Optimal Risk Sharing with Heterogeneous Investment Horizons and Recursive Preferences

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Abstract

I examine the role of heterogeneous investment horizons for the wealth distribution among agents with nonseparable recursive preferences, and the resulting equilibrium asset prices. The novelty of this study is introducing a new type of heterogeneity in terms of different investment horizons in a class of models with recursive preferences that can match asset pricing data much closer. I construct a model with short-term and long-term investors, and find that over time long-term investors, who take larger amount of information for their utility optimization outcrowd myopic investors in terms of wealth share. In the presence of long-run risk, however, short-term agents, who are less averse to persistent shocks to consumption growth rate may dominate over the long-term agents, driving the equity premium below the empirically observed one and posing a challenge for long-run risk models.

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1. Introduction

In their seminal paper Bansal and Yaron (2004) demonstrate that long-run risks represented by the persistence in the growth prospects help explain many features of asset market data regarded as puzzles. Investors with different horizons, however, may be concerned about transitory and long-run risks to a different extent (Bansal, Dittmar, and Kiku, 2009). From high and medium frequency traders such as hedge funds and mutual funds to longer horizon traders such as pension funds, investors' preferences and objectives can be influenced by the liquidity constraints they face and their investment horizon. Therefore, recognizing the existence and interaction of investors with heterogeneous horizons and the variation of their sensitivity to long-run risks is important for analyzing their market participation. The difference in risk pricing and saving behavior related to the extent that short-term and long-term investors care about long-run risks can affect wealth distribution and stock prices. In the absence of long-run risk differences in the horizon can also influence the wealth distribution between agents. The choice between a myopic and a dynamic long-term investment strategy (where the investor considers a larger set of information in the latter) can lead to uneven wealth accumulation, even if each agent chooses her own individual optimal strategy. This paper focuses on understanding the economics behind the wealth distribution between agents with recursive preferences and different investment horizons in an economy with and without long-run risks and its effect on equilibrium asset prices.

The novelty of this study is exploring a new line of heterogeneity in terms of different investment horizons in a class of asset pricing models featuring recursive preferences that have shown potential in resolving asset pricing puzzles. Even though accounting for heterogeneity is plausible and can lead to interesting interactions among agents, these models typically analyze a representative agent. The reason, as emphasized by Collin-Dufresne, Johannes, and Lochstoer (2015), is the complexity in solving such models and so far there are only a few studies incorporating agents of different types (Borovicka, 2016; Collin-Dufresne, Johannes, and Lochstoer, 2017; Garleanu and Panageas, 2015; Pohl, Schmedders, and Wilms, 2016, among others). These papers have mainly focused on studying investors who differ in their beliefs and preferences. However, the effect of investor horizon heterogeneity in such models has not been investigated yet.

To study these effects I construct models with two agents who have a short and a long investor horizon when there are either iid consumption growth rate shocks or long-run risks as in the setting of Bansal and Yaron (2004). Since the consumption sharing rule depends on the value functions that I try to solve for, I employ the numerical method of Collin-Dufresne, Johannes, and Lochstoer (2015) using backwards recursion in order to find a solution in the general model. I also study some special cases with closed-form solutions which allow us to draw implications about the wealth distributions between agents with different horizons. I find that long-term investors accumulate larger wealth shares over time compared to myopic short-term agents. Even though each agent chooses her optimal strategy, the investor with long-term strategy that incorporates more information in her decisions rapidly outcrowds the myopic agent who only cares about the close future when she selects her optimal portfolio. I show that the behavior of the myopic agent resembles the one of an impatient agent or an agent who has a distinct preference for early resolution of uncertainty. In contrast, in the presence of long-run risks, the myopic agent may sell insurance against these persistent risks to the long-term agent who is more concerned about them. Since the short-term agent would bear a larger part of the risk in the economy, she would accumulate a higher premium and wealth share. As the consumption share of the myopic agent who is not as concerned about the long-run risk increases, the importance of long-run risk in the economy drops, leading to a decrease in the equity premium and a challenge to the long-run risk models.

Allowing for nonseparable recursive preferences when studying the relation between heterogeneity of investment horizons and asset prices is important since it provides the opportunity to explore this link in a setting that matches the stylized facts from asset pricing data much more accurately. First, under recursive preferences, risk aversion can be separated from the elasticity of intertemporal substitution. Heterogeneous investment horizons and liquidity objectives can entail different degrees of distaste for intertemporal fluctuations, independent from the risk aversion levels of investors. This can lead to interesting implications for risk premia. Second, under recursive preferences long-run risks that are characterized as persistent shocks to the growth rate can be included as a state variable. The long-run risk model proposed by Bansal and Yaron (2004) can justify many of the features of asset pricing data. In the model news regarding the expected growth rates affect the volatility of the price-dividend ratio and positively covary with the marginal rates of substitution of investors, leading to higher risk premia. The higher the persistence of the shocks, the larger the risk premia become. Taking into account the corresponding sensitivity of agents with different investment horizons to long-run risks can lead to interesting dynamics in their risk sharing behavior and wealth distribution that can affect asset prices in the short and the long run.

Previous studies that have addressed the importance of investment horizon have shown that this characteristic matters once the economic setup becomes more realistic. Merton (1969) shows that under the assumption of no transaction costs and no changing investment opportunity sets portfolio allocation of investors is independent of their horizon. However, focusing on more realistic settings including market illiquidity and hedge demands brings evidence that short-term and long-term decisions can be substantially different. On the one hand, capital liquidity is important for portfolio choices of investors with different horizons since their investment goals vary with respect to their liquidity objectives. For instance, Beber, Driessen, and Tuijp (2012) show that the heterogeneity of investor horizons determines the liquidity premium of assets and the way liquidity affects asset prices. On the other hand, Merton (1971) finds that state variables that account for investors' intertemporal hedging demands can influence their decisions at different horizons. Accounting for recursive preferences and the degree of sensitivity of investors with heterogeneous investment horizons to long-run risks in a model that justifies asset pricing data features much closer can shed more light on the distribution of wealth and the way it affects risk premia.

The rest of the paper is organized as follows. Section 2 presents the model setup and the economy in which I study the relation between investment horizon heterogeneity, market participation of investors, and asset prices. Section 3 presents the analytical results of the paper. Section 4 outlines the numerical method used to find the optimal wealth allocation between the agents. Section 5 focuses on the role of heterogenous investment horizons in the presence of long-run risk and section 6 concludes the paper.

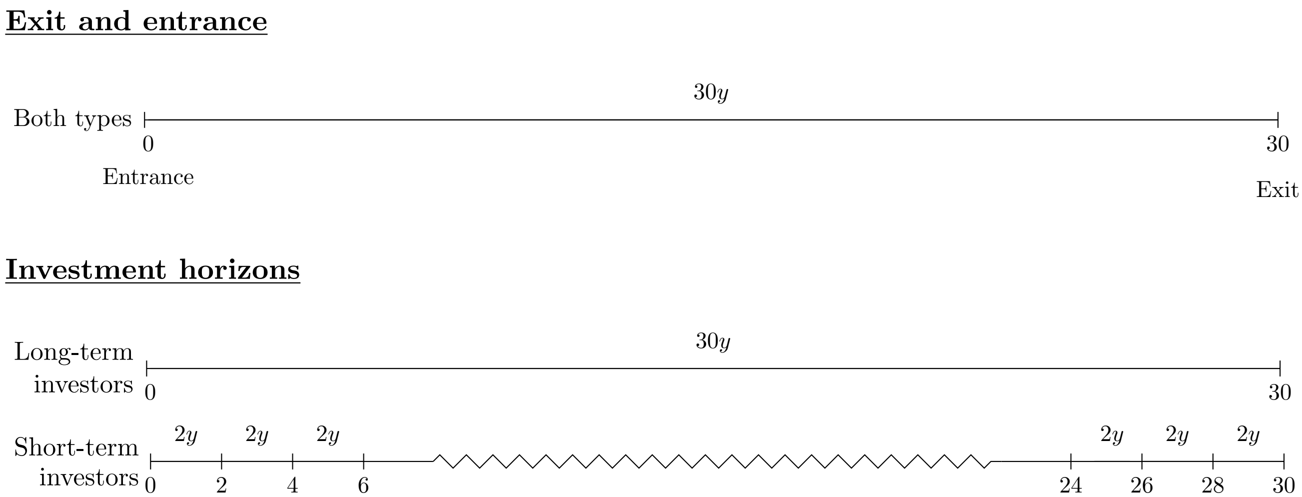
2. The model

2.1 Setup and economy

In this section I describe a model with two agents $i = \{A, B\}$ who have different investment horizons and can also differ in terms of their preferences. Long-term investors are denoted by $i = A$ and are assumed to have an investor horizon of $\tau_A = 30$ years, while short-term investors are myopic, denoted by $i = B$, have a horizon of $\tau_B = 2$ years (see Figure 1). The investors of each type enter the market and invest for a period of 30 years and after 30 years they exit the market. While the long-term investors have long-run goals and optimize over a period of 30 years, short-term investors are myopic, with an investment horizon of 2 years, and thus optimize over a period of 2 years. After these 2 years they optimize for another 2 and continue in the same way until they exit the market. There is no uncertainty about the length of the investment horizon of each agent and no hedging demands related to such an uncertainty are needed. The calibration of the model parameters is monthly. The lengths of horizons of the two types of investors are varied from 30 years (360 months) to 100 years (1200 months), and 500 years (6000 months) in order to study the long-term wealth distribution effects of investment horizon in the economy.

At every point in time there is one agent of each type and both agents that are currently investing entered the market at the same time. These assumptions imply a two-agent representation in the economy and minimize the number of state variables in terms of the relative consumption share of each investor. The method extends conceptually to N agents, but this would scale the problem in N and would require using $N - 1$ endogenous state variables to describe the wealth distribution among the agents. Since this makes the problem more computationally intensive, I consider the case with only two types of agents.

Figure 1: Model timeline



Aggregate consumption is given by C_t, while individual consumption of investors A and B at time t is denoted by $C_{A,t}$ and $C_{B,t}$. The market clearing condition is:

$$C_{A,t} + C_{B,t} = C_t. \tag{1}$$

I first consider an economy where the only shock to consumption growth rate is an iid shock ε_t:

$$g_{t+1} = \ln\left(\frac{C_{t+1}}{C_t}\right) = \mu + \sigma\varepsilon_{t+1} \tag{2}$$

Both investors have Epstein-Zin preferences suggested by Koopmans (1960), Kreps-Porteus (1978), Epstein and Zin (1989), and Weil (1989), and given by:

$$V_{i,t} = V_{i,t}(C_{i,t}, V_{i,t+1}) = \left[(1-\beta_i)C_{i,t}^{\rho_i} + \beta_i E_t(V_{i,t+1}^{\alpha_i})^{\frac{\rho_i}{\alpha_i}}\right]^{\frac{1}{\rho_i}}, \tag{3}$$

where $E_t(V_{i,t+1}^{\alpha_i})^{\frac{1}{\alpha_i}}$ is the certainty equivalent of all future consumptions.

