Intrinsic risk measures

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Abstract

Monetary risk measures classify a financial position by the minimal amount of external capital that must be added to the position to make it acceptable.
We propose a new concept: intrinsic risk measures. The definition via external capital is avoided and only internal resources appear. An intrinsic risk measure is defined by the smallest percentage of the currently held financial position which has to be sold and reinvested in an eligible asset such that the resulting position becomes acceptable.
We show that this approach requires less nominal investment in the eligible asset to reach acceptability. It provides a more direct path from unacceptable positions towards the acceptance set and implements desired properties such as monotonicity and quasi-convexity solely through the structure of the acceptance set. We derive a representation on cones and a dual representation on convex acceptance sets and we detail the connections of intrinsic risk measures to their monetary counterparts.

Keywords: intrinsic risk measures, monetary risk measures, acceptance sets, coherence, conicity, quasi-convexity, value at risk

1. Introduction

Risk measures associated with acceptance criteria as introduced by P. Artzner, F. Delbaen, J. Eber, and D. Heath [1] are maps $\rho_{\mathcal{A},r}$ from a function space $\mathcal{X} \subseteq \mathbb{R}^\Omega$ to $\mathbb{R}$ of the form

$$\rho_{\mathcal{A},r}(X_T) = \inf \{m \in \mathbb{R} \mid X_T + mr\mathbf{1}_\Omega \in \mathcal{A}\} . \tag{1.1}$$

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These maps are means to measure the 'risk' of a financial position $X_T \in \mathcal{X}$ with respect to certain acceptability criteria and a risk-free investment. The latter are specified as a subset $\mathcal{A} \subset \mathcal{X}$, the *acceptance set*, and the risk-free return rate $r > 0$, respectively. Geometrically[1], the risk of an unacceptable position $X_T \in \mathcal{X} \setminus \mathcal{A}$ in Equation (1.1) is defined as a scalar 'distance' to the acceptance set in direction $r\mathbf{1}_\Omega$. Such risk measures are known as *cash-additive* risk measures. Evidently, the acceptance set forms the primary object, whereas the risk-free asset contributes only a constant factor. More recent research has revisited the original idea using eligible assets with random return rates $r : \Omega \to \mathbb{R}_{>0}$, as for example P. Artzner, F. Delbaen, and P. Koch-Medina [2] and D. Konstantinides and C. Kountzakis [10]. W. Farkas, P. Koch-Medina, and C. Munari [7], [8] focus on general eligible assets $r : \Omega \to \mathbb{R}_{\geq 0}$, revealing significant shortcomings of the simplified constant approach. They point out that an appropriate interplay between eligible assets and acceptance sets is crucial for a consistent and successful risk measurement. They incorporate eligible assets as traded assets $S = (S_0, S_T)$ with initial unitary price $S_0 \in \mathbb{R}_{>0}$ and random payoff $S_T : \Omega \to \mathbb{R}_{\geq 0}$, and replace $r\mathbf{1}_\Omega$ in Equation (1.1) by the random return S_T/S_0. This alteration yields the extended definition

$$\rho_{\mathcal{A},S}(X_T) = \inf \left\{ m \in \mathbb{R} \,\middle|\, X_T + \tfrac{m}{S_0} S_T \in \mathcal{A} \right\}. \tag{1.2}$$

Beside the geometric interpretation of $\frac{m}{S_0} S_T$ as a 'vector' it is economically interpreted as the payoff of $\frac{m}{S_0}$ units of asset S.
The more general definition in (1.2) can be consistently reduced to (1.1) if S_T is bounded away from zero, this means if $S_T \geq \varepsilon$, for some $\varepsilon > 0$.[2] This constitutes the basis for the simplified approach with constant return. However, payoffs of relevant financial instruments such as defaultable bonds and options do not satisfy this condition, and thus, the generalisation to *S-additive* risk measures in (1.2) is justified.

Referring to eligible assets, P. Artzner, F. Delbaen, J. Eber, and D. Heath suggest in [1], Section 2.1, p. 205 that

> *The current cost of getting enough of this or these [commonly accepted] instrument(s) is a good candidate for a measure of risk of the initially unacceptable position.*'

Both cash-additive and S-additive risk measures are conceptually in line with this suggestion, and we broadly refer to them as *monetary risk measures*[3]. This is a suitable name as these risk measures are defined as actual money which can be used to buy the eligible asset. Hence, they can be interpreted as more than just measurement tools. Referring to cash-additivity (or Axiom T), P. Artzner, F. Delbaen, J. Eber, and D. Heath claim in [1], Remark 2.7, p. 209 that

> '*By insisting on references to cash and to time, [...] our approach goes much further than the interpretation [...] that "the main function of a risk measure is to properly rank risks."*'

[1] See Figure 1a for a visual example.
[2] See [7], Section 1, p. 146ff. for a detailed discussion.
[3] A definition is given in Section 2.2.

The application of this approach requires to raise the monetary amount $\rho_{\mathcal{A},S}(X_T)$ and carry it in the eligible asset S. However, the possible acquisition of additional capital is not completely accounted for by monetary risk measures. This raises the questions as to what effect this has on the risk measure and to which extent this method is applicable in reality.
Another approach is to restructure the portfolio and directly raise capital from the current position to invest it in the eligible asset, as was already mentioned in [1], Section 2.1, p. 205:

'*For an unacceptable risk [...] one remedy may be to alter the position.*'

The aim of this article is to reflect about this thought and develop it towards a new class of risk measures, which we will call *intrinsic risk measures*. For great adaptability, we develop our approach based on acceptance sets $\mathcal{A} \subset \mathcal{X}$ as primary objects and the extended framework of general eligible assets $S = (S_0, S_T) \in \mathbb{R}_{>0} \times \mathcal{A}$.

In the 'future wealth' approach described in [1], p. 205, it is not possible to change the current financial position, representing the principle of 'bygones are bygones'. The authors argue that the knowledge of the initial value of the position is not needed. So the risk measure is only used to determine the size of the buffer with respect to the eligible asset which sufficiently absorbs losses of this fixed position. However, we believe that a reconstruction of the financial position is possible and beneficial, since losses are not absorbed but essentially reduced as the eligible asset becomes part of the position. The intention to sell part of the current position requires the knowledge of the initial value. So while monetary risk measures are defined on $\mathcal{X}$, intrinsic risk measures take the initial value $X_0 \in \mathbb{R}_{>0}$ into account and are defined on $\mathbb{R}_{>0} \times \mathcal{X}$. For financial positions $X = (X_0, X_T)$ the intrinsic risk measure is given by

$$R_{\mathcal{A},S}(X) = \inf \left\{ \lambda \in [0,1] \,\middle|\, (1-\lambda)X_T + \lambda \tfrac{X_0}{S_0} S_T \in \mathcal{A} \right\}. \tag{1.3}$$

In words, we search for the smallest $\lambda \in [0,1]$ such that selling the fraction λ of our initial position and investing the monetary amount λX_0 in the eligible asset S yields an acceptable position. Using the convex combination $(1-\lambda)X_T + \lambda \frac{X_0}{S_0} S_T$, $\lambda \in [0,1]$, instead of $X_T + \frac{m}{S_0} S_T$, $m \in \mathbb{R}$, changes the form of risk measures and suggests a new way to shift unacceptable positions towards the acceptance set.[4] Furthermore, standard properties such as monotonicity and, in contrast to monetary risk measures, also quasi-convexity are imposed solely through the structure of the underlying acceptance set.

