

A Theory of Government Bailouts in a Heterogeneous Banking System

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Abstract

How should a government bail out a heterogeneous banking system subject to systemic self-fulfilling runs? To answer this question, we develop a theory of banking with multiple groups of depositors of different size and wealth, where systemic self-fulfilling runs emerge as a consequence of a global game, and a government uses a public good to bailout banks through liquidity injections. In this framework, we characterize the endogenous probability of a systemic self-fulfilling run, and the conditions under which a full bailout cannot be part of the equilibrium. The optimal bailout strategy should target those banks whose bailout has the largest marginal impact on the probability of a systemic self-fulfilling run, and whose depositors are at the lower end of the wealth distribution.

JEL: D81, G01, G21, G28

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1. Introduction

In 2007-2009, the US economy experienced one of the largest financial crises ever seen, that came to be interpreted as a systemic self-fulfilling bank run (Gorton and Metrick 2012). The conventional policy response against bank runs has generally been through the government commitment to insure deposits up to a certain limit. Yet, in this period, we also observed one of the largest government bailouts of the banking system ever seen: in October 2008, the Treasury announced the “Paulson’s Plan”, which included an equity infusion of US\$125 billion in the ten largest banks of the country, and a three-year government guarantee on their issuance of new unsecured debt. Moreover, the Treasury invested more than US\$400 billion in the “Troubled Asset Relief Program”, to rescue several financial corporations deemed “systemically important”, and the Federal Reserve, through its liquidity facilities, extended credit to the US financial system for around US\$1.5 trillion. As the crisis gained an international dimension, the US government was not left alone in rescuing the global financial system: in Germany, the federal government guaranteed banks’ equity for around EUR480 billion; in the UK, the government put into place two rescue packages and loan guarantees totalling GBP550 billion for the nine largest banks of the country; in Switzerland, UBS alone received funds for around EUR40 billion. Finally, after the systemic banking and sovereign crisis of 2012, the EU recognized the need to strengthen the Economic and Monetary Union by implementing a banking union, with the objective of further enhancing financial stability and risk sharing, and weakening the link between banks and national sovereign debts. However, while the first two pillars of the banking union (namely, the Single Supervisory Mechanism and the Single Resolution Mechanism) are already operational, there is still the need to implement the third one (the European Deposit Insurance Scheme) and, in particular, to understand its redistributive implications.

These narratives raise some compelling questions. How should a government bail out a heterogeneous banking system subject to systemic (i.e. economy-wide) self-fulfilling runs? What are the criteria to choose which banks are systemically important and worth bailing out? And what are the effects of such a government intervention on the banking system itself, in particular in terms of redistribution and risk sharing? The aim of the present paper is to provide an answer to these questions, by developing a theory of banking and government bailouts in a heterogeneous banking system.

To this end, our starting point is the three-period model by Diamond and Dybvig (1983). This is the standard workhorse model for both positive and normative analyses of the banking system, as it rationalizes its existence as a mechanism to decentralize the constrained-efficient allocation of resources, in an economy subject to idiosyncratic shocks that force the agents to consume in an interim period (i.e. before their investments mature). We modify this framework by assuming that the economy is populated by a continuum of risk-averse

agents, divided into a given number of groups that are heterogeneous with respect to initial per-capita wealth and size. The economy is also populated by a large number of banks, operating in a competitive market with free entry. The banks collect the heterogeneous initial wealths of the agents/depositors and, being the heterogeneity ex-ante observable, offer them a group-specific (or – equivalently – wealth-specific) deposit contract. Thus, we focus our attention on the behavior of one representative bank for each wealth group, homogeneously serving the depositors within it. So, we indifferently talk about heterogeneous wealth groups or heterogeneous banks.

We assume that all banks in the economy invest the deposits, which are the only liabilities in their balance sheets, into a common productive asset. With some positive probability, which represents the aggregate state of the economy, this asset yields a positive return that negatively depends on the total number of depositors who are withdrawing in the interim period in the whole economy. In that sense, the productive asset features a cross-group investment externality, in the spirit of Romer (1990) and Morris and Shin (2000). Moreover, this way of modeling the investment externality is qualitatively equivalent to a pecuniary externality generated by banks fire-selling their assets during a systemic self-fulfilling run, in the presence of the cash-in-the-market pricing of Allen and Gale (1994).

Due to the presence of strategic complementarities in the depositors' decisions to withdraw in the interim period, the economy exhibits two equilibria: one in which only the depositors who are hit by the idiosyncratic shock withdraw in the interim period, and one in which all depositors withdraw, thus starting a run. To characterize a unique equilibrium, we follow the literature on “global games” (Carlsson and van Damme 1993; Morris and Shin 1998) and assume that each depositor in each wealth group observes a private noisy signal about the realization of the aggregate state, based on which she forms posterior beliefs about the true state and the signals of all other depositors, and ultimately decides whether to run on her bank. As in Goldstein and Pauzner (2005), an equilibrium in threshold strategies emerges, in which two types of runs can start, depending on the signals that the depositors receive: a “fundamental” run, whenever the signal is so low that all depositors withdraw from their banks, irrespective of the behavior of the other depositors, and a “self-fulfilling” run, whenever the signal is below a threshold that makes the depositors indifferent between running or not, taking into account their posterior beliefs, and that fully depends on the terms of the deposit contract.

Differently from Goldstein and Pauzner (2005), however, the threshold signals for a self-fulfilling run are wealth-specific, due to the presence of both within- and between-group strategic complementarities in the depositors' decisions to run. While the former are already well-understood, the latter are a novelty in the existing literature for two reasons. First, it is not obvious how to model them: our chosen strategy is to introduce the investment externality, so that the return on the productive asset depends on how many depositors

withdraw in the interim period in the whole economy. Second, the presence of strategic complementarities in a global game with heterogeneous agents represents a theoretical challenge: in fact, Frankel *et al.* (2003) prove the existence and uniqueness of the equilibrium in a similar environment, but also that its characterization depends on the solution of a system of possibly nonlinear indifference conditions (in our case, one for each wealth group), and is manageable only under some suitable specifications. We address this issue by showing that, as the volatility of the noisy signals goes to zero, the wealth-specific threshold signals, below which a self-fulfilling run starts, tend to cluster around a unique value. This means that, if the depositors get a signal with low noise about a common aggregate state, they tend to run in accordance with the same threshold strategy: in other words, a self-fulfilling run is systemic. The common threshold signal fully depends on the terms of the deposit contract that each representative bank chooses for her own wealth group, and we characterize it by employing the “Belief Constraint” of Sakovics and Steiner (2012). According to it, the “Laplacian Property” of Morris and Shin (1998) holds on average: each depositor, given her beliefs and the fact that she knows that all depositors (including herself) play a threshold strategy, can infer the distribution function of the total number of depositors running, in her own wealth group as well as in the whole economy. This result allows us to characterize the common threshold signal for a systemic self-fulfilling run by studying the average indifference condition for a depositor between running or not.

With this tool in hand, we first solve for the decentralized banking equilibrium, and then for the banking equilibrium with bailouts. As far as the first one is concerned, each representative bank offers a deposit contract satisfying a distorted Euler equation: the ratio between the marginal rate of substitution between consumption in the interim period and consumption in the final period and the expected return on the productive asset (which is equal to 1 in an equilibrium with perfect information) is distorted by the fact that each representative bank internalizes that its deposit contract affects the common threshold signal and, as a consequence, also the welfare differential that its own depositors enjoy from avoiding a systemic self-fulfilling run.

As far as the banking equilibrium with bailouts is concerned, we instead assume the existence of an economy-wide government authority, operating as a social planner, who expropriates a public good and redistributes it to the banks with full commitment, through lump-sum liquidity injections, whenever the fraction of depositors withdrawing is higher than the known fraction of early consumers. Under these assumptions, a bailout always lowers the threshold signal below which a systemic self-fulfilling run starts: on the one hand, the expectation of a future liquidity injection makes the depositors less afraid of a bank’s bankruptcy, thus increasing their incentives to run and increasing the common threshold signal; on the other hand, a liquidity injection allows the banks to serve in the final period more depositors who do not run, before

declaring bankruptcy. Thus, under the assumption that the utility function satisfies the Inada conditions, the second effect dominates, as some depositors who do not run move from zero to a positive consumption. Hence, the total effect of the bailout on the common threshold signal is to lower it.

Under some mild conditions on the exogenous parameters of the model, we further show that a full bailout of the banking system, that completely rules out systemic self-fulfilling runs, cannot be part of the equilibrium. This result emerges as a consequence of the interaction between the government budget constraint and bank incentives to provide risk sharing to their depositors, as represented by the equilibrium conditions of the banking problem. In fact, the wealth-specific injection-to-contribution ratio can only take two values, for the government budget constraint to clear: it can either be equal to 1 in all wealth groups or, if it is higher than 1 for one group, there must be at least one other group for which it is lower than 1. While the latter case never satisfies the equilibrium conditions of the banking problem, we find the sufficient conditions under which also the former does not satisfy it. Interestingly, these conditions rule out a full bailout of the banking system, even when feasible. As a full bailout is feasible, also a partial bailout is feasible. Therefore, under the sufficient conditions, the equilibrium government bailout can only be partial, and always implies a redistribution of resources across wealth groups.

Under a partial government bailout, we characterize the optimal allocation of liquidity injections. In particular, the partial government bailout should inject liquidity so as to maximize the depositors' marginal benefits of the bailout. These marginal benefits are determined by a sufficient statistics, that accounts for two bank characteristics: first, how poor the wealth group served by the bank is, so as to maximize the effect of the liquidity injection on the depositors' marginal utility in the case of bankruptcy; second, how systemic the bank is, in terms of the effects that a liquidity injection through it have on the common threshold signal and on the total expected welfare differential (for the whole economy) from avoiding a systemic self-fulfilling run. These two characteristics have some direct counterparts in the real world, and thus represent a fully theory-based but readily applicable way to bailout banks facing systemic self-fulfilling runs. First, the bailout should target those groups at the lower end of the wealth distribution, as deposit insurance does in the real world. Second, it should target those wealth groups that impose higher externalities on the whole economy. In that sense, our result provides a rationale for the "contribution approach" (Staum 2012; Drehmann and Tarashev 2013; Tarashev *et al.* 2016) to the measurement of the systemic relevance of a financial institution, in contrast to the "participation approach" of Acharya *et al.* (2017). Finally, such a government intervention, albeit partial, is still beneficial for the whole economy, even for those wealth groups who finance the bailout scheme but whose banks do not receive any liquidity injection: this happens because

the liquidity injections lower the common threshold signal for a systemic self-fulfilling run, and thus allow the banks in all wealth groups to provide better risk sharing to their depositors against the idiosyncratic shocks.

The present paper contributes to the literature on banking and financial crises in many respects. First, by developing a theory of a heterogeneous banking system, where the probability of a systemic self-fulfilling run and banks' risk taking behavior are jointly and endogenously determined, this paper is the first, to the best of our knowledge, to explicitly study the role of wealth heterogeneity, which some new evidence suggests to be a key driver of depositors' behavior and runs (Iyer *et al.* 2015). To this end, the paper models systemic self-fulfilling runs as global games among heterogeneous depositors, and solves them by adapting to the Diamond and Dybvig (1983) framework some novel results from economic theory (Sakovics and Steiner 2012). Second, the paper contributes to the economics of government intervention during financial crises. In a recent working paper, Allen *et al.* (2017) extend the homogeneous economy of Goldstein and Pauzner (2005) by introducing a benevolent regulator, who provides a bank guarantee in fixed amount. However, this environment is not suitable to analyze the heterogeneity of a banking system and the redistributive implications of a common bailout scheme. Cooper and Kempf (2016) instead develop a banking model where the depositors are heterogeneous with respect to wealth, and formally study taxation and redistribution after a self-fulfilling run, in the absence of a committed regulator. However, they only analyze self-fulfilling runs as sunspot-driven coordination failures among the depositors. In other words, in their environment the probability of a systemic self-fulfilling run is exogenous by assumption, and depends only indirectly on the level of banks' risk taking.

The rest of the paper is organized as follows: in section 2, we lay down the environment of the model; in section 3, we study the strategic complementarities among the depositors in the decision about whether to join a systemic self-fulfilling run or not; in section 4, we characterize the decentralized banking equilibrium; in section 5, we characterize the government bailout scheme; finally, section 6 concludes.

2. A Model of a Banking System

2.1. Preferences and Endowments

The economy lives for three periods, labeled $t = 0, 1, 2$, and is populated by a unitary continuum of agents, divided into G groups, indexed by j , each of dimension m^j . The groups are heterogeneous with respect to their endowments: the agents in group j receive an initial group-specific endowment e^j at date 0, and a further endowment of a public good \bar{e} , equal for all groups, at date 1. Although being all ex-ante equal, at date 1 the agents are hit by a private

idiosyncratic shock θ , that takes value 0 with probability $1 - \pi$ and 1 with probability π . The shocks affect the point in time at which the agents want to consume, in accordance with the welfare function:

$$U(c_1^j, c_2^j, \theta) = \theta u(c_1^j) + (1 - \theta)u(c_2^j) + \bar{e}. \quad (1)$$

If $\theta = 1$, the agents only want to consume at date 1, while, if $\theta = 0$, they only want to consume at date 2. Thus, in line with the literature, we call type-0 and type-1 agents late (or “patient”) consumers, and early (or “impatient”) consumers, respectively. The law of large numbers holds, so π and $1 - \pi$ are the fractions of agents in the whole economy who turn out to be early or late consumers. The utility function $u(c)$ is twice continuously differentiable, increasing, concave and with a coefficient of relative risk aversion greater than 1. Moreover, $u(0) = 0$ and the Inada conditions hold: $\lim_{c \rightarrow 0} u'(c) = +\infty$ and $\lim_{c \rightarrow +\infty} u'(c) = 0$.¹ The public good \bar{e} enters the welfare function linearly and in an additive-separable fashion without loss of generality.