Since the investment horizon τ_i of each type of agents is finite, I assume that she consumes her terminal wealth before exiting the market. Thus, the terminal value function at time $T = 30$ takes the following form:

$$U_{i,T} = (1-\beta_i)C_{i,T}. \tag{4}$$

For the rest of the periods before time T the long-term investor optimizes dynamically, taking into account all future expected utilities until her exit (as in equation (3)). The short-term agent realizes she will invest and consume until the terminal date T and therefore weighs and discounts the expected future value functions proportionally to the remaining time until the terminal period. However, she is myopic and only takes into account the current and following periods' information for her optimization. Thus, while the short-term investor's value functions at time T and $T-1$ take the same form as the ones of he long-term investor, for all the periods from $t = T-2$ until $t = 0$, the value function of the myopic investor is given by:

$$V_{B,T-t} = \left[(1-\beta_B) C_{B,T-t}^{\rho_B} + (\beta_B + ... + \beta_B^t) E_{T-t} \left(U^{\alpha_B}(C_{B,T-t+1}) \right)^{\frac{\rho_B}{\alpha_B}} \right]^{\frac{1}{\rho_B}}, \tag{5}$$

where $U(C_{B,t}) = (1-\beta_B)C_{B,t}$ is the utility of agent B from consumption at time t. Thus, at the end of each of her 2-year investment horizons the myopic agent of type $i = B$ transfers her wealth to the next period during the time they are in the market, but does not optimize dynamically for more than a period ahead, taking less information into account for her optimization than agent A.

Initially, I consider the case when investors do not differ in terms of their preference parameters in order to study the pure effect of investment horizon in the model. Afterwards, I allow the preference parameters to differ across agents of type A and B and denote the time discount factor as β_i, $\rho_i = 1 - 1/\psi_i$, where ψ_i is the elasticity of intertemporal substitution (EIS), and $\alpha_i = 1 - \gamma_i$, where γ_i is the risk aversion level. I assume a concave form of the utility functions of both types of agents and thus the risk aversion γ_i and EIS ψ_i parameters are assumed to take only positive values. Hence, $\rho_i \leq 1$ and $\alpha_i \leq 1$. The time discount factor β_i ranges between 0 and 1.

2.2 Pareto problem

The two-agent Pareto problem can be represented as the optimization of a social planner who maximizes the weighted sum of utilities of the investors of both types at time $t = 0$ subject to the market clearing condition:

$$\max_{\{C_{A,t}, C_{B,t}\}_{t=0}^{T}} w_t V_{A,0} + (1-w_t) V_{B,0} \tag{6}$$
$$\text{s.t. } C_{A,t} + C_{B,t} = C_t \text{ for all states and time.}$$

Even though the individual utility functions are recursive, the social planner utility is not recursive. However, as shown by Lucas and Stokey (1984), Kan (1995), and Backus, Routledge, and Zin (2009) a recursive formulation exists. Applying Theorem 3 from Lucas and Stockey (1984) it follows that the Pareto optimal allocation is given by the following Bellman equation:

$$J(C_t, V_{B,t}) = \max_{\{C_{A,t}, V_{B,t+1}\}} \left[(1-\beta_A) C_{A,t}^{\rho_A} + \beta_A E_t [J(C_{t+1}, V_{B,t+1})^{\alpha_A}]^{\frac{\rho_A}{\alpha_A}} \right]^{\frac{1}{\rho_A}} \tag{7}$$
$$\text{s.t. } V_{B,t}(C_{B,t}, V_{B,t+1}) \geq V_{B,t}$$
$$C_{A,t} + C_{B,t} = C_t,$$

where $V_{B,t}$ is the so-called promised utility to agent B at time t and $J(C_t, V_{B,t}) = V_{A,t}$. Since there is monotonicity in preferences, the utility-promise constraint is binding and hence, the constraint can be replaced by $V_{B,t}(C_{B,t}, V_{B,t+1}) = V_{B,t}$.

The problem of choosing a feasible allocation between the two agents can be viewed as a problem of maximizing the utility of agent A at time t over her own consumption $C_{A,t}$ and the promised utility to agent B at time $t+1$, $V_{B,t+1}$, that is the aggregate utility over the remaining horizon that agent A promises to agent B. Agent A can increase her consumption at time t up to the point where the utility of agent B at time t does not fall below the promised utility $V_{B,t}$. Thus, agent A can either choose to have higher consumption $C_{A,t}$ at time t and lower utility $V_{A,t+1}$ at time $t+1$ by promising higher utility to agent B at time $t+1$, or alternatively agent A can choose to have lower consumption $C_{A,t}$ at time t and higher utility $V_{A,t+1}$ at time $t+1$ by promising lower utility $V_{B,t+1}$ to agent B at time $t+1$. The promised utility $V_{B,t+1}$ that was chosen at time t will serve as a constraint for the minimum utility agent B will receive at time $t+1$, and so on until the terminal date T. In the optimization problem (7), C_t is an exogenous and $V_{B,t}$ is an endogenous state variable.

I solve a normalized version of the model with all variables divided by aggregate consumption. I denote $v_{i,t} = V_{i,t}/C_t$ and $c_{i,t} = C_{i,t}/C_t$ and hence, the value functions can be written as:

$$v_{i,t} = \left[(1-\beta_i)c_{i,t}^{\rho_i} + \beta_i E_t\left[v_{i,t+1}^{\alpha_i}(C_{t+1}/C_t)^{\alpha_i}\right]^{\frac{\rho_i}{\alpha_i}}\right]^{\frac{1}{\rho_i}}, \tag{8}$$

where the market clearing condition is $c_{A,t}+c_{B,t} = 1$. As shown in Epstein and Zin (1989) the stochastic discount factor for agent i would be:

$$\begin{aligned} &E_t[M_{t+1}^i R_{t+1}^j] = 1 \text{ for all times } t \text{ and states } \omega_{t+j} \\ &M_{t+1}^i = \beta_i\left(\frac{c_{i,t+1}}{c_{i,t}}\right)^{\rho_i-1}\left(\frac{C_{t+1}}{C_t}\right)^{\alpha_i-1}\left(\frac{v_{i,t+1}}{E_t\left[v_{i,t+1}^{\alpha_i}(C_{t+1}/C_t)^{\alpha_i}\right]^{\frac{1}{\alpha_i}}}\right)^{\alpha_i-\rho_i}. \end{aligned} \tag{9}$$

Under the assumption of frictionless complete markets the equilibrium requirement (presented here in a normalized form) for solving the maximization problem is given by the first-order condition - the marginal intertemporal rates of substitution of the two agents must be equal for each state and over each time period (Collin-Dufresne, Johannes, and Lochstoer, 2015):

$$\begin{aligned} &\beta_A\left(\frac{c_{A,t+1}}{c_{A,t}}\right)^{\rho_A-1}\left(\frac{C_{t+1}}{C_t}\right)^{\alpha_A-1}\left(\frac{v_{A,t+1}}{E_t\left[v_{A,t+1}^{\alpha_A}(C_{t+1}/C_t)^{\alpha_A}\right]^{\frac{1}{\alpha_A}}}\right)^{\alpha_A-\rho_A} = \\ &= \beta_B\left(\frac{c_{B,t+1}}{c_{B,t}}\right)^{\rho_B-1}\left(\frac{C_{t+1}}{C_t}\right)^{\alpha_B-1}\left(\frac{v_{B,t+1}}{E_t\left[v_{B,t+1}^{\alpha_B}(C_{t+1}/C_t)^{\alpha_B}\right]^{\frac{1}{\alpha_B}}}\right)^{\alpha_B-\rho_B}. \end{aligned} \tag{10}$$

Given a Pareto-optimal allocation, we can find the equilibrium prices and hence, all competitive equilibria can be determined. However, we can only estimate the initial endowments given an equilibrium condition, but we cannot find the equilibrium given the initial endowments. The reason is that the current period value functions depend on the future value functions and consumption allocations that are unknown. The initial endowments of the two agents are implicitly determined by the utility V_B that agent A promises to agent B.

Since the consumption sharing rule depends on the value functions that I try to solve for, the general model does not have a closed-form solution, so I employ the numerical method of Collin-Dufresne, Johannes, and Lochstoer (2015) using backwards recursion in order to find a solution.

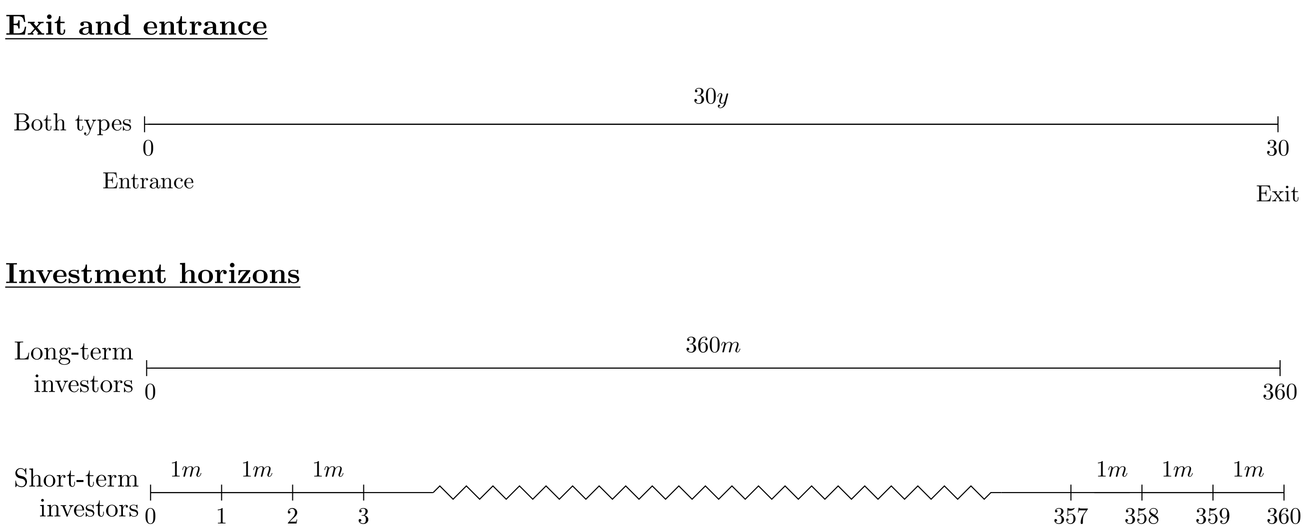
3. Analytical results

Even though the general model does not have a closed-form solution, in this section I show some special cases with analytical solutions and their implications for the wealth distribution between agents with different investment horizons.

3.1 Equilibrium wealth distribution with heterogeneous investment horizons

I first consider the special case when there are only iid shocks to consumption growth rate and the risk aversion levels of the two agents are equal ($\alpha_A = \alpha_B$). In order to focus on the effect of investment horizon for wealth distribution I assume that the short-term agent, denoted by $i = B$ has the shortest possible horizon using monthly calibration of $\tau_B = 1$ month and CRRA utility preferences. The long-term investor, denoted by $i = A$, has a horizon of $\tau_A = 30$ years. I also consider alternative lengths of the long-term investor horizon of 100 years and 500 years in order to study the long-term effects of horizon on wealth distribution (see Figure 2). I show that in this case the shocks to consumption growth rate ε_t do not influence the consumption sharing in the economy and we can find an analytical solution for the share of each agent that has implications for the effect of investment horizon on risk sharing.