The subsequent work has grown from the master's thesis of A. Smirnow [12]. We will introduce acceptance sets and traditional risk measures, give economic motivation, and review important properties in Section 2 to build a foundation for comparison. In Section 3, we define the new class of intrinsic risk measures and we derive basic properties. We derive an alternative representation on cones and show that intrinsic

[4] See Figure b.

risk measures require less investment in the eligible asset to yield acceptable positions. Finally, we study a dual representation of intrinsic risk measures on convex acceptance sets.

2. Terminology and preliminaries

In this section, we establish the foundations on which we can build our framework. Common terminology such as acceptance sets and traditional risk measures are introduced and discussed.
Throughout this chapter we work on an atomless probability space $(\Omega, \mathcal{F}, \mathbb{P})$. For the sake of exposition we consider financial positions on the space of essentially bounded random variables $\mathcal{X} = L^\infty(\Omega, \mathcal{F}, \mathbb{P})$ endowed with the $\mathbb{P}$-almost sure order and the $\mathbb{P}$-essential supremum norm. The majority of our results can be stated on arbitrary ordered real topological vector spaces.

2.1. *Acceptance sets*

In the financial world, it is a central task to hold positions that satisfy certain acceptability criteria, may they represent own preferences or be of regulatory nature. These criteria can be brought into a mathematical framework via what is known as acceptance sets.

Definition 2.1. *A subset $\mathcal{A} \subset \mathcal{X}$ is called an* acceptance set *if it satisfies*

1. Non-triviality*: $\mathcal{A} \neq \emptyset$ and $\mathcal{A} \subsetneq \mathcal{X}$, and*
2. Monotonicity*: $X_T \in \mathcal{A}$, $Y_T \in \mathcal{X}$, and $Y_T \geq X_T$ imply $Y_T \in \mathcal{A}$.*

An element $X_T \in \mathcal{A}$ is called $\mathcal{A}$-acceptable*, or just* acceptable *if the reference to $\mathcal{A}$ is clear. Similarly, we say $X_T \notin \mathcal{A}$ is* ($\mathcal{A}$-)unacceptable.

Non-triviality is mathematically important and also representative of real world requirements, as generally not every situation is acceptable and any event requires near-term reactions. Monotonicity implements the idea that any financial position dominating an acceptable position must be acceptable. These two axioms constitute the basis for acceptance sets and reflect the 'minimal' human rationale.
Depending on the context, it is often necessary to impose further structure and we recall three relevant properties.

Definition 2.2. *An acceptance set $\mathcal{A} \subset \mathcal{X}$ is called*

- *a* cone *or* conic *if $X_T \in \mathcal{A}$ implies for all $\lambda > 0 : \lambda X_T \in \mathcal{A}$,*
- convex *if $X_T, Y_T \in \mathcal{A}$ implies for all $\lambda \in [0,1] : \lambda X_T + (1-\lambda) Y_T \in \mathcal{A}$,*
- closed *if $\mathcal{A} = \bar{\mathcal{A}}$.*

The cone property allows for arbitrary scaling of financial positions invariant of their acceptability status. Convexity represents the principle of diversification: given two acceptable positions, any convex combination of these will be acceptable. In Section 2.2, we will see how these two properties translate to monetary risk measures. Finally, closedness is of mathematical importance when considering limits of sequences of acceptable positions. Apart from this, it is economically motivated as it prohibits arbitrarily small perturbations to make unacceptable positions acceptable.

The next lemma summarises some useful properties of acceptance sets, which will be used in subsequent sections.

Lemma 2.3. *Let $\mathcal{A} \subset \mathcal{X}$ be an acceptance set. Then*

1. $\mathcal{A}$ contains sufficiently large constants but no sufficiently small constants.
2. $S_T \in \operatorname{int}(\mathcal{A})$ if and only if there exists an $\varepsilon > 0$ such that $S_T - \varepsilon \mathbf{1}_\Omega \in \mathcal{A}$.
3. The interior $\operatorname{int}(\mathcal{A})$ and the closure $\bar{\mathcal{A}}$ are both acceptance sets, and $\operatorname{int}(\mathcal{A}) = \operatorname{int}(\bar{\mathcal{A}})$.
4. If $\mathcal{A}$ is a cone, then $\operatorname{int}(\mathcal{A})$ and $\bar{\mathcal{A}}$ are cones, and $0 \notin \operatorname{int}(\mathcal{A})$ and $0 \in \bar{\mathcal{A}}$.

Proof. 1. Since $\mathcal{A}$ is a nonempty, proper subset of $\mathcal{X}$, the first assertion follows from monotonicity of $\mathcal{A}$.
2. The second assertion also follows directly from monotonicity of $\mathcal{A}$.
3. The proof of the third assertion goes along the lines of the proof of Lemma 2.3 in [8], p. 60 and is omitted here.
4. Given $S_T \in \operatorname{int}(\mathcal{A})$, Assertion 2 together with the cone property imply $\lambda(S_T - \varepsilon \mathbf{1}_\Omega) \in \mathcal{A}$, for some $\varepsilon > 0$ and all $\lambda > 0$. The other direction of Assertion 2 implies $\lambda S_T \in \operatorname{int}(\mathcal{A})$. Given $S_T \in \bar{\mathcal{A}}$, take a sequence $\{S_T^n\}_{n\in\mathbb{N}} \subset \mathcal{A}$ with limit S_T. Then conicity implies $\{\lambda S_T^n\}_{n\in\mathbb{N}} \subset \mathcal{A}$, for any $\lambda > 0$, and we conclude that λS_T belongs to $\bar{\mathcal{A}}$. The last two claims follow by similar arguments. □

We conclude this section with the well-known example of the *Value-at-Risk acceptance set.*

Example 2.4 (Value-at-Risk acceptance)**.** *For any probability level $\alpha \in \left(0, \frac{1}{2}\right)$ the set*

$$\mathcal{A}_\alpha = \{X_T \in \mathcal{X} \,|\, \mathbb{P}[X_T < 0] \le \alpha\}$$

defines a closed, conic acceptance set which, in general, is not convex.

Indeed, a few calculations show that $\mathcal{A}_\alpha$ is a conic acceptance set. For closedness in $L^\infty(\mathbb{P})$ consider a sequence $\{X_T^n\}_{n\in\mathbb{N}} \subset \mathcal{A}_\alpha$ converging to some X_T. For any $\delta > 0$ and any $n \in \mathbb{N}$ the following inequality holds,

$$\begin{aligned}\mathbb{P}[X_T < -\delta] &= \mathbb{P}[X_T < -\delta\,, X_T^n < -\tfrac{\delta}{2}] + \mathbb{P}[X_T < -\delta\,, X_T^n \ge -\tfrac{\delta}{2}] \\ &\le \alpha + \mathbb{P}[|X_T^n - X_T| > \tfrac{\delta}{2}]\,.\end{aligned}$$

Since norm convergence implies convergence in probability, letting $n \to \infty$ we get $\mathbb{P}[X_T < -\delta] \le \alpha$. It follows $\mathbb{P}[X_T < 0] = \lim_{\delta\to 0} \mathbb{P}[X_T < -\delta] \le \alpha$. To show that $\mathcal{A}_\alpha$ is not convex, we use its conicity to reduce the problem to finding $X_T, Y_T \in \mathcal{A}_\alpha$ such that $X_T + Y_T \notin \mathcal{A}_\alpha$. For two disjoint subsets $A, B \in \mathcal{F}$ with $\mathbb{P}[A] = \mathbb{P}[B] = \alpha$ the choices $X_T = -\mathbf{1}_A$ and $Y_T = -\mathbf{1}_B$ yield the desired inequality.

2.2. *Traditional risk measures*

Traditional risk measures, commonly known as just risk measures, are instruments to measure risk in the financial world. Acceptance sets determine the meaning of 'good' and 'bad', acceptable or not. Traditional risk measures refine this differentiation and allow us to rank financial positions with respect to their distance to the acceptance set. To clearly distinguish between these risk measures and intrinsic risk measures, we define the broad class of traditional risk measures following [1], Definition 2.1, p. 207.