2.2. Banks and Technologies

The economy is also populated by a large number of banks, operating in a competitive market with free entry. At date 0, the banks collect the endowments of the agents, and invest them so as to maximize their profits, subject to agents’ participation. Perfect competition and free entry ensure that the banks solve an equivalent dual problem: they maximize the expected welfare of their depositors, subject to budget constraints. To this end, the banks offer a group-specific (or – equivalently – wealth-specific) standard deposit contract $\{d^j, d_L^j(A)\}$, stating the uncontingent amount that the depositors can withdraw at date 1, and the state-dependent amount that they can withdraw at date 2.² As the realizations of the idiosyncratic types are private information, the depositors must have the incentives to truthfully report their types. This implies that the deposit contracts must satisfy the incentive compatibility constraint $d^j \leq d_L^j(A)$, for every group j . The assumption that the banks offer a wealth-specific deposit contract comes at no loss of generality, as the ex-ante wealth heterogeneity is observable.

1. A typical utility function satisfying these assumption is the CRRA function $u(c) = ((c + \psi)^{1-\gamma} - \psi^{1-\gamma}) / (1 - \gamma)$, with $\gamma > 1$. The constant ψ can be interpreted as an exogenous consumption that the agents enjoy, and can be chosen to be arbitrarily close to zero so as to satisfy the Inada conditions.

2. In order to rule out uninteresting run equilibria, the amount of early consumption d^j must be smaller than $\min\{1/\pi, R\}$. The relationship between depositors and banks is exclusive, in the sense that the formers can only deposit their endowments into a bank, and cannot interact one with each other. The fact that the banks have to offer a standard deposit contract here is assumed. However, Farhi *et al.* (2009) show that a standard deposit contract, with an uncontingent amount of early consumption, endogenously emerge as part of the banking equilibrium, in the presence of non-exclusive deposit contracts.

To finance the deposit contract, the banks invest the deposits – which are the only liability on their balance sheets – in a productive asset that, for each unit invested at date 0, yields a stochastic return A at date 2. This stochastic return takes values $R(1 - \ell)$ with probability p , and 0 with probability $1 - p$, where ℓ is the total fraction of depositors who withdraw at date 1 in the whole economy. The probability of success of the productive asset p represents the aggregate state of the economy, and is distributed uniformly over the interval $[0, 1]$, with $(1 - \pi)\mathbb{E}[p]R > 1$. Moreover, the productive asset can be liquidated at date 1, i.e. before its natural maturity, and yield 1 unit of consumption for each unit liquidated. Intuitively, this asset represents a productive investment opportunity, whose return in case of success depends on how much of the initial investment reaches maturity in the whole economy. Put differently, the productive asset exhibits an investment externality. In Appendix A, we show that this mechanism is qualitatively equivalent to an environment with an explicit secondary market, where the banks sell the productive assets in order to finance the early withdrawals, and a pecuniary externality originates financial contagion.

At date 1, in accordance to the deposit contract chosen at date 0, the banks pay d^j to all the depositors who try to withdraw, and finance these early withdrawals by liquidating the productive asset until their resources are exhausted. When this happens, and the banks are not able to fulfil their contractual obligations any more, they instead go into bankruptcy, in which case they must liquidate all the productive assets in portfolio, and serve their depositors according to an “equal service constraint”, i.e. such that all depositors get an equal share of the available resources. Finally, at date 2 the depositors who have not withdrawn at date 1 are residual claimants of an equal share of the remaining resources.

We assume that the depositors cannot observe the true value of the realization of the fundamental p , but receive at date 1 a noisy private signal $\sigma = p + \eta$. The term η is an idiosyncratic noise, indistinguishable from the true value of p , that is uniformly distributed over the interval $[-\varepsilon, +\varepsilon]$, where ε is a positive but negligible constant. Given the signal received, each late consumer decides whether to withdraw from her bank at date 2, as the realization of her idiosyncratic shock would command, or “run on her bank” and withdraw at date 1, in accordance to the scheme to be described in the incoming section.

2.3. Timing and Definitions

The timing of actions is the following: at date 0, the banks collect the initial endowments, and choose the deposit contracts $\{d^j, d_L^j(A)\}$; at date 1, all agents get to know their private types and signals, and the early consumers withdraw, while the late consumers, once observed the signals, decide whether to run on their banks or not; finally, at date 2, those late consumers who have not withdrawn at date 1 withdraw an equal share of the available resources.

As wealth heterogeneity is perfectly observable at date 0, the banks will offer a wealth-specific deposit contract. Hence, we focus on the behavior of G representative banks, each serving one wealth group. We solve the model by backward induction, and characterize a pure-strategy Bayesian Nash equilibrium, where the banks choose the same wealth-specific deposit contracts and the depositors decide whether to run in accordance with the threshold strategy:

$$a^j(\sigma) = \begin{cases} \text{wait} & \text{if } \sigma \geq \sigma^{j*}, \\ \text{run} & \text{if } \sigma < \sigma^{j*}. \end{cases} \quad (2)$$

Selecting threshold strategies comes at no loss of generality, as Goldstein and Pauzner (2005) show in a similar environment that every equilibrium strategy is a threshold strategy. The definition of equilibrium is as follows:

DEFINITION 1. *Given the distributions of the idiosyncratic and aggregate shocks and of the individual signals, a decentralized banking equilibrium is a deposit contract $\{d^j, d_L^j(A)\}$ and depositors' decisions to run in each group $j = 1, \dots, G$ such that, for every realization of signals and idiosyncratic types $\{\sigma, \theta\}$:*

- *the depositors' decisions to run maximize their expected welfare;*
- *the deposit contract maximizes the depositors' expected welfare, subject to budget constraints.*

2.4. Banking Equilibrium with Perfect Information

As a benchmark for the results that follow, we start our analysis with the characterization of the banking equilibrium with perfect information, where a social planner who can observe the realization of the private idiosyncratic shocks hitting the depositors maximizes their expected (or aggregate) welfare subject to budget constraints. More formally, for each group j the social planner solves:

$$\max_{d^j} \pi u(d^j) + (1 - \pi) \int_0^1 pu \left(R(1 - \pi) \frac{e^j - \pi d^j}{1 - \pi} \right) dp + \bar{e}. \quad (3)$$

The planner knows that, with probability π , a depositor in group j will turn out to be an early consumer and consume d^j and, with probability $1 - \pi$, she will turn out to be a late consumer.³ In this case, the total amount of available resources to a bank in group j in period 2 depends on the realization of the aggregate state p , on the total number of late consumers in the whole economy, equal to $1 - \sum_j m_j \pi = 1 - \pi$, and on the amount of productive assets that are

3. In equilibrium, by the Inada conditions, both early and late consumption must be positive.

not liquidated to pay early consumption, $e^j - \pi d^j$. The first-order condition with respect to early consumption d^j gives the equilibrium condition:

$$u'(d^j) = (1 - \pi)\mathbb{E}[p]Ru'(R(e^j - \pi d^j)). \quad (4)$$

Intuitively, this result shows that the planner provides an allocation satisfying an Euler equation, i.e. so that the marginal rate of substitution between early and late consumption is equal to the expected return of the productive asset (equivalent to the marginal rate of transformation of a production technology). Moreover, as the utility function $u(c)$ is concave, d^{j*} and $d_L^{j*} = R(e^j - \pi d^{j*})$ are both nondecreasing in e^j .⁴ Moreover, the concavity of the utility function and the assumption that $(1 - \pi)\mathbb{E}[p]R > 1$ imply that the incentive compatibility constraint is satisfied, hence this allocation is equivalent to a constrained efficient allocation, in which the social planner has to induce truth-telling among the depositors. In what follows, we assume that the parameters of the model are such that this condition always holds.

3. Systemic Self-fulfilling Runs

We now move to the analysis of the banking equilibrium in the presence of private signals regarding the aggregate state of the economy. As we will show, these signals force the depositors to coordinate their actions: run under some range of signals, and not run under another. This will allow us to determine the probability of occurrence of a systemic self-fulfilling run for given deposit contracts and – later on – the optimal bailout scheme as the best response of the government to the strategic behavior of the depositors. The effect of the signals is twofold: they provide private information about the state of the fundamental and about the signals of the other depositors, which allows an inference regarding their actions. Intuitively, obtaining a high signal increases the incentives for a late consumer to wait until date 2 and not withdraw (i.e. “run on her bank”) at date 1, because it induces the belief that the realization of the aggregate state is good, and the signals of the other depositors are also high (under the assumption that the volatility of the signal is sufficiently small).

More formally, a late consumer in group j receives a private signal σ at date 1, and takes as given the deposit contract fixed at date 0. Based on this, she creates her posterior beliefs about how many depositors are withdrawing at date 1 in her own group as well as in the whole economy, and the probability of the realization of the aggregate state A , and decides whether to withdraw or not.

4. To see this, notice that the objective function of the social planner exhibits increasing differences in (e^j, d^j) . Also, with a simple change of variable, namely letting $x^j = d^j - \frac{e^j}{\pi}$ we can show that the objective function has decreasing differences in (x^j, e^j) , which is equivalent to having d_L^j increasing in e^j .

We assume the existence of two regions of extremely high and extremely low signals, where the decision of a late consumer is independent of her posterior beliefs. In the “lower dominance region”, the signal is so low that a late consumer always runs, irrespective of the behavior of the others. This happens below the threshold $\underline{\sigma}^j$, that makes her indifferent between withdrawing or not, and is defined by:

$$u(d^j) = \underline{\sigma}^j u(R(e^j - \pi d^j)), \quad (5)$$

from where it is easy to see that the threshold $\underline{\sigma}^j$ is increasing in the early consumption d^j : the more the bank promises to an early consumer in group j , the larger is the set of signals below which the depositors in that group run irrespective of what the other late consumers do. In the “upper dominance region”, instead, the signal is so high that a late consumer always wait until date 2 to withdraw. Following Goldstein and Pauzner (2005), we assume that this happens above a threshold $\bar{\sigma}^j$, where the investment is safe, i.e. $p = 1$, and gives the same return $R(1 - \pi)$ at date 1 and date 2. In this way, a late consumer is sure to get $R(e^j - \pi d^j)$ at date 2, irrespective of the behavior of all the other late consumers, and prefers to wait for any possible realization of the aggregate state.

The existence of the lower and upper dominance regions, regardless of their size, ensures the existence of an equilibrium in the intermediate region $[\underline{\sigma}^j, \bar{\sigma}^j]$, where the late consumers decide whether to run or not based on their posterior beliefs. In this region, a late consumer runs if her signal is lower than a threshold σ^{j*} , which is the value of the signal that makes her indifferent between running or not given her beliefs. More formally, define the utility advantage of waiting versus running as:

$$v^j(p, n, n^j) = \begin{cases} \sigma u\left(R(1 - n)\frac{e^j - n^j d^j}{1 - n^j}\right) - u(d^j) & \text{if } \pi \leq n^j < \frac{e^j}{d^j}, \\ -u\left(\frac{e^j}{n^j}\right) & \text{if } \frac{e^j}{d^j} \leq n^j < 1, \end{cases} \quad (6)$$

where n^j and n are the total number of depositors who are withdrawing at date 1 in group j and in the whole economy, respectively. These are given by:

$$n^j = \pi + (1 - \pi)\text{prob}(\sigma \leq \sigma^{j*}), \quad (7)$$

$$n = \sum_k m^k n^k = \pi + (1 - \pi) \sum_k m^k \text{prob}\{\sigma \leq \sigma^{k*}\}, \quad (8)$$

where it is clear that the number of depositors withdrawing at date 1 is given by the sum of the π early consumers plus those among the $1 - \pi$ late consumers who get a signal below the threshold σ^{j*} . Importantly, as the signals σ are random variables, the Laplacian Property (Morris and Shin 1998) ensures that their cumulative distribution functions are uniformly distributed over the interval $[0, 1]$. Hence, n^j is uniformly distributed over the interval $[\pi, 1]$ and its probability distribution function is the constant $f(n^j) = 1/(1 - \pi)$.

The expression for $v^j(p, n, n^j)$ highlights that, when the number of depositors running is between π (i.e., when there is no run) and e^j/d^j (i.e. the maximum number of depositors that the banks can serve according to the contract with the available resources), a late consumer receiving a signal σ holds the belief that the productive asset will turn out productive with probability $\mathbb{E}[p] = \mathbb{E}[\sigma - \eta] = \sigma$. In that case, if she waits until date 2, she consumes $d_L^j(R, n, n^j) = R(1-n)\frac{e^j - n^j d^j}{1-n^j}$ or $d_L^j(0, n, n^j) = 0$ otherwise, and if she withdraws she consumes d^j . In contrast, when the number of depositors running is higher than e^j/d^j , the representative bank of group j goes into bankruptcy: it is forced to liquidate all productive assets and equally share the proceeds among the depositors. Hence, a late consumer gets zero if she waits, and e^j/n^j if she withdraws.

The function $v^j(p, n, n^j)$ exhibits both between- and within-group strategic complementarities. To see that, calculate:

$$\frac{\partial v^j}{\partial n^{\ell \neq j}} = \begin{cases} -R\sigma u'(d_L^j(R, n, n^j))m^\ell \frac{e^j - n^j d^j}{1-n^j} & \text{if } \pi \leq n^j < \frac{e^j}{d^j}, \\ 0 & \text{if } \frac{e^j}{d^j} \leq n^j < 1, \end{cases} \quad (9)$$

and notice that the derivative in the first interval is always negative. As far as the within-group strategic complementarity, instead:

$$\frac{\partial v^j}{\partial n^j} = \begin{cases} R\sigma u'(d_L^j(R, n, n^j)) \left[-m^j \frac{e^j - n^j d^j}{1-n^j} + (1-n) \frac{e^j - d^j}{(1-n^j)^2} \right] & \text{if } \pi \leq n^j < \frac{e^j}{d^j}, \\ u' \left(\frac{e^j}{n^j} \right) \frac{e^j}{n^{j2}} > 0 & \text{if } \frac{e^j}{d^j} \leq n^j < 1. \end{cases} \quad (10)$$

Interestingly, the derivative in the first interval is negative (i.e. we have one-sided strategic complementarity) only if $d^j > \Phi^j(n)e^j$, where:

$$\Phi^j(n) = \frac{(1-n) - m^j(1-n^j)}{(1-n) - m^j n^j(1-n^j)}, \quad (11)$$

and $\Phi^j(n)$ is lower than 1.⁵

Given the function $v^j(p, n, n^j)$, we derive the threshold signal σ^{j*} as the value of the signal such that $\mathbb{E}[v^j(p, n, n^j)|\sigma^{j*}] = 0$, or the one solving:

$$\begin{aligned} \int_{\pi}^1 \int_{\pi}^{\frac{e^j}{d^j}} \sigma^{j*} u \left(R(1-n) \frac{e^j - n^j d^j}{1-n^j} \right) dn^j dn &= \\ = \int_{\pi}^1 \left[\int_{\pi}^{\frac{e^j}{d^j}} u(d^j) dn^j + \int_{\frac{e^j}{d^j}}^1 u \left(\frac{e^j}{n^j} \right) dn^j \right] dn. \end{aligned} \quad (12)$$

5. For the rest of this section, we guess that this condition is satisfied. When characterizing the equilibrium deposit contract, we show that $d^j > e^j$, thus confirming our conjecture.