Figure 2: Model timeline: Case 2



When the short-term investor has CRRA utility, her risk aversion and elasticity of intertemporal substitution parameters become equal ($\alpha_B = \rho_B$). Thus, we can write the equilibrium condition in frictionless complete markets as follows:

$$\beta_A \left(\frac{c_{A,t+1}}{c_{A,t}}\right)^{\rho_A - 1} \left(\frac{C_{t+1}}{C_t}\right)^{\alpha_A - 1} \left(\frac{v_{A,t+1}}{E_t\left[v_{A,t+1}^{\alpha_A}(C_{t+1}/C_t)^{\alpha_A}\right]^{\frac{1}{\alpha_A}}}\right)^{\alpha_A - \rho_A} = \beta_B \left(\frac{c_{B,t+1}}{c_{B,t}}\right)^{\rho_B - 1} \left(\frac{C_{t+1}}{C_t}\right)^{\alpha_B - 1} \tag{11}$$

$$\Leftrightarrow \beta_A \left(\frac{c_{A,t+1}}{c_{A,t}}\right)^{\rho_A - 1} \left(e^{\mu + \sigma\varepsilon_{t+1}}\right)^{\alpha_A - 1} \left(\frac{v_{A,t+1}}{E_t\left[v_{A,t+1}^{\alpha_A}(C_{t+1}/C_t)^{\alpha_A}\right]^{\frac{1}{\alpha_A}}}\right)^{\alpha_A - \rho_A} =$$

$$= \beta_B \left(\frac{c_{B,t+1}}{c_{B,t}}\right)^{\rho_B - 1} \left(e^{\mu + \sigma\varepsilon_{t+1}}\right)^{\alpha_B - 1}. \tag{12}$$

Since the risk aversion levels of the two agents are equal, it follows that $\left(\frac{C_{t+1}}{C_t}\right)^{\alpha_A - 1} = \left(\frac{C_{t+1}}{C_t}\right)^{\alpha_B - 1}$. Thus we can solve analytically for the consumption share of the long-term agent (c_A) using the following equilibrium condition, and the consumption share of the short-term investor will then be $c_B = 1 - c_A$:

$$\beta_A \left(\frac{c_{A,t+1}}{c_{A,t}}\right)^{\rho_A - 1} \left(\frac{v_{A,t+1}}{E_t\left[v_{A,t+1}^{\alpha_A}(C_{t+1}/C_t)^{\alpha_A}\right]^{\frac{1}{\alpha_A}}}\right)^{\alpha_A - \rho_A} = \beta_B \left(\frac{c_{B,t+1}}{c_{B,t}}\right)^{\rho_B - 1}. \tag{13}$$

We can see that the consumption allocation does not depend on the random shocks ε_t and it is a deterministic function of time. Hence, the consumption share of agent $c_{A,t}$ is only a function of her past consumption share $c_{A,t-1}$: $c_{A,t} = c_{A,t}(c_{A,t-1})$. The value function of the agent with Epstein-Zin preferences is predictable at time t and we prove by recursion that it takes the form $j_{t+1} = E_t[j_{t+1}]$. This leads to the following theorem (see the proof in Appendix B):

Theorem 1. *Suppose agent A has Epstein-Zin preferences, agent B has CRRA preferences and both agents have equal risk aversion and time discounting parameters. Then, the consumption sharing between them is a deterministic function of time, that is, the consumption share does not depend on any of the past shocks to consumption growth rate. The value functions are also deterministic. In particular, the following condition holds:* $\frac{\left(\frac{c_{A,t+1}}{c_{A,t}}\right)^{\rho_A - 1}}{\left(\frac{1-c_{A,t+1}}{1-c_{A,t}}\right)^{\alpha-1}} = e^{\mu+\frac{1}{2}\alpha\sigma^2}$, *where* $e^{\mu+\frac{1}{2}\alpha\sigma^2}$ *is constant. In case* $\mu > -\frac{1}{2}\alpha\sigma^2$, *the consumption share of the investor with Epstein-Zin preferences increases over time, and* $\lim_{t\to\infty} c_{A,t} = 1$ *such that she outcrowds the CRRA agent. In case* $\mu < -\frac{1}{2}\alpha\sigma^2$, *the consumption share of the investor with Epstein-Zin preferences decreases over time.*

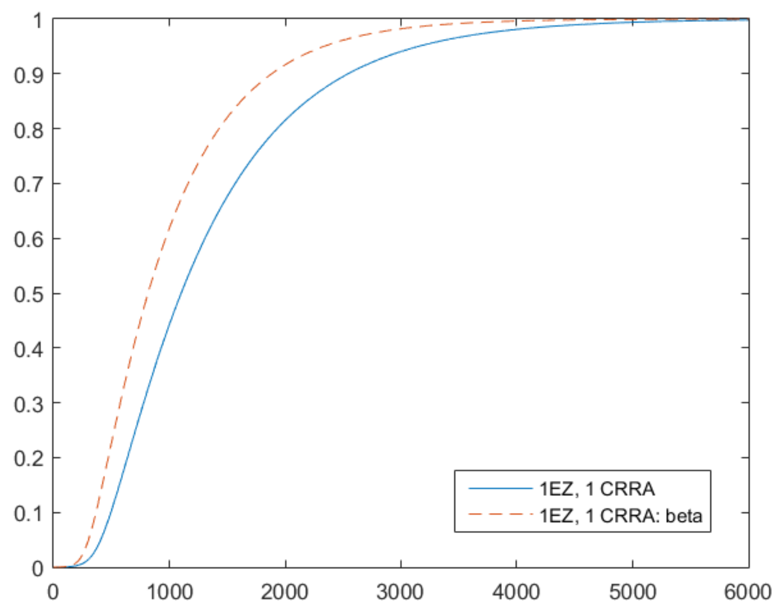
Merton (1969) and Samuelson (1969) show that under CRRA preferences myopia, or behaving as if each period were the last one before retirement, is optimal. Since the solution of the pareto problem gives the optimal consumption allocation between agents based on each agent's optimal strategy, the CRRA investor will be myopic. Thus, solving the special case with one CRRA and one Epstein-Zin agent sheds light on the equilibrium wealth distribution between a myopic short-term investor and a long-term investor who optimizes her utility dynamically. Using the monthly calibration of Bansal and Yaron (2004), where $\mu = 0.0015$ and $\sigma = 0.0078$, (or alternative reasonable parameters), we can show that the first case $\mu > -\frac{1}{2}\alpha\sigma^2$ is normally satisfied and observed in the data. Hence, the investor with Epstein-Zin preferences who has a longer horizon dominates over the myopic CRRA investors and accumulates a larger wealth share over time. The wealth distribution between myopic and non-myopic agents who have equal risk aversion and time discounting preferences is not clear ex ante, but the results confirm the intuition that the dynamic optimization that takes into account a larger set of information for more periods ahead leads to a larger wealth accumulation.

To see how fast the long-term agent accumulates wealth and whether she grows large enough wealth share to dominate over the short-term agent over time, I explore the evolution of consumption shares of the two agents for the investment horizons of 30, 100, and 500 years (360, 1200, and 6000 months) ahead, when the long-term agent's initial consumption share is $c_A = 0.0001$. Figure 3 shows the results in the case when one agent has CRRA preferences, while the second one has Epstein-Zin preferences and both agents have the same risk aversion and time discounting parameters ($\alpha_A = \alpha_B = -9$ and $\beta_A = \beta_B = 0.998$). We observe that after 30 years the long-term investor accumulates a share of about 3% of the total aggregate consumption and the short-term agent holds 97%. Over time, however, the share of the long-term investor increases rapidly compared to the one of the short-term agent. In particular, only after 100 years, the long-term investor accumulates over half of the wealth in the economy (54.98%) and after 500 years she holds 99.81%. Thus, the consumption share of the long-term agent who takes into account more information for her utility optimization converges to 1 in the future and she outcrowds the short-term myopic agent.

Figure 3 also shows the evolution of the consumption path when the two agents have different time preference parameters. The long-term agent is more patient and has $\beta_A = 0.999$, while the short-term agent is less patient with $\beta_B = 0.995$. We observe that the long-term agent who is additionally more patient accumulates wealth faster compared to a long-term investor who is as patient as the short-term agent. Thus, the effect of horizon and the fact that the long-term investor outcrowds the myopic investor is even more pronounced when the long-term agent is also more patient than the short-term one.

Figure 3: Long-term investor's consumption share evolution

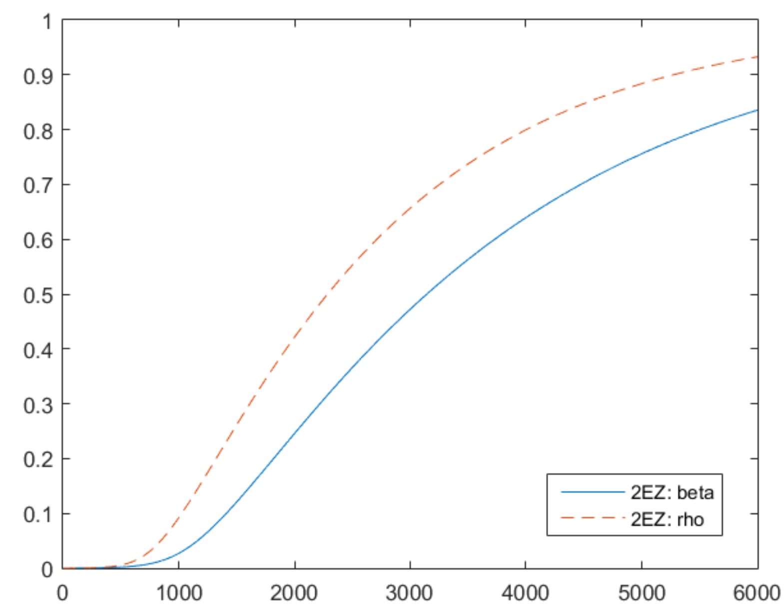
The figure plots the average of 10000 simulated paths of consumption share of the long-term agent (A) with Epstein-Zin preferences when the short-term agent (B) has myopic CRRA preferences. The initial endowment of agent (A) is set to $c_{A,0} = 0.0001$ and the consumption share is estimated for the next 6000 months (500 years). The solid line represents the case when the two agents have the same time discounting parameters ($\beta_A = \beta_B = 0.998$) and the dashed line to the case when the long-term agent is more patient ($\beta_A = 0.999$ and $\beta_B = 0.995$).



