An Accelerated Approach to Static Hedging Barrier Options: Richardson Extrapolation Techniques

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January, 2018

ABSTRACT

We propose an accelerated static replication approach for pricing barrier options under a variety of models by employing Richardson extrapolation techniques. This approach is first introduced to accelerate the computational efficiency and accuracy of the method of Derman, Ergener, and Kani (1995) (the DEK method) in static hedging a European up-and-out call option under the model of Black and Scholes (1973). The error estimation of this approach aids to determine how many replication matched points should be considered for attaining to a desired accuracy. The application of this approach is further generalized to the constant elasticity of variance (CEV) model and we compare the performance of alternative methods such as the modified DEK method of Chung, Shih, and Tsai (2010). This approach is also extended to improve the method of Fink (2003) under Heston's stochastic volatility model.

JEL Classification: G13

Keywords：Barrier options; Constant elasticity of variance; Error estimation; Richardson extrapolation; Romberg sequence; Static hedging; Stochastic Volatility

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1. Introduction

Hedging exotic options is an important work for the financial institutions to manage risks. The static hedge and the dynamic hedge approaches are two main hedging methods in the financial literature. Compared with the dynamic hedging approach, the static hedge may be cheaper when the transaction frequency is high and the transaction cost is large. There two categories of static hedging approaches developed. The first approach, proposed by Bowie and Carr (1994), Carr and Chou (1997), and Carr, Ellis, and Gupta (1998), is to construct static replication portfolio in a continuum of standard European options with the same maturity as the exotic option but different strike prices. The second approach, developed by Derman, Ergener, and Kani (DEK, 1995) utilize a standard European option to match the boundary at maturity of the exotic option and a continuum of standard European options with the same strike price but different maturities to match the boundary before the maturity of the exotic option. Chung, Shih, and Tsai (2010) show the modified DEK model improving performance of the static hedging method for barrier options significantly. The replication errors of the DEK model results from the non-zero value of the static replication portfolio on the barrier except at some discrete time points. It is able to increase the number of time points to enhance the accuracy of static hedging. However, increasing the number of time points will result in computationally time consuming. Therefore, we employ a new method to improve the efficiency of static hedging in the second category and show its superiority in static hedging. Pricing and hedging Barrier options is a major topic in the financial research.

In this paper, we provide a general accelerated static replication approach with the repeated Richardson extrapolation for pricing barrier options. This article provides an error estimation method to obtain the appropriate number of matched-points given an error tolerance level. Richardson extrapolation technique is often employed to derive an efficient computational formula for evaluating the values of options. For example, Geske and Johnson (1984) firstly use Richardson extrapolation technique to price compound options. Chang, Chung, and Stapleton (2007) also use Richardson extrapolation technique to price American-style options. Chang, Guo, and Hung (2016) further employ Richardson extrapolation technique to derive a more general and computationally efficient formula for American options. However, while Richardson extrapolation technique succeeds in deriving efficient approximation formulae for options, its application in static hedging is seldom explored. Because the computationally time consuming problem of static hedging, it is important to enhance its computational efficiency and accuracy with Richardson extrapolation techniques. Employing Richardson extrapolation techniques to derive an approximation may also provide a way to determine the accuracy of the approximation.[1] One question is how many options or time points we have to choose to achieve a given level of accuracy. Farago, Havasi, and Zlatev (2010) give a reliable error estimation method by applying Richardson extrapolation. We employ their method to determine the number of matched-points or options that should be included into a static replication portfolio to achieve a given desired replication accuracy.

The rest of this paper is organized as follow: In the second section, we introduce the repeated Richardson extrapolation technique. Section 3 shows the application in static hedging a European up-and-out call option under the Black-Scholes model and its performance. In Section 4, we describe an error estimation method with Richardson extrapolation techniques. Section 5 further generalizes its application to the CEV model and compares the performance of alternative

[1] Ibanez (2002) obtains the correct order for the error term by applying the Richardson extrapolation for pricing American option. Chang, Chung, and Stapleton (2007) also use Richardson extrapolation technique to provide reliable error estimation for pricing American-style options.

methods. Section 6 extends its application to the method of Fink (2003) under Heston's stochastic volatility model. Finally, the conclusion is given in section 7.

2. Repeated-Richardson Extrapolation

Consider the problem of calculating an unknown quantity, $f(0)$, for which an analytical formula is not provided. In place of the unknown solution $f(0)$, take a discrete approximation $f(h)$ with the step-size $h > 0$ to be a calculable function yielded by some numerical scheme[2], such that $\lim_{h\to 0} f(h) = f(0) = a_0$. Based on the existence of an asymptotic expansion, we assume that $f(h)$ is a sufficiently smooth function written as:

$$f(h) = a_0 + a_1 h^{p_1} + a_2 h^{p_2} + a_3 h^{p_3} + \cdots + a_k h^{p_k} + O(h^{p_{k+1}}) \qquad (1)$$

with unknown parameters $a_0, a_1, \cdots$, and $0 < p_1 < p_2 < \cdots$, where $h \in [0, H]$ for some basic step $H > 0$ and $O(h^{p_{k+1}})$ denotes a quantity whose size is proportional to $h^{p_{k+1}}$, or possibly smaller. The idea of the Richardson extrapolation is that we can find a linear combination of two different step-sizes. For the sequence of the estimations $f(h_1), f(h_2), \cdots, f(h_n)$, we compute the function $f(h)$ a number of times with successively smaller step-sizes, $h_1 > h_2 > \cdots > 0$. According to Schmidt (1968), we can establish the following recursion when $p_m = p \times m$, $m = 1, \cdots, k$.

Recursion:

For $j = 1, 2, 3, 4, \cdots, i$, the Richardson extrapolation and its repeated version are defined by

$$f_{i,j} = f_{i+1,j-1} + \frac{f_{i+1,j-1} - f_{i,j-1}}{\left(\frac{h_i}{h_{i+j}}\right)^p - 1}, \qquad (2)$$

[2] Arciniega and Allen (2004) apply Richardson extrapolation to increase the order of accuracy of the fully implicit and Crank-Nicolson difference schemes for solving option prices under the model of Black and Scholes (1973).

where $f_{i,0} = f(h_i)$. In the following, we restrict our attention to the case of $p = 1$, which is one of the commonly-used extrapolation schemes. We can establish the following extrapolation tableau stopped at the n–th order:

$$
\begin{array}{llllll}
f(h_1) = f_{1,0} & f_{1,1} & f_{1,2} & f_{1,3} & \cdots & f_{1,n} \\
f(h_2) = f_{2,0} & f_{2,1} & f_{2,2} & & & \\
f(h_3) = f_{3,0} & f_{3,1} & & & & \\
f(h_4) = f_{4,0} & & & & & \\
\vdots & & & & & \\
f(h_{n+1}) = f_{n+1,0} & & & & &
\end{array}
$$

The sequence $\{f(h_i)\}$ is taken as the first column in the extrapolation tableau. Each quantity $f_{i,j}$ is computed in terms of two successive approximations. The two-point Richardson extrapolation technique can be repeated, resulting in a numerical scheme which is extremely fast and can dramatically improve accuracy. The idea behind recursion (2) is to provide two mechanisms for enhancing the accuracy: by increasing i one obtains a reduction in the step-size parameter, while taking j large implies more accurate approximations. Both mechanisms work simultaneously, which indicates that the quantities $f_{n,n-1}$ are those of most interest. This provides us with the possibility of order control. The repeated Richardson extrapolation technique avoids complicated calculation and provides a better result of estimation. Therefore, the repeated Richardson extrapolation can be used to enhance the computational efficiency and accuracy of the static hedging.

