Transient and Persistent Factor Structure in Equity Options

Hamed Ghanbari*†

Comments are greatly appreciated.

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†Address for correspondence: Hamed Ghanbari, Assistant Professor Finance, Faculty of Management, University of Lethbridge, 4401 University Drive, Lethbridge, Alberta, T1K 3M4, CANADA. E-mail: hamed.ghanbari@uleth.ca.

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Abstract

We introduce a model for derivations of individual equity option prices that captures persistent and transient factor structure in equity options. We derive a closed-form equation for pricing individual equity options and show how transient and persistent factor loadings affect the instantaneous expected returns of equity options. For the firms listed on Dow Jones Index, we find that the option-implied transient betas are always greater than those of persistent betas, implying that large capitalization firms listed in the Dow Jones index co-move more with transient (larger) variations in market. The different sensitivities to the transient and persistent systematic risks are important for a portfolio manager who hedges her portfolio exposure to transient versus persis-tent systematic market variations. Our closed-form sensitivity expressions make this analysis readily available. In cross-sectional analysis, our model predicts that firms with higher transient beta have a steeper term structure of implied volatility and a steeper implied volatility moneyness slope. Our model also predicts that variances risk premiums have more significant effect on the equity option skew when the transient beta is higher. On the empirical front, for the firms listed on the Dow Jones index, our model provides a good fit to the observed equity option prices. At the market index level, we use data from the S&P 500 index and options markets and obtain negative prices for persistent and transient variance components, implying that investors are willing to pay for insurance against increases in volatility risk, even if those increases have little persistence. Our empirical results indicate that unlike stochastic volatility model, join restrictions do not lead to the poor performance of two-factor SV model, measured by Vega-weighted root mean squared errors.

JEL Classification: G10; G12; G13
Keywords: equity options; factor structure; skewness; variance risk premium; two-factor stochastic volatility; joint estimations;

1 Introduction

The dynamics of index return volatility and their role in pricing options have had a long history following the classic early works by Wiggins [1987] and Heston [1993], that recognized the volatility's stochastic nature and managed to derive closed form expressions for the resulting European options. Related early contributions were also by Duan [1995], Duan et al. [1999], and Heston and Nandi [2000] under GARCH return dynamics, with option prices derived either by numerical methods or with closed form expressions. More recent studies, however, have pointed out that a single factor stochastic volatility (SV) or GARCH is not sufficient to represent both the underlying (P) and the risk neutral (Q) measures of the joint dynamics of returns and variances for the key S&P 500 index and its options.[1] In particular, these studies show that one-factor models are incapable of simultaneously fitting the persistence of volatility and the volatility of volatility, and that two volatility factors (one with persistent dynamics and one with transient dynamics) are needed to explain return volatility dynamics; similar considerations apply also to option-based risk neutral returns.

At the market index level, this paper examines index option pricing under two SV factors, where aggregate market volatility is decomposed into a more persistent volatility component, which has nearly a unit root, and a transitory volatility component, which has more rapid time decay. Building up on Christoffersen et al. [2009] model, we adopt an affine two-factor SV process for the underlying index returns and introduce an admissible pricing kernel to derive the risk-neutral index dynamics and to price European options.[2]

We investigate empirically the pricing performance of our two-factor SV model in S&P 500 options by estimating the joint dynamics of returns and variances under the P and Q measures.[3] First, we filter two vectors of daily spot variances using the Particle Filter (PF) method.[4] We follow the conventional filtration procedure of similar studies but provide a novel and methodologically important solution for the challenging issue of how to separate the two variance components' paths. We then use a likelihood-based loss function that combines a return-based and an option-based likelihood functions to obtain a consistent set of structural parameters for the two-factor SV model.[5] The resulting parameter estimates

[1]See, for instance, Bollerslev and Zhou [2002], Alizadeh et al. [2002], and Chernov et al. [2003] for the P-returns and Bates [2000], Christoffersen et al. [2008], and Christoffersen et al. [2009] for the option-based Q-distribution.

[2]Note that the extracted risk-neutral dynamics are not restricted to the introduced admissible pricing kernel, where investor's variance risk preference is distinguished from her equity risk preference. We obtain the same risk-neutral dynamics by assuming a standard stochastic discount factor in Appendix F.

[3]Joint estimation appropriately weights returns and option data and simultaneously address the model's ability to fit the time-series of returns and cross-section of option prices. The importance of joint estimation of the structural parameters of the underlying returns and volatility dynamics has been addressed in Bates [1996], Chernov and Ghysels [2000], Pan [2002], Eraker [2004], and Broadie et al. [2007] among others.

[4]For the application of PF in estimating the model parameters see Gordon et al. [1993], Johannes et al. [2009], Johannes and Polson [2009], Christoffersen et al. [2010], and Boloorforoosh [2014].

[5]The main challenge in such an efficient joint estimation procedure is its heavy computational burden. To overcome this challenge, previous studies mostly focused on a very short time-series and/or weekly/monthly option dataset, See Pan [2002] and Eraker [2004]. However, efficient programming and parallel computing

are therefore consistent with the return data and option data. Further, joint estimation allow us to obtain two separate variance risk premiums; a transient variance risk premium and a persistent variance risk premium. To the best of our knowledge, this is the first study that estimates consistent P- and Q-parameters from underlying index return and option data[6] and reports variance risk premium for a persistent and a transient component.

At the individual equity level, we extend the insights of two factor stochastic volatility models into the pricing of equity options, formulates the simultaneous equilibrium of both equity underlying and option markets, and tests empirically the derived results. In particular, we examine how individual equity option prices respond to the existence of two volatility components and so to the transient and persistent factor loadings. We find that the existence of multiple volatility components in the dynamics of index has significant implications for equity option prices.

We extend the one-volatility-factor model in Christoffersen et al. [2015] and assume that individual equity returns are related to the market index with two distinct systematic components (two constant factor loadings), and an idiosyncratic component which is stochastic and follows the standard square root process. Hence, equity returns are related to the market index with two distinct betas; a transient beta and a persistent beta. We find that option-implied transient beta is always higher than option-implied persistent beta, implying that for large capitalization firms listed in the Dow Jones index, transient (larger) variations in market tends to be related to the proportionally larger systematic price reactions than persistent and smaller variation in the market index.

In empirical analysis of the index model, we find that one of the volatility factors is highly persistent (persistent component) while the immediate impact of volatility shocks on the other volatility factor is bigger but short-lived (transient component). We obtain negative correlation between shocks to the market returns and each variance component, implying that both components are important in capturing the so-called leverage effect. We find the point estimate of the transient correlation parameter is less negative ($\rho_2 = -0.2173$) compared to that of the persistent correlation ($\rho_1 = -0.6918$) and therefore has a less significant effect on the skewness and kurtosis dynamics of index and on the volatility smirk.

We find negative prices for both variance components, namely $\lambda_1 = -1.0798$ and $\lambda_2 = -1.0355$. Our finding implies that investors are willing to pay for an insurance against an increase in volatility risk, even if that increase has little persistence. To the best of our knowledge none of the previous studies of two-factor stochastic volatility models reports the price of the variance risk factors as they either focused on the options market data or the underlying returns data. The negative variance risk premium for both transient and persistent variance components are consistent with the findings in Adrian and Rosenberg [2008]. Using a large cross-section of stock returns data, they find negative and significant

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techniques allow us to keep a large time-series of returns and the entire cross-section of daily option prices over the same time span.

[6] According to Christoffersen et al. [2009, Section 6], "an integrated analysis of multifactor models using option data as well as underlying returns out to be done."

prices for both short-run and long-run volatility components.[7]

At the individual equity level, we estimate the structural parameters and filter spot idiosyncratic variance for the firms listed in the Dow Jones index. We find that proposed option pricing model provides a good fit to the observed equity option prices across all of the 27 firms, both in-sample and out-of-sample. Further, the in-sample performance of our model over the one-factor structure of Christoffersen et al. [2015] together with its cross-sectional implications regarding IV term-structure, moneyness slope, and equity option skew support the importance of transient and persistent factor loadings in pricing equity options. Our estimation results show that the transient and persistent betas have quite different values across all the firms: in our sample of 27 firms, the transient beta has values ranging from 1.01 to 1.35, while the persistent beta is about half the value, range from 0.34 to 0.68.

The proposed factor structure has a number of important cross-sectional implications for equity options. Our model predicts that firms with transient betas have higher implied volatilities. It also predicts that firms with higher transient betas have steeper term structures of implied volatility while the persistent beta has a marginal effect on the implied volatility term structure. It also predicts that the implied volatility moneyness slope is steeper for the firms with the higher transient betas while the persistent beta has a much less significant effect on the moneyness slope. Consistent with previous studies, we find that the variance risk premium has a significant effect on the equity option skew. More to the point, our model predicts that it is the transient variance risk premium that mainly drives the slope of equity implied volatility smile for individual equities.

Our models' framework is especially important for a portfolio manager who hedges her portfolio's exposure to the systematic risk factors in the portfolio of stocks and options.[8] Our proposed factor structure and closed-form option pricing equation make this analysis readily available and yields similar closed-form expressions for the exposure of equity options to the transient and persistent market variance components in addition to its exposure to the overall market returns. We also obtain a closed-form expression for the expected equity option returns and show that exposures to the level of market index and market variance components affect the expected equity option returns. In other words, we are able to disentangle the effect of market risk premium from those of persistent and transient variance risk premiums on the expected equity option returns.

Our proposed factor structure in equity options is motivated by the extensive empirical evidence that supports the presence of two variance components in the dynamics of the market index.[9] In the P-distribution domain, they document that two volatility factors are needed to explain the volatility dynamics, since one-factor models are incapable of simultaneously fitting the persistence of volatility and the volatility of volatility in the dynamics of the

[7] Note that unlike discrete time model of Adrian and Rosenberg [2008] we do not impose any restrictions on the variance dynamics other than independence of variance shocks.

[8] The proposed framework is equally important for risk managers and dispersion traders.

[9] The aggregate market volatility is decomposed into two independent components, one with persistent dynamics and the other one with transient dynamics.

market index. Chernov et al. [2003] suggest that the addition of a second volatility factor breaks the link between tail thickness and volatility persistence and leads to a significant improvement relative to a single SV models in capturing the return dynamics. Bollerslev and Zhou [2002] and Alizadeh et al. [2002] documents the importance of two volatility components in capturing the dynamics of exchange rates. According to Dai and Singleton [2000; 2002] multifcator volatility models are needed to model the term structure of the interest rate.

Extensive empirical evidence in the Q-distribution domain also point toward the existence of two variance components. Egloff et al. [2010] and Mencía and Sentana [2013] find that two-factor SV models have more flexibility to fit the term structure of the volatility and to control the level and the slope of the volatility smirk in cross-sections of option prices. Egloff et al. [2010, Page 1289] show that the upward sloping autocorrelation term structure of variance swap rate quotes points to the existence of multiple variance risk factors. Christoffersen et al. [2009] show that multiple SV models can better capture the time-varying nature of the smirk as ut can generate sufficient amounts of conditional skewness and kurtosis. In a model free framework, they show that the first two principal components of the Black-Scholes implied variances on a sample of S&P 500 index options together explain more than 95% of the variation in the implied variances.

Similar inconsistencies in the joint estimation of the SV model are illustrated by Broadie et al. [2007]. They note the failure of SV model to reconcile the P- and Q-estimates of certain structural parameters of the SV model, namely the correlation coefficient and volatility of volatility, and conclude that the SV model is basically misspecified. They also show that the joint restrictions on the returns and volatility dynamics under the P and Q measures leads to the poor performance of the SV model, measured by the high level of IVRMSE. However, in our empirical analysis, we find that joint restrictions on the P and Q dynamics does not lead to the poor performance of our two-factor SV model.

Although our study is not the first one to examine multifactor SV models, it is the only one to present consistent P- and Q-parameter estimates both theoretically and empirically. For instance Bates [2000] examined a multifactor specification in option pricing by relying on the Q-distribution only. Christoffersen et al. [2008] introduced a two-component GARCH model, which can generate more flexible skewness and volatility of volatility dynamics in capturing the dynamics of the S&P 500 index returns and in pricing European S&P 500 call options. They document that the empirical performance of the volatility component model is significantly better than that of the benchmark GARCH(1,1) model, both in-sample and out-of-sample. They also find that the proposed volatility component specification could better capture the volatility term structure. Nonetheless, the absence of an explicit pricing kernel linking the P- and Q-distributions in that study necessitated either the use of an arbitrary price of volatility risk or the estimation of the risk neutral parameters by relying on the Q-distribution only. Christoffersen et al. [2009] explore multiple variance factors model under Q-distribution only and find that it can generate stochastic correlation between total instantaneous volatility and stock returns.

The paper proceeds as follows. Section 2 presents the theoretical model for pricing index options and individual equity options. In Section 3 we discuss the properties and implications of the model. Section 4 contains the description of the data sets. In Section 5 we discuss the estimation methodologies for both index and equity options. Section 6 contains the estimation results and parameter estimates for index and 27 individual equities and investigate the performance of the models. Section 7 nvestigate the performance of the model and report goodness-of-fit statistics. Section 8 examines the stability of the model and reports the out-of-sample performance. Section 9 concludes. The appendix provides the proofs of the theoretical results and further details on discretization of the model and the Particle Filter Methods.

2 Model Setup

We start by a multiple-factor stochastic volatility dynamics that governs the market index returns under the P-distributions and then introduce a pricing kernel that links the P-dynamics to their risk-neutral counterparts by imposing appropriate martingale's restrictions on pricing kernel. We complete the the index model by deriving a closed-form pricing equation for index options. We then describe the dynamics of individual equity returns under P distribution and introduce an appropriate stochastic discount factor (SDF) to find the equity dynamics under Q measure. Last, we derive a closed-form equation that gives the price of individual equity options.

2.1 The Multifactor Stochastic Volatility Model

We assume the following two-factor stochastic volatility process governing the dynamics of the market index returns and variance under the physical distributions.

$$\begin{aligned} dS_t/S_t &= (r + \mu_1 v_{1,t} + \mu_2 v_{2,t})dt + \sqrt{v_{1,t}}dz_{1,t} + \sqrt{v_{2,t}}dz_{2,t} \\ dv_{1,t} &= \kappa_1(\theta_1 - v_{1,t})dt + \sigma_1\sqrt{v_{1,t}}dw_{1,t} \\ dv_{2,t} &= \kappa_2(\theta_2 - v_{2,t})dt + \sigma_2\sqrt{v_{2,t}}dw_{2,t} \end{aligned} \tag{1}$$

with two independent variance components as described in the following stochastic structure.

$$\begin{aligned} &\langle dw_{1,t}, dz_{1,t}\rangle = \rho_1 dt, \ -1 \leq \rho_1 \leq +1 \\ &\langle dw_{2,t}, dz_{2,t}\rangle = \rho_2 dt, \ -1 \leq \rho_2 \leq +1 \\ &\langle dw_{1,t}, dw_{2,t}\rangle = 0 \\ &\rho_1^2 + \rho_2^2 \leq +1 \\ &\rho_1^2 + \rho_2^2 \leq +1 \end{aligned} \tag{2}$$

As in the Heston [1993] SV model: θ_1 and θ_2 are the unconditional average variances of persistent and transient components, κ_1 and κ_2 capture the speed of mean reversion of each variance components, and σ_1 and σ_2 measure the volatility of variance components. The market equity risk premiums are denoted by $\mu_1 v_{1,t}$ and $\mu_2 v_{2,t}$. Following Bollerslev and Zhou [2006] we expect that μ_1 and μ_2 measure the persistent and transient "continuous-time" volatility feedback effects or risk-return trade-offs. The instantaneous correlation between shocks to the market returns and shocks to the persistent variance component is measured by ρ_1 and the instantaneous correlation between shocks to market returns and shocks to the transient variance component is given by ρ_2. As in Bollerslev and Zhou [2006], we expect that ρ_1 and ρ_2 account for persistent and transient "continuous-time" leverage (asymmetry) effect.

Note that (2) implies that the total return variance and the correlation between return and total variance are as follows.

$$
\begin{aligned}
&\mathrm{Var}_t[dS_t/S_t] = v_{1,t}dt + v_{2,t}dt = v_t dt \\
&\mathrm{Corr}_t[dS_t/S_t, dV_t] = \frac{\rho_1\sigma_1 v_{1,t} + \rho_2\sigma_2 v_{2,t}}{\sqrt{\sigma_1^2 v_{1,t} + \sigma_2^2 v_{2,t}}\,\sqrt{v_{1,t} + v_{2,t}}}dt
\end{aligned} \tag{3}
$$

We may then prove the following result.

Proposition 1. *The market index has the following dynamics under the risk-neutral measure:*

$$
\begin{aligned}
dS_t/S_t &= r dt + \sqrt{v_{1,t}}d\tilde{z}_{1,t} + \sqrt{v_{2,t}}d\tilde{z}_{2,t}\ , \\
dv_{1,t} &= \tilde{\kappa}_1(\tilde{\theta}_1 - v_{1,t})dt + \sigma_1\sqrt{v_{1,t}}d\tilde{w}_{1,t}\ , \\
dv_{2,t} &= \tilde{\kappa}_2(\tilde{\theta}_2 - v_{2,t})dt + \sigma_2\sqrt{v_{2,t}}d\tilde{w}_{2,t}\ ,
\end{aligned} \tag{4}
$$

where, $\tilde{\kappa}_1 = \kappa_1 + \lambda_1$, $\tilde{\kappa}_2 = \kappa_2 + \lambda_2$, $\tilde{\theta}_1 = \frac{k_1\theta_1}{k_1+\lambda_1}$, $\tilde{\theta}_2 = \frac{k_2\theta_2}{k_2+\lambda_2}$. *The market prices of risk factors are*

$$
\begin{aligned}
\psi_{1,t} &= \frac{\sigma_1\mu_1 - \rho_1\lambda_1}{\sigma_1(1-\rho_1^2)}\sqrt{v_{1,t}}\ ,\ \psi_{2,t} = \frac{\sigma_2\mu_2 - \rho_2\lambda_2}{\sigma_2(1-\rho_2^2)}\sqrt{v_{2,t}}\ , \\
\psi_{3,t} &= \frac{\lambda_1 - \rho_1\sigma_1\mu_1}{\sigma_1(1-\rho_1^2)}\sqrt{v_{1,t}}\ ,\ \psi_{4,t} = \frac{\lambda_2 - \rho_2\sigma_2\mu_2}{\sigma_2(1-\rho_2^2)}\sqrt{v_{2,t}}\ .
\end{aligned} \tag{5}
$$

One admissible pricing kernel that links the physical dynamics in (1) to the risk-neutral dynamics in (4) takes the following exponential affine form.

$$
\frac{M_t}{M_0} = \left(\frac{S_t}{S_0}\right)^{\phi} \exp\left[\delta t + \eta_1\int_0^t v_{1,s}ds + \eta_2\int_0^t v_{2,s}ds + \zeta_1(v_{1,t} - v_{1,0}) + \zeta_2(v_{2,t} - v_{2,0})\right] \tag{6}
$$

As in the Christoffersen et al. [2013], $\{\delta,\ \eta_1,\ \eta_2\}$ governs the time-preferences, while $\{\phi,\ \zeta_1,\ \zeta_2\}$ governs the respected risk aversion to the index and variance risk factors, all of which are defined in the appendix.

Proof. See Appendix A. □

We note that the introduced nonlinear log pricing kernel in (6) is one way of "completing the market" and linking P- to Q- dynamics, where ζ_1, ζ_2 capture the nonlinearity of the log pricing kernel.[10] Transforming the physical dynamics in (1) into the risk neutral dynamics in (4) can also be done by assuming the following standard stochastic discount factor and without explicit assumptions about the investor's variance preferences. The proof of such a transformation together with a more general SDF that also includes the price of risk factors for individual equities are provided in appendix F.

$$\frac{dM_t}{M_t} = -rdt - \psi_t' dW_t \ , \tag{7}$$

where $\psi_t \equiv [\psi_{1,t}, \psi_{2,t}, \psi_{3,t}, \psi_{4,t}]$ is the vector of market price of risk factors and $W_t \equiv [z_{1,t}, z_{2,t}, w_{1,t}, w_{2,t}]$ is the vector of innovations in return and variance.

To embed the options market data into the estimation of structural parameters, we determine a closed-from expression for the price of the European call options, with strike price K and time to maturity τ, by inverting the conditional characteristic function of the log spot index prices, $x_t = \ln(S_t)$.

$$C_t(S_t, K, v_{1,t}, v_{2,t}, \tau) = S_t P_1 - K e^{-r\tau} P_2 \ , \tag{8}$$

where,

$$\begin{aligned} P_1 &= \frac{1}{2} + \frac{1}{\pi} \frac{1}{S_t e^{r\tau}} \int_0^\infty \Re\Big[\frac{e^{-i\phi \ln K} \tilde{f}(v_{1,t}, v_{2,t}, \tau, \phi - i)}{i\phi}\Big] d\phi \ , \\ P_2 &= \frac{1}{2} + \frac{1}{\pi} \int_0^\infty \Re\Big[\frac{e^{-i\phi \ln K} \tilde{f}(v_{1,t}, v_{2,t}, \tau, \phi)}{i\phi}\Big] d\phi \ , \end{aligned} \tag{9}$$

and where the risk-neutral conditional characteristic function of the natural logarithm of the index price at expiration, $x_{t+\tau}$, is

$$\tilde{f}(v_{1,t}, v_{2,t}, \tau, \phi) \equiv \mathrm{E}_t^Q \left[\exp(i\phi x_{t+\tau}) \mid x_t\right] . \tag{10}$$

[10] Note also that ζ_1, ζ_2 affect a wedge between physical and risk neutral structural parameters of volatility dynamics.

Since the two-factor SV model in (4) is an affine process, following Heston [1993], the conditional risk-neutral characteristic function in (10) has the following affine exponential form.[11]

$$\tilde{f}(v_{1,t}, v_{2,t}, \tau, \phi) = \exp\left[i\phi x_t + i\phi r\tau + A_1(\tau,\phi) + A_2(\tau,\phi) + B_1(\tau,\phi)v_{1,t} + B_2(\tau,\phi)v_{2,t}\right], \quad (11)$$

where[12] for every $j = \{1, 2\}$

$$\begin{aligned}
A_j(\tau,\phi) &= \frac{\tilde{\kappa}_j\tilde{\theta}_j}{\sigma_j^2}\Big[(\tilde{\kappa}_j - \rho_j\sigma_j i\phi - d_j)\tau - 2\ln\big[\frac{1 - c_j e^{-d_j\tau}}{1 - c_j}\big]\Big] \\
B_j &= \frac{\tilde{\kappa}_j - \rho_j\sigma_j i\phi - d_j}{\sigma_j^2}\big[\frac{1 - e^{-d_j\tau}}{1 - c_j e^{-d_j\tau}}\big] \\
c_j &= \frac{\tilde{\kappa}_j - \rho_j\sigma_j i\phi - d_j}{\tilde{\kappa}_j - \rho_j\sigma_j i\phi + d_j} \\
d_j &= \sqrt{(\tilde{\kappa}_j - \rho_j\sigma_j i\phi)^2 + \sigma_j^2\phi(\phi + i)}\,.
\end{aligned} \quad (12)$$

2.2 The Individual Equity Model

For individual equities, we assume that equity returns are related to the market returns with two distinct systematic risk factors and two constant factor loadings β_1^i and β_2^i. Following Bakshi et al. [2003] we assume that idiosyncratic shocks to equity returns ξ_t^i follows a standard square-root process. This assumption allows us to characterize the differences in the moments' dynamics of individual equity and index options. [13]

$$\begin{aligned}
&dS_t^i/S_t^i = \mu^i dt + \beta_1^i(\mu_1 v_{1,t}dt + \sqrt{v_{1,t}}dz_{1,t}) + \beta_2^i(\mu_2 v_{2,t}dt + \sqrt{v_{2,t}}dz_{2,t}) + \sqrt{\xi_t^i}dz_t^i \\
&d\xi_t^i = \kappa^i(\theta^i - \xi_t^i)dt + \sigma^i\sqrt{\xi_t^i}dw_t^i
\end{aligned} \quad (13)$$

where κ^i, θ^i, and σ^i can be defined as for their market counterparts. ρ^i is the correlation coefficient between idiosyncratic return innovations and idiosyncratic variance innovations for every individual equity i. This parameter derives an asymmetry in the relation between

[11] Note that the conditional risk-neutral characteristic function of the natural logarithm of return, $x_{t+\tau} - x_t = \ln(S_{t+\tau}/S_t)$, can be defined with the same expression as (11) but without the first component, $i\phi x_t$.

[12] Following Duffie et al. [2000], the coefficients A_1, A_2, B_1, and B_2 are the solutions of a system of Riccati equations subject to appropriate boundary conditions. For the ease of computation we modify these solutions based on the little Heston trap formulation of Albrecher et al. [2006].

[13] Our model can be extended to examine the idiosyncratic variance risk premium while incorporating two factor structure in the dynamics of equity returns. We discuss the implications of priced idiosyncratic variance in the following section.

idiosyncratic volatility and returns for individual equities.[14] Given the specification (13), the total instantaneous variance for stock i at time t under physical measure is given by

$$v_t^i \equiv (\beta_1^i)^2 v_{1,t} + (\beta_2^i)^2 v_{2,t} + \xi_t^i \tag{14}$$

Proposition (2) gives the risk neutral dynamics of an individual equity by assuming a conventional stochastic discount factor, given the physical dynamics (1) and (13). As in the index model, we also assume that the prices of market variance components are proportional to the spot volatility components.[15].

Proposition 2. *Using a conventional stochastic discount factor similar to (11) and given the dynamics of the individual equity returns under P-measure (13), the following dynamics govern its Q-measure counterparts.*

$$\begin{aligned} dS_t^i/S_t^i &= rdt + \beta_1^i\sqrt{v_{1,t}}d\tilde{z}_{1,t} + \beta_2^i\sqrt{v_{2,t}}d\tilde{z}_{2,t} + \sqrt{\xi_t^i}d\tilde{z}_t^i \\ d\xi_t^i &= \kappa^i(\theta^i - \xi_t^i)dt + \sigma^i\sqrt{\xi_t^i}dw_t^i \end{aligned} \tag{15}$$

The market prices of risk factors are

$$\begin{aligned} \psi_{1,t} &= \frac{\sigma_1\mu_1 - \rho_1\lambda_1}{\sigma_1(1-\rho_1^2)}\sqrt{v_{1,t}}\ ,\ \psi_{2,t} = \frac{\sigma_2\mu_2 - \rho_2\lambda_2}{\sigma_2(1-\rho_2^2)}\sqrt{v_{2,t}}\ , \\ \psi_{1,t}^i &= \frac{\mu^i - r}{\sqrt{\xi_t^i}(1-(\rho^i)^2)}\ ,\ \psi_{2,t}^i = -\frac{\mu^i - r}{\sqrt{\xi_t^i}}\frac{\rho^i}{1-(\rho^i)^2} \end{aligned} \tag{16}$$

Proof. See Appendix B. □

As the dynamics of individual equities are affine, the conditional risk-neutral characteristic function of the natural logarithm of the equity price i is derived analytically in the following proposition. We may then compute a closed-from pricing equation for European equity call options with strike price K and time to maturity τ. See also Appendix C.

Proposition 3. *Given the dynamics of the individual equity returns under the Q-measure (15), the risk-neutral conditional characteristic function of the natural logarithm of individual equity price i, $x_{t+\tau}^i = \ln(S_{t+\tau}^i)$, is:*

[14] Following Andersen et al. [2001] we expect that the observed asymmetry should be weaker but still present for individual equities.

[15] We can simply extend our model and consider the priced idiosyncratic variance risk by assuming that idiosyncratic variance risk is also proportional to the spot idiosyncratic volatility. In this case, $\tilde{\kappa}^i = \kappa^i + \lambda^i$, $\tilde{\theta}^i = \frac{k^i\theta^i}{k^i+\lambda^i}$. Further details are provided in the proof of the Proposition (1).

$$
\begin{aligned}
\tilde{f}^i(x_t^i, v_{1,t}, v_{2,t}, \xi^i, \beta_1^i, \beta_2^i, \tau, \phi) &\equiv \mathrm{E}_t^Q\left[\exp(i\phi x_{t+\tau}^i) \mid x_t^i\right] \\
&= \exp\left[i\phi x_t^i + i\phi r\tau - A_1(\tau,\phi) - A_2(\tau,\phi) - B(\tau,\phi)\right. \\
&\quad \left. + C_1(\tau,\phi)v_{1,t} + C_2(\tau,\phi)v_{2,t} + D(\tau,\phi)\xi_t^i\right] ,
\end{aligned} \tag{17}
$$

where, the expressions for $A_1(\tau,\phi)$, $A_2(\tau,\phi)$, $B(\tau,\phi)$, $C_1(\tau,\phi)$, $C_2(\tau,\phi)$, and $D(\tau,\phi)$ are provided within the proof. Then, individual equity option prices may be found as follows.

$$
C_t^i(S_t^i, K, \tau) = S_t^i P_1^i - Ke^{-r\tau} P_2^i , \tag{18}
$$

where,

$$
\begin{aligned}
P_1^i &= \frac{1}{2} + \frac{1}{\pi}\frac{1}{S_t^i e^{r\tau}}\int_0^\infty \Re\Big[\frac{e^{-i\phi \ln K}\tilde{f}^i(v_{1,t}, v_{2,t}, \xi_t^i, \tau, \phi - i)}{i\phi}\Big] d\phi , \\
P_2^i &= \frac{1}{2} + \frac{1}{\pi}\int_0^\infty \Re\Big[\frac{e^{-i\phi \ln K}\tilde{f}^i(v_{1,t}, v_{2,t}, \xi_t^i, \tau, \phi)}{i\phi}\Big] d\phi .
\end{aligned} \tag{19}
$$

Proof. See Appendix C. □

3 Model Properties and Implications

This section explores, both theoretically and numerically, some of the implications of the proposed two-factor structure in the dynamics of equity returns. In particular, we examine the relative importance of the transient and persistent factors on the sensitivity of the equity option prices with respect to the level of the market index and with respect to each variance component. We also investigate the effects of factor loadings β_1^i and β_2^i and their importance on the instantaneous expected returns of individual equity options. We close this section by exploring a number of important cross-sectional implications of two-factor structure in equity options, some of which shed some lights on the relations between the systematic risk factors and moments of the conditional distribution of equity returns.