In order to test whether the investment behavior and consumption path of the myopic agent resemble the one of an impatient agent or an agent who prefers early resolution of uncertainty less, I proceed the following way. I estimate the evolution of consumption paths when both investors in the economy have Epstein-Zin utility, but differ in terms of their time discounting and elasticity of intertemporal substitution parameters. I start with the case when agents have different time discounting parameters ($\beta_A = 0.999$ and $\beta_B = 0.995$) and plot the consumption share of the more patient one (solid line, Figure 4). We can see that the more patient agent accumulates larger wealth over time compared to the less patient one. Intuitively, the patient agent prefers to postpone her consumption for the future, and as a result she consumes less today, more tomorrow, and her share becomes larger over time. Starting with a consumption share of $c_A = 0.0001$, the patient agent accumulates 0.0086% of total wealth in 30 years, 5.67% after 100 years, and 83.55% after 500 years. Thus, even though her consumption share grows over time, it does so slower compared to the case when the patient agent has as well a long-term investment horizon and the less patient agent is myopic. We can find, however, that increasing the difference between time discounting parameters of the two agents, such that $\beta_B < 0.995$ leads to a wealth accumulation of the patient agent as fast as the one of the long-term agent. Hence, a myopic investment strategy can be equivalent to the strategy of an impatient agent who prefers to consume more today and accumulate less wealth in the future.

Figure 4: Consumption share evolution with different preference parameters

The figure plots the average of 10000 simulated paths of consumption share of two Epstein-Zin agents: one who is more patient (solid line, $\beta_A = 0.999, \beta_B = 0.995$) and one who has lower preference for early resolution of uncertainty (dashed line, $\rho_A = 0.5, \rho_B = -4$), while the rest of the parameters are equal. The initial endowment of the agents is set to $c_{A,0} = 0.0001$ and the consumption share is estimated for the next 6000 months (500 years).



I also consider the case when both agents have Epstein-Zin preferences, but different elasticity of intertemporal substitution parameters to see whether the investment behavior of a myopic agent resembles the one of an agent who prefers early resolution of uncertainty less ($\rho_A = 0.5$ and $\rho_B = -4$). In Figure 4 (dashed line) I plot the consumption path of the agent who has lower preference for early resolution of uncertainty. Starting with $c_A = 0.0001$ the investor accumulates 1.9% in 30 years, 15.72% in 100 years, and 93.29% in 500 years. Thus the investment behavior of a myopic agent is close to the one of an agent who prefers early resolution of uncertainty. In particular, if $\alpha_B = \rho_B$ the consumption path of the two agents are identical.

3.2 Equilibrium wealth distribution with CRRA utility

It is useful to revisit the problem of determining the wealth distribution between agents with CRRA utility, which can be solved analytically. Appendix B presents the proofs of Propositions 1 - 3 that show the consumption share evolutions when both agents have the same preference parameters, when they have different time discounting parameters and different risk aversion parameters.

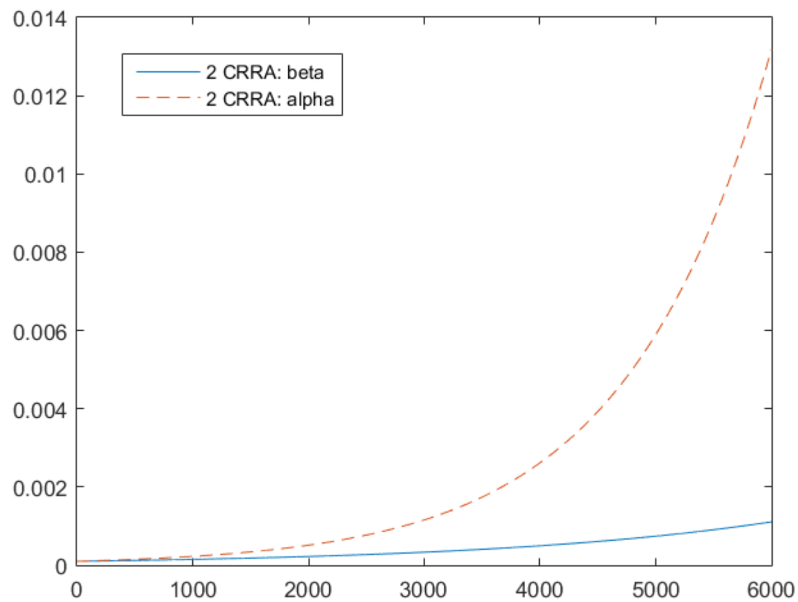
When both agents have the same preference parameters the consumption shares of the two remain constant over time and are equal to their initial endowments (Proposition 1).

If the two investors have different time discounting (β) parameters, I show that the more patient investor accumulates larger consumption share over time compared to the less patient agent (Proposition 2). This result is intuitive since the more patient agent prefers to postpone her consumption to the future and thus she grows a larger wealth share. Figure 5 plots the evolution of the consumption share of the more patient agent, which increases over time, but much slower compared to the case when both agents have Epstein-Zin preferences. For instance, the more patient agent with initial consumption share of 0.01% accumulates 0.2% of total aggregate consumption in 500 years, while her share increases from 0.01% to over 82% in 500 years when the agents have Epstein-Zin preferences and the same investment horizons, and to 99% when the patient investor has longer-term horizon than the impatient one.

In Proposition 3 I show that when the two investors have different risk aversion levels, the more risk loving investor increases her share over time from 0.01% to 1.3% in 500 years in case $\varepsilon_{t+1} > -\mu/\sigma$ (the case generally observed in the data). Intuitively the risk loving agent invests more in the risky asset and since the expected return on the risky asset is higher than the one on the risk-free asset, the risk loving agent accumulates a larger consumption share. In other words, the more risk loving agent would bear larger proportion of the total risk in the economy and sell insurance to the more risk averse agent, for which she requires a higher compensation and thus grows a larger wealth share over time. Since both investors have CRRA preferences and are myopic and consider only 1 period ahead for their optimization, these cases do not have direct implications for the role of investment horizon.

Figure 5: Consumption share evolution with CRRA utility and different preferences

The figure plots the average of 10000 simulated paths of consumption share of two CRRA agents: one who is more patient (solid line, $\beta_A = 0.999, \beta_B = 0.995$) and one who is more risk loving (dashed line, $\alpha_A = -4, \alpha_B = -9$), while the rest of the parameters are equal. The initial endowment of the agents is set to $c_{A,0} = 0.0001$ and the consumption share is estimated for the next 6000 months (500 years).



4. Numerical Method

The special cases described in the previous section have closed-form solutions, but the general risk sharing problem with agents with recursive preferences does not since the consumption shares depend on the value functions we need to solve for. Therefore, I solve the optimization problem (7) numerically using backward recursion, starting from time $t = T$ (Section **4.1**) and subsequently iterating (Section **4.2**).

4.1 Time $t = T$

At Time $t = T$, the economy ends and the optimization problem (7) in normalized form becomes

$$\begin{aligned} j^*(v_{B,T}) \quad &= \quad \max_{c_{A,T}} \left[(1-\beta_A)c_{A,T}^{\rho_A}\right]^{\frac{1}{\rho_A}} \\ \text{s.t.} \quad &\left[(1-\beta_B)c_{B,T}^{\rho_B}\right]^{\frac{1}{\rho_B}} = v_{B,T}, \\ &c_{A,T} + c_{B,T} = 1. \end{aligned} \tag{14}$$

Thus, the value functions at time $t = T$ are given analytically as functions of the relative consumption of agent A:

$$v_{A,T} = (1-\beta_A)^{\frac{1}{\rho_A}} c_{A,T} \tag{15}$$

$$v_{B,T} = (1-\beta_B)^{\frac{1}{\rho_B}} (1-c_{A,T}). \tag{16}$$

We will use these functions when we are solving for the optimal $c_{A,T}$ at time $t = T-1$.

4.2 Recursion at time t

In this step, it is convenient to use $c_{A,t}$ as the endogenous state variable and x_t and σ_t as the exogenous state variables. Thus, we can write the value function of agent i at time t as a function of the state variables at time t: $v_{i,t} = v_{i,t}(x_t, \sigma_t, c_{A,t}, \varepsilon_{t+1})$. Since we are solving the problem backwards, we may assume that it is already solved at time $t+1$ and we are given the value functions:

$$j^*_{A,t+1} = \frac{J^*_{A,t+1}}{C_{t+1}} = j^*_{A,t+1}(x_{t+1}, \sigma_{t+1}, c_{A,t+1}, \varepsilon_{t+2}), \tag{17}$$

$$v^*_{B,t+1} = \frac{V^*_{B,t+1}}{C_{t+1}} = v^*_{B,t+1}(x_{t+1}, \sigma_{t+1}, c_{A,t+1}, \varepsilon_{t+2}). \tag{18}$$

In the numerical implementation, the functions will not be known for any possible value of the state variables they depend on, $c_{A,t+1}$, x_{t+1}, and σ_{t+1}. That is, they will only be known at a number of (bivariate) grid points. Intermediate values have to be calculated by interpolation.

The problem to solve now reads as follows. Note that I make the dependence of the social planner's value function j_t on the state of the economy x_t and σ_t and the consumption share of agent A, $c_{A,t}$, explicit; the dependence on x_t and σ_t enters through the conditional expectation E_t. I also replace $C_{t+1}/C_t = e^{\mu + x_t + \sigma_t \varepsilon_{t+1}}$ that follows from equation (33):

$$\begin{aligned} j^*_t(v_{B,t}|x_t, \sigma_t, c_{A,t}) = \max_{\{c_{A,t}, v_{B,t+1}\}} & \left[(1-\beta_A)c_{A,t}^{\rho_A} + \beta_A E_t\left[j^*_{t+1}(v_{B,t+1}|x_{t+1}, \sigma_{t+1}, c_{A,t+1})^{\alpha_A} e^{(\mu + x_t + \sigma_t \varepsilon_{t+1})\alpha_A}\right]^{\frac{\rho_A}{\alpha_A}}\right]^{\frac{1}{\rho_A}} \\ \text{s.t. } v_{B,t} = & \left[(1-\beta_B)c_{B,t}^{\rho_B} + \beta_B E_t\left[\left(v_{B,t+1}\left(x_{t+1}, \sigma_{t+1}, c_{A,t+1}\right)\right)^{\alpha_B}\left(e^{(\mu + x_t + \sigma_t \varepsilon_{t+1})\alpha_B}\right)\right]^{\frac{\rho_B}{\alpha_B}}\right]^{\frac{1}{\rho_B}} \\ & c_{A,t} + c_{B,t} = 1. \end{aligned} \tag{19}$$