Definition 2.5. *A* traditional risk measure *is a map from* $\mathcal{X}$ *to* $\mathbb{R}$*.*

In Section 3, we will see that intrinsic risk measures are defined on $\mathbb{R}_{>0} \times \mathcal{X}$.
In what follows we recall some well-known traditional risk measures. For the remainder of this section, let X_T, Y_T, Z_T and $\mathbf{r} = r\mathbf{1}_\Omega$ be elements of $\mathcal{X}$, and let ρ denote a traditional risk measure.

2.2.1. *Coherent risk measures*

Coherent risk measures form the historical foundation of modern risk measure theory. P. Artzner, F. Delbaen, J. Eber, and D. Heath define them in [1], Definition 2.4, p. 210 by the following set of axioms. A traditional risk measure is called *coherent* if it satisfies

- *Decreasing Monotonicity*: $X_T \geq Y_T$ implies $\rho(X_T) \leq \rho(Y_T)$,
- *Cash-additivity*: for $m \in \mathbb{R}$ we have $\rho(X_T + m\mathbf{r}) = \rho(X_T) - m$,
- *Positive Homogeneity*: for $\lambda \geq 0$ we have $\rho(\lambda X_T) = \lambda\rho(X_T)$, and
- *Subadditivity*: $\rho(X_T + Y_T) \leq \rho(X_T) + \rho(Y_T)$.

Monotonicity allows us to rank financial positions according to their risk. It is cash-additivity that constitutes the basis for the interpretation of a risk measure as an additionally required amount of capital. Adding this capital to the financial position, its risk becomes 0, since by cash-additivity, $\rho(X_T + \rho(X_T)\mathbf{r}) = 0$. These assumptions seem natural in the context of capital requirements and they are truly characterised by the term *monetary risk measures*, as coined by H. Föllmer and A. Schied in [9], Definition 4.1, p. 153.

2.2.2. *Convex risk measures*

Positive homogeneity, however, may not be satisfied, as risk can behave in non-linear ways. A possible variation is the following property around which H. Föllmer and A. Schied [9] base their discussion of risk measures.

- *Convexity*: for all $\lambda \in [0, 1]$ we have

$$\rho(\lambda X_T + (1 - \lambda)Y_T) \leq \lambda\rho(X_T) + (1 - \lambda)\rho(Y_T).$$

A short calculation reveals that under positive homogeneity, subadditivity and convexity are equivalent. H. Föllmer and A. Schied decide in [9], Definition 4.4, p. 154 to drop the homogeneity axiom and replace subadditivity by convexity, and call the result a *convex measure of risk* – a convex monetary risk measure.

The axioms we have seen so far form a canonical connection to our acceptance sets.

Proposition 2.6. *Any monetary risk measure $\rho : \mathcal{X} \to \mathbb{R}$ defines via*

$$\mathcal{A}_\rho = \{X_T \in \mathcal{X} \,|\, \rho(X_T) \leq 0\} \tag{2.1}$$

an acceptance set. Moreover, if ρ is positive homogeneous, then $\mathcal{A}_\rho$ is a cone, and if ρ is convex, then $\mathcal{A}_\rho$ is convex.

On the other hand, each acceptance set $\mathcal{A}$ defines a monetary risk measure

$$\rho_{\mathcal{A}}(X_T) = \inf\{m \in \mathbb{R} \,|\, X_T + m\mathbf{r} \in \mathcal{A}\}. \tag{2.2}$$

Similarly, if $\mathcal{A}$ is a cone, then $\rho_{\mathcal{A}}$ is positive homogeneous, and if $\mathcal{A}$ is convex, then $\rho_{\mathcal{A}}$ is convex.
In particular, this means $\rho_{\mathcal{A}_\rho} = \rho$ and $\mathcal{A} \subseteq \mathcal{A}_{\rho_{\mathcal{A}}}$, with equality $\mathcal{A} = \mathcal{A}_{\rho_{\mathcal{A}}}$ if the acceptance set is closed.

Proof. The proof goes along the lines of the proofs of Proposition 4.6 and Proposition 4.7 in [9], p. 155f. for bounded measurable functions on $(\Omega, \mathcal{F})$, and is omitted here. □

Proposition 2.6 allows us to define acceptance sets via known risk measures and vice versa. Example 2.7 illustrates how properties can be inferred. A more general version of Proposition 2.6 is stated in Proposition 2.10.

Example 2.7 (Value at Risk acceptance)**.** *For a given probability level $\alpha \in \left(0, \frac{1}{2}\right)$ we define the risk measure* Value-at-Risk (VaR_α) *for all random variables on* $(\Omega, \mathcal{F})$ *by*

$$\mathrm{VaR}_\alpha(X_T) = \inf\{m \in \mathbb{R} \,|\, \mathbb{P}[X_T + m < 0] \leq \alpha\},$$

the negative of the α-quantile of X_T. Corresponding to Proposition 2.6, the VaR_α-*acceptance set is given by*

$$\mathcal{A}_{\mathrm{VaR}_\alpha} = \{X_T \in \mathcal{X} \,|\, \mathrm{VaR}_\alpha(X_T) \leq 0\}.$$

Recalling the closed, conic set $\mathcal{A}_\alpha = \{X_T \in \mathcal{X} \,|\, \mathbb{P}[X_T < 0] \leq \alpha\}$ from Example 2.4, we find that it defines the Value-at-Risk via Equation (2.2)*. So with Proposition 2.6 we conclude that $\mathcal{A}_\alpha = \mathcal{A}_{\mathrm{VaR}_\alpha}$ and that* VaR_α *is a positive homogeneous monetary risk measure which, in general, is not convex, and thus, not coherent.*

Convexity also allows for an alternative treatment of risk measures. The rich literature on convex functional analysis finds convenient application in the theory of risk measures. And risk measures are enriched with a dual representation and more possibilities of interpretation.
We recall two important results for completeness and for the comparison to the intrinsic dual representation in Section 3.4. The first one is given in [9], Theorem 4.31, p. 172.

Theorem 2.8. *Let $\mathcal{M}_\sigma(\mathbb{P}) = \mathcal{M}_\sigma(\Omega, \mathcal{F}, \mathbb{P})$ be the set of all σ-additive probability measures on $\mathcal{F}$ which are absolutely continuous with respect to $\mathbb{P}$. Let $\mathcal{A} \subset \mathcal{X}$ be a convex, $\sigma(L^\infty, L^1)$-closed (weak*-closed) acceptance set. Let $\rho_\mathcal{A}$ be defined as in Equation (2.2) with $\mathbf{r} = \mathbf{1}_\Omega$. The risk measure has the representation*

$$\rho_\mathcal{A}(X_T) = \sup_{\mathbb{Q} \in \mathcal{M}_\sigma(\mathbb{P})} \big\{ \mathbb{E}_\mathbb{Q}[-X_T] - \alpha_{\min}(\mathbb{Q}, \mathcal{A}) \big\}, \tag{2.3}$$

with the minimal penalty function $\alpha_{\min}$ *defined for all $\mathbb{Q} \in \mathcal{M}_\sigma(\mathbb{P})$ by*

$$\alpha_{\min}(\mathbb{Q}, \mathcal{A}) = \sup_{X_T \in \mathcal{A}} \mathbb{E}_\mathbb{Q}[-X_T]. \tag{2.4}$$

Theorem 2.8 can now be directly applied to coherent risk measures, which of course are convex and positive homogeneous. But one can additionally show that with positive homogeneity we can restrict the supremum to a subset $\mathcal{M} \subset \mathcal{M}_\sigma(\mathbb{P})$ on which $\alpha_{\min}(\cdot, \mathcal{A}) = 0$. For further details see [9], Corollary 4.18 and Corollary 4.34, p. 165 and p. 175.