In the numerical analysis, we fix parameters as follows; structural parameters for the market index model are from Christoffersen et al. [2009], for individual equities the parameters are set to replicate the observed patterns in the one-factor model of Christoffersen et al. [2015]. Further, these parameter values highlight the importance of two-factor structure relative to one-factor structure in examining the properties and cross-sectional implications of factor structure in equity options. Since we are interested in the role of the persistent beta, β_1^i, and transient beta, β_2^i, we explore the model properties for different sets of betas while keeping the total unconditional risk-neutral equity variance constant.

The total unconditional risk-neutral equity variance is evaluated at its mean reverting value equal to $\tilde{v}^i \equiv (\beta_1^i)^2\tilde{\theta}_1 + (\beta_2^i)^2\tilde{\theta}_2 + \theta^i = 0.11$. Note that we fix the total unconditional risk-neutral market variance to 0.05, with its persistent component $\tilde{\theta}_1 = 0.006$ and transient component $\tilde{\theta}_2 = 0.044$. Therefore, for every set of betas, the unconditional idiosyncratic equity variance can be defined by $\theta^i = \tilde{v}^i - (\beta_1^i)^2\tilde{\theta}_1 - (\beta_2^i)^2\tilde{\theta}_2$. The spot market persistent and transient variance components are set to $v_{1,t} = 0.012$ and $v_{2,t} = 0.048$ respectively and the total spot equity variance is set to $v_t^i = 0.05$. Consequently, for different sets of betas, the spot idiosyncratic variance under the physical measure can be defined as $\xi_t^i = v_t^i - (\beta_1^i)^2 v_{1,t} - (\beta_2^i)^2 v_{2,t}$. We choose the remaining structural parameters of the market and equity dynamics as follows: $\{\tilde{\kappa}_1 = 0.18, \tilde{\kappa}_2 = 2.8, \sigma_1 = 3.6, \sigma_2 = 0.29, \rho_1 = -0.96, \rho_2 = -0.83\}$ and $\{\tilde{\kappa}^i = 0.8, \sigma^i = 0.2, \rho^i = 0\}$. We fix the risk-free rate at 4% per year and examine at-the-money equity options with 3 months to maturity. We explore the model properties and their cross-sectional implications by assuming the ratio of spot index price over spot equity price as $S_t^i/S_t = 0.1$.

The proposed two-factor structure explicitly shows how changes in the level of the spot market index are translated into the equivalent changes in the equity option prices. It also allow us to examine how equity option prices respond to variations in the persistent and transient market variance components. The following proposition establishes these relations and creates a basis for further sensitivity analysis.

Proposition 4. *Given the closed-form equity option pricing expression in Proposition (3), the sensitivity of the individual equity call option prices C_t^i with respect to the level of the market index S_t may be given by:*

$$\frac{\partial C_t^i}{\partial S_t} = \frac{\partial C_t^i}{\partial S_t^i}\frac{S_t^i}{S_t}(\beta_1^i + \beta_2^i)\ . \tag{20}$$

Further, the sensitivity of the individual equity call option prices C_t^i with respect to the market variance components $v_{1,t}$ and $v_{2,t}$ are:

$$\begin{aligned} \frac{\partial C_t^i}{\partial v_{1,t}} &= \frac{\partial C_t^i}{\partial v_t^i}(\beta_1^i)^2\ , \\ \frac{\partial C_t^i}{\partial v_{2,t}} &= \frac{\partial C_t^i}{\partial v_t^i}(\beta_2^i)^2\ . \end{aligned} \tag{21}$$

where the total spot variance for equity i is $v_t^i = (\beta_1^i)^2 v_{1,t} + (\beta_2^i)^2 v_{2,t} + \xi_t^i$.

Proof. See Appendix D. □

We interpret the expression (20) as the "market delta" and the expressions (21) as the "persistent market vega" and "transient market vega" for call options on equity i. Figure (1) shows the market sensitivity of the model-implied equity call option prices, given the structural parameter values defined above. We plot the market delta for different sets of betas to examine the relative importance of transient and persistent factors. Consistent with Christoffersen et al. [2015], we find that firms with different sets of betas have different sensitivities to changes in the level of the market index. Consistent with Proposition (4), we observe that firm's with higher transient (persistent) beta are more sensitive to the changes in the level of the market index when we keep persistent (transient) beta constant. The same is also true for firms with higher average beta. Although, we cannot distinguish between the effect of transient and persistent betas on market delta per se, we observe that at-the-moeny equity call option prices are relatively more sensitive to the transient beta. Note that the top panel of figure (1) replicates the market delta following the calibration in the one-factor model of Christoffersen et al. [2015].

[Figure (1) about here]

Figures (2) and (3) plot the sensitivity of the model-implied equity call option prices with respect to the persistent and transient market variance components using the parameter values described above. Christoffersen et al. [2015] find that firms with higher betas are more sensitive to changes in the market volatility. Our model predicts the same pattern with respect to the total market volatility. More to the point, we find that firms with higher persistent betas are more sensitive to changes in the persistent variance component while the effect of the transient beta on the persistent market vega is marginal but reverse. Further, firms with higher transient betas are more sensitive to changes in the transient variance factor while the effect of the persistent beta on the transient market vega is reverse but significant. In other words, persistent beta has an important effect on the transient market vega across different levels of moneyness (See Figure (3)). This distinctive property of our model allows a portfolio manager to better examine the exposure of her portfolio to the variations in market returns,[16] a feature that is absent in the single factor structure of Christoffersen et al. [2015]. Comparing the level of transient market vega and persistent market vega, our model predicts that equity call option prices are more sensitive to the transient volatility component compared to the persistent volatility component.

[Figure (2) about here]

[Figure (3) about here]

Our two-factor structure and closed-form equity option pricing formula allow us to shed some light on the relation between the expected returns of individual equity options and the characteristics of market returns and variance components as expressed in Proposition (5)

[16] Remember that option market vega is the amount of money per underlying share that the option value will gain or loose as market volatilities rise or fall by 1%. It is also important as value of some option strategies are partially sensitive to changes in volatility.

below. This result allows us to disentangle the effect of the market risk premium from those of variance component risk premiums on the equity option returns. It also shows how equity betas play a direct role on the equity option returns. In particular, the second component in the right-hand-side (RHS) of equation (22), which is related to the market risk premium, affects the equity option returns through the market delta by an adjustment factor which includes the persistent and transient betas. Moreover, the third component in the RHS of (22), which is related to variances risk premiums, shows how equity betas affect the equity option returns through the total market vega of equity options. Note that $\partial C_t^i / \partial v_t$ measures the total market vega of equity options.

Proposition 5. *Given the closed-form equity option pricing expression (18)-(19), the dynamics of the market index (1) and individual equity returns (13), the instantaneous expected excess returns on individual equity call options under the physical measure can be characterized as follows.*

$$\begin{aligned} \frac{1}{dt} E_t^P \Big[\frac{dC_t^i}{C_t^i} - r dt \Big] = & \Big[(\mu^i - r) \frac{S_t^i}{C_t^i} \Big] \frac{\partial C_t^i}{\partial S_t^i} \\ & + \Big[\frac{\beta_1^i \mu_1 v_{1,t} + \beta_2^i \mu_2 v_{1,t}}{\beta_1^i + \beta_2^i} \frac{S_t}{C_t^i} \Big] \frac{\partial C_t^i}{\partial S_t} \\ & + \Big[\frac{(\beta_1^i)^2 \lambda_1 v_{1,t} + (\beta_2^i)^2 \lambda_2 v_{2,t}}{(\beta_1^i)^2 + (\beta_2^i)^2} \frac{1}{C_t^i} \Big] \frac{\partial C_t^i}{\partial v_t} \end{aligned} \tag{22}$$

Proof. See Appendix D. □

Our proposed two-factor structure has also important cross-sectional implications for equity options. Christoffersen et al. [2015] document that firms with higher betas have a steeper term structure of implied volatility. However, our model moves further and provides a novel term structure effect. In particular, we show how the term structure of implied volatility responds differently to the transient and persistent variations in market returns. Using the parameter values introduced at the beginning of this section, we show how β_1^i and β_2^i have different and non-trivial effects on the implied volatility term structures of individual equity options. Figure (4) plots the model implied volatility for at-the-money equity call options with respect to time-to-maturity for different sets of betas. Consistent with the finding in Christoffersen et al. [2015] (the top LHS panel), the higher the average betas the steeper the term structure of the implied volatility of equity options (the top RHS panel). In particular, our model predicts that the term structure of implied volatility of equity options is more sensitive to the Transient beta (the bottom LHS panel) while the impact of the persistent beta on the term structures of implied volatility of equity options is marginal (the bottom RHS panel).[17] In other words, firms with higher transient betas have a term structure of implied volatility that co-moves more with the market term structure of IV.

[17]Note that in all the graphs the total unconditional equity variance under the risk neutral measure is fixed at $\tilde{v}^i = (\beta_1^i)^2 \tilde{\theta}_1 + (\beta_2^i)^2 \tilde{\theta}_2 + \theta^i = 0.11$.

[Figure (4) about here]

We close this section by discussing the implications of two-factor structure on the relation between the market variance risk premiums and the equity option skew. Figure (6) plots the difference between the model implied volatility for three-month equity call options with respect to the moneyness (S/K) for different sets of betas. The implied volatility difference is computed as the difference between equity call option IV when we increase variance component risk premiums from $\lambda_1 = \lambda_2 = -0.5$ to $\lambda_1 = \lambda_2 = 0$. As expected, the variance risk premiums have a more significant effect on the implied volatility of equity call options when the beta is higher (the top RHS panel). In particular, we observe that the transient beta has a more significant effect on the slope of equity implied volatility smile (the bottom LHS panel) compared to the persistent beta (the bottom RHS panel). In other words, in-the-money equity call options are getting relatively more expensive for firms with higher transient betas when we increase variance risk premiums. Note that for all the graphs the total unconditional equity variance is fixed $\tilde{v}^i = (\beta_1^i)^2\tilde{\theta}_1 + (\beta_2^i)^2\tilde{\theta}_2 + \theta^i = 0.11$. Note also that the top LHS panel replicates the same pattern following the calibration in the one-factor model of Christoffersen et al. [2015].

[Figure (5) about here]

We close this section by discussing the implications of two-factor structure on the relation between the market variance risk premiums and the equity option skew. Figure (6) plots the difference between the model implied volatility for three-month equity call options with respect to the moneyness (S/K) for different sets of betas. The implied volatility difference is computed as the difference between equity call option IV when we increase variance component risk premiums from $\lambda_1 = \lambda_2 = -0.5$ to $\lambda_1 = \lambda_2 = 0$. As expected, the variances risk premiums have more significant effect on the implied volatility of equity call option when the beta is higher (the top RHS panel). In particular, we observe that the transient beta has more significant effect on the slope of equity implied volatility smile (the bottom LHS panel) compared to the persistent beta (the bottom RHS panel). In other words, in-the-money equity call options are getting relatively more expensive for firms with higher transient beta when we increase variance risk premiums. Note that for all the graphs the total unconditional equity variance is fixed $\tilde{v}^i = (\beta_1^i)^2\tilde{\theta}_1 + (\beta_2^i)^2\tilde{\theta}_2 + \theta^i = 0.11$. Note also that the top LHS panel replicates the same pattern following the calibration in one-factor model of Christoffersen et al. [2015].

[Figure (6) about here]

4 Data

We obtain daily prices of S&P 500 index options from the OptionMetrics volatility surface data set, which is based on the midpoint of bid-ask quotes. Our sample of S&P 500 index options is from January 4, 1996 through December 29, 2011. We follow the data cleaning routine commonly used in the empirical option pricing literature: we remove options with implied volatility less than 5% and greater than 150%; we also follow the filtering rules in Bakshi et al. [1997] to remove options that violate various no-arbitrage conditions. We focus on out-of-the-money (OTM) options with maturity up to and including one-year and with 10% moneyness (spot price over strike price).[18,19] Our option-based optimization function minimizes the squared deviations between model and market option prices and therefore may put greater weight on expensive in-the-money (ITM) and long-maturity options.[20] Moreover, ITM S&P 500 call options are less liquid than OTM call options. To prevent such biases in our optimization, we discard all ITM options and use OTM S&P 500 put options and convert them into ITM call options. After cleaning, we have 345,710 S&P 500 index option quotes together with daily underlying returns. This is the data set that we use to filter daily spot variances and to estimate a set of structural parameters.

For individual equities, we choose all the firms listed in the Dow Jones Industrial Average index and collect equity options data from OptionMetrics.[21] We keep all options up to 10% moneyness and with maturity up to and including 1 year. Note that options on individual equities are American, the price of which could be affected by early exercise premium. To prevent any bias in the estimation of the structural parameters of equities and daily spot idiosyncratic variance, the loss function needs to be defined based on the implied volatility as implied volatilities and deltas for the equity options reported in OptionMetrics are computed by the Cox et al. [1979] binomial tree model. Otherwise, if the loss function is based on mean-squared option pricing errors, we either need to restrict our sample to out-of-the-money equity options that are less sensitive to early exercise premium or have to covert the American-style equity options into European-style equity options by taking into account the early exercise premium. Due to the computational burden of such adjustments and considering the closed-from European option pricing equation in Proposition (3), we focus on OTM equity options.[22]

[18] This range of moneyness implies that we keep OTM call options with moneyness less than 1.1 and OTM put options with moneyness greater than 0.9.

[19] As discussed in previous section, multiple-factor SV models could better capture the slope and the level of smirk compare to single-factor SV models. Therefore, unlike similar analysis, we undertake a more extensive calibration exercise by incorporating the information content of options on longer maturity horizons and wider moneyness ranges. For instance, Ait-Sahalia and Kimmel [2007, Section 7] only include short-maturity at-the-money S&P 500 Index Options; Eraker [2004] use 3,270 call options contracts recorded over 1,006 trading days; Jones [2003] models are estimated using a sample of 3537 S&P 100 index options from January 1986 to June 2000.

[20] See Huang and Wu [2004].

[21] Note that we drop the Bank of America, the Kraft Foods Incorporation, and the Travelers Companies Incorporation.

[22] See Bakshi et al. [2003] and Christoffersen et al. [2015].

To filter daily spot market transient and persistent variance components, we use data from S&P 500 index and option markets. We obtain S&P 500 index option prices from the OptionMetrics volatility surface data set from January 4, 1996 through December 29, 2011. We follow the data cleaning routine commonly used in the empirical option pricing literature: we remove options with implied volatility less than 5% and greater than 150%; we also follow the filtering rules in Bakshi et al. [1997] to remove options that violate various no-arbitrage conditions. We focus on out-of-the-money (OTM) option contract with maturity up to and including one-year and with moneyness (spot over strike price) up to 10%.[23] After cleaning, our sample contains 345,710 S&P 500 index option contracts.

The data for daily equity prices, equity returns, daily index level, index returns, and the dividend yields are from CRSP. In the empirical analysis, we first adjust daily equity prices and index level with dividend yields and then compute option prices using the dividend-adjusted returns. Risk-free interest rates for all maturities are estimated by linear interpolation between the closest zero-coupon rates of the Zero Coupon Yield Curve from OptionMetrics.

Table (1) presents the descriptive statistics of the index call option contracts in our sample sorted by moneyness (stock price over strike price) and day-to-maturity (DTM). Note that we focus on OTM option contracts, which means S/K is below 1 for OTM call contracts. After cleaning, we have 208,098 out-of-the-money call option contracts with an average day-to-maturity of 143 days, an average price of \$35.59, an average implied volatility of 20.64%, and an average delta of 0.37. Table (2) reports the descriptive statistics of the index put option contracts in our sample sorted by moneyness and day-to-maturity. After cleaning, we use 137,612 out-of-the-money (S/K is above 1) put option contracts with an average day-to-maturity of 136 days, an average price of \$32.11, an average implied volatility of 24.34%, and an average delta of -0.29. Note that Panel C in Tables (1) and (2) reflect the well-known volatility smirk in index options, as implied volatility is larger for OTM put options (Table (2), Panel C) compared to the OTM call options (Table (1), Panel C).

[Table (1) about here]

[Table (2) about here]

Table (3) presents the descriptive statistics of the equity option contracts that are used to filter daily spot idiosyncratic variance, and to estimate the structural parameters for individual equities and market index. This table reports the number of available call and put option contracts for each firm after data cleaning. For every firm, we also report the average number of days-to-maturity and average implied volatility of option contracts in our sample. Overall, we have 4,241,990 equity call options and 3,209,990 equity put options with an average days-to-maturity of 135 days. On average, for every firm we have 275,999 option contracts with an average implied volatility of 28.52%.

[23] See Ghanbari [2016] for detailed description of the S&P 500 index options data set and its summary statistics.

[Table (3) about here]

Tables (4) and (5) provide further details regarding equity call options and put options. On average we observe that equity call options in our sample are more expensive (2.688 for calls versus 2.344 for puts), more sensitive to underlying equity prices and volatilties, have lower implied volatility (27.32% for calls and 29.73% for puts), and have a greater number of days-to-maturity (137 days for calls and 134 days for puts.)

[Table (4) about here]

[Table (5) about here]

5 Estimation Methodology

Our estimation methodology is twofold. At the market index level, we do a joint-estimation by filtering the vectors of daily spot variance components and estimating a set of structural parameters for the market index dynamics. Then, for every individual equity, we filter spot idiosyncratic variance and structural parameters, given the filtered transient and persistent spot variance components of the market index.

5.1 Estimation of the Index Model

We follow the approach in the Appendix (E) and estimate structural parameters and filter persistent and transient daily spot variance components of the market index model by combining information from underlying index and option markets (known as joint estimation). We use a two-component likelihood function, a return-based component and an option-based component, to impose consistency between structural parameters under P and Q distributions. To filter unobserved transient and persistent spot variance components, we use the sampling-importance-resampling (SIR) implementation of the Particle Filter (PF) methods.[24]

Our optimization function is as follows.

$$\max_{\Theta,\tilde{\Theta}} \big(LLR + LLO\big) \tag{23}$$

[24]See Appendix E for implantation of PF in the context of two-factor stochastic volatility model. See Pitt [2002] for a detailed description of the PF algorithm.

where LLR is the return-based and LLO is the option-based likelihood functions and Θ is the set of structural parameters of the market index model under P-measure and $\tilde{\Theta}$ is the equivalent set under Q-measure.

$$LLR \propto \sum_{t=1}^{T} \ln\Big(\frac{1}{N}\sum_{j=1}^{N} \breve{W}_t^j(\Theta)\Big) \tag{24}$$

where $\breve{W}_t^j$ is the normalized weight of particle j at time t, N is the number of daily particles, and $\Theta \equiv \{\kappa_1, \kappa_2, \theta_1, \theta_2, \sigma_1, \sigma_2, \rho_1, \rho_2, \lambda_1, \lambda_2\}$.

$$LLO \propto -\frac{1}{2}\Big(M\ln(2\pi) + \sum_{n=1}^{M}\big(\ln(s^2) + \eta_n^2/s^2\big)\Big)\ , \tag{25}$$

where M is the total number of index option contracts and η_n is the Vega-weighted loss function for option n.

$$\eta_n = (C_n^O - C_n^M(\tilde{\Theta}, \hat{v}_1^Q, \hat{v}_2^Q, S_t, K, \tau))/Vega_n\ , \qquad n = 1, \ldots, M \tag{26}$$

where C_n^O is the observed price of call option n and $C_n^M(\tilde{\Theta}, \hat{v}_1^Q, \hat{v}_2^Q, S_t, K, \tau)$ is the model price of call option n. $Vega_n$ is the Black and Scholes [1973] option Vega for the same option contract. Note that we obtain daily persistent ($\hat{v}_{1,t}^Q$) and transient ($\hat{v}_{2,t}^Q$) spot variance components under Q measure as the average of smoothly re-sampled particles of daily variance components.

$$\hat{v}_{1,t}^Q = \frac{1}{N}\sum_{j=1}^{N} v_{1,t}^j\ , \qquad \hat{v}_{2,t}^Q = \frac{1}{N}\sum_{j=1}^{N} v_{2,t}^j \tag{27}$$

Our index optimization algorithm is iterative. Each iteration starts with an initial set of structural parameters, which then will be used to filter transient and persistent daily spot variance components using the information content of index returns. Then, given spot variance components, structural parameters of the index, and observed option prices, the next set of optimal parameters can be reached by minimizing the option pricing errors over the entire sample. The procedure iterates until an optimal set of structural parameters is reached and thereby we obtain the final vectors of transient and persistent spot variance components.

5.2 Estimation of the Individual Equity Model

We estimate a set of structural parameters $\tilde{\Theta}^i \equiv \{\kappa^i, \theta^i, \sigma^i, \rho^i, \beta_1^i, \beta_2^i\}$ and a vector of daily spot idiosyncratic variances $\{\xi_t^i\}$ for each individual equity in our sample following the two-step iterative approach of Bates [2000] and Huang and Wu [2004]. In the first step, given a set of initial structural parameters for each equity, $\tilde{\Theta}_0^i$, we estimate a vector of daily spot idiosyncratic variance conditional on a set of risk-neutral structural parameters of the market model, $\hat{\tilde{\Theta}}$, and filtered daily risk-neutral spot variance components, $\{\hat{v}_{1,t}^Q, \hat{v}_{2,t}^Q\}$. Using a Vega-weighted loss function, the set of daily spot idiosyncratic variance $\hat{\xi}_t^i$ for every firm i can be obtained as the solution to the following optimization problem, which minimizes the Vega-weighted daily mean-squared option pricing errors.

$$\hat{\xi}_t^i = \arg\min_{\xi_t^i} \sum_{n=1}^{M_t^i} (C_{n,t}^{i,O} - C_{n,t}^{i,M_t^i}(\tilde{\Theta}_0^i, \hat{\tilde{\Theta}}, \hat{v}_{1,t}^Q, \hat{v}_{2,t}^Q, \xi_t^i))^2 / (Vega_{n,t}^i)^2 \ , \qquad t = 1, \ldots, T \ , \tag{28}$$

where M_t^i is the total number of available option contracts for the equity i on day t, $C_{n,t}^{i,O}$ is the observed price of equity option n for stock i on day t, $C_{n,t}^{i,M_t^i}$ is the model price for the same option obtained from equity pricing equation (18), and $Vega_{n,t}^i$ is the Black-Scholes option Vega for the same equity option contract. Note that we repeat the optimization in (28) every day and for every equity to estimate a vector of spot idiosyncratic variances over the entire sample.

The second step estimates the structural parameters $\tilde{\Theta}^i$ for firm i, by minimizing sum of daily Vega-weighted mean-squared option pricing errors over the entire sample, given filtered daily spot idiosyncratic variance obtained in the first step, the dynamics of the market index and the filtered daily spot variance components. We may then solve the the following optimization problem.

$$\hat{\tilde{\Theta}}^i = \arg\min_{\xi_t^i} \sum_{n=1}^{M^i} (C_n^{i,O} - C_n^{i,M^i}(\tilde{\Theta}^i, \hat{\tilde{\Theta}}, \hat{v}_{1,t}^Q, \hat{v}_{2,t}^Q, \hat{\xi}_t^i))^2 / (Vega_n^i)^2 \ , \tag{29}$$

where $M^i \equiv \sum_{t=1}^T M_t^i$ is the total number of available option contracts for equity i. For every equity, the procedure iterates between the optimizations in (28) and (29) to minimize the pricing error until the change in the RMSE of the estimation in the second step is no longer significant. Note that every new iteration starts based on the structural parameters of the previous iteration, $\tilde{\Theta}_0^i = \hat{\tilde{\Theta}}^i$.

6 Parameter Estimation Results

This section first reports the filtered daily spot variance components together with the structural parameter estimates for the two-factor SV model. We use a long time-series of daily S&P 500 index returns and the entire cross-section of S&P 500 option prices that span the period from January 4, 1996 to December 29, 2011. Given the slow mean-reversion in the dynamic of market volatility, it is important to let the data set span a long time series. This is in particular important in our analysis as we decompose the overall market volatility into two independent components and would like to characterize the dynamics of transient and persistent variance components. We also report structural parameters and daily spot idiosyncratic variance for 27 firms listed in the Dow Jones Industrial Average Index. The parameter estimates and latent idiosyncratic variances are conditional on the transient and persistent spot variance components $\hat{v}^Q_{1,t}$ and $\hat{v}^Q_{2,t}$ and structural parameters $\hat{\tilde{\Theta}}$ of the index model.

To provide a basis for further comparison and to examine the model fit under the joint-estimation, we also report the structural parameters of the market model, estimated only from option data.

6.1 Parameter Estimates, The Market Index Model

In what follows we set the market risk premium μ is set to the sample average daily index returns. We use OTM index options with up to 10% moneyness and then convert the OTM puts into ITM calls through put-call parity. Table (6) reports structural parameter estimates (under P measure) that characterize the dynamics of index returns and its persistent and transient variance components. Panel A provides result of the joint estimation; a consistent set of parameters under P and Q measures. Therefore, the speeds of mean reversion and the unconditional mean of the persistent and transient variance components under Q-measure are linked to their P-measure equivalents through the market prices of the volatility risk factors ($\tilde{\kappa}_1 = \kappa_1 + \lambda_1$, $\tilde{\kappa}_2 = \kappa_2 + \lambda_2$, $\tilde{\theta}_1 = \frac{k_1\theta_1}{k_1+\lambda_1}$, $\tilde{\theta}_2 = \frac{k_2\theta_2}{k_2+\lambda_2}$).[25] To provide a basis for further comparison and to examine the goodness of fit of the two-factor SV model under the joint-estimation, we also estimate structural parameters using only option data. This result is provided in Panel C. Note that we assume that the transient and persistent beta coefficients are the same under P and Q measures following Serban et al. [2008].

[Table (6) about here]

As discussed, the purpose of two-factor stochastic volatility model is to capture independent movements in the underlying returns and option prices over time. Consistent with previous studies in both discrete time GARCH models and continuous time stochastic volatility models, we find that one of the volatility factors is highly persistent and the other one is highly mean-reverting. In joint-estimation, we find that the first variance component is slowly mean-reverting with $\kappa_1 = 1.4271$ under physical measure while the rate of mean reversion in the second variance component is much higher with $\kappa_2 = 3.5874$ under the physical measure.[26] The point estimate of mean reversion parameters from option-based estimation is

[25] See Proposition (1).

[26] These value correspond to a daily variance persistence of $1 - 1.4271/365 = 0.9961$ for the first component

similar to those from joint estimation. Using options data only, we find that $\tilde{\kappa}_1 = 0.2267$ and $\tilde{\kappa}_2 = 2.9137$, which is consistent with the speed of mean reversion from joint estimation where under Q-measure $\tilde{\kappa}_1 = 0.3473$ and $\tilde{\kappa}_2 = 2.5520$.

To gain a better intuition about persistent and transient variance components we define the half-life ($T_{1/2}$) of a variance component as the number of weeks that it takes for auto-correlation of each variance component to decay to half of its weekly autocorrelation level. Half-life can be computed as $T_{1/2} = \ln(\phi/2)/\ln(\phi)$ where $\Delta t = 7/365$ and $\phi = \exp(-\kappa\Delta t)$, denoting weekly autocorrelation of time-series each variance component. The risk neutral point estimate of mean reversion speed in transient variance component implies a half-life around 15 weeks while it is 105 weeks in the persistent variance component, almost 7 times larger than its transient counterpart. These values confirm that first variance component is highly persistent while the second one is highly auto-correlated and thus the immediate impact of variance shocks on this component is larger but short-lived.

We observe that the unconditional persistent variance under P-measure is $\theta_1 = 0.0026$, which is much less than the unconditional transient variance $\theta_2 = 0.0171$. The unconditional risk neutral persistent and transient variance components are $\tilde{\theta}_1 = 0.0106$ and $\tilde{\theta}_2 = 0.0240$ which correspond to 10.30% and 15.49% volatility per year. Note that the unconditional variance of both components are consistent with the average filtered daily spot persistent variance and daily spot transient variance over the entire sample.

Consistent with our intuition, we observe a wide spread between the volatility of variance in the persistent and transient variance components. As a result of joint estimation we find that $\sigma_1 = 0.0855$ and $\sigma_2 = 0.3496$. This result is consistent with the option-based estimation where we find that transient variance component is much more volatile with $\sigma_2 = 0.5678$ compared to the persistent variance component with $\sigma_1 = 0.0958$. Higher level of volatility of variance in option-based estimation compared to the joint estimation is consistent with previous studies.[27]

We find negative prices for both variance components where $\lambda_1 = -1.0798$ and $\lambda_2 = -1.0355$. These negative prices imply that investors are willing to pay for an insurance against an increase in volatility risk, even if that increase has little persistence. To the best of our knowledge none of the previous studies of two-factor stochastic volatility models in option market reports the prices of the variance risk factors as they either focused on the options market data or the underlying index returns data. Our negative prices for both variance components is consistent with asset pricing studies where the short-run and the long-run volatility components are priced cross-sectional asset pricing factors. Adrian and Rosenberg [2008] use a a large cross-section of individual stocks over a very long period and find that prices of both short-run and long-run variance components are negative and highly significant. Therefore, our join estimation result confirm that there is a consensus of opinions about the price of transient and persistent variance components among option traders and

and $1 - 3.5874/365 = 0.9901$ for the second component.

[27] For instance, Bates [2000] reports that option-based estimates of volatility of variance is larger than the one obtained from time-series-based estimates.

equity traders.

