# Assets' Dependence Structure Implications for Portfolio Insurance*

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**Abstract**

Portfolio insurance strategies that control benchmark-underperformance risk require estimating the maximum multiplier of the risk budget, which determines the allocation to the performance-seeking asset (PSA) at each point in time. We explore the implications of taking into account the expected co-movements of the PSA and the benchmark asset for the estimation of the multiplier of these portfolio insurance strategies. We illustrate these implications with a maximum relative-drawdown strategy investing in the equal-weighted S&P 500 index as the PSA and in the cap-weighted S&P 500 index as the benchmark asset. Through Monte Carlo simulations we find that the multiplier almost doubles in size across scenarios, and the long-term returns of the strategy using this approach are superior relative to the strategy with a multiplier that ignores expected co-movements according to stochastic dominance tests.

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# 1 Introduction

Cushion-based portfolio insurance strategies are popular among investment practitioners, and have gained increasing attention in the academic literature. For instance, management fees in hedge funds are often defined in terms of a *high water mark*, which implies that controlling drawdowns is crucial for professional asset managers. Indeed, Lan, Wang, and Yang (2013) indicate that this kind of strategies are optimal, if one assumes the fund is liquidated when the assets under management fall below a given fraction of its high water-mark (see also Goetzmann, Ingersoll, and Ross, 2003, for a related theoretical argument). Furthermore, Dichtl and Drobetz (2011) run Monte Carlo simulations for popular cushion-based portfolio insurance strategies and benchmark strategies and find that most portfolio insurance strategies are the preferred investment strategy for a prospect theory investor.

Cushion-based portfolio insurance strategies need reliable estimates of the upper bound of their multiplier parameter. Former research have provided several approaches to estimate the multiplier upper bound of portfolio insurance strategies using a locally riskless asset paying a constant interest rate as their reserve asset. However, strategies using locally risky reserve assets are more relevant in practical applications. This paper presents an estimation methodology of the multiplier upper bound that takes into account the dependence structure of the two assets composing the strategy, in the general case of a risky reserve asset.

Portfolio insurance strategies using a locally riskless asset paying a constant interest rate as their reserve asset (often approximated as cash) are unsuitable for investors with long horizons or long-term 'commitments', such as pension funds, because cash presents a large duration miss-match with their long-term liabilities. This kind of investors would rather use an insurance strategy in which the riskless asset is a liability hedging portfolio matching the pensions fund's liabilities. The reason for this is that losses in such context are measured in relative terms with respect to the present value of the pension fund's liabilities.

Another relevant application in which losses are measured in relative terms with respect to a benchmark, is investing in the so-called 'smart-beta' or alternative equity indices. Indeed, there is a very large and growing offer of exchange traded funds (ETFs) tracking 'alternative beta' indices (also called 'smart beta') that offer systematic diversification strategies different from the standard market-capitalization weighted indices (see Amenc, Goltz, and Lodh, 2012; Hsu, Chow, Kalesnik, and Little, 2011). These alternative indices tend to have higher returns in the long run than their benchmarks, but also represent significant deviations from the cap-weighted benchmark indices. Measuring and controlling the relative losses related to the risk of deviating from the benchmark is important for agents investing in the index trackers of alternative indices.

Cushion-based portfolio insurance strategies that can use locally risky assets different from cash, in which the riskless asset is a benchmark portfolio, include the Constant Proportion Portfolio Insurance (CPPI) introduced in Perold (1986), and Black and Jones (1987) (see also Black and Perold, 1992; Perold and Sharpe, 1995), the relative drawdown control (RDD) and the excess drawdown control strategies (EDD) discussed in detail in Mantilla-Garcia (2014). The latter two strategies are extensions of the maximum drawdown control strategy, also known as Time Invariant Portfolio Protection (see Estep and Kritzman, 1988; Grossman and Zhou, 1993; Cvitanic and Karatzas, 1995).

Most former studies on the estimation of the multiplier upper bound of this kind of strategies, such as Bertrand and Prigent (2002), Cont and Tankov (2009), Ben Ameur and Prigent (2013), Hamidi, Jurczenko, and Maillet (2009), Hamidi, Maillet, and Prigent (2008, 2009a, 2014) assume that the reserve asset is a locally riskless asset paying a constant rate of return. In such case there is no dependence with the risky asset (since the correlation with a constant is null). This paper presents a formula of the multiplier for portfolio insurance strategies in the general case with a locally risky reserve or benchmark asset, and an estimation methodology that takes into account the dependence structure of the two assets composing the strategy. We find that ignoring the dependence structure of the two assets can potentially imply large underestimations of the multiplier upper bound and hence a significant opportunity cost related to the underspending of the risk budget of the portfolio insurance strategy.

## 2 Portfolio Insurance Multiplier of Dependent Assets

The CPPI, RDD, and EDD risk-control strategies aim to *insure* that the portfolio respects a given performance constraint by following an asset allocation rule that prevents the value of the portfolio, denoted hereafter $A$, to fall below a Floor value $F$ at all times. In order to achieve so, this kind of strategies dynamically allocate wealth to a risky performance-seeking asset $S$ at every time $t$ equal to,

$$\omega_S(t) = m_t \times (A(t) - F(t)), \quad m_t > 0, \tag{1}$$

while the remaining wealth is invested in the reserve or benchmark asset $B$. The pair, reserve asset and floor type, must be chosen such that the reserve asset super-replicates the Floor value process. Whenever $A$ approaches $F$, caused by underperformance of the risky asset, wealth is reallocated towards the reserve asset to maintain a positive *cushion*, i.e. $C(t) = A(t) - F(t) > 0$, at all times. The multiplier parameter that determines the risk exposure of the strategy per unit of available cushion may be a positive constant or an adapted non-negative time-varying process.

While the allocation of type (1) strategies changes continuously over time in theory, trading only happens in discrete time. Hence, in order to set weights back to the current target, reallocations are triggered whenever the actual portfolio exposure and its target drift apart beyond a given percentage. More precisely, let the *implied* multiplier of type (1) strategies be defined as

$$\tilde{m}_t := \frac{\omega_S(t)}{c(t)} \tag{2}$$

where $c(t) := 1 - \frac{F(t)}{A(t)}$. No trading takes place whenever the implied multiplier is inside the no-trading band, $\tilde{m}_t \in (m_t(1-\tau), m_t(1+\tau))$, and reallocations are *triggered* every time $\tilde{m}_t$ exits the no-trading band. In our illustrations below, we set $\tau = 0.2$ as in Hamidi, Maillet, and Prigent (2009b), and we enforce a no-leverage upper bound on $\omega_S(t)$ i.e., $\omega_S(t) = \min\left\{1, m_t \times \left(1 - \frac{F(t)}{A(t)}\right)\right\}$. In our tests, we assume 3 basis points (bps) of trading costs per share (a reasonable figure given the high liquidity of S&P500 ETFs), in order to account for the impact that trading has in the returns of these dynamic strategies.

Assuming continuous-time trading and prices, type (1) portfolio insurance strategies with properly defined Floors satisfy $C_t > 0$ at all $t$ for all possible values of $m$. However, in practice trading can only happen in discrete time and asset prices present "jumps". Thus, it is important to determine the maximum value of $\tilde{m}_t$ that would allow the Cushion to remain positive even if the worst possible scenario happens between $t$ and $t+1$, before the portfolio manager can reallocate assets.