3. Richardson Extrapolation Technique in Static Hedging under the Black-Scholes Model

In this section, the well-known Black-Scholes model is considered to be a basic model for the illustration of how to employ Richardson extrapolation technique to improve the efficiency and accuracy of static hedging barrier options. One reason for the Black-Scholes model is that the existing analytical formulae of barrier options can be treated as benchmarks. The basic model under the risk-neutral measure is given by

$$dS_t = (r - d)S_t dt + \sigma S_t dW_t, \tag{3}$$

where S_t is the stock price at time t, r is the risk-free interest rate, d is the dividend yield, σ is the volatility, and W_t denotes a Wiener process.

To solve the possible problem of non-uniform convergence, Omberg (1987) and Chang, Chung, and Stapleton (2007) suggest using geometric-spaced exercise points generated by successively doubling the number of uniformly-spaced exercise dates. Therefore, we also employ Romberg sequence to implement the repeated Richardson extrapolation technique. Romberg sequence: $\{1, 2,\ 4,\ 8,\ 16,\ 32,\ 64,\ 128, \cdots,\ 2^n\}$. One reason for Romberg sequence is to increase the possibility that $f(h_{\mathrm{i}})$ could monotonically converge to $f(0)$ by ensuring each matched replication set nesting the previous one. In addition to Romberg sequence, we also consider other three sequences: harmonic sequence: {1, 2, 3, 4, 5, 6, 7, 8, $\cdots$, n , $\cdots$}; double harmonic sequence: {2, 4, 6, 8, 10, 12, 14, 16, $\cdots$, $2n$, $\cdots$ }; Burlisch sequence: {2, 3, 4, 6, 8, 12, 16, 24, 32, $\cdots$, $2n_{k-2}$, $\cdots$ } (for $k \geq 4$). Figure 1 shows the price convergence of a European up-and-out call (UOC) option (parameters: current stock price $S_0 = 105$, strike price $K = 100$, risk-free interest rate $r = 0.055$, dividend yield $d = 0.025$, barrier $B = 110$, volatility $\sigma = 0.2$, and time to maturity $T = 1$) under the DEK model with four different sequences respectively, including harmonic sequence, double harmonic sequence, Burlisch sequence, and Romberg sequence.

[Figure 1 is here]

All the above sequences show the uniform price convergence as the number of matched points N increases under the DEK model. In our study, the geometric-spaced matched points (i.e. Romberg sequence) and the arithmetic-spaced matched points (i.e. harmonic sequence) for pricing European UOC options are not necessary incurring the problem of non-uniform convergence.[3]

[Table 1 is here]

However, the convergence speed of four different sequences may be quite different (see Table 1). For a large error tolerance level, *TOL*=10%, the convergence speeds of four sequences are not different from each other very much. As the error tolerance level being smaller, the harmonic sequence and the double harmonic sequences could become markedly computational time consuming. The computation time for the harmonic sequence could be approximately 110 times longer than that for Romberg sequence. Because the computation time of geometric-spaced matched points could be much less than that of the arithmetic-spaced matched points, Romberg sequence is suggested when employing repeated Richardson extrapolation technique in static hedging barrier options. Hence, given a specific step size $h_i = \Delta t = T/2^i$ where T is the time to maturity of the barrier option, let $f_{i,j}$ denote its corresponding estimation value after applying j-times of Richardson extrapolation to the DEK method. Table 2 indicates results after employing repeated Richardson extrapolation technique in the DEK method of static hedging a European

[3] Chang, Chuang and Stapleton (2007) find that exercise points of harmonic sequence may cause non-uniform price convergence for some American-style options and incur an efficiency reduction of Richardson extrapolation technique.

UOC option with parameters as those in Table 1. The benchmark value for this barrier option is 0.0611 derived from the existing analytical formula under the Black-Scholes model.[4]

[Table 2 is here]

Table 2 shows that repeated Richardson extrapolation technique could improve the DEK method to provide more efficient and accurate estimators for static hedging European UOC options under the traditional Black-Scholes model.

4. Error Estimation

One advantage of the repeated Richardson extrapolation technique is that we are able to acquire the error bounds of the approximation. The repeated Richardson extrapolation can determine the accuracy of the approximation and how many matched points should be considered for attaining to a given desired accuracy.

Farago, Havasi, and Zlatev (2010) show that the error of estimation based on the Richardson extrapolation method can be calculated by ERROR = $\left|\frac{\omega_n - Z_n}{2^p - 1}\right|$, where ω_n and Z_n are estimators and Z_n has the double step-size of ω_n . The estimation value for the specific step-size h is given by:

$$Z_n = f(h) = a_0 + a_1 h^{p_1} + \mathrm{O}(h^{p_2}). \quad (4)$$

For choosing another step-size,

$$\omega_n = f\left(\frac{h}{2}\right) = a_0 + a_1 (\tfrac{h}{2})^{p_1} + \mathrm{O}(h^{p_2}). \quad (5)$$

By omitting the high order term and eliminating a_0, we are able to obtain

[4] Refer to Hull (2012).

$$a_1 = \frac{2^{p_1}(f\left(\frac{h}{2}\right) - f(h))}{h^{p_1}(2^{p_1}-1)}. \quad (6)$$

Substitute Eq. (6) into Eq. (5):

$$f\left(\frac{h}{2}\right) - a_0 = \frac{f\left(\frac{h}{2}\right) - f(h)}{2^{p_1}-1} + \mathrm{O}(h^{p_2}). \quad (7)$$

The left side denotes the accurate error and the right side represents an estimation error. Based on the method of Farago, Havasi, and Zlatev (2010) and under the assumption of $(p_1, p_2) = (1,2)$, the estimated error of $f_{i+1,j}$ would be by $|f_{i+1,j} - f_{i,j}|$ and the accurate error is $|f_{i+1,j} - a_0|$, where a_0 is the theoretical value of the barrier option. We are able to show how the step size can be chosen to meet the error tolerance level in advance.

[Table 3 is here]

The validity of the error estimation method may depend on the high order term $\mathrm{O}(h^2)$. We test the validity of our error estimation method over 1458 options with different parameters (S_0= 70, 80, 90; B =S_0+10, S_0+20, S_0+30; K= B-10, B-20, B-30; r=0.15, 0.2; d=0.01, 0.015, 0.02; σ= 0.1, 0.2, 0.3). The numerical testing results can be shown in Table 3. In Table 3, the denominator represents the number of options whose price estimators match $|f_{i+1,j} - f_{i,j}|$< the desired errors and the numerator is the number of options whose price estimators match $|f_{i+1,j} - f_{i,j}| <$ the desired errors and $|f_{i+1,j} - a_0|$ <the desired errors. In general, the error estimation model works well especially when the sum of i and j increases. This finding is consistent with the result of Chang, Chuang and Richard (2007), who utilize the Schmidt inequality (1968). For example, when $(i-1, i)$ = (2, 3) and j =3, there are 424 options out of 425 options which satisfy the actual error less than the desired error (1% of the theoretical value) given the estimation error less than the desired error. Hence, the probability of estimates satisfying the required accuracy is more than 99 percent. Therefore, we can further provide a quite accurate estimator of the number of

options which would be used to consist of the static hedging portfolio to achieve the required replication accuracy.