Our joint estimation results show that correlation between shocks to the index returns and shocks to the persistent variance component is $\rho_1 = -0.6918$. The correlation between shocks to the index returns and shocks to the transient variance component is $\rho_2 = -0.2173$. ρ_1 and ρ_2 captures asymmetry in the response of persistent and transient variance components to positive versus negative return shocks and can be considered as the persistent and transient continuous time leverage (asymmetry) effect. The leverage effect induces negative skewness in index returns and thus yields a volatility smirk. Our results show that that leverage effect is more significant in the persistent variance component compared to the transient variance component. Therefore, persistent variance component has more significant effect on the dynamic of index skewness. Using the data from option market only, we find that $\rho_1 = -0.91$ and $\rho_1 = -0.49$. The higher absolute level of option implied correlation coefficients compared to those of joint estimation is partly related to the well documented fact that risk neutral distribution is more negatively skewed.

Our persistent and transient correlation coefficients are almost consistent with those of previous studies in option market. The average correlation coefficients in Christoffersen et al. [2009, Table 3] are $\rho_1 = -0.96$ for their first variance component and $\rho_2 = -0.83$ for their second variance component.[28] Bates [2000] also reports the structural parameter estimates of a two-factor SV model using 1988-1993 S&P 500 futures option prices. He obtains one set of structural parameters over the entire sample where $\rho_1 = -0.78$ and $\rho_2 = -0.38$. To provide a basis for comparison, we also estimate structural parameters using options data only over the same sample period and find $\rho_1 = -0.91$ and $\rho_2 = -0.49$. There are potential explanations for differences between the reported estimates of the correlation coefficients in these studies, not in the least, the very different data set and the very different time span. Despite differences in the magnitude of the coefficients, the point estimates for the correlation coefficients are negative for both persistent and transient variance components across all these studies. Further, the transient variance component has lower (in absolute value) level of correlation compared to the persistent variance components in all these studies.

To provide some empirical evidence on the difference between persistent and transient variance components over time, we plot the paths of filtered variance components. Figure (7) plots filtered time series of risk-neutral spot variance component of S&P 500 index based on our two-factor stochastic volatility model. Panel A shows time series of persistent variance component and Panel B shows time series of transient variance component. The blue plots are based on the Particle Filter method using data from both S&P 500 index and option markets (joint estimation) and the red plots are filtered spot variances using only S&P 500 options data.

[Figure (7) about here]

[28] Christoffersen et al. [2009] use data on European S&P 500 call option quotes over the period 1990-2004. Note that they estimate a separate set of structural parameters for every year in their sample.

Naturally, the overall patterns of persistent and transient variance components filtered from joint estimation are consistent with those filtered from options data only. However, option implied variance components are more volatile in the sense that when variance increases, it tends to do more sharply compared to the one filtered based on joint estimation and thus exhibit more spikes. In particular, this pattern in more pronounced in the transient variance component (Panel B). The observed sharper spikes in option-based filtered variance in the two-factor SV model is consistent with previous studies of one-factor SV model. The smoother variance paths in joint-estimation is partly due to smooth resampling procedure in SIR PF method and partly due to imposed consistency between parameter estimates under P and Q measures.

To provide more intuition about the total risk neutral variance in our two-factor SV model, Figure (8) combines persistent and transient variance components and plots time series of total spot variance versus model-free option-implied VIX volatility index. As we expect, the time series of option implied total spot variance is closely related to the VIX volatility index. Further, the time series of total spot variance from joint estimation follow the same pattern as the VIX volatility index. However, due to joint restrictions, the total spot variance from joint estimation do not exhibits volatility spikes as large as those observed in the VIX volatility index.

[Figure (8) about here]

6.2 Parameter Estimates, The Individual Equity Model

The data for individual equities starts from June 1, 1996 rather than January 1, 1996. Note that we drop the first 5 months of each equity's data set to prevent any estimation bias, as the filtered spot market variance components are noisy in the first months of the estimation period. Note also that S&P 500 Index options are European style while the individual equity options are American style, the price of which might be affected by early exercise premium. To reduce the bias in the calculation of equity option prices using the closed-form pricing equation in Proposition (3) we focus on OTM options.[29,30]

Table (7) reports the structural parameter estimates that characterize the dynamics of the

[29] Bakshi et al. [2003] show that for OTM S&P 100 American options the early exercise premium is negligible. They estimate two separate implied volatilities: the implied volatility that equates the option price to the American option price from binomial tree model, and the implied volatility that equates the option price to the Black-Scholes price where the discounted dividends are subtracted from the spot price. They find that although American option implied volatility is smaller than its Black-Scholes counterparts, the difference is negligible and within the bid-ask spread.

[30] Using the data of the firms listed on Dow Jones Index, Christoffersen et al. [2015] show that the early exercise premium is negligible for equity call options. As a robustness test, we also estimate the equity model by using only the equity call options rather than OTM calls and puts. We find that the point estimates of structural parameters are quite similar to our base case estimation where we use OTM put and call option contracts. This result is available from the author upon request.

individual equity returns and idiosyncratic variance under the Q measure. The table also contains the point estimates of the persistent and transient betas for 27 firms in our sample.

[Table (7) about here]

The speed of mean reversion for risk-neutral idiosyncratic variance ranges from $\tilde{\kappa}^i = 0.3920$ for Coca Cola to $\tilde{\kappa}^i = 1.7078$ for 3M. This range of $\tilde{\kappa}^i$ is implies that most of the firms in our sample have highly persistent idiosyncratic variance with average speed of mean reversion 0.8055. In other words, the average half-life of idiosyncratic variance for the firms in our sample is almost 46 weeks, implying that it takes 46 weeks for the idiosyncratic variance autocorrelation to decay to half of its weekly autocorrelation. We also find that most of the firms in our sample have an idiosyncratic variance that is more persistent than the overall market variance.

The unconditional risk neutral idiosyncratic variance of the firms in our sample starts from $\tilde{\theta}^i = 0.0093$ for General Electric and increases up to $\tilde{\theta}^i = 0.0756$ for Hewlett-Packard. The point estimates for the volatility of the idiosyncratic variance range from $\sigma^i = 0.0670$ for General Electric to $\sigma^i = 0.3967$ for Hewlett-Packard. For all the firms in our sample, the average point estimates for the volatility of the idiosyncratic variance is 0.1823. The correlation between shocks to equity returns and shocks to idiosyncratic variance is negative for all the equities (except for Verizon) and ranges from $\rho^i = -0.99$ for JP Morgan to $\rho^i = 0.512$ for Verizon.

The betas estimates are novel and to the best of our knowledge this is the first study that reports the option-implied persistent beta and transient beta for individual equities and thus there is no benchmark for further comparisons. However, we find that firms respond differently to transient and persistent variations in market index returns. The persistent beta ranges from $\beta_1^i = 0.3430$ for American Express to $\beta_1^i = 0.6798$ for IBM. The transient beta starts from $\beta_2^i = 1.0125$ for Procter & Gamble and increases to $\beta_2^i = 1.3466$ for JP Morgan. The average persistent beta is 0.4899 and the average transient beta is 1.2284. Across all 27 firms in our sample the transient beta is always greater than the persistent beta, implying that for the large capitalization firms listed in the Dow Jones index, transient and larger variations in the market tend to be related to the proportionally larger systematic price reactions across equities than persistent and smaller variations in the market index.

Our point estimates of the transient and persistent option-implied betas are similar to the continuous beta and jump beta of Todorov and Bollerslev [2010] who introduce a framework to separate and identify continuous and discontinuous systematic risks. Using high frequency data from a large cross-section of forty large-capitalized individual stocks, they find that the average jump betas are larger than the continuous betas with few exceptions. Although we only use option data and estimate ad-hoc constant beta over the entire sample, we observe a similar pattern as theirs between our transient and persistent betas.[31]

[31]The assumption of constant transient and persistent betas allow us to keep the affine specification of the dynamics of individual equity and derive a closed-form equity option pricing equation. We can, however,

As discussed in Section 3, the proposed two-factor structure has important implications for equity option market deltas, market Vegas, and instantaneous expected returns of equity options. We also show how this two-factor structure affects the slope of the term structure and moneyness of implied volatility of individual equity options. Along these lines, our findings of different sensitivities to the systematic transient and persistent risk factors may corroborate the theoretical implications of our model. The beta estimates have further implications for portfolio management, suggesting the importance of different strategies for hedging transient versus persistent systematic market variations.

We close this section by providing more intuition about the idiosyncratic variance across the firms in our sample by presenting the distributional properties of the filtered spot idiosyncratic variance. Table (8) reports the mean, median, standard deviation, and the maximum of the filtered spot idiosyncratic variances for every firm conditional on the structural parameters of the two-factor SV model of index and the filtered market spot variance components. We observe that for all the firms the median is significantly lower than the mean, implying that the mean estimates of the filtered spot idiosyncratic volatilities are driven by outliers that may be common to all firms.

[Table (8) about here]

7 Model Performance and In-Sample Fit

We measure the goodness of fit of the market index model using the following Vega-weighted root mean squared option pricing errors (Vega RMSE) as it is consistent with the loss function that we used in the optimization routine.

$$\text{Vega RMSE} \equiv \sqrt{\frac{1}{N}\sum_{n,t}^{M}\big(\frac{C^O_{n,t} - C^M_{n,t}(\hat{\hat{\Theta}}, \hat{v}^Q_{1,t}, \hat{v}^Q_{2,t})}{Vega_{n,t}}\big)^2}\,, \tag{30}$$

where, $C^O_{n,t}$ is the observed price of index option n on day t, $C^M_{n,t}$ is the model price for the same index option on the same day, and $Vega_{n,t}$ is the Black-Scholes option Vega for the same option contract on the same day. To provide a reference for comparison, we also report the implied volatility root mean squared error (IVRMSE) of option pricing model.

estimate time-varying betas by modifying our estimation procedure. We can fix the structural parameters of the market and individual equities and estimate conditional betas and spot idiosyncratic variance on a daily basis, given the transient and persistent spot variance components using a loss function very similar to 29.

$$\text{IVRMSE} \equiv \sqrt{\frac{1}{N}\sum_{n,t}^{M}\left(IV_{n,t}^{O} - IV(C_{n,t}^{M}(\hat{\tilde{\Theta}}, \hat{v}_{1,t}^{Q}, \hat{v}_{2,t}^{Q}))\right)^2}\,, \tag{31}$$

where, $IV_{n,t}^{O}$ is the Black-Scholes implied volatility of observed index option n on day t and $IV(C_{n,t}^{M}(\hat{\tilde{\Theta}}, \hat{v}_{1,t}^{Q}, \hat{v}_{2,t}^{Q}))$ is the Black-Scholes implied volatility of the model option price for the same index option on the same day.

For individual equities, Vega RMSEs and IVRMSEs are computed with equations similar to (30) and (31) while replacing $C_{n,t}^{O}$ with $C_{n,t}^{O,i}$, $C_{n,t}^{M}(\hat{\tilde{\Theta}}, \hat{v}_{1,t}^{Q}, \hat{v}_{2,t}^{Q})$ with $C_{n,t}^{i,M^i}(\tilde{\Theta}^i, \hat{\tilde{\Theta}}, \hat{v}_{1,t}^{Q}, \hat{v}_{2,t}^{Q}, \hat{\xi}_t^i)$, $Vega_{n,t}$ with $Vega_{n,t}^{i}$, $IV_{n,t}^{O}$ with $IV_{n,t}^{O,i}$, and $IV(C_{n,t}^{M}(\hat{\tilde{\Theta}}, \hat{v}_{1,t}^{Q}, \hat{v}_{2,t}^{Q}))$ with $IV(C_{n,t}^{i,M^i}(\tilde{\Theta}^i, \hat{\tilde{\Theta}}, \hat{v}_{1,t}^{Q}, \hat{v}_{2,t}^{Q}, \hat{\xi}_t^i))$.

Table (9) reports in-sample goodness-of-fit statistics for the two-factor stochastic volatility model over the entire sample, 1996 through 2011 for various maturities. Panels A and B report in-sample fit statistcis for calls and puts separately. The left panel reports model fit based on the joint estimation while the right panel gives reports option-based fit. We find that the overall Vega-weighted RMSE of joint estimation and option-based estimation are 2.56% and 0.98% respectively. Note that the overall IVRMSE are 2.59% and 0.99% respectively, which means that Vega-weighted RMSE could be used as an approximation of IVRMSE. Overall, our two-factor SV model provides a better fit to call option contracts compared to put option contracts, which is consistent with the findings in one-factor stochastic volatility model.

Note that joint estimation imposes a consistency between physical and risk neutral parameters which are otherwise not identical. Such a restriction is not required in option-based estimation which could partly explain the better in-sample fit of option-based estimation compared to joint estimation. However, the reported RMSEs confirms that unlike stochastic volatility model, joint restrictions on return and variance dynamics under P and Q measures does not lead to the poor performance of the two-factor SV model.

Broadie et al. [2007] refer to the inconsistency between the option-based estimates of certain structural parameters in SV model and the parameter estimates from underlying time-series of returns and indicate that the SV model is basically misspecified. In particular, they state that the point estimates of the correlation coefficient and volatility of volatility are incompatible under the P and Q measures. They also show that the joint restrictions on the returns and volatility dynamics under the P and Q measures lead to the poor performance of the stochastic volatility model, measured by high level of RMSE. Using S&P 500 returns and futures options data over the period of 1987 through 2003, they find IVRMSE of 1.1% for the option-based estimation and 8.73% while imposing time-series consistency.

They note that this poor performance of SV model indicates the inability of the SV models to generate sufficient amounts of conditional skewness and kurtosis. This drawback in standard SV models is mainly attributed to the fact that the estimated conditional higher moments are highly correlated with the estimated conditional variance. By contrast, in-sample fit of our two-factor SV model is significantly improved relative to the Heston SV model. Further, the spread between Vega-weighted RMSE of joint estimation and option-based estimation is reduced significantly in the two-factor SV model versus the Heston SV model. The better performance of two-factor SV model is due to the fact that it can generate stochastic correlation between volatility and stock returns. This feature enables the two-factor SV model to better capture the conditional skewness and kurtosis of the index dynamics.[32]

Table (14) provides goodness-of-fit statistics for 27 the firms in our sample, both in-sample and out-of-sample. Using option data over the period 1996-2011, we find that all the firms in our sample has a Vega RMSE below 2 except for Cisco and Chevron. We find similar in-sample performance when the goodness-of-fit is measure by IVRMSE. The average Vega RMSEs and IVRMSEs across all the firms are 1.61% and 1.59% respectively. The average relative IVRMSE, measured as the ratio of IVRMSE over the average Black-Scholes IV, is 5.66%. We find that Boeing has the best fit with IVRMSE of 1.35% and Cisco has the worst fit with IVRMSE of 2.12%; however, the fit is quite similar across the firms. Overall we conclude that the model provides a reasonably good fit for all 27 firms.

We find that our model has a relatively better in-sample fit compared to the one-factor structure model. For the firms listed on Dow Jones index, Christoffersen et al. [2015, Table 4] find that the average IVRMSE is 1.66%.[33] Further, comparing goodness-of-fits in our model with those of Heston model for the same firms, reported in Christoffersen et al. [2015, Table A.2], also supports the performance of our model. Overall, the in-sample performance of our model over the one-factor structure together with its cross-sectional implications regarding IV term-structure, moneyness slope, and equity option skew support the importance of transient and persistent factor loadings in pricing equity options.

8 Model Stability and Out-of-Sample Performance

In order to examine the stability of the two-factor SV model of the market index and its out-of-sample performance, we divide the dataset into two subsample periods. The first subsample is from January 1996 through December 2003 and contains 169,800 daily option contracts. The second one is from January 2004 to December 2011 which contains 175,910 daily option contracts. Using both daily returns and option data we filter spot daily persistent variance path and transient variance path and repeat the joint estimation routine within each subsample. Table (10) reports the parameter estimates within each subsample (Panels A and B). For the sake of comparison, Panels C and D also report the parameter estimates from option-based estimation. The main results of the subsample tests are as follows.

First, we find that the PF method is a reliable filtering technique even within shorter sample

[32]Previous studies show that using the option data only two factor SV model improves on the benchmark SV model both in-sample and out-of-sample, see Christoffersen et al. [2009, Section 3.1].

[33]Note that their sample span the period 1996 to 2010.

period of 8 years. We observe that the time series of total spot daily variances under risk neutral measure is largely consistent with the time series of the VIX option implied volatility index within each subsample period.

Second, the parameter estimates within each subsample period is largely inline with those obtained from whole-sample estimates. Moreover, within each subsample period, the joint estimation results is also consistent with option-based parameter estimates. We find that point estimate for the transient mean reversion parameter is higher in the second subsample period while the opposite is true for the persistent mean reversion speed. Overall, the level and the order of parameter estimates are almost consistent within both subsample periods and also across both estimation methods (joint estimation and option-based estimation).[34]

Third, the correlation coefficients between transient and persistent variance shocks and return shocks within subsample periods remain consistent with the ones estimated over the entire sample period and those reported in previous studies[35] in the sense that the magnitude of persistent correlation coefficient is higher than its transient counterpart. Further, persistent and transient variance risk premiums remain negative with the same order within two subsample periods, confirming our previous findings that investors are willing to pay to avoid transient and highly mean reverting volatility shocks.

Fourth, we evaluate our model fit within both subsample periods and report Vega RMSEs and IVRMSEs separately for calls and puts and for different maturities. Entries in Table (11) and Table (12) are inline with model fit over the entire sample period, reported in Table (9). Our joint estimation result show a better in-sample fit over the second subsample period as Vega RMSEs and IVRMSEs are reduced.

Fifth, in order to measure the out-of-sample performance of the two-factor SV model in capturing the behaviour of S&P 500 index options, we use the parameter estimates form the first subsample (1996-2003). Given the parameter estimates from the first subsample period, we use Particle Filter methods to filter risk neutral spot daily persistent and transient variance components over the second subsample period and then compute the IVRMSEs and Vega RMSE over the second subsample (2004-2011). Table (13) reports the summary statistics of the out-of-sample performance for different maturities and for calls and puts separately. Comparing out-of-sample entries in (13) with those of in-sample in (12) over the same period supports the stable performance of the two-factor SV model either in joint-estimation or in option-based estimation.

Entries in the last column of Table (14) reports out-of-sample performance of the equity model. We divide the data set into two subsample periods. using data from 1996 to 2003

[34] Christoffersen et al. [2009, Table 3] report annual risk neutral parameter estimates for the two-factor SV model over the period 1990 through 2004 using data from S&P 500 index option data. Our option-based subsample parameter estimates are mostly consistent with their average annual result except for the volatility of volatility parameter. Apart from differences in the size of sample, this difference in point estimates may partly be explained by the fact that the annual parameter estimates in Christoffersen et al. [2009] does not satisfy the Feller condition. Feller [1951] shows that a square root process is strictly positive if $2\kappa\theta > \sigma^2$.

[35] See Section 6.

we estimate structural parameters for the index model, for every individual equity, and filter persistent and transient daily spot index variance components, and spot idiosyncratic variance for all the firms. In the next step we filter spot idiosyncratic variance for all the firms over the period 2004 to 2011, given spot variance components and structural parameters in the first subsample period. Note that we use an optimization function similar to (28). We find that the model provides good out-of-sample fit. For most of the firms, the out-of-sample Vega RMSEs are consistent with their in-sample Vega RMSEs. Overall, the average Vega RMSE is 1.81% across all 27 firms.[36]

9 Concluding Remarks

In this paper we investigate a two-factor stochastic volatility model where the aggregate market volatility is decomposed into a persistent and a transient volatility components. We extend the pricing kernel in Christoffersen et al. [2013], where investor's equity preference is distinguished from her variance preference, and introduce an admissible pricing kernel that links the proposed market dynamics under P and Q measures. We also discuss alternative pricing kernel for risk neutralization without separating equity and variance preferences. As the proposed two-factor specification is affine, we obtain a closed-from pricing expression for European call options. We use a long time-series of daily S&P 500 index returns and the entire cross-section of S&P 500 option prices over the same time span. We filter time series of persistent and transient spot variance components and simultaneously estimate a set of structural parameters that characterizes the dynamics of index return and variance components.

Motivated by the extensive empirical evidence that supports the existence of two volatility components in the dynamics of index, we examine how individual equity option prices respond to transient and persistent factor loadings. We adopt a two-factor stochastic volatility model as in Ghanbari [2016] where aggregate market volatility is decomposed into two independent volatility components, a transient component and a persistent component. Then we extend the model in Christoffersen et al. [2015] and assume that individual equity returns are related to market index returns with two distinct systematic components and an idiosyncratic component, which is stochastic and follows a standard square root process. We derive a closed form pricing equation for individual equity call options where equity option prices depend on two constant factor loadings, a transient beta and a persistent beta.

In empirical analysis, we show that the proposed decomposition of volatility can be characterized by different sensitivity of the variance components to the volatility shocks and different

[36]The out-of-sample performance can also be examined with spot idiosyncratic variance obtained from one-day ahead $(t+1)$ forecast of idiosyncratic variance for individual equity i given the in-sample structural parameter estimates and time t spot idiosyncratic variance. One-day ahead $(t+1)$ forecast of idiosyncratic variance may be computed as $\hat{\xi}^i_{t+1|t} \equiv \mathrm{E}_t[\xi^i_{t+1}] = \theta^i + (\xi^i_t - \theta^i)(1 - \exp(-\frac{\kappa^i}{252}))$. However, this approach may be more suitable for instance if in-sample fit is based on a Wednesday options and then out-of-sample fit can be examined based on the Thursday options.

persistence in variance components. Consistent with the previous studies in both discrete time GARCH models and continuous time stochastic volatility models, we find that one of the volatility component is highly persistent and the other one is highly mean-reverting, where immediate impact of volatility shocks on the transient volatility component is bigger but short-lived. We obtain negative risk premium for both variance components, implying that investors are willing to pay for insurance against increases in volatility risk, even if such increases have little persistence. The negative risk premiums of both variance components are consistent with the findings in equity market where Adrian and Rosenberg [2008] find that short-run and long-run variance components are priced factors with negative risk premium. We also obtain negative correlations between shocks to the index returns and shocks to the transient and persistent variance components. In particular, we observe that the persistent correlation coefficient has more significant effect on the dynamics of index skewness.

Our model provides good fit to observed option prices both in- and out-of-sample, measured by Vega-weighted root mean squared option pricing errors and implied volatility root mean squared errors. More to the point, we find that unlike stochastic volatility model, joint restrictions on return and variance dynamics under P and Q measures does not lead to the poor performance of our two-factor SV model.

For the firms listed on Dow Jones Index, we estimate structural parameters and filter spot idiosyncratic variances, which together characterize the dynamics of the individual equity under the risk-neutral measure. Given the level of IVRMSEs, we find that our model provides a good-fit both in-sample and out-of-sample. We also report the point estimates of transient and persistent betas for 27 firms. We find that for all the firms, the transient beta is always greater than the persistent beta, implying that for large capitalization firms listed in the Dow Jones index, transient and larger variations in the market tends to be related to the proportionally larger systematic price reactions across equities than persistent and smaller variations in the market index. It also supports the presence of a two-factor structure in our model. Along this line, the different sensitivities to the systematic transient and persistent risks may corroborate the theoretical implication of our model. The beta estimates have further implications for portfolio management, suggesting the importance of different strategies for hedging transient versus persistent systematic market variations.

Our equity option pricing model sheds some lights on the impact of systematic price changes on the equity option prices. We find closed-form expressions for the sensitivity of the equity option prices to the changes in the index level (market delta) and changes in the persistent and transient variance components (persistent and transient market vega) and show how transient and persistent betas may affect the expected returns of individual equity options through market delta and vegas. Our closed-form pricing equation and proposed factor structure allow a portfolio manager to hedge her portfolio exposure to the level of the market index, and to the persistent and transient variations in the market index.

We show that the proposed two-factor structure has important cross-sectional implications for equity options. Consistent with the findings of Duan and Wei [2009], our model predicts that firms with a higher beta have a higher implied volatility. More to the point, we find that firms with a higher transient beta have a steeper term structure of implied volatility and a steeper implied volatility moneyness slope. We also observe that the variance risk premium has a more significant effect on the implied volatility smile of equity options (equity option skew) when the transient beta is higher. Overall, the in-sample performance of our model over the one-factor structure, its out-of-sample performance, together with its cross-sectional implications regarding IV term structure, moneyness slope, and equity option skew support the importance of transient and persistent factor loadings in pricing equity options.

Appendix

A Proof of Proposition 1

We impose the condition that the product of the price of any traded asset and the pricing kernel under physical measure is a martingale. We also impose the condition that the discounted price of any traded asset under risk neutral measure is also a martingale. We show that the two-factor stochastic volatility process under physical measure in (1) are linked to its risk-neutral counterpart in (4) by the unique arbitrage free pricing kernel introduced in (6) and deduce restrictions on the time-preference parameters, $\{\delta, \eta_1, \eta_2\}$, risk-aversion (equity aversion) parameter, ϕ, and variance preference parameters (variance aversion), $\{\zeta_1, \zeta_2\}$. We close this proof by showing how physical Wiener processes $\{z_{1,t}, z_{2,t}, w_{1,t}, w_{2,t}\}$ are linked to risk neutral Wiener processes $\{\tilde{z}_{1,t}, \tilde{z}_{2,t}, \tilde{w}_{1,t}, \tilde{w}_{2,t}\}$ by equity premium $\{\mu_1, \mu_2\}$ and variance premium $\{\lambda_1, \lambda_2\}$ parameters.

