 2.2 with $\delta = r_f$, $\mu = r_f - q_f$, $\sigma = \|\sigma_f\|$, and $K = K_f$. $\square$

In the previous proposition we have shown that the price of the American quanto put option in Equation (2.2) coincides with the foreign American put option price converted at the current spot exchange rate. Intuitively, if the domestic investor buys the option (2.2), she has the right to the get the foreign-denominated payoff $(K_f - S_f(\tau))^+$ whenever exercised at τ with $t \leq \tau \leq T$. The domestic denominated value of the payoff $(K_f - S_f(\tau))^+$ exercised at τ is $X(\tau)(K_f - S_f(\tau))^+$. The same right is obtained by entering a long position on the foreign put with $(K_f - S_f(\tau))^+$ at exercise date τ, whose price at time t in domestic currency units is $X(t) \cdot v(t, S_f(t); r_f, \|\sigma_f\|, r_f, K_f)$.

In Proposition 2.3 we have characterized the optimal exercise policies for the American quanto option defined in Equation (2.2), whose behavior depends only on the foreign riskless rate r_f.

In the next section we focus on American quanto put options defined in Equations (2.3), (2.4), (2.5), (2.6), whose behavior depends on *both* the domestic rate r_d and the foreign rate r_f.

3 American quanto options and the interplay with the sign of the riskless rates

Our first step consists in reducing American quanto put options (2.3), (2.4), (2.5) and (2.6) to American put options. This characterization, which is done in the following lemma, will allow us to work out the optimal exercise policies for American quanto options in Propositions 3.1 and 3.2.

Lemma 3.1 *The no-arbitrage price of the option* (2.3) *can be computed as*

$$\begin{aligned} V(t) &= \sup_{t\le\tau\le T} \mathbf{E}^Q \left[e^{-r_d(T-t)} \left(K_d - X(\tau)S_f(\tau)\right)^+ \middle| \mathcal{F}_t \right] \\ &= v(t, X(t)S_f(t); r_d - q_f, \|\sigma_X + \sigma_f\|, r_d, K_d) \end{aligned}$$

The option in (2.4) *can be computed as*

$$V(t) = v(t, X_0 \cdot S_f(t); \mu_f^Q, \|\sigma_f\|, r_d, K_d)$$

and the option (2.5) *as*

$$V(t) = v(t, F_0 \cdot S_f(t); \mu_f^Q, \|\sigma_f\|, r_d, K_d),$$

and (2.6) *as*

$$\begin{aligned} V(t) &= v(t, X_0 \cdot S_f(t); \mu_f^Q, \|\sigma_f\|, r_d, X_0 \cdot K_f) \\ &= X_0 v(t, S_f(t); \mu_f^Q, \|\sigma_f\|, r_d, K_f). \end{aligned}$$

Proof. The underlying of the option in Equation (2.3) is the lognormal $S_f^*(t) = X(t)S_f(t)$, whose domestic risk neutral drift is r_d and whose volatility vector is $\sigma_X + \sigma_f$ as from Equation (2.1). This yields the formula for (2.3). In the remaining options (2.4) and (2.5) the underlying S_f is denominated in the domestic currency by using the constant initial spot exchange rate X_0 in option (2.4), and the constant initial forward exchange rate F_0 in option (2.5). Thus the domestic risk neutral underlying drift is μ_f^Q, its volatility is σ_f and its initial level is multiplied by the constant X_0 in option (2.4), and the constant initial forward exchange rate F_0 in option (2.5). In the last option (2.6), both the strike price and the foreign security are denominated in the domestic currency by using the constant initial spot exchange rate. Therefore, the domestic risk neutral underlying S_f drift is μ_f^Q, its volatility is σ_f, and both the initial underlying value and the strike price are then multiplied by X_0. Because of the put payoff homogeneity, we then obtain $V(t) = v(t, X_0 \cdot S_f(t); \mu_f^Q, \|\sigma_f\|, r_d, X_0 \cdot K_f)$ $= X_0 v(t, S_f(t); \mu_f^Q, \|\sigma_f\|, r_d, K_f)$. □

In order to describe in detail the optimal exercise policies for options (2.3) , (2.4) , (2.5)and (2.6) we distinguish two main cases.

- Case 1: $r_d \geq 0$.

  This is the traditional assumption on non-negative domestic interest rate. We solve the problem in Proposition 3.1.