Most former studies about the estimation of the multiplier upper bound such as Bertrand and Prigent (2002), Cont and Tankov (2009), Ben Ameur and Prigent (2013), Hamidi et al. (2009), Hamidi et al. (2008, 2009a, 2014) assume that $B$ is a locally riskless asset paying a constant rate of return considered "relatively small" compared to the worst possible loss in the risky asset. In that particular case, neglecting the return of $B$, the upper bound of the multiplier is:

$$m_t \leq \mathbb{M}_t := \frac{-1}{r_S(t, t+1)}, \tag{3}$$

for every $t$ and $t+1$ such that $r_S(t,t+1) < 0$, where $r_S(t,t+1) = \frac{S_{t+1}}{S_t} - 1$. In equation (3) the left tail of the risky asset is the only matter of concern to estimate the upper bound, hence an univariate approach is sufficient to estimate the multiplier upper bound.

Neglecting the potential impact of $r_B$ in the multiplier upper bound is a reasonable assumption in the particular case in which the reserve asset is cash, which is the case of the MDD strategy, as in Hamidi et al. (2014). However, in all applications of type (1) strategies in which the reserve asset $B$ is locally risky, neglecting the potential impact of $B$ in the multiplier upper bound is imprudent from a risk-management standpoint. For instance, in cases such as a Dynamic Core-Satellite (i.e., a CPPI strategy adapted to a relative risk control context) as in Amenc, Malaise, and Martellini (2004), an asset-liability management floor protection strategy as in Martellini and Milhau (2009), or a relative drawdown control relative to an equity benchmark as in Mantilla-Garcia (2014), the reserve asset $B$ can be a benchmark equity index, a portfolio of zero-coupon bonds with long maturity matching future cash-flow needs (i.e. a pension fund's liability), or a "safe enough" asset different from cash, such as a bonds portfolio with short maturity.

Hamidi et al. (2014) and Mantilla-Garcia (2014) showed that in the general case in which the reserve asset $B$ is stochastic, the discrete-time trading upper bound of the multiplier, $\mathbb{M}_t$, that can guarantee in general the Cushion's positivity condition is:

$$m_t \leq \mathbb{M}_t := \frac{-(1 + r_B(t, t+1))}{r_S(t, t+1) - r_B(t, t+1)}, \tag{4}$$

for every $t$ and $t+1$ at which the condition $(r_S(t, t+1) - r_B(t, t+1)) < 0$ is satisfied (there is no condition or upper bound otherwise). Mantilla-Garcia (2014) showed that, if $(r_S(t, t+1) - r_B(t, t+1)) < 0$, then[2]

$$\begin{cases} \frac{-1}{r_S(t,t+1) - r_B(t,t+1)} \leq \mathbb{M}_t, & \text{if } r_B(t, t+1) > 0 \\ \frac{-1}{r_S(t,t+1)} \leq \mathbb{M}_t, & \text{if } r_B(t, t+1) \leq 0 \end{cases}$$

Hence in general,

$$\overline{m}_t := \min\left\{ \frac{-1}{r_S(t, t+1)}, \frac{-1}{r_S(t, t+1) - r_B(t, t+1)} \right\} \leq \mathbb{M}_t. \tag{5}$$

Mantilla-Garcia (2014) used the following conservative estimate of $\overline{m}_t$ given by,

$$\hat{m}_t^u := \frac{-1}{LT_S(t) - RT_B(t)} \leq \frac{-1}{LT_S(t)} \leq \mathbb{M}_t, \tag{6}$$

where $LT_S(t) < 0$ and $RT_B(t) > 0$ are univariate estimates of the left and right tails of the return distributions of assets $S$ and $B$ at time $t$ respectively[3]. In the remainder of the paper, we refer to this estimate of the multiplier upper bound as the "Univariate Multiplier", as it does not take into account the dependence between the two assets of the portfolio insurance strategies. This multiplier constitutes the benchmark for the bivariate approach to estimate the multiplier that we propose hereafter.

[2]See Appendix in Mantilla-Garcia (2014) for a detailed proof.

[3]Notice that expressions (5) and (6) show that in cases in which the reserve asset $B$ is locally risky, the right tail of the return distribution of $B$ is also relevant to estimate the upper bound of the multiplier, as a sudden significant *increase* in its value may also cause a Floor violation. In other words, the right tail of the distribution of the reserve asset can be of critical importance for the estimation of the upper bound of the multiplier in several applications.

Notice that estimate (6) calculates $LT_S(t)$ and $RT_B(t)$ independently, ignoring the potential dependence between assets $S$ and $B$, which implicitly assumes that the most negative value of $r_S(t, t+1)$ would arrive at the same time as the largest value of $r_B(t, t+1)$. In the particular case in which $S$ and $B$ present a correlation $\rho_{SB}$ close to $-1$, such approach would not be too conservative, but for any other value of $\rho_{SB} >> -1$, this approach might be too conservative.

Thus, given inequality (5), we propose a more general method to estimate $\overline{m}_t$, as

$$\hat{m}_t = \min\left\{\frac{-1}{LT_S(t)}, \frac{-1}{LT_W(t)}\right\} \leq \mathbb{M}_t, \tag{7}$$

where $LT_W(t)$ denotes the left tail of the distribution of $W(t) = r_S(t, t+1) - r_B(t, t+1)$, in which the distribution of $W$ is a function of a general dependence structure between $S$ and $B$ modeled with a copula, and given marginal distribution functions.

# 3 An application to the Relative MaxDrawdown Strategy

Hereafter we consider a relevant application in practice of a portfolio insurance strategy in which the performance-seeking asset $S$ is an alternative equity index and the benchmark asset $B$ is a market cap-weighted equity index for the same universe of stocks. In such case, the correlation $\rho_{SB}$ is in fact very close to 1, and as a consequence the bivariate approach integrating the dependence between the two assets yields an upper bound estimate for the multiplier that is materially larger than the estimate using the conservative univariate approach (6) used in previous studies, which we take as the benchmark strategy.

A common practice to deal with benchmark underperformance risk is to impose *tracking error* constraints on the performance of the portfolio, which is measured as the standard deviation of the differences in periodic (say daily or monthly) returns. From the standpoint of investors, a more sensible measure of risk would be the cumulative underperformance with respect to the benchmark. In fact, if a manager accumulates too much relative losses with respect to the benchmark, in general there is no guarantee that the portfolio will be able to recover back the lost ground in terms of wealth with respect to its benchmark. Hence, limiting the relative underperformance at all times, can limit the potential regret of the investor. Indeed, severe underperformance relative to a given benchmark, such as standard equity market indices, is one of the main risks faced by portfolio managers and institutional investors seeking to outperform their benchmarks by investing in active investment strategies or 'alternative betas'.

Equal-weighted portfolios constitute the simplest alternative diversification form of indexation. DeMiguel, Garlappi, and Uppal (2009) report that equal-weighted portfolios significantly outperform the market cap-weighted portfolios over long periods of time for a universe of stocks covered by the S&P 500 index (they also find it performs better than portfolio optimization techniques that are prone to parameter estimation error). Hence, an investor seeking to outperform the benchmark S&P 500 cap-weighted index (CW) but with a predefined limit on its potential underperformance could implement a relative max-drawdown control strategy using the allocation formula (1) for portfolio insurance, using the S&P 500 equal-weighted index (EW) as the performance-seeking asset $S$ and the benchmark cap-weighted (CW) index as the reserve asset $B$.