Corollary 2.9. *Let $\mathcal{A}$ be a conic, convex, $\sigma(L^\infty, L^1)$-closed acceptance set. Define the subset $\mathcal{M} = \{\mathbb{Q} \in \mathcal{M}_\sigma(\mathbb{P}) \,|\, \alpha_{\min}(\mathbb{Q}, \mathcal{A}) = 0\}$. Then the coherent risk measure $\rho_\mathcal{A} : \mathcal{X} \to \mathbb{R}$ can be written as*

$$\rho_\mathcal{A}(X_T) = \sup_{\mathbb{Q} \in \mathcal{M}} \mathbb{E}_\mathbb{Q}[-X_T].$$

2.2.3. *Cash-subadditivity and quasi-convexity of risk measures*

N. El Karoui and C. Ravanelli [6] point out that in presence of *stochastic interest rates* a financial position must be discounted before a cash-additive risk measure is applied. Consequently, the axiom of cash-additivity relies on the assumption that the discounting process does not carry additional risk. To relax this restriction they suggest the property of *cash-subadditivity*, where the equality in the cash-additivity condition is changed to the inequality '$\geq$'. However, S. Cerreia-Vioglio, F. Maccheroni, M. Marinacci and L. Montrucchio [3] explain that under cash-subadditivity, convexity is not a rigorous representative of the diversification principle, which translates into the following requirement for risk measures.

- *Diversification Principle*: if $\rho(X_T), \rho(Y_T) \leq \rho(Z_T)$ is satisfied, then

$$\text{for all } \lambda \in [0,1] : \rho(\lambda X_T + (1-\lambda) Y_T) \leq \rho(Z_T).$$

Substituting $\rho(Z_T)$ by $\max\{\rho(X_T), \rho(Y_T)\}$ yields the equivalent and recently importance gaining property of

- *Quasi-convexity*: for all $\lambda \in [0,1]$ we have

$$\rho(\lambda X_T + (1-\lambda)Y_T) \leq \max\{\rho(X_T), \rho(Y_T)\}\,.$$

Interestingly, quasi-convexity is equivalent to convexity under cash-additivity. Indeed, for any two positions with $\rho(X_T) \leq \rho(Y_T)$ we find an $m \in \mathbb{R}_{\geq 0}$ such that $\rho(X_T - m\mathbf{r}) = \rho(Y_T)$ so that for any $\lambda \in [0,1]$ we get

$$\begin{aligned}\rho(\lambda X_T + (1-\lambda)Y_T) + \lambda m &\leq \max\{\rho(X_T - m\mathbf{r}), \rho(Y_T)\} \\ &= \lambda\rho(X_T) + (1-\lambda)\rho(Y_T) + \lambda m\,.\end{aligned}$$

This equivalence does not hold under cash-subadditivity as shown in [12], Example 2.10, p. 12, resulting in the necessity to explicitly implement the diversification principle and thus, in the introduction of cash-subadditive, quasi-convex risk measures.

2.2.4. *General monetary risk measures*

Stochastic interest rates can also be directly addressed through risk measures of the form

$$\rho_{\mathcal{A},S}(X_T) = \inf\Big\{m \in \mathbb{R} \,\Big|\, X_T + \tfrac{m}{S_0} S_T \in \mathcal{A}\Big\}, \tag{2.5}$$

as introduced in [7] and [8]. This approach avoids implicit discounting, since the stochastic eligible asset is now part of the risk measure. C. Munari provides a broad discussion of the discounting argument, revealing fundamental issues with discounting in the context of acceptance sets in [11], Section 1.3, p. 26.

Equation (2.5) defines a generalised monetary risk measure which satisfies the following property for its defining eligible asset $S = (S_0, S_T)$,

- *S-additivity*: for $m \in \mathbb{R}$ we have $\rho_{\mathcal{A},S}(X_T + mS_T) = \rho_{\mathcal{A},S}(X_T) - mS_0$.

This general setup also yields the equivalence of quasi-convexity and convexity, and it exhibits a similar correspondence between acceptance sets and risk measures. The following result extends Proposition 2.6 to stochastic eligible assets.

Proposition 2.10. *Proposition 2.6 holds true if we replace $L^\infty(\Omega, \mathcal{F}, \mathbb{P})$ by any real ordered topological vector space, cash-additivity by S-additivity, and Equation* (2.2) *by Equation* (2.5)*, for any eligible asset $S = (S_0, S_T) \in \mathbb{R}_{>0} \times \mathcal{A}$.*

Proof. See the proofs of propositions 3.2.3, 3.2.4, 3.2.5, and 3.2.8 in [11], p. 87f.. The second claim in Proposition 2.6 follows from two short calculations. □

3. Intrinsic risk measures

The risk measures in the previous section all yield the same procedure to make an unacceptable position X_T acceptable – raise the required 'minimal' capital $\rho_{\mathcal{A},S}(X_T)$ and get the acceptable position $X_T^\rho := X_T + \frac{\rho_{\mathcal{A},S}(X_T)}{S_0} S_T$. A procedure to acquire the required capital-level and the risk of failing to obtain it are not addressed by these risk measures. But what if we do not use external capital?

3.1. *Fundamental concepts*

In this section, we explore a different procedure to obtain acceptable positions. We suggest to sell part of the risky position and invest the acquired capital in the acceptable eligible asset. Hereby, the distance to the acceptance set is directly reduced and therefore also the risk.

In order to sell our original position we require the knowledge of the initial value $X_0 \in \mathbb{R}_{>0}$. Following the definition of general eligible assets $S = (S_0, S_T) \in \mathbb{R}_{>0} \times \mathcal{A}$ in Section 2.2.4, we consider financial positions $X = (X_0, X_T)$ on the product space $\mathbb{R}_{>0} \times \mathcal{X}$. The main object in this approach is the net worth of the convex combination of the risky position and a multiple of the eligible asset

$$X_T^{\lambda,S} := (1-\lambda)X_T + \lambda \frac{X_0}{S_0} S_T \in \mathcal{X}\,,\ \lambda \in [0,1].$$

The notation $X_T^{\lambda,S}$ is convenient and we extend it to the whole position $X \in \mathbb{R}_{>0} \times \mathcal{X}$ as

$$X^{\lambda,S} := (X_0, X_T^{\lambda,S}) \in \mathbb{R}_{>0} \times \mathcal{X}.$$

Hence, $X^{\lambda,S}$ describes a position with initial value X_0 which is split in $(1-\lambda)X_0$ and λX_0 and is then invested to get $(1-\lambda)X_T$ and $\lambda \frac{X_0}{S_0} S_T$, respectively. We aim to find the smallest λ such that $X_T^{\lambda,S}$ is acceptable, this defines the intrinsic risk measure.