- Case 2: $r_d < 0$.

  The case of a negative domestic interest rate has attracted the interest of the financial literature, due to the persistence of negative interest rates in the European and in the Japanese markets. We address it in Proposition 3.2.

Proposition 3.1 *Suppose* $r_d \geq 0$. *Then*

1. *For the option* (2.3) *there exists a critical price at* t, $S^c(t)$ *such that early exercise is optimal at* t *if*

$$X(t)S_f(t) \leq S^c(t).$$

If $r_d > q_f$ *then*

$$\lim_{t \to T} S^c(t) = K_d$$

$$\lim_{t \to T} \frac{K_d - S^c(t)}{\|\sigma_f + \sigma_X\| K_d \sqrt{(T-t) \ln \frac{\gamma}{(T-t)}}} = 1 \text{ where } \gamma = \frac{\|\sigma_f + \sigma_X\|^2}{8\pi (r_d - q_f)^2}.$$

If $q_f = r_d$

$$\lim_{t \to T} \frac{K_d - S^c(t)}{\|\sigma_f + \sigma_X\| K_d \sqrt{(T-t) \ln \frac{1}{(T-t)}}} = \sqrt{2}.$$

If $q_f > r_d > 0$ *we have that*

$$S^c(T^-) = \lim_{t\to T} S^c(t) = \frac{r_d}{q_f} K_d < K_d$$

with

$$\lim_{t\to T} \frac{S^c(T^-) - S^c(t)}{S^c(T^-)\,\|\sigma_f + \sigma_X\|\,\sqrt{(T-t)}} = y^*,$$

where $y^* \approx -0.638$ *is defined in* (2.12).

2. *For the option* (2.4) *there exists a critical price at* t, $S^c(t)$, *such that early exercise is optimal at* t *if*

$$X_0 S_f(t) \le S^c(t).$$

Set $q_* = q_f + r_d - r_f + \sigma_f \sigma_X$. *If* $r_d > q_*$ *i.e.* $q_f - r_f + \sigma_f \sigma_X < 0$ *then*

$$\lim_{t\to T} S^c(t) = K_d$$

with

$$\lim_{t\to T} \frac{K_d - S^c(t)}{\|\sigma_f\|\,K_d\sqrt{(T-t)\ln\frac{\gamma}{(T-t)}}} = 1 \text{ where } \gamma = \frac{\|\sigma_f\|^2}{8\pi\left(r_f - q_f - \sigma_X\sigma_f\right)^2}.$$

If $q_* = r_d$,

$$\lim_{t\to T} \frac{K_d - S^c(t)}{\|\sigma_f\|\,K_d\sqrt{(T-t)\ln\frac{1}{(T-t)}}} = \sqrt{2}$$

If $q_* > r_d > 0$, *i.e.* $q_f - r_f + \sigma_f\sigma_X > 0$, *we have that*

$$S^c(T^-) = \lim_{t\to T} S^c(t) = \frac{r_d}{q_*} K_d < K_d$$

with

$$\lim_{t\to T} \frac{S^c(T^-) - S^c(t)}{S^c(T^-)\,\|\sigma_f\|\,\sqrt{(T-t)}} = y^*,$$

where $y^* \approx -0.638$ *is defined in* (2.12).

3. *For the option* (2.5) *there exists a critical price at* t, $S^c(t)$, *such that early exercise is optimal at* t *if*

$$F_0 S_f(t) \leq S^c(t).$$

The limits and the asymptotics for $S^c(t)$ *at maturity coincide with the ones described for the option* (2.4) *in Point 2.*

4. *For the option* (2.6) *there exists a critical price at* t, $S^c(t)$, *such that early exercise is optimal at* t *if*

$$S_f(t) \leq S^c(t).$$

The limits and the asymptotics for $S^c(t)$ *at maturity coincide with the ones described for the option* (2.4) *in Point 2.*

Proof. The proof follows by applying Theorem 2.2 and Lemma 3.1. In particular Point 1 follows with

$$\delta = r_d,\ \mu = r_d - q_f,\ \sigma = \left\| \sigma_f + \sigma_X \right\|,\ K = K_f.$$

Points 2, 3 and 4 follow with

$$\delta = r_f,\ \mu = r_f - q_f - \sigma_X \sigma_f,\ \sigma = \left\| \sigma_f \right\|,\ K = K_f. \qquad \square$$

Proposition 3.2 *Suppose* $r_d < 0$.