As mentioned before, we actually solve numerically the induced first-order conditions:

$$\beta_A\left(\frac{c_{A,t+1}}{c_{A,t}}\right)^{\rho_A-1}\left(e^{\mu+x_t+\sigma_t\varepsilon_{t+1}}\right)^{\alpha_A-1}\left(\frac{j_{t+1}(v_{B,t+1}|x_{t+1},\sigma_{t+1},c_{A,t+1})}{E_t[(j_{t+1}(v_{B,t+1}|x_{t+1},\sigma_{t+1},c_{A,t+1}))^{\alpha_A}e^{(\mu+x_t+\sigma_t\varepsilon_{t+1})\alpha_A}]^{\frac{1}{\alpha_A}}}\right)^{\alpha_A-\rho_A}-$$

$$\beta_B\left(\frac{1-c_{A,t+1}}{1-c_{A,t}}\right)^{\rho_B-1}\left(e^{\mu+x_t+\sigma_t\varepsilon_{t+1}}\right)^{\alpha_B-1}\left(\frac{v_{B,t+1}}{E_t[(v_{B,t+1}\,(x_{t+1},\sigma_{t+1},c_{A,t+1}))^{\alpha_B}e^{(\mu+x_t+\sigma_t\varepsilon_{t+1})\alpha_B}]^{\frac{1}{\alpha_B}}}\right)^{\alpha_B-\rho_B}=0 \tag{20}$$

$$v_{B,t}=\left[(1-\beta_B)c_{B,t}^{\rho_B}+\beta_B E_t[v_{B,t+1}^{\alpha_B}(e^{(\mu+x_t+\sigma_t\varepsilon_{t+1})\alpha_B})]^{\frac{\rho_B}{\alpha_B}}\right]^{\frac{1}{\rho_B}} \tag{21}$$

$$c_{A,t}+c_{B,t}=1. \tag{22}$$

Instead of solving the problem for the decision variables $c_{A,t}$, $c_{B,t}$, and $v_{B,t+1}$ implied by the system of equations (19), it is convenient to use a grid for $c_{A,t}$, in addition to the grids for the exogenous state variables x_t, and σ_t, and determine the optimal $c_{A,t+1}$, and the utilities j_{t+1} and $v_{B,t+1}$ for all combinations of the grid points. In order to do this we first define:

$$k_t=\frac{E_t[(v_{B,t+1}\,(x_{t+1},\sigma_{t+1},c_{A,t+1}(k_t)))^{\alpha_B}\,e^{(\mu+x_t+\sigma_t\varepsilon_{t+1})\alpha_B}]^{\frac{\rho_B-\alpha_B}{\alpha_B}}}{E_t[(j_{t+1}(v_{B,t+1}|x_{t+1},\sigma_{t+1},c_{A,t+1}(k_t)))^{\alpha_A}e^{(\mu+x_t+\sigma_t\varepsilon_{t+1})\alpha_A}]^{\frac{\rho_A-\alpha_A}{\alpha_A}}} \tag{23}$$

Rewriting equation (20) and substituting k_t in it we get:

$$\frac{c_{A,t+1}^{\rho_A-1}}{(1-c_{A,t+1})^{\rho_B-1}}\frac{(j_{t+1}(v_{B,t+1}|x_{t+1},\sigma_{t+1},c_{A,t+1}))^{\alpha_A-\rho_A}}{v_{B,t+1}^{\alpha_B-\rho_B}}=k_t\frac{\beta_B}{\beta_A}\frac{c_{A,t}^{\rho_A-1}}{(1-c_{A,t})^{\rho_B-1}}\left(e^{\mu+x_t+\sigma_t\varepsilon_{t+1}}\right)^{\alpha_B-\alpha_A} \tag{24}$$

It is evident that the evolution of the consumption share from time t to $t+1$ depends on k_t. As Collin-Dufresne, Johannes, and Lochstoer (2015) show, k_t uniquely determines $c_{A,t+1}$ since $c_{A,t+1}\in(0,1)$ is decreasing in k_t (when $\alpha_i-1<0$, which is the case we consider). However, from equation (23) we only know k_t as a function of $c_{A,t+1}$. Thus, we can solve equations (23) and (24) jointly for k_t as a fixed point problem. Once we know the solution for k_t we can find the corresponding $c_{A,t+1}$ for each combination of the grid points of the endogenous and exogenous state variables.

It is important to point out that $c_{A,t+1}$ and $v_{B,t+1}$ will depend on the exogenous evolution of C_{t+1} and thus on the shock to consumption growth rate ε_{t+1}. Therefore, I choose 5 different grid points for ε_{t+1}, and for each combination of state variables I solve equations (23) and (24) for each of the values of this shock.

As already mentioned, due to the backwards recursion method I use in order to solve the problem numerically, the optimal j_{t+1}^* and $v_{B,t+1}^*$ will be known from the previous recursion on a grid of different possible values of the state variables $c_{A,t+1}$, x_{t+1}, and σ_{t+1}. Since x_{t+1}, and σ_{t+1} depend on the shocks e_{t+1} and ω_{t+1} I choose 5 different grid points for each of these shocks. Thus, at time t I interpolate both $v^*_{A,t+1}$ and $v^*_{B,t+1}$ for the values of the optimal $c^*_{A,t+1}$ that I solve for and the values of x_{t+1} and σ_{t+1}, as functions of the grids that I use for the state variables x_t and σ_t and the shocks e_{t+1} and ω_{t+1}:

$$x_{t+1} = \rho_x x_t + \varphi_e \sigma_t e_{t+1} \tag{25}$$

$$\sigma^2_{t+1} = \max\ [\sigma^2 + \rho_\sigma(\sigma^2_t - \sigma^2) + \sigma_\omega \omega_{t+1},\ \underline{\upsilon}] \tag{26}$$

The grids for the exogenous, x_t, σ^2_t, and endogenous $c_{A,t}$ state variables are chosen as follows. I use 6 grids for both x_t and σ^2_t and spline interpolation between the grid points which range in the intervals:

$$\sigma^2 = \left[\underline{\upsilon},\ \sigma^2 + 4\sigma_\omega / \sqrt{1-\rho^2_\sigma}\right] \tag{27}$$

$$x = \left[-2\varphi_e \sqrt{\left(\sigma^2 + 4\sigma_\omega/\sqrt{1-\rho^2_\sigma}\right)/(1-\rho^2_x)},\ 2\varphi_e \sqrt{\left(\sigma^2 + 4\sigma_\omega/\sqrt{1-\rho^2_\sigma}\right)/(1-\rho^2_x)}\right]. \tag{28}$$

I use 25 grid points for the endogenous state variable c_A:

$$c_A = [0.0001, 0.9999]. \tag{29}$$

The shocks ε, e, and ω are approximated using Gaussian quadrature. The terminal time T is set far in the future at 500 years. I use the model parameters estimated by Bansal and Yaron (2004) and set $\mu = \mu_d = 0.0015$, $\rho_x = 0.979$, $\varphi_e = 0.044$, $\sigma = 0.0078$, $\rho_\sigma = 0.987$, $\sigma_\omega = 2.3e^{-6}$, $\phi_x = 3$, and $\varphi_d = 4.5$. I assume that the risk aversion level is $\gamma_i = 10$ and thus the risk aversion parameter equals $\alpha_i = -9$. The elasticity of intertemporal substitution is set to $\psi_i = 1.5$ and the EIS parameter is $\rho_i = 1/3$. The time discount factor is set to $\beta_i = 0.998$. I also allow the preference parameters of the two investors to differ and provide sensitivity analysis in order to understand the role of the agents' preferences for the relation between investment horizon and risk premia.

At the end of time t the problem is solved and we know the optimal $c^*_{A,t+1}$ and the utilities j^*_t and $v^*_{B,t}$ on a grid for the state variables $v_{B,t}$, x_t, and σ_t, and the shock ε_{t+1}:

$$j^*_{A,t} = \frac{J^*_{A,t}}{C_t} = j^*_{A,t}(c_{A,t}, x_t, \sigma_t, \varepsilon_{t+1}), \tag{30}$$

$$v^*_{B,t} = \frac{V^*_{B,t}}{C_t} = v^*_{B,t}(c_{A,t}, x_t, \sigma_t, \varepsilon_{t+1}). \tag{31}$$

For the following recursion at time $t-1$ we interpolate their values corresponding to the grids of the state variables $c_{A,t-1}$, x_{t-1}, and σ_{t-1}, and the shock ε_t and we solve the problem for the decision variable $c_{A,t}$.

4.3 Time $t = 0$

At time 0 the equilibrium condition can be written as follows:

$$\beta_A \left(\frac{c_{A,1}}{c_{A,0}}\right)^{\rho_A - 1} \left(e^{\mu + x_0 + \sigma_0 \varepsilon_1}\right)^{\alpha_A - 1} \left(\frac{j_1(v_{B,1}|x_1, \sigma_1, c_1)}{E_1[j_1(v_{B,1}|x_1, \sigma_1, c_1)^{\alpha_A} e^{(\mu + x_0 + \sigma_0\varepsilon_1)\alpha_A}]^{\frac{1}{\alpha_A}}}\right)^{\alpha_A - \rho_A} =$$
$$= \beta_B \left(\frac{1 - c_{A,1}}{1 - c_{A,0}}\right)^{\rho_B - 1} \left(e^{\mu + x_0 + \sigma_0 \varepsilon_1}\right)^{\alpha_B - 1} \left(\frac{v_{B,1}}{E_0[(v_{B,1}\,(\varepsilon_1, v_{B,0}|x_0, \sigma_0, c_0))^{\alpha_B} e^{(\mu + x_0 + \sigma_0\varepsilon_1)\alpha_B}]^{\frac{1}{\alpha_B}}}\right)^{\alpha_B - \rho_B} \tag{32}$$

In this period I solve for the optimal $c^*_{A,1}$ (and $c^*_{B,1} = 1 - c^*_{A,1}$) and I determine the optimal utilities $j^*_{A,0}$ and $v^*_{B,0}$ of the two agents on a grid of the state variables x_0, σ_0, and $c_{A,0}$. Due to the backwards recursion we assume that we know the optimal value functions at time $t + 1$, $j^*_{A,1}$ and $v^*_{B,1}$, and thus, the initial utilities of the two agents will depend on the promised utility by agent A to agent B, $v_{B,1}$. It is important to note that we can determine these initial utilities and the initial endowments of the two agents, $c_{A,0}$ and $c_{B,0}$, given the equilirbia for all the periods ahead until the terminal date T, but we cannot find the possible equilibria knowing only the initial endowments.

The estimations using the described numerical method will show the consumption distribution of the two agents over time and will determine the optimal investment strategies of the two agents. Analyzing the conditions under which one of the two agents is allocated a larger consumption share and dominates the economy will shed light on the market interaction between the investors and their hedge demands. The risk premia they require and the fraction of the market they dominate can have important implications for the resulting risk premia in the market.

5. Heterogeneous investment horizons and long-run risk

In this section I consider a Bansal and Yaron (2004) economy where the dynamics of the log-dividend $g_{d,t+1}$ and log-consumption g_{t+1} growth rates contain a persistent and predictable component x_t and are determined as follows:

$$\begin{aligned} g_{t+1} &= \ln\left(\frac{C_{t+1}}{C_t}\right) = \mu + x_t + \sigma_t \varepsilon_{t+1} \\ x_{t+1} &= \rho_x x_t + \varphi_e \sigma_t e_{t+1} \\ \sigma^2_{t+1} &= \max\,[\sigma^2 + \rho_\sigma(\sigma^2_t - \sigma^2) + \sigma_\omega \omega_{t+1},\ \underline{v}] \\ g_{d,t+1} &= \mu_d + \phi_x x_t + \varphi_d \sigma_t u_{t+1}. \end{aligned} \tag{33}$$

The shocks ε_{t+1}, e_{t+1}, ω_{t+1}, and u_{t+1} are iid, mutually independent and standard normally distributed. The parameters $\varphi_d > 1$ and $\phi_x > 1$ allow for calibration of the dividend volatility and its correlation with consumption. As in Abel (1999) ϕ_x represents the leverage ratio on expected consumption growth.