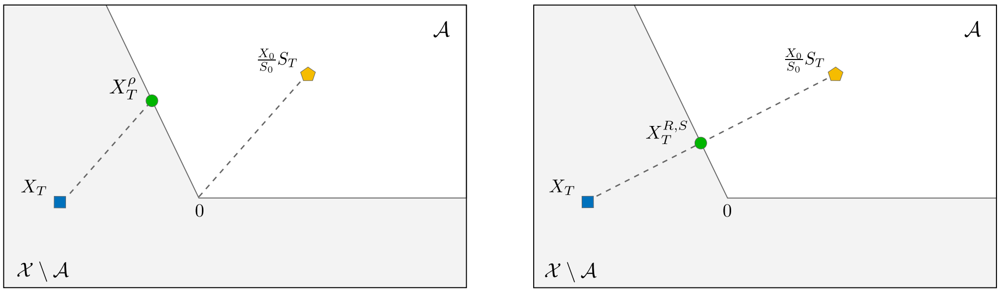
Definition 3.1 (Intrinsic Risk Measure)**.** *For an acceptance set $\mathcal{A} \subset \mathcal{X}$ and an eligible asset $S \in \mathbb{R}_{>0} \times \mathcal{A}$ the intrinsic risk measure is a map $R_{\mathcal{A},S} : \mathbb{R}_{>0} \times \mathcal{X} \to [0,1]$ defined by*

$$R_{\mathcal{A},S}(X) = \inf \left\{ \lambda \in [0,1] \,\middle|\, X_T^{\lambda,S} \in \mathcal{A} \right\}. \tag{3.1}$$

For well-definedness two short considerations yield that the acceptance set must either be a cone or that 0 must be contained in it[5]. In both cases, $\lambda \frac{X_0}{S_0} S_T$ is acceptable for $\lambda \in (0,1]$, or $\lambda \in [0,1]$ if $\mathcal{A}$ is closed. This means selling all of the original position leaves us always with an acceptable net worth $\frac{X_0}{S_0} S_T$.

A brief comparison of the intrinsic approach and the traditional monetary approach is provided below. Consider the conceptual Figure 1 and imagine that $\mathcal{A}$ is an arbitrary closed acceptance set.

[5]The assumption $0 \in \mathcal{A}$ is widely used in the financial literature, as for example the equivalent Axiom 2.1 in [1], p. 206 or, if $\mathcal{A}$ is closed, the *normalisation property* $\rho(0) = 0$ in [9], above Remark 4.2, p. 154.



(a) Traditional approach (b) Traditional approach

Figure 1: The payoff of the eligible asset (yellow ⬠) is used to make the unacceptable position (blue □) acceptable (green ○).

While the monetary approach, illustrated in Figure a, yields the position $X_T^\rho := X_T + \frac{\rho_{\mathcal{A},S}(X_T)}{S_0} S_T$, the intrinsic approach, illustrated in Figure b, gives us

$$X_T^{R_{\mathcal{A},S}(X),S} := (1 - R_{\mathcal{A},S}(X))X_T + R_{\mathcal{A},S}(X)\frac{X_0}{S_0}S_T,$$

which we abbreviate with $X_T^{R,S}$ if the reference to $\mathcal{A}, S$, and X is clear.

1. We notice that, since $\mathcal{A}$ is closed, both risk measures are strictly positive if and only if $X_T \notin \mathcal{A}$. In this case, and if $S_T \in \operatorname{int}(\mathcal{A})$, both altered positions X_T^ρ and $X_T^{R,S}$ lie on the boundary of the acceptance set. Moreover, if $\mathcal{A}$ is either a cone or convex with $0 \in \mathcal{A}$, then the set $\{X_T^{\lambda,S} \,|\, \lambda \in [R_{\mathcal{A},S}(X), 1]\}$ belongs to $\mathcal{A}$. A similar results holds true for monetary risk measures.

2. If we assume a conic acceptance set as in Figure 1, we intuit that $X_T^{R,S}$ must be a multiple of X_T^ρ. And indeed, in Corollary 3.7 we will derive the relation

$$X_T^{R_{\mathcal{A},S}(X),S} = (1 - R_{\mathcal{A},S}(X))X_T^\rho. \tag{3.2}$$

3. By Definition 3.1, it is apparent that intrinsic risk measures cannot attain infinite values as opposed to traditional risk measures. W. Farkas, P. Koch-Medina, and C. Munari have shown in [8], Theorem 3.3 and Corollary 3.4, p. 62 that on closed, conic acceptance sets

$$\rho_{\mathcal{A},S} \text{ is finite if and only if } S_T \in \operatorname{int}(\mathcal{A}).$$

For a graphical illustration imagine that in Figure 1, $S_T \in \partial\mathcal{A}$. Then in Figure a, a possible X_T^ρ would move along a line 'parallel' to the boundary, thus it would never reach $\mathcal{A}$. Consequently, $\rho_{\mathcal{A},S}(X_T) = +\infty$ and X_T^ρ is actually not defined.
In contrast, one can show[6] that on closed, conic acceptance sets

$$R_{\mathcal{A},S} < 1 \text{ on } \mathbb{R}_{>0} \times \mathcal{X} \setminus \mathcal{A} \text{ if and only if } S_T \in \operatorname{int}(\mathcal{A})\,.$$

[6]For a direct proof one can use Lemma 2.3 and the fact that $X_T^{R,S} \in \mathcal{A}$. For a proof via monetary risk measures consider Theorem 3.3 and Corollary 3.5 below.

Hence, if $S_T \in \partial\mathcal{A}$ in Figure b, then $X_T^{R,S}$ and $\frac{X_0}{S_0}S_T$ coincide on the boundary with $R_{\mathcal{A},S}(X) = 1$.

Having established a basic intuition for this approach, we will now take a deeper look at some of its properties. For this we introduce the notions of monotonicity and convexity on $\mathbb{R}_{>0} \times \mathcal{X}$.

1. The monotonicity of $\mathcal{A}$ should be reflected by the corresponding intrinsic risk measure. So we need to extend the ordering on $\mathcal{X}$ to $\mathbb{R}_{>0} \times \mathcal{X}$. Two possible orderings are *element-wise* and *return-wise* defined respectively by

$$X \geqslant_{\text{el}} Y \;\text{ if }\; X_0 \geq Y_0 \text{ and } X_T \geq Y_T, \;\text{ and}$$
$$X \geqslant_{\text{re}} Y \;\text{ if }\; \frac{X_T}{X_0} \geq \frac{Y_T}{Y_0}.$$

2. On $\mathbb{R}_{>0} \times \mathcal{X}$, we think of convex combinations element-wise as

$$\alpha X + (1-\alpha)Y := (\alpha X_0 + (1-\alpha)Y_0\,,\, \alpha X_T + (1-\alpha)Y_T) \in \mathbb{R}_{>0} \times \mathcal{X}.$$

We can now show monotonicity and quasi-convexity of intrinsic risk measures with respect to these rules.

Proposition 3.2 (Monotonicity, Quasi-convexity)**.** *Let $\mathcal{A}$ be an acceptance set containing* 0*, let $S \in \mathbb{R}_{>0} \times \mathcal{A}$ be an eligible asset and let $X, Y \in \mathbb{R}_{>0} \times \mathcal{X}$.*

1. The orders $X \geqslant_{el} Y$ and, on conic acceptance sets, $X \geqslant_{re} Y$, imply $R_{\mathcal{A},S}(X) \leq R_{\mathcal{A},S}(Y)$.