This gives:

$$\sigma^{j*} = \frac{(1 - \pi) \left[\int_{\pi}^{\frac{e^j}{d^j}} u(d^j) dn^j + \int_{\frac{e^j}{d^j}}^1 u\left(\frac{e^j}{n^j}\right) dn^j \right]}{\int_{\pi}^1 \int_{\pi}^{\frac{e^j}{d^j}} u\left(R(1 - n) \frac{e^j - n^j d^j}{1 - n^j}\right) dn^j dn}, \quad (13)$$

for every wealth group $j = 1, \dots, G$. Following Frankel et al. (2003) and Sakovics and Steiner (2012), we can prove the following:

PROPOSITION 1. *The Bayesian Nash equilibrium in threshold strategies characterizing the withdrawing decisions of the depositors is unique. As the volatility of the noise ε goes to zero, all threshold signals σ^{j*} converge to a common limit σ^* , which is characterized by the average indifference condition:*

$$\sum_j m^j \mathbb{E}[v^j(p, n, n^j) | \sigma^*] = 0, \quad (14)$$

and gives:

$$\sigma^*(\mathbf{d}) = \frac{(1 - \pi) \sum_j m^j \left[\int_{\pi}^{\frac{e^j}{d^j}} u(d^j) dn^j + \int_{\frac{e^j}{d^j}}^1 u\left(\frac{e^j}{n^j}\right) dn^j \right]}{\sum_j m^j \int_{\pi}^1 \int_{\pi}^{\frac{e^j}{d^j}} u\left(R(1 - n) \frac{e^j - n^j d^j}{1 - n^j}\right) dn^j dn}, \quad (15)$$

where $\mathbf{d} = \{d^j\}_{j=1}^G$.

Proof. In Appendix B. □

In order to show uniqueness, we rewrite the problem in terms of the difference between the group-specific threshold signals σ^{j*} and threshold signal σ^* of a reference group, namely in terms of $\Delta^j = (\sigma^{j*} - \sigma^*)$. The reasoning behind this rescaling is as follows: on the one hand, if the signal lies above all threshold signals, all groups wait; on the other hand, if the signal is below all threshold signals, all groups run. It is only when the signal falls within the cluster formed by the group threshold signals that there is strategic uncertainty: certain groups run and others wait, and the depositors do not know how many are choosing each action. The magnitude of the strategic uncertainty depends on Δ , i.e. the vector of the Δ^j -s. Facing this strategic uncertainty, the agents in each group must form a belief about how many agents in each group – and overall – are running. In order to do that, we resort to the concept of “Belief Constraint” of Sakovics and Steiner (2012). The Belief Constraint highlights that the Laplacian Property holds on average,

meaning that the average cumulative distribution function of a random variable is uniformly distributed over the interval $[0, 1]$. This powerful results yields that the total number of depositors n withdrawing in the whole economy is also uniformly distributed, over the interval $[\pi, 1]$. Then, the unique equilibrium in threshold strategies is given by the solution to the system of G equations $H^j(\sigma^*, \Delta)(\varepsilon) = \mathbb{E}[v^j(\sigma^*, \Delta)] = 0$ for every group j . We prove the uniqueness of the solution by showing that if two solutions exist, a contradiction arises. The second part of the Proposition instead shows that, whenever the volatility of the noise of the signals is sufficiently low, the system of indifference equations $H^j(\sigma^*, \Delta)(\varepsilon) = 0$ is well approximated by $H^j(\sigma^*, \Delta)(0) = 0$. Therefore, also the solution to the system for a small ε lies in the neighbourhood of the solution to the system when ε is zero. This last solution yields a unique threshold signal $\sigma^*(\mathbf{d})$, that solves $\sum_j m^j H^j(\sigma^*, \Delta)(0) = 0$.

This result is crucial. Once the volatility of the noise is small, the group threshold signals cluster around a common threshold $\sigma^*(\mathbf{d})$, which uniquely determines the probability of a systemic self-fulfilling run occurring in the economy. This common threshold signal depends on the deposit contracts chosen by the representative banks in each group. The following corollary sheds light on this relationship.

COROLLARY 1. *The threshold $\sigma^*(\mathbf{d})$ is increasing in every d^j .*

Proof. In Appendix B. □

This result highlights the channels of financial contagion from one wealth group to the rest of the economy: as the representative bank of group j promises a higher amount of early consumption, its depositors anticipate that it might not be able to serve them all, in the case of a systemic self-fulfilling run. In addition to that, also the depositors in the other groups internalize the fact that a run in one group might reduce the return on the productive asset, and force their banks to go bankrupt. Hence, the range of signals for which a systemic self-fulfilling run occurs increases with the early consumption offered by each bank.

4. Decentralized Banking Equilibrium

Having characterized the endogenous threshold strategy played by the late consumers at date 1, in this section we determine the optimal contract offered by the representative banks in each wealth group at date 0. To this end, a bank in group j solves the following problem:

$$\max_{d^j} \int_0^{\sigma^*(\mathbf{d})} u(e^j) dp + \int_{\sigma^*(\mathbf{d})}^1 [\pi u(d^j) + (1 - \pi) p u(R(e^j - \pi d^j))] dp + \bar{e}. \quad (16)$$

Whenever the signal is between 0 and $\sigma^*(\mathbf{d})$ (remember that the noise term is positive but negligible), a systemic run happens, either fundamental or self-fulfilling, and all depositors receive the per-capita return from the liquidation of the whole productive assets available in portfolio. When instead the signal is between $\sigma^*(\mathbf{d})$ and 1, no systemic self-fulfilling run happens, and the depositors turn out to be early consumers with probability π and late consumers with probability $1 - \pi$, as in the banking equilibrium with perfect information.⁶ To complete the characterization of the decentralized banking equilibrium, define the expected welfare gain from avoiding a run in group j as:

$$\Delta U^j = \pi u(d^j) + (1 - \pi)\sigma^*(\mathbf{d})u(R(e^j - \pi d^j)) - u(e^j). \quad (17)$$

Then, the first-order condition with respect to d^j implicitly determines the optimal contract:

$$\pi \int_{\sigma^*}^1 [u'(d^j) - (1 - \pi)pRu'(R(e^j - \pi d^j))] dp = \frac{\partial \sigma^*(\mathbf{d})}{\partial d^j} \Delta U^j. \quad (18)$$

This distorted Euler equation highlights that the endogeneity of the threshold signal $\sigma^*(\mathbf{d})$ forces the banks to impose a wedge between the marginal rate of substitution between early and late consumption and the expected return on the productive asset. To see that more clearly, rewrite (18) in terms of the marginal rate of substitution:

$$\begin{aligned} MRS &\equiv \frac{u'(d^j)}{u'(R(e^j - \pi d^j))} = \\ &= \frac{1}{\pi(1 - \sigma^*(\mathbf{d}))} \frac{1}{u'(R(e^j - \pi d^j))} \frac{\partial \sigma^*(\mathbf{d})}{\partial d^j} \Delta U^j + (1 - \pi) \frac{1 + \sigma^*(\mathbf{d})}{2} R. \end{aligned} \quad (19)$$

The right-hand side of (19) is higher than the expected return on the productive asset, namely $\frac{(1-\pi)R}{2}$, which is equal to the marginal rate of substitution between early and late consumption in the banking equilibrium with perfect information. In other words, the endogeneity of the threshold signal $\sigma^*(\mathbf{d})$ forces the banks to increase the marginal rate of substitution, i.e. lower the amount of early consumption offered, with respect to the banking equilibrium with perfect information. Yet, it can be proved that the decentralized banking equilibrium still Pareto-dominates an autarkic equilibrium.

LEMMA 1. *In the decentralized banking equilibrium, $d^j > e^j$ for every group $j = 1, \dots, G$.*

Proof. In Appendix B. □

6. By the Inada conditions, the non-negativity constraints on early and late consumption are always slack.

The proof of this Lemma is based on showing that $d^j = e^j$ for every group j leaves some marginal benefits unexploited, and that if $d^\ell > e^\ell$ for at least one group ℓ , then it is not optimal to have $d^k = e^k$ for any group $k \neq \ell$. The Lemma highlights that, despite the possible emergence of fundamental or systemic self-fulfilling runs, the banking system provides better risk sharing than an autarkic equilibrium, where the agents cannot access the banking system and have to independently choose their own investments. In fact, in such a case, the agents would invest all their endowments e^j in the productive asset; then, an early consumer would liquidate all of it and consume $c_1^j = e^j$, while a late consumer would instead keep the investment, and consume either $c_2^j = Re^j$ or $c_2^j = 0$. In other words, the banking system compresses the ex-post income profiles of the risk-averse depositors, and improves welfare. However, it should be noted that the amount of risk sharing that the banks offer in the decentralized banking equilibrium is still lower than what they would offer in the banking equilibrium with perfect information, as they internalize the fact that a high value of early consumption has the negative consequence of increasing the threshold signal $\sigma^*(\mathbf{d})$.

5. Government Bailouts

Having characterized the decentralized banking equilibrium of the heterogeneous economy, in this section we study the optimal allocation of a government bailout scheme, and how this affects in turns the amount of risk sharing provided by the banks against the idiosyncratic risk. To this end, we assume the existence of an economy-wide government authority, with the ability to expropriate the public good \bar{e} and use it to attribute group-specific lump-sum subsidies s^j whenever the fraction of depositors withdrawing is higher than π . In that sense, the government authority operates as a social planner, who chooses a liquidity injection scheme to maximize the expected welfare of the depositors, subject to limited available resources and fiscal instruments. Importantly, the scheme is established with full commitment at the beginning of date 1, and is implemented at the end of the same period. For the sake of clarity, here we summarize the timing of actions: at date 0, the banks in each wealth group collect the initial endowments, and choose the deposit contracts; at date 1, the government authority chooses the liquidity injection scheme; then, all agents get to know their private types and signals, and the late consumers decide whether to run on their banks; finally, the liquidity injection scheme is implemented; at date 2, those late consumers who did not withdraw at date 1 withdraw an equal share of the available resources.

As in the previous section, we solve for the banking equilibrium with bailouts by backward induction. Hence, we start by studying the decisions of the late consumers about whether to join a run or not, depending on the deposit contract and on the government bailout scheme. In the case of government

intervention, the budget constraint of a bank in group j at date 1 reads:

$$X^j + s^j = n^j d^j, \quad (20)$$

where X^j is the amount of productive assets that needs to be liquidated. Thus, the amount of productive assets that gets to maturity is equal to $e^j - X^j$, and affects the utility that the late consumers get if they do not withdraw at date 1:

$$v^j(p, n, n^j, s^j) = \begin{cases} \sigma u \left(R(1-n) \frac{e^j + s^j - n^j d^j}{1-n^j} \right) - u(d^j) & \text{if } \pi \leq n^j < \frac{e^j + s^j}{d^j}, \\ -u \left(\frac{e^j + s^j}{n^j} \right) & \text{if } \frac{e^j + s^j}{d^j} \leq n^j \leq 1. \end{cases} \quad (21)$$

The subsidy s^j influences the advantage of waiting versus running in three ways: (i) it increases the amount of liquidity available to the banks; (ii) it increases the maximum fraction of depositors that can be served before the banks go into bankruptcy; (iii) it increases the consumption of all depositors at bankruptcy. Again, by the Belief Constraint, we characterize the endogenous threshold signal below which all late consumers run from the average indifference condition between running or not, and derive:

$$\sigma^*(\mathbf{d}, \mathbf{s}) = \frac{(1-\pi) \sum_j m^j \left[\int_{\pi}^{\frac{e^j + s^j}{d^j}} u(d^j) dn^j + \int_{\frac{e^j + s^j}{d^j}}^1 u \left(\frac{e^j + s^j}{n^j} \right) dn^j \right]}{\sum_j m^j \int_{\pi}^1 \int_{\pi}^{\frac{e^j + s^j}{d^j}} u \left(R(1-n) \frac{e^j + s^j - n^j d^j}{1-n^j} \right) dn^j dn}, \quad (22)$$

where $\mathbf{s} = \{s^j\}_{j=1, \dots, G}$. From here, we can calculate the effect of a marginal increase of a subsidy s^j on the common threshold signal $\sigma^*(\mathbf{d}, \mathbf{s})$:

$$\begin{aligned} \frac{\partial \sigma^*(\mathbf{d}, \mathbf{s})}{\partial s^j} &= \frac{(1-\pi)m^j}{\sum_j m^j \int_{\pi}^1 \int_{\pi}^{\frac{e^j + s^j}{d^j}} u \left(R(1-n) \frac{e^j + s^j - n^j d^j}{1-n^j} \right) dn^j dn} \times \\ &\times \left[\int_{\frac{e^j + s^j}{d^j}}^1 u' \left(\frac{e^j + s^j}{n^j} \right) \frac{1}{n^j} dn^j + \right. \\ &\left. - \sigma^*(\mathbf{d}, \mathbf{s}) \int_{\pi}^1 \int_{\pi}^{\frac{e^j + s^j}{d^j}} u'(d_L^j(R, n, n^j)) \frac{R(1-n)}{1-n^j} dn^j dn \right], \quad (23) \end{aligned}$$

where $d_L^j(R, n, n^j) = R(1-n) \frac{e^j + s^j - n^j d^j}{1-n^j}$. At a first sight, the sign of this expression seems undetermined, as the subsidy has both a positive effect on the incentives to run (it increases consumption at bankruptcy) and a negative effect (it increases late consumption) However, by the Inada conditions:

$$\lim_{n^j \rightarrow \frac{e^j + s^j}{d^j}} u'(d_L^j(R, n, n^j)) = \lim_{c \rightarrow 0} u'(c) = +\infty. \quad (24)$$

Hence, (23) is negative.⁷ This is an important result, because it shows that the effect of the subsidy is to reduce the threshold signal, and therefore the endogenous probability of a self-fulfilling run. Crucially, this result is a consequence of the assumption that the government commits to intervene whenever the fraction of depositors running is above π . In fact, in section 5.3, we show that a government who cannot commit to intervene, and only bails out banks ex post, actually increases the incentives of the depositors to run, and therefore the threshold signal to start a systemic self-fulfilling run.