The equation for dividend growth will not be of importance to the analysis in this paper. The time-varying economic uncertainty in consumption growth rate is given by σ_{t+1} and it is required to have a minimum value of $\underline{\upsilon} = e^{-8}$ as in Collin-Dufresne, Johannes, and Lochstoer (2015). Thus, the economy is Markovian and the distribution of aggregate consumption C_{t+1} conditionally on the information at time t depends on a vector of exogenously simulated state variables $X_t = [x_t \;\; \sigma_t^2]'$. The two competitive agents with different investment horizons agree on the aggregate endowment process.

The solution of the model will show the consumption distribution between the agents over time and will determine their optimal investment strategies. Analyzing the conditions under which one of the two types of agents is allocated a larger consumption share and dominates the economy will shed light on the market interaction between the investors and the resulting risk premia on the market. As a more realistic model with long-run risk that takes into account the differences in investment horizons, this model also has implications about the equity term structure at different horizons.

There are two alternative channels that could affect the wealth distribution between agents – risk premium and saving. Under the risk premium channel short-term agents who are less averse to long-run risks are expected to sell insurance to long-term investors. Thus, compared to long horizon investors, short-term investors will invest more in the risky assets than in the risk-free asset. As a result, they will benefit more from the higher risk premia of these assets and will accumulate larger wealth. Since short-term investors care less about long-run risks (that help in resolving the equity premium puzzle), the importance of these shocks for pricing assets drops and hence, the model generated risk premium decreases. On the other hand, however, long-term agents who care about long-run risk more may choose a higher saving rate and accumulate more wealth, driving risk premia up. Thus, determining which channel prevails will have important implications about wealth distribution and equilibrium asset prices.

6. Conclusion

This paper studies the role of heterogeneous investment horizons for the risk sharing and consumption allocation among agents with nonseparable recursive preferences, and the equilibrium asset prices. Thus, I focus on a novel heterogeneity in models with recursive preference and its effect on wealth distribution and asset prices. Constructing models with short-term and long-term agents that feature iid shocks to consumption growth rate, I find analytically that the long-term agents outcrowd short-term myopic agents in terms of wealth share. The investment behavior of the myopic agent resembles that of an impatient agent or one that prefers early resolution of uncertainty. In the presence of long-run risk, however, short-term agents may dominate, which would lead to a drop in the share of investors who are concerned about and price long-run risk. As long-run risk becomes less important for pricing assets, the equilibrium risk premium would drop.

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A Pareto Problem and Equilibrium Derivation

The two-agent Pareto problem can be represented as the optimization of a social planner who maximizes the weighted sum of utilities of the investors of both types at time $t = 0$ subject to the market clearing condition:

$$\max_{\{C_{A,t},C_{B,t}\}_{t=0}^{T}} w_0 V_{A,0} + (1 - w_0) V_{B,0} \tag{34}$$
$$\text{s.t. } C_{A,t} + C_{B,t} = C_t \text{ for all states and time.}$$

Even though the individual utility functions are recursive, the social planner utility is not recursive. However, as shown by Lucas and Stokey (1984), Kan (1995), and Backus, Routledge, and Zin (2009) a recursive formulation exists. Applying Theorem 3 from Lucas and Stockey (1984) it follows that the Pareto optimal allocation is given by the following Bellman equation:

$$J(C_t, V_{B,t}) = \max_{\{C_{A,t},V_{B,t+1}\}} \left[(1 - \beta_A) C_{A,t}^{\rho_A} + \beta_A E_t[J(C_{t+1}, V_{B,t+1})^{\alpha_A}]^{\frac{\rho_A}{\alpha_A}}\right]^{\frac{1}{\rho_A}} \tag{35}$$
$$\text{s.t. } V_{B,t}(C_{B,t}, V_{B,t+1}) \geq V_{B,t} \tag{36}$$
$$C_{A,t} + C_{B,t} = C_t, \tag{37}$$

where $V_{B,t}$ is the so-called promised utility to agent B at time t. The resulting value function for agent A is then given by $V_{A,t} = J(C_t, V_{B,t})$. Since there is monotonicity in preferences, the utility-promise constraint is binding and hence, constraint (36) can be replaced by $V_{B,t}(C_{B,t}, V_{B,t+1}) = V_{B,t}$.

The problem of choosing a feasible allocation between the two agents can be viewed as a problem of maximizing the utility of agent A at time t over her own consumption $C_{A,t}$ and the promised utility to agent B at time $t+1$, $V_{B,t+1}$, that is the aggregate utility over the remaining horizon that agent A promises to agent B. Agent A can increase her consumption at time t up to the point where the utility of agent B at time t does not fall below the promised utility at time t, $V_{B,t}$. Thus, agent A can either choose to have higher consumption $C_{A,t}$ at time t and lower utility $V_{A,t+1}$ at time $t+1$ by promising higher utility $V_{B,t+1}$ to agent B at time $t+1$, or alternatively agent A can choose to have lower consumption $C_{A,t}$ at time t and higher utility $V_{A,t+1}$ at time $t+1$ by promising lower utility $V_{B,t+1}$ to agent B at time $t+1$. The promised utility $V_{B,t+1}$ that was chosen at time t will serve as a constraint for the minimum utility agent B will receive at time $t+1$, and so on until the terminal date T. In the optimization problem (35), C_t is an exogenous and $V_{B,t}$ is an endogenous state variable.

Given a Pareto-optimal allocation, we can find the equilibrium prices and hence, all competitive equilibria can be determined. However, we can only estimate the initial endowments given an equilibrium condition, but we cannot find the equilibrium given the initial endowments (Lucas and Stokey, 1984). The reason is that the current period value functions depend on the future value functions and consumption allocations that are unknown. The initial endowments of the two agents are implicitly determined by the utility V_B that agent A promises to agent B.

To derive the equilibrium condition we formulate and maximize the Lagrangian on the constraints in equation (35):

$$\mathcal{L} = \max_{\{C_{A,t}, V_{B,t+1}, \lambda_t\}} \Big[(1-\beta_A)C_{A,t}^{\rho_A} + \beta_A E_t[J(C_{t+1}, V_{B,t+1})^{\alpha_A}]^{\frac{\rho_A}{\alpha_A}}\Big]^{\frac{1}{\rho_A}} + \lambda_t\bigg(\Big[(1-\beta_B)C_{B,t}^{\rho_B} + \beta_B E_t[V_{B,t+1}^{\alpha_B}]^{\frac{\rho_B}{\alpha_B}}\Big]^{\frac{1}{\rho_B}} - V_{B,t}\bigg) \tag{38}$$

where λ_t is the Lagrange multiplier. Now, given the state variables C_t and $V_{B,t}$, we find the decision variables $C_{A,t}$ and $V_{B,t+1}$ such that the first-order conditions are satisfied:

$$\frac{\partial \mathcal{L}}{\partial C_{A,t}} = \frac{1}{\rho_A} J(C_t, V_{B,t})^{1-\rho_A} \rho_A (1-\beta_A) C_{A,t}^{\rho_A - 1} - \lambda_t \frac{1}{\rho_B} V_{B,t}^{1-\rho_B} \rho_B (1-\beta_B) C_{B,t}^{\rho_B - 1} = 0 \tag{39}$$

$$\begin{aligned}\frac{\partial \mathcal{L}}{\partial V_{B,t+1}} &= \frac{1}{\rho_A} J(C_t, V_{B,t})^{1-\rho_A} \beta_A \frac{\rho_A}{\alpha_A}\Big(E_t[J(C_{t+1}, V_{B,t+1})^{\alpha_A}]^{\frac{1}{\alpha_A}}\Big)^{\rho_A - \alpha_A} \alpha_A J(C_{t+1}, V_{B,t+1})^{\alpha_A - 1}\Big(\frac{\partial J_{t+1}}{\partial V_{B,t+1}}\Big) \\ &\quad + \lambda_t \frac{1}{\rho_B} V_{B,t}^{1-\rho_B} \beta_B \frac{\rho_B}{\alpha_B}\Big(E_t[V_{B,t+1}^{\alpha_B}]^{\frac{1}{\alpha_B}}\Big)^{\rho_B - \alpha_B} \alpha_B V_{B,t+1}^{\alpha_B - 1} = 0\end{aligned} \tag{40}$$

$$\frac{\partial \mathcal{L}}{\partial \lambda_t} = \Big[(1-\beta_B)C_{B,t}^{\rho_B} + \beta_B E_t[V_{B,t+1}^{\alpha_B}]^{\frac{\rho_B}{\alpha_B}}\Big]^{\frac{1}{\rho_B}} - V_{B,t}, \tag{41}$$

where $\frac{\partial J_{t+1}}{\partial V_{B,t+1}} = -\lambda_{t+1}$, as the function J_{t+1} depends on the promised utility to agent B, $V_{B,t+1}$ only through constraint (36). Thus, changing $V_{B,t+1}$ by a certain amount leads to a decrease in J_{t+1} that equals that amount times λ_{t+1}.

From equation (39) we get:

$$\lambda_t = \frac{J(C_t, V_{B,t})^{1-\rho_A}(1-\beta_A)C_{A,t}^{\rho_A - 1}}{V_{B,t}^{1-\rho_B}(1-\beta_B)C_{B,t}^{\rho_B - 1}} \text{ at time t} \tag{42}$$

$$\lambda_{t+1} = \frac{J(C_{t+1}, V_{B,t+1})^{1-\rho_A}(1-\beta_A)C_{A,t+1}^{\rho_A - 1}}{V_{B,t+1}^{1-\rho_B}(1-\beta_B)C_{B,t+1}^{\rho_B - 1}} \text{ at time t+1.} \tag{43}$$

Then we simplify equation (40) and substitute equations (42) and (43) in it:

$$J(C_t, V_{B,t})^{1-\rho_A}\beta_A\Big(E_t[J(C_{t+1}, V_{B,t+1})^{\alpha_A}]^{\frac{1}{\alpha_A}}\Big)^{\rho_A-\alpha_A} J(C_{t+1}, V_{B,t+1})^{\alpha_A-1}(-\lambda_{t+1})+$$
$$+\lambda_t V_{B,t}^{1-\rho_B}\beta_B\Big(E_t[V_{B,t+1}^{\alpha_B}]^{\frac{1}{\alpha_B}}\Big)^{\rho_B-\alpha_B} V_{B,t+1}^{\alpha_B-1}=0$$
$$J(C_t, V_{B,t})^{1-\rho_A}\beta_A\Big(E_t[J(C_{t+1}, V_{B,t+1})^{\alpha_A}]^{\frac{1}{\alpha_A}}\Big)^{\rho_A-\alpha_A} J(C_{t+1}, V_{B,t+1})^{\alpha_A-1}\frac{J(C_{t+1}, V_{B,t+1})^{1-\rho_A}(1-\beta_A)C_{A,t+1}^{\rho_A-1}}{V_{B,t+1}^{1-\rho_B}(1-\beta_B)C_{B,t+1}^{\rho_B-1}}=$$
$$=\frac{J(C_t, V_{B,t})^{1-\rho_A}(1-\beta_A)C_{A,t}^{\rho_A-1}}{V_{B,t}^{1-\rho_B}(1-\beta_B)C_{B,t}^{\rho_B-1}} V_{B,t}^{1-\rho_B}\beta_B\Big(E_t[V_{B,t+1}^{\alpha_B}]^{\frac{1}{\alpha_B}}\Big)^{\rho_B-\alpha_B} V_{B,t+1}^{\alpha_B-1}$$

$$\Leftrightarrow\ \beta_A\left(\frac{C_{A,t+1}}{C_{A,t}}\right)^{\rho_A-1}\left(\frac{J(C_{t+1}, V_{B,t+1})}{E_t[J(C_{t+1}, V_{B,t+1})^{\alpha_A}]^{1/\alpha_A}}\right)^{\alpha_A-\rho_A} = \beta_B\left(\frac{C_{B,t+1}}{C_{B,t}}\right)^{\rho_B-1}\left(\frac{V_{B,t+1}}{E_t[V_{B,t+1}^{\alpha_B}]^{1/\alpha_B}}\right)^{\alpha_B-\rho_B} \tag{44}$$