Below we describe in detail the relative drawdown control strategy, which is equivalent to an absolute maximum drawdown control strategy, also known as TIPP, after a change of numeraire from dollars to shares of the benchmark asset. First we introduce some notation to define the notion of maximum drawdown. The maximum drawdown (MDD) of an investment is defined as the largest value loss from a *peak* to a *bottom* observed at current time $t$. More precisely, for a value process $A$, the drawdown at time $s$, denoted $D_s(A)$, is the percentage loss experienced by $A$ with respect to its *running maximum* observed since time $t_0$, denoted $M^A_{t_0}(s)$, attained for the last time at $s^\star_{t_0,A}$. Therefore, the maximum drawdown observed since inception $t_0 = 0$ to current time $t$, denoted by $\bar{D}_{t_0,t}(A)$, is defined as follows:

$$\bar{D}_{t_0,t}(A) := \sup_{t_0 \le s \le t} D_s(A) \tag{8}$$

$$\text{where } D_s(A) := -r^A\left(s^\star_{t_0,A}, s\right) \tag{9}$$

$$M^A_{t_0}(s) := \sup_{t_0 \le q \le s} \{A_q, M^A_{t_0}\} \tag{10}$$

$$\text{and } s^\star_{t_0,A} := \sup_{t_0 \le q \le s} \{q : A_q \ge M^A_{t_0}(s)\}, \tag{11}$$

where $r^A(t_1, t_2)$ denotes the return of the value process $A$ between the two instants $t_1$ and $t_2$, for any $t_0 \le t_1 \le t_2$. For simple returns, $r^A\left(s^\star_{t_0,A}, s\right) := \left(\frac{A(s)}{A(s^\star_{t_0,A})} - 1\right)$.

Now, let the *relative value* process of any given portfolio $A$ with respect to benchmark $B$ be denoted as $Z = \frac{A}{B}$. This is equivalent to a change of numeraire where the value of the portfolio is measured in *shares of the benchmark asset* instead of dollars.

The relative value $Z$ increases (decreases) when portfolio $A$ outperforms (underperforms) benchmark $B$. Notice that for log returns, $\log\left(\frac{Z(t)}{Z(s)}\right) = r^A(s,t) - r^B(s,t)$ for any $s \in [0,t]$ and $t \in [s, \infty)$. For instance, the process $\log\left(\frac{Z(t)}{Z(s)}\right)$ is defined as the *relative return* process by Fernholz (2002) (page 16). Thus, the *relative* drawdown, and the RDD Floor are defined as follows.

**Definition** $\left(\overline{RD}_{t_0,t}(A,B)\right)$: *The relative drawdown of portfolio A with respect to the bench-*

*mark $B$ at time $s$ is defined as*

$$RD_s(A,B) := D_s(Z),$$

*and the maximum relative drawdown at time $t$ is defined as*

$$\overline{RD}_{t_0,t}(A,B) := \bar{D}_{t_0,t}(Z),$$

*for* $Z(s) = \frac{A(s)}{B(s)} \qquad \forall s \in [t_0, t]$.

**Definition** (*RDDFloor*): *Let the Relative Drawdown Floor value process for a type* (1) *strategy be defined as*

$$F(t) = k\frac{A(t^{\star}_{t_0,Z})}{B(t^{\star}_{t_0,Z})}B(t), \tag{12}$$

*for all $t \in [t_0, \infty)$, where $A$ is the value of the portfolio, $B$ the value of the benchmark, $Z := A/B$ and $t^{\star}_{t_0,Z}$ is defined such that: $Z(t^{\star}_{t_0,Z}) = \sup_{s\in[t_0,t]} Z(s)$.*

Mantilla-Garcia (2014) also showed that if the value of the RDD Floor of the portfolio is always above the RDD Floor (12), then its maximum relative drawdown is lower than the risk budget $x$. Additionally, it showed that the underperformance to the benchmark asset, measured as the maximum difference in log returns between any two times $s$ and $t$ such that $s \leq t$, is also limited to $\tilde{x} = -\log(1-x)$. Formally,

$$\begin{aligned} \overline{RD}_{t_0,t}(A,B) &\leq x \\ r^A(s,t) - r^B(s,t) &\geq -\tilde{x} \end{aligned}$$

for all $s \leq t$ and $t \in [s, \infty)$.

Notice that the formula of RMDD Floor (12), when measured in shares of the benchmark asset, i.e. dividing its value by $B(t)$, yields the TIPP Floor formula when the interest rate is assumed to be zero[4].

# 4 Estimation of multiplier for dependent asset returns

In this section we discuss the methodological aspects around the estimation of the multiplier of portfolio insurance strategies taking into account the dependence structure of $r_S$ and $r_B$ using copulas. For the sake of clarity, we first present an estimation of the multiplier that introduces the dependence structure, but assumes constant parameters for the assets'

[4] The TIPP Floor formula assuming zero interest rate for the risk-free asset is $F(t) = kA(t^{\star}_{t_0,Z})$, i.e. a fraction $k$ of the running maximum of the value of the portfolio.

return distribution, and then we present an estimation of the multiplier using a model with time-varying volatilities.

A general approach to modeling the dependence structure of two variables $X$ and $Y$ is to use the concept of copula. The concept of copula dates back to Sklar (1959) but has been successfully used in finance, as in Embrechts, McNeil, and Straumann (1999). We shall assume that the reader is already familiar to some extent with copulas and refer to the works of Nelsen (2007) or Joe (1997) for a thorough review. Sklar's theorem (see for instance Nelsen, 2007) states that any joint distribution function $G$ can be decomposed into a series of marginal distribution functions and a unique copula function $C$ coupling the marginal distribution functions (dfs) as follows (in the two-dimensional case),

$$G(x,y) = C\left(F_X(x), F_Y(y)\right), \quad x, y \in \mathbb{R}, \tag{13}$$

for continuous marginal distribution functions $F_X$ and $F_Y$. The copula is a df on $[0,1]^2$ with uniform marginal dfs.

Fischer, Köck, Schlüter, and Weigert (2009) show that even though copulas have been thoroughly understood and well studied in the bivariate case, the higher-dimensional case still offers several open issues where it is still unclear how to construct copulas which capture the characteristics of financial returns. They show that traditional elliptical copulas (i.e. Gaussian and Student-t copula) dominate both empirical and practical applications. This is why we use both the Gaussian and Student-t copulas in the rest of this work.

## 4.1 Unconditional Multiplier Estimates

Embrechts and Puccetti (2007) introduced a computationally efficient method with quasi-analytical solutions to estimate quantiles of the distribution function $F_{X+Y}$ of the sum of two[5] dependent random variables variables $X$ and $Y$, given marginal distributions and a copula dependence structure. We use their algorithm to produce estimates of low-probability quantiles that composes the multiplier of the distribution, with a change of variables to calculate the difference of two dependent random variables in order to compute the quantile of the tracking error process $W = r_S - r_B$.

Embrechts and Puccetti (2007)'s algorithm assumes that $X$ and $Y$ are non-negative random variables with a constant dependence copula structure $C$. Hence, in order to obtain an unconditional estimate of low probability quantiles of $W$, we set $X = 1 - r_B$ and $Y = 1 + r_S$. We fit the marginal dfs $F_X$, $F_Y$, and copula function $C$ to the historical data and then perform a numerical inversion of $P(X+Y < s_q) = q$, to obtain $s_q$, with $q = 0.01\%$.