On the other hand, it is reasonable to control the step-size of the repeated Richardson extrapolation given an error tolerance parameter that makes the estimation error meet our requirement. Farago, Havasi, and Zlatev (2010) suggest a step-size control method which is given by

$$h_{new} = \omega \times \sqrt[p]{\frac{TOL}{ERROR}} h_{old} \tag{8}$$

where ω = 0.9 in experimental practices, p=1, and

$$h_n = {}^{T}/_{2^n}. \tag{9}$$

Substituting h from Eq. (9) into Eq. (8) ,

$$n_{new} = n_{old} + Log_2\left(\frac{ERROR}{TOL}\right) - Log_2(\omega)\ . \tag{10}$$

Given a specific error tolerance level (*TOL*) and the initial value of n_{old}, we could obtain a better estimator of n_{new} to achieve the required accuracy based on Eq. (10). However, the computation time of finding the best step-size may be time consuming if an improper initial parameter (n) is used. In order to examine the robustness and the best initial value of n by the repeated Richardson extrapolation method, we employ 324 different options (S_0= 90, 95, 105; B =S_0+5, S_0+10, S_0+15; K= B-5, B-10, B-15; r=0.15; d=0.01, 0.015, 0.02; σ= 0.1, 0.2; $T = 1, 2$) and three different-magnitude error tolerance level (*TOL*=10%, 1%, and 0.1%) to test the step-size control method proposed by Farago, Havasi, and Zlatev (2010).

[Table 4 is here]

We define two measures as follows. One is the average-reset-times index (*ARTI*) which represents average times of n being reset by Eq. (10) using the statistical results of the above 324 options. Given a proper initial n_{old} , the best situation is that the step-size control method of

Farago, Havasi, and Zlatev (2010) always finds the best n_{new} at the first time and $ARTI$ will be one. The other measure is the average-excess index (AEI) which represents the average difference between the best n and the n obtained by Eq. (10). The implication of AEI is to show how much resource is wasted to calculate an overestimated n. Under the optimization situation, the parameter AEI should be zero. For example, in case of pricing the European UOC option mentioned above with a required TOL = 1% and specifying initial value of n =5. Based on Eq. (10), we obtain the new n = 8 and the corresponding error = 0.36%. But the best n is 7 with the error =0.76%. In this numerical example, the variable n is reset once, and the difference between the n computed by Eq. (10) and the best n is 8-7 = 1. Hence, the index $ARTI$ is 1 and AEI is also 1.

The statistical results of the above 324 options have been shown in Table 4. In these numerical results, the step-size control method of Farago, Havasi, and Zlatev (2010) works quite well for pricing European UOC options with Richardson extrapolation of the DEK method under the Black-Scholes model. The average reset times are always less than twice in most cases. Except these cases of TOL=0.1% with initial n =6 or 7 and TOL =1% with n = 4, the average-reset-times index is not larger than one (i.e. $ARTI \leq 1$). The step-size control method works quite well for pricing European UOC options with Richardson extrapolation of the DEK method under the Black-Scholes model. We can choose a proper initial n for a required error tolerance level. For TOL=10%, 1%, and 0.1%, the best choice of the initial n are those with the lowest sum of $ARTI$ and AEI (shown in gray grids) which are n =5, 7, and 7, respectively.

5. A Generalization to Constant Elasticity of Variance Model

In the constant elasticity of variance (CEV) model of Cox (1975) and Cox and Ross (1976) and under the risk-neutral probability measure, the stock price is given by

$$\mathrm{d}S_t = (r-d)S_t dt + \sigma S_t^{\beta/2} dW_t \tag{11}$$

with $S_0 > 0$ and where W_t is a standard Brownian motion. The volatility σ, the risk-free rate r, and the dividend yield d are assumed constant. Schroder (1989) extended the CEV model by expressing the pricing formulae of a European call option with strike K and time to maturity T for $\beta < 2$ as follows:

$$C(S_0, K, \sigma, r, d, T, \beta) = S_0 e^{-dT} Q\left(2y; 2 + \frac{2}{(2-\beta)}, 2x\right) - Ke^{-rT}\left(1 - Q\left(2x; \frac{2}{(2-\beta)}, 2y\right)\right) \tag{12}$$

with $Q(\omega; v, \lambda)$ being the complementary distribution function of a non-central chi-square law with v degrees of freedom and non-centrality parameter λ, and where $y = kK^{(2-\beta)}$, $x = kS_0^{(2-\beta)} e^{((r-d)(2-\beta)T)}$, and $k = \frac{2(r-d)}{\sigma^2 (2-\beta)(e^{(r-d)(2-\beta)T} - 1)}$. Even though we will concentrate our analysis in the case of $\beta < 2$, the corresponding CEV call option formula for $\beta > 2$ is also reproduced below:

$$C(S_0, K, \sigma, r, d, T, \beta) = S_0 e^{-dT} Q\left(2x; \frac{2}{(2-\beta)}, 2y\right) - Ke^{-rT}\left(1 - Q\left(2y; 2 + \frac{2}{(2-\beta)}, 2x\right)\right). \tag{13}$$

To facilitate the implementation of the improved DEK approach of Chung, Shih, and Tsai. (2010), we also reproduce the theta formulae of the European call option under the CEV model from Tsai (2014):

$$\frac{\partial C}{\partial t}(S_0, K, \sigma, r, d, T, \beta) =$$

$$rC(S_0, K, \sigma, r, d, T, \beta) - (r-d)S_0 \frac{\partial C}{\partial S}(S_0, K, \sigma, r, d, T, \beta) \frac{-1}{2} \sigma^2 S_0^{\beta} \frac{\partial^2 C}{\partial S^2}(S_0, K, \sigma, r, d, T, \beta) \tag{14}$$

with

$$\frac{\partial C}{\partial S}(S_0,K,\sigma,r,d,T,\beta)=e^{-dT}Q\left(2y;2+\frac{2}{(2-\beta)},2x\right)$$

$$+S_0^{2-\beta}e^{-dT}(2-\beta)ke^{(r-d)(2-\beta)T}\left[Q\left(2y;4+\frac{2}{(2-\beta)},2x\right)-Q\left(2y;2+\frac{2}{(2-\beta)},2x\right)\right]$$

$$-2Ke^{-rT}(2-\beta)kS_0^{1-\beta}e^{(r-d)(2-\beta)T}p\left(2x;\tfrac{2}{2-\beta},2y\right) \qquad (15)$$

$$\frac{\partial^2 C}{\partial {S_0}^2}(S_0,K,\sigma,r,d,T,\beta)=$$

$$e^{-dT}(2-\beta)(3-\beta)kS_0^{1-\beta}e^{(r-d)(2-\beta)T}\left[Q\left(2y;4+\frac{2}{(2-\beta)},2x\right)-Q\left(2y;2+\frac{2}{(2-\beta)},2x\right)\right]$$

$$+S_0^{3-2\beta}e^{-dT}\left[(2-\beta)ke^{(r-d)(2-\beta)T}\right]^2$$

$$\left[Q\left(2y;6+\frac{2}{(2-\beta)},2x\right)-Q\left(2y;4+\frac{2}{(2-\beta)},2x\right)+Q\left(2y;2+\frac{2}{(2-\beta)},2x\right)\right]$$

$$-2Ke^{-rT}(2-\beta)(1-\beta)kS_0^{-\beta}e^{(r-d)(2-\beta)T}p\left(2x;\frac{2}{2-\beta},2y\right)$$

$$+2Ke^{-rT}(2-\beta)^2k^2S_0^{2-2\beta}e^{2(r-d)(2-\beta)T}\left[p\left(2x;\tfrac{2}{2-\beta},2y\right)-p\left(2x;\tfrac{2}{2-\beta}-2,2y\right)\right] \qquad (16)$$

Substituting Eqs. (12), (15), and (16) into (14) yields the theta of a European call option under the CEV model.