Equation (44) gives the equilibrium condition. I solve a normalized version of the model with all variables divided by aggregate consumption. I denote the value functions as $v_{i,t} = V_{i,t}/C_t$ and consumption shares as $c_{i,t} = C_{i,t}/C_t$. Hence, the equilibrium condition can be written as:

$$\beta_A\left(\frac{c_{A,t+1}C_{t+1}}{c_{A,t}C_t}\right)^{\rho_A-1}\left(\frac{j(v_{B,t+1})C_{t+1}}{E_t[j(v_{B,t+1})^{\alpha_A}C_{t+1}^{\alpha_A}]^{1/\alpha_A}}\right)^{\alpha_A-\rho_A} = \beta_B\left(\frac{c_{B,t+1}C_{t+1}}{c_{B,t}C_t}\right)^{\rho_B-1}\left(\frac{v_{B,t+1}C_{t+1}}{E_t[v_{B,t+1}^{\alpha_B}C_{t+1}^{\alpha_B}]^{1/\alpha_B}}\right)^{\alpha_B-\rho_B} \tag{45}$$

$$\Leftrightarrow \beta_A\left(\frac{c_{A,t+1}C_{t+1}}{c_{A,t}C_t}\right)^{\rho_A-1}\left(\frac{j(v_{B,t+1})C_{t+1}}{E_t[j(v_{B,t+1})^{\alpha_A}C_{t+1}^{\alpha_A}]^{1/\alpha_A}}\right)^{\alpha_A-\rho_A}\frac{C_t^{\alpha_A-\rho_A}}{C_t^{\alpha_A-\rho_A}} =$$
$$= \beta_B\left(\frac{c_{B,t+1}C_{t+1}}{c_{B,t}C_t}\right)^{\rho_B-1}\left(\frac{v_{B,t+1}C_{t+1}}{E_t[v_{B,t+1}^{\alpha_B}C_{t+1}^{\alpha_B}]^{1/\alpha_B}}\right)^{\alpha_B-\rho_B}\frac{C_t^{\alpha_B-\rho_B}}{C_t^{\alpha_B-\rho_B}} \tag{46}$$

$$\Leftrightarrow \beta_A\left(\frac{c_{A,t+1}}{c_{A,t}}\right)^{\rho_A-1}\left(\frac{C_{t+1}}{C_t}\right)^{\rho_A-1}\left(\frac{C_{t+1}}{C_t}\right)^{\alpha_A-\rho_A}\left(\frac{j(v_{B,t+1})}{E_t[j(v_{B,t+1})^{\alpha_A}(C_{t+1}/C_t)^{\alpha_A}]^{1/\alpha_A}}\right)^{\alpha_A-\rho_A} =$$
$$= \beta_B\left(\frac{c_{B,t+1}}{c_{B,t}}\right)^{\rho_B-1}\left(\frac{C_{t+1}}{C_t}\right)^{\rho_B-1}\left(\frac{C_{t+1}}{C_t}\right)^{\alpha_B-\rho_B}\left(\frac{v_{B,t+1}}{E_t[v_{B,t+1}^{\alpha_B}(C_{t+1}/C_t)^{\alpha_B}]^{1/\alpha_B}}\right)^{\alpha_B-\rho_B} \tag{47}$$

$$\Leftrightarrow \beta_A\left(\frac{c_{A,t+1}}{c_{A,t}}\right)^{\rho_A-1}\left(\frac{C_{t+1}}{C_t}\right)^{\alpha_A-1}\left(\frac{j(v_{B,t+1})}{E_t[j(v_{B,t+1})^{\alpha_A}(C_{t+1}/C_t)^{\alpha_A}]^{1/\alpha_A}}\right)^{\alpha_A-\rho_A} =$$
$$= \beta_B\left(\frac{c_{B,t+1}}{c_{B,t}}\right)^{\rho_B-1}\left(\frac{C_{t+1}}{C_t}\right)^{\alpha_B-1}\left(\frac{v_{B,t+1}}{E_t[v_{B,t+1}^{\alpha_B}(C_{t+1}/C_t)^{\alpha_B}]^{1/\alpha_B}}\right)^{\alpha_B-\rho_B} \tag{48}$$

B Proofs of analytical results

Theorem 1. *Suppose agent A has Epstein-Zin preferences, agent B has CRRA preferences and both agents have equal risk aversion and time discounting parameters. Then, the consumption sharing between them is a deterministic function of time, that is, the consumption share does not depend on any of the past shocks to consumption growth rate. The value functions are also deterministic. In particular, the following condition holds:* $\frac{\left(\frac{c_{A,t+1}}{c_{A,t}}\right)^{\rho_A-1}}{\left(\frac{1-c_{A,t+1}}{1-c_{A,t}}\right)^{\alpha-1}} = e^{\mu+\frac{1}{2}\alpha\sigma^2}$*, where* $e^{\mu+\frac{1}{2}\alpha\sigma^2}$ *is constant. In case* $\mu > -\frac{1}{2}\alpha\sigma^2$*, the consumption share of the investor with Epstein-Zin preferences increases over time, and* $\lim_{t\to\infty} c_{A,t} = 1$ *such that she outcrowds the CRRA agent. In case* $\mu < -\frac{1}{2}\alpha\sigma^2$*, the consumption share of the investor with Epstein-Zin preferences decreases over time.*

Proof.

$$\beta_A\left(\frac{c_{A,t+1}}{c_{A,t}}\right)^{\rho_A-1}\left(e^{\mu+\sigma\varepsilon_{t+1}}\right)^{\alpha_A-1}\left(\frac{j_{t+1}(v_{B,t+1}|\sigma, c_{A,t+1})}{E_t[(j_{t+1}(v_{B,t+1}|\sigma, c_{A,t+1}))^{\alpha_A} e^{(\mu+\sigma\varepsilon_{t+1})\alpha_A}]^{\frac{1}{\alpha_A}}}\right)^{\alpha_A-\rho_A} =$$
$$\beta_B\left(\frac{1-c_{A,t+1}}{1-c_{A,t}}\right)^{\rho_B-1}\left(e^{\mu+\sigma\varepsilon_{t+1}}\right)^{\alpha_B-1}$$

Since $\alpha_A = \alpha_B$, the consumption sharing does not depend on the shock to consumption growth rate ε_{t+1}, i.e. the consumption share of agent A, $c_{A,t}$ is a function of the past consumption share $c_{A,t-1}$: $c_{A,t} = c_{A,t}(c_{A,t-1})$ only. Using backwards induction we show that $c_{A,t} = E_{t-1}[c_{A,t}]$ and $j_t = E_{t-1}[j_t]$ at any time t.

At time T $j_T = \left((1-\beta_A)c_{A,T}^{\rho_A}\right)^{\frac{1}{\rho_A}}$. Hence, since $c_{A,T} = E_{T-1}[c_{A,T}]$, the value function of the agent with Epstein-Zin preferences is deterministic and $j_T = E_{T-1}[j_T]$.

At time $T-1$, we assume that $c_{A,T-1}$ and j_{T-1} are deterministic, and hence $c_{A,T-1} = E_{T-2}[c_{A,T-1}]$ and $j_{T-1} = E_{T-2}[j_{T-1}]$. Thus, the equilibrium condition at time $T-2$ takes the following form:

$$\beta_A\left(\frac{c_{A,T-1}}{c_{A,T-2}}\right)^{\rho_A-1}\left(e^{\mu+\sigma\varepsilon_{T-1}}\right)^{\alpha_A-1}\left(\frac{j_{T-1}(v_{B,T-1}|\sigma, c_{A,T-1})}{E_{T-2}[(j_{T-1}(v_{B,T-1}|\sigma, c_{A,T-1}))^{\alpha_A} e^{(\mu+\sigma\varepsilon_{T-1})\alpha_A}]^{\frac{1}{\alpha_A}}}\right)^{\alpha_A-\rho_A} =$$
$$\beta_B\left(\frac{1-c_{A,T-1}}{1-c_{A,T-2}}\right)^{\rho_B-1}\left(e^{\mu+\sigma\varepsilon_{T-1}}\right)^{\alpha_B-1}$$
$$\Rightarrow \beta_A\left(\frac{c_{A,T-1}}{c_{A,T-2}}\right)^{\rho_A-1}\left(\frac{j_{T-1}(v_{B,T-1}|\sigma, c_{A,T-1})}{j_{T-1}(v_{B,T-1}|\sigma, c_{A,T-1})E_{T-2}[e^{(\mu+\sigma\varepsilon_{T-1})\alpha_A}]^{\frac{1}{\alpha_A}}}\right)^{\alpha_A-\rho_A} = \beta_B\left(\frac{1-c_{A,T-1}}{1-c_{A,T-2}}\right)^{\rho_B-1} \tag{49}$$

Then, we can conclude that $c_{A,T-2}$ is deterministic and $c_{A,T-2} = E_{T-3}[c_{A,T-2}]$, as it does not depend on any random shocks. To check whether j_{T-2} is deterministic we substitute $c_{A,T-2}$ and j_{T-1}

in $E_{T-3}[j_{T-2}]$:

$$E_{T-3}[j_{T-2}] = E_{T-3}\left[\left(\beta_A c_{A,T-2}^{\rho_A} + (1-\beta_A)E_{T-2}[j_{T-1}^{\alpha_A}]^{\frac{\rho_A}{\alpha_A}}\right)^{\frac{1}{\rho_A}}\right] =$$

$$= \left(\beta_A c_{A,T-2}^{\rho_A} + (1-\beta_A)E_{T-2}[j_{T-1}^{\alpha_A}]^{\frac{\rho_A}{\alpha_A}}\right)^{\frac{1}{\rho_A}} = j_{T-2}. \tag{50}$$

Thus, if we assume that $c_{A,T-1}$ and j_{T-1} are deterministic, it follows that $c_{A,T-2}$ and j_{T-2} are also deterministic. Therefore, by backwards induction we can conclude that $c_{A,t}$ and j_t are deterministic at any time t and we get that:

$$\left(\frac{c_{A,t+1}}{c_{A,t}}\right)^{\rho_A-1}\left(\frac{j_{t+1}(v_{B,t+1}|\sigma, c_{A,t})}{(j_{t+1}(v_{B,t+1}|\sigma, c_{A,t+1}))E_t[e^{(\mu+\sigma\varepsilon_{t+1})\alpha}]^{\frac{1}{\alpha}}}\right)^{\alpha-\rho_A} = \left(\frac{1-c_{A,t+1}}{1-c_{A,t}}\right)^{\rho_B-1}$$

$$\Rightarrow \frac{\left(\frac{c_{A,t+1}}{c_{A,t}}\right)^{\rho_A-1}}{\left(\frac{1-c_{A,t+1}}{1-c_{A,t}}\right)^{\alpha-1}} = E_t[e^{(\mu+\sigma\varepsilon_{t+1})\alpha}]^{\frac{\alpha-\rho_A}{\alpha}} = [e^{\alpha\mu+\frac{1}{2}\alpha^2\sigma^2}]^{\frac{\alpha-\rho_A}{\alpha}} = [e^{\mu+\frac{1}{2}\alpha\sigma^2}]^{\alpha-\rho_A}.$$

Hence, the ratio of the changes in consumption shares of the two agents stays constant over time.