1. *Consider the option* (2.3)*. Its underlying drift is* $r_d - q_f < 0$ *and the American quanto option is optimally exercised at maturity only.*

2. *Consider the option* (2.4). *If* $\mu_f^Q = r_f - q_f - \sigma_f \sigma_X < 0$, *then the American quanto option is optimally exercised at maturity only.*

   If $\mu_f^Q - \frac{\|\sigma_f\|^2}{2} = r_f - q_f - \sigma_f \sigma_X - \frac{\|\sigma_f\|^2}{2} > 0$, *and*

$$\left(r_f - q_f - \sigma_f \sigma_X - \frac{\|\sigma_f\|^2}{2}\right)^2 + 2r_d \|\sigma_f\|^2 > 0, \tag{3.1}$$

   holds (or, resp., if there exists $\overline{x} > 0$ *such that the finite-maturity American quanto put option* (2.4) *is optimally exercised at* $\overline{t} \in (0,T)$ *when* $X_0 S_f(\overline{t}) = \overline{x}$*), then there exist two critical prices at* t, $l(t) < u(t)$, *such that early exercise is optimal at* t *if*

$$l(t) \leq X_0 S_f(t) \leq u(t),$$

   and continuation is optimal at t *if*

$$X_0 S_f(t) < l(t) \text{ or } X_0 S_f(t) > u(t),$$

   for all $t \in [0,T]$ *(resp. for all* $t \in \left[\overline{t}, T\right]$*). Moreover*

$$u(t) - K_d \sim -K_d \|\sigma_f\| \sqrt{(T-t) \ln \frac{\|\sigma_f\|^2}{8\pi (T-t) \left(r_f - q_f - \sigma_f \sigma_X\right)^2}}.$$

   For $t \to T$, *the lower free boundary satisfies*

$$l(t) - \frac{r_d K_d}{r_d - \left(r_f - q_f - \sigma_f \sigma_X\right)} \sim \frac{r_d K_d}{r_d - \left(r_f - q_f - \sigma_f \sigma_X\right)} \left(-y^* \|\sigma_f\| \sqrt{(T-t)}\right),$$

   where $y^* \approx -0.638$ *is defined in* (2.12).

3. *Consider the option* (2.5). *If* $\mu_f^Q = r_f - q_f - \sigma_f \sigma_X < 0$, *then the American quanto option is optimally exercised at maturity only.*

   If $\mu_f^Q = r_f - q_f - \sigma_f \sigma_X > 0$, *and* (3.1) *holds (or, resp., if there exists* $\overline{x} > 0$ *such that the finite-maturity American quanto put option* (2.5) *is optimally exercised at* $\bar{t} \in (0,T)$ *when* $F_0 S_f(\bar{t}) = \bar{x}$*), then there exist two critical prices at* t, $l(t) < u(t)$, *such that early exercise is optimal at* t *if*

$$l(t) \leq F_0 S_f(t) \leq u(t),$$

and continuation is optimal at t *if*

$$F_0 S_f(t) < l(t) \text{ or } F_0 S_f(t) > u(t),$$

for all $t \in [0,T]$ *(resp. for all* $t \in [\bar{t}, T]$*). The limits and the asymptotics of the critical prices at maturity coincide with the ones computed for the option* (2.4)*.*

4. *Consider the option* (2.6)*. If* $\mu_f^Q = r_f - q_f - \sigma_f \sigma_X < 0$, *then the American quanto option is optimally exercised at maturity only.*

   If $\mu_f^Q = r_f - q_f - \sigma_f \sigma_X > 0$, *and* (3.1) *holds (or, resp., if there exists* $\bar{x} > 0$ *such that the finite-maturity American quanto put option* (2.6) *is optimally exercised at* $\bar{t} \in (0,T)$ *when* $S_f(\bar{t}) = \bar{x}$*), then there exist two critical prices at* t, $l(t) < u(t)$, *such that early exercise is optimal at* t *if*

$$l(t) \leq S_f(t) \leq u(t),$$

   and continuation is optimal at t *if*

$$S_f(t) < l(t) \text{ or } S_f(t) > u(t),$$

   for all $t \in [0,T]$ *(resp. for all* $t \in [\bar{t}, T]$*). The limits and the asymptotics of the critical prices at maturity coincide with the previous ones computed for the options* (2.4) *and* (2.5)*.*

Proof. The proof follows by applying Theorem 2.2 and Lemma 3.1, as explained in the proof of Proposition 3.1. □

We observe that the critical prices described in Propositions 3.1 and 3.2 for the American quanto options of Equations (2.3) , (2.4) and (2.5) are all expressed in domestic currency. Early exercise occurs at t if $S_f(t)$, converted in the domestic currency according to the payoff definition, enters the early exercise region determined by the (domestic) critical price. On the contrary, for the American quanto option of Equation (2.6) the critical price is expressed in foreign currency units.