To replicate the European up-and-out call (UOC) option in the CEV model written at time t_0 with strike price K, expiration date T, and barrier value B, we follow Tsai (2014) to select a European call option with the same strike and maturity as the UOC option. This process ensures that both the static replication portfolio and the target UOC option have the same payoff at maturity, and the remaining work forces the static hedge portfolio with values matching the boundary conditions of the UOC before expiration at n evenly spaced discrete points, that is $t_0 = 0 < t_1 < t_2 < \cdots < t_{n-1} = T - \Delta t$, where $t_i - t_{i-1} = \Delta t = T/n$ for all i. For example, at time t_{n-1} when the stock price is equal to the barrier price B, the value matching condition suggests that

$$C(B, K, \sigma, r, d, T - t_{n-1}) + W_{n-1} C(B, B, \sigma, r, d, T - t_{n-1}) = 0 \tag{17}$$

where the number of units of the standard European option, W_i, at time t_i can be determined by working backward using similar procedures. After solving all W_is, the value of the n-point static hedge portfolio $C_{n,up-and-out}$ at time 0 is obtained as follows:

$$\begin{gathered} C_{n,up-and-out} = \\ C(S_0, K, \sigma, r, d, T) + W_{n-1} C(S_0, B, \sigma, r, d, T) \\ +W_{n-2} C(S_0, B, \sigma, r, d, t_{n-1}) + \ldots + W_0 C(S_0, B, \sigma, r, d, t_1). \end{gathered} \tag{18}$$

This is the way that Derman, Ergener, and Kani. (1995) employ at-the-money call options with different maturities to form a hedging portfolio. However, the value of such a call option may be sensitive to time changes, especially when the time to maturity approaches zero. Chung, Shih, and Tsai (2010) show the use of the DEK method to form the static replication portfolio may generate serious mismatches at the barrier and apply the improved DEK method by matching both values and thetas of the portfolio on the boundary condition of the UOC option before maturity at n evenly spaced discrete points:

$$\begin{gathered} C(B, K, \sigma, r, d, T - t_{n-1}) + W_{n-1} C(B, B, \sigma, r, d, T - t_{n-1}) \\ +\widehat{W}_{n-1} Bin(B, B, \sigma, r, d, T - t_{n-1}) = 0 \end{gathered} \tag{19}$$

$$\frac{\partial C(B,K,\sigma,r,d,T-t_{n-1})}{\partial t} + W_{n-1} \frac{\partial C(B,B,\sigma,r,d,T-t_{n-1})}{\partial t} + \widehat{W}_{n-1} \frac{\partial Bin(B,B,\sigma,r,d,T-t_{n-1})}{\partial t} = 0 \,, \tag{20}$$

where $Bin(S,K,\sigma,r,d,T)$ denotes a European cash or nothing binary call option with the payoff defined as one dollar if $S_T \geq K$. Tsai (2014) gives the pricing formula of the binary option and its Greeks by

$$Bin(S_0,K,\sigma,r,d,T,\beta) = e^{-rT}\left(1 - Q\left(2x;\frac{2}{(2-\beta)},2y\right)\right) \tag{21}$$

$$\frac{\partial Bin}{\partial S}(S_0,K,\sigma,r,d,T,\beta) = 2e^{-rT}(2-\beta)kS_0^{1-\beta}e^{(r-d)(2-\beta)T}p\left(2x;\frac{2}{2-\beta},2y\right) \tag{22}$$

$$\frac{\partial^2 Bin}{\partial {S_0}^2}(S_0,K,\sigma,r,d,T,\beta) =$$

$$2e^{-rT}(2-\beta)(1-\beta)kS_0^{-\beta}e^{(r-d)(2-\beta)T}p\left(2x;\frac{2}{2-\beta},2y\right)$$

$$-2e^{-rT}(2-\beta)^2k^2S_0^{2-2\beta}e^{2(r-d)(2-\beta)T}\left[p\left(2x;\frac{2}{2-\beta},2y\right) - p\left(2x;\frac{2}{2-\beta}-2,2y\right)\right]. \tag{23}$$

Similarly, the theta of the binary option can be computed from the delta, gamma, and the price formulae with the equation (14). W_i and $\widehat{W}_i$ are then solved using the value-matching and theta-matching conditions. Recursive procedures are used in the backward direction to determine the number of units of the options, W_i and $\widehat{W}_i$. All solutions of W_i and $\widehat{W}_i$ will yield the value of the n-point improved static hedge portfolio $C_{n,up-and-out}^{modified}$ at time 0:

$$\begin{gathered} C_{n,up-and-out}^{modified} = C(S_0,X,\sigma,r,d,T) \\ +W_{n-1}C(S_0,B,\sigma,r,d,T) + \widehat{W}_{n-1}Bin(S_0,B,\sigma,r,d,T) \\ +W_{n-2}C(S_0,B,\sigma,r,d,t_{n-1}) + \widehat{W}_{n-2}Bin(S_0,B,\sigma,r,d,t_{n-1}) \\ +\ldots+W_0C(S_0,B,\sigma,r,d,t_1) + \widehat{W}_0Bin(S_0,B,\sigma,r,d,t_1). \end{gathered} \tag{24}$$

We start to compare the numerical results of static replications for the DEK, improved DEK, DEK Richardson extrapolation, and improved DEK Richardson extrapolation methods.

[Table 5 is here]

Assume that the underlying price follows model of (11) with $\beta = 0$, $\sigma S_0^{\beta/2-1} = 0.25$, $r = 0.10$, and $d = 0$. A closed form solution to the value of an up and out call option with $S_0 = 100$, $K = 100$, $B = 120$, and $T = 1$ is not yet known under the CEV model. The benchmark value of the UOC is 0.871 calculated by Tsai (2014) from the transformed trinomial tree of Boyle and Tian (1999) with 100,000 time-steps. In Table 5, the first column shows the parameter (i) that determines the time period ($h_i = \Delta t = T/2^i$) between adjacent matched time points along the barrier boundary of the UOC option and the second column shows the estimation values obtained from the DEK static hedging. Numbers in brackets are time consumption measured by seconds. The third column shows results after employing Richardson extrapolation technique once ($j = 1$). The fourth column shows results after employing Richardson extrapolation technique twice ($j = 2$) and so on. The benefit to employ Richardson extrapolation technique is significant especially when there are only a few time points matched in static hedging. For example, when $i = 4$, there are only 17 ($17 = 4^2 + 1$) matched time points along the barrier boundary. Applying static hedging with the DEK method gives $f_{4,0} = 1.041$ and $f_{3,0} = 1.222$. After employing Richardson extrapolation once, we have $f_{3,1} = f_{4,0} + (f_{4,0} - f_{3,0})/(2-1) = 0.861$ by Eq. (2). A reduction of error in percentage is $(|f_{4,0} - a_0| - |f_{3,1} - a_0|)/|f_{4,0} - a_0| = 94.12\%$ where $a_0 = 0.871$. If employing Richardson extrapolation twice, we will have $f_{2,2} = 0.869$ and a further reduction of error in percentage is $(|f_{4,0} - a_0| - |f_{2,2} - a_0|)/|f_{4,0} - a_0| = 98.82\%$.