[5] Arbenz, Embrechts, and Puccetti (2011) improved and generalized it to calculate the sum of $n$ variables.

Finally we obtain $LT_W^q = s_q - 2$, and $m^q = \min\{\frac{-1}{LT_S^q}, \frac{-1}{LT_W^q}\}$, where $LT_S^q$ is obtained using the fitted distribution $F_Y$.

Hereafter we present the results of the aforementioned procedure applied on the daily returns series of the equal-weighted and cap-weighted S&P 500 total return index including dividend distributions from 1980-01-02 to 2010-12-31, retrieved from the CRSP database S&P 500 index file, and estimate $LT_W^q$, and $m^q$.

First, we fit t-student marginal distributions and two different Copula functions to the data. The estimated degrees of freedom of the marginals and the t-student copula were all both equal to 3 (within rounding error), implying fat tails for both returns and extreme co-movements. The estimated Pearson correlation between $X$ and $Y$ is $-0.9606$ (hence $\rho_{SB} = 0.9606$) suggesting that the univariate approach ignoring the dependence structure of equation (7) might be too conservative in this case.

Indeed, the unconditional estimate of the multiplier upper bound obtained with the bivariate approach, using the t-student marginals and Copula is $\hat{m} = \min\{6.50, 21.66\} = 6.50$, which is almost twice the upper bound obtained with the conservative univariate approach, $\hat{m} = \frac{-1}{-0.15-0.13} = 3.57$. This suggests that using the bivariate approach can yield important opportunity cost reductions for investors using portfolio insurance strategies.

In order to check whether this large difference comes from the dependence structure, we set the correlation parameter of the t-student copula to $\approx 1$ (hence $\rho_{SB} \approx -1$), and 0. The multiplier value obtained with $\rho_{SB} \approx -1$, is $\hat{m} = \min\{6.50, 3.58\} = 3.58$, which approximately matches the multiplier obtained with the univariate approach (3.57). On the other hand, even under the independence assumption ($\rho_{SB} = 0$), which is very conservative in this case, a much larger value ($\hat{m} = 5.13$) than the univariate approach is obtained.

## 4.2 Multipliers Estimates with Conditional Variance

There is a large amount of evidence that the conditional variance of stock returns varies over time, and hence extreme distribution quantile estimates should also be adjusted accordingly. Furthermore, Zieling, Mahayni, and Balder (2014) showed that a cushion-based portfolio insurance strategies using a multiplier that varies over time with the level of volatility is superior to the respective strategies using a constant multiple. Hence, we now consider a model in which the dependence structure between the variables considered is constant over time, but their marginal distributions present time-varying second moments following a GARCH(1,1) model (see Engle, 1982; Bollerslev, 1986), and assume that the residuals of the GARCH models follow t-student distributions with degrees of freedom $\nu$ as in Bollerslev (1987):

$$r_{it}|\mathfrak{F}_{t-1} \sim t\left(\mu_i, h_{it}, \nu_i\right), \tag{14}$$

for $i = \{S, B\}$. In equation (14), $\mathfrak{F}_{t-1}$ denotes the information set available at time $t-1$, the returns' mean $\mu$ is assumed constant, and the detrended return process denoted $u_t := r_t - \mu$ has conditional variance with dynamics

$$h_{it} = \alpha_{i0} + \alpha_{i1}u_{it}^2 + \alpha_{i2}h_{it-1}, \tag{15}$$

where $\alpha_{i0}$, $\alpha_{i1}$ and $\alpha_{i2}$ are the GARCH(1,1) model parameters. Given the t-student distribution assumption of the residuals, it is possible to fit the parameters of the GARCH model to data using maximum likelihood. We use the 'rugarch' package from R to fit the model to the S&P 500 cap-weighted and equal-weighted indices daily log-returns. The parameters are presented in Table 1. We then fit a copula to the standardized residuals $\epsilon_{i,t} = u_{it}/\sqrt{h_{it}}$ using the 'copula' package from R. We consider a t-student copula as well as a normal copula to model the dependence structure of the bivariate distribution of the two assets, and obtain a correlation coefficient of 0.9669 and 4 degrees of freedom for the t-student copula. For the normal copula we find a correlation parameter of 0.9651.

Using the GARCH and copula parameters, we generate 10,000 Monte Carlo simulations of 10 years of daily log-returns[6] of $S$ and $B$ and convert them to simple returns (i.e. $R = \exp(r) - 1$). We present a summary of the distribution of the annualized returns of the simulated scenarios over the 10 years period for both indices in Tables 2 and 5, for the normal and t-student copulas respectively. As observed in these tables, the median excess annualized return of the equal-weighted index relative to the cap-weighted index is 4.77% for the simulations with the normal copula and 4.82% for the scenarios from the t-student copula. The excess annualized return over the historical sample was 3.9%. However, the 1% quantile of the distribution of the 10 year annualized return is $-1.4\%$ and $-1.14\%$ for the normal and t-copulas, which represents a very large potential relative risk, despite the very high correlation between the two indices.

Tables 3 and 6 present summaries of the simulated indices' annualized volatility, absolute maxdrawdown and relative maxdrawdown distributions across the 10,000 scenarios, for the normal and t-copulas respectively. In these tables we observe that in terms of absolute risk (volatility and maxdrawdown), both indices have very similar values across scenarios. Furthermore, we observe a relative maxdrawdown for the equal-weighted index, relative to the cap-weighted becnhmark that ranges from 4% for the 1% quantile up to almost 50% for the 99% quantiles, which represents a very important risk of underperformance for investors

[6] We assume 260 trading days per year.

seeking to outperform the cap-weighted benchmark. These figures are also consistent with the relative maxdrawdown of the historical equal-weighted index of 30.9%.

Table 1: GARCH(1,1) parameters fitted to S&P 500 cap-weighted and equal-weighted indices daily returns from 1980-01-02 to 2010-12-31.

| | $\mu$ | $\alpha_0$ | $\alpha_1$ | $\alpha_2$ | $\nu$ |
|---|---|---|---|---|---|
| CW index | 7.010E-04 | 6.442E-07 | 5.562E-02 | 9.396E-01 | 6.499E+00 |
| EW index | 8.503E-04 | 8.813E-07 | 6.630E-02 | 9.268E-01 | 6.497E+00 |

In order to estimate the multiplier upper bound at each point in time for each of the simulated scenarios using formula (7), we need to estimate the quantiles, at a given confidence level $1-q$, of the distributions of the return difference $W = r_S - r_B$ and of the returns of $S$. Embrechts and Puccetti (2007)'s algorithm assumes that the two random variables are i.i.d., hence in this section we use instead a Monte Carlo simulation approach to estimate the quantiles of the return differences as follows. Notice that the quantiles of the distributions of $r_S$ and $W$ will vary at each point in time of each scenario, because the model for the assets returns above accommodates for time-varying variance. Hence, for every simulated value of the variances pair $(h_{St}, h_{Bt})$ for each scenario $j$ at time point $t$, we will estimate the quantiles of $W$ and $r_S$, by multiplying the 10,000 standardized shocks pairs $(\epsilon_S, \epsilon_B)$ generated in at time step $t$ with the Copula function in order to generate a distribution of the possible values of $W$ and $r_S$ for every simulated pair level of variances $(h_{St}, h_{Bt})$ at each scenario $j$ and time $t$. In other words, for every variance simulation $(h_{St}, h_{Bt})$ at time $t$ of scenario $j$, we will have 10,000 possible values of $r_S$ and $r_B$, from which we can calculate the quantiles $LT_S^q(t,j)$ and $LT_W^q(t,j)$ at a confidence level $1-q$, and replace them in formula (7) to obtain a multiplier level at each point in time for each scenario.