Finite maturity American quanto options may display optimal early exercise opportunities, even if the corresponding perpetual American quanto options do not admit finite perpetual free boundaries. This happens when assumption (3.1) is not verified, but the necessary condition translating condition (2.24) for the different payoff's specifications holds true. We state the condition here follows for the quanto options.

Proposition 3.3 ***(Necessary condition for early exercise of American quanto options when when*** $r_d < 0$ ***and*** $\mu_f^Q = r_f - q_f - \sigma_f \sigma_X > 0$***)*** *A necessary condition for the optimal exercise of the finite-maturity American quanto put option* (2.4)*,* (2.5) *and* (2.6) *is*

$$\mathcal{N}^{-1}\left(e^{r_d T}\right) - \mathcal{N}^{-1}\left(e^{\left(r_d - \mu_f^Q\right)T}\right) \geq \|\sigma_f\| \sqrt{T}. \tag{3.2}$$

where $\sigma_f = \|\sigma_f\|$.

Proof. The necessary condition 2.24 found in Proposition 2.5 in Battauz, De Donno and Sbuelz [3] requires the European put option to fall below its payoff at t for some value of the underlying. For the quanto options this corresponds to the existence of x_m such that

$$v_e(t, x_m; \mu_f^Q, \|\sigma_f\|, r_d, K) = (K - x_m)^+$$

where $K = K_d$ and $x_m = X_0 \cdot S_f(t)$ for the American quanto option (2.4) , $K = K_d$ and $x_m = F_0 \cdot S_f(t)$ for the American quanto option (2.5) , and

$K = K_f$ and $x_m = S_f(t)$ for the American quanto option (2.6). Then the remaining part of the proof of Proposition 2.5 in Battauz, De Donno and Sbuelz [3] follows. Assumption (3.2) is necessary for the existence of optimal exercise opportunities at date $t = 0$. If early exercise is optimal at any date $t \in [0, T]$ for some x_m, it is also optimal for all future dates for the same x_m, as the American quanto options (2.4), (2.5), and (2.6) are decreasing with respect to t. $\square$

Remark 3.1 *In this paper we have focused on American quanto put options. Via the American put-call symmetry, our results do also apply to symmetric American quanto call options (see for instance Proposition 3.1 in Battauz, De Donno and Sbuelz [3]).*

4 Numerical Examples

In this section we provide examples of American quanto options to show how the domestic interest rate contributes in shaping their free boundaries. To streamline our analysis we focus on the American quanto option with payoff (2.6), that can be reduced to the American put option

$$\begin{aligned} V(t) &= \sup_{t \le \tau \le T} \mathbf{E}^Q \left[e^{-r_d(T-t)}\, X_0 \cdot (K_f - S_f(\tau))^+ \middle| \mathcal{F}_t \right] \\ &= X_0 v(t, S_f(t); \mu_f^Q, \|\sigma_f\|, r_d, K_f), \end{aligned}$$

following Lemma 3.1. We first introduce an example with a positive interest rate, and then move to the case of a domestic negative interest rate. Option prices are computed via binomial approximation (see Hull, 2018), setting the upwards and downards coefficients

$$u = e^{\|\sigma_f\| \sqrt{\Delta t}}, \quad d = e^{-\|\sigma_f\| \sqrt{\Delta t}}$$

and the risk-neutral probability of an upwards movement

$$\mathbf{q} = \frac{e^{\mu_f^Q \Delta t} - d}{u - d}$$

We fix

$$r_d = +0.90\%, \quad r_f = 2\%, \quad X(0) = 0.94, \quad \|\sigma_X\| = 7.8\%$$
$$\|\sigma_f\| = 10\%, \quad \rho = -1\%, \quad \text{and } q_f = 0,$$

that deliver

$$\mu_f^Q = r_f - q_f - \rho \|\sigma_f\| \|\sigma_X\| = 2.008\%$$

For an American quanto put option (2.6) deeply in-the-money at inception with $S_f(0) = 0.5$ and $K_f = 1$, maturity $T = 6$ months and $N = 125$ time steps we obtain

$$\Delta t = 0.004, \quad u = 1.006, \quad d = 0,994, \quad \text{and } \mathbf{q} = 50.47\%.$$