Consider that index return under physical and risk-neutral measures follows the dynamics (A.1) and (A.2).

$$
\begin{aligned}
dS_t/S_t &= (r + \mu_1 v_{1,t} + \mu_2 v_{2,t})dt + \sqrt{v_{1,t}}dz_{1,t} + \sqrt{v_{2,t}}dz_{2,t} \\
dv_{1,t} &= \kappa_1(\theta_1 - v_{1,t})dt + \sigma_1\sqrt{v_{1,t}}(\rho_1 dz_{1,t} + \sqrt{1-\rho_1^2}dB_{1,t}) \\
dv_{2,t} &= \kappa_2(\theta_2 - v_{2,t})dt + \sigma_2\sqrt{v_{2,t}}(\rho_2 dz_{2,t} + \sqrt{1-\rho_2^2}dB_{2,t})
\end{aligned} \tag{A.1}
$$

$$
\begin{aligned}
dS_t/S_t &= rdt + \sqrt{v_{1,t}}d\tilde{z}_{1,t} + \sqrt{v_{2,t}}d\tilde{z}_{2,t} \\
dv_{1,t} &= \tilde{\kappa}_1(\tilde{\theta}_1 - v_{1,t})dt + \sigma_1\sqrt{v_{1,t}}(\rho_1 d\tilde{z}_{1,t} + \sqrt{1-\rho_1^2}d\tilde{B}_{1,t}) \\
dv_{2,t} &= \tilde{\kappa}_2(\tilde{\theta}_2 - v_{2,t})dt + \sigma_2\sqrt{v_{2,t}}(\rho_2 d\tilde{z}_{2,t} + \sqrt{1-\rho_2^2}d\tilde{B}_{2,t})
\end{aligned} \tag{A.2}
$$

Then, following Christoffersen et al. [2013], we show that the pricing kernel links the physical and risk neutral measures has the following exponential affine form.

$$
\frac{M_t}{M_0} = \big(\frac{S_t}{S_0}\big)^{\phi} \exp\Big[\delta t + \eta_1 \int_0^t v_{1,s}ds + \eta_2 \int_0^t v_{2,s}ds + \zeta_1(v_{1,t} - v_{1,0}) + \zeta_2(v_{2,t} - v_{2,0})\Big] \tag{A.3}
$$

Note that in the sprite of Cox et al. [1985] and Heston [1993] we assume that the market price of each variance risk factor is proportional to spot variance. Therefore, the risk neutral process in (A.2) can be defined as follows.

$$
\begin{aligned}
dS_t/S_t &= rdt + \sqrt{v_{1,t}}d\tilde{z}_{1,t} + \sqrt{v_{2,t}}d\tilde{z}_{2,t} \\
dv_{1,t} &= (\kappa_1(\theta_1 - v_{1,t}) - \lambda_1 v_1)dt + \sigma_1\sqrt{v_{1,t}}d\tilde{w}_{1,t} \\
dv_{2,t} &= (\kappa_2(\theta_2 - v_{2,t}) - \lambda_2 v_2)dt + \sigma_2\sqrt{v_{2,t}}d\tilde{w}_{2,t}
\end{aligned} \tag{A.4}
$$

The log stock price process under physical measure and log pricing kernel process have the following dynamics respectively.

$$
d(\log(S_t)) = (r + \mu_1 v_{1,t} + \mu_2 v_{2,t} - \frac{1}{2}v_{1,t} - \frac{1}{2}v_{2,t})dt + \sqrt{v_{1,t}}dz_{1,t} + \sqrt{v_{2,t}}dz_{2,t} \tag{A.5}
$$

$$
d(\log(M_t)) = \phi \cdot d(\log(S_t)) + (\delta + \eta_1 v_{1,t} + \eta_2 v_{2,t})dt + \zeta_1 dv_{1,t} + \zeta_2 dv_{2,t} \tag{A.6}
$$

Replacing (A.5) and (A.1) into (A.6) we have:

$$
\begin{aligned}
d(\log(M_t)) =& \big[\phi(r + \mu_1 v_{1,t} + \mu_2 v_{2,t} - \frac{1}{2}v_{1,t} - \frac{1}{2}v_{2,t}) + \delta + \eta_1 v_{1,t} + \eta_2 v_{2,t} \\
&+ \zeta_1\kappa_1(\theta_1 - v_{1,t}) + \zeta_2\kappa_2(\theta_2 - v_{2,t})\big]dt \\
&+ \big[\phi\sqrt{v_{1,t}} + \zeta_1\rho_1\sigma_1\sqrt{v_{1,t}}\big]dz_{1,t} + \big[\phi\sqrt{v_{2,t}} + \zeta_2\rho_2\sigma_2\sqrt{v_{2,t}}\big]dz_{2,t} \\
&+ \big[\zeta_1\sigma_1\sqrt{v_{1,t}}\sqrt{1-\rho_1^2}\big]dB_{1,t} + \big[\zeta_2\sigma_2\sqrt{v_{2,t}}\sqrt{1-\rho_2^2}\big]dB_{2,t}.
\end{aligned} \tag{A.7}
$$

As $dM_t/M_t = d(\log(M_t)) + \frac{1}{2}[d(\log(M_t))]^2$ we have

$$
\begin{aligned}
dM_t/M_t =& \big[\phi(r + \mu_1 v_{1,t} + \mu_2 v_{2,t} - \frac{1}{2}v_{1,t} - \frac{1}{2}v_{2,t}) + \delta + \eta_1 v_{1,t} + \eta_2 v_{2,t} \\
&+ \zeta_1\kappa_1(\theta_1 - v_{1,t}) + \zeta_2\kappa_2(\theta_2 - v_{2,t}) + \frac{1}{2}\phi^2(v_{1,t} + v_{2,t}) \\
&+ \phi(\zeta_1\rho_1\sigma_1 v_{1,t} + \zeta_2\rho_2\sigma_2 v_{2,t}) + \frac{1}{2}\zeta_1^2\sigma_1^2 v_{1,t}^2 + \frac{1}{2}\zeta_2^2\sigma_2^2 v_{2,t}^2\big]dt \\
&+ \big[\phi\sqrt{v_{1,t}} + \zeta_1\rho_1\sigma_1\sqrt{v_{1,t}}\big]dz_{1,t} + \big[\phi\sqrt{v_{2,t}} + \zeta_2\rho_2\sigma_2\sqrt{v_{2,t}}\big]dz_{2,t} \\
&+ \big[\zeta_1\sigma_1\sqrt{v_{1,t}}\sqrt{1-\rho_1^2}\big]dB_{1,t} + \big[\zeta_2\sigma_2\sqrt{v_{2,t}}\sqrt{1-\rho_2^2}\big]dB_{2,t}.
\end{aligned} \tag{A.8}
$$

The first restriction on the pricing kernel is that the product of the money market account, $B_t = B_0\exp(rt)$, and the pricing kernel, M_t, should be a martingale under physical measure. Therefore, $E[d(B_t \cdot M_t)] = 0$ or $E[dM_t/M_t] = -rdt$.

$$\Big[\phi(r+\mu_1 v_{1,t}+\mu_2 v_{2,t}-\frac{1}{2}v_{1,t}-\frac{1}{2}v_{2,t})+\delta+\eta_1 v_{1,t}+\eta_2 v_{2,t}+\zeta_1\kappa_1(\theta_1-v_{1,t})+\zeta_2\kappa_2(\theta_2-v_{2,t})$$
$$+\frac{1}{2}\phi^2(v_{1,t}+v_{2,t})+\phi(\zeta_1\rho_1\sigma_1 v_{1,t}+\zeta_2\rho_2\sigma_2 v_{2,t})+\frac{1}{2}\zeta_1^2\sigma_1^2 v_{1,t}^2+\frac{1}{2}\zeta_2^2\sigma_2^2 v_{2,t}^2\Big]dt=-rdt \tag{A.9}$$

As (A.9) holds for $v_{1,t}=v_{2,t}=0$,

$$\delta=-r(\phi+1)-\zeta_1\kappa_1\theta_1-\zeta_2\kappa_2\theta_2. \tag{A.10}$$

(A.9) also holds for $v_{1,t}=v_{2,t}=\infty$.

$$\begin{aligned}\eta_1&=-\phi\mu_1+1/2\phi+\zeta_1\kappa_1-1/2(\phi^2+\zeta_1^2\sigma_1^2+2\phi\zeta_1\sigma_1\rho_1)\\ \eta_2&=-\phi\mu_2+1/2\phi+\zeta_2\kappa_2-1/2(\phi^2+\zeta_2^2\sigma_2^2+2\phi\zeta_2\sigma_2\rho_2)\end{aligned} \tag{A.11}$$

The second restriction on the pricing kernel is based on the fact that $[S_t.M_t]$ is also a martingale under physical measure. Therefore, $E[d(S_t\cdot M_t)]=0$. As a result of this restriction we have

$$\begin{aligned}&v_{1,t}(\mu_1+\phi+\zeta_1\sigma_1\rho_1)+v_{2,t}(\mu_2+\phi+\zeta_2\sigma_2\rho_2)=0,\\ &\phi=\frac{-1}{v_{1,t}+v_{2,t}}\big[(\mu_1+\zeta_1\sigma_1\rho_1)v_{1,t}+(\mu_2+\zeta_2\sigma_2\rho_2)v_{2,t}\big].\end{aligned} \tag{A.12}$$

If we impose the restriction that $\mu_1+\zeta_1\sigma_1\rho_1\equiv\mu_2+\zeta_2\sigma_2\rho_2$, then (A.12) can be simplified as follows.

$$\phi=-(\mu_1+\zeta_1\sigma_1\rho_1)=-(\mu_2+\zeta_2\sigma_2\rho_2) \tag{A.13}$$

We impose the third restriction on pricing kernel so that for any asset $U\equiv U(S,v_1,v_2,t)$, $[U(t).M_t]$ is also a martingale under P-distribution. Therefore, $E[d(U\cdot M_t)]=E[dU.M_t+U.dM_t+dU.dM_t]=0$. Replacing M_t and dM_t into this equation we have the following restriction where $U_S=\partial U(S,v_1,v_2,t)/\partial S$, $U_{v_1}=\partial U(S,v_1,v_2,t)/\partial v_1$, and $U_{v_2}=\partial U(S,v_1,v_2,t)/\partial v_2$.

$$
\begin{aligned}
& -rU + U_t + U_S(r + \mu_1 v_{1,t} + \mu_2 v_{2,t})S + U_{v_{1,t}}\kappa_1(\theta_1 - v_{1,t}) + U_{v_{2,t}}\kappa_2(\theta_2 - v_{2,t}) \\
& + \frac{1}{2}U_{SS}(v_{1,t} + v_{2,t}) + \frac{1}{2}U_{v_{1,t}v_{1,t}}\sigma_1^2 v_{1,t} + \frac{1}{2}U_{v_{2,t}v_{2,t}}\sigma_2^2 v_{2,t} + U_{Sv_{1,t}}\rho_1\sigma_1 v_{1,t} + U_{Sv_{2,t}}\rho_2\sigma_2 v_{2,t} \\
& + (U_S S\sqrt{v_{1,t}} + U_{v_{1,t}}\rho_1\sigma_1\sqrt{v_{1,t}})(\phi\sqrt{v_{1,t}} + \zeta_1\rho_1\sigma_1\sqrt{v_{1,t}}) \\
& + (U_S S\sqrt{v_{2,t}} + U_{v_{2,t}}\rho_2\sigma_2\sqrt{v_{2,t}})(\phi\sqrt{v_{2,t}} + \zeta_2\rho_2\sigma_2\sqrt{v_{2,t}}) \\
& + U_{v_{1,t}}\zeta_1\sigma_1^2 v_{1,t}(1 - \rho_1^2) + U_{v_{2,t}}\zeta_2\sigma_2^2 v_{2,t}(1 - \rho_2^2) = 0
\end{aligned} \tag{A.14}
$$

The last restriction is based on the fact that discounted price process should be a martingale under risk neutral measure. Therefore, for any asset, $U(S, v_1, v_2, t)$, whose payoff depends on the state variables $\{S, v_1, v_2\}$, U/B_t is a Q-martingale. This restriction implies that $E^Q[d(U/B_t)] = 0$ or equivalently $E^Q[d(U(S, v_1, v_2, t))] = rU(S, v_1, v_2, t)$.

$$
\begin{aligned}
& U_t + rSU_S + U_{v_{1,t}}(\kappa_1(\theta_1 - v_{1,t}) - \lambda_1 v_{1,t}) + U_{v_{2,t}}(\kappa_1(\theta_1 - v_{1,t}) - \lambda_2 v_{2,t}) + \frac{1}{2}U_{SS}(v_{1,t} + v_{1,t}) \\
& + \frac{1}{2}U_{v_{1,t}v_{1,t}}\sigma_1^2 v_{1,t} + \frac{1}{2}U_{v_{2,t}v_{2,t}}\sigma_2^2 v_{2,t} + U_{Sv_{1,t}}\rho_1\sigma_1 v_{1,t} + U_{Sv_{2,t}}\rho_2\sigma_2 v_{2,t} = rU.
\end{aligned} \tag{A.15}
$$

Replace (A.15) from the last restriction into (A.14) from the third restriction.

$$
\begin{aligned}
& U_S(\mu_1 v_{1,t} + \mu_2 v_{2,t})S + U_{v_{1,t}}\lambda_1 v_{1,t} + U_{v_{2,t}}\lambda_2 v_{2,t} \\
& + (U_S S\sqrt{v_{1,t}} + U_{v_{1,t}}\rho_1\sigma_1\sqrt{v_{1,t}})(\phi\sqrt{v_{1,t}} + \zeta_1\rho_1\sigma_1\sqrt{v_{1,t}}) \\
& + (U_S S\sqrt{v_{2,t}} + U_{v_{2,t}}\rho_2\sigma_2\sqrt{v_{2,t}})(\phi\sqrt{v_{2,t}} + \zeta_2\rho_2\sigma_2\sqrt{v_{2,t}}) \\
& + U_{v_{1,t}}\zeta_1\sigma_1^2 v_{1,t}(1 - \rho_1^2) + U_{v_{2,t}}\zeta_2\sigma_2^2 v_{2,t}(1 - \rho_2^2) = 0
\end{aligned}
$$

$$
\begin{aligned}
& U_S(\mu_1 v_{1,t} + \mu_2 v_{2,t})S + U_{v_{1,t}}\lambda_1 v_{1,t} + U_{v_{2,t}}\lambda_2 v_{2,t} \\
& + U_S S\phi v_{1,t} + U_S S\zeta_1\rho_1\sigma_1 v_{1,t} + U_{v_{1,t}}\rho_1\sigma_1\phi v_{1,t} + U_{v_{1,t}}\zeta_1\sigma_1^2 v_{1,t} \\
& + U_S S\phi v_{2,t} + U_S S\zeta_2\rho_2\sigma_2 v_{2,t} + U_{v_{2,t}}\rho_2\sigma_2\phi v_{2,t} + U_{v_{2,t}}\zeta_2\sigma_2^2 v_{2,t} = 0
\end{aligned} \tag{A.16}
$$

From the second restriction in (A.12) we know that $\mu_1 v_{1,t} + \mu_2 v_{2,t} = -\phi v_{1,t} - \zeta_1\rho_1\sigma_1 v_{1,t} - \phi v_{2,t} - \zeta_2\rho_2\sigma_2 v_{2,t}$. Therefore, we can further simplify (A.16).

$$
U_{v_{1,t}}\left(\rho_1\sigma_1\phi + \lambda_1 + \zeta_1\sigma_1^2\right)v_{1,t} + U_{v_{2,t}}\left(\rho_2\sigma_2\phi + \lambda_2 + \zeta_2\sigma_2^2\right)v_{2,t} = 0 \tag{A.17}
$$

One admissible solution for (A.17) would be:

$$\begin{aligned} \rho_1\sigma_1\phi + \lambda_1 + \zeta_1\sigma_1^2 &= 0 \\ \rho_2\sigma_2\phi + \lambda_2 + \zeta_2\sigma_2^2 &= 0 \end{aligned} \tag{A.18}$$

If we combine restrictions in (A.18) with those introduced in (A.13) and replace them back into (A.13) we have ϕ, ζ_1, and ζ_2.

$$\begin{aligned} \zeta_1 &= \frac{\rho_1\sigma_1\mu_1 - \lambda_1}{\sigma_1^2(1-\rho_1^2)} \\ \zeta_2 &= \frac{\rho_2\sigma_2\mu_2 - \lambda_2}{\sigma_2^2(1-\rho_2^2)} \end{aligned} \tag{A.19}$$

$$\phi = -\mu_1 - \frac{\rho_1^2\sigma_1^2\mu_1 - \lambda_1\rho_1\sigma_1}{\sigma_1^2(1-\rho_1^2)} = -\mu_2 - \frac{\rho_2^2\sigma_2^2\mu_2 - \lambda_2\rho_2\sigma_2}{\sigma_2^2(1-\rho_2^2)} \tag{A.20}$$

Therefore, an admissible pricing kernel linking the P and Q dynamics in (A.1) and (A.2) is as follows.

$$\frac{dM_t}{M_t} = -rdt - \mu_1\sqrt{v_{1,t}}dz_{1,t} - \mu_2\sqrt{v_{2,t}}dz_{2,t} + \frac{\rho_1\sigma_1\mu_1 - \lambda_1}{\sigma_1^2(1-\rho_1^2)}dB_{1,t} + \frac{\rho_2\sigma_2\mu_2 - \lambda_2}{\sigma_2^2(1-\rho_2^2)}dB_{2,t} \tag{A.21}$$

This is the pricing kernel introduced in (1).

Now, we show that how physical shocks are linked to risk neutral shocks through equity premium $\{\mu_1, \mu_2\}$ and variance premium $\{\lambda_1, \lambda_2\}$ parameters.

$$\begin{aligned} d\tilde{z}_{1,t} &= dz_{1,t} + (\psi_{1,t} + \rho_1\psi_{3,t})dt \\ d\tilde{z}_{2,t} &= dz_{2,t} + (\psi_{2,t} + \rho_2\psi_{4,t})dt \\ d\tilde{w}_{1,t} &= dw_{1,t} + (\psi_{3,t} + \rho_1\psi_{1,t})dt \\ d\tilde{w}_{2,t} &= dw_{2,t} + (\psi_{4,t} + \rho_2\psi_{2,t})dt \end{aligned} \tag{A.22}$$

Replace physical shocks in return dynamics (1) by risk neutral shocks introduced in (A.22).

$$\begin{aligned} dS_t/S_t &= (r + \mu_1 v_{1,t} + \mu_2 v_{2,t})dt \\ &\quad + \sqrt{v_{1,t}}d\tilde{z}_{1,t} - (\psi_{1,t} + \rho_1\psi_{3,t})\sqrt{v_{1,t}}dt + \sqrt{v_{2,t}}d\tilde{z}_{2,t} - (\psi_{2,t} + \rho_2\psi_{4,t})\sqrt{v_{2,t}}dt \end{aligned} \tag{A.23}$$

As a result of risk neutralization in (A.23), the expected stock returns in (A.23) should be equal to the risk free rate of returns. Therefore, we have the following restriction.

$$(\mu_1 v_{1,t} + \mu_2 v_{2,t})dt = (\psi_{1,t} + \rho_1 \psi_{3,t})\sqrt{v_{1,t}}dt + (\psi_{2,t} + \rho_2 \psi_{4,t})\sqrt{v_{2,t}}dt \tag{A.24}$$

One possible solution of (A.24) is as follows.

$$\begin{aligned} \mu_1\sqrt{v_{1,t}} &= \psi_{1,t} + \rho_1\psi_{3,t} \\ \mu_2\sqrt{v_{2,t}} &= \psi_{2,t} + \rho_2\psi_{4,t} \end{aligned} \tag{A.25}$$

Similarly, we replace the proposed transformation in (A.22) into the dynamics of volatilities in (1).

$$\begin{aligned} dv_{1,t} &= \kappa_1(\theta_1 - v_{1,t})dt + \sigma_1\sqrt{v_{1,t}}d\tilde{w}_{1,t} - \sigma_1\sqrt{v_{1,t}}(\psi_{3,t} + \rho_1\psi_{1,t})dt \\ dv_{2,t} &= \kappa_2(\theta_2 - v_{2,t})dt + \sigma_2\sqrt{v_{2,t}}d\tilde{w}_{2,t} - \sigma_2\sqrt{v_{2,t}}(\psi_{4,t} + \rho_2\psi_{2,t})dt \end{aligned} \tag{A.26}$$

The risk-neutral variance dynamics in (A.26) should be equivalent to those in (A.4), where the market price of variance risk factors is proportional to spot variance. Therefore, we have following restrictions:

$$\begin{aligned} \sigma_1\sqrt{v_{1,t}}(\psi_{3,t} + \rho_1\psi_{1,t}) &= \lambda_1 v_{1,t} \\ \sigma_2\sqrt{v_{2,t}}(\psi_{4,t} + \rho_2\psi_{2,t}) &= \lambda_2 v_{2,t} \end{aligned} \tag{A.27}$$

Combining the restrictions in (A.25) and (A.27), we have the following results, which link the physical distribution (1) to the risk neutral distribution (4).

$$\begin{aligned} \psi_{1,t} &= \frac{\sigma_1\mu_1 - \rho_1\lambda_1}{\sigma_1(1-\rho_1^2)}\sqrt{v_{1,t}} \\ \psi_{2,t} &= \frac{\sigma_2\mu_2 - \rho_2\lambda_2}{\sigma_2(1-\rho_2^2)}\sqrt{v_{2,t}} \\ \psi_{3,t} &= \frac{\lambda_1 - \rho_1\sigma_1\mu_1}{\sigma_1(1-\rho_1^2)}\sqrt{v_{1,t}} \\ \psi_{4,t} &= \frac{\lambda_2 - \rho_2\sigma_2\mu_2}{\sigma_2(1-\rho_2^2)}\sqrt{v_{2,t}} \end{aligned} \tag{A.28}$$

B Proof of Proposition 2

We transform the physical dynamics of individual equity returns (B.1) to its risk neutral counterparts (B.2) by assuming an appropriate stochastic discount factor (SDF).

$$\begin{aligned} dS_t^i/S_t^i &= \mu^i dt + \beta_1^i(\mu_1 v_{1,t}dt + \sqrt{v_{1,t}}dz_{1,t}) + \beta_2^i(\mu_2 v_{2,t}dt + \sqrt{v_{2,t}}dz_{2,t}) + \sqrt{\xi_t^i}dz_t^i \\ d\xi_t^i &= \kappa^i(\theta^i - \xi_t^i)dt + \sigma^i\sqrt{\xi_t^i}dw_t^i \end{aligned} \tag{B.1}$$

$$\begin{aligned} dS_t^i/S_t^i &= rdt + \beta_1^i\sqrt{v_{1,t}}d\tilde{z}_{1,t} + \beta_2^i\sqrt{v_{2,t}}d\tilde{z}_{2,t} + \sqrt{\xi_t^i}d\tilde{z}_t^i \\ d\xi_t^i &= \kappa^i(\theta^i - \xi_t^i)dt + \sigma^i\sqrt{\xi_t^i}dw_t^i \end{aligned} \tag{B.2}$$

where

$$\begin{aligned} \langle dz_t^i, dw_t^i\rangle &= \rho^i dt \\ \langle dz_t^i, dw_t^j\rangle &= 0\ \forall(i \neq j) \end{aligned} \tag{B.3}$$

As individual equity returns are linked to the market index returns with a two-factor model and two constant factor loadings β_1 and β_2, the proposed SDF should jointly specify the risk neutral distributions of the market index and individual equity returns. Remember that the dynamics of market index returns under the P- and Q-measure are as follows.

$$\begin{aligned} dS_t/S_t &= (r + \mu_1 v_{1,t} + \mu_2 v_{2,t})dt + \sqrt{v_{1,t}}dz_{1,t} + \sqrt{v_{2,t}}dz_{2,t} \\ dv_{1,t} &= \kappa_1(\theta_1 - v_{1,t})dt + \sigma_1\sqrt{v_{1,t}}(\rho_1 dz_{1,t} + \sqrt{1-\rho_1^2}dB_{1,t}) \\ dv_{2,t} &= \kappa_2(\theta_2 - v_{2,t})dt + \sigma_2\sqrt{v_{2,t}}(\rho_2 dz_{2,t} + \sqrt{1-\rho_2^2}dB_{2,t}) \end{aligned} \tag{B.4}$$

$$\begin{aligned} dS_t/S_t &= rdt + \sqrt{v_{1,t}}d\tilde{z}_{1,t} + \sqrt{v_{2,t}}d\tilde{z}_{2,t} \\ dv_{1,t} &= \tilde{\kappa}_1(\tilde{\theta}_1 - v_{1,t})dt + \sigma_1\sqrt{v_{1,t}}(\rho_1 d\tilde{z}_{1,t} + \sqrt{1-\rho_1^2}d\tilde{B}_{1,t}) \\ dv_{2,t} &= \tilde{\kappa}_2(\tilde{\theta}_2 - v_{2,t})dt + \sigma_2\sqrt{v_{2,t}}(\rho_2 d\tilde{z}_{2,t} + \sqrt{1-\rho_2^2}d\tilde{B}_{2,t}) \end{aligned} \tag{B.5}$$

where

$$\begin{aligned} &\langle dw_{1,t}, dz_{1,t}\rangle = \rho_1 dt,\ -1 \leq \rho_1 \leq +1 \\ &\langle dw_{2,t}, dz_{2,t}\rangle = \rho_2 dt,\ -1 \leq \rho_2 \leq +1 \\ &\langle dw_{1,t}, dw_{2,t}\rangle = 0 \\ &\rho_1^2 + \rho_2^2 \leq +1 \end{aligned} \tag{B.6}$$

We assume the following standard SDF.

$$\frac{dM_t}{M_t} = -rdt - \psi_t' dW_t \ , \tag{B.7}$$

where $\psi_t \equiv [\psi_{1,t}, \psi_{2,t}, \psi_{3,t}, \psi_{4,t}, \psi^i_{1,t}, \psi^i_{2,t}]$ $i = \{1, 2, \cdot\cdot\cdot, n\}$ is the vector of market price of risk factors and $W_t \equiv [z_{1,t}, z_{2,t}, w_{1,t}, w_{2,t}, z^i_t, w^i_t]$ $i = \{1, 2, \cdot\cdot\cdot, n\}$ is the vector of innovations in market return, market variance components, equity i return, and equity i idiosyncratic variance. Given the SDF in (B.7), the change-of-measure from P- to Q-distribution has the following exponential form.

$$\frac{dQ}{dP}(t) \equiv M_t \exp(rt) = \exp\Big[-\int_0^t \psi_u^{'} dW_u - \frac{1}{2}\int_0^t \psi_u^{'} d\langle W, W^{'}\rangle_u \psi_u \Big] \tag{B.8}$$

where $\langle W, W^{'}\rangle$ is the covariance operator.

We follow the notion of Doléans-Dade exponential (stochastic exponential) and define the stochastic exponential $\varepsilon(\cdot)$ as follow.

$$\varepsilon\Big(\int_0^t \vartheta_u^{'} dW_u\Big) \equiv \exp\Big[\int_0^t \vartheta_u^{'} dW_u - \frac{1}{2}\int_0^t \vartheta_u^{'} d\langle W, W^{'}\rangle_u \vartheta_u\Big] \tag{B.9}$$

Therefore, the change-of-measure (B.8) can be expressed in term of stochastic exponential as

$$\frac{dQ}{dP}(t) = \varepsilon\Big(\int_0^t -\psi_u^{'} dW_u\Big) \tag{B.10}$$

Applying Ito's lemma, for every individual equity i, we have the following dynamic under the physical measure.

$$\begin{aligned} \log\big(\frac{S^i_t}{S^i_0}\big) = & \big[\mu^i + \beta^i_1 \mu_1 v_{1,t} + \beta^i_2 \mu_2 v_{2,t} - \frac{1}{2}(\beta^i_1)^2 v_{1,t} - \frac{1}{2}(\beta^i_2)^2 v_{2,t} - \frac{1}{2}\xi^i_t\big]t \\ & + \beta^i_1 \int_0^t \sqrt{v_{1,u}} dz_{1,u} + \beta^i_2 \int_0^t \sqrt{v_{2,u}} dz_{2,u} + \int_0^t \sqrt{\xi^i_u} dz^i_u \end{aligned} \tag{B.11}$$

Given (B.11) and definition of stochastic exponential (B.9) we have

$$\frac{S^i_t}{S^i_0} = \exp\big[(\mu^i + \beta^i_1 \mu_1 v_{1,t} + \beta^i_2 \mu_2 v_{2,t})t\big] \varepsilon\Big(\int_0^t \beta^i_1 \sqrt{v_{1,u}} dz_{1,u} + \int_0^t \beta^i_2 \sqrt{v_{2,u}} dz_{2,u} + \int_0^t \sqrt{\xi^i_u} dz^i_u\Big) \tag{B.12}$$

Note that

$$\varepsilon\Big(\int_0^t \beta_1^i \sqrt{v_{1,u}} dz_{1,u}\Big) = \exp\Big[\int_0^t \beta_1^i \sqrt{v_{1,u}} dz_{1,u} - \frac{1}{2}\int_0^t (\beta_1^i)^2 v_{1,u} du\Big] \tag{B.13}$$

To find the market prices of risk we impose the restriction that the product of the price of any individual equity and the pricing kernel under physical measure is a P-martingale. Given the change-of-measure (B.10), for every individual equity i, the following process $N(t)$ should be a P-martingale.

$$N(t) \equiv \frac{S_t^i}{S_0^i}\frac{dQ}{dP}(t)\exp(-rt) \tag{B.14}$$

where

$$\begin{aligned} N(t) =& \exp\left[(-r + \mu^i + \beta_1^i \mu_1 v_{1,t} + \beta_2^i \mu_2 v_{2,t})t\right] \\ & \varepsilon\Big(\int_0^t \beta_1^i \sqrt{v_{1,u}} dz_{1,u}\Big)\varepsilon\Big(-\int_0^t \psi_{1,u} dz_{1,u} - \int_0^t \psi_{3,u} dw_{1,u}\Big) \\ & \varepsilon\Big(\int_0^t \beta_2^i \sqrt{v_{2,u}} dz_{2,u}\Big)\varepsilon\Big(-\int_0^t \psi_{2,u} dz_{2,u} - \int_0^t \psi_{4,u} dw_{2,u}\Big) \\ & \varepsilon\Big(\int_0^t \sqrt{\xi_u^i} dz_u^i\Big)\varepsilon\Big(-\int_0^t \psi_{1,u}^i dz_u^i - \int_0^t \psi_{2,u}^i dw_u^i\Big) \\ & \varepsilon\Big(-\sum_{j\notin i}\int_0^t \psi_{1,u}^j dz_u^j - \sum_{j\notin i}\int_0^t \psi_{2,u}^j dw_u^j\Big) \end{aligned} \tag{B.15}$$

We decompose $N(t)$ into two orthogonal components $N(t) \equiv I(t)L(t)$ and then make sure that $I(t)$ and $L(t)$ are a P-martingale.

$$\begin{aligned} I(t) =& \exp\left[(-r + \mu^i + \beta_1^i \mu_1 v_{1,t} + \beta_2^i \mu_2 v_{2,t})t\right] \\ & \varepsilon\Big(\int_0^t \beta_1^i \sqrt{v_{1,u}} dz_{1,u}\Big)\varepsilon\Big(-\int_0^t \psi_{1,u} dz_{1,u} - \int_0^t \psi_{3,u} dw_{1,u}\Big) \\ & \varepsilon\Big(\int_0^t \beta_2^i \sqrt{v_{2,u}} dz_{2,u}\Big)\varepsilon\Big(-\int_0^t \psi_{2,u} dz_{2,u} - \int_0^t \psi_{4,u} dw_{2,u}\Big) \\ & \varepsilon\Big(\int_0^t \sqrt{\xi_u^i} dz_u^i\Big)\varepsilon\Big(-\int_0^t \psi_{1,u}^i dz_u^i - \int_0^t \psi_{2,u}^i dw_u^i\Big) \end{aligned} \tag{B.16}$$

$$L(t) = \varepsilon\Big(-\sum_{j\notin i}\int_0^t \psi_{1,u}^j dz_u^j - \sum_{j\notin i}\int_0^t \psi_{2,u}^j dw_u^j\Big) \tag{B.17}$$

From the definition of a stochastic exponential we know that $\varepsilon(\cdot)$ are P-martingales and so does $L(t)$. Therefore, we only need to make sure that $I(t)$ is also a P-martingale. Using the properties of a stochastic exponential $\varepsilon(\cdot)$, $\varepsilon(X_t)\varepsilon(Y_t) = \varepsilon(X_t + Y_t)\exp(\langle X, Y\rangle_t)$ and the correlation structure (B.3) and (B.6) we can rewrite the process of $I(t)$ as follows.

$$\begin{aligned} I(t) =& \exp\left[(-r + \mu^i + \beta_1^i \mu_1 v_{1,t} + \beta_2^i \mu_2 v_{2,t})t\right] \\ & \varepsilon\Big(\int_0^t (\beta_1^i \sqrt{v_{1,u}} - \psi_{1,u}) dz_{1,u} - \int_0^t \psi_{3,u} dw_{1,u}\Big) \exp\Big[-\int_0^t \beta_1^i \sqrt{v_{1,u}}(\psi_{1,u} + \rho_1 \psi_{3,u}) du\Big] \\ & \varepsilon\Big(\int_0^t (\beta_2^i \sqrt{v_{2,u}} - \psi_{2,u}) dz_{2,u} - \int_0^t \psi_{4,u} dw_{2,u}\Big) \exp\Big[-\int_0^t \beta_2^i \sqrt{v_{2,u}}(\psi_{2,u} + \rho_2 \psi_{4,u}) du\Big] \\ & \varepsilon\Big(\int_0^t (\sqrt{\xi_u^i} - \psi_{1,u}^i) dz_u^i - \int_0^t \psi_{2,u}^i dw_u^i\Big) \exp\Big[-\int_0^t \sqrt{\xi_u^i}(\psi_{1,u}^i + \rho^i \psi_{2,u}^i) du\Big] \end{aligned} \tag{B.18}$$