Tables 4 and 7 present the distribution of the multiplier using the bivariate and univariate approaches across time and scenarios. The multiplier from the bivariate approach presents values in all quantiles of their distribution that are about twice the corresponding values of the estimates obtained with the univariate approach, under both the normal and the t-student copula simulations. This result represents an important reduction in opportunity costs for the portfolio insurance strategy, in the sense that the strategy can invest twice as much in the risky performance-seeking asset, for the same level of maximum relative drawdown across scenarios. Indeed, the portfolio insurance strategy limits the relative maxdrawdown across all scenarios to be below the corresponding risk budget, as we can observe in Tables 9 and 15 for the 10% risk budget (for normal and t-student copula simulations), Tables 11 and 17 for the 15% risk budget, and Tables 13 and 19 for the 20% risk budget.

Tables 8 presents a summary of the distribution of the annualized return over the 10 year

Table 2: Annualized return of the cap-weighted and equal-weighted indices. Issued from the simulation of 10000 scenarios of 10 years of daily returns. The dependence of the two indices was model with a normal copula.

| | 1% | 5% | 10% | 25% | 50% | Mean | 75% | 90% | 95% | 99% |
|---|---|---|---|---|---|---|---|---|---|---|
| Return Equal-Weighted Stock Index | 8.46 | 13.94 | 16.31 | 20.28 | 24.72 | 24.90 | 29.28 | 33.60 | 36.62 | 43.02 |
| Return Cap-Weighted Stock Index (Bench) | 3.66 | 9.37 | 11.67 | 15.55 | 20.02 | 20.15 | 24.40 | 28.61 | 31.71 | 38.39 |
| Return Difference by scenario | -1.39 | 1.05 | 1.93 | 3.34 | 4.77 | 4.75 | 6.20 | 7.54 | 8.45 | 10.51 |

Table 3: Risk of the cap-weighted and equal-weighted indices. Issued from the simulation of 10000 scenarios of 10 years of daily returns. The dependence of the two indices was model with a normal copula.

| | 1% | 5% | 10% | 25% | 50% | Mean | 75% | 90% | 95% | 99% |
|---|---|---|---|---|---|---|---|---|---|---|
| Vol EW Stock Index | 12.33 | 13.24 | 13.81 | 14.89 | 16.53 | 17.51 | 18.84 | 22.00 | 24.71 | 34.00 |
| Vol CW Stock Index (Bench) | 12.19 | 13.14 | 13.74 | 14.96 | 16.76 | 17.89 | 19.39 | 22.99 | 26.20 | 36.23 |
| MDD EW Stock Index | 11.70 | 14.24 | 15.88 | 19.41 | 24.89 | 27.86 | 33.13 | 43.71 | 52.11 | 70.90 |
| MDD CW Stock Index (Bench) | 12.38 | 15.30 | 17.08 | 20.91 | 26.90 | 29.97 | 35.69 | 47.10 | 55.03 | 74.75 |
| RMDD EW Stock Index | 4.48 | 5.58 | 6.37 | 8.11 | 10.98 | 13.31 | 15.69 | 22.43 | 28.98 | 47.23 |
| RMDD CW Stock Index (Bench) | -0.00 | -0.00 | -0.00 | -0.00 | -0.00 | 0.00 | -0.00 | -0.00 | -0.00 | -0.00 |

Table 4: Distribution of simulated conditional multiplier upper bound using a normal-copula. Issued from the simulation of 10000 scenarios of 2600 daily returns.

| | 1% | 5% | 10% | 25% | 50% | Mean | 75% | 90% | 95% | 99% |
|---|---|---|---|---|---|---|---|---|---|---|
| Bivariate Multiplier normal-copula | 4.87 | 7.67 | 9.48 | 12.96 | 17.37 | 17.87 | 22.27 | 26.91 | 29.74 | 35.06 |
| Univariate Multiplier normal-copula | 2.31 | 3.67 | 4.54 | 6.14 | 8.13 | 8.29 | 10.27 | 12.25 | 13.44 | 15.64 |

Table 5: Annualized return of the cap-weighted and equal-weighted indices. Issued from the simulation of 10000 scenarios of 10 years of daily returns. The dependence of the two indices was model with a t copula.

| | 1% | 5% | 10% | 25% | 50% | Mean | 75% | 90% | 95% | 99% |
|---|---|---|---|---|---|---|---|---|---|---|
| Return Equal-Weighted Stock Index | 8.32 | 13.81 | 16.36 | 20.27 | 24.67 | 24.90 | 29.35 | 33.79 | 36.58 | 42.87 |
| Return Cap-Weighted Stock Index (Bench) | 3.54 | 9.16 | 11.66 | 15.59 | 19.95 | 20.13 | 24.38 | 28.74 | 31.88 | 38.43 |
| Return Difference by scenario | -1.14 | 1.32 | 2.16 | 3.42 | 4.82 | 4.77 | 6.11 | 7.40 | 8.29 | 10.23 |

Table 6: RiskAssets metrix of the cap-weighted and equal-weighted indices. Issued from the simulation of 10000 scenarios of 10 years of daily returns. The dependence of the two indices was model with a t copula.

| | 1% | 5% | 10% | 25% | 50% | Mean | 75% | 90% | 95% | 99% |
|---|---|---|---|---|---|---|---|---|---|---|
| Vol EW Stock Index | 12.33 | 13.25 | 13.80 | 14.93 | 16.51 | 17.54 | 18.79 | 22.02 | 24.89 | 33.86 |
| Vol CW Stock Index (Bench) | 12.19 | 13.18 | 13.78 | 14.97 | 16.76 | 17.95 | 19.34 | 23.02 | 26.04 | 37.28 |
| MDD EW Stock Index | 11.84 | 14.35 | 16.03 | 19.46 | 24.98 | 27.83 | 32.80 | 43.32 | 51.46 | 71.77 |
| MDD CW Stock Index (Bench) | 12.50 | 15.23 | 17.05 | 21.04 | 27.03 | 30.01 | 35.68 | 46.64 | 55.56 | 75.46 |
| RMDD EW Stock Index | 4.14 | 5.19 | 5.91 | 7.41 | 9.98 | 12.15 | 14.32 | 20.42 | 26.38 | 43.47 |
| RMDD CW Stock Index (Bench) | -0.00 | -0.00 | -0.00 | -0.00 | -0.00 | 0.00 | -0.00 | -0.00 | -0.00 | -0.00 |

Table 7: Distribution of simulated conditional multiplier upper bound using a t-copula. Issued from the simulation of 10000 scenarios of 2600 daily returns.

| | 1% | 5% | 10% | 25% | 50% | Mean | 75% | 90% | 95% | 99% |
|---|---|---|---|---|---|---|---|---|---|---|
| Bivariate Multiplier t-copula | 4.91 | 7.72 | 9.53 | 13.01 | 17.45 | 17.94 | 22.35 | 26.99 | 29.82 | 35.17 |
| Univariate Multiplier t-copula | 2.31 | 3.70 | 4.56 | 6.18 | 8.16 | 8.32 | 10.30 | 12.29 | 13.48 | 15.67 |

period for the two strategies across scenarios, one using the bivariate multplier and one under the univariate multiplier, for the 10% risk budget and the normal copula simulations.The strategy using the bivariate multiplier presents a return above the strategy with univariate multiplier of around 1% across all the quantiles of the distribution (notice also that the return of the strategy is about 2% above the return of the benchmark index across quantiles). Table 14 presents similar results for the t-copula simulations. These results suggest that the bivariate strategy might present stochastic dominance (SD) over the strategy with the univariate multiplier[7].