At the beginning of date 1, the government authority, given the deposit contract chosen by the banks at date 0, and taking into account the best responses of the depositors, maximizes their expected welfare, subject to its own budget constraint. More formally, it solves the following problem:

$$\max_{s^j} \sum_j m^j \left[\int_0^{\sigma^*(\mathbf{d}, \mathbf{s})} u(e^j + s^j) dp + \int_{\sigma^*(\mathbf{d}, \mathbf{s})}^1 \left[\pi u(d^j) + (1 - \pi) pu(R(e^j - \pi d^j)) + \bar{e} \right] dp \right], \quad (25)$$

subject to the budget constraint:

$$\sum_j m^j s^j = \bar{e}, \quad (26)$$

and to the lower and upper bounds for the subsidy $s^j \geq 0$ and $s^j \leq d^j - e^j$, where the second is the value of s^j such that $\frac{e^j + s^j}{d^j} = 1$, i.e. the bank in group j is “run-proof”. As in the previous section, we characterize a pure-strategy symmetric Bayesian Nash equilibrium in threshold strategies.

DEFINITION 2. *Given the distributions of the idiosyncratic and aggregate shocks and of the individual signals, a banking equilibrium with bailouts is a deposit contract $\{d^j, d_L^j(A)\}$, depositors’ decisions to run and a scheme of subsidies $\{s^j\}$ for each group $j = 1, \dots, G$ such that, for every realization of signals and idiosyncratic types $\{\sigma, \theta\}$:*

- *the depositors’ decisions to run maximize their expected welfare;*
- *the subsidies maximize the depositors’ expected welfare, subject to the government budget constraints.*
- *the deposit contract maximizes the depositors’ expected welfare, subject to budget constraints;*

7. Notice that the function $v^j(p, n, n^j, s^j)$ has a kink at $n^j = \frac{e^j + s^j}{d^j}$. In other words, it is not differentiable in that point, hence the second integral of (23) takes a large but still bounded value.

The first-order conditions of the government problem allow us to characterize the following Proposition:

PROPOSITION 2. *The government bailout scheme targets subsidies on the groups with the largest statistics:*

$$\Psi^j = -\frac{\partial \sigma^*(\mathbf{d}, \mathbf{s})}{\partial s^j} \sum_k m^k \Delta U_B^k + \sigma^*(\mathbf{d}, \mathbf{s}) u'(e^j + s^j), \quad (27)$$

where:

$$\Delta U_B^k = (1 - \pi) \left[\sigma^* u(R(e^k - \pi d^k)) - u(d^k) \right] + \bar{e}. \quad (28)$$

There exists a unique group \hat{j} for which $\Psi^{\hat{j}} = 1$, such that (i) those groups with $\Psi^j > 1$ are fully subsidized and get $s^j = d^j - e^j > 0$, (ii) those groups with $\Psi^j < 1$ are not subsidized and get $s^j = 0$, and (iii) those groups for which $\Psi^j = 1$ get $s^j \in [0, d^j - e^j]$.

Proof. In Appendix B. □

The proof of the Proposition is based on the following lines of reasoning: as the per capita marginal cost of subsidizing the representative bank of each group is the same across groups, and equal to the Lagrange multiplier on the government budget constraint, in equilibrium it is optimal to allocate subsidies so that the marginal benefits of the subsidies are equal across groups. These are equal to the sum of three parts: first, the difference between the Lagrange multipliers of the upper and lower bounds of s^j , that regulates whether a bank is fully, partially or not subsidized; second, the marginal effect that a subsidy to the representative bank of group j has on the common threshold signal $\sigma^*(\mathbf{d}, \mathbf{s})$, and as a consequence on the total welfare differential from avoiding a systemic run (the first element of (27)) that also takes into account the cost that each group incurs when financing the bailout scheme, in terms of not enjoying the direct consumption of the public good; third, the marginal utility of a subsidy to the representative bank of group j in the case of bankruptcy (the second element of (27)). In equilibrium, it is optimal for the government to rank the banks according to (27), and always fully subsidize the first one of the ranking, i.e. $s^{(1)} = d^{(1)} - e^{(1)}$, where (j) represents the bank of the j -th group in the ranking. Moreover, as the ranking is monotonic, it is also optimal to fully subsidize all banks, until the government budget constraint clears. Consequently, there can only be a unique threshold group (\hat{j}) in the ranking, above which all groups are fully subsidized and above which all groups get zero. In other words, either all banks are fully subsidized, whenever the threshold group is the (G) -th, or some banks are fully subsidized at the top of the ranking, and some others get zero.

5.1. The Banking Equilibrium with Bailouts

Given the optimal allocation of the subsidies, characterized in the previous section, we conclude the analysis of this economy with the characterization of the banking equilibrium with bailouts.

We start our analysis by guessing that the threshold group is the (G) -th, or $\hat{j} = G$. In this case, Proposition 2 states that all banks are fully subsidized, meaning that, in the case of a run, they would be able to serve all depositors. As a consequence, no systemic self-fulfilling run occurs in equilibrium. Yet, this does not rule out fundamental runs, still happening in the lower dominance regions of each group. The subsidies affect the threshold signals $\underline{\sigma}^j$ below which a fundamental run occurs in group j :

$$\underline{\sigma}^j = \frac{u(d^j)}{u(R(e^j + s^j - \pi d^j))} = \frac{u(d^j)}{u(R(1 - \pi)d^j)}. \quad (29)$$

This threshold is increasing in d^j . To see that, calculate:

$$\frac{\partial \underline{\sigma}^j}{\partial d^j} = \frac{u'(d^j)u(R(1 - \pi)d^j) - (1 - \pi)Ru'(R(1 - \pi)d^j)u(d^j)}{(u(R(1 - \pi)d^j))^2}, \quad (30)$$

and notice that the numerator is positive, because $(1 - \pi)R > p(1 - \pi)R$, which is larger than 1 by assumption, and because of the coefficient of relative risk aversion being larger than 1.⁸ Then, a bank in group j at date 0 solves the problem:

$$\max_{d^j} \int_0^{\underline{\sigma}^j} u(d^j) dp + \int_{\underline{\sigma}^j}^1 [\pi u(d^j) + (1 - \pi)pu(R(e^j - \pi d^j)) + \bar{e}] dp, \quad (31)$$

and the first-order condition gives:

$$\pi \int_{\underline{\sigma}^j}^1 [u'(d^j) - (1 - \pi)pRu'(R(e^j - \pi d^j))] dp = \frac{\partial \underline{\sigma}^j}{\partial d^j} \Delta U_B^j - \underline{\sigma}^j u'(d^j), \quad (32)$$

where:

$$\Delta U_B^j = (1 - \pi) \left[\underline{\sigma}^j u(R(e^j - \pi d^j)) - u(d^j) \right] + \bar{e}. \quad (33)$$

As in the decentralized banking equilibrium, it can be proved that, under some mild conditions, the banking equilibrium with bailouts is better than an autarkic equilibrium.

LEMMA 2. *Assume that $u(c)$ is CRRA and $e^j \geq \bar{e}$. Then, in the banking equilibrium with full bailouts, $d^j > e^j$ for every $j = 1, \dots, G$.*

8. See footnote 10.

Proof. In Appendix B. □

This lemma shows that, in an equilibrium with full bailouts, the representative bank of group j still offers more insurance against the idiosyncratic shock than what the depositors would get in autarky, despite the fact that increasing early consumption also increases the threshold signal below which there is a fundamental run. From here onwards, we assume that the conditions of Lemma 2 hold, and conclude the characterization of the banking equilibrium with bailouts.

PROPOSITION 3. *Assume that a full bailout is feasible and $e^j \geq \bar{e}$ for every group j , and that the utility function $u(c)$ is CRRA with a coefficient of relative risk aversion bounded from above. Then, there cannot exist a banking equilibrium with full bailouts.*

Proof. In Appendix B. □

Proposition 3 states the sufficient conditions that rule out a full government bailout from the banking equilibrium. The intuition for this result is the following. For a full bailout to be an equilibrium, the government budget constraint must clear, meaning that $\sum_j m^j (d^j - e^j) / \bar{e} = 1$. Two cases are possible: either the subsidy-to-contribution ratio $(d^j - e^j) / \bar{e}$ is larger than 1 for some groups j and lower than 1 for the remaining groups $k \neq j$, or it is equal to 1 for all groups. The first case, in particular when $d^j - e^j < \bar{e}$, does not satisfy the equilibrium condition (32) of the banking problem. Assume instead that $d^j - e^j = \bar{e}$ for all groups j . In this case, early consumption d^j has the double effect of positively affecting (i) the threshold for a fundamental run $\underline{\sigma}^j$ and (ii) the consumption of all depositors in group j when a fundamental run takes place (the right-hand side of (32)). As the second effect dominates the first, the sum of the two imposes a negative distortion between the marginal rate of substitution between early and late consumption and the expected return on the productive asset in the bank optimality condition (the left-hand side of (32)). For the marginal rate of substitution to be lower than the expected return on the productive asset, given that the agents are risk-averse, early consumption must be high. Then, a sufficient condition for this not to be an equilibrium is that the coefficient of relative risk aversion is sufficiently low (but still higher than 1). Hence, under these conditions, a full bailout is not an equilibrium.

As the full bailout is feasible, also a partial bailout is feasible. Therefore, the direct consequence of the previous Proposition is that, under some sufficient conditions, the equilibrium government bailout can only be partial. In other words, all groups see the public good expropriated, but only few groups get subsidized. Hence, the implementation of an optimal partial government bailout always implies a wealth redistribution of resources across groups.

In the unsubsidized groups, the banking problem reads as in the decentralized environment without bailouts in (16), with the two key differences that the depositors enjoy the public good only when the signal is above the threshold signal $\sigma^*(\mathbf{d}, \mathbf{s})$ (i.e. if no run happens) and the threshold signal $\sigma^*(\mathbf{d}, \mathbf{s})$ itself is lower:

$$\max_{d^j} \int_0^{\sigma^*(\mathbf{d}, \mathbf{s})} u(e^j) dp + \int_{\sigma^*(\mathbf{d}, \mathbf{s})}^1 [\pi u(d^j) + (1 - \pi)pu(R(e^j - \pi d^j)) + \bar{e}] dp. \quad (34)$$

The following corollary shows that, despite the expropriation of the public good, the unsubsidized banks are still able to increase the amount of risk sharing against the idiosyncratic shocks with respect to the decentralized banking equilibrium, and therefore to increase the expected welfare of their depositors.

COROLLARY 2. *In the banking equilibrium with partial government bailouts, the unsubsidized banks offer higher early consumption than in the decentralized banking equilibrium, or $d_B^j > d^j$ for every $j = 1, \dots, G$.*

Proof. In Appendix B. □

The result of the Corollary shows that, despite losing the public good without receiving any subsidy, all unsubsidized banks are able, through the lower probability of a systemic self-fulfilling run, to improve risk sharing for their depositors. A similar but stronger result also emerges for the subsidized banks. In fact, the equilibrium condition of their banking problem under partial bailout would read:

$$\pi \int_{\sigma^*(\mathbf{d}, \mathbf{s})}^1 [u'(d^j) - (1 - \pi)pRu'(R(e^j - \pi d^j))] dp = \frac{\partial \sigma^*(\mathbf{d}, \mathbf{s})}{\partial d^j} \Delta U_B^j + \sigma^*(\mathbf{d}, \mathbf{s})u'(d^j), \quad (35)$$

where:

$$\Delta U_B^j = (1 - \pi) \left[\sigma^*(\mathbf{d}, \mathbf{s})u(R(e^j - \pi d^j)) - u(d^j) \right] + \bar{e}, \quad (36)$$

and:

$$\frac{\partial \sigma^*(\mathbf{d}, \mathbf{s})}{\partial d^j} = \frac{(1 - \pi)m^j}{DEN} \left[(1 - \pi)u'(d^j) - \sigma^*(\mathbf{d}, \mathbf{s}) \int_{\pi}^1 u'(R(1 - n)d^j)R(1 - n)dn \right], \quad (37)$$

which is positive.⁹ The equilibrium condition in (35) resembles the distorted Euler equation in (32), with the key difference that now, as the bailout is only

9. In the expression above, DEN is the positive denominator of threshold signal $\sigma^*(\mathbf{d}, \mathbf{s})$. To see that the threshold signal $\sigma^*(\mathbf{d}, \mathbf{s})$ is increasing in d^j , we observe that, by the Inada conditions, $\lim_{n \rightarrow 1} u'(R(1 - n)d^j)R(1 - n) = \lim_{n \rightarrow 1} u''(R(1 - n)d^j)Rd^j = -\infty$, where we used the l'Hôpital Rule for the second equality. Hence, the term in the square brackets is positive.

partial, self-fulfilling runs can happen, but only below the lower threshold signal $\sigma^*(\mathbf{d}, \mathbf{s}) < \sigma^*(\mathbf{d}, \mathbf{0})$. The expression in (35) makes clear that the subsidized banks have one further reason, with respect to the unsubsidized banks, to increase early consumption above the level of the decentralized banking equilibrium: in order to get more subsidies in the case of a systemic self-fulfilling run. In other words, it must also be the case that $d_B^j > d^j$ for all those j groups that are fully subsidized. To sum up, a government bailout scheme, albeit partial, always allows better risk sharing for every group in the economy.

5.2. Discussion

Propositions 2 and 3 suggest a practical criterion to optimally bailout a heterogeneous banking system subject to systemic self-fulfilling runs. In fact, for given available resources to redistribute, a government authority should:

1. calculate the statistics Ψ^j for all groups j ;
2. rank the groups in decreasing order of Ψ^j ;
3. fully bailout the banks of the groups with the highest Ψ^j , until the resources are exhausted.

The statistics Ψ^j , representing the total expected marginal benefit from the subsidy in group j , is the sufficient criterion to implement the optimal bailout. It is made of two parts, each highlighting one group characteristic that the scheme should target:

- High $\frac{\partial \sigma^*(\mathbf{d}, \mathbf{s})}{\partial s^j} \sum_k m^k \Delta U_B^k$;
- High $\sigma^* u'(e^j + s^j)$.

