VALUE OF THE FIRM WITH CORPORATE TAXES AND CONTINGENT VALUATION APPROACH

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ABSTACT

In my opinion, this working paper presents a novel way of considering corporate taxes in firm valuation. In classical valuation theory (or practice), the value of the firm is considered as the present value of the after tax cash flows to equity holders in a unlevered firm, discounted at the weighted average risk adjusted cost of financing (debt and equity), where the cost of debt is expressed in after tax terms. The referred cash flow is known as the free cash flow to the firm or FCFF. Free cash flow is the cash available to equity holders and bondholders (the latter, in after tax terms). Here I consider cash flows of the firm before paying taxes, and this cash flows are available to pay taxes, equity holders, and bondholders. I consider the pre-tax value of the firm, with three claimholders: government, stockholders, and bond holders. The introduction of government as a third claimholder has important implications on the firm value and how resulting values are related.

This working paper presents a step by step guide to the valuation of a firm, more precisely, of the valuation of the claims involved in a firm, in the context of what is known as Merton's Structural Model. The novelty is that I use a numerical integration valuation approach, similar in nature with the valuation through binomial trees, and solve for the risk adjusted returns of every and each

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component of the firm value. Risk adjusted returns are obtained for: debt, equity, unlevered free cash flow, tax savings, and taxes (with and without debt).

Two fundamental equations in firm value are, V = D + E (asset value is equal to debt value plus equity value), and $V = V_U + VTS$ (asset value is equal to value of unlevered assets plus value of tax savings from debt). In the financial theory there is no apparent solution or agreement on the correct discount rate to obtain VTS. There is agreement on how to value V_U instead.

This work may represent a starting point for a solution to the apparently unsolved problem in valuation, which is the appropriate risk adjusted discount rate for tax savings from debt. If this is the case, this work would be a first important step to come with a solution for this risk adjusted return of tax savings from debt, in a multi-period setting.

Besides this, I present an adjusted solution to the weighted average cost of capital of the firm. This solution is fully compatible with other risk adjusted returns I derive in this paper. The adjusted aspect of the weighted average cost of capital is in the approach through which I obtain the after-tax cost of debt. This is thought to be the pretax cost of debt, adjusted by the marginal tax rate of the firm; but in this paper I obtain quite different results.

If I would have two choose two main contributions to valuation theory possibly and hopefully included in my paper these would be: risk adjusted cost function for tax savings from debt cash flows; calculation of the so called after-tax cost of debt, an essential piece of the weighted average cost of capital, explicitly considering future expected tax savings from debt to do the adjustment.

ACKNOWLEGMENTS

The inspiration to carry on this work for more than a decade, I owe it to Professor Thomas E. Copeland (and Fred Weston as well) reading in its well-known text-book Financial Theory and Corporate Policy, about the consistency between the valuation of the firm through option pricing, and the valuation grounded on the Capital Asset Pricing Model, CAPM. Also I feel greatly indebted to Aswath Damodaran, Ignacio Velez-Pareja, Harry De-Angelo, as great friends and constant sources of inspiration. Another great source of inspiration has been the enormous work on this matter by Professor Robert Merton. I am especially thankful to Professor Merton, for authorizing me to include in this paper a personal answer he gave me several years ago about YTM of a bond; his kind gesture was a great incentive for this work. A special mention deserves Pablo Fernandez (IESE) whose work has also been determinant in this working paper. Pablo's favorable comments on this paper were a great pivot and thrust at moments when I felt quite dubious about it. I also feel largely indebted to Aswath Damodaran and his encouraging to present this paper to European Financial Management Association (EFMA) 2019 Annual Meetings. Professor John C. Hull also provided me with helpful orientation about numerical integration as a valid method for contingent claims valuation. The idea of modeling V_T at maturity with risk-neutral and with actuarial probabilities, wouldn't have been possible to me without professor Hull's clear exposition in his well-known