In Table 20 we present the results of two formal tests of stochastic dominance. The first three columns are the output of the stochastic dominance test from the R package 'dunn.test', which computes Dunn's test (Dunn (1964)) for stochastic dominance and reports the results among multiple pairwise comparisons after a Kruskal-Wallis test for stochastic dominance among k groups (Kruskal and Wallis, 1952). The 'dunn.test' makes pairwise comparisons based on Dunn's z-test-statistic approximations to the actual rank statistics. The null hypothesis for each pairwise comparison is that the probability of observing a randomly selected value from the first group that is larger than a randomly selected value from the second group equals one half; this null hypothesis corresponds to that of the Wilcoxon-Mann-Whitney rank-sum test. As we can see from the p-values in the third column of Table 20, the null hypothesis is clearly rejected at 1% level for all risk budgets in both the normal and t-student copulas simulations. The last four columns of the table present the percentage of the observations for which the stochastic dominance statistics indicate that the bivariate approach dominates the univariate approach at the 1st to 4th SD levels. These SD statistics were calculated using the function stochdom2 from the R package 'generalCorr' (Vinod and Fordham University, Vinod and Fordham University). The dominating distribution is superior in terms of local mean, variance, skewness and kurtosis respectively, representing dominance orders 1 to 4 (see Vinod, 2004, 2008, sec. 4.3 for details). For all risk budgets, the percentage of stochastic dominance of the bivariate approach over the univariate approach is above 99.9% in the simulations from both, the normal and the t-student copula.

## 5 Conclusion

There are several applications in finance in which losses are measured in relative terms, with respect to a benchmark index. In such cases the benchmark index becomes the numeraire or unit of value, as opposed to current dollar terms. This paper studies the parametrization

[7]Stochastic dominance means that the probability that a randomly drawn observation from one group will be greater than a randomly drawn observation from another.

of a portfolio insurance strategy that limits cumulative losses, measured in relative terms, to a predefined limit set ex-ante. In order to achieve this objective, the strategy allocates at every point in time to a performance-seeking asset, a multiple of the distance between the current value of the portfolio, and the value of a Floor process (the rest is allocated to the bechmark asset or portfolio). Hence, the estimation of the multiple is crucial for the performance of the strategy. On the one hand, the multiplier parameter cannot be too high, so that the relative performance of the two assets before the next rebalancing of the portfolio does not cause the value of portfolio to fall below its Floor value. On the other hand, if the multiplier is too low, the insurance protection becomes more expensive, because the strategy under-spends its risk budget, having a lower access to the upside potential of the performance-seeking asset of the strategy.

Former research has mainly focused on the estimation of the multiplier upper bound of portfolio insurance strategies investing in a locally riskless asset paying a constant interest rate as their reserve asset. In such case, there is no dependence structure between the risky asset and a constant return (since the latter is not a stochastic process). However, strategies relying in a locally risky reserve or benchmark asset, are more relevant in practice. This paper presents an estimation approach of the multiplier upper bound that takes into account the dependence structure of the two assets composing the strategy.

We illustrate the advantages of the proposed methodology over the alternative approach which ignores the dependence structure of the two assets, with an application that has become increasingly relevant for practitioners. We present the results of Monte Carlo simulations of a strategy controlling the relative drawdown of an alternative equity index with respect to a standard market-cap weighted benchmark index. Indeed, several 'smart-beta' indices that offer systematic diversification strategies different from the rather concentrated market-cap weighted indices. Such alternative indices tend to have higher returns in the long run than their benchmarks, but also represent significant deviations from them. Hence, investing in alternative indices presents an underperformance risk relative to the benchmark index that is not limited a priori. In this sense, the relative drawdown control strategy allows investors to bet on any alternative investment of their choice, but to limit cumulative relative losses at every horizon to a chosen level set ex-ante.

In our tests we find that, in this context, taking into account the correlation between the two assets, almost doubles the level of the multiplier, without presenting any Floor violations, and hence maintaining the maximum level of relative underperformance equal. This increase in the multiplier is reflected in a very consistent benefit in terms of the long-term average return of the strategy. Indeed, in our simulations we find that the strategy using the proposed multiplier estimation method, which takes into account the dependence structure of the two assets, presents stochastic dominance relative to the strategy using the multiplier that ignores the dependence structure of the assets.

Table 8: Annualized return of the relative drawdown portfolio insurance strategies with Risk Budget of 0.1. One using the proposed bivariate multiplier, and the strategy using the univariate (benchmark) multiplier. Issued from the simulation of 10000 scenarios of 10 years of daily returns. The dependence of the two indices was model with a normal copula.

| | 1% | 5% | 10% | 25% | 50% | Mean | 75% | 90% | 95% | 99% |
|---|---|---|---|---|---|---|---|---|---|---|
| Return Strategy with Bivariate Multiplier | 6.58 | 12.22 | 14.71 | 18.72 | 23.24 | 23.30 | 27.73 | 32.01 | 34.81 | 41.38 |
| Return Strategy with Univariate Multiplier | 5.94 | 11.39 | 13.85 | 17.74 | 22.20 | 22.31 | 26.66 | 30.84 | 33.76 | 40.21 |

Table 9: Risk metrics of the relative drawdown portfolio insurance strategies with Risk Budget of 0.1. One using the proposed bivariate multiplier, and the strategy using the univariate (benchmark) multiplier. Issued from the simulation of 10000 scenarios of 10 years of daily returns. The dependence of the two indices was model with a normal copula.

| | 1% | 5% | 10% | 25% | 50% | Mean | 75% | 90% | 95% | 99% |
|---|---|---|---|---|---|---|---|---|---|---|
| Vol of Strategy with Bivariate Multiplier | 12.26 | 13.17 | 13.74 | 14.82 | 16.48 | 17.57 | 18.90 | 22.20 | 25.36 | 35.43 |
| Vol of Strategy Univariate Multiplier | 12.18 | 13.12 | 13.70 | 14.82 | 16.51 | 17.61 | 19.00 | 22.39 | 25.58 | 35.41 |
| MaxDrawdown of Strategy with Bivariate Multiplier | 11.79 | 14.37 | 16.13 | 19.81 | 25.27 | 28.33 | 33.70 | 44.52 | 52.83 | 71.75 |
| MaxDrawdown of Strategy Univariate Multiplier | 12.07 | 14.57 | 16.33 | 20.09 | 25.65 | 28.77 | 34.21 | 45.10 | 53.48 | 72.66 |
| RMDD of Strategy with Bivariate Multiplier | 4.26 | 5.13 | 5.66 | 6.68 | 7.96 | 7.79 | 9.07 | 9.68 | 9.85 | 9.98 |
| RMDD of Strategy Univariate Multiplier | 2.73 | 3.20 | 3.49 | 4.08 | 4.90 | 5.06 | 5.89 | 6.89 | 7.49 | 8.57 |