These two characteristics have some direct counterparts in practice. First, the bailout should target the groups that impose the highest externalities on the whole economy: Those should be the banks that are more interconnected to, or impose the highest losses onto, the other banks in the economy. In that sense, our result does not support the “participation approach” of Acharya *et al.* (2017), that measures the systemic importance of a financial institution in terms of its participation to the total systemic losses incurred by that financial institution in the case of a financial crisis. In contrast, our result provides a further rationale for the “contribution approach” of Staum (2012), Drehmann and Tarashev (2013) and Tarashev *et al.* (2016), according to which the systemic importance of a financial institution depends on its contribution to the emergence of systemic risk for the whole economy. Second, the optimal bailout scheme should target the banks of those wealth groups where the marginal benefit of the subsidy is the largest. The government can achieve that by targeting the subsidies at the lower end of the wealth distribution, exactly as a common deposit insurance scheme does. In other words, our result rationalizes at the same time the bailout of the most systemic banks and of the poorest depositors.

Finally, it is worth mentioning that Propositions 2 and 3, despite being quite general, are based on some sensible assumptions. First, the introduction of a public good available in equal supply for all groups allows us to separate in a parsimonious way the financing problem of the government from the banking problem, as in Allen et al. (2017). Moreover, the assumption that the endowments e^j of each group are larger than the available amount of the public good \bar{e} reflects the observation that, in modern economies, total banks' liabilities are generally higher than 100 per cent of a country's GDP. Second, the hypothesis of full expropriation of the public good can be relaxed without altering the results: for example, an alternative self-financed scheme, where some banks get negative subsidies to finance the positive subsidies to the others, would yield similar results, and in particular characterize the threshold group in the ranking below which the banks have to finance the subsidies to the others. Third, our assumption that the government scheme is established after the banks' decisions about the deposit contracts has no impact on the optimal allocative mechanism of the subsidies. To see that, consider an alternative timing, in which the government chooses the subsidy scheme at date $t = 0$, before the banks determine the deposit contracts. Formally, the government would maximize the total expected welfare of the depositors, by solving the problem in (25), subject to its own budget constraint in (26) and to the condition that the banks' behavior is optimal, i.e. it satisfies the distorted Euler equation in (18). The solution to this problem, once again, would have the government ranking the groups in decreasing order of marginal benefit of the subsidy, and fully subsidizing them starting from the top of the ranking, until its budget clears.

5.3. Ex-Post Government Bailouts with Discretion

In the previous sections, we crucially assume that the government could commit to a bailout scheme at the beginning of date 1, after the banks have chosen the deposit contracts, but before they start liquidating the productive assets and the revelation of the depositors' idiosyncratic types and signals. In this extension, we relax this hypothesis, and instead assume that the government authority lacks commitment, and can only act *ex post*, after a systemic self-fulfilling run has already taken place. For the sake of clarity, here we summarize the modified timing of actions: at date 0, the banks in each wealth group collect the initial endowments, and choose the deposit contracts; at date 1, all agents get to know their private types and signals, and the late consumers decide whether to run or not; then, if a run takes place, the government intervenes with the subsidy scheme; finally, at date 2, all late consumers who did not withdraw at date 1 withdraw.

We solve for the equilibrium by backward induction. The government authority chooses the allocation of subsidies that maximizes the *ex post* welfare

of the whole economy if a run has taken place:

$$\max_{s^j} \sum_j m^j \left[u(e^j + s^j) \right], \quad (38)$$

subject to the budget constraint:

$$\sum_j m^j s^j \leq \bar{e} \quad (39)$$

and to the non-negativity constraint $s^j \geq 0$. The equilibrium allocation of the subsidies satisfies the first-order condition:

$$u'(e^j + s^j) + \lambda^j = u'(e^\ell + s^\ell) + \lambda^\ell, \quad (40)$$

for every group j, ℓ , where λ^j is the Lagrange multiplier on the non-negativity constraint of s^j . Assuming an interior solution, this condition boils down to $e^j + s^j = e^\ell + s^\ell$: it is optimal to equalize the ex post consumption of all depositors in the economy, by engaging in a redistribution of the public good from the wealth-rich groups to the poor ones.

Having characterized the optimal subsidy scheme, we analyze how it affects the incentives of a late consumer to join a run. To this end, as in the previous sections, we study the advantage of waiting versus running, which is given by:

$$v^j(p, n, n^j, s^j) = \begin{cases} \sigma u \left(R(1-n) \frac{e^j - n^j d^j}{1-n^j} \right) - u(d^j) & \text{if } \pi \leq n^j < \frac{e^j}{d^j}, \\ -u \left(\frac{e^j + s^j}{n^j} \right) & \text{if } \frac{e^j}{d^j} \leq n^j \leq 1. \end{cases} \quad (41)$$

From the previous expression, it is clear that the subsidy scheme only affects the utility of the depositors after bankruptcy. Then, applying the Belief Constraint as in the previous sections, we derive the threshold signal below which a late consumer runs as:

$$\sigma^*(\mathbf{d}, \mathbf{s}) = \frac{(1-\pi) \sum_j m^j \left[\int_{\pi}^{\frac{e^j}{d^j}} u(d^j) dn^j + \int_{\frac{e^j}{d^j}}^1 u \left(\frac{e^j + s^j}{n^j} \right) dn^j \right]}{\sum_j m^j \int_{\pi}^1 \int_{\pi}^{\frac{e^j}{d^j}} u \left(R(1-n) \frac{e^j - n^j d^j}{1-n^j} \right) dn^j dn}. \quad (42)$$

The threshold signal $\sigma^*(\mathbf{d}, \mathbf{s})$ is now increasing in any subsidy s^j : higher subsidies increase the incentives of a late consumer to run, as she internalizes that the ex-post bailout increases her consumption in the case of bankruptcy. This effect is also present in the bailout scheme with full commitment of the previous sections, but is counterbalanced by the fact that the banks can employ the subsidy also to liquidate a lower amount of productive assets, and serve more late consumers who do not run before bankruptcy.

This result highlights the crucial difference that commitment makes in the case of government bailouts: an anticipated ex-post intervention, while redistributing optimally across wealth groups and increasing welfare if a run happens, has the unintended consequence of increasing the probability of a self-fulfilling run ex ante.

6. Concluding Remarks

With the present paper, we aimed at studying how a government should bail out a heterogeneous banking system subject to systemic self-fulfilling runs, what criteria to use in order to choose which banks are systemically important, and what the effects of such a government intervention are on the banking system itself, in terms of redistribution and risk sharing. To this end, we develop a theory of heterogeneous banking, where systemic self-fulfilling runs endogenously emerge as a consequence of a global game among heterogeneous depositors. This framework represents a challenge because the characterization of the endogenous threshold strategy in the presence of heterogeneous depositors' incentives to run is non-trivial: yet, we are able to address this issue by applying – to the best of our knowledge, first in the literature on self-fulfilling bank runs – the concept of Belief Constraint. With this tool in hand, we show that the optimal bailout strategy involves some degree of wealth redistribution, i.e. the full subsidization only of a limited group of banks, and should target those whose bailout has the largest marginal impact on the probability of a systemic self-fulfilling run, and are the poorest. Finally, we also find that, albeit partial, the optimal government bailout scheme is still beneficial for the whole economy, as all depositors enjoy a lower probability of a systemic self-fulfilling run and better risk sharing. In other words, we provide a clear criterion, which can be easily and directly implemented in the real world, to improve expected welfare by bailing out banks during periods of financial fragility.

The analysis presented here can be easily extended in two directions. First, it would be worthwhile analyzing how alternative financing schemes for the bailout fund might affect financial fragility and risk sharing: for example, we might compare the centralized scheme analyzed here, that involves some degree of wealth redistribution across groups, to one that maximizes the expected welfare of the whole economy, but without a common budget. In this way, we might shed some light on the impact of wealth redistribution, which is a heated matter of debate in the discussion about the European Deposit Insurance Scheme.

Second, the present environment is a natural place to study financial regulation. In fact, the banks in equilibrium do not internalize their contribution to financial fragility, in the sense that they choose how to serve their depositors without taking into account the effect that this has on the the total

welfare differential from avoiding a systemic self-fulfilling run for all the other depositors in the economy. This failure of the first theorem of welfare economics calls for the introduction of bank-specific limits to risk sharing, in the form of an upper bound on the amount of early consumption that the banks can offer. In addition to that, it should be noted that the implementation of a partial bailout scheme might generate a “fiscal distortion”: in fact, a subsidized bank does not take into account that, by increasing the amount of early consumption offered to its depositors, also increases the amount of subsidies that is going to receive, thus generating a further tightening of the government budget constraint. This effect, which is reminiscent of the fiscal externality of Davila and Goldstein (2016), should be addressed with some further regulatory intervention, in order to limit the amount of risk sharing provided by the subsidized banks. This argument calls for a differential regulatory treatment between subsidized and unsubsidized banks, and in particular for tighter limits to risk sharing for the first category. We leave a formal analysis of these issues for future work.

References

- Acharya, Viral V., Lasse H. Pedersen, Thomas Philippon, and Matthew Richardson (2017). “Measuring Systemic Risk.” The Review of Financial Studies, 30(1), 2.
- Allen, Franklin, Elena Carletti, Itay Goldstein, and Agnese Leonello (2017). “Government Guarantees and Financial Stability.” ECB Working Paper Series No. 2032.
- Allen, Franklin and Douglas Gale (1994). “Limited Market Participation and Volatility of Asset Prices.” American Economic Review, 84(4), 933–955.
- Carlsson, Hans and Eric van Damme (1993). “Global Games and Equilibrium Selection.” Econometrica, 61(5), 989–1018.
- Cooper, Russell and Hubert Kempf (2016). “Deposit Insurance and Orderly Liquidation without Commitment: Can we Sleep Well?” Economic Theory, 61(2), 365–392.
- Davila, Eduardo and Itay Goldstein (2016). “Optimal Deposit Insurance.” Mimeo.
- Diamond, Douglas W. and Philip H. Dybvig (1983). “Bank Runs, Deposit Insurance, and Liquidity.” Journal of Political Economy, 91(3), 401–419.
- Drehmann, Mathias and Nikola Tarashev (2013). “Measuring the systemic importance of interconnected banks.” Journal of Financial Intermediation, 22(4), 586 – 607.
- Farhi, Emmanuel, Michael Golosov, and Aleh Tsyvinski (2009). “A Theory of Liquidity and Regulation of Financial Intermediation.” Review of Economic Studies, 76, 973–992.
- Frankel, David M., Stephen Morris, and Ady Pauzner (2003). “Equilibrium selection in global games with strategic complementarities.” Journal of Economic Theory, 108(1), 1–44.
- Goldstein, Itay and Ady Pauzner (2005). “Demand-Deposit Contracts and the Probability of Bank Runs.” Journal of Finance, 60(3), 1293–1327.
- Gorton, Gary B. and Andrew Metrick (2012). “Getting up to Speed on the Financial Crisis: A One-Weekend-Reader’s Guide.” NBER Working Paper No. 17778.
- Iyer, Rajkamal, Thais Jensen, Niels Johannesen, and Adam Sheridan (2015). “The Run for Safety: Financial Fragility and Deposit Insurance.” Mimeo.
- Morris, Stephen and Hyun Song Shin (1998). “Unique Equilibrium in a Model of Self-Fulfilling Currency Attacks.” American Economic Review, 88(3), 587–597.
- Morris, Stephen and Hyun Song Shin (2000). “Rethinking Multiple Equilibria in Macroeconomic Modeling.” NBER Macroeconomics Annual, 15, 139–182.
- Romer, Paul M. (1990). “Endogenous Technological Change.” Journal of Political Economy, 98(5), S71–S102.
- Sakovics, Jozsef and Jakub Steiner (2012). “Who Matters in Coordination Problems?” American Economic Review, 102(7), 3439–3461.

- Staum, Jeremy (2012). “Systemic Risk Components and Deposit Insurance Premia.” Quantitative Finance, 12(4), 651–662.
- Tarashev, Nikola, Kostas Tsatsaronis, and Claudio Borio (2016). “Risk Attribution Using the Shapley Value: Methodology and Policy Applications.” Review of Finance, 20(3), 1189–1213.
- Topkis, Donald M. (1998). Supermodularity and Complementarity. Princeton University Press, Princeton, NJ.

Appendix A: Pecuniary Externalities

In this section, we show an alternative way to model the strategic complementarities in a banking model with heterogeneous depositors. In particular, we introduce in the model a pecuniary externality in the spirit of the cash-in-the-market pricing of Allen and Gale (1994), and show that it brings about results that are qualitatively similar to the investment externality that we model in the main text.

To this end, extend the environment of the main text in the following directions. The economy is populated also by a continuum of measure 1 of agents with endowment $e = \{0, 1, 0\}$ and utility:

$$u(c_1, c_2) = c_2. \quad (\text{A.1})$$

These agents do not have access to any storage technology, but they can access a secondary market for the productive assets, where they can buy those that the banks sell in order to finance the depositors' withdrawals. Then, the clearing condition in this secondary market reads:

$$1 = \mathcal{P} \sum_k m^k n^k d^k, \quad (\text{A.2})$$

where the cash-in-the-market price of the productive asset:

$$\mathcal{P} = \frac{1}{\sum_k m^k n^k d^k} \quad (\text{A.3})$$

equalizes their supply, coming from the banks liquidating them in order to finance early withdrawals, to their demand, coming from the risk-neutral buyers. With this in hand, further define X^j as the amount of productive assets that the bank in group j has to liquidate. Then, the bank budget constraint at date 1 reads:

$$\mathcal{P} X^j = n^j d^j. \quad (\text{A.4})$$

Finally, assume that, if the bank goes bankrupt, it cannot access the secondary market, and has to liquidate the productive asset by using a costly liquidation technology with recovery rate $r < 1$.