Thus, given $\varepsilon(\cdot)$ are P-martingales, the process $I(t)$ is a P-martingale when the following restriction holds.

$$\begin{aligned} & \exp\left[(-r + \mu^i + \beta_1^i \mu_1 v_{1,t} + \beta_2^i \mu_2 v_{2,t})t\right] \\ & \exp\Big[-\int_0^t \beta_1^i \sqrt{v_{1,u}}(\psi_{1,u} + \rho_1 \psi_{3,u}) du\Big] \exp\Big[-\int_0^t \beta_2^i \sqrt{v_{2,u}}(\psi_{2,u} + \rho_2 \psi_{4,u}) du\Big] \\ & \exp\Big[-\int_0^t \sqrt{\xi_u^i}(\psi_{1,u}^i + \rho^i \psi_{2,u}^i) du\Big] = 1 \end{aligned} \tag{B.19}$$

The restriction (B.19) holds if the following conditions for the market index, (B.20), and for every individual equity i, (B.21), hold.

$$\begin{aligned} \mu_1 v_{1,t} t - \sqrt{v_{1,t}}(\psi_{1,t} + \rho_1 \psi_{3,t})t = 0 \\ \mu_2 v_{2,t} t - \sqrt{v_{2,t}}(\psi_{3,t} + \rho_2 \psi_{4,t})t = 0 \end{aligned} \tag{B.20}$$

$$-rt + \mu^i t - \sqrt{\xi_t^i}(\psi_{1,t}^i + \rho^i \psi_{2,t}^i)t = 0 \tag{B.21}$$

To fully specify the market prices of risk we assume that market price of variance risk factors are proportional to spot volatility components, following Heston [1993].

$$\begin{aligned} (\psi_{3,t} + \rho_1 \psi_{1,t}) &= \frac{v_{1,t}}{\sigma_1 \sqrt{v_{1,t}}} \lambda_1 \\ (\psi_{4,t} + \rho_2 \psi_{2,t}) &= \frac{v_{2,t}}{\sigma_2 \sqrt{v_{2,t}}} \lambda_2 \end{aligned} \tag{B.22}$$

If we assume that the idiosyncratic variance is also a priced risk factor, then its price is also proportional to the spot idiosyncratic volatility for every individual equity i. Otherwise, $\lambda^i = 0$.

$$(\psi^i_{2,t} + \rho^i \psi^i_{1,t}) = \frac{\xi^i_t}{\sigma^i \sqrt{\xi^i_t}} \lambda^i \tag{B.23}$$

Combining the restrictions in (B.20) and (B.22), we have the following market price of risk factors.

$$\begin{aligned} \psi_{1,t} &= \frac{\sigma_1 \mu_1 - \rho_1 \lambda_1}{(1-\rho_1^2)} \frac{\sqrt{v_{1,t}}}{\sigma_1} \\ \psi_{2,t} &= \frac{\sigma_2 \mu_2 - \rho_2 \lambda_2}{(1-\rho_2^2)} \frac{\sqrt{v_{2,t}}}{\sigma_2} \\ \psi_{3,t} &= \frac{\lambda_1 - \rho_1 \sigma_1 \mu_1}{(1-\rho_1^2)} \frac{\sqrt{v_{1,t}}}{\sigma_1} \\ \psi_{4,t} &= \frac{\lambda_2 - \rho_2 \sigma_2 \mu_2}{(1-\rho_2^2)} \frac{\sqrt{v_{2,t}}}{\sigma_2} \end{aligned} \tag{B.24}$$

Combining the restrictions in (B.21) and (B.23) and given that idiosyncratic variance is not priced, we have the following results for every individual equity.

$$\begin{aligned} \psi^i_{1,t} &= \frac{\mu^i - r}{\sqrt{\xi^i_t}(1-(\rho^i)^2)} \\ \psi^i_{2,t} &= \left(-\frac{\mu^i - r}{\sqrt{\xi^i_t}} + \frac{\xi^i \lambda^i}{\sigma^i}\right) \frac{\rho^i}{1-(\rho^i)^2} \end{aligned} \tag{B.25}$$

Given the market prices of risk factors (B.24) (B.25), we apply the Girsanov's theorem to transform physical innovations of the market index dynamics (B.4) and individual equity dynamics (B.1) to their risk neutral counterparts in (B.5) and (B.2). Note that we assume idiosyncratic variance is not priced and thus $\lambda^i = 0$.

$$\begin{aligned} d\tilde{z}^i_t &= dz^i_t + \psi^i_{1,t} dt + \rho^i \psi^i_{2,t} dt \\ d\tilde{z}_{1,t} &= dz_{1,t} + \psi_{1,t} dt + \rho_1 \psi_{3,t} dt \\ d\tilde{z}_{2,t} &= dz_{2,t} + \psi_{2,t} dt + \rho_2 \psi_{4,t} dt \\ d\tilde{w}_{1,t} &= dw_{1,t} + \psi_{3,t} dt + \rho_1 \psi_{1,t} dt \\ d\tilde{w}_{2,t} &= dw_{2,t} + \psi_{4,t} dt + \rho_2 \psi_{2,t} dt \end{aligned} \tag{B.26}$$

With some algebra we have the following transformations.

$$
\begin{aligned}
d\tilde{z}_t^i &= dz_t^i + (\mu^i - r)dt/\sqrt{\xi_t^i} \\
d\tilde{z}_{1,t} &= dz_{1,t} + \mu_1\sqrt{v_{1,t}}dt \\
d\tilde{z}_{2,t} &= dz_{2,t} + \mu_2\sqrt{v_{2,t}}dt \\
d\tilde{w}_{1,t} &= dw_{1,t} + (\lambda_1/\sigma_1)\sqrt{v_{1,t}}dt \\
d\tilde{w}_{2,t} &= dw_{2,t} + (\lambda_2/\sigma_2)\sqrt{v_{2,t}}dt
\end{aligned} \tag{B.27}
$$

Replacing $dz_t^i, dw_t^i, dz_{1,t}, dz_{2,t}, dw_{1,t}, dw_{2,t}$ from (B.27) into the physical dynamics in (B.1) and (B.4) and knowing that $\tilde{\kappa}_1 = \kappa_1 + \lambda_1$, $\tilde{\kappa}_2 = \kappa_2 + \lambda_2$, $\tilde{\theta}_1 = \frac{k_1\theta_1}{k_1+\lambda_1}$, $\tilde{\theta}_2 = \frac{k_2\theta_2}{k_2+\lambda_2}$ we obtain risk neutral return and variance components dynamics.

$$
\begin{aligned}
dS_t^i/S_t^i &= \mu^i dt + \beta_1^i(\mu_1 v_{1,t}dt + \sqrt{v_{1,t}}dz_{1,t}) + \beta_2^i(\mu_2 v_{2,t}dt + \sqrt{v_{2,t}}dz_{2,t}) + \sqrt{\xi_t^i}dz_t^i \\
&= \mu^i dt + \beta_1^i(\mu_1 v_{1,t}dt + \sqrt{v_{1,t}}(d\tilde{z}_{1,t} - \mu_1\sqrt{v_{1,t}}dt)) \\
&\quad + \beta_2^i(\mu_2 v_{2,t}dt + \sqrt{v_{2,t}}(d\tilde{z}_{2,t} - \mu_2\sqrt{v_{2,t}}dt)) + \sqrt{\xi_t^i}(d\tilde{z}_t^i - (\mu^i - r)dt/\sqrt{\xi_t^i}) \\
&= rdt + \beta_1^i\sqrt{v_{1,t}}d\tilde{z}_{1,t} + \beta_2^i\sqrt{v_{2,t}}d\tilde{z}_{2,t} + \sqrt{\xi_t^i}d\tilde{z}_t^i
\end{aligned} \tag{B.28}
$$

$$
\begin{aligned}
dS_t/S_t &= (r + \mu_1 v_{1,t} + \mu_2 v_{2,t})dt + \sqrt{v_{1,t}}dz_{1,t} + \sqrt{v_{2,t}}dz_{2,t} \\
&= (r + \mu_1 v_{1,t} + \mu_2 v_{2,t})dt + \sqrt{v_{1,t}}(d\tilde{z}_{1,t} - \mu_1\sqrt{v_{1,t}}dt) + \sqrt{v_{2,t}}(d\tilde{z}_{2,t} - \mu_2\sqrt{v_{2,t}}dt) \\
&= rdt + \sqrt{v_{1,t}}d\tilde{z}_{1,t} + \sqrt{v_{2,t}}d\tilde{z}_{2,t}
\end{aligned} \tag{B.29}
$$

$$
\begin{aligned}
dv_{1,t} &= \kappa_1(\theta_1 - v_{1,t})dt + \sigma_1\sqrt{v_{1,t}}(d\tilde{w}_{1,t} - (\lambda_1/\sigma_1)\sqrt{v_{1,t}}dt) \\
&= (\kappa_1\theta_1 - (\kappa_1 + \lambda_1)v_{1,t})dt + \sigma_1\sqrt{v_{1,t}}d\tilde{w}_{1,t} \\
&= \tilde{\kappa}_1(\tilde{\theta}_1 - v_{1,t})dt + \sigma_1\sqrt{v_{1,t}}d\tilde{w}_{1,t}
\end{aligned} \tag{B.30}
$$

$$
\begin{aligned}
dv_{2,t} &= \kappa_2(\theta_2 - v_{2,t})dt + \sigma_2\sqrt{v_{2,t}}(d\tilde{w}_{2,t} - (\lambda_2/\sigma_2)\sqrt{v_{2,t}}dt) \\
&= (\kappa_2\theta_2 - (\kappa_2 + \lambda_2)v_{2,t})dt + \sigma_2\sqrt{v_{2,t}}d\tilde{w}_{2,t} \\
&= \tilde{\kappa}_2(\tilde{\theta}_2 - v_{2,t})dt + \sigma_2\sqrt{v_{2,t}}d\tilde{w}_{2,t}
\end{aligned} \tag{B.31}
$$

C Proof of Proposition 3

Given the Q dynamics of index returns and individual equities returns in (4) and (15), applying Ito's lemma on x_t^i, delivers the following expression.

$$
\begin{aligned}
x_{t+\tau}^i - x_t^i &= r\tau - \frac{1}{2}\left[{\beta_1^i}^2 v_{1,t:t+\tau} + {\beta_2^i}^2 v_{2,t:t+\tau} + \xi_{t:t+\tau}^i\right]\tau \\
&\quad + \beta_1^i\int_t^{t+\tau}\sqrt{v_{1,u}}d\tilde{z}_{1,u} + \beta_2^i\int_t^{t+\tau}\sqrt{v_{2,u}}d\tilde{z}_{2,u} + \int_t^{t+\tau}\sqrt{\xi_t^i}d\tilde{z}_u^i
\end{aligned} \tag{C.1}
$$

For the ease of notations we define:

$$\tilde{z}_{v_{1,\tau}} \equiv \int_t^{t+\tau} \sqrt{v_{1,u}} d\tilde{z}_{1,u} \ ,$$
$$\tilde{z}_{v_{2,\tau}} \equiv \int_t^{t+\tau} \sqrt{v_{2,u}} d\tilde{z}_{2,u} \ ,$$
$$\tilde{z}_{\xi^i_\tau} \equiv \int_t^{t+\tau} \sqrt{\xi^i_u} d\tilde{z}^i_u \ .$$

By the definition of risk-neutral conditional characteristic function of log-returns in (17) we have:[37]

$$\tilde{f}^i(\tau,\phi) = E^Q_t\Big[\exp\big[i\phi(r\tau - \frac{1}{2}({\beta^i_1}^2 v_{1,t:t+\tau} + {\beta^i_2}^2 v_{2,t:t+\tau} + \xi^i_{t:t+\tau})\tau + \beta^i_1\tilde{z}_{v_{1,\tau}} + \beta^i_2\tilde{z}_{v_{2,\tau}} + \tilde{z}_{\xi^i_\tau})\big]\Big]. \tag{C.2}$$

Define the stochastic exponential $\zeta(\cdot)$ as follows.

$$\zeta\Big(\int_0^t w'_u dW_u\Big) \equiv \exp\big[\int_0^t w'_u dW_u - \frac{1}{2}\int_0^t w'_u d\langle W, W'\rangle w_u\big] \tag{C.3}$$

Therefore,

$$\begin{aligned}\zeta\big(i\phi\beta^i_1\ \tilde{z}_{v_{1,\tau}}\big) &= \exp\big[i\phi\beta^i_1\ \tilde{z}_{v_{1,\tau}} - \frac{1}{2}(i\phi\beta^i_1)^2\langle\tilde{z}_{v_{1,\tau}}, \tilde{z}_{v_{1,\tau}}\rangle\big] \\ &= \exp\big[i\phi\beta^i_1\ \tilde{z}_{v_{1,\tau}} + \frac{1}{2}\phi^2{\beta^i_1}^2 v_{1,t:1+\tau}\big].\end{aligned} \tag{C.4}$$

Similar to (C.4), define $\zeta\big(i\phi\beta^i_2\ \tilde{z}_{v_{2,\tau}}\big)$ and $\zeta\big(i\phi\ \tilde{z}_{\xi^i_\tau}\big)$ and then combine these three stochastic exponential with (C.2) to get the following risk-neutral conditional characteristic function.

$$\tilde{f}^i(\tau,\phi) = e^{i\phi r\tau}E^Q_t\Big[\zeta\big(i\phi\beta^i_1\ \tilde{z}_{v_{1,\tau}}\big)\zeta\big(i\phi\beta^i_2\ \tilde{z}_{v_{2,\tau}}\big)\zeta\big(i\phi\ \tilde{z}_{\xi^i_\tau}\big)\exp\big[-g_1 v_{1,t:t+\tau} - g_2 v_{2,t:t+\tau} - g_3\xi^i_{t:t+\tau}\big]\Big] \tag{C.5}$$

where, $g_1 = \frac{1}{2}i\phi{\beta^i_1}^2(1-i\phi)$, $g_2 = \frac{1}{2}i\phi{\beta^i_2}^2(1-i\phi)$, and $g_3 = \frac{1}{2}i\phi(1-i\phi)$. Following Carr and Wu [2004], we define a new change-of-measure from Q-measure to C-measure as follows.[38]

[37]For compactness, the dependence of risk-neutral conditional characteristic function to x^i_t, $v_{1,t}$, $v_{2,t}$, ξ^i_t, β^i_1, and β^i_2 is suppressed in (C.2).

[38]As the Radon-Nikodym derivatives in(C.6) is defined based on the stochastic exponential $\zeta(\cdot)$, it is Martingale by definition.

$$\frac{dC}{dQ}(t) \equiv \zeta\big(i\phi\beta_1^i\ \tilde{z}_{v_{1,\tau}}\big)\zeta\big(i\phi\beta_2^i\ \tilde{z}_{v_{2,\tau}}\big)\zeta\big(i\phi\ \tilde{z}_{\xi^i_\tau}\big) \tag{C.6}$$

The Radon-Nikodym derivatives of C with respect to Q in (C.6) allows to write (C.5) as

$$\begin{aligned}\tilde{f}^i(\tau,\phi) &= e^{i\phi r\tau}E_t^Q\Big[\frac{\frac{dC}{dQ}(T)}{\frac{dC}{dQ}(t)}\exp\big[-g_1v_{1,t:t+\tau}-g_2v_{2,t:t+\tau}-g_3\xi^i_{t:t+\tau}\big]\Big]\\ &= e^{i\phi r\tau}E_t^C\Big[\exp\big[-g_1v_{1,t:t+\tau}-g_2v_{2,t:t+\tau}-g_3\xi^i_{t:t+\tau}\big]\Big].\end{aligned} \tag{C.7}$$

Accordingly, we transform the risk-neutral shocks to index returns volatlities and to the idiosyncratic returns volatility to their C-measure counterparts by applying the extension of Grisanov's theorem within the complex plane.

$$\begin{aligned}d\tilde{w}_{1,t} &= dw_{1,t}^C + (i\phi\rho_1\beta_1^i\sqrt{v_{1,t}})dt\\ d\tilde{w}_{2,t} &= dw_{2,t}^C + (i\phi\rho_2\beta_2^i\sqrt{v_{2,t}})dt\\ d\tilde{w}_t^i &= dw_t^{i,C} + (i\phi\rho^i\sqrt{\xi_t^i})dt\end{aligned} \tag{C.8}$$

As a results, the index volatilities dynamics and idiosyncratic volatility dynamics of individual equity under the C-measure are

$$\begin{aligned}dv_{1,t} &= \kappa_1^C(\theta_1^C - v_{1,t})dt + \sigma_1\sqrt{v_{1,t}}dw_{1,t}^C\ ,\\ dv_{2,t} &= \kappa_2^C(\theta_2^C - v_{2,t})dt + \sigma_2\sqrt{v_{2,t}}dw_{2,t}^C\ ,\\ d\xi_t^i &= \kappa^{i,C}(\theta^{i,C} - \xi_t^i)dt + \sigma^i\sqrt{\xi_t^i}dw_t^{i,C},\end{aligned} \tag{C.9}$$

where,

$$\begin{aligned}\kappa_1^C &= \tilde{\kappa}_1 - i\phi\rho_1\beta_1^i\sigma_1 \quad & \theta_1^C &= \tilde{\kappa}_1\tilde{\theta}_1/\kappa_1^C\ ,\\ \kappa_2^C &= \tilde{\kappa}_2 - i\phi\rho_2\beta_2^i\sigma_2 \quad & \theta_2^C &= \tilde{\kappa}_2\tilde{\theta}_2/\kappa_2^C\ ,\\ \kappa^{i,C} &= \kappa^i - i\phi\rho^i\sigma^i \quad & \theta^{i,C} &= \kappa^i\theta^i/\kappa^{i,C}\ .\end{aligned}$$

Using the closed-form solution of the moment generating functions of $E_t^C[\exp(-g_1v_{1,t:t+\tau})]$, and $E_t^C[\exp(-g_2v_{2,t:t+\tau})]$, and $E_t^C[\exp(-g_3\xi^i_{t:t+\tau})]$, the risk-neutral conditional characteristic function of log individual equity prices has the following affine form.

$$\begin{aligned}\tilde{f}^i(v_{1,t},v_{2,t},\xi_t^i,\tau,\phi) = \exp\big[&i\phi x_t^i + i\phi r\tau - A_1(\tau,\phi) - A_2(\tau,\phi) - B(\tau,\phi)\\ &- C_1(\tau,\phi)v_{1,t} - C_2(\tau,\phi)v_{2,t} - D(\tau,\phi)\xi_t^i\big],\end{aligned} \tag{C.10}$$

$$
\begin{aligned}
A_1(\tau,\phi) &= \frac{\tilde{\kappa}_1\tilde{\theta}_1}{\sigma_1^2}\Big[2\ln\big[1-\frac{d_1-\kappa_1^C}{2d_1}(1-e^{-d_1\tau})\big]+(d_1-\kappa_1^C)\tau\Big],\\
A_2(\tau,\phi) &= \frac{\tilde{\kappa}_2\tilde{\theta}_2}{\sigma_2^2}\Big[2\ln\big[1-\frac{d_2-\kappa_2^C}{2d_2}(1-e^{-d_2\tau})\big]+(d_2-\kappa_2^C)\tau\Big],\\
B(\tau,\phi) &= \frac{\tilde{\kappa}^i\tilde{\theta}^i}{{\sigma^i}^2}\Big[2\ln\big[1-\frac{d^i-\kappa^{i,C}}{2d^i}(1-e^{-d^i\tau})\big]+(d^i-\kappa^{i,C})\tau\Big],\\
C_1(\tau,\phi) &= \frac{2g_1(1-e^{-d_1\tau})}{2d_1-(d_1-\kappa_1^C)(1-e^{-d_1\tau})},\\
C_2(\tau,\phi) &= \frac{2g_2(1-e^{-d_2\tau})}{2d_2-(d_2-\kappa_2^C)(1-e^{-d_2\tau})},\\
D(\tau,\phi) &= \frac{2g^i(1-e^{-d^i\tau})}{2d^i-(d^i-\kappa^{i,C})(1-e^{-d^i\tau})},\\
d_1 &= \sqrt{(\kappa_1^C)^2+2\sigma_1^2g_1}\ ,\\
d_2 &= \sqrt{(\kappa_2^C)^2+2\sigma_2^2g_2}\ ,\\
d^i &= \sqrt{(\kappa^{i,C})^2+2{\sigma^i}^2g^i}\ ,\\
g_1 &= \frac{1}{2}i\phi{\beta_1^i}^2(1-i\phi)\ ,\\
g_2 &= \frac{1}{2}i\phi{\beta_2^i}^2(1-i\phi)\ ,\\
g^i &= \frac{1}{2}i\phi(1-i\phi)\ .
\end{aligned}
\tag{C.11}
$$

We determine the price of a European call option on an individual equity with the strike price K and the time-to-maturity τ by inverting the risk-neutral conditional characteristic function of log-returns.[39]

$$
C_t^i(S_t^i,K,\tau) = S_t^iP_1^i - Ke^{-r\tau}P_2^i\ , \tag{C.12}
$$

where,

$$
\begin{aligned}
P_1^i &= \frac{1}{2}+\frac{1}{\pi}\frac{1}{S_t^ie^{r\tau}}\int_0^\infty \Re\Big[\frac{e^{-i\phi\ln K}\tilde{f}^i(v_{1,t},v_{2,t},\xi_t^i,\tau,\phi-i)}{i\phi}\Big]d\phi\ ,\\
P_2^i &= \frac{1}{2}+\frac{1}{\pi}\int_0^\infty \Re\Big[\frac{e^{-i\phi\ln K}\tilde{f}^i(v_{1,t},v_{2,t},\xi_t^i,\tau,\phi)}{i\phi}\Big]d\phi\ .
\end{aligned}
\tag{C.13}
$$

[39]Note that the risk-neutral conditional characteristic function of the logarithm of individual equity returns, $x_{t+\tau}^i - x_t^i = \ln(S_{t+\tau}^i/S_t^i)$, can be defined with the same expression as (C.10) but without the first component, $i\phi x_t^i$.

D Appendix D

Proofs of Proposition (4) and Proposition (5) are available upon request.

E Estimation of the Index Model - Discretization and Particle Filter Methods

To estimate the parameters of two-factor stochastic volatility model of the index we follow the literature on the estimation of stochastic volatility models, where the main challenge is the estimation of unobserved latent volatilities. There are several approaches to estimate stochastic volatility model. Our own approach combines the information from underlying index and option markets to impose consistency between structural parameters under P and Q distributions, known as joint estimation. Therefore, we use a likelihood function that contains a return-based component and an option-based component, as in Santa-Clara and Yan [2010] and Christoffersen et al. [2013].[40] Here we do a joint-estimation by filtering the two vectors of daily spot variances, $\{v_{1,t}, v_{2,t}\}$, and simultaneously estimating a set of structural parameters of the dynamics of index returns and variances, including the market price of each variance component, $\Theta \equiv \{\kappa_1, \kappa_2, \theta_1, \theta_2, \sigma_1, \sigma_2, \rho_1, \rho_2, \lambda_1, \lambda_2\}$. Note that joint estimation allow us to have reliable prices of variance risk factors, as we can get a consistent set of structural parameters between the P and Q distributions.

Since the market variances are unobserved state variables, we first extract daily instantaneous persistent and transient variance components using the Particle Filter (PF) method. This optimal filtering methodology provides a tool for learning about unobserved shocks and states from discretely observed prices generated by continuous-time models.[41] Although we generally follow the conventional filtration procedure in the literature, we provide a novel approach to the challenge of filtering the two separate variance paths. Our proposed solution is not trivial and to the best of our knowledge is novel and constitutes a methodological contribution to the option pricing literature.

To define the return-based likelihood function and filter spot variances, we start by discretizing the returns dynamics (1). Applying Ito's lemma to equation (1), gives the dynamics of logarithm of stock prices as follows.

[40] Consistency can also be imposed through moment-based and simulation-based methods; see Ait-Sahalia and Kimmel [2007], Eraker [2004], Jones [2003], Chernov and Ghysels [2000], and Pan [2002]. Other approaches use only option-based data to estimate only the Q distribution; Bakshi et al. [1997], Bates [2000], Huang and Wu [2004], and Christoffersen et al. [2009].

[41] For the application of PF in estimating the model parameters see Gordon et al. [1993], Johannes et al. [2009], Johannes and Polson [2009], Christoffersen et al. [2010], and Boloorforoosh [2014].

$$
\begin{aligned}
& d\ln(S_t) = (\mu - \frac{1}{2}(v_{1,t} + v_{2,t}))dt + \sqrt{v_{1,t}}dz_{1,t} + \sqrt{v_{2,t}}dz_{2,t} \ , \\
& dv_{1,t} = \kappa_1(\theta_1 - v_{1,t})dt + \sigma_1\sqrt{v_{1,t}}dw_{1,t} \ , \\
& dv_{2,t} = \kappa_2(\theta_2 - v_{2,t})dt + \sigma_2\sqrt{v_{2,t}}dw_{2,t} \ ,
\end{aligned}
\tag{E.1}
$$

where, $\mu \equiv r + \mu_1 v_{1,t} + \mu_2 v_{2,t}$. We discretize (E.1) using the Euler scheme.[42] Equation (E.2) models the relation between observed index prices and unobserved variances at time $t + \Delta t$ conditional on the time t variances.

$$
\begin{aligned}
& \ln(S_{t+\Delta t}) - \ln(S_t) = (\mu - \frac{1}{2}(v_{1,t} + v_{2,t}))\Delta t + \sqrt{v_{1,t}\Delta t}\ z_{1,t+\Delta t} + \sqrt{v_{2,t}\Delta t}\ z_{2,t+\Delta t} \ , \\
& v_{1,t+\Delta t} = v_{1,t} + \kappa_1(\theta_1 - v_{1,t})\Delta t + \sigma_1\sqrt{v_{1,t}\Delta t}\ w_{1,t+\Delta t} \ , \\
& v_{2,t+\Delta t} = v_{2,t} + \kappa_2(\theta_2 - v_{2,t})\Delta t + \sigma_2\sqrt{v_{2,t}\Delta t}\ w_{2,t+\Delta t} \ .
\end{aligned}
\tag{E.2}
$$

Brownian shocks $z_{1,t+\Delta t}$, $z_{2,t+\Delta t}$, $w_{1,t+\Delta t}$, and $w_{2,t+\Delta t}$ are normal random variables with mean zero and variance one. From the first equation in (E.2) we use the observed daily index log-prices ($\ln(S_t)$, $\ln(S_{t+\Delta t})$) to first filter the daily return's shocks ($z_{1,t+\Delta t}$, $z_{2,t+\Delta t}$) and then, using the filtered shocks in returns and the last two equation in (E.2), we filter daily spot variances ($v_{1,t+\Delta t}$, $v_{2,t+\Delta t}$). Note that we filter filter the summation of return shocks $z_{1,t+\Delta t} + z_{2,t+\Delta t}$ as we cannot separate the daily observed shocks into two components, $z_{1,t+\Delta t}$ and $z_{2,t+\Delta t}$. Therefore, we rewrite the underlying dynamics as (E.3), given that the return shocks are uncorrelated and then discretize this dynamics.

$$
\begin{aligned}
& d\ln(S_t) = (\mu - \frac{1}{2}(v_{1,t} + v_{2,t}))dt + \sqrt{v_{1,t} + v_{2,t}}dz_t \ , \\
& dv_{1,t} = \kappa_1(\theta_1 - v_{1,t})dt + \sigma_1\sqrt{v_{1,t}}dw_{1,t} \ , \\
& dv_{2,t} = \kappa_2(\theta_2 - v_{2,t})dt + \sigma_2\sqrt{v_{2,t}}dw_{2,t} \ ,
\end{aligned}
\tag{E.3}
$$

with the correlation structure:

$$
\begin{aligned}
& \langle dw_{1,t}, dz_{1,t}\rangle = \rho_1 dt, \ -1 \leq \rho_1 \leq +1 \\
& \langle dw_{2,t}, dz_{2,t}\rangle = \rho_2 dt, \ -1 \leq \rho_2 \leq +1 \\
& \langle dw_{1,t}, dw_{2,t}\rangle = 0 \\
& \rho_1^2 + \rho_2^2 \leq +1 \\
& \rho_1^2 + \rho_2^2 \leq +1 \ .
\end{aligned}
\tag{E.4}
$$

We decompose the variance shocks into orthogonal components as in (E.5) and then discretize the return dynamics (E.3) using the Euler scheme and shock's decomposition (E.5).[43]

[42] According to Eraker [2004] and Li et al. [2008] the discretization bias of the Euler scheme is negligible for daily data.

[43] Note that the quadratic variations of the transformed using the proposed shocks decomposition (E.5) should remain the same as $\sqrt{dt}$.

$$
\begin{aligned}
dw_{1,t} &= \rho_1 dz_t + \sqrt{1-\rho_1^2}\, dB_{1,t} \\
dw_{2,t} &= \rho_2 dz_t - \frac{\rho_1\rho_2}{\sqrt{1-\rho_1^2}}\, dB_{1,t} + \sqrt{\frac{1-\rho_1^2-\rho_2^2}{1-\rho_1^2}}\, dB_{2,t} \\
\langle dB_{1,t}\,, dB_{2,t}\rangle &= 0
\end{aligned} \tag{E.5}
$$

$$
\begin{aligned}
\ln(S_{t+\Delta t}) - \ln(S_t) &= (\mu - \frac{1}{2}(v_{1,t}+v_{2,t}))\Delta t + \sqrt{(v_{1,t}+v_{2,t})\Delta t}\; z_{t+\Delta t}\,, \\
v_{1,t+\Delta t} &= v_{1,t} + \kappa_1(\theta_1 - v_{1,t})\Delta t + \sigma_1\sqrt{v_{1,t}\Delta t}\; w_{1,t+\Delta t}\,, \\
v_{2,t+\Delta t} &= v_{1,t} + \kappa_2(\theta_2 - v_{2,t})\Delta t + \sigma_2\sqrt{v_{2,t}\Delta t}\; w_{2,t+\Delta t}\,,
\end{aligned} \tag{E.6}
$$

where, $z_{t+\Delta t}$, $w_{1,t+\Delta t}$, and $w_{2,t+\Delta t}$ are all $N(0,1)$. Now, using daily index log-returns, we proceed to filter the spot variances from the discretized model in (E.6) given the correlation structure in (E.5).