Table 10: Annualized return of the relative drawdown portfolio insurance strategies with Risk Budget of 0.15. One using the proposed bivariate multiplier, and the strategy using the univariate (benchmark) multiplier. Issued from the simulation of 10000 scenarios of 10 years of daily returns. The dependence of the two indices was model with a normal copula.

| | 1% | 5% | 10% | 25% | 50% | Mean | 75% | 90% | 95% | 99% |
|---|---|---|---|---|---|---|---|---|---|---|
| Return Strategy with Bivariate Multiplier | 7.59 | 13.25 | 15.63 | 19.59 | 24.13 | 24.27 | 28.73 | 33.01 | 35.78 | 42.17 |
| Return Strategy with Univariate Multiplier | 6.94 | 12.43 | 14.87 | 18.86 | 23.35 | 23.45 | 27.82 | 32.10 | 34.96 | 41.51 |

Table 11: Risk metrics of the relative drawdown portfolio insurance strategies with Risk Budget of 0.15. One using the proposed bivariate multiplier, and the strategy using the univariate (benchmark) multiplier. Issued from the simulation of 10000 scenarios of 10 years of daily returns. The dependence of the two indices was model with a normal copula.

| | 1% | 5% | 10% | 25% | 50% | Mean | 75% | 90% | 95% | 99% |
|---|---|---|---|---|---|---|---|---|---|---|
| Vol of Strategy with Bivariate Multiplier | 12.32 | 13.23 | 13.79 | 14.87 | 16.52 | 17.56 | 18.85 | 22.01 | 25.14 | 35.00 |
| Vol of Strategy Univariate Multiplier | 12.30 | 13.19 | 13.77 | 14.85 | 16.51 | 17.57 | 18.92 | 22.18 | 25.33 | 34.82 |
| MaxDrawdown of Strategy with Bivariate Multiplier | 11.69 | 14.30 | 15.99 | 19.62 | 25.01 | 28.04 | 33.25 | 43.93 | 52.04 | 71.12 |
| MaxDrawdown of Strategy Univariate Multiplier | 12.01 | 14.40 | 16.18 | 19.85 | 25.34 | 28.40 | 33.71 | 44.54 | 52.75 | 72.02 |
| RMDD of Strategy with Bivariate Multiplier | 4.48 | 5.52 | 6.23 | 7.77 | 9.96 | 10.07 | 12.48 | 14.20 | 14.66 | 14.96 |
| RMDD of Strategy Univariate Multiplier | 3.73 | 4.48 | 4.88 | 5.78 | 7.06 | 7.32 | 8.60 | 10.16 | 11.11 | 12.81 |

Table 12: Annualized return of the relative drawdown portfolio insurance strategies with Risk Budget of 0.2. One using the proposed bivariate multiplier, and the strategy using the univariate (benchmark) multiplier. Issued from the simulation of 10000 scenarios of 10 years of daily returns. The dependence of the two indices was model with a normal copula.

| | 1% | 5% | 10% | 25% | 50% | Mean | 75% | 90% | 95% | 99% |
|---|---|---|---|---|---|---|---|---|---|---|
| Return Strategy with Bivariate Multiplier | 8.06 | 13.62 | 16.00 | 19.97 | 24.48 | 24.63 | 29.04 | 33.34 | 36.33 | 42.58 |
| Return Strategy with Univariate Multiplier | 7.50 | 13.09 | 15.55 | 19.54 | 24.01 | 24.16 | 28.58 | 32.83 | 35.75 | 42.23 |

Table 13: Risk metrics of the relative drawdown portfolio insurance strategies with Risk Budget of 0.2. One using the proposed bivariate multiplier, and the strategy using the univariate (benchmark) multiplier. Issued from the simulation of 10000 scenarios of 10 years of daily returns. The dependence of the two indices was model with a normal copula.

| | 1% | 5% | 10% | 25% | 50% | Mean | 75% | 90% | 95% | 99% |
|---|---|---|---|---|---|---|---|---|---|---|
| Vol of Strategy with Bivariate Multiplier | 12.33 | 13.24 | 13.81 | 14.88 | 16.54 | 17.55 | 18.87 | 21.92 | 24.94 | 34.54 |
| Vol of Strategy Univariate Multiplier | 12.32 | 13.23 | 13.80 | 14.89 | 16.53 | 17.56 | 18.87 | 22.04 | 25.12 | 34.66 |
| MaxDrawdown of Strategy with Bivariate Multiplier | 11.70 | 14.27 | 15.94 | 19.51 | 24.99 | 27.96 | 33.32 | 43.89 | 51.60 | 70.35 |
| MaxDrawdown of Strategy Univariate Multiplier | 11.72 | 14.34 | 16.12 | 19.77 | 25.11 | 28.18 | 33.33 | 44.09 | 52.33 | 71.41 |
| RMDD of Strategy with Bivariate Multiplier | 4.49 | 5.57 | 6.35 | 8.03 | 10.63 | 11.36 | 14.34 | 17.89 | 19.21 | 19.92 |
| RMDD of Strategy Univariate Multiplier | 4.23 | 5.12 | 5.68 | 6.85 | 8.61 | 9.05 | 10.80 | 13.10 | 14.48 | 16.93 |

Table 14: Annualized return of the relative drawdown portfolio insurance strategies with Risk Budget of 0.1. One using the proposed bivariate multiplier, and the strategy using the univariate (benchmark) multiplier. Issued from the simulation of 10000 scenarios of 10 years of daily returns. The dependence of the two indices was model with a t copula.

| | 1% | 5% | 10% | 25% | 50% | Mean | 75% | 90% | 95% | 99% |
|---|---|---|---|---|---|---|---|---|---|---|
| Return Strategy with Bivariate Multiplier | 6.37 | 12.29 | 14.81 | 18.90 | 23.29 | 23.43 | 27.84 | 32.16 | 34.87 | 41.18 |
| Return Strategy with Univariate Multiplier | 5.63 | 11.39 | 13.85 | 17.85 | 22.20 | 22.39 | 26.74 | 31.10 | 33.97 | 40.40 |

Table 15: Risk metrics of the relative drawdown portfolio insurance strategies with Risk Budget of 0.1. One using the proposed bivariate multiplier, and the strategy using the univariate (benchmark) multiplier. Issued from the simulation of 10000 scenarios of 10 years of daily returns. The dependence of the two indices was model with a t copula.

| | 1% | 5% | 10% | 25% | 50% | Mean | 75% | 90% | 95% | 99% |
|---|---|---|---|---|---|---|---|---|---|---|
| Vol of Strategy with Bivariate Multiplier | 12.30 | 13.20 | 13.75 | 14.85 | 16.47 | 17.62 | 18.83 | 22.29 | 25.38 | 36.89 |
| Vol of Strategy Univariate Multiplier | 12.22 | 13.15 | 13.71 | 14.83 | 16.50 | 17.67 | 18.99 | 22.45 | 25.50 | 36.79 |
| MaxDrawdown of Strategy with Bivariate Multiplier | 12.04 | 14.47 | 16.17 | 19.78 | 25.41 | 28.34 | 33.47 | 44.12 | 52.60 | 73.64 |
| MaxDrawdown of Strategy Univariate Multiplier | 12.17 | 14.56 | 16.35 | 20.06 | 25.80 | 28.82 | 34.22 | 44.98 | 53.55 | 74.50 |
| RMDD of Strategy with Bivariate Multiplier | 3.96 | 4.77 | 5.25 | 6.27 | 7.53 | 7.45 | 8.76 | 9.51 | 9.77 | 9.96 |
| RMDD of Strategy Univariate Multiplier | 2.57 | 2.99 | 3.25 | 3.79 | 4.56 | 4.73 | 5.48 | 6.46 | 7.08 | 8.18 |