Under these assumptions, the advantage of waiting versus running for a late consumer in group j is given by:

$$v^j(p, n, n^j) = \begin{cases} \sigma u \left(R \frac{e^j - (\sum_k m^k n^k d^k) n^j d^j}{1 - n^j} \right) - u(d^j) & \text{if } \pi \leq n^j < \frac{e^j}{d^j \sum_k m^k n^k d^k} \\ -u \left(\frac{r e^j}{n^j} \right) & \text{if } \frac{e^j}{d^j \sum_k m^k n^k d^k} \leq n^j < 1 \end{cases} \quad (\text{A.5})$$

It is easy to see the sign of the cross-group strategic complementarities: $v^j(p, n, n^j)$ is decreasing in $n^{\ell \neq j}$ in the first interval, and $= 0$ in the second. The within-group strategic complementarities, however, are more complex. Clearly,

$v^j(p, n, n^j)$ is increasing in n^j in the second interval. In the first interval, instead:

$$\frac{\partial v^j}{\partial n^j} = \frac{\sigma u'(d_L^j(R, n, n^j))R}{(1 - n^j)^2} \left[- \left[m^j n^j d^{j2} + d^j \left(\sum_k m^k n^k d^k \right) \right] (1 - n^j) + e^j + \left(\sum_k m^k n^k d^k \right) n^j d^j \right]. \quad (\text{A.6})$$

This derivative is negative whenever:

$$d^j > \frac{1}{2} \left[- \frac{\sum_{k \neq j} m^k n^k d^k}{m^j n^j (2 - n^j)} + \sqrt{\left(\frac{\sum_{k \neq j} m^k n^k d^k}{m^j n^j (2 - n^j)} \right)^2 + \frac{4e^j}{m^j n^j (2 - n^j)}} \right]. \quad (\text{A.7})$$

This expression, despite being more complex, is essentially similar to Equation (11) in the main text. In other words, the assumption of a investment externality allows us to convey the same message of a pecuniary externality, regarding how strategic complementarities arise in this economy, but in a more parsimonious and elegant way.

Appendix B: Proofs

Proof of Proposition 1. We start by proving the first part of the Proposition. The utility advantage of waiting versus running is:

$$v^j(p, n, n^j) = \begin{cases} \sigma u \left(R(1 - n) \frac{e^j - n^j d^j}{1 - n^j} \right) - u(d^j) & \text{if } \pi \leq n^j < \frac{e^j}{d^j} \\ -u \left(\frac{e^j}{n^j} \right) & \text{if } \frac{e^j}{d^j} \leq n^j < 1, \end{cases} \quad (\text{B.1})$$

where n^j and n are the aggregate actions, i.e. the total number of depositors who are withdrawing at date 1 in group j and in the whole economy, respectively. These are given by:

$$n^j = \pi + (1 - \pi) \text{prob}(\sigma \leq \sigma^{j*}), \quad (\text{B.2})$$

$$n = \sum_k m^k n^k = \pi + (1 - \pi) \sum_k m^k \text{prob}\{\sigma \leq \sigma^{k*}\}. \quad (\text{B.3})$$

Define $\Delta^j = (\sigma^{j*} - \sigma^*)$ as the difference between the threshold signal σ^{j*} of group j and the threshold signal σ^* of a generic group (which will turn out to be the unique equilibrium threshold). Given this definition, we can rescale the aggregate actions as:

$$\tilde{n}^j = \pi + (1 - \pi)(1 - F(\sigma^{j*} - p)) = \pi + (1 - \pi)(1 - F(\Delta^j - \zeta)) \equiv \tilde{n}^j(\zeta, \Delta^j), \quad (\text{B.4})$$

$$\tilde{n} = \sum_k m^k n^k = \pi + (1 - \pi) \sum_k m^k (1 - F(\sigma^{k*} - p)) \equiv \tilde{n}(\zeta, \Delta), \quad (\text{B.5})$$

where Δ is the vector of Δ^j -s. Moreover, define $\vartheta(\tilde{n}, \Delta)$ as the inverse of $\tilde{n}(\zeta, \Delta)$ with respect to ζ . Finally, define:

$$H^j(\sigma^*, \Delta) = \mathbb{E}[v^j(\sigma^* + \vartheta(\tilde{n}, \Delta), \tilde{n}(\zeta, \Delta), \tilde{n}^j(\zeta, \Delta^j))]. \quad (\text{B.6})$$

We follow Frankel *et al.* (2003) and prove by contradiction that the solution to the system of indifference conditions:

$$H^j(\sigma^*, \Delta) = 0, \quad (\text{B.7})$$

for all $j = 1, \dots, G$ is unique. Assume there exist two distinct solutions, namely (σ^*, Δ^*) and $(\sigma^{*'}, \Delta^{*'})$. We distinguish two cases: $\Delta^* = \Delta^{*'}$ and $\Delta^* \neq \Delta^{*'}$. Suppose first that $\Delta^* = \Delta^{*'}$, then it must be that $\sigma^* \neq \sigma^{*'}$ and without loss of generality, $\sigma^* < \sigma^{*'}$. Since $H^j(\sigma^*, \Delta)$ is increasing in σ^* , this implies that $H(\sigma^*, \Delta^*) < H(\sigma^{*'}, \Delta^{*'})$. However, given that both (σ^*, Δ^*) and $(\sigma^{*'}, \Delta^{*'})$ are solutions to the system, we should have that $H(\sigma^*, \Delta^*) = H(\sigma^{*'}, \Delta^{*'}) = 0$, and that is a contradiction.

Now suppose that $\Delta^* \neq \Delta^{*'}$ and $\sigma^* \leq \sigma^{*'}$. Choose $h \in \arg \max_j (\Delta^{j*'} - \Delta^{j*})$ and let $D = \max_j (\Delta^{j*'} - \Delta^{j*}) \geq 0$. Observe that $\Delta^{h*'} - \Delta^{j*'} \geq \Delta^{h*} - \Delta^{j*}$, for all $j = 1, \dots, G$, with strict inequality for at least one j . Define $\tilde{\sigma} = \sigma^{*' + D} > \sigma^{*' \geq \sigma^*$, hence:

$$H^h(\tilde{\sigma}, \Delta^*) \geq H^h(\sigma^*, \Delta^*) = 0. \quad (\text{B.8})$$

In order to prove the contradiction, we have to show that:

$$H^h(\tilde{\sigma}, \Delta) \geq H^h(\sigma^{*'}, \Delta^{*'}) = 0. \quad (\text{B.9})$$

To this end, rewrite:

$$\begin{aligned} H^h(\tilde{\sigma}, \Delta^*) &= \int_0^1 \int_0^1 v^h(p, n, n^h) dn^h dn = \\ &= \int_{-\varepsilon}^{\varepsilon} v^h(\tilde{\sigma}^h - \eta^h, \tilde{n}(\Delta^{h*} - \eta^h, \Delta^*), n^h(\Delta^{h*} - \eta^h, \Delta^{h*})) d\eta^h, \end{aligned} \quad (\text{B.10})$$

where $\tilde{\sigma}^h = \tilde{\sigma} + \Delta^{h*}$, and:

$$H^h(\sigma^{*'}, \Delta^{*'}) = \int_{-\varepsilon}^{\varepsilon} v^h(\sigma^{h*'} - \eta^h, \tilde{n}(\Delta^{h*'} - \eta^h, \Delta^{*'}), n^h(\Delta^{h*'} - \eta^h, \Delta^{h*'})) d\eta^h, \quad (\text{B.11})$$

where $\sigma^{h*'} = \sigma^{*' + \Delta^{h*'}}$. It is easy to see that $\sigma^{h*'} = \tilde{\sigma}^h$, as $\tilde{\sigma}^h = \tilde{\sigma} + \Delta^{h*} = \sigma^{*' + D} + \Delta^{h*} = \sigma^{*' + \Delta^{h*' - \Delta^{h*}} + \Delta^{h*} = \sigma^{*' + \Delta^{h*'}}$. Moreover:

$$\tilde{n}(\Delta^{h*'} - \eta^h, \Delta^{*'}) \geq \tilde{n}(\Delta^{h*} - \eta^h, \Delta^*), \quad (\text{B.12})$$

for all η^h , as:

$$\sum_j m^j (1 - F(\Delta^{j*'} - \Delta^{h*'} + \eta^h)) \geq \sum_j m^j (1 - F(\Delta^{j*} - \Delta^{h*} + \eta^h)) \quad (\text{B.13})$$

holds due to the observation above. Similarly:

$$F(\Delta^{j*'} - \Delta^{h*'} + \eta^h) \leq F(\Delta^{j*} - \Delta^{h*} + \eta^h) \quad (\text{B.14})$$

for all η^h . Hence, $H^j(\tilde{\sigma}, \Delta) \geq H^h(\sigma^{*'}, \Delta^{*'})$ because $H^j(\sigma, \Delta)$ is decreasing in $\tilde{n}(\zeta, \Delta)$ and $\tilde{n}^j(\zeta, \Delta)$. This gives a contradiction, and concludes the proof of the first part of the Lemma.

As far as the second part of the Proposition is concerned, we start by showing that, when ε is small, the system of indifference conditions $H^j(\sigma^*, \Delta)(\varepsilon) = 0$ is well approximated by $H^j(\sigma^*, \Delta)(0) = 0$. Notice that, as $\varepsilon \rightarrow 0$, we have that $\zeta = 0$ and $\vartheta(\tilde{n}, \Delta) = 0$. Hence:

$$\begin{aligned} H^j(\sigma^*, \Delta)(\varepsilon) &= \int_0^1 \int_{\pi}^{\frac{e^j}{d^j}} \left[(\sigma^* + \vartheta(\tilde{n}, \Delta)) u \left(\frac{R(1 - \tilde{n}(\zeta, \Delta))e^j - \tilde{n}^j(\zeta, \Delta)d^j}{1 - \tilde{n}^j(\zeta, \Delta)} \right) + \right. \\ &\quad \left. - u(d^j) \right] d\tilde{n}^j(\zeta, \Delta) d\tilde{n}(\zeta, \Delta) - \int_0^1 \int_{\frac{e^j}{d^j}}^1 u \left(\frac{e^j}{d^j} \right) d\tilde{n}^j(\zeta, \Delta) d\tilde{n}(\zeta, \Delta), \end{aligned} \quad (\text{B.15})$$

$$\begin{aligned} H^j(\sigma^*, \Delta)(0) &= \int_0^1 \int_{\pi}^{\frac{e^j}{d^j}} \left[\sigma^* u \left(\frac{R(1 - \tilde{n}(0, \Delta))e^j - \tilde{n}^j(0, \Delta)d^j}{1 - \tilde{n}^j(0, \Delta)} \right) + \right. \\ &\quad \left. - u(d^j) \right] d\tilde{n}^j(0, \Delta) d\tilde{n}(0, \Delta) - \int_0^1 \int_{\frac{e^j}{d^j}}^1 u \left(\frac{e^j}{d^j} \right) d\tilde{n}^j(0, \Delta) d\tilde{n}(0, \Delta). \end{aligned} \quad (\text{B.16})$$

The intervals of integration of the two functions are the same. Moreover, the integrands are both Lipschitz continuous in σ^* . Hence, there exists a constant C_1 such that:

$$|H^j(\sigma^*, \Delta)(\varepsilon) - H^j(\sigma^*, \Delta)(0)| \leq C_1 \varepsilon. \quad (\text{B.17})$$

In other words, as ε goes to zero, the two systems of equations coincide. To see that also the solutions of the two systems of equations coincide, let σ^* and Δ^* be the solution of the system of indifference conditions $H^j(\sigma^*, \Delta)(0) = 0$. Given any neighbourhood N of (σ^*, Δ^*) , the function $H^j(\sigma^*, \Delta)(0)$ is uniformly bounded from 0 by some ι on $S \setminus N$. Choosing $\bar{\varepsilon}$ such that $|H^j(\sigma^*, \Delta)(\varepsilon) - H^j(\sigma^*, \Delta)(0)| \leq \iota$ for all $\varepsilon < \bar{\varepsilon}$, the system of equations $H^j(\sigma^*, \Delta)(\varepsilon) = 0$ has no solution outside of N .

Finally, to characterize the unique threshold signal $\sigma^*(\mathbf{d})$ in (15), we premultiply all group-specific indifference conditions (B.16) by m^j , and sum

across j :

$$\begin{aligned} \sum_j m^j H^j(\sigma^*, \Delta)(0) &= \sum_j m^j \left[\int_0^1 \int_\pi^{\frac{e^j}{d^j}} \left[\sigma^* u \left(\frac{R(1 - \tilde{n}(0, \Delta))e^j - \tilde{n}^j(0, \Delta)d^j}{1 - \tilde{n}^j(0, \Delta)} \right) + \right. \right. \\ &\quad \left. \left. - u(d^j) \right] d\tilde{n}^j(0, \Delta) d\tilde{n}(0, \Delta) + \right. \\ &\quad \left. - \int_0^1 \int_{\frac{e^j}{d^j}}^1 u \left(\frac{e^j}{d^j} \right) d\tilde{n}^j(0, \Delta) d\tilde{n}(0, \Delta) \right]. \end{aligned} \quad (\text{B.18})$$

By the Laplacian Property, $\tilde{n}^j(0, \Delta) \sim U[\pi, 1]$, hence the probability distribution $f(\tilde{n}^j(0, \Delta)) = \frac{1}{1-\pi}$ is independent of Δ . In a similar way, by the Belief Constraint (Sakovics and Steiner 2012), the Laplacian Property holds on average, meaning that also $\tilde{n}(0, \Delta) \sim U[\pi, 1]$, therefore the probability distribution $f(\tilde{n}(0, \Delta)) = \frac{1}{1-\pi}$ is independent of Δ . Thus, the average indifference condition takes the form:

$$\begin{aligned} &\sum_j m^j \int_\pi^1 \left[\int_\pi^{\frac{e^j}{d^j}} \sigma^* u \left(R(1 - n) \frac{e^j - n^j d^j}{1 - n^j} \right) dn^j \right] dn = \\ &= \sum_j m^j \int_\pi^1 \left[\int_\pi^{\frac{e^j}{d^j}} u(d^j) dn^j + \int_{\frac{e^j}{d^j}}^1 u \left(\frac{e^j}{n^j} \right) dn^j \right] dn. \end{aligned} \quad (\text{B.19})$$

Rearranging this expression, we get threshold signal $\sigma^*(\mathbf{d})$ in (15). This ends the proof. \square