We take an initial in-the-money underlying value because we want to investigate what happens within our binomial model in the deeply in-the-money region during the option life. In Figure 1 the binomial tree S_f is delimited by the grey diamonds. We compute the upper (and unique) free boundary by taking at any t the maximum underlying value within the early exercise region at t, namely

$$u(t) = \max\left(S_f(t) : X_0 (K_f - S_f(t))^+ = V(t)\right),$$

among the binomial realizations of $S_f(t)$ at t. The upper free boudary is plotted in Figure 1 with the blue dots, starting after 103 consequent upwards movements, where the continuation region begins. The asymptotic approximation for the free boundary at maturity obtained in Proposition 3.1 is plotted with blue stars and is very closed to the binomial upper free boundary.

The flat perpetual boundary,valued 0.78, is dotted with blue circles in Figure 1. As the initial underlying value $S_f(0)$ is below both the perpetual and the finite maturity free boundary, the initial value of both the perpetual and the finite maturity binomial option is $X_0 \cdot (K_f - S_f(0))^+ = 0.94 \cdot 0.5 = 0.47$.

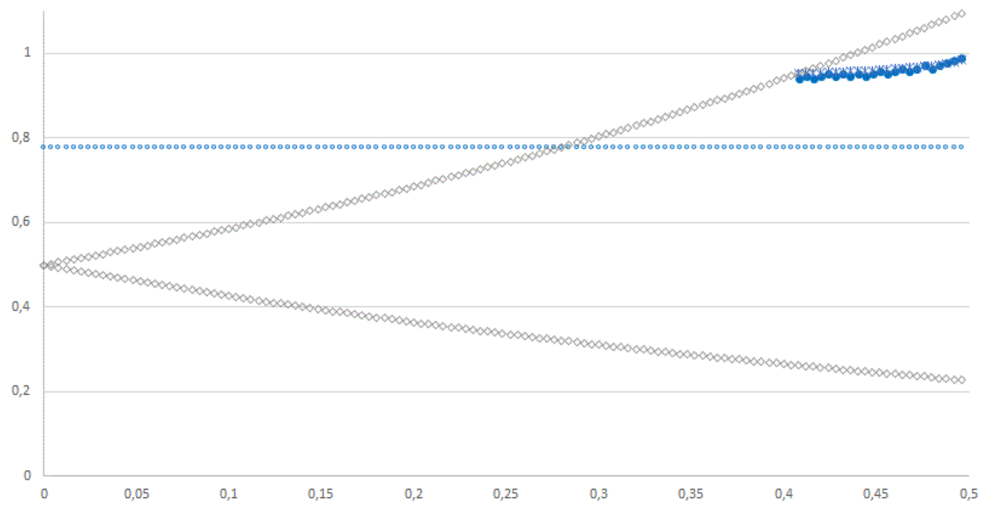

Figure 1. The free boundary when r_d is positive.

We then assign to the domestic riskless interest rate a opposite negative value, $r_d = -0.90\%$, keeping all the other parameters unchanged.

In this case, conditions (3.1) and (3.2) are met. A double continuation region appears. Its existence is remarkable, because it violates the usual property of down-connectedness of the exercise region of put options, that has been established in quite general settings (see Detemple and Tian, 2002). The perpetual lower and upper free boundaires are, resp., 0.45 and 0.68. Again, the price of the perpetual option coincides with its immediate payoff 0.47. The underlying binomial tree is unchanged, as the domestic riskless interest rate $r_d = -0.90\%$ does not enter the domestic risk-neutral dynamics of the foreign risky security. The binomial upper (resp. lower) free boundary is computed by taking at any t the maximum (resp. the minimum) underlying value within the early exercise region at t, namely

$$u(t) = \max\left(S_f(t) : X_0 (K_f - S_f(t))^+ = V(t)\right)$$
$$l(t) = \min\left(S_f(t) : X_0 (K_f - S_f(t))^+ = V(t)\right)$$