Table 16: Annualized return of the relative drawdown portfolio insurance strategies with Risk Budget of 0.15. One using the proposed bivariate multiplier, and the strategy using the univariate (benchmark) multiplier. Issued from the simulation of 10000 scenarios of 10 years of daily returns. The dependence of the two indices was model with a t copula.

| | 1% | 5% | 10% | 25% | 50% | Mean | 75% | 90% | 95% | 99% |
|---|---|---|---|---|---|---|---|---|---|---|
| Return Strategy with Bivariate Multiplier | 7.46 | 13.13 | 15.78 | 19.77 | 24.17 | 24.35 | 28.76 | 33.15 | 35.88 | 42.01 |
| Return Strategy with Univariate Multiplier | 6.68 | 12.41 | 14.94 | 19.04 | 23.39 | 23.56 | 27.95 | 32.32 | 35.15 | 41.50 |

Table 17: Risk metrics of the relative drawdown portfolio insurance strategies with Risk Budget of 0.15. One using the proposed bivariate multiplier, and the strategy using the univariate (benchmark) multiplier. Issued from the simulation of 10000 scenarios of 10 years of daily returns. The dependence of the two indices was model with a t copula.

| | 1% | 5% | 10% | 25% | 50% | Mean | 75% | 90% | 95% | 99% |
|---|---|---|---|---|---|---|---|---|---|---|
| Vol of Strategy with Bivariate Multiplier | 12.32 | 13.25 | 13.78 | 14.91 | 16.49 | 17.60 | 18.79 | 22.04 | 25.02 | 36.42 |
| Vol of Strategy Univariate Multiplier | 12.30 | 13.21 | 13.77 | 14.86 | 16.48 | 17.63 | 18.86 | 22.28 | 25.26 | 36.70 |
| MaxDrawdown of Strategy with Bivariate Multiplier | 11.85 | 14.38 | 16.02 | 19.58 | 25.18 | 28.01 | 33.06 | 43.34 | 52.04 | 72.99 |
| MaxDrawdown of Strategy Univariate Multiplier | 12.04 | 14.50 | 16.14 | 19.82 | 25.42 | 28.44 | 33.70 | 44.35 | 52.88 | 74.15 |
| RMDD of Strategy with Bivariate Multiplier | 4.15 | 5.15 | 5.79 | 7.14 | 9.15 | 9.48 | 11.84 | 13.78 | 14.46 | 14.92 |
| RMDD of Strategy Univariate Multiplier | 3.54 | 4.10 | 4.52 | 5.34 | 6.54 | 6.81 | 7.99 | 9.55 | 10.49 | 12.20 |

Table 18: Annualized return of the relative drawdown portfolio insurance strategies with Risk Budget of 0.2. One using the proposed bivariate multiplier, and the strategy using the univariate (benchmark) multiplier. Issued from the simulation of 10000 scenarios of 10 years of daily returns. The dependence of the two indices was model with a t copula.

| | 1% | 5% | 10% | 25% | 50% | Mean | 75% | 90% | 95% | 99% |
|---|---|---|---|---|---|---|---|---|---|---|
| Return Strategy with Bivariate Multiplier | 7.74 | 13.48 | 16.09 | 20.07 | 24.50 | 24.67 | 29.11 | 33.54 | 36.28 | 42.41 |
| Return Strategy with Univariate Multiplier | 7.35 | 13.05 | 15.62 | 19.66 | 24.06 | 24.24 | 28.66 | 33.05 | 35.88 | 42.23 |

Table 19: Risk metrics of the relative drawdown portfolio insurance strategies with Risk Budget of 0.2. One using the proposed bivariate multiplier, and the strategy using the univariate (benchmark) multiplier. Issued from the simulation of 10000 scenarios of 10 years of daily returns. The dependence of the two indices was model with a t copula.

| | 1% | 5% | 10% | 25% | 50% | Mean | 75% | 90% | 95% | 99% |
|---|---|---|---|---|---|---|---|---|---|---|
| Vol of Strategy with Bivariate Multiplier | 12.33 | 13.25 | 13.80 | 14.92 | 16.51 | 17.59 | 18.80 | 22.03 | 25.00 | 35.67 |
| Vol of Strategy Univariate Multiplier | 12.33 | 13.25 | 13.80 | 14.89 | 16.49 | 17.61 | 18.80 | 22.10 | 25.04 | 36.30 |
| MaxDrawdown of Strategy with Bivariate Multiplier | 11.86 | 14.37 | 16.05 | 19.51 | 25.08 | 27.90 | 33.04 | 43.18 | 51.57 | 72.46 |
| MaxDrawdown of Strategy Univariate Multiplier | 12.02 | 14.41 | 16.07 | 19.68 | 25.25 | 28.19 | 33.22 | 43.91 | 52.40 | 73.60 |
| RMDD of Strategy with Bivariate Multiplier | 4.15 | 5.19 | 5.90 | 7.37 | 9.74 | 10.58 | 13.28 | 17.01 | 18.74 | 19.83 |
| RMDD of Strategy Univariate Multiplier | 3.95 | 4.73 | 5.25 | 6.33 | 7.90 | 8.37 | 9.98 | 12.21 | 13.61 | 16.13 |

Table 20: Tests of Stochastic Dominance of RMDD strategy with bivariate multiplier over the strategy using the univariate (benchmark) multiplier. Issued from the simulation of 10000 scenarios.

| | Chi2 | Dunn Z | P-value | sd1 | sd2 | sd3 | sd4 |
|---|---|---|---|---|---|---|---|
| SD with normal copula and RB 0.1 | 1.0595E+02 | 1.0293E+01 | 3.7771E-25 | 9.9975E-01 | 9.9960E-01 | 9.9950E-01 | 9.9935E-01 |
| SD with normal copula and RB 0.15 | 7.0537E+01 | 8.3986E+00 | 2.2583E-17 | 9.9960E-01 | 9.9940E-01 | 9.9925E-01 | 9.9915E-01 |
| SD with normal copula and RB 0.2 | 2.1859E+01 | 4.6754E+00 | 1.4671E-06 | 9.9960E-01 | 9.9950E-01 | 9.9940E-01 | 9.9930E-01 |
| SD with t copula and RB 0.1 | 1.1678E+02 | 1.0806E+01 | 1.6050E-27 | 9.9970E-01 | 9.9975E-01 | 9.9995E-01 | 9.9995E-01 |
| SD with t copula and RB 0.15 | 6.5302E+01 | 8.0810E+00 | 3.2123E-16 | 9.9965E-01 | 9.9970E-01 | 9.9980E-01 | 9.9995E-01 |
| SD with t copula and RB 0.2 | 1.8114E+01 | 4.2560E+00 | 1.0404E-05 | 9.9990E-01 | 9.9995E-01 | 9.9995E-01 | 9.9995E-01 |

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