We follow Pitt [2002][44] and adopt a particular implementation of the PF, which is referred to as the sampling-importance-resampling (SIR) PF. This implementation of PF method allow us to approximate the true density of the persistent variance component ($v_{1,t}$) and the transient variance component ($v_{2,t}$) using two sets of particles that are updated recursively through equations (E.6). In other words, we recursively simulate next period particles of each variance component until we have the empirical distributions of each variance factor over the entire sample. That is, given N particles of $\{v^j_{1,t}\}^N_{j=1}$, N particles of $\{v^j_{2,t}\}^N_{j=1}$, simulated return shocks, and $w_{1,t+\Delta t}$ and $w_{2,t+\Delta t}$ we generate the next period particles, N particles $\{v^j_{1,t+\Delta t}\}^N_{j=1}$ and another N particles $\{v^j_{2,t+\Delta t}\}^N_{j=1}$ at any time $t+\Delta t$.

We start by simulating return's shocks $z^j_{t+\Delta t}$ given the initial value of structural parameters Θ_0 and current variance particles $\{v^j_{1,t}, v^j_{2,t}\}$, on every day t and for every particle $j = 1, 2, ..., N$, according to (E.7). Then using (E.8) we simulate volatility shocks $w^j_{1,t+\Delta t}$ and $w^j_{1,t+\Delta t}$. Note that $\epsilon^j_{1,t+\Delta t}$ and $\epsilon^j_{2,t+\Delta t}$ are independent standard normal random variables.

$$
z^j_{t+\Delta t} = \Big[\ln(S_{t+\Delta t}/S_t) - (\mu - \frac{1}{2}(v^j_{1,t}+v^j_{2,t}))\Delta t\Big] / \sqrt{(v^j_{1,t}+v^j_{2,t})\Delta t} \tag{E.7}
$$

$$
\begin{aligned}
w^j_{1,t+\Delta t} &= \rho_1 z^j_{t+\Delta t} + \sqrt{1-\rho_1^2}\;\epsilon^j_{1,t+\Delta t} \\
w^j_{2,t+\Delta t} &= \rho_2 z^j_{t+\Delta t} - \frac{\rho_1\rho_2}{\sqrt{1-\rho_1^2}}\;\epsilon^j_{1,t+\Delta t} + \sqrt{\frac{1-\rho_1^2-\rho_2^2}{1-\rho_1^2}}\,\epsilon^j_{2,t+\Delta t}
\end{aligned} \tag{E.8}
$$

[44] See Pitt [2002], Christoffersen et al. [2010], and Boloorforoosh [2014] for a detailed description of the PF algorithm.

Then, given the simulated return's shocks $\{z^j_{t+\Delta t}\}^N_{j=1}$ and simulated shocks to the persistent and transient variance components $\{w^j_{1,t+\Delta t}\}^N_{j=1}$ and $\{w^j_{2,t+\Delta t}\}^N_{j=1}$, we simulate next period variance particles $\{\tilde{v}^j_{1,t+\Delta t}\}$ and $\{\tilde{v}^j_{2,t+\Delta t}\}$, for every day t according to (E.9).

$$\begin{aligned} \tilde{v}^j_{1,t+\Delta t} &= v^j_{1,t} + \kappa_1(\theta_1 - v_{1,t})\Delta t + \sigma_1\sqrt{v_{1,t}\Delta t}\, w_{1,t+\Delta t} \\ \tilde{v}^j_{2,t+\Delta t} &= v^j_{2,t} + \kappa_2(\theta_2 - v_{2,t})\Delta t + \sigma_2\sqrt{v_{2,t}\Delta t}\, w_{2,t+\Delta t} \end{aligned} \tag{E.9}$$

This is the "Sampling Step," at the end of which we generate N possible daily values for the persistent variance component $v_{1,t+\Delta t}$ and another N possible daily values for the transient variance component $v_{2,t+\Delta t}$ over the entire sample. In the next step, "Importance Step," we evaluate importance of the sampled daily particles by assigning appropriate weights $\tilde{W}^j_{t+\Delta t}$ to the simulated daily particles using a multivariate normal distribution. Intuitively, these weights, $\tilde{W}^j_{t+\Delta t}$, are likelihood that the next day return at $t + 2\Delta t$ is generated by this set of particles. Then, the probability of each daily particle can be defined by normalizing the weights within each day according to (E.12). Note that these weights are the basis of our likelihood function under the P distribution.

$$(r_{t+2\Delta t}|\{\tilde{v}_{1,t+\Delta t}\ ,\tilde{v}_{2,t+\Delta t}\}) \sim N\big[(\mu - \frac{1}{2}(\tilde{v}_{1,t+\Delta t} + \tilde{v}_{2,t+\Delta t}))\Delta t\ ,(\tilde{v}_{1,t+\Delta t} + \tilde{v}_{2,t+\Delta t})\Delta t\big] \tag{E.10}$$

$$\tilde{W}^j_{t+\Delta t} = \frac{1}{\sqrt{2\pi(\tilde{v}^j_{1,t+\Delta t} + \tilde{v}^j_{2,t+\Delta t})\Delta t}} \cdot \exp\Big(-\frac{1}{2}\frac{\big(\ln(\frac{S_{t+2\Delta t}}{S_{t+\Delta t}}) - (\mu - \frac{1}{2}(\tilde{v}^j_{1,t+\Delta t} + \tilde{v}^j_{2,t+\Delta t}))\Delta t\big)^2}{(\tilde{v}^j_{1,t+\Delta t} + \tilde{v}^j_{2,t+\Delta t})\Delta t}\Big) \tag{E.11}$$

$$\breve{W}^j_{t+\Delta t} = \frac{\tilde{W}^j_{t+\Delta t}}{\sum_{j=1}^N \tilde{W}^j_{t+\Delta t}} \tag{E.12}$$

Note that combining independent shocks $z_{1,t}$ and $z_{2,t}$ in (E.3) imposes a restriction on the weights of daily variance particles. Therefore, the importance probability is assigned to the summation of return's shocks. However, estimation results show that the path of filtered spot persistent variance component and transient variance component in our two-factor SV model are not sensitive to this assumption. We investigate the sensitivity of our result to this weighting assumption by estimating daily spot variances using the two-step iterative approach, following Huang and Wu [2004]. We do not observe significant difference between filtered spot variances in two-step iterative approach and those filtered with particle filter method.

In the last step, "Resampling Step," we find the empirical distribution of smoothly resampled daily particles. Following the Pitt [2002] algorithm, we draw smoothed daily particles by assigning uniform distributions to the raw daily particles for persistent and transient variance components. As in the sampling step, we start from the beginning of the sample period and recursively simulate the next period daily particles using the smoothly resampled daily particles. The procedure continues until we have the empirical distributions of the persistent and transient variance components over the entire sample.

Given the appropriate weights (E.12), we define the return-based likelihood function as follows.

$$LLR \propto \sum_{t=1}^{T} \ln \Big(\frac{1}{N} \sum_{j=1}^{N} \breve{W}_t^j(\Theta) \Big) \tag{E.13}$$

Our implementation uses the maximum likelihood importance sampling (MLIS) methodology to maximize LLR criterion. Note that return-based likelihood function (E.13) is a function of the structural parameters of the market model under P measure, $\Theta \equiv \{\kappa_1, \kappa_2, \theta_1, \theta_2, \sigma_1, \sigma_1, \rho_1, \rho_2\}$. Note also that the filtered daily spot persistent variance component $v_{1,t}^P$ and transient variance component $v_{2,t}^P$ can be defined as the average of the smoothly resampled particles.

$$\hat{v}_{1,t}^P = \frac{1}{N} \sum_{j=1}^{N} v_{1,t}^j \ , \qquad \hat{v}_{2,t}^P = \frac{1}{N} \sum_{j=1}^{N} v_{2,t}^j \tag{E.14}$$

F Risk Neutral Distribution

Risk neutral distribution in (4) can also be extracted by assuming the following standard stochastic discount factor, without explicit assumptions about the investor's variance preferences.

$$\frac{dM_t}{M_t} = -rdt - \psi_t' dW_t \ , \tag{F.1}$$

where $\psi_t \equiv [\psi_{1,t}, \psi_{2,t}, \psi_{3,t}, \psi_{4,t}]$ is the vector of market price of risk factors and $W_t \equiv [z_{1,t}, z_{2,t}, w_{1,t}, w_{2,t}]$ is the vector of innovations in market index return and variance components. Given the SDF in (F.1), the change-of-measure from P to Q distribution has the following exponential form.

$$\frac{dQ}{dP}(t) \equiv M_t \exp(rt) = \exp \Big[- \int_0^t \psi_u' dW_u - \frac{1}{2} \int_0^t \psi_u' d\langle W, W' \rangle_u \psi_u \Big] \tag{F.2}$$

where $\langle W, W' \rangle$ is the covariance operator.

We follow the notion of Doléans-Dade exponential (stochastic exponential) and define the stochastic exponential $\varepsilon(\cdot)$ as follow.

$$\varepsilon\Big(\int_0^t \vartheta'_u dW_u\Big) \equiv \exp\Big[\int_0^t \vartheta'_u dW_u - \frac{1}{2}\int_0^t \vartheta'_u d\langle W, W'\rangle_u \vartheta_u\Big] \tag{F.3}$$

Therefore, the change-of-measure (F.2) can be expressed in term of stochastic exponential as

$$\frac{dQ}{dP}(t) = \varepsilon\Big(\int_0^t -\psi'_u dW_u\Big) \tag{F.4}$$

Applying Ito's lemma, we get the following dynamic for the log stock price process under physical measure.

$$\log\big(\frac{S_t}{S_0}\big) = (r + \mu_1 v_{1,t} + \mu_2 v_{2,t})t - \frac{1}{2}v_{1,t}t + \int_0^t \sqrt{v_{1,u}}dz_{1,u} - \frac{1}{2}v_{2,t} + \int_0^t \sqrt{v_{2,u}}dz_{2,u} \tag{F.5}$$

Given (F.5) and definition of stochastic exponential (F.3) we have

$$\frac{S_t}{S_0} = \exp\big[(r + \mu_1 v_{1,t} + \mu_2 v_{2,t})t\big]\varepsilon\Big(\int_0^t \sqrt{v_{1,u}}dz_{1,u}\Big)\varepsilon\Big(\int_0^t \sqrt{v_{2,u}}dz_{2,u}\Big) \tag{F.6}$$

To find the market prices of risk we impose the restriction that the product of the price of any traded asset and the pricing kernel under physical measure is a P-martingale. Given the change-of-measure (F.2), the following process, $N(t)$, should be a P-martingale.

$$N(t) \equiv \frac{S_t}{S_0}\frac{dQ}{dP}(t)\exp(-rt) \tag{F.7}$$

where

$$\begin{aligned} N(t) = &\exp\big[(\mu_1 v_{1,t} + \mu_2 v_{2,t})t\big] \\ &\varepsilon\Big(\int_0^t \sqrt{v_{1,u}}dz_{1,u}\Big)\varepsilon\Big(-\int_0^t \psi_{1,u}dz_{1,u} - \int_0^t \psi_{3,u}dw_{1,u}\Big) \\ &\varepsilon\Big(\int_0^t \sqrt{v_{2,u}}dz_{2,u}\Big)\varepsilon\Big(-\int_0^t \psi_{2,u}dz_{2,u} - \int_0^t \psi_{4,u}dw_{2,u}\Big) \end{aligned} \tag{F.8}$$

Using the properties of a stochastic exponential $\varepsilon(\cdot)$, $\varepsilon(X_t)\varepsilon(Y_t) = \varepsilon(X_t + Y_t)\exp(\langle X, Y\rangle_t)$ we can rewrite the process of $N(t)$ as follows.

$$
\begin{aligned}
N(t) = & \exp\left[(\mu_1 v_{1,t} + \mu_2 v_{2,t})t\right] \\
& \varepsilon\Big(\int_0^t \big(\sqrt{v_{1,u}} - \psi_{1,u}\big)dz_{1,u} - \int_0^t \psi_{3,u}dw_{1,u}\Big) \exp\Big[-\int_0^t \sqrt{v_{1,u}}(\psi_{1,u} + \rho_1\psi_{3,u})du\Big] \\
& \varepsilon\Big(\int_0^t \big(\sqrt{v_{2,u}} - \psi_{2,u}\big)dz_{2,u} - \int_0^t \psi_{4,u}dw_{2,u}\Big) \exp\Big[-\int_0^t \sqrt{v_{2,u}}(\psi_{2,u} + \rho_2\psi_{4,u})du\Big]
\end{aligned} \tag{F.9}
$$

From the definition of a stochastic exponential we know that $\varepsilon(\cdot)$ are P-martingales. Thus, the process $N(t)$ is a P-martingale when the following restriction holds.

$$
\exp\left[(\mu_1 v_{1,t} + \mu_2 v_{2,t})t\right] \exp\left[-\int_0^t \sqrt{v_{1,u}}(\psi_{1,u} + \rho_1\psi_{3,u})du\right] \exp\Big[-\int_0^t \sqrt{v_{2,u}}(\psi_{2,u} + \rho_2\psi_{4,u})du\Big] = 1 \tag{F.10}
$$

The restriction in (F.10) can be satisfied if

$$
\begin{aligned}
\mu_1 v_{1,t}t - \sqrt{v_{1,t}}(\psi_{1,t} + \rho_1\psi_{3,t})t = 0 \\
\mu_2 v_{2,t}t - \sqrt{v_{2,t}}(\psi_{3,t} + \rho_2\psi_{4,t})t = 0
\end{aligned} \tag{F.11}
$$

To fully specify the market prices of risk we assume that market price of variance risk factors are proportional to spot volatilites, following Heston [1993].

$$
\begin{aligned}
(\psi_{3,t} + \rho_1\psi_{1,t}) &= \frac{v_{1,t}}{\sigma_1\sqrt{v_{1,t}}}\lambda_1 \\
(\psi_{4,t} + \rho_2\psi_{2,t}) &= \frac{v_{2,t}}{\sigma_2\sqrt{v_{2,t}}}\lambda_2
\end{aligned} \tag{F.12}
$$

Combining the restrictions in (F.11) and (F.12), we have the following market price of risk factors. Note that these prices are the same as those we find in Proposition (1).

$$
\begin{aligned}
\psi_{1,t} &= \frac{\sigma_1\mu_1 - \rho_1\lambda_1}{(1-\rho_1^2)}\frac{\sqrt{v_{1,t}}}{\sigma_1} \\
\psi_{2,t} &= \frac{\sigma_2\mu_2 - \rho_2\lambda_2}{(1-\rho_2^2)}\frac{\sqrt{v_{2,t}}}{\sigma_2} \\
\psi_{3,t} &= \frac{\lambda_1 - \rho_1\sigma_1\mu_1}{(1-\rho_1^2)}\frac{\sqrt{v_{1,t}}}{\sigma_1} \\
\psi_{4,t} &= \frac{\lambda_2 - \rho_2\sigma_2\mu_2}{(1-\rho_2^2)}\frac{\sqrt{v_{2,t}}}{\sigma_2}
\end{aligned} \tag{F.13}
$$

Given the market price of risk factors (F.13), we can apply Girsanov's theorem to find transform physical innovations in (1) to its risk neutral counterpart in (4).

$$
\begin{aligned}
d\tilde{z}_{1,t} &= dz_{1,t} + \psi_{1,t}dt + \rho_1\psi_{3,t}dt \\
d\tilde{z}_{2,t} &= dz_{2,t} + \psi_{2,t}dt + \rho_2\psi_{4,t}dt \\
d\tilde{w}_{1,t} &= dw_{1,t} + \psi_{3,t}dt + \rho_1\psi_{1,t}dt \\
d\tilde{w}_{2,t} &= dw_{2,t} + \psi_{4,t}dt + \rho_2\psi_{2,t}dt
\end{aligned} \tag{F.14}
$$

With some algebra we have the following transformations.

$$
\begin{aligned}
d\tilde{z}_{1,t} &= dz_{1,t} + \mu_1\sqrt{v_{1,t}}dt \\
d\tilde{z}_{2,t} &= dz_{2,t} + \mu_2\sqrt{v_{2,t}}dt \\
d\tilde{w}_{1,t} &= dw_{1,t} + (\lambda_1/\sigma_1)\sqrt{v_{1,t}}dt \\
d\tilde{w}_{2,t} &= dw_{2,t} + (\lambda_2/\sigma_2)\sqrt{v_{2,t}}dt
\end{aligned} \tag{F.15}
$$

Replacing $dz_{1,t}, dz_{2,t}, dw_{1,t}, dw_{2,t}$ from (F.15) into the physical dynamics in (1) and knowing that $\tilde{\kappa}_1 = \kappa_1 + \lambda_1$, $\tilde{\kappa}_2 = \kappa_2 + \lambda_2$, $\tilde{\theta}_1 = \frac{k_1\theta_1}{k_1+\lambda_1}$, $\tilde{\theta}_2 = \frac{k_2\theta_2}{k_2+\lambda_2}$ we obtain risk neutral return and variance dynamics.

$$
\begin{aligned}
dS_t/S_t &= (r + \mu_1 v_{1,t} + \mu_2 v_{2,t})dt + \sqrt{v_{1,t}}dz_{1,t} + \sqrt{v_{2,t}}dz_{2,t} \\
&= (r + \mu_1 v_{1,t} + \mu_2 v_{2,t})dt + \sqrt{v_{1,t}}(d\tilde{z}_{1,t} - \mu_1\sqrt{v_{1,t}}dt) + \sqrt{v_{2,t}}(d\tilde{z}_{2,t} - \mu_2\sqrt{v_{2,t}}dt) \\
&= rdt + \sqrt{v_{1,t}}d\tilde{z}_{1,t} + \sqrt{v_{2,t}}d\tilde{z}_{2,t}
\end{aligned} \tag{F.16}
$$

$$
\begin{aligned}
dv_{1,t} &= \kappa_1(\theta_1 - v_{1,t})dt + \sigma_1\sqrt{v_{1,t}}(d\tilde{w}_{1,t} - (\lambda_1/\sigma_1)\sqrt{v_{1,t}}dt) \\
&= (\kappa_1\theta_1 - (\kappa_1 + \lambda_1)v_{1,t})dt + \sigma_1\sqrt{v_{1,t}}d\tilde{w}_{1,t} \\
&= \tilde{\kappa}_1(\tilde{\theta}_1 - v_{1,t})dt + \sigma_1\sqrt{v_{1,t}}d\tilde{w}_{1,t}
\end{aligned} \tag{F.17}
$$

$$
\begin{aligned}
dv_{2,t} &= \kappa_2(\theta_2 - v_{2,t})dt + \sigma_2\sqrt{v_{2,t}}(d\tilde{w}_{2,t} - (\lambda_2/\sigma_2)\sqrt{v_{2,t}}dt) \\
&= (\kappa_2\theta_2 - (\kappa_2 + \lambda_2)v_{2,t})dt + \sigma_2\sqrt{v_{2,t}}d\tilde{w}_{2,t} \\
&= \tilde{\kappa}_2(\tilde{\theta}_2 - v_{2,t})dt + \sigma_2\sqrt{v_{2,t}}d\tilde{w}_{2,t}
\end{aligned} \tag{F.18}
$$

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Table 1: S&P 500 Index Call Option Data Characteristics by Moneyness and Maturity

Panel A: Number of call option contracts

	DTM≤30	30<DTM≤91	91<DTM≤182	DTM>182	All
S/K≤0.92	152	3,371	12,690	8,782	24,995
0.92<S/K≤0.94	642	8,220	17,345	8,342	34,549
0.94<S/K≤0.96	4,033	14,436	18,557	8,096	45,122
0.96<S/K≤0.98	10,761	17,202	17,000	7,167	52,130
S/K>0.98	13,052	16,137	15,628	6,485	51,302
All	28,640	59,366	81,220	38,872	208,098

Panel B: Average price of call option contracts

	DTM≤30	30<DTM≤91	91<DTM≤182	DTM>182	All
S/K≤0.92	13.6200	15.5478	23.0998	47.0797	24.8368
0.92<S/K≤0.94	11.7434	16.1440	26.2574	56.2993	27.6110
0.94<S/K≤0.96	9.9935	18.0151	34.2459	69.4400	32.9236
0.96<S/K≤0.98	11.5532	24.4015	44.6126	82.1867	40.6885
S/K>0.98	18.5235	35.5330	57.9296	95.6642	51.9126
All	13.0867	21.9283	37.2290	70.1340	35.5945

Panel C: Average implied volatility of call option contracts

	DTM≤30	30<DTM≤91	91<DTM≤182	DTM>182	All
S/K≤0.92	0.4071	0.2299	0.1894	0.1791	0.2514
0.92<S/K≤0.94	0.3163	0.2034	0.1760	0.1831	0.2197
0.94<S/K≤0.96	0.2213	0.1792	0.1770	0.1881	0.1914
0.96<S/K≤0.98	0.1784	0.1741	0.1833	0.1958	0.1829
S/K>0.98	0.1715	0.1829	0.1900	0.2028	0.1868
All	0.2589	0.1939	0.1831	0.1898	0.2064

Panel D: Average delta of call option contracts

	DTM≤30	30<DTM≤91	91<DTM≤182	DTM>182	All
S/K≤0.92	0.2316	0.2302	0.2724	0.3726	0.2767
0.92<S/K≤0.94	0.2329	0.2549	0.3121	0.4268	0.3067
0.94<S/K≤0.96	0.2381	0.2984	0.3832	0.4827	0.3506
0.96<S/K≤0.98	0.2996	0.3843	0.4608	0.5319	0.4191
S/K>0.98	0.4422	0.4976	0.5377	0.5771	0.5136
All	0.2889	0.3331	0.3932	0.4782	0.3733

Note to Table: This table reports the summary statistics of out-of-the-money S&P 500 call option contracts in our sample, from January 1, 1996 to December 31, 2011. The implied volatilities and the deltas are from the OptionMetrics volatility surface data set. S denotes the price of the S&P 500 index, K is the option strike price, and DTM is the number of calandar days to maturity.

Table 2: S&P 500 Index Put Option Data Characteristics by Moneyness and Maturity

Panel A: Number of put option contracts

	DTM≤30	30<DTM≤91	91<DTM≤182	DTM>182	All
S/K≤1.02	10,776	13,499	13,463	5,904	43,642
1.02<S/K≤1.04	7,163	10,951	12,018	5,008	35,140
1.04<S/K≤1.06	3,699	8,083	10,399	5,317	27,498
1.06<S/K≤1.08	1,248	5,334	8,105	3,908	18,595
S/K>1.08	385	3,173	5,591	3,588	12,737
All	23,271	41,040	49,576	23,725	137,612

Panel B: Average price of put option contracts

	DTM≤30	30<DTM≤91	91<DTM≤182	DTM>182	All
S/K≤1.02	18.7121	30.3521	44.9423	63.5550	39.3904
1.02<S/K≤1.04	13.9689	25.4113	40.1731	59.5418	34.7738
1.04<S/K≤1.06	12.7334	21.7862	34.1231	55.3294	30.9930
1.06<S/K≤1.08	14.0224	20.8254	30.5229	44.3883	27.4397
S/K>1.08	16.1005	20.9994	30.9259	43.7921	27.9545
All	15.1075	23.8749	36.1375	53.3213	32.1103

Panel C: Average implied volatility of put option contracts

	DTM≤30	30<DTM≤91	91<DTM≤182	DTM>182	All
S/K≤1.02	0.1929	0.1933	0.1992	0.2121	0.1994
1.02<S/K≤1.04	0.2194	0.2134	0.2158	0.2127	0.2153
1.04<S/K≤1.06	0.2646	0.2314	0.2233	0.2313	0.2376
1.06<S/K≤1.08	0.3342	0.2599	0.2367	0.2200	0.2627
S/K>1.08	0.4255	0.2904	0.2583	0.2343	0.3021
All	0.2873	0.2377	0.2266	0.2221	0.2434

Panel D: Average delta of put option contracts

	DTM≤30	30<DTM≤91	91<DTM≤182	DTM>182	All
S/K≤1.02	-0.3931	-0.3988	-0.3931	-0.3631	-0.3870
1.02<S/K≤1.04	-0.2860	-0.3221	-0.3403	-0.3334	-0.3204
1.04<S/K≤1.06	-0.2348	-0.2699	-0.2932	-0.3060	-0.2760
1.06<S/K≤1.08	-0.2194	-0.2395	-0.2579	-0.2612	-0.2445
S/K>1.08	-0.2175	-0.2209	-0.2431	-0.2547	-0.2341
All	-0.2702	-0.2902	-0.3055	-0.3037	-0.2924

Note to Table: This table reports the summary statistics of out-of-the-money S&P 500 put option contracts in our sample, from January 1, 1996 to December 31, 2011. The implied volatilities and delta are from the OptionMetrics volatility surface data set. S denotes the price of the S&P 500 index, K is the option strike price, and DTM is the number of calandar days to maturity.

Table 3: Data Sample Summary

Company	Ticker	Call	Put	All Options	Avg DTM	Avg IV
S&P 500 Index	**SPX**	208,098	137,612	345,710	141	22.49%
Alcoa	**AA**	134,112	106,732	240,844	130	35.16%
American Express	**AXP**	143,880	109,422	253,302	132	31.62%
Boeing	**BA**	149,949	116,967	266,916	131	30.52%
Caterpillar	**CAT**	145,951	113,189	259,140	130	32.04%
Cisco	**CSCO**	127,223	100,605	227,828	128	36.92%
Chevron	**CVX**	178,737	132,901	311,638	135	24.56%
Dupont	**DD**	162,592	122,417	285,009	135	27.43%
Disney	**DIS**	145,656	114,062	259,718	138	29.84%
General Electric	**GE**	151,825	112,771	264,596	141	27.74%
Home Depot	**HD**	145,260	113,691	258,951	134	30.92%
Hewlett-Packard	**HPQ**	127,524	101,302	228,826	131	35.36%
IBM	**IBM**	164,543	125,043	289,586	135	27.09%
Intel	**INTC**	123,444	98,783	222,227	135	36.09%
Johnson & Johnson	**JNJ**	189,496	137,546	327,042	140	21.83%
JP Morgan	**JPM**	149,895	110,342	260,237	132	31.60%
Coca Cola	**KO**	178,611	131,747	310,358	141	23.03%
McDonald's	**MCD**	163,946	126,156	290,102	138	26.05%
3M	**MMM**	176,339	131,127	307,466	135	24.82%
Merck	**MRK**	160,622	120,662	281,284	134	27.68%
Microsoft	**MSFT**	138,523	106,266	244,789	140	30.69%
Pfizer	**PFE**	145,288	112,830	258,118	141	28.63%
Procter & Gamble	**PG**	186,969	137,111	324,080	139	22.12%
AT&T	**T**	174,932	123,359	298,291	135	25.85%
United Technologies	**UTX**	166,534	126,111	292,645	134	26.64%
Verizon	**VZ**	167,457	117,498	284,955	138	26.02%
Walmart	**WMT**	165,015	127,833	292,848	138	25.74%
Exxon Mobil	**XOM**	177,667	133,517	311,184	137	24.07%
Average		157,111	118,889	275,999	135	28.52%
Minimum		123,444	98,783	222,227	128	21.83%
Maximum		189,496	137,546	327,042	141	36.92%

Note to Table: his table reports the number of available call and put options for index and for each firm in our sample. Our sample contains options with moneyness up to 10% and maturity up to and including 1 year over the period 1996-2011. We rely on the implied volatility surface data set provided by OptionMetrics. For each firm, we also report the average number of days-to-maturity (Avg DTM) and the average Black-Scholes implied volatility (Avg IV) of available contracts.

Table 4: Data Sample Summary - Call Options

Ticker	Avg Price	Min Price	Max Price	Avg IV	Min. IV	Max IV	Avg Delta	Avg Vega	Avg DTM
SPX	35.59	1.876	195.53	20.64%	7.03%	74.98%	0.373	251.02	143
AA	2.256	0.110	14.121	34.20%	16.93%	153.65%	0.442	8.385	130
AXP	3.218	0.375	27.372	30.28%	12.72%	148.17%	0.436	12.612	133
BA	3.022	0.375	14.928	29.57%	16.06%	89.57%	0.429	13.062	131
CAT	3.351	0.376	15.375	30.98%	16.01%	103.28%	0.432	13.882	131
CSCO	2.364	0.093	32.268	35.87%	15.93%	107.08%	0.441	7.251	129
CVX	3.196	0.375	15.509	23.45%	12.79%	94.43%	0.416	16.718	137
DD	2.319	0.375	13.407	26.25%	12.29%	92.26%	0.427	10.961	136
DIS	1.899	0.375	17.498	28.56%	6.95%	95.86%	0.441	8.422	139
GE	2.385	0.375	27.865	26.38%	6.90%	148.93%	0.438	10.855	143
HD	2.215	0.375	15.933	29.72%	14.84%	100.91%	0.435	9.111	136
HPQ	2.869	0.375	46.162	34.47%	15.32%	97.89%	0.445	9.303	132
IBM	4.976	0.361	36.790	25.83%	11.93%	86.82%	0.416	23.901	136
INTC	2.946	0.375	28.764	35.20%	17.34%	90.86%	0.455	9.389	136
JNJ	2.391	0.375	14.911	20.44%	9.66%	70.84%	0.409	14.260	142
JPM	2.759	0.131	19.016	30.02%	11.19%	160.94%	0.431	11.158	133
KO	2.080	0.375	10.651	21.73%	8.27%	69.30%	0.416	11.767	143
MCD	2.008	0.375	13.560	24.80%	11.58%	78.87%	0.429	10.308	139
MMM	3.608	0.375	17.730	23.66%	12.51%	79.62%	0.413	18.890	136
MRK	2.797	0.375	23.758	26.56%	14.29%	85.20%	0.432	12.354	136
MSFT	3.143	0.375	29.554	29.44%	12.22%	87.86%	0.450	11.448	141
PFE	2.175	0.375	22.262	27.57%	14.20%	100.98%	0.441	8.982	143
PG	2.770	0.375	19.779	20.77%	9.28%	64.34%	0.409	16.262	142
T	1.611	0.075	9.373	24.41%	10.04%	82.25%	0.432	7.657	137
UTX	3.247	0.375	22.284	25.34%	13.16%	82.34%	0.417	16.273	135
VZ	2.078	0.375	12.448	24.58%	9.22%	86.98%	0.444	9.779	141
WMT	2.199	0.375	17.836	24.52%	11.16%	67.26%	0.418	11.103	140
XOM	2.688	0.375	15.079	22.92%	12.58%	84.79%	0.414	14.474	139
Avg.	2.688	0.334	20.527	27.32%	12.42%	96.71%	0.430	12.169	137

Note to Table: This table reports the number of available call option contracts for the index and for each firm in our sample. Our sample contains call options with moneyness up to 10% and maturity up to and including 1 year over the period 1996-2011. We rely on the implied volatility surface data set provided by OptionMetrics. For each firm, we also report the average number of days-to-maturity (Avg DTM), the average Black-Scholes implied volatility (Avg IV), the average Black-Scholes delta (Avg Delta), and the average Black-Scholes vega (Avg Vega) of available contracts.