Case 1: $e^{\mu+\frac{1}{2}\alpha\sigma^2} > 1 \Rightarrow \mu + \frac{1}{2}\alpha\sigma^2 > 0 \Rightarrow \mu > -\frac{1}{2}\alpha\sigma^2$

Since $e^{\mu+\frac{1}{2}\alpha\sigma^2} > 1$ and $\alpha - \rho_A < 0 \Rightarrow [e^{\mu+\frac{1}{2}\alpha\sigma^2}]^{\alpha-\rho_A} < 1 \Rightarrow$

$$\frac{c_{A,t+1}}{c_{A,t}}^{\rho_A-1} < \frac{1-c_{A,t+1}}{1-c_{A,t}}^{\alpha-1}$$

We have $\rho_A - 1 < 0,\ \alpha - 1 < 0,$ and $\alpha - 1 < \rho_A - 1 \Rightarrow$

$$\frac{c_{A,t+1}}{c_{A,t}} > \frac{1-c_{A,t+1}}{1-c_{A,t}}$$

$$c_{A,t+1} - c_{A,t}c_{A,t+1} > c_{A,t} - c_{A,t}c_{A,t+1}$$

$\Rightarrow c_{A,t+1} > c_{A,t}$ and $c_{B,t+1} < c_{B,t}$

Case 2: $e^{\mu+\frac{1}{2}\alpha\sigma^2} < 1 \Rightarrow \mu + \frac{1}{2}\alpha\sigma^2 < 0 \Rightarrow \mu < -\frac{1}{2}\alpha\sigma^2$

Since $e^{\mu+\frac{1}{2}\alpha\sigma^2} < 1$ and $\alpha - \rho_A < 0 \Rightarrow [e^{\mu+\frac{1}{2}\alpha\sigma^2}]^{\alpha-\rho_A} > 1 \Rightarrow$

$$\frac{c_{A,t+1}}{c_{A,t}}^{\rho_A-1} > \frac{1-c_{A,t+1}}{1-c_{A,t}}^{\alpha-1}$$

We have $\rho_A - 1 < 0,\ \alpha - 1 < 0,$ and $\alpha - 1 < \rho_A - 1 \Rightarrow$

$$\frac{c_{A,t+1}}{c_{A,t}} < \frac{1-c_{A,t+1}}{1-c_{A,t}}$$

$$c_{A,t+1} - c_{A,t}c_{A,t+1} < c_{A,t} - c_{A,t}c_{A,t+1}$$

$\Rightarrow c_{A,t+1} < c_{A,t}$ and $c_{B,t+1} > c_{B,t}$

Thus, if $\mu > -\frac{1}{2}\alpha\sigma^2$ and $\alpha_A = \alpha_B$, the consumption sharing is deterministic and the investor with Epstein-Zin preferences accumulates larger share than the agent with CRRA preferences. If $\mu < -\frac{1}{2}\alpha\sigma^2$, the investor with CRRA preferences accumulates larger consumption share than the longer-term Epstein-Zin agent. Since Case 1 is the one which is satisfied using reasonable parameters (and using the calibration of Bansal and Yaron (2004)) and since CRRA preferences imply a shorter (myopic) investment horizon than Epstein-Zin preferences, a longer-term horizon investor will accumulate larger consumption share over time than a myopic investor.

□

Proposition 1. *If two agents have CRRA preferences and equal risk aversion and time discounting parameters, their consumption shares remain constant over time and the wealth distribution in the economy is determined by the initial endowments.*

Proof.

$$\begin{aligned}
\beta_A\left(\frac{c_{A,t+1}}{c_{A,t}}\right)^{\alpha_A-1}(e^{\mu+\sigma\varepsilon_{t+1}})^{\alpha_A-1} &= \beta_B\left(\frac{c_{B,t+1}}{c_{B,t}}\right)^{\alpha_B-1}(e^{\mu+\sigma\varepsilon_{t+1}})^{\alpha_B-1} \\
\beta\left(\frac{c_{A,t+1}}{c_{A,t}}\right)^{\alpha-1} &= \beta\left(\frac{1-c_{A,t+1}}{1-c_{A,t}}\right)^{\alpha-1} \\
c_{A,t+1} - c_{A,t}c_{A,t+1} &= c_{A,t} - c_{A,t}c_{A,t+1} \\
\Rightarrow c_{A,t} &= c_{A,t+1}
\end{aligned}$$

□

Proposition 2. *If two agents have CRRA preferences and equal risk aversion parameters but different time discounting parameters, the consumption share of the more patient agent (B) increases over time, such that she dominates over the less patient agent (A).*

Proof.

$$\beta_A \left(\frac{c_{A,t+1}}{c_{A,t}}\right)^{\alpha_A - 1} (e^{\mu+\sigma\varepsilon_{t+1}})^{\alpha_A - 1} = \beta_B \left(\frac{c_{B,t+1}}{c_{B,t}}\right)^{\alpha_B - 1} (e^{\mu+\sigma\varepsilon_{t+1}})^{\alpha_B - 1}, \beta_A < \beta_B$$

$$\frac{\left(\frac{c_{B,t+1}}{c_{B,t}}\right)^{\alpha-1}}{\left(\frac{c_{A,t+1}}{c_{A,t}}\right)^{\alpha-1}} = \frac{\beta_A}{\beta_B}$$

$$\frac{\left(\frac{1-c_{A,t+1}}{1-c_{A,t}}\right)}{\left(\frac{c_{A,t+1}}{c_{A,t}}\right)} = \left(\frac{\beta_A}{\beta_B}\right)^{\frac{1}{\alpha-1}} > 1$$

$$\frac{1-c_{A,t+1}}{1-c_{A,t}} > \frac{c_{A,t+1}}{c_{A,t}}$$

$$c_{A,t} - c_{A,t}c_{A,t+1} > c_{A,t+1} - c_{A,t}c_{A,t+1}$$

$$\Rightarrow c_{A,t+1} < c_{A,t}$$

Thus, if $\beta_A < \beta_B$, then $c_{A,t+1} < c_{A,t}$ and $c_{B,t+1} > c_{B,t}$, i.e. the consumption share of the more patient agent increases over time. In particular, we can show that the change in consumption share of the more patient investor is $\frac{\beta_A}{\beta_B}^{\frac{1}{\alpha-1}}$ times larger than the decrease in consumption share of the more impatient agent.

$$\left(\frac{1-c_{A,t+1}}{1-c_{A,t}}\right) = \left(\frac{\beta_A}{\beta_B}\right)^{\frac{1}{\alpha-1}} \left(\frac{c_{A,t+1}}{c_{A,t}}\right)$$

□

Proposition 3. *Suppose two agents have CRRA preferences and equal time discounting parameters but different risk aversion parameters. In case the realized shock in the economy is $\varepsilon_{t+1} > -\mu/\sigma$, the consumption share of the less risk averse agent (B) increases over time, such that she dominates over the more risk averse agent (A). In case the realization falls below the bound $\varepsilon_{t+1} < -\mu/\sigma$, the consumption share of the less risk averse agent decreases over time.*

Proof.

$$\beta_A \left(\frac{c_{A,t+1}}{c_{A,t}}\right)^{\alpha_A - 1} (e^{\mu+\sigma\varepsilon_{t+1}})^{\alpha_A - 1} = \beta_B \left(\frac{1-c_{B,t+1}}{c_{B,t}}\right)^{\alpha_B - 1} (e^{\mu+\sigma\varepsilon_{t+1}})^{\alpha_B - 1},\ \alpha_A < \alpha_B$$

$$\frac{\left(\frac{c_{A,t+1}}{c_{A,t}}\right)^{\alpha_A-1}}{\left(\frac{1-c_{A,t+1}}{1-c_{A,t}}\right)^{\alpha_B-1}} = (e^{\mu+\sigma\varepsilon_{t+1}})^{\alpha_B-\alpha_A}$$

Case 1: $e^{\mu+\sigma\varepsilon_{t+1}} > 1 \Rightarrow \mu + \sigma\varepsilon_{t+1} > 0 \Rightarrow \varepsilon_{t+1} > -\mu/\sigma$

$$\Rightarrow \frac{c_{A,t+1}}{c_{A,t}}^{\alpha_A - 1} > \frac{1 - c_{A,t+1}}{1 - c_{A,t}}^{\alpha_B - 1}$$

Since $\alpha_A - 1 < 0$, $\alpha_B - 1 < 0$, and $\alpha_A - 1 < \alpha_B - 1$ we need

$$\frac{c_{A,t+1}}{c_{A,t}} < \frac{1 - c_{A,t+1}}{1 - c_{A,t}}$$

$$c_{A,t+1} - c_{A,t}c_{A,t+1} < c_{A,t} - c_{A,t}c_{A,t+1}$$

$$\Rightarrow c_{A,t+1} < c_{A,t} \text{ and } c_{B,t+1} > c_{B,t}$$

Case 2: $e^{\mu+\sigma\varepsilon_{t+1}} < 1 \Rightarrow \mu + \sigma\varepsilon_{t+1} < 0 \Rightarrow \varepsilon_{t+1} < -\mu/\sigma$

$$\Rightarrow \frac{c_{A,t+1}}{c_{A,t}}^{\alpha_A - 1} < \frac{1 - c_{A,t+1}}{1 - c_{A,t}}^{\alpha_B - 1}$$

Since $\alpha_A - 1 < 0$, $\alpha_B - 1 < 0$, and $\alpha_A - 1 < \alpha_B - 1$ we need

$$\frac{c_{A,t+1}}{c_{A,t}} > \frac{1 - c_{A,t+1}}{1 - c_{A,t}}$$

$$c_{A,t+1} - c_{A,t}c_{A,t+1} > c_{A,t} - c_{A,t}c_{A,t+1}$$

$$\Rightarrow c_{A,t+1} > c_{A,t} \text{ and } c_{B,t+1} < c_{B,t}$$

Thus, in case $\varepsilon_{t+1} > -\mu/\sigma$ and $\alpha_A < \alpha_B$ and $\gamma_A > \gamma_B$, then $c_{B,t+1} > c_{B,t}$ i.e. the consumption share of the less risk averse agent increases over time. If $\varepsilon_{t+1} < -\mu/\sigma$, i.e. ε_{t+1} is a negative shock with a very negative realization, then the less risk averse agent incurs large losses and her consumption share decreases over time. □