2. Let $\mathcal{A}$ be additionally convex. Then $R_{\mathcal{A},S}$ is quasi-convex, that means for all $\alpha \in [0,1]$, and any $X, Y \in \mathbb{R}_{>0} \times \mathcal{X}$

$$R_{\mathcal{A},S}(\alpha X + (1-\alpha)Y) \leq \max\{R_{\mathcal{A},S}(X), R_{\mathcal{A},S}(Y)\}.$$

Proof. 1. If $X \geqslant_{\text{el}} Y$, then $X_T^{\lambda,S} \geq Y_T^{\lambda,S}$ and thus, by monotonicity of the acceptance set, $R_{\mathcal{A},S}(X) \leq R_{\mathcal{A},S}(Y)$. Similarly, $X \geqslant_{\text{re}} Y$ implies $X_T^{\lambda,S} \geq \frac{X_0}{Y_0} Y_T^{\lambda,S}$. By conicity we have $\frac{X_0}{Y_0} Y_T^{R(Y),S} \in \mathcal{A}$ and again by monotonicity we get $X_T^{\lambda,S} \in \mathcal{A}$.
2. Assume without loss of generality $R_{\mathcal{A},S}(X) \leq R_{\mathcal{A},S}(Y)$. As mentioned above, since $\mathcal{A}$ is convex, $\{X_T^{\lambda,S} \,|\, \lambda \in [R_{\mathcal{A},S}(X), 1]\} \subset \mathcal{A}$. Hence, if $\lambda \in [R_{\mathcal{A},S}(Y), 1]$, then the convex combinations $Y_T^{\lambda,S}, X_T^{\lambda,S}$ lie in $\mathcal{A}$ and also their convex combinations $\alpha X_T^{\lambda,S} + (1-\alpha)Y_T^{\lambda,S} \in \mathcal{A}$, for all $\alpha \in [0,1]$. But these convex combinations commutate so that

$$\begin{aligned} R_{\mathcal{A},S}(\alpha X + (1-\alpha)Y) &= \inf\left\{\lambda \in [0,1] \,\middle|\, \alpha X_T^{\lambda,S} + (1-\alpha)Y_T^{\lambda,S} \in \mathcal{A}\right\} \\ &\leq R_{\mathcal{A},S}(Y) = \max\left\{R_{\mathcal{A},S}(X), R_{\mathcal{A},S}(Y)\right\}, \end{aligned}$$

showing quasi-convexity of the intrinsic risk measure. □

So while monotonicity of $\mathcal{A}$ is passed on to underlying intrinsic risk measures, convexity of the acceptance set implies quasi-convexity and not convexity of the measures as we have seen in Proposition 2.6 for monetary risk measures. A counter-example to convexity can be constructed with the transition property for unacceptable X and $\alpha \in [0, R_{\mathcal{A},S}(X)]$,

$$R_{\mathcal{A},S}(X^{\alpha,S}) = \frac{R_{\mathcal{A},S}(X) - \alpha}{1-\alpha}\,,$$

which can be derived using the bijection $[0,1] \to [\alpha, 1]$ with $\lambda \mapsto (1-\lambda)\alpha + \lambda$, and the fact that $(1-\beta)X + \beta X^{\alpha,S} = X^{\alpha\beta,S}$. With help of Example 2.4 it can be shown that convexity of $\mathcal{A}$ is necessary for quasi-convexity of the intrinsic risk measure. Finally, a similar argument yields quasi-convexity with respect to eligible assets $S^1, S^2 \in \mathbb{R}_{>0} \times \mathcal{A}$ with same initial price $S_0^1 = S_0^2$,

$$R_{\mathcal{A},\alpha S^1+(1-\alpha)S^2}(X) \leq \max\{R_{\mathcal{A},S^1}(X), R_{\mathcal{A},S^2}(X)\}\,.$$

3.2. *Representation on conic acceptance sets*

In this section, we will use cash- or S-additivity of monetary risk measures to derive an alternative representation of intrinsic risk measures on cones. This representation allows us to apply important results from monetary to intrinsic risk measures.

Theorem 3.3 (Representation on cones)**.** *Let $\rho_{\mathcal{A},S} : \mathcal{X} \to \mathbb{R}$ be a monetary risk measure defined by a closed, conic acceptance set $\mathcal{A}$ and an eligible asset $S \in \mathbb{R}_{>0} \times \mathcal{A}$. Then the intrinsic risk measure with respect to $\mathcal{A}$ and S can be written as*

$$R_{\mathcal{A},S}(X) = \frac{(\rho_{\mathcal{A},S}(X_T))^+}{X_0 + \rho_{\mathcal{A},S}(X_T)}. \tag{3.3}$$

Proof. Since $\mathcal{A}$ is closed, we can use Proposition 2.10 to write

$$R_{\mathcal{A},S}(X) = \inf\{\lambda \in [0,1] \,|\, X_T^{\lambda,S} \in \mathcal{A}\} = \inf\{\lambda \in [0,1] \,|\, \rho_{\mathcal{A},S}(X_T^{\lambda,S}) \leq 0\}.$$

But $\rho_{\mathcal{A},S}$ is S-additive and positive homogeneous, so that we have

$$R_{\mathcal{A},S}(X) = \inf\big\{\lambda \in [0,1] \,|\, \rho_{\mathcal{A},S}(X_T) \leq \lambda\big(X_0 + \rho_{\mathcal{A},S}(X_T)\big)\big\}.$$

If $\rho_{\mathcal{A},S}(X_T) > 0$, then we can solve for λ to get the form in Equation (3.3). If $\rho_{\mathcal{A},S}(X_T) \leq 0$, then $X_T \in \mathcal{A}$ and therefore $R_{\mathcal{A},S}(X) = 0$. We abbreviate these two cases with $(\rho_{\mathcal{A},S}(X_T))^+$ in the numerator. □

Example 3.4. *For continuous X_T and constant eligible assets $S_T = rS_0\mathbf{1}_\Omega > 0$ we can directly derive the representation in Equation* (3.3) *on the conic Value-at-Risk acceptance set $\mathcal{A}_\alpha = \{X_T \in \mathcal{X} \,|\, \mathbb{P}[X_T < 0] \leq \alpha\}$ from Example 2.7. Let F_X be the* *continuous cumulative distribution function of X_T with inverse F_X^{-1}. For $X_T \notin \mathcal{A}_\alpha$, this means $F_X^{-1}(\alpha) < 0$, we get*

$$\begin{aligned} R_{\mathcal{A}_\alpha,S}(X) &= \inf\left\{\lambda \in (0,1) \,|\, \mathbb{P}[X_T^{\lambda,S} < 0] \leq \alpha\right\} \\ &= \inf\left\{\lambda \in (0,1) \,|\, F_X(-(1-\lambda)^{-1}\lambda r X_0) \leq \alpha\right\} \\ &= \frac{F_X^{-1}(\alpha)}{F_X^{-1}(\alpha) - rX_0} = \frac{VaR_\alpha(X_T)}{rX_0 + VaR_\alpha(X_T)}, \end{aligned}$$

an expression similar to Equation (3.3). Of course, while we use the constant eligible asset $S_T = rS_0\mathbf{1}_\Omega$, the Value-at-Risk is of the form $\rho_{\mathcal{A}_\alpha}(X) = \inf\{m \in \mathbb{R} \,|\, X_T + m\mathbf{1}_\Omega \in \mathcal{A}_\alpha\}$ with $r = 1$.

In our opinion, Theorem 3.3 is a very convenient result that allows us to draw connections to traditional risk measures. This is true for all conic acceptance sets, including the commonly used Value-at-Risk and Expected Shortfall acceptance sets. In particular, some important results from traditional risk measures can be directly applied to intrinsic risk measures.