[Table 6 is here]

In Table 6, we consider the modified DEK method proposed by Chung, Shih, and Tsai (2010) in static hedging UOC options with the theta-match condition. The first column shows the parameter (i) that determines the time period ($h_i = \Delta t = T/2^i$) between adjacent matched time points along the barrier boundary of the UOC option and the second column shows the estimation values obtained from the modified DEK method. Numbers in brackets are time consumption measured by seconds. The third column shows results if we further employ Richardson extrapolation technique once ($j = 1$) in the modified DEK method. The fourth column shows results after employing Richardson extrapolation technique twice ($j = 2$) and so on. Note that the benefit to employ Richardson extrapolation technique becomes less significant in the modified DEK method. For example, $f_{4,0} = 0.883$ and $f_{3,0} = 0.902$. After employing Richardson extrapolation once, we have $f_{3,1} = f_{4,0} + (f_{4,0} - f_{3,0})/(2-1) = 0.863$. A reduction of error in percentage is $(|f_{4,0} - a_0| - |f_{3,1} - a_0|) \,/\, |f_{4,0} - a_0| = 33.33\%$ where $a_0 = 0.871$. If employing Richardson extrapolation twice, we will have $f_{2,2} = 0.866$. The error reduction in percentage increases to 58.33 % (= $(|f_{4,0} - a_0| - |f_{2,2} - a_0|)/|f_{4,0} - a_0|)$. Although it may seem appropriate for employing the Richardson extrapolation in the modified DEK method, the benefit become less significant than that in the case of the DEK method. For instance, the Richardson extrapolation of the DEK method gives a more accurate value of $f_{2,2}$=0.869 in Table 5 than that of $f_{2,2}$=0.866 in Table 6. Meanwhile, the computation time of $f_{2,2}$=0.869 in Table 5 with the Richardson extrapolation of the DEK method is 0.8514 seconds and that of $f_{2,2}$=0.866 in Table 6 with the Richardson extrapolation of the modified DEK method is 2.4019 seconds. Therefore, the Richardson extrapolation of the DEK method is more preferred an efficient approach to the Richardson extrapolation of the modified DEK method.

[Table 7 is here]

Table 7 compares the performance of alternative methods in static replication of an UOC option in terms of computation time and accuracy. Numbers in Columns 2 and 3 show that the theta-matching condition in the modified DEK method actually improves the accuracy of value estimation of the DEK method especially when the number of discrete points, $2^i + 1$, is small . For instance, when $i = 3$, the modified DEK method gives a more accurate estimated value of 0.902 with the computation time of 0.4278 seconds than that of the DEK method, which gives an estimated value of 1.222 with the computation time of 0.1428 seconds. To achieve a compatible accuracy, the DEK method must increase i to six and its computation time is 7.4001 seconds. Therefore, the theta-match condition of the modified DEK method does improve the efficiency of the original DEK method. However, Table 7 also shows that our proposed DEK Richardson extrapolation method is superior to the modified DEK method in improving the efficiency of the DEK method. For instance, the accuracy of value estimation in gray grids is compatible with each other but their computation time is very different. When the Richardson extrapolation is applied to the DEK method for three times, the computation time of $f_{2,3}$ will be 3.9872 seconds. Meanwhile, the modified DEK method spends 108.48 seconds in order to achieve the compatible accuracy.

6. An Extension to Heston's Stochastic Volatility Model

In this section, we also extend the proposed approach to static hedging UOC options under Heston's stochastic volatility model. We assume that the stochastic process that drives the price of the underlying asset is given by

$$dS_t = (r - d)S_t dt + \sqrt{V_t} S_t (\rho dW_t^V + \sqrt{1 - \rho^2} dW_t^S), \tag{25}$$

$$dV_t = \kappa(\theta - V_t)dt + \sigma\sqrt{V_t} dW_t^V, \tag{26}$$

where κ determines the rate of mean reversion of the variance, θ is its long run mean, and σ determines the volatility of the variance. σ is often referred to as "volatility of volatility." W_t^V and W_t^S are independent Wiener processes. Heston (1993) derives a semi-analytical formula for the value of a European call option when the stock price follows this process. Fink (2003) construct a static hedging whose value equals that of a barrier option for a set of points along the boundary for not just one volatility state, but for a set of volatility states. As before, to create a static hedging we require a portfolio that is equal in value to the up and out call at all points both on the barrier and at the terminal date. One may approximate such a hedge by matching not only at n different points in time, but also at each of n_V different volatility states. The approach is to construct a portfolio equal in value to the up and out call on the grid of points and having no payoff in the interior.

[Table 8 is here]

Assume that the underlying price follows model of (25) and (26) with $V_0 = 0.04$, $\kappa = 1.5$, $\theta = 0.04$, $\rho = -0.5$, $\sigma = 0.2$, $r = 0.055$, and $d = 0.025$. A closed form solution to the value of an up and out call option with $S_0 = 100$, $K = 100$, $B = 110$, and $T = 1$ is not yet known. The benchmark price is provided by Monte Carlo simulation. Fink (2003) shows the simulation yielded a price for the UOC option of \$1.604 with a standard error of 0.005238. In Table 8, the first column shows the parameter (i) that determines the time period ($h_i = \Delta t = T/2^i$) between adjacent matched time points along the boundary for each one of volatility states and the second column shows the estimation values obtained from this static hedging with two volatility states ($n_V = 2$, $v_1 = 0.04$, and $v_2 = 1.00$). Numbers in brackets are time consumption measured by seconds. The third column shows results after employing Richardson extrapolation technique once ($j = 1$). The fourth column shows results after employing Richardson extrapolation technique twice ($j = 2$) and so on. The benefit to employ Richardson extrapolation technique is significant especially when there are only a few time points matched in static hedging. For example, when $i = 2$, there are only 5 ($5 = 2^2 + 1$) matched time points along the boundary for each one of volatility states. Applying static hedging with two volatility states gives $f_{3,0} = 1.7222$ and $f_{2,0} = 1.8062$. After employing Richardson extrapolation once, we have $f_{2,1} = f_{3,0} + (f_{3,0} - f_{2,0})/(2-1) = 1.6382$ by Eq. (2). A reduction of error in percentage is $(|f_{3,0} - a_0| - |f_{2,1} - a_0|)/|f_{3,0} - a_0| = 71.07\%$ where $a_0 = 1.604$. If employing Richardson extrapolation twice, we have $f_{1,2} = 1.6255$ and a further reduction of error in percentage is $(|f_{3,0} - a_0| - |f_{1,2} - a_0|)/|f_{3,0} - a_0|$ =81.81%. However, our proposed method of Richardson extrapolation of Fink shows its ability to improve the accuracy without the cost of computation time. For instance, the accuracy of value estimation in gray grids is compatible with each other but their computation time is very different. If the Richardson extrapolation is applied to the Fink method for two times, its computation time is 5.27 seconds. Meanwhile, the Fink method costs 114.73 seconds in order to achieve the compatible accuracy.