among the binomial realizations of $S_f(t)$ at t. In Figure 2 the binomial tree is delimited by grey diamonds. The standard part of the continuation region appears in the upper region of the tree after 103 upwards movements, that push the American Quanto put option towards the out-of-the-money region. The non-standard part of the continuation region appears in the very deeply in-the-money region, below the perpetual lower free boundary, after 73 downwards movements. In Figure 2 we plot with blue (resp. red) dots the upper (resp. lower) binomial free boundary. The asymptotic approximation for the upper (resp. lower) free boundary obtained in Proposition 3.2 are plotted with blue (resp. red) stars. The flat perpetual upper (resp. lower) boundary is plotted with blue (resp. red) circles. As the initial value $S_f(0)$ is within the early exercise region, the initial price of the finite-maturity option coincides with its immediate payoff 0.47.

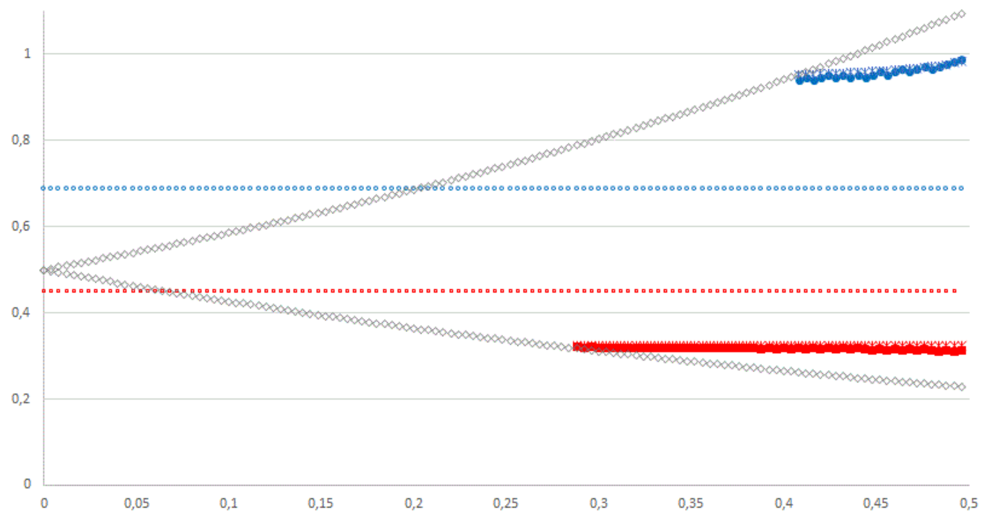

Figure 2. The double free boundaries when r_d is negative.

Finally, we focus on US underlying assets, calibrating our parameters over the period December, 15th, 2015 to December, 15th, 2016, and evaluating Quanto options on December 14th 2016 (data source: Bloomberg). The euro yield curve is negative and the US yield curve is positive, thus fitting into assumptions of 3.2. In particular, on December 14th 2016, the euro and the US yield curves are reported in Table 1.

Table 1: The EU and US Yield Curves

TTM	EU Percent. Yield	US Percent. Yield
30d	-0.97%	0.64%
90d	-0.91%	0.54%
180d	-0.80%	0.69%
270d	-0.80%	
360d	-0.80%	0.89%

Since our model allows us only constant interest rates, we fix the rates at the intermediate level $r_d = -0.80\%$ for the euro, and $r_f = 0.69\%$ for the USD. On the same date, the exchange spot rate $X(0) = 0.94$ (the inverse eurodollar is $1/0.94 = 1.06$). The volatility of the exchange rate X is $\|\sigma_X\| = 8\%$. We select US stocks from different sectors, Johnson and Johnson's (JNJ), Microsoft (MSFT), Amazon (AMZN), and Apple (AAPL.0). The stocks display different levels of volatility and correlation with the exchange rate X over the period under investigation, that are reported in the first two rows of Table 2. The third row displays the stock risk neutral drift $\mu_f^Q = r_f - q_f - \rho \|\sigma_f\| \|\sigma_X\|$, under the assumption $q_f = 0$.