Corollary 3.5. *Let $\mathcal{A}$ be a closed, conic acceptance set.*

1. *$R_{\mathcal{A},S} < 1$ on $\mathbb{R}_{>0} \times \mathcal{X} \setminus \mathcal{A}$ if and only if $S_T \in \operatorname{int}(\mathcal{A})$.*
2. *If $S_T \in \operatorname{int}(\mathcal{X}_+)$, then $R_{\mathcal{A},S}$ is continuous on $\mathbb{R}_{>0} \times \mathcal{X}$.*
3. *If $\mathcal{A}$ is additionally convex, then $S_T \in \operatorname{int}(\mathcal{A})$ implies continuity of $R_{\mathcal{A},S}$.*
4. *$R_{\mathcal{A},S}$ is scale-invariant, meaning $R_{\mathcal{A},S}(\alpha X) = R_{\mathcal{A},S}(X)$, for $\alpha > 0$.*

Proof. 1. With the representation in Theorem 3.3 and the finiteness result in [8], Theorem 3.3, p. 62 the assertion follows directly.
2. By [7], Proposition 3.1, p. 154, if $S_T \in \operatorname{int}(\mathcal{X}_+)$, then $\rho_{\mathcal{A},S}$ is continuous. The map $f : (x_0, x) \mapsto \frac{x^+}{x_0+x}$ is jointly continuous on $\mathbb{R}_{>0} \times \mathbb{R}$. Therefore, as the composition of two continuous maps the intrinsic risk measures is continuous on $\mathbb{R}_{>0} \times \mathcal{X}$.
3. In this case, [7], Theorem 3.16, p. 159 gives us continuity of $\rho_{\mathcal{A},S}$. The assertion follows as in the second part.
4. If $X_T \in \mathcal{A}$, then so is αX_T and thus, $R_{\mathcal{A},S}(\alpha X) = R_{\mathcal{A},S}(X) = 0$. If $X_T \notin \mathcal{A}$, then $\rho_{\mathcal{A},S}(X_T) > 0$ and the assertion follows from positive homogeneity of $\rho_{\mathcal{A},S}$ and Theorem 3.3. □

Another version of Theorem 3.3 is the representation of monetary risk measures on $\mathcal{X} \setminus \mathcal{A}$ in terms of intrinsic risk measures.

Corollary 3.6. *Let $\mathcal{A}$ be a closed, conic acceptance set, $S \in \mathbb{R}_{>0} \times \operatorname{int}(\mathcal{A})$ and $X = (X_0, X_T) \in \mathbb{R}_{>0} \times \mathcal{X} \setminus \mathcal{A}$. Then*

$$\rho_{\mathcal{A},S}(X_T) = \frac{X_0 R_{\mathcal{A},S}(X)}{1 - R_{\mathcal{A},S}(X)}. \tag{3.4}$$

Proof. We have $\rho_{\mathcal{A},S}(X_T) > 0$ on $\mathcal{X} \setminus \mathcal{A}$ and by Corollary 3.5, $S_T \in \operatorname{int}(\mathcal{A})$ implies $R_{\mathcal{A},S} < 1$ on $\mathbb{R}_{>0} \times \mathcal{X} \setminus \mathcal{A}$. Setting $X = (X_0, X_T)$, for any $X_0 > 0$, and rearranging Equation (3.3) yields the assertion. □

With this representation we confirm our claim that $X_T^\rho = X_T + \frac{\rho_{\mathcal{A},S}(X_T)}{S_0} S_T$ is a multiple of $X_T^{R,S}$.

Corollary 3.7. *In the setting of Corollary 3.6, we have*

$$X_T^{R_{\mathcal{A},S}(X),S} = (1 - R_{\mathcal{A},S}(X)) X_T^\rho . \tag{3.5}$$

Proof. Dividing $X_T^{R,S}$ by $1 - R_{\mathcal{A},S}(X)$ and using Equation (3.4) yields the desired relation. □

The representation in (3.3) does not hold for convex, non-conic acceptance sets. However, it does give us an upper bound.

Proposition 3.8. *Let $\mathcal{A}$ be a closed, convex acceptance set containing* 0*, which is not a cone. Then the following inequality holds,*

$$R_{\mathcal{A},S}(X) \leq \frac{(\rho_{\mathcal{A},S}(X_T))^+}{X_0 + \rho_{\mathcal{A},S}(X_T)}. \tag{3.6}$$

Proof. Using Proposition 2.10, we establish with S-additivity, and then convexity and the fact that $\rho_{\mathcal{A},S}(0) \leq 0$ the inequality

$$\rho_{\mathcal{A},S}(X_T^{\lambda,S}) = \rho_{\mathcal{A},S}((1-\lambda)X_T) - \lambda X_0 \leq (1-\lambda)\rho_{\mathcal{A},S}(X_T) - \lambda X_0.$$

With this we arrive at the inclusion

$$\{\lambda \in [0,1] \,|\, (1-\lambda)\rho_{\mathcal{A},S}(X_T) - \lambda X_0 \leq 0\} \subseteq \{\lambda \in [0,1] \,|\, \rho_{\mathcal{A},S}(X_T^{\lambda,S}) \leq 0\},$$

which implies (3.6). □

3.3. *Efficiency of the intrinsic approach*

In the previous section, we have derived all necessary results to compare the intrinsic and the traditional approach on a monetary basis. We find that on conic or convex acceptance sets the intrinsic approach requires less investment in eligible assets. But on cones it yields positions with the same performance.

Corollary 3.9. *Let $\mathcal{A}$ be a closed acceptance set, either conic or convex. For an unacceptable position $X = (X_0, X_T)$ and an eligible asset S we have*

$$X_0 R_{\mathcal{A},S}(X) \leq \rho_{\mathcal{A},S}(X_T).$$

Proof. With Theorem 3.3 for conic acceptance sets, and Proposition 3.8 for the convex case we establish $X_0 R_{\mathcal{A},S}(X) \leq X_0 \frac{\rho_{\mathcal{A},S}(X_T)}{X_0 + \rho_{\mathcal{A},S}(X_T)}$. For unacceptable X_T the inequality $X_0 \frac{\rho_{\mathcal{A},S}(X_T)}{X_0 + \rho_{\mathcal{A},S}(X_T)} \leq \rho_{\mathcal{A},S}(X_T)$ holds true, proving the assertion. □

So while the magnitude of the initial value X_0 controls the required monetary amount, Corollary 3.9 shows us that the amount $X_0 R_{\mathcal{A},S}(X)$ is always less than $\rho_{\mathcal{A},S}(X_T)$. This means using the intrinsic approach, less capital is transitioned to the eligible asset.

But since less money is invested in the eligible asset, one could think that the intrinsic approach yields worse acceptable positions compared to the traditional approach. However, comparing the resulting positions in terms of returns, for example with the (revised) Sharpe ratio, shows otherwise.
Given a financial position $X = (X_0, X_T)$, a monetary risk measure yields the acceptable position $X_T^\rho = X_T + \frac{\rho_{\mathcal{A},S}(X_T)}{S_0} S_T$. This means that at inception, the initial value must be $X_0^\rho := X_0 + \rho_{\mathcal{A},S}(X_T)$. On the other hand, an intrinsic risk measure does not change the initial value X_0 to get the acceptable position $X_T^{R,S}$. Interestingly, the returns of these positions are equal on cones.

Corollary 3.10. *Let $\mathcal{A}$ be a closed, conic acceptance set, X an unacceptable position, and S an eligible asset. The returns of the positions $(X_0, X_T^{R(X),S})$ and (X_0^ρ, X_T^ρ) are equal.*

Proof. Dividing both sides of Equation (3.5) by X_0 and using Equation (3.3) yield the assertion. □

3.4. *Dual representations on convex acceptance sets*

Referring to duality results of convex and coherent risk measures stated in Section 2.2.2, we derive a dual representation of intrinsic risk measures. The derivation is based on a representation of convex acceptance sets by $\mathcal{M}_\sigma(\mathbb{P})$, the set of σ-additive, absolutely continuous probability measures $\mathbb{Q} \ll \mathbb{P}$, similar to that of S. Drapeau and M. Kupper in [4], Lemma 2, p. 52.