7. Conclusion

This paper proposes an accelerated static replication approach for pricing barrier options by employing Richardson extrapolation techniques. This approach is first examined for barrier options under the Black-Scholes model with the method of Derman, Ergener, and Kani (1995). We employ four different sequences including harmonic sequence, double harmonic sequence, Burlisch sequence, and Romberg sequence to examine their convergence properties and speeds. All the above sequences indicate the uniform convergence. However, the computational time of geometric-spaced exercise points (Romberg sequence) is efficiently less than that of the arithmetic-spaced exercise points (harmonic sequence). Based on the method of Farago, Havasi, and Zlatev (2010), our proposed model can provide a reliable error estimation method and acquire the specific replication portfolio under any given error tolerance level by utilizing the repeated Richardson extrapolation technique. Numerical results demonstrate that the error estimation method works well and aids to determine how many replication matched points should be considered for attaining to a given desired accuracy.

Then, the application of our approach is further generalized to the constant elasticity of variance model. Our approach could perform better than alternative methods in static hedging European up-and-out call option in terms of accuracy and computational time. Finally, this approach is extended to improve the method of Fink (2003) in static hedging barrier options under Heston's stochastic volatility model. Numerical results demonstrate that there is a significant reduction of error in percentage after employing our approach especially when there are only a few time points matched in static hedging. A further extension of our proposed approach to other option static hedging problems is left for future studies.

Reference

Black, F., Scholes, M., 1973. The pricing of options and corporate liability, Journal of Political Economy 81, 637–654.

Bowie, J., Carr, P., 1994. Static simplicity, Risk Magazine 7, 45–49.

Boyle, P., Tian, Y.S., 1999. Pricing lookback and barrier options under the CEV process, Journal of Financial and Quantitative Finance 34, 241–264.

Carr, P., Chou, A., 1997. Breaking barriers: static hedging of barrier securities, Risk Magazine 10, 139–145.

Carr, P., Ellis, K., Gupta, V., 1998. Static hedging of exotic options, Journal of Finance 3, 1165–1190.

Chang, C., Chuang, S., Stapleton, R., 2007. Richardson extrapolation techniques for the pricing of American-style options, Journal of Futures Markets 27, 791–187.

Chang, L, Guo, J., Hung, M., 2016. A generalization of the recursive integration method for the analytic valuation of American options, Journal of Futures Markets, 36, 887-901.

Chung, S., Shih, P., Tsai, W., 2010. A modified static hedging method for continuous barrier options, Journal of Futures Markets 30, 1150–1166.

Geske, R., Johnson, H., 1979. The valuation of compound options, Journal of Financial Economics 7, 63–81.

Chou, A., Georgiev, G., 1998. A uniform approach to static replication, Journal of Risk 1, 73–87.

Cox, J., 1975. Notes on option pricing I: constant elasticity of variance diffusions. Working Paper, Stanford University.

Cox, J., and Ross S., 1976. The valuation of options for alternative stochastic process, Journal of Financial Economics, 3, 145–166.

Derman, E., Ergener, D., Kani, I., 1995. Static options replication, Journal of Derivatives 2, 78–95.

Farago, I., Havasi, A., Zlatev, Z., 2010. Efficient implementation of stable Richardson extrapolation algorithms, Computers and Mathematics with Applications 60, 2309–2325.

Fink, J., 2003. An examination of the effectiveness of static hedging in the presence of stochastic volatility, Journal of Futures Markets 23, 859–890.

Geske, R., Johnson, H., 1979. The valuation of compound options, Journal of Financial Economics 7, 63–81.

Heynen, P., Kat, H., 1996. Discrete partial barrier options with a moving barrier, Journal of Financial Engineering 5, 199–209.

Heston, S., 1993. A closed-form solution for options with stochastic volatility with application to bond and currency option. Review of Financial Studies 6, 327–343.

Hull, J., 2012. Options, Futures, and Other Derivatives, 8th ed, (Pearson: Harlow)

Ibanez, A., 2002. Robust pricing of the American put option: a note on Richardson extrapolation and the early exercise premium, Management Science 49, 1210–1228.

Larguinho, M., Dias, C., Braumann, C., 2013. On the computation option prices and Greeks under the CEV model, Quantitative Finance 13, 907 –917.

Ndogmo, J., Ntwiga, D., 2011. High-order accurate implicit methods for barrier option pricing, Applied Mathematics and Computation 218, 2210–2224.

Omberg, E., 1987. A note on the convergence of the binomial pricing and compound option models, Journal of Finance 42, 463–469.

Schmidt, J., 1968. Asymptomatic approximation: an acceleration convergence method, Numerical Mathematics 11, 53–56.

Schroder, M., 1989. Computing the constant elasticity of variance option pricing formula, Journal of Finance, 44, 211–219.

Tsai, W., 2014. Improved method for static replication under the CEV model, Finance Research Letters, 11, 194–202.

Table 1 Computation Time of Four Sequences with Different Error Tolerance

Sequences \ TOL	TOL=10%	TOL=1%	TOL=0.1%
harmonic	0.0311	4.8958	2383.4222
double harmonic	0.0209	1.8489	919.3173
Burlisch sequence	0.0302	0.8696	33.7024
Romberg sequence	0.0337	1.4474	21.5560

The computation time of four sequences respectively employed in static hedging a European UOC option with S_0=105, K= 100, B=105, r=0.055, d=0.025, σ=0.2, and T=1. The computation time is measured by seconds. For a large error tolerance level, TOL=10%, the convergence speeds of four sequences are not different from each other very much. As the error tolerance level being smaller, the harmonic sequence and the double harmonic sequences could become markedly computational time consuming. The computation time for the harmonic sequence could be approximately 110 times longer than that for Romberg sequence.

Table 2 Numerical Results of Repeated Richardson Extrapolation Employed in the DEK method under the Black-Scholes Model

$f_{i,j}$	DEK	Richardson Extrapolation of DEK						
	$j = 0$	1	2	3	4	5	6	7
$i = 1$	0.2073	0.1016	0.0775	0.0685	0.0646	0.0628	0.0619	0.0615
2	0.1280	0.0805	0.0691	0.0647	0.0628	0.0619	0.0615	
3	0.0924	0.0705	0.0650	0.0629	0.0619	0.0615		
4	0.0760	0.0657	0.0630	0.0620	0.0615			
5	0.0683	0.0634	0.0620	0.0615				
6	0.0646	0.0622	0.0615					
7	0.0628	0.0616						
8	0.0619							

Numerical results of repeated Richardson extrapolation employed in the DEK method of static hedging a standard European up-and-out call option with model parameters: $S_0 = 105$, $K = 100$, $r = 0.055$, $d = 0.025$, $B = 110$, $\sigma = 0.2$, and $T = 1$. The price of this option is 0.0611 obtained from the analytical formula under the Black-Scholes model. Note that $f_{1,j}$ quickly converges to the theoretical benchmark value 0.0611 as j increases.