Proof of Corollary 1. We study the sign of:

$$\begin{aligned} \frac{\partial \sigma^*}{\partial d^j} &= \frac{1}{\left[\sum_j m^j \int_\pi^1 \int_\pi^{\frac{e^j}{d^j}} u \left(R(1 - n) \frac{e^j - n^j d^j}{1 - n^j} \right) dn^j dn \right]^2} \times \\ &\times \left[\left[(1 - \pi) m^j u'(d^j) \left(\frac{e^j}{d^j} - \pi \right) \right] \times \right. \\ &\times \left[\sum_j m^j \int_\pi^1 \int_\pi^{\frac{e^j}{d^j}} u \left(R(1 - n) \frac{e^j - n^j d^j}{1 - n^j} \right) dn^j dn \right] + \\ &\left. - \left[m^j \int_\pi^1 \left[- \int_\pi^{\frac{e^j}{d^j}} u'(d_L^j(R, n, n^j)) R(1 - n) \frac{n^j}{1 - n^j} dn^j \right] dn \right] \times \right. \\ &\left. \times (1 - \pi) \sum_j m^j \left[\int_\pi^{\frac{e^j}{d^j}} u(d^j) dn^j + \int_{\frac{e^j}{d^j}}^1 u \left(\frac{e^j}{n^j} \right) dn^j \right] \right] \end{aligned} \quad (\text{B.20})$$

This is clearly positive, as the utility function $u(c)$ is increasing in c and e^j/d^j is larger than π . This ends the proof. \square

Proof of Lemma 1. We prove this Lemma by contradiction. We start by assuming that $d^j = e^j$ for all $j = 1, \dots, G$. In this case, no self-fulfilling run can happen in any group, because the representative bank is able to serve all depositors, even in the case of a run. Hence, each group j is left only with runs happening in their group-specific lower dominance region, below the thresholds $\underline{\sigma}^j$. Then, the first-order conditions of the banking problems at $d^j = e^j$ read:

$$FOC^j = \pi(1 - \underline{\sigma}^j) \left[u'(e^j) - \frac{1 - \pi}{2}(1 + \underline{\sigma}^j)Ru'(R(1 - \pi)e^j) \right] + \\ - \frac{u'(e^j) - \pi \underline{\sigma}^j Ru'(R(1 - \pi)e^j)}{u(R(1 - \pi)e^j)}(1 - \pi) [\underline{\sigma}^j u(R(e^j - \pi d^j)) - u(e^j)], \quad (\text{B.21})$$

For every j . By definition of the lower dominance region, the last parenthesis of (B.21) is exactly equal to zero. Moreover, as the coefficient of relative risk aversion is larger than 1,¹⁰ we have that:

$$\frac{u'(e^j)}{u'(R(1 - \pi)e^j)} \geq \frac{R(1 - \pi)e^j}{e^j}. \quad (\text{B.22})$$

Hence:

$$FOC^j \geq \pi(1 - \underline{\sigma}^j)R(1 - \pi)u'(R(1 - \pi)e^j) \left[1 - \frac{1 + \underline{\sigma}^j}{2} \right] \quad (\text{B.23})$$

which is strictly positive, as $\underline{\sigma}^j$ is smaller than 1. This proves that there must be at least one group ℓ for which $d^\ell > e^\ell$. We now study the first-order condition of the remaining groups $j \neq \ell$ for which $d^j = e^j$, taking into account that, as $d^\ell > e^\ell$, a systemic self-fulfilling run is possible, below the threshold signal $\sigma^*(\mathbf{d})$:

$$FOC^j = \pi(1 - \sigma^*(\mathbf{d})) \left[u'(e^j) - \frac{1 - \pi}{2}(1 + \sigma^*(\mathbf{d}))Ru'(R(1 - \pi)e^j) \right] + \\ - \frac{\partial \sigma^*(\mathbf{d})}{\partial d^j}(1 - \pi) [\sigma^*(\mathbf{d})u(R(1 - \pi)e^j) - u(e^j)]. \quad (\text{B.24})$$

By definition of the lower dominance region, $\underline{\sigma}^j = u(e^j)/u(R(1 - \pi)e^j) < \sigma^*(\mathbf{d})$, meaning that the second part of (B.24) is positive. For the same lines of

10. The assumption about the coefficient of relative risk aversion is crucial for this result to hold. To see this, rewrite $-\frac{u''(c)c}{u'(c)} > 1$ as $-\frac{u''(c)}{u'(c)} > \frac{1}{c}$. This, in turn, means that $-(\log[u'(c)])' > (\log[c])'$. Integrate between z_1 and $z_2 > z_1$ so as to obtain $\log[u'(z_1)] - \log[u'(z_2)] > \log[z_2] - \log[z_1]$. Once taken the exponent, the last expression gives $\frac{u'(z_1)}{u'(z_2)} > \frac{z_2}{z_1}$. If $z_1 > z_2$, the inequality is reversed.

reasoning above, also the first part of (B.24) is positive. Hence, $FOC^j > 0$, meaning that also $d^j > e^j$ for all $j \neq \ell$. To sum up, $d^j > e^j$ for all groups $j = 1, \dots, G$. \square

Proof of Proposition 2. Attach the Lagrange multiplier ξ to the government budget constraint (26), and $m^j \lambda^j$ and $m^j \chi^j$ to the upper and lower bounds of s^j , respectively. The first-order condition with respect to s^j reads:

$$-\frac{\partial \sigma^*(\mathbf{d}, \mathbf{s})}{\partial s^j} \sum_k m^k \Delta U_B^k + \sigma^*(\mathbf{d}, \mathbf{s}) u'(e^j + s^j) + \lambda^j - \chi^j = \xi \quad (\text{B.25})$$

for all $j = 1, \dots, G$, where ΔU_B^k is defined as in (28). Then, the government bailout scheme satisfies the equilibrium condition:

$$\begin{aligned} & -\frac{\partial \sigma^*(\mathbf{d}, \mathbf{s})}{\partial s^j} \sum_k m^k \Delta U^k + \sigma^*(\mathbf{d}, \mathbf{s}) u'(e^j + s^j) + \lambda^j - \chi^j = \\ & = -\frac{\partial \sigma^*(\mathbf{d}, \mathbf{s})}{\partial s^\ell} \sum_k m^k \Delta U^k + \sigma^*(\mathbf{d}, \mathbf{s}) u'(e^\ell + s^\ell) + \lambda^\ell - \chi^\ell, \end{aligned} \quad (\text{B.26})$$

for any two group j and ℓ . Calculate Ψ^j according to (27), which is obviously positive for every group j , because we proved that $\sigma^*(\mathbf{d}, \mathbf{s})$ is a decreasing function of the subsidy s^j and $u(c)$ is increasing. Then, rank the groups by decreasing Ψ^j . For the condition (B.26) to hold, it must be the case that:

$$\lambda^{(1)} - \chi^{(1)} < \lambda^{(2)} - \chi^{(2)} < \dots < \lambda^{(G)} - \chi^{(G)}, \quad (\text{B.27})$$

where (j) indicates the j -th group in the rank. Assume that $\lambda^{(1)} - \chi^{(1)} > 0$. For this to be true, it must be that $\lambda^{(1)} > 0$ and $\chi^{(1)} = 0$, meaning that the group with the highest Ψ^j gets the lowest possible subsidy, or $s^{(1)} = 0$. But if $\lambda^{(1)} - \chi^{(1)} > 0$, also $\lambda^j - \chi^j > 0$ for all groups j , meaning that all groups get the lowest possible subsidy, or $s^j = 0$ for all j . Clearly, this cannot be an equilibrium, because the threshold signal $\sigma^*(\mathbf{d}, \mathbf{s})$ is decreasing in the subsidies s^j , so the economy would be better off by using the proceeds from taxation to positively subsidize at least one group. Hence, we must have that $\lambda^{(1)} - \chi^{(1)} < 0$, meaning that $\lambda^{(1)} = 0$ and $\chi^{(1)} > 0$: in other words, the group with the highest Ψ^j has to be fully subsidized, or $s^{(1)} = d^{(1)} - e^{(1)}$. On the contrary, assume that $\lambda^{(G)} - \chi^{(G)} < 0$. Then $\chi^{(G)} > 0$ and $\lambda^{(G)} = 0$, implying that $s^{(G)} = d^{(G)} - e^{(G)} > 0$. However, if $\lambda^{(G)} - \chi^{(G)} < 0$, also $\lambda^j - \chi^j < 0$ for all j , and $s^j > 0$ for all j . This can be an equilibrium only if the budget constraint is satisfied, i.e. if (26) holds. Otherwise, the only possible equilibrium left is one where some groups are fully subsidized and some others are not. This implies that there exists a threshold group for which there is indifference. This ends the proof. \square

Proof of Lemma 2. We substitute $d^j = e^j$ in the equilibrium condition of the subsidized banking problem. Notice that, in this case:

$$\underline{\sigma}^j = \frac{u(e^j)}{u(R(1-\pi)e^j)}. \quad (\text{B.28})$$

We now show that the first-order condition (32) is always strictly positive:

$$\begin{aligned} FOC &= \pi(1 - \underline{\sigma}^j) \left[u'(e^j) - \frac{1-\pi}{2}(1 + \underline{\sigma}^j)Ru'(R(1-\pi)e^j) \right] + \\ &\quad - \frac{u'(e^j) - (1-\pi)\underline{\sigma}^j Ru'(R(1-\pi)e^j)}{u(R(1-\pi)e^j)} \times \\ &\quad \times \left[(1-\pi) \left[\frac{u(e^j)}{u(R(1-\pi)e^j)}u(R(1-\pi)e^j) - u(e^j) \right] + \bar{e} \right] + \\ &\quad + \frac{u(e^j)}{u(R(1-\pi)e^j)}u'(e^j). \end{aligned} \quad (\text{B.29})$$

As the coefficient of relative risk aversion is larger than 1, we have that:

$$\frac{u'(e^j)}{u'(R(1-\pi)e^j)} \geq \frac{R(1-\pi)e^j}{e^j}. \quad (\text{B.30})$$

Hence:

$$\begin{aligned} FOC &\geq \pi(1 - \underline{\sigma}^j)R(1-\pi)u'(R(1-\pi)e^j) \left[1 - \frac{1 + \underline{\sigma}^j}{2} \right] + \\ &\quad + \frac{(1-\pi)\underline{\sigma}^j Ru'(R(1-\pi)e^j)}{u(R(1-\pi)e^j)}\bar{e} + \frac{u'(e^j)}{u(R(1-\pi)e^j)} \left[u(e^j) - \bar{e} \right], \end{aligned} \quad (\text{B.31})$$

which is strictly larger than 0, as $\underline{\sigma}^j < 1$ and $u(e^j) > e^j \geq \bar{e}$ with CRRA utility. We provide a simple numerical example that proves this last point. Assume $u(c) = \frac{(c+\zeta)^{1-\gamma} - \zeta^{1-\gamma}}{1-\gamma}$, where ζ is a small but positive constant such that $u(0) = 0$. The smaller ζ is, the higher is the value of x at which $u(x) = x$. The example in Figure B.1 shows that, for ζ sufficiently small, $u(c)$ is always larger than c . This ends the proof. \square

Proof of Proposition 3. To start the proof, notice that the government budget constraint with full bailouts can be written as:

$$\sum_{j=1}^G m^{(j)} \frac{d^{(j)} - e^{(j)}}{\bar{e}} = 1. \quad (\text{B.32})$$

Since the left-hand side of this expression is a weighted average of the subsidy-to-contribution ratio $\frac{d^{(j)} - e^{(j)}}{\bar{e}}$, there are only two possible cases that clear the budget:

1. $d^j - e^j = \bar{e}$ for every group j ;

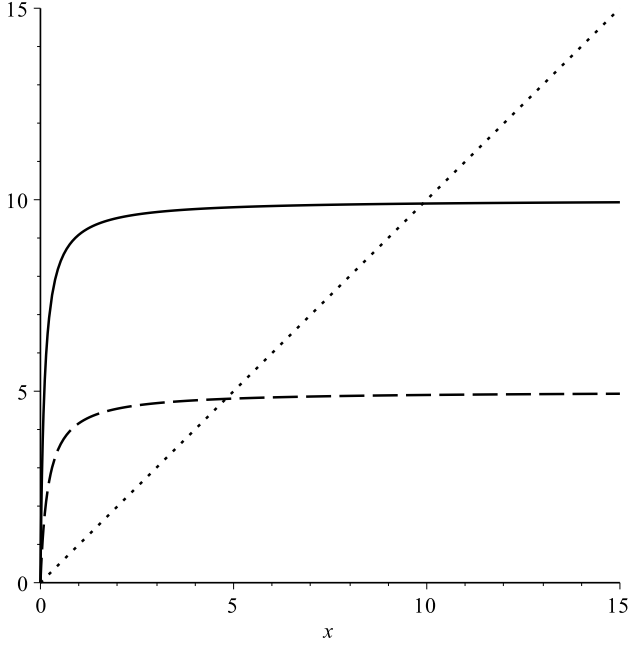


FIGURE B.1: A CRRA utility function with $\gamma = 2$ and $\zeta = 0.1$ (solid line) and $\zeta = 0.2$ (dashed line) versus the linear function $u(x) = x$ (dotted line).

2. $d^j - e^j > \bar{e}$ for some group j and $d^\ell - e^\ell < \bar{e}$ for some other group $\ell \neq j$.