Lemma 3.11. *Let $\mathcal{A}$ be a $\sigma(L^\infty, L^1)$-closed, convex acceptance set. Then $X_T \in \mathcal{A}$ if and only if for all probability measures $\mathbb{Q} \in \mathcal{M}_\sigma(\mathbb{P})$*

$$\inf_{Y_T \in \mathcal{A}} \mathbb{E}_\mathbb{Q}[Y_T] \leq \mathbb{E}_\mathbb{Q}[X_T].$$

Proof. The 'only if' implication is evidently true. We outline the proof of the 'if' direction. Using a version of the Hahn-Banach Separation Theorem, see for example N. Dunford and J. T. Schwartz [5], Theorem V.2.10, p. 417, one shows that for any $X_T \in \mathcal{X} \setminus \mathcal{A}$ there is a linear functional ℓ in the topological dual space $\mathcal{X}^*$ such that $\inf_{y \in \mathcal{A}} \ell(y) > \ell(x)$. The structure of $\mathcal{A}$ implies that ℓ is positive on the positive cone $\{X_T \in \mathcal{X} \mid X_T \geq 0\}$. Under the weak*-topology $\sigma(L^\infty, L^1)$, using the Radon-Nikodým Theorem, as for example stated in [5], Theorem III.10.2, p. 176, these linear functionals can be identified with expectations with respect to σ-additive, absolutely continuous probability measures $\mathbb{Q} \ll \mathbb{P}$ in $\mathcal{M}_\sigma(\mathbb{P})$. □

Using this result we can now derive a dual representation for intrinsic risk measures.

Theorem 3.12 (Dual representation)**.** *Let $\mathcal{A}$ be a $\sigma(L^\infty, L^1)$-closed, convex acceptance set containing* 0 *and let S be an eligible asset. For $\mathbb{Q} \in \mathcal{M}_\sigma(\mathbb{P})$ define the* penalty function[7] $\alpha(\mathbb{Q}, \mathcal{A}) = \inf_{X_T \in \mathcal{A}} \mathbb{E}_\mathbb{Q}[X_T]$*. The intrinsic risk measure can be written as*

$$R_{\mathcal{A},S}(X) = \sup_{\mathbb{Q}\in\mathcal{M}_\sigma(\mathbb{P})} \frac{(\alpha(\mathbb{Q},\mathcal{A}) - \mathbb{E}_\mathbb{Q}[X_T])^+}{\frac{X_0}{S_0}\mathbb{E}_\mathbb{Q}[S_T] - \mathbb{E}_\mathbb{Q}[X_T]}. \tag{3.7}$$

Proof. By Lemma 3.11, we have the equivalence $X_T^{\lambda,S} \in \mathcal{A}$ if and only if for all $\mathbb{Q} \in \mathcal{M}_\sigma(\mathbb{P}) : \mathbb{E}_\mathbb{Q}\big[X_T^{\lambda,S}\big] \geq \alpha(\mathbb{Q},\mathcal{A})$, or rewritten,

$$\lambda \mathbb{E}_\mathbb{Q}\big[\tfrac{X_0}{S_0} S_T - X_T\big] \geq \alpha(\mathbb{Q},\mathcal{A}) - \mathbb{E}_\mathbb{Q}[X_T].$$

For $X_T \in \mathcal{A}$, Lemma 3.11 directly implies that the infimum over λ is equal to 0, for all $\mathbb{Q} \in \mathcal{M}_\sigma(\mathbb{P})$. For $X_T \notin \mathcal{A}$, Lemma 3.11 gives the inequality $\mathbb{E}_\mathbb{Q}[\frac{X_0}{S_0}S_T] - \mathbb{E}_\mathbb{Q}[X_T] \geq \alpha(\mathbb{Q},\mathcal{A}) - \mathbb{E}_\mathbb{Q}[X_T] > 0$ so that we can solve for λ and get

$$\begin{aligned} R_{\mathcal{A},S}(X) &= \inf \left\{ \lambda \in [0,1] \,\middle|\, \forall \mathbb{Q} \in \mathcal{M}_\sigma(\mathbb{P}) :\, \lambda \geq \frac{\alpha(\mathbb{Q},\mathcal{A}) - \mathbb{E}_\mathbb{Q}[X_T]}{\frac{X_0}{S_0}\mathbb{E}_\mathbb{Q}[S_T] - \mathbb{E}_\mathbb{Q}[X_T]} \right\} \\ &= \sup_{\mathbb{Q}\in\mathcal{M}_\sigma(\mathbb{P})} \frac{\alpha(\mathbb{Q},\mathcal{A}) - \mathbb{E}_\mathbb{Q}[X_T]}{\frac{X_0}{S_0}\mathbb{E}_\mathbb{Q}[S_T] - \mathbb{E}_\mathbb{Q}[X_T]}. \end{aligned}$$

From here the representation in (3.7) follows. □

It is interesting to find the same terms in the numerator in Equation (3.7) and the expression in Equation (2.3). But here, the numerator is normalised by an expected distance between financial position and eligible asset before the supremum over $\mathcal{M}_\sigma(\mathbb{P})$ is taken.
In case of a conic acceptance set and a constant eligible asset, we can link Theorem 3.12 via the dual representation of coherent risk measures in Corollary 2.9 to Theorem 3.3.

Corollary 3.13. *Let $\mathcal{A}$ be a $\sigma(L^\infty, L^1)$-closed, convex cone and $S_T = S_0\mathbf{1}_\Omega$. Then we recover the representation in Equation* (3.3)*.*

Proof. A short calculation confirms that on cones, $\alpha(\mathbb{Q},\mathcal{A}) = \lambda\alpha(\mathbb{Q},\mathcal{A})$ is satisfied for all $\lambda > 0$, and thus, $\alpha(\mathbb{Q},\mathcal{A}) \in \{0, \pm\infty\}$. Using Theorem 3.12, but taking the supremum over $\mathcal{M} = \{\mathbb{Q} \in \mathcal{M}_\sigma(\mathbb{P}) \,|\, \alpha(\mathbb{Q},\mathcal{A}) = 0\}$, yields

$$R_{\mathcal{A},S}(X) = \sup_{\mathbb{Q}\in\mathcal{M}} \frac{(\mathbb{E}_\mathbb{Q}[-X_T])^+}{X_0 + \mathbb{E}_\mathbb{Q}[-X_T]}.$$

But for any constant $c > 0$ the map $x \mapsto \frac{x}{c+x}$ is increasing on $\mathbb{R}_{\geq 0}$ and therefore, we can split the supremum and then use the dual representation of coherent risk measures from Corollary 2.9 to get

$$R_{\mathcal{A},S}(X) = \frac{\sup_{\mathbb{Q}\in\mathcal{M}}(\mathbb{E}_\mathbb{Q}[-X_T])^+}{X_0 + \sup_{\mathbb{Q}\in\mathcal{M}} \mathbb{E}_\mathbb{Q}[-X_T]} = \frac{(\rho_{\mathcal{A},S}(X_T))^+}{X_0 + \rho_{\mathcal{A},S}(X_T)},$$

the representation of intrinsic risk measures on cones from Theorem 3.3. □

[7] The negative of the minimal penalty function $\alpha_{\min}$ in Equation (2.4).

4. Conclusion

In this article, we have extended the methodology of risk measurement with a new type of risk measure: the intrinsic risk measure. We argued that since traditional risk measures are defined via hypothetical external capital, it is natural to consider risk measures that only allow the usage of internal capital contained in the financial position.

We discussed basic properties of intrinsic risk measures and provided some examples. We derived an alternative representation on conic acceptance sets, such as the ones associated with Value-at-Risk and Expected Shortfall. With this we showed that the intrinsic approach requires less investment in the eligible asset, and at the same time yields acceptable positions with the same performance. As the representation on cones does not hold on convex acceptance sets, we established a dual representation in terms of σ-additive probability measures.

Finally, we mention two ideas for further studies. First of all, the extension to general ordered topological vector spaces is necessary to provide greater adaptivity. The setting with multiple financial positions and multiple eligible assets should be studied in the context of portfolio rearrangement and how the intrinsic risk measure could help the process of optimisation.

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