Table 5: Data Sample Summary - Put Options

Ticker	Avg Price	Min Price	Max Price	Avg IV	Min IV	Max IV	Avg Delta	Avg Vega	Avg DTM
SPX	32.11	2.640	195.53	24.34%	8.90%	82.74%	-0.292	227.67	136
AA	1.908	0.110	14.121	36.13%	17.39%	159.25%	-0.342	7.840	129
AXP	2.821	0.375	27.372	32.95%	12.20%	149.37%	-0.340	11.851	130
BA	2.604	0.375	14.928	31.47%	17.43%	93.33%	-0.339	12.114	130
CAT	2.981	0.376	15.375	33.11%	17.86%	104.41%	-0.340	12.959	130
CSCO	2.120	0.093	32.268	37.97%	16.34%	112.08%	-0.351	6.862	128
CVX	2.754	0.375	15.509	25.67%	11.68%	98.59%	-0.327	15.499	134
DD	1.978	0.375	13.407	28.61%	13.70%	94.19%	-0.333	10.133	133
DIS	1.618	0.375	17.498	31.11%	14.31%	99.48%	-0.343	7.738	137
GE	2.018	0.375	27.865	29.09%	7.10%	149.59%	-0.337	10.048	140
HD	1.946	0.375	15.933	32.12%	14.03%	103.50%	-0.343	8.508	133
HPQ	2.368	0.375	46.162	36.25%	16.45%	94.06%	-0.350	8.721	129
IBM	4.535	0.361	36.790	28.35%	12.38%	90.96%	-0.336	22.422	134
INTC	2.596	0.375	28.764	36.97%	16.35%	92.03%	-0.353	9.103	134
JNJ	2.081	0.375	14.911	23.22%	9.61%	77.42%	-0.327	13.112	137
JPM	2.471	0.131	19.016	33.19%	11.99%	169.06%	-0.337	10.568	131
KO	1.827	0.375	10.651	24.34%	9.52%	67.51%	-0.330	10.878	139
MCD	1.727	0.375	13.560	27.30%	12.47%	74.29%	-0.336	9.455	136
MMM	3.175	0.375	17.730	25.99%	13.82%	86.39%	-0.329	17.609	134
MRK	2.316	0.375	23.758	28.80%	9.07%	88.64%	-0.334	11.504	132
MSFT	2.821	0.375	29.554	31.94%	11.20%	94.44%	-0.349	11.241	139
PFE	1.864	0.375	22.262	29.68%	13.95%	75.78%	-0.343	8.501	140
PG	2.435	0.375	19.779	23.47%	9.58%	74.12%	-0.327	15.103	137
T	1.400	0.075	9.373	27.30%	10.25%	86.45%	-0.334	7.206	134
UTX	2.904	0.375	22.284	27.94%	13.62%	87.87%	-0.333	15.167	133
VZ	1.728	0.375	12.448	27.45%	10.94%	89.81%	-0.330	9.118	135
WMT	1.979	0.375	17.836	26.97%	11.44%	72.69%	-0.335	10.324	136
XOM	2.309	0.375	15.079	25.22%	12.79%	97.18%	-0.329	13.299	136
Avg.	2.344	0.334	20.527	29.73%	12.87%	99.35%	-0.337	11.366	134

Note to Table: This table reports the number of available put option contracts for the index and for each firm in our sample. Our sample contains put options with moneyness up to 10% and maturity up to and including 1 year over the period 1996-2011. We rely on the implied volatility surface data set provided by OptionMetrics. For each firm, we also report the average number of days-to-maturity (Avg DTM), the average Black-Scholes implied volatility (Avg IV), the average Black-Scholes delta (Avg Delta), and the average Black-Scholes vega (Avg Vega) of available contracts.

Table 6: Market Parameter Estimates

Panel A: Parameter Estimates (Physical) - Joint Estimation

κ_1	κ_2	θ_1	θ_2	σ_1	σ_2	ρ_1	ρ_2	λ_1	λ_2
1.4271	3.5874	0.0026	0.0171	0.0855	0.3496	-0.6918	-0.2173	-1.0798	-1.0355

Panel B: Parameter Estimates (Risk Neutral) - Options-based Estimation

$\tilde{\kappa}_1$	$\tilde{\kappa}_2$	$\tilde{\theta}_1$	$\tilde{\theta}_2$	σ_1	σ_2	ρ_1	ρ_2
0.2267	2.9137	0.0590	0.0100	0.0958	0.5678	-0.9135	-0.4934

Note to Table: This table reports the structural parameter estimates of the S&P 500 Index for the two-factor stochastic volatility model. The reported results in Panel A are from the joint estimation using the daily S&P 500 index returns and options data. Structural parameters in Panel B are estimated using only options data. In both panels, we use OTM call and put options with moneyness up to 10% over the period 1996-2011. As in Proposition (2), $\tilde{\kappa}_1 = \kappa_1 + \lambda_1$, $\tilde{\kappa}_2 = \kappa_2 + \lambda_2$, $\tilde{\theta}_1 = \frac{k_1\theta_1}{k_1+\lambda_1}$, $\tilde{\theta}_2 = \frac{k_2\theta_2}{k_2+\lambda_2}$. Therefore, risk neutral parameters from the joint estimation are $\tilde{\kappa}_1 = 0.3473$, $\tilde{\kappa}_2 = 2.5520$, $\tilde{\theta}_1 = 0.0106$, $\tilde{\theta}_2 = 0.0240$.

Table 7: Individual Equity Parameter Estimates

Company	Ticker	$\tilde{\kappa}$	$\tilde{\theta}$	σ	ρ	β_1	β_2
Alcoa	**AA**	0.7253	0.0202	0.1612	-0.87	0.3850	1.3159
American Express	**AXP**	0.7663	0.0128	0.1009	-0.91	0.3430	1.3203
Boeing	**BA**	0.7692	0.0235	0.1757	-0.97	0.4108	1.3046
Caterpillar	**CAT**	0.6354	0.0291	0.1984	-0.84	0.3608	1.3215
Cisco	**CSCO**	0.6804	0.0653	0.3599	-0.81	0.4420	1.2508
Chevron	**CVX**	0.9390	0.0097	0.0913	-0.88	0.5816	1.1538
Dupont	**DD**	0.8702	0.0137	0.1310	-0.92	0.4949	1.2888
Disney	**DIS**	0.6995	0.0247	0.1841	-0.89	0.4462	1.2854
General Electric	**GE**	0.5694	0.0093	0.0670	-0.85	0.4968	1.3111
Home Depot	**HD**	0.6912	0.0340	0.2379	-0.83	0.4278	1.3097
Hewlett-Packard	**HPQ**	0.6159	0.0756	0.3967	-0.64	0.4432	1.2458
IBM	**IBM**	0.7717	0.0186	0.1676	-0.78	0.6798	1.2853
Intel	**INTC**	0.8160	0.0295	0.2123	-0.84	0.4322	1.2652
Johnson & Johnson	**JNJ**	0.6492	0.0238	0.2015	-0.95	0.5574	1.0197
JP Morgan	**JPM**	0.8606	0.0193	0.1836	-0.99	0.4483	1.3466
Coca Cola	**KO**	0.3920	0.0291	0.1895	-0.87	0.6077	1.0897
McDonald's	**MCD**	0.9305	0.0262	0.2109	-0.97	0.4754	1.1359
3M	**MMM**	1.7078	0.0107	0.1569	-0.86	0.5886	1.1752
Merck	**MRK**	1.2259	0.0105	0.1073	-0.89	0.5018	1.2276
Microsoft	**MSFT**	0.7777	0.0108	0.0710	-0.81	0.4513	1.2739
Pfizer	**PFE**	0.8957	0.0210	0.1724	-0.88	0.5067	1.2166
Procter & Gamble	**PG**	0.5107	0.0470	0.3056	-0.85	0.5782	1.0125
AT&T	**T**	0.6972	0.0098	0.0830	-0.93	0.5116	1.2126
United Technologies	**UTX**	0.9778	0.0271	0.2606	-0.83	0.5221	1.2668
Verizon	**VZ**	0.8423	0.0102	0.0970	0.51	0.4719	1.1838
Walmart	**WMT**	0.6533	0.0314	0.2136	-0.86	0.4695	1.1724
Exxon Mobil	**XOM**	1.0785	0.0148	0.1849	-0.94	0.5925	1.1764
Average		0.8055	0.0244	0.1823	-0.820	0.4899	1.2284
Min		0.3920	0.0093	0.0670	-0.990	0.3430	1.0125
Max		1.7078	0.0756	0.3967	0.512	0.6798	1.3466

Note to Table: This table reports the risk-neutral structural parameter estimates for individual equities conditional on the structural parameters of the S&P 500 index and the vectors of filtered spot market variance components. This table also reports the persistent beta β_1^i and the transient beta β_2^i for individual equity i. The market parameters and spot variance components are estimated using OTM call and put options over the period 1996-2011 with moneyness up to 10%. For individual equities, we use OTM call and put options with moneyness up to 10% over the period 1996-2011, where we drop the first five months.

Table 8: Distributional Properties of Spot Idiosyncratic Volatility

Company	Ticker	Mean	Std dev	Max	Median
Alcoa	**AA**	0.1259	0.1387	0.6879	0.0900
American Express	**AXP**	0.1068	0.1489	0.7138	0.0692
Boeing	**BA**	0.0633	0.0442	0.2484	0.0521
Caterpillar	**CAT**	0.0783	0.0628	0.4395	0.0587
Cisco	**CSCO**	0.1497	0.1328	0.8274	0.0987
Chevron	**CVX**	0.0293	0.0267	0.2126	0.0260
Dupont	**DD**	0.0460	0.0476	0.2526	0.0292
Disney	**DIS**	0.0636	0.0515	0.2661	0.0460
General Electric	**GE**	0.0618	0.0938	0.6134	0.0413
Home Depot	**HD**	0.0741	0.0600	0.3230	0.0510
Hewlett-Packard	**HPQ**	0.1250	0.1231	0.4893	0.0903
IBM	**IBM**	0.0439	0.0482	0.2620	0.0260
Intel	**INTC**	0.1206	0.0882	0.6408	0.0927
Johnson & Johnson	**JNJ**	0.0225	0.0257	0.2340	0.0116
JP Morgan	**JPM**	0.1070	0.1325	0.9138	0.0786
Coca Cola	**KO**	0.0268	0.0308	0.1729	0.0133
McDonald's	**MCD**	0.0389	0.0345	0.1638	0.0277
3M	**MMM**	0.0297	0.0304	0.1645	0.0180
Merck	**MRK**	0.0438	0.0367	0.2189	0.0358
Microsoft	**MSFT**	0.0749	0.0614	0.4605	0.0647
Pfizer	**PFE**	0.0490	0.0425	0.2021	0.0356
Procter & Gamble	**PG**	0.0256	0.0326	0.2411	0.0103
AT&T	**T**	0.0522	0.0532	0.5365	0.0359
United Technologies	**UTX**	0.0399	0.0374	0.2126	0.0258
Verizon	**VZ**	0.0428	0.0438	0.3520	0.0280
Walmart	**WMT**	0.0436	0.0550	0.2870	0.0193
Exxon Mobil	**XOM**	0.0234	0.0210	0.1556	0.0204
Average		0.0633	0.0631	0.3812	0.0443
Minimum		0.0225	0.0210	0.1556	0.0103
Maximum		0.1497	0.1489	0.9138	0.0987

Note to Table: This table reports the mean, median, standard deviation, and maximum of spot idiosyncratic variance for every firm i conditional on the structural parameters of the S&P 500 index and filtered spot market variance components. The reported results are based on OTM call and put index option and individual equity option contracts with moneyness up to 10% over the period 1996-2011.

Table 9: Goodness of Fit

	Option Based Estimation				Joint Estimation		
	Number of Obs.	**Vega RMSE**	**IV RMSE**	**IVRMSE/ Avg. IV**	**Vega RMSE**	**IV RMSE**	**IVRMSE/ Avg. IV**
	Panel A: Goodness of Fit - Call Option Contracts						
DTM≤30	28,640	1.2956			2.7171		
30<DTM≤91	59,366	0.8695			2.5104		
91<DTM≤182	81,220	0.6913			2.3505		
DTM>182	38,872	0.8943			2.6032		
All	208,098	0.8846	0.9132	4.4244	2.5299	2.5637	12.4210
	Panel B: Goodness of Fit - Put Option Contracts						
DTM≤30	23,271	1.6193			2.8857		
30<DTM≤91	41,040	1.0712			2.4509		
91<DTM≤182	49,576	0.8342			2.4941		
DTM>182	23,725	1.0440			2.5256		
All	137,612	1.1064	1.1167	4.5879	2.5877	2.6389	10.8418
	Panel C: Goodness of Fit - All Option Contracts						
DTM≤30	51,911	1.4497			2.7946		
30<DTM≤91	100,406	0.9571			2.4835		
91<DTM≤182	130,796	0.7486			2.4180		
DTM>182	62,597	0.9538			2.5665		
All	345,710	0.9790	0.9992	4.4428	2.5566	2.5939	11.5335

Note to Table: This table reports goodness-of-fit statistics for individual equity options. In-sample statistics are computed using options over the entire sample, 1996-2011. All numbers are in percentage points. We compute the Vega-weighted root mean squared error (Vega RMSE) along with the implied volatility root mean squared error (IVRMSE). We also report the ratio of IVRMSE over the average Black-Scholes implied volatility. We also report out-of-sample Vega RMSE over the period 2004-2011, given the in-sample parameter estimates, market spot variance components, and spot idiosyncratic variance over the period 1996-2003.

Table 10: Subsample Parameter Estimates

κ_1	κ_2	θ_1	θ_2	σ_1	σ_2	ρ_1	ρ_2	λ_1	λ_2
Panel A: Joint Estimation:1996 - 2003									
1.2138	3.2780	0.0033	0.0195	0.0855	0.3220	-0.6514	-0.2985	-1.1008	-0.9755
Panel B: Joint Estimation (2003 - 2011)									
1.1274	4.2337	0.0069	0.0289	0.0793	0.4675	-0.5102	-0.3086	-1.0684	-1.0351
Panel C: Options-based Estimation (1996-2003)									
0.1794	2.6176	0.0437	0.0104	0.0912	0.3732	-0.8891	-0.4434		
Panel D: Options-based Estimation (2003-2011)									
0.1117	3.4731	0.0623	0.0247	0.0837	0.6692	-0.7550	-0.6497		

Note to Table: This table reports the structural parameter estimates of the S&P 500 Index for the two-factor stochastic volatility model over two subsample period. The first subsample is from January 1996 to December 2003 and the second one is from January 2004 to December 2011. The point estimates in Panel A and Panel B are from the joint estimation using the daily S&P 500 index returns and options data. Entries in Panel C and Panel D are estimated using only options data. In both panels, we use OTM call and put options with moneyness up to 10% over the period 1996-2011.

Table 11: Subsample Goodness of Fit (1996-2003)

	Option Based Estimation				**Joint Estimation**		
	Number of Obs.	**Vega RMSE**	**IV RMSE**	**IVRMSE/ Avg. IV**	**Vega RMSE**	**IV RMSE**	**IVRMSE/ Avg. IV**
Panel A: Subsample Goodness of Fit (1996-2003) - Call Option Contracts							
DTM≤30	14,267	1.2355			2.9061		
30<DTM≤91	30,414	0.8397			2.8784		
91<DTM≤182	39,160	0.7194			2.7826		
DTM>182	18,237	0.7593			3.0274		
All	102,078	0.8514	0.8846	4.5041	2.8787	2.9137	12.8697
Panel B: Subsample Goodness of Fit (1996-2003) - Put Option Contracts							
DTM≤30	11,775	1.5167			3.3108		
30<DTM≤91	20,282	1.1038			2.9729		
91<DTM≤182	24,137	0.8742			2.9596		
DTM>182	11,528	1.0111			2.9025		
All	67,722	1.1006	1.1067	4.7416	3.0462	3.1389	11.9169
Panel C: Subsample Goodness of Fit (1996-2003) - All Option Contracts							
DTM≤30	26,042	1.3698			3.1091		
30<DTM≤91	50,696	0.9542			2.9218		
91<DTM≤182	63,297	0.7820			2.8691		
DTM>182	29,765	0.8655			2.9682		
All	169,800	0.9586	0.9792	4.5567	2.9592	3.0055	12.2725

Note to Table: This table reports in-sample goodness-of-fit statistics for our two-factor stochastic volatility model over the entire sample, 1996 through 2011 for various maturities. We also report in-sample fit for calls and puts separately. All numbers are in percentage points. We compute the Vega-weighted root mean squared error (Vega RMSE) along with the implied volatility root mean squared error (IVRMSE). We also report the ration of IVRMSE over the average implied volatility. To provide a basis for caparison the left panel reports pricing errors based on the option data and the right panel reports those of joint estimation.

Table 12: Subsample Goodness of Fit (2004-2011)

	Option Based Estimation				Joint Estimation		
	Number of Obs.	**Vega RMSE**	**IV RMSE**	**IVRMSE/ Avg. IV**	**Vega RMSE**	**IV RMSE**	**IVRMSE/ Avg. IV**
Panel A: Subsample Goodness of Fit (2004-2011) - Call Option Contracts							
DTM≤30	14,373	1.3526			2.5715		
30<DTM≤91	28,952	0.8998			2.1570		
91<DTM≤182	42,060	0.6640			1.9298		
DTM>182	20,635	0.9985			2.0532		
All	106,020	0.9155	0.9471	4.1833	2.2014	2.3017	10.1665
Panel B: Subsample Goodness of Fit (2004-2011) - Put Option Contracts							
DTM≤30	11,496	1.7181			2.4266		
30<DTM≤91	20,758	1.0383			1.9112		
91<DTM≤182	25,439	0.7944			1.9656		
DTM>182	12,197	1.0741			2.0348		
All	69,890	1.1121	1.1437	4.3421	2.0802	2.1294	8.0843
Panel C: Subsample Goodness of Fit (2004-2011) - All Option Contracts							
DTM≤30	25,869	1.5259			2.5109		
30<DTM≤91	49,710	0.9601			2.0487		
91<DTM≤182	67,499	0.7159			1.9459		
DTM>182	32,832	1.0273			2.0445		
All	175,910	0.9982	1.0297	4.2046	2.1480	2.2348	9.1255

Note to Table: This table reports goodness-of-fit statistics for our two-factor stochastic volatility model over the subsample from January 2004 through December 2011 for various maturities. We also report in-sample fit for calls and puts separately. All numbers are in percentage points. We compute Vega-weighted root mean squared error (Vega RMSE) along with implied volatility root mean squared error (IVRMSE). We also report the ration of IVRMSE over the average implied volatility. To provide a basis for caparison the left panel reports pricing errors based on the option data and the right panel reports those of joint estimation.

Table 13: Out of Sample Goodness of Fit (2004-2011)

	Option Based Estimation				Joint Estimation		
	Number of Obs.	**Vega RMSE**	**IV RMSE**	**IVRMSE/ Avg. IV**	**Vega RMSE**	**IV RMSE**	**IVRMSE/ Avg. IV**
Panel A: Out of Sample Goodness of Fit (2004-2011) - Call Option Contracts							
DTM≤30	14,373	1.4764			2.7853		
30<DTM≤91	28,952	0.9372			2.2801		
91<DTM≤182	42,060	0.6902			1.9978		
DTM>182	20,635	1.0797			2.1189		
All	106,020	0.9753	0.9985	4.4103	2.2201	2.3907	10.5596
Panel B: Out of Sample Goodness of Fit (2004-2011) - Put Option Contracts							
DTM≤30	11,496	1.8064			2.5780		
30<DTM≤91	20,758	1.1048			1.9984		
91<DTM≤182	25,439	0.8359			1.9856		
DTM>182	12,197	1.1153			2.1478		
All	69,890	1.1708	1.2142	4.6097	2.1259	2.2087	8.3853
Panel C: Out of Sample Goodness of Fit (2004-2011) - All Option Contracts							
DTM≤30	25,869	1.6313			2.6952		
30<DTM≤91	49,710	1.0105			2.1670		
91<DTM≤182	67,499	0.7485			1.9932		
DTM>182	32,832	1.0931			2.1297		
All	175,910	1.0573	1.0893	4.4480	2.1831	2.3201	9.4737

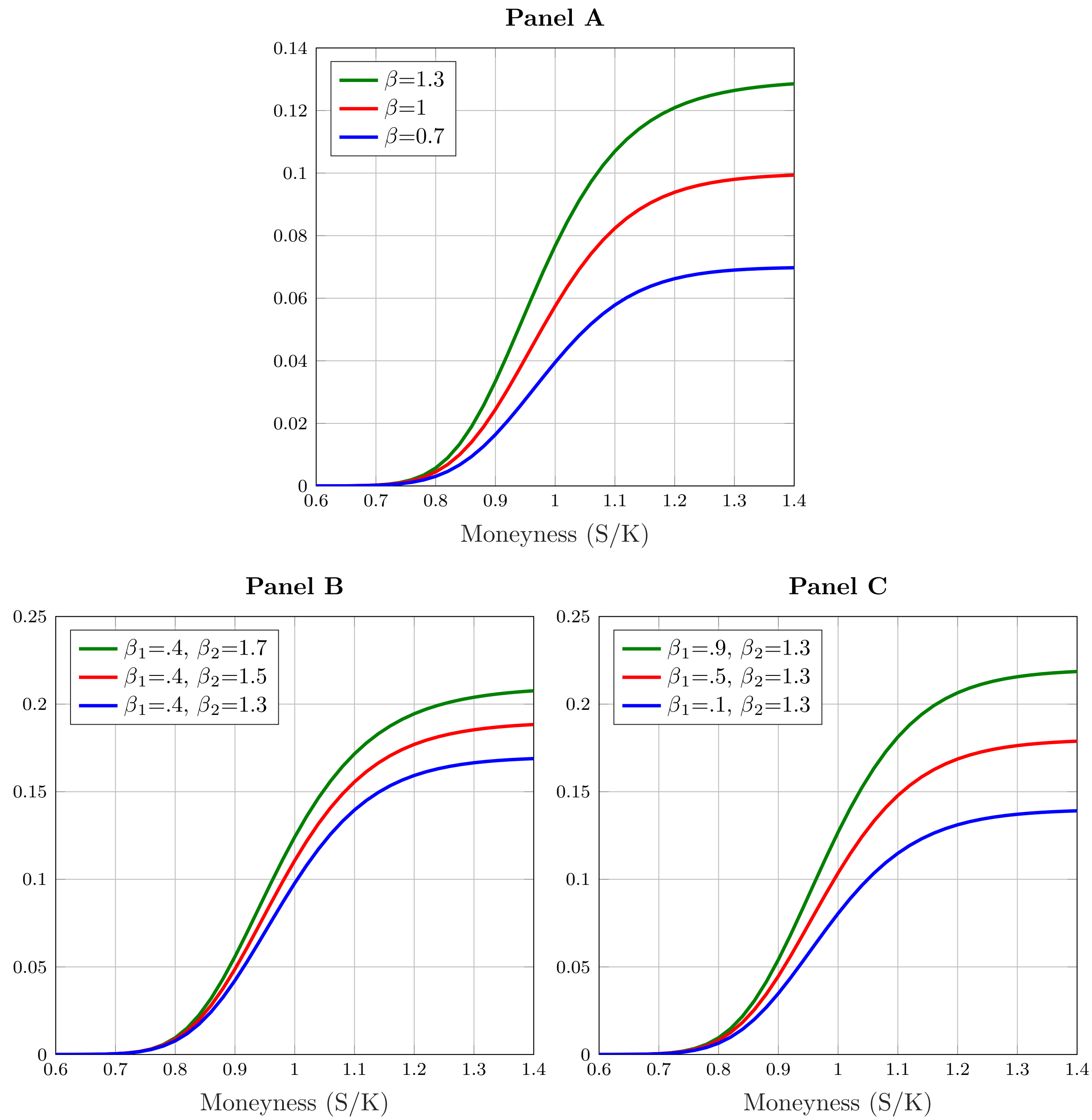
Note to Table: This table reports out-of-sample goodness-of-fit statistics for our two-factor stochastic volatility model over the period from January 2004 through December 2011 for various maturities. We also report out-of-sample fit for calls and puts separately. All numbers are in percentage points. Out-of-sample daily spot persistent and transient variance components are filtered with Particle Filter method given the in-sample structural parameter estimates over the period January 1996 through December 2003. The Vega RMSE along with the IVRMSE are computed given in-sample structural parameters and filtered variance components. We also report the ratio of IVRMSE over the average implied volatility. To provide a basis for capari-son the left panel reports pricing errors based on the option data and the right panel reports those of joint estimation.

Table 14: Goodness of Fit - Individual Equities

	In-Sample			Out-of-Sample
Ticker	**Vega RMSE**	**IV RMSE**	**IVRMSE/ Avg. IV**	**Vega RMSE**
AA	1.84	1.87	5.32	2.24
AXP	1.82	1.79	5.66	2.14
BA	1.41	1.35	4.42	1.97
CAT	1.50	1.47	4.59	1.68
CSCO	2.14	2.12	5.74	2.23
CVX	2.02	1.95	7.94	2.24
DD	1.42	1.41	5.14	1.53
DIS	1.75	1.69	5.66	1.97
GE	1.84	1.86	6.71	1.93
HD	1.58	1.54	4.98	1.72
HPQ	1.53	1.53	4.33	1.87
IBM	1.46	1.42	5.24	1.61
INTC	1.56	1.58	4.38	1.68
JNJ	1.42	1.40	6.41	1.65
JPM	1.85	1.82	5.76	2.08
KO	1.54	1.46	6.34	1.62
MCD	1.34	1.33	5.11	1.59
MMM	1.41	1.39	5.60	1.74
MRK	1.36	1.41	5.09	1.46
MSFT	1.67	1.64	5.34	1.75
PFE	1.49	1.46	5.10	1.73
PG	1.39	1.37	6.19	1.39
T	1.98	1.96	7.58	2.21
UTX	1.48	1.44	5.41	1.54
VZ	1.56	1.55	5.96	1.59
WMT	1.57	1.55	6.02	1.76
XOM	1.66	1.63	6.77	1.82
Average	1.61	1.59	5.66	1.81

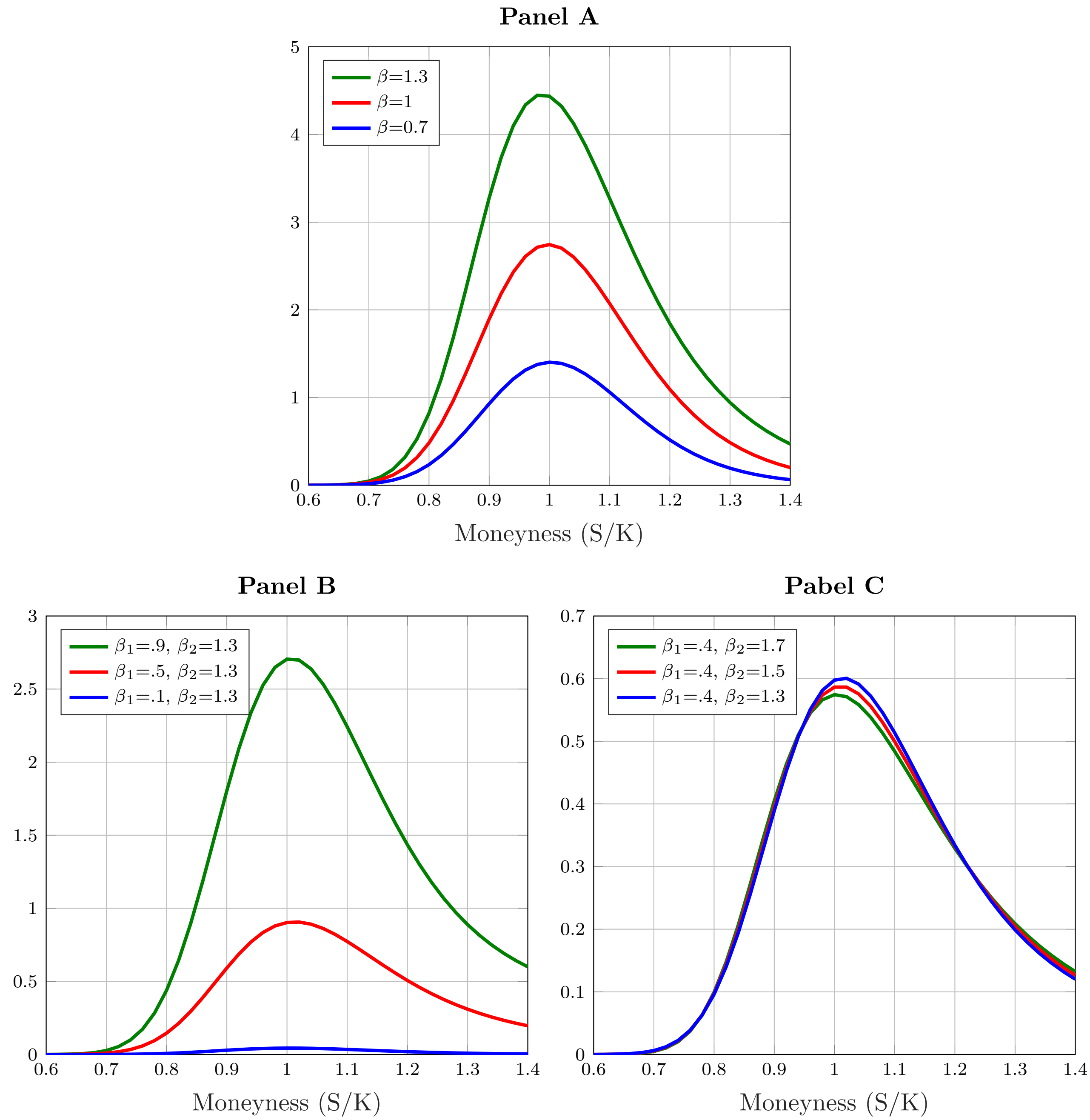
Note to Table: This table reports goodness-of-fit statistics for individual equity options. In-sample results are over the entire sample, 1996 through 2011. All numbers are in percentage points. We compute the Vega-weighted root mean squared error (Vega RMSE) along with the implied volatility root mean squared error (IVRMSE). We also report the ratio of IVRMSE over the average implied volatility. We also report out-of-sample Vega RMSE over the period of 2004 to 2011, given the in-sample parameter estimates, market variance components, and equity idiosyncratic variance over the period of 1996 to 2003.