We analyze the two cases separately. For the first case, assume that $d^j = e^j + \bar{e}$ is an equilibrium. Analyze the equilibrium condition of the subsidized banking problem. The right-hand side of (32) reads:

$$\begin{aligned}
 RHS &= \frac{u'(e^j + \bar{e}) - (1 - \pi)\underline{\sigma}^j Ru'(R(1 - \pi)(e^j + \bar{e}))}{u(R(1 - \pi)(e^j + \bar{e}))} \times \\
 &\times \left[(1 - \pi)u(e^j + \bar{e}) \left[\frac{u(R(e^j - \pi(e^j + \bar{e})))}{u(R(1 - \pi)(e^j + \bar{e}))} - 1 \right] + \bar{e} \right] - \underline{\sigma}^j u'(e^j + \bar{e}) = \\
 &= \Theta + u'(e^j + \bar{e}) \frac{\bar{e} - u(e^j + \bar{e})}{u(R(1 - \pi)(e^j + \bar{e}))}, \tag{B.33}
 \end{aligned}$$

where Θ is a sum of negative values. Then, as the utility function $u(c)$ is CRRA:

$$u(e^j + \bar{e}) > e^j + \bar{e} > \bar{e} \tag{B.34}$$

hence the right-hand side of (32) is negative. Consequently, it must also be the case that the left-hand side is negative. This implies that:

$$(1 - \underline{\sigma}^j)u'(e^j + \bar{e}) < (1 - \pi)\frac{1 - \underline{\sigma}^{j2}}{2}Ru'(R(e^j - \pi(e^j + \bar{e}))), \tag{B.35}$$

which can be rewritten as:

$$\frac{1-\pi}{2} \left[1 + \frac{u(e^j + \bar{e})}{u(R(1-\pi)(e^j + \bar{e}))} \right] R > \frac{u'(e^j + \bar{e})}{u'(R(e^j - \pi(e^j + \bar{e})))} \quad (\text{B.36})$$

However, as the coefficient of relative risk aversion is larger than 1:

$$\frac{1-\pi}{2} \left[1 + \frac{u(e^j + \bar{e})}{u(R(1-\pi)(e^j + \bar{e}))} \right] > \frac{e^j - \pi(e^j + \bar{e})}{e^j + \bar{e}} = \frac{e^j}{e^j + \bar{e}} - \pi. \quad (\text{B.37})$$

Hence, a sufficient condition to rule out that $d^j = e^j + \bar{e}$ is that:

$$\pi + \frac{1-\pi}{2} \left[1 + \frac{u(e^j + \bar{e})}{u(R(1-\pi)(e^j + \bar{e}))} \right] - \frac{e^j}{e^j + \bar{e}} \leq 0 \quad (\text{B.38})$$

We numerically evaluate (B.38), assuming that $u(c) = \frac{(c+\zeta)^{1-\gamma} - \zeta^{1-\gamma}}{1-\gamma}$, where again ζ is a small but positive constant such that $u(0) = 0$.¹¹ Figure B.2 shows that the condition (B.38) is negative whenever the coefficient of relative risk aversion γ is close to but higher than 1, is positive for high values of γ , and monotonically increasing. Thus, there exists a unique threshold value of γ below which $d^j = e^j + \bar{e}$ is ruled out as a solution.

Now we move to the second case. Assume that $d^j < e^j + \bar{e}$ is an equilibrium. Again, analyze the equilibrium condition of the subsidized banking problem. The left-hand side of (32) reads:

$$LHS = \pi(1 - \underline{\sigma}^j) \left[u'(d^j) - (1-\pi) \frac{1 + \underline{\sigma}^j}{2} Ru'(R(e^j - \pi d^j)) \right]. \quad (\text{B.39})$$

As the coefficient of relative risk aversion is larger than 1, notice that:

$$\frac{u'(d^j)}{u'(R(e^j - \pi d^j))} \geq \frac{R(e^j - \pi d^j)}{d^j}. \quad (\text{B.40})$$

Hence:

$$LHS \geq \pi(1 - \underline{\sigma}^j) Ru'(R(e^j - \pi d^j)) \left[\frac{e^j}{d^j} - \left(\pi + (1-\pi) \frac{1 + \underline{\sigma}^j}{2} \right) \right]. \quad (\text{B.41})$$

Remember that $\underline{\sigma}^j$ is increasing in d^j . Thus:

$$\pi + (1-\pi) \frac{1 + \underline{\sigma}^j(d^j)}{2} < \pi + (1-\pi) \frac{1 + \underline{\sigma}^j(e^j + \bar{e})}{2} < \frac{e^j}{e^j + \bar{e}}, \quad (\text{B.42})$$

where the second inequality comes from the condition in (B.38). Plugging this into the left-hand side, we get:

$$LHS > (1 - \underline{\sigma}^j) Ru'(R(e^j - \pi d^j)) \left[\frac{e^j}{d^j} - \frac{e^j}{e^j + \bar{e}} \right], \quad (\text{B.43})$$

11. We assign the following values for the remaining parameters: $\pi = .01$, $R = 2.1$, $\zeta = .2$, $e^j = 1$, $\bar{e} = .9$. The numerical results are robust to changes in any of these values.

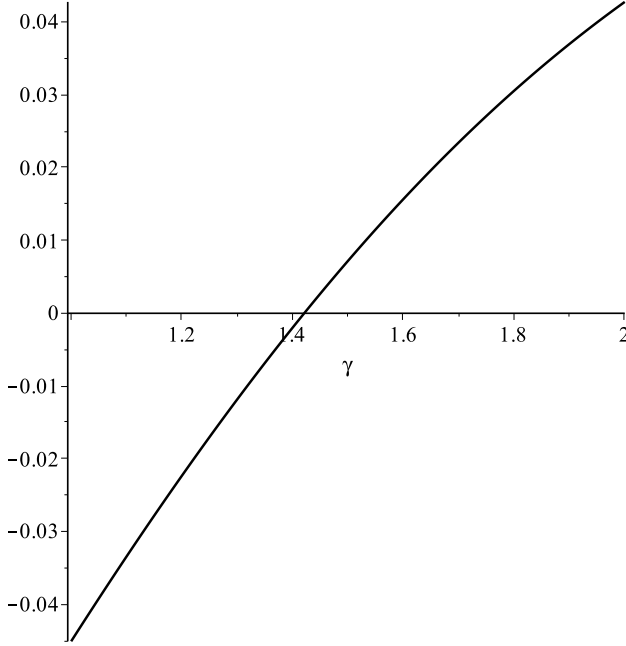


FIGURE B.2: The condition (B.38) that rules out that a subsidy-to-contribution ratio equal to 1 is an equilibrium, as a function of the coefficient of relative risk aversion (on the x-axis).

which is strictly positive, again as $d^j < e^j + \bar{e}$. Finally, we go to the right-hand side of the equilibrium condition (32):

$$\begin{aligned}
 RHS &= \frac{u'(d^j) - (1 - \pi)\underline{\sigma}^j R u'(R(1 - \pi)d^j)}{u(R(1 - \pi)d^j)} \times \\
 &\quad \times \left[(1 - \pi)u(d^j) \left[\frac{u(R(e^j - \pi d^j))}{u(R(1 - \pi)d^j)} - 1 \right] + \bar{e} \right] - \underline{\sigma}^j u'(d^j) = \\
 &= \Theta + u'(d^j) \frac{\bar{e} - u(d^j)}{u(R(1 - \pi)d^j)}, \tag{B.44}
 \end{aligned}$$

where Θ is a sum of negative values, as $d^j \geq e^j$. Then, as $u(c)$ is CRRA:

$$u(d^j) > d^j > \bar{e} \tag{B.45}$$

hence the right-hand side of (32) is negative. To sum up, we have that, under $d^j < e^j + \bar{e}$, the left-hand side of (32) is positive and the right-hand side is negative, which is impossible. Hence, $d^j < e^j + \bar{e}$ cannot be an equilibrium. This ends the proof. \square

Proof of Corollary 2. In order to prove the Lemma, we want to show that the objective function of the unsubsidized bank in (34) is supermodular. In fact, if

that is the case, the equilibrium early consumption d_B^j is an increasing function of the subsidy s^k , for any j and k (Topkis 1998): in other words, a positive subsidy to any group would increase early consumption in all unsubsidized groups.

As the bank objective function is the sum of twice continuously differentiable functions, it is a twice continuously differentiable function itself. Hence, to prove that it is supermodular it suffices to show that $\partial^2 f / \partial d^j \partial s^k$ and $\partial^2 f / \partial s^k \partial d^j$ are both non-negative. The first-order conditions of (34) with respect to d^j and s^j read:

$$\begin{aligned} \frac{\partial f}{\partial d^j} &= \pi \int_{\sigma^*(\mathbf{d}, \mathbf{s})}^1 \left[u'(d^j) - (1 - \pi)pRu'(R(e^j - \pi d^j)) \right] + \\ &\quad - \frac{\partial \sigma^*(\mathbf{d}, \mathbf{s})}{\partial d^j} \left[\pi u(d^j) + (1 - \pi)\sigma^*(\mathbf{d}, \mathbf{s})u(R(e^j - \pi d^j)) + \bar{e} - u(e^j) \right], \end{aligned} \quad (\text{B.46})$$

$$\frac{\partial f}{\partial s^k} = - \frac{\partial \sigma^*(\mathbf{d}, \mathbf{s})}{\partial s^k} \left[\pi u(d^j) + (1 - \pi)\sigma^*(\mathbf{d}, \mathbf{s})u(R(e^j - \pi d^j)) + \bar{e} - u(e^j) \right]. \quad (\text{B.47})$$

Hence, from (B.46) we get:

$$\begin{aligned} \frac{\partial^2 f}{\partial d^j \partial s^k} &= -\pi \frac{\partial \sigma^*(\mathbf{d}, \mathbf{s})}{\partial s^k} \left[u'(d^j) - (1 - \pi)\sigma^*(\mathbf{d}, \mathbf{s})Ru'(R(e^j - \pi d^j)) \right] + \quad (\text{B.48}) \\ &\quad - (1 - \pi) \frac{\partial \sigma^*(\mathbf{d}, \mathbf{s})}{\partial d^j} \frac{\partial \sigma^*(\mathbf{d}, \mathbf{s})}{\partial s^k} u(R(e^j - \pi d^j)) + \\ &\quad - \frac{\partial^2 \sigma^*(\mathbf{d}, \mathbf{s})}{\partial d^j \partial s^k} \left[\pi u(d^j) + (1 - \pi)\sigma^*(\mathbf{d}, \mathbf{s})u(R(e^j - \pi d^j)) + \bar{e} - u(e^j) \right]. \end{aligned} \quad (\text{B.49})$$

The first two terms are positive, as the threshold signal $\sigma^*(\mathbf{d}, \mathbf{s})$ is decreasing in s^k . Regarding the last term, notice that its sign depends on the sign of the cross derivative:

$$\begin{aligned} \frac{\partial^2 \sigma^*}{\partial d^j \partial s^k} &= \frac{m^j}{\left[\sum_j m^j \int_{\pi}^1 \int_{\pi}^{\frac{e^j + s^j}{d^j}} u \left(R(1 - n) \frac{e^j + s^j - n^j d^j}{1 - n^j} \right) dn^j dn \right]^2} \times \\ &\quad \times \left[\left[(1 - \pi) \frac{u'(d^j)}{d^j} + \frac{\partial \sigma^*(\mathbf{d}, \mathbf{s})}{\partial s^k} \right] \times \right. \\ &\quad \times \left[\int_{\pi}^1 \int_{\pi}^{\frac{e^j + s^j}{d^j}} u'(d_L^j(R, n, n^j)) R(1 - n) \frac{n^j}{1 - n^j} dn^j dn \right] + \\ &\quad \left. + \sigma^*(\mathbf{d}, \mathbf{s}) \int_{\pi}^1 \int_{\pi}^{\frac{e^j + s^j}{d^j}} u''(d_L^j(R, n, n^j)) (R(1 - n))^2 \frac{n^j}{(1 - n^j)^2} dn^j dn \right] \times \end{aligned}$$

$$\begin{aligned}
& \times \left[\sum_j m^j \int_{\pi}^1 \int_{\pi}^{\frac{e^j+s^j}{d^j}} u \left(R(1-n) \frac{e^j+s^j-n^j d^j}{1-n^j} \right) dn^j dn \right] + \\
& - \frac{\partial DEN}{\partial s^k} \left[(1-\pi) u'(d^j) \left(\frac{e^j+s^j}{d^j} \right) + \right. \\
& \left. + \sigma^*(\mathbf{d}, \mathbf{s}) \int_{\pi}^1 \int_{\pi}^{\frac{e^j+s^j}{d^j}} u'(d_L^j(R, n, n^j)) R(1-n) \frac{n^j}{1-n^j} dn^j dn \right]. \tag{B.50}
\end{aligned}$$

where DEN is the denominator of $\sigma^*(\mathbf{d}, \mathbf{s})$. As $\sigma^*(\mathbf{d}, \mathbf{s})$ is decreasing in s^k , and its denominator is clearly increasing, we have that the cross derivative is negative. Therefore $\frac{\partial^2 f}{\partial d^j \partial s^k}$ is strictly positive.

As far as the second cross derivative is concerned, we make use of (23) and from (B.47) we calculate:

$$\begin{aligned}
\frac{\partial^2 f}{\partial s^k \partial d^j} = & - \left[\frac{(1-\pi)m^j}{DEN^2} \left[-\frac{u'(d^j)}{d^j} - \frac{\partial \sigma^*(\mathbf{d}, \mathbf{s})}{\partial d^j} \times \right. \right. \\
& \times \int_{\pi}^1 \int_{\pi}^{\frac{e^j+s^j}{d^j}} u'(d_L^j(R, n, n^j)) \frac{R(1-n)}{1-n^j} dn^j dn + \\
& \left. + \sigma^*(\mathbf{d}, \mathbf{s}) \int_{\pi}^1 \int_{\pi}^{\frac{e^j+s^j}{d^j}} u''(d_L^j(R, n, n^j)) \left(\frac{R(1-n)}{1-n^j} \right)^2 n^j dn^j dn \right] \times DEN + \\
& - \frac{\partial DEN}{\partial d^j} \left[\int_{\frac{e^j+s^j}{d^j}}^1 u' \left(\frac{e^j+s^j}{n^j} \right) \frac{1}{n^j} dn^j + \right. \\
& \left. - \sigma^*(\mathbf{d}, \mathbf{s}) \int_{\pi}^1 \int_{\pi}^{\frac{e^j+s^j}{d^j}} u'(d_L^j(R, n, n^j)) \frac{R(1-n)}{1-n^j} dn^j dn \right] \times \\
& \times \left[\pi u(d^j) + (1-\pi) \sigma^*(\mathbf{d}, \mathbf{s}) u(R(e^j - \pi d^j)) + \bar{e} - u(e^j) \right] + \\
& - \frac{\partial \sigma^*(\mathbf{d}, \mathbf{s})}{\partial s^k} \left[\pi u'(d^j) + (1-\pi) \frac{\partial \sigma^*(\mathbf{d}, \mathbf{s})}{\partial d^j} u(R(e^j - \pi d^j)) + \right. \\
& \left. - (1-\pi) \pi \sigma^*(\mathbf{d}, \mathbf{s}) R u'(R(e^j - \pi d^j)) \right]. \tag{B.51}
\end{aligned}$$

Again, as $\sigma^*(\mathbf{d}, \mathbf{s})$ is decreasing in s^k and DEN is decreasing in d^j , this cross derivative is positive. This ends the proof. \square