Table 2: Stock parameters values

	JNJ	MSFT	AMZN	AAPL.0
$\|\sigma_f\|$	14%	23%	30%	13.7%
ρ	−0.5%	6%	−1, 7%	−0.09%
μ_f^Q	0.7%	0.6%	0.7%	0.7%

The correlation is slightly negative, but in the Microsoft case, where is positive. Interestingly, Assumption (2.13) that ensures the boundedness of the perpetual American quanto option is never true. On the contrary, the necessary finite- maturity condition (3.2) is satisfied over the 6 months option maturity. American quanto put options (2.6) with 6 months maturity on these stocks display a non-standard deeply in the money continuation region. We provide here the details for an American quanto put option (2.6) on the JNJ stock. We set the maturity $T = 6$, and in-the-money initial values $S_f(0) = 1$ and $K_f = 1.15$. With $N = 125$ time steps we obtain

$$\Delta t = 0.004, \quad u = 1.009, \quad d = 0,992, \quad \text{and } \mathbf{q} = 49.94\%.$$

In Figure 3 the binomial tree for the JNJ stock is delimited by the grey diamonds. The blue (resp. red) dots denote the upper (resp. the lower) binomial free boundary. As from Proposition 3.2, the upper free boundary converges at maturity to the strike price $K_f = 1.15$. The left-limit of the lower free boundary at T is 0.6. The asymptotic approximation for the upper (resp. lower) free boundary obtained in Proposition 3.2 are plotted with blue (resp. red) stars. The asymptotic approximations for both the upper and lower free boundary are very closed to the binomial boundaries over the entire option life. There exist no perpetual constant barriers in this case, as Assumption (2.13) is violated and the perpetual option is unbounded.

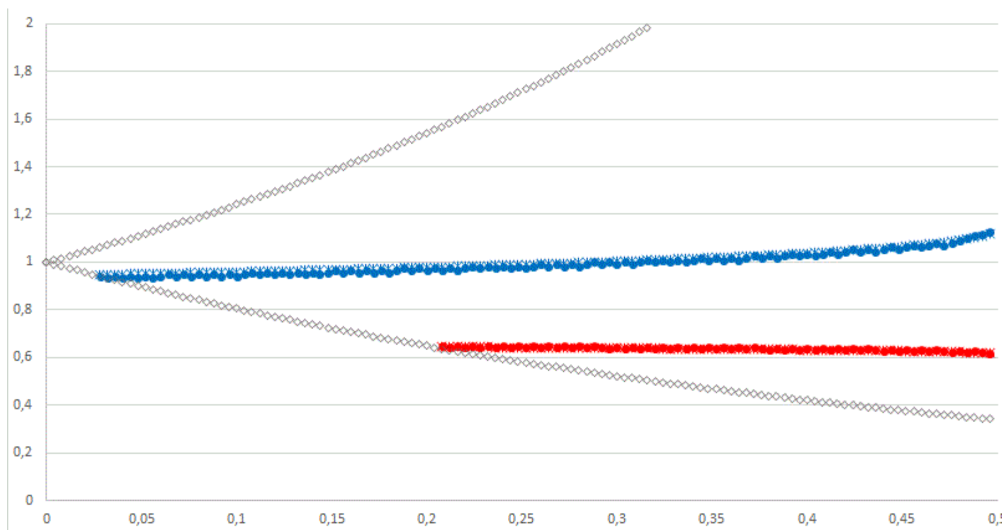

Figure 3. Quanto put option on JNJ stock. The early exercise region is delimited between the dotted blue and red lines (the blue and red stars denote the asymptotic approximations provided in Proposition 3.2).

These examples show how a non-standard double continuation region appears for finite maturity American quanto put options under real circumstances. Interestingly, when the maturity of these options tends to infinite, the value of the perpetual American quanto put options becomes unbounded, because the tendence to infinite postponement, due to negative domestic interest rates, prevails. Hence in the perpetual case early exercise is never optimal, the early exercise region is empty, and there is no free boundary.

5 Conclusion

In a diffusive currency market model we have studied the interplay of the signs of the domestic and the foreign riskfree rate with the optimal exercise policies for American quanto options. In particular, we have shown that, given a positive foreign riskless rate, a negative domestic riskless rate (as it is currently true for the European and the Japanese markets), may lead to the existence of a double continuation region for American quanto options written on a foreign risky security. We have also shown examples of finite maturity American quanto put options that exhibit a double continuation region surrounding a non-empty early exercise region even if the perpetual early exercise region is empty and the value of the perpetual option is unbounded. In this case, we have also provided accurate asymptotic approximations for the free boundaries at maturity.

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