Figure 1: Market Delta of Equity Call Options



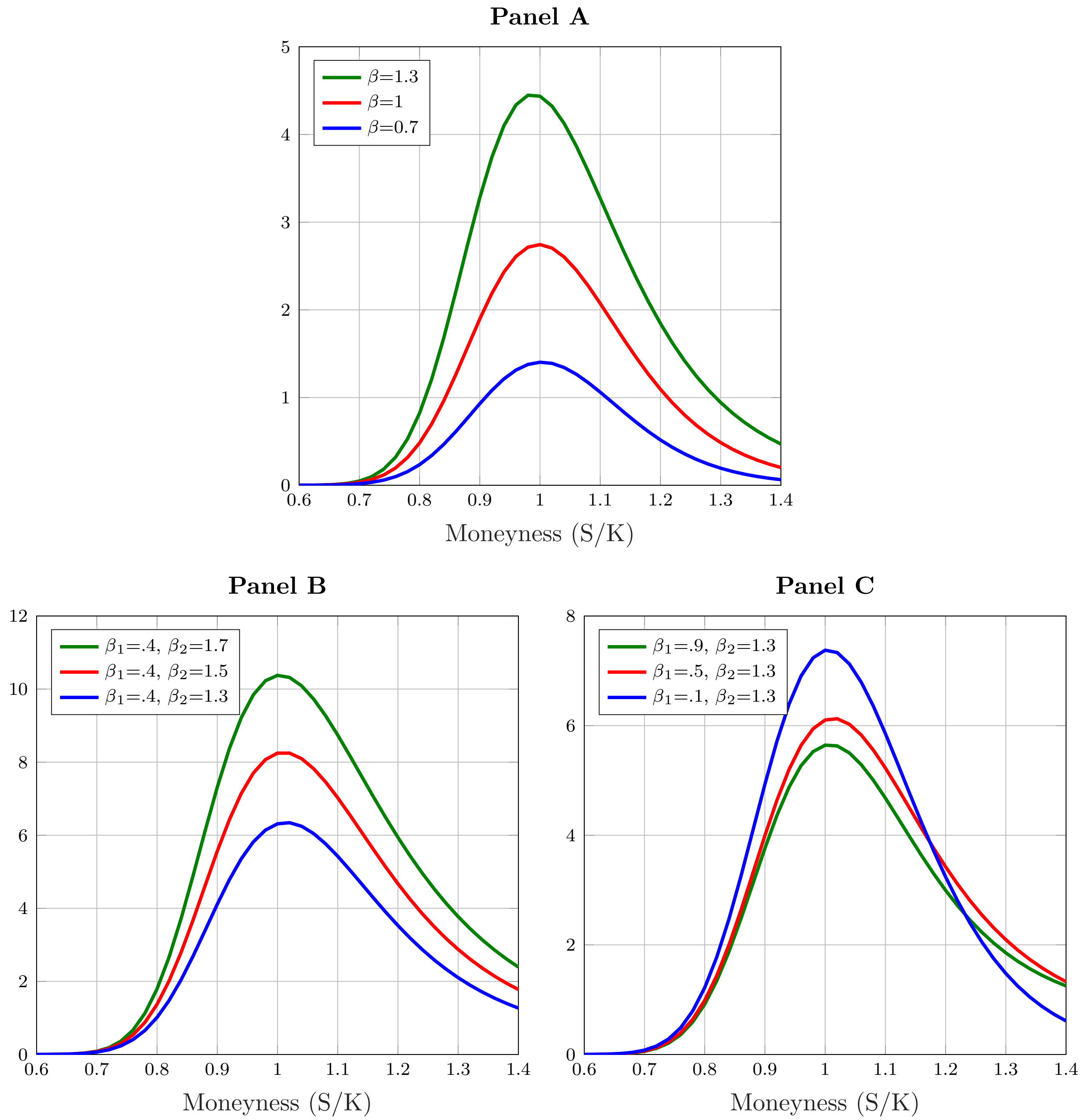
Note to Figure: This figure plots the sensitivity of the model-implied equity call option prices with respect to the level of market index for different sets of betas. Panel A shows this sensitivity following the calibration in in one-factor structure model of Christoffersen et al. [2015] while Panels B and C are the sensitivity in our two-factor structure model. Panel B, shows market delta when persistent beta is constant and Panel C is market delta when transient beta is constant.

Figure 2: Persistent Market Vega of Equity Call Options



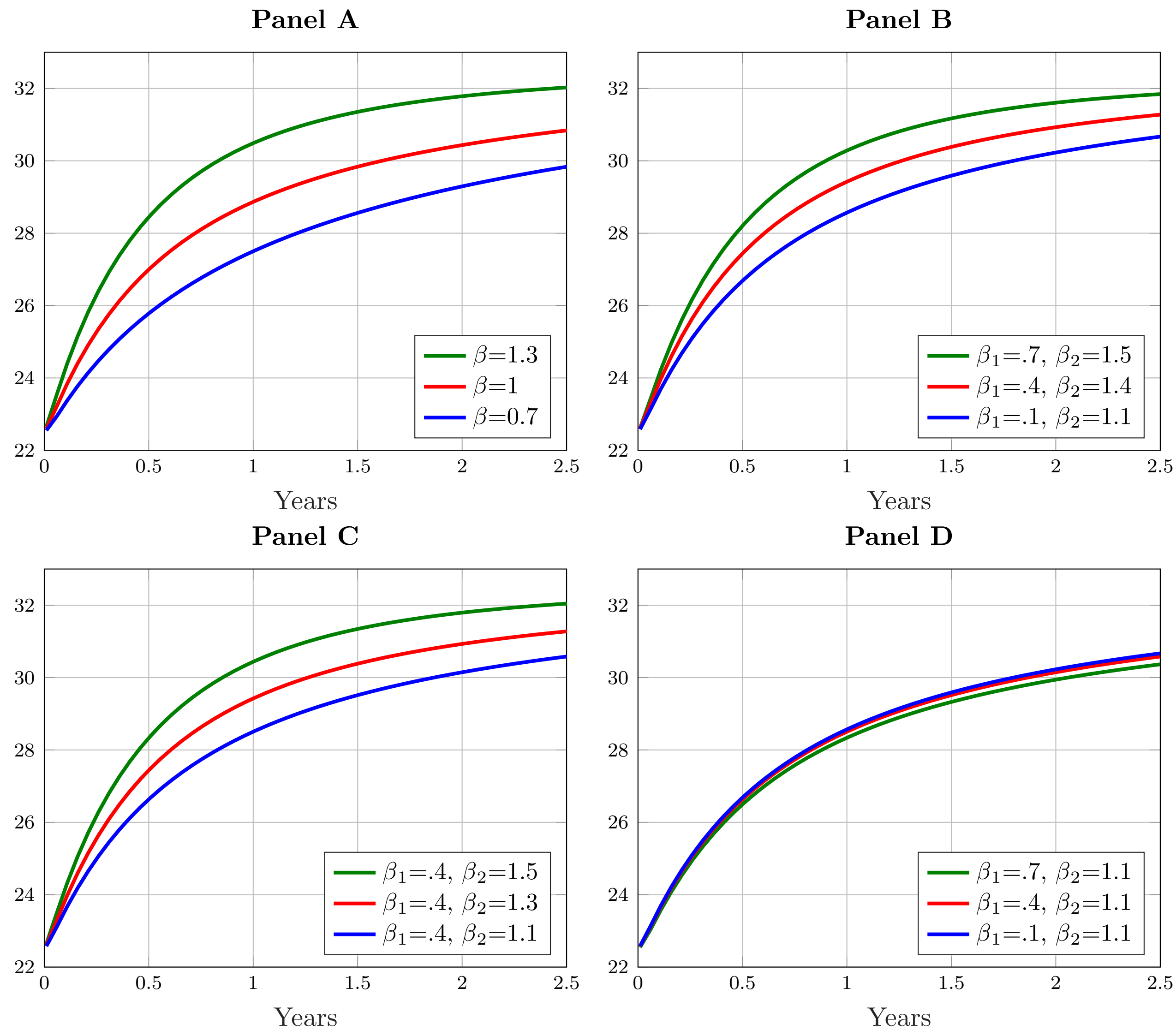
Note to Figure: This figure plots the sensitivity of the model-implied equity call option prices with respect to the persistent variance component for different sets of betas. Panel A shows this sensitivity following the calibration in in one-factor structure model while Panels B and C are the sensitivity in our two-factor structure model. Panel B, shows the persistent market vega when transient beta is constant and Panel C is the persistent market vega when persistent beta is constant. Note also that for all the graphs the total unconditional equity variance is fixed, $\tilde{v}^i = (\beta_1^i)^2\tilde{\theta}_1 + (\beta_2^i)^2\tilde{\theta}_2 + \theta^i = 0.11$.

Figure 3: Transient Market Vega of Equity Call Options



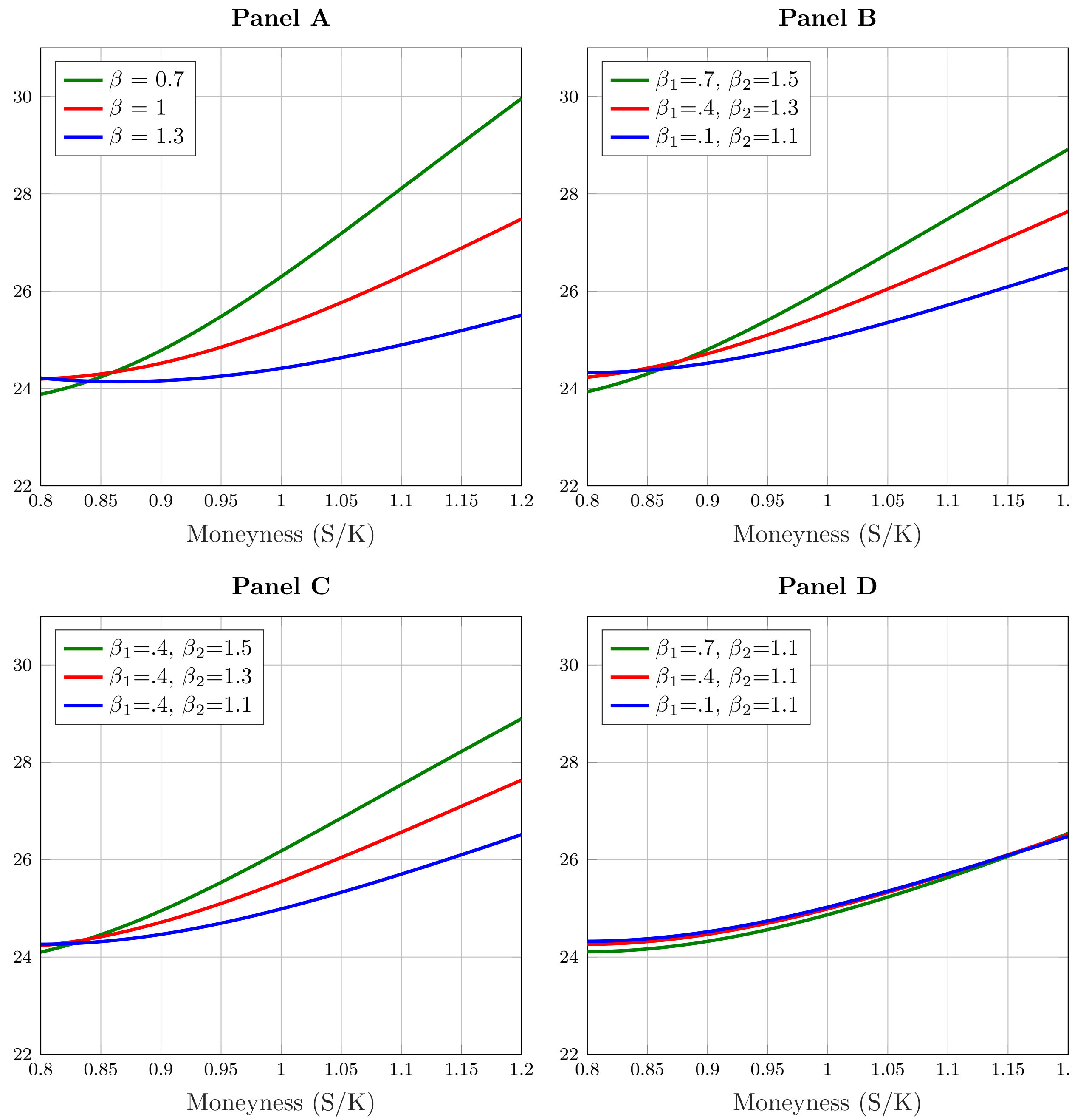
Note to Figure: This figure plots the sensitivity of the model-implied equity call option prices with respect to the transient variance component for different sets of betas. Panel A shows this sensitivity following the calibration in in one-factor structure model while Panels B and C are the sensitivity in our two-factor structure model. Panel B, shows the transient market vega when persistent beta is constant and Panel C is the transient market vega when transient beta is constant. Note also that for all the graphs the total unconditional equity variance is fixed, $\tilde{v}^i = (\beta_1^i)^2\tilde{\theta}_1 + (\beta_2^i)^2\tilde{\theta}_2 + \theta^i = 0.11$.

Figure 4: Persistent and Transient Betas and Implied Volatility Term Structure



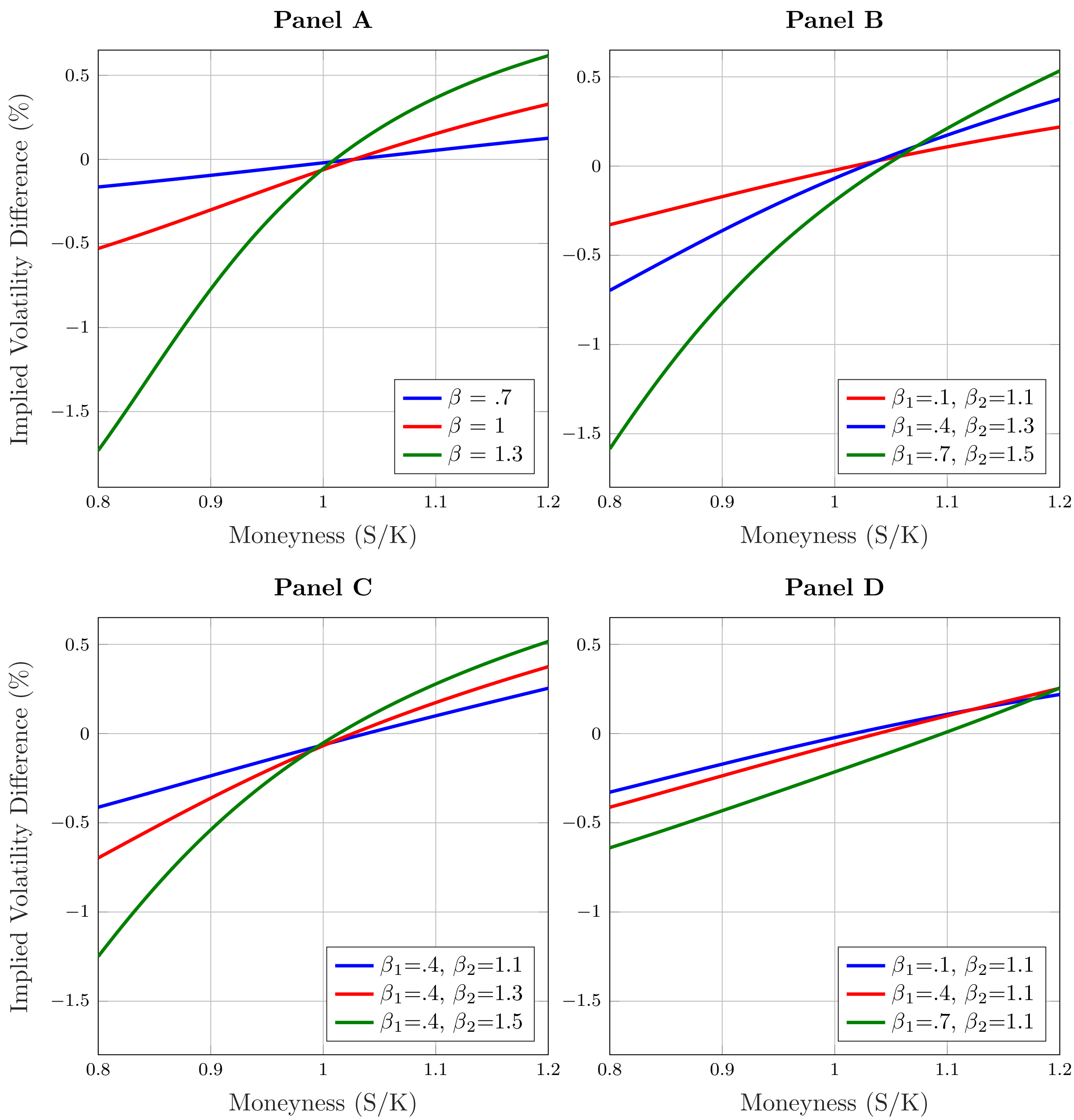
Note to Figure: This figure plots the model-implied volatility for at-the-money equity call options with respect to the time-to-maturity for different sets of betas. Panel A shows the term-structure effect following the one-factor structure model and Panel B replicates the same IV structure with our two-factor structure model. Panels C shows IV term structure when persistent beta β_1^i is constant and Panel D shows IV term structure when transient beta β_2^i is constant. Note that for all the graphs the total unconditional equity variance is fixed, $\tilde{v}^i = (\beta_1^i)^2\tilde{\theta}_1 + (\beta_2^i)^2\tilde{\theta}_2 + \theta^i = 0.11$. We also fix the total unconditional risk-neutral market variances to 0.05, with $\tilde{\theta}_1 = 0.006$ and $\tilde{\theta}_2 = 0.044$. Therefore, the unconditional idiosyncratic equity variance for every set of betas can be defined by $\theta^i = \tilde{v}^i - (\beta_1^i)^2\tilde{\theta}_1 - (\beta_2^i)^2\tilde{\theta}_2$. The spot market variance components are set equal to $v_{1,t} = 0.012$ and $v_{2,t} = 0.048$ and the total spot equity variance is $v_t^i = 0.05$. Consequently, we define the spot idiosyncratic variance for different sets of betas as $\xi_t^i = v_t^i - (\beta_1^i)^2 v_{1,t} - (\beta_2^i)^2 v_{2,t}$. We choose the remaining structural parameters of the market and equity dynamics as follows: $\{\tilde{\kappa}_1 = 0.18, \tilde{\kappa}_2 = 2.8, \sigma_1 = 3.6, \sigma_2 = 0.29, \rho_1 = -0.96, \rho_2 = -0.83\}$ and $\{\tilde{\kappa}^i = 0.8, \sigma^i = 0.2, \rho^i = 0\}$. We keep the risk-free rate at 4% per year and the ratio of spot index price over spot equity price is equal to $S_t^i/S_t = 0.1$. Note that the Y axis is Implied Volatility.

Figure 5: Persistent and Transient Betas and Implied Volatility Across Moneyness



Note to Figure: This figure plots the model-implied volatility for three-month equity call options with respect to the moneyness (S/K) for different sets of betas. Panel A shows the IV moneyness slope following the one-factor structure model and Panel B replicates the same IV moneyness slope with our two-factor structure model. Panels C shows IV moneyness slope when persistent beta β_1^i is constant and Panel D shows IV moneyness slope when transient beta β_2^i is constant. Note that for all the graphs the total unconditional equity variance is fixed at $\tilde{v}^i = (\beta_1^i)^2\tilde{\theta}_1 + (\beta_2^i)^2\tilde{\theta}_2 + \theta^i = 0.11$. Note also that the Y axis is Implied Volatility.

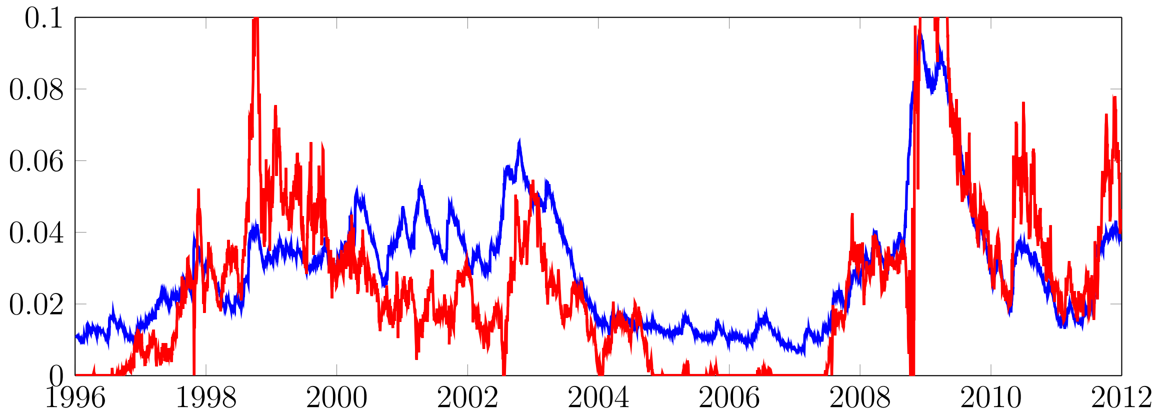
Figure 6: Persistent and Transient Variances Risk Premiums and Implied Volatility Smile



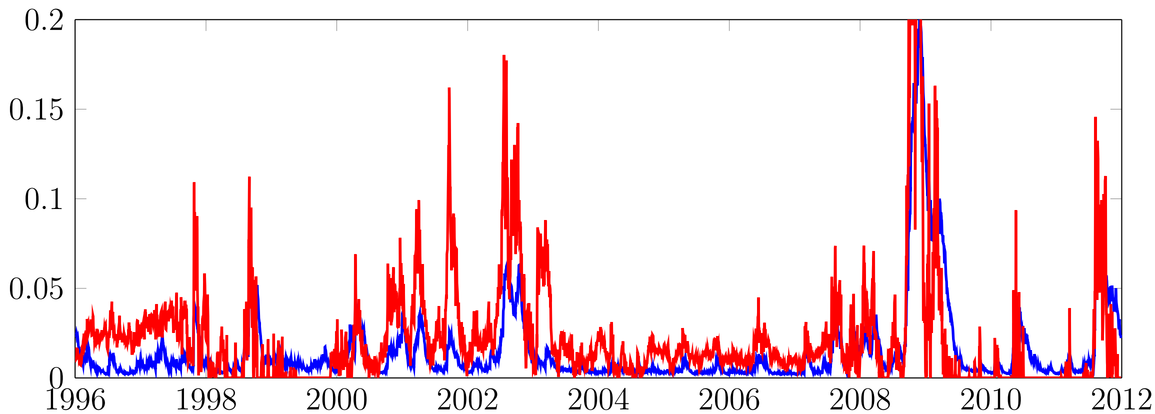
Note to Figure: This figure plots the difference between model-implied volatility for three-month equity call options with respect to the moneyness (S/K) for different sets of betas. The implied volatility difference is the difference between IV when $\lambda_1 = \lambda_2 = -0.5$ and when $\lambda_1 = \lambda_2 = 0$. Panel A shows the effect of market variance risk premium on equity option skew (slope of IV curve) following the calibration in one-factor structure model while Panel B replicates the same effect in our two-factor structure model. Panels C shows IV difference when persistent beta β_1^i is constant and Panel D shows IV difference when transient beta β_2^i is constant. Note that for all the graphs the total unconditional equity variance is fixed, $\tilde{v}^i = (\beta_1^i)^2\tilde{\theta}_1 + (\beta_2^i)^2\tilde{\theta}_2 + \theta^i = 0.11$.

Figure 7: The S&P 500 Index Spot Variance Components Paths

Panel A: Persistent Variance Component



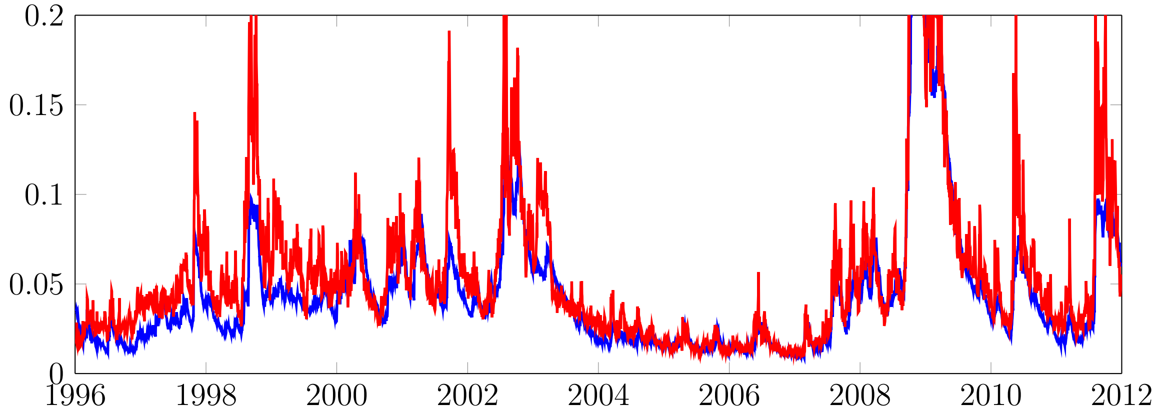
Panel B: Transient Variance Component



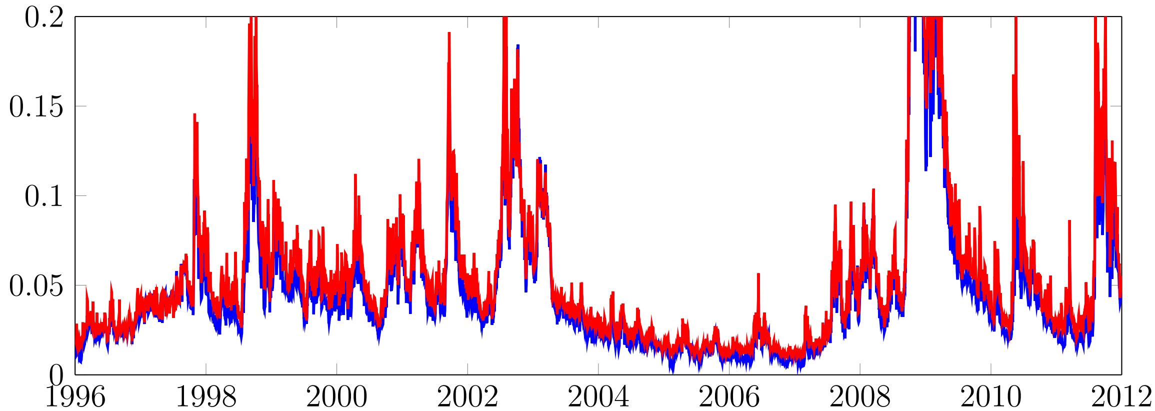
Note to Figure: We plot time series of risk-neutral spot variances for the S&P 500 index in the two-factor stochastic volatility model. Panel A shows time series of persistent variance component and Panel B shows time series of transient variance component. The blue plots are based on the Particle Filter method using data from both S&P 500 index and option markets (joint estimation). The red plots are filtered spot variances using data from S&P 500 option market only.

Figure 8: The S&P 500 Index Total Spot Variance Path Versus VIX

Panel A: Joint Estimation Versus VIX



Panel B: Option-Based Estimation Versus VIX



Note to Figure: We plot time series of risk-neutral total spot variance for the S&P 500 index by combining persistent and transient variance components of the two-factor stochastic volatility model. The blue plots in Panel A is based on the Particle Filter method using data from both S&P 500 index and option markets (joint estimation). The blue plot in Panel B is based on data from S&P 500 option market only. Red plots in both panels are time series of the VIX option implied volatility index.