Table 3 Statistics Results for Error Estimation

(i-1, i) \ j	1	2	3	4	5	6	7
Panel A : Desired error = 1%*theoretical value							
(1,2)	46/56 0.8214	72/79 0.9114	149/154 0.9675	441/443 0.9955	869/870 0.9989	1159/1160 0.9991	
(2,3)	66/72 0.9167	138/142 0.9718	424/425 0.9976	856/858 0.9977	1155/1155 1		
(3,4)	121/122 0.9918	389/392 0.9923	836/838 0.9976	1148/1150 0.9983			
(4,5)	311/316 0.9842	790/792 0.9975	1142/1143 0.9991				
(5,6)	699/700 0.9986	1125/1125 1					
(6,7)	1063/1066 0.9971						
Panel B : Desired error = 0.2%*theoretical value							
(1,2)	22/24 0.9167	31/33 0.9394	45/45 1	66/66 1	126/126 1	334/337 0.9911	
(2,3)	28/33 0.8485	45/45 1	66/66 1	123/125 0.984	333/334 0.997		
(3,4)	42/44 0.9545	64/66 0.9697	117/119 0.9832	321/325 0.9877			
(4,5)	62/62 1	104/105 0.9905	311/312 0.9968				
(5,6)	93/96 0.9688	283/284 0.9965					
(6,7)	225/226 0.9956						
Panel C : Desired error = 0.05%*theoretical value							
(1,2)	12/12 1	13/17 0.7647	22/24 0.9167	32/32 1	45/45 1	66/66 1	
(2,3)	12/16 0.75	21/23 0.913	31/32 0.9688	45/45 1	66/66 1		
(3,4)	20/21 0.9524	29/31 0.9355	45/45 1	65/66 0.9848			
(4,5)	27/28 0.9643	45/45 1	64/65 0.9846				
(5,6)	42/42 1	63/64 0.9844					
(6,7)	61/62 0.9839						

We test the validity of the error estimation method over 1458 options with different parameters (S_0= 70, 80, 90; B =S_0+10, S_0+20, S_0+30; K= B-10, B-20, B-30; r=0.15, 0.2; d=0.01, 0.015, 0.02; σ= 0.1, 0.2, 0.3). In Table 3, the denominator represents the number of options whose price estimators match $|f_{i+1,j}-f_{i,j}|$< the desired errors and the numerator is the number of options whose price estimators match $|f_{i+1,j}-f_{i,j}| <$ the desired errors and $|f_{i+1,j} - a_0| <$the desired errors. In general, the error estimation model works well especially when the sum of i and j increases.

Table 4 Statistics Results for Step-size Control Method

Initial n	TOL=10%	TOL=1%	TOL=0.1%
n =4	(0.62,0.35)	(1.01,0.94)	(1.08,1.00)
n =5	(0.15,0.41)	(0.96,0.16)	(1.12,0.10)
n =6	(0.03,1.23)	(0.76,0.30)	(1.00,0.22)
n =7	(0.00,2.21)	(0.51,0.51)	(1.00,0.18)

In order to examine the robustness and the best initial value of n by the repeated Richardson extrapolation method, we employ 324 different options (S_0= 90, 95, 105; B =S_0+5, S_0+10, S_0+15; K= B-5, B-10, B-15; r=0.15; d=0.01, 0.015, 0.02; σ= 0.1, 0.2; $T = 1, 2$) and three different-magnitude error tolerance level (*TOL*=10%, 1%, 0.1%) to test the step-size control method proposed by Farago, Havasi, and Zlatev (2010). Results for the step-size control method are given in the form of (*ARTI*, *AEI*). The average-reset-times index (*ARTI*) which represents average times of n being reset by Eq. (9) using the statistical results of the above 324 options.Tthe average-excess index (*AEI*) which represents the average difference between the best n and the n obtained by Eq. (9). The average reset times are always less than twice in most cases. Except these cases of *TOL*=0.1% with initial n =6 or 7 and *TOL* =1% with n = 4, the average-reset-times index is not larger than one (i.e. *ARTI*$\leq$1). The step-size control method works quite well for pricing European UOC options with Richardson extrapolation of the DEK method under the Black-Scholes model. We can choose a proper initial n for a required error tolerance level. For *TOL*=10%, 1%, and 0.1%, the best choice of the initial n are those with the lowest sum of *ARTI* and *AEI* (shown in gray grids) which are n =5, 7, and 7.

Table 5 Numerical Results of Repeated Richardson Extrapolation Employed in the DEK method under the CEV Model

$f_{i,j}$ i	DEK $j = 0$	Richardson Extrapolation of DEK 1	 2	 3	 4	 5	 6
2	1.606 (0.0426)	0.837 (0.1855)	0.869 (0.8514)	0.871 (3.9872)	0.871 (18.939)	0.871 (92.205)	0.871 (456.04)
3	1.222 (0.1428)	0.861 (0.6660)	0.870 (3.1357)	0.871 (14.952)	0.871 (73.266)	0.871 (363.83)	
4	1.041 (0.5232)	0.868 (2.4698)	0.871 (11.817)	0.871 (58.313)	0.871 (290.57)		
5	0.955 (1.9466)	0.870 (9.3467)	0.871 (46.497)	0.871 (232.25)			
6	0.912 (7.4001)	0.871 (37.150)	0.871 (185.76)				
7	0.892 (29.750)	0.871 (148.61)					
8	0.881 (118.86)						

Numerical results arise from employing repeat Richardson extrapolation with the DEK method in static hedging an up- and-out call option with $S_0 = 100$, $K = 100$, $B = 120$, $T = 1$, $\beta = 0$, $\sigma S_0^{\beta/2-1} = 0.25$, $r = 0.10$, and $d = 0$. The benchmark value of the UOC is 0.871 calculated by Tsai (2014) from the transformed trinomial tree of Boyle and Tian (1999) with 100,000 time-steps. The first column shows the parameter (i) that determines the time period ($h_i = \Delta t = T/2^i$) between adjacent matched time points along the barrier boundary of the UOC option and the second column shows the estimation values obtained from this static hedging. Numbers in brackets are time consumption measured by seconds. The third column shows results after employing Richardson extrapolation technique once ($j = 1$). The fourth column shows results after employing Richardson extrapolation technique twice ($j = 2$) and so on.

Table 6 Numerical Results of Repeated Richardson Extrapolation Employed in the modified DEK method under the CEV Model

$f_{i,j}$	Modified DEK	Richardson Extrapolation of Modified DEK					
i	$j = 0$	1	2	3	4	5	6
2	0.951 (0.1081)	0.854 (0.5360)	0.866 (2.4019)	0.870 (11.691)	0.871 (60.533)	0.871 (315.69)	0.871 (1627.4)
3	0.902 (0.4278)	0.863 (1.8659)	0.869 (9.2891)	0.870 (48.842)	0.871 (255.15)	0.871 (1311.8)	
4	0.883 (1.4381)	0.868 (7.4231)	0.870 (39.553)	0.871 (206.31)	0.871 (1056.6)		
5	0.875 (5.9850)	0.870 (32.130)	0.871 (166.76)	0.871 (850.30)			
6	0.873 (26.145)	0.870 (134.63)	0.871 (683.54)				
7	0.871 (108.48)	0.871 (548.91)					
8	0.871 (440.43)						

Numerical results arise from employing repeat Richardson extrapolation with the modified DEK method in static hedging an up- and-out call option with $S_0 = 100$, $K = 100$, $B = 120$, $T = 1$, $\beta = 0$, $\sigma S_0^{\beta/2-1} = 0.25$, $r = 0.10$, and $d = 0$. The benchmark value of the UOC is 0.871 calculated by Tsai (2014) from the transformed trinomial tree of Boyle and Tian (1999) with 100,000 time-steps. The first column shows the parameter (i) that determines the time period ($h_i = \Delta t = T/2^i$) between adjacent matched time points along the barrier boundary of the UOC option and the second column shows the estimation values obtained from this static hedging. Numbers in brackets are time consumption measured by seconds. The third column shows results after employing Richardson extrapolation technique once ($j = 1$). The fourth column shows results after employing Richardson extrapolation technique twice ($j = 2$) and so on.

Table 7 Performance Comparison of Alternative Methods of Static Replication under the CEV Model

$f_{i,j}$ i	Modified DEK	DEK	Richardson Extrapolation of DEK					
			j =1	2	3	4	5	6
2	0.951 (0.1081)	1.606 (0.0426)	0.837 (0.1855)	0.869 (0.8514)	0.871 (3.9872)	0.871 (18.939)	0.871 (92.205)	0.871 (456.04)
3	0.902 (0.4278)	1.222 (0.1428)	0.861 (0.6660)	0.870 (3.1357)	0.871 (14.952)	0.871 (73.266)	0.871 (363.83)	
4	0.883 (1.4381)	1.041 (0.5232)	0.868 (2.4698)	0.871 (11.817)	0.871 (58.313)	0.871 (290.57)		
5	0.875 (5.9850)	0.955 (1.9466)	0.870 (9.3467)	0.871 (46.497)	0.871 (232.25)			
6	0.873 (26.145)	0.912 (7.4001)	0.871 (37.150)	0.871 (185.76)				
7	0.871 (108.48)	0.892 (29.750)	0.871 (148.61)					
8	0.871 (440.43)	0.881 (118.86)						

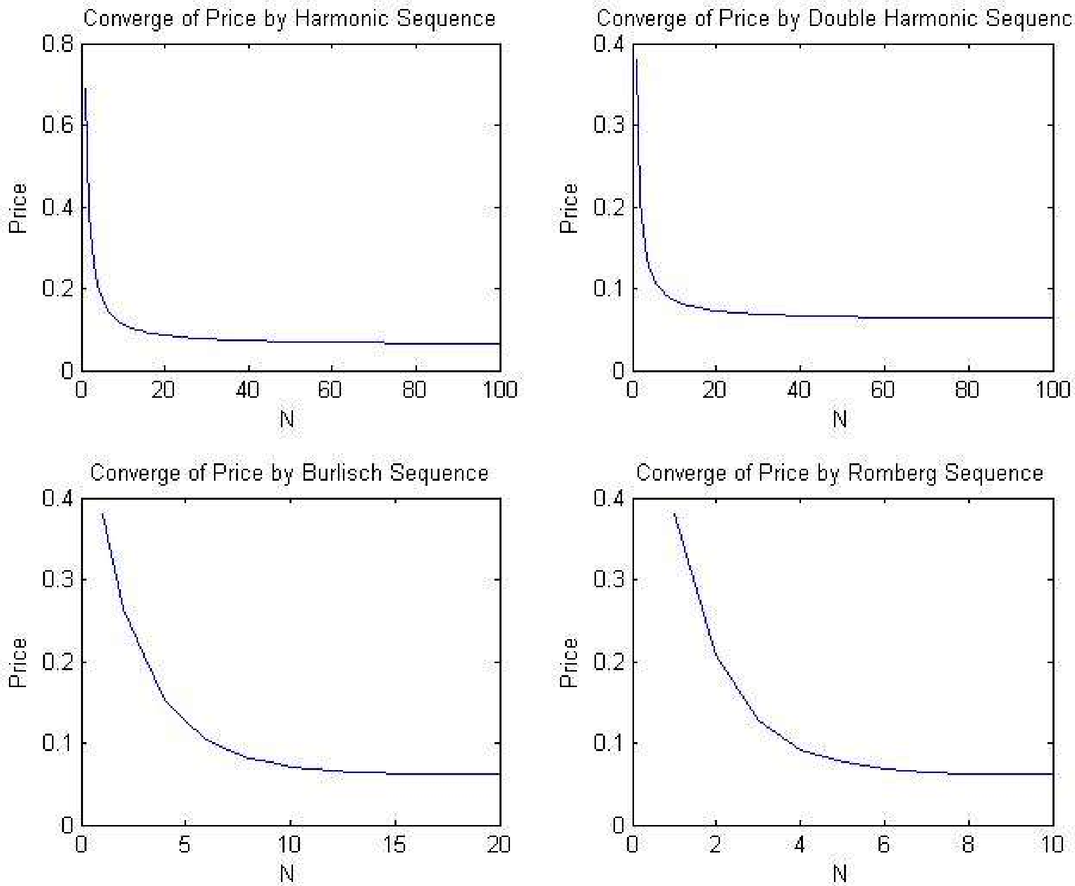
Numerical results arise from employing repeat Richardson extrapolation with the DEK method and its modified version in static hedging an up- and-out call option with $S_0 = 100$, $K = 100$, $B = 120$, $T = 1$, $\beta = 0$, $\sigma S_0^{\beta/2-1} = 0.25$, $r = 0.10$, and $d = 0$. The benchmark value of the UOC is 0.871 calculated by Tsai (2014) from the transformed trinomial tree of Boyle and Tian (1999) with 100,000 time-steps. The first column shows the parameter (i) that determines the time period ($h_i = \Delta t = T/2^i$) between adjacent matched time points along the barrier boundary of the UOC option. Other columns show the estimation values of the UOC option obtained from alternative methods of static replication. Numbers in brackets are time consumption measured by seconds.

Table 8 Numerical Results of Repeated Richardson Extrapolation Employed in the DEK method under Heston's Stochastic Volatility Model

$f_{i,j}$	Fink	Richardson Extrapolation of Fink					
i	$j=0$	1	2	3	4	5	6
1	1.9361 (0.77)	1.6763 (2.04)	1.6255 (5.27)	1.6165 (15.42)	1.6147 (48.34)	1.6141 (163.07)	1.6140 (599.54)
2	1.8062 (1.28)	1.6382 (4.50)	1.6177 (14.65)	1.6148 (47.57)	1.6141 (162.30)	1.6140 (598.77)	
3	1.7222 (3.22)	1.6228 (13.38)	1.6152 (46.30)	1.6142 (161.02)	1.6140 (597.50)		
4	1.6725 (10.15)	1.6171 (43.08)	1.6143 (157.80)	1.6140 (594.28)			
5	1.6448 (32.92)	1.6150 (147.65)	1.6141 (584.12)				
6	1.6299 (114.73)	1.6143 (551.20)					
7	1.6221 (436.47)						

Numerical results arise from employing repeat Richardson extrapolation in static hedging an up and out call option with $S_0 = 100$, $K = 100$, $B = 110$, $T = 1$, $V_0 = 0.04$, $\kappa = 1.5$, $\theta = 0.04$, $\rho = -0.5$, $\sigma = 0.2$, $r = 0.055$, and $d = 0.025$. A closed form solution to the value of this up and out call option is not yet known. The benchmark price is found via Monte Carlo simulation. Fink (2003) shows the simulation yielded a price for the option of $1.604 with a standard error of 0.005238. In Table 7, the first column shows the parameter (i) that determines the time period ($h_i = \Delta t = T/2^i$) between adjacent matched time points along the barrier boundary for each one of volatility states and the second column shows the estimation values obtained from this static hedging with two volatility states ($n_V = 2$, $v_1 = 0.04$, and $v_2 = 1.00$). Numbers in brackets are time consumption measured by seconds. The third column shows results after employing Richardson extrapolation technique once ($j = 1$). The fourth column shows results after employing Richardson extrapolation technique twice ($j = 2$) and so on.

Figure 1. Price Convergence of Static Replication of a European UOC Option Using the DEK Method with Four Different Sequences



Price convergence of four different sequences (harmonic sequence, double harmonic sequence, Burlisch sequence, and Romberg sequence) for static replication of a European UOC option with parameters: S_0 =105, K = 100, B =105, r =0.055, d =0.025, σ =0.2, and T =1. All the above sequences show the uniform price convergence to its theoretical value as the number of matched time points (N) increases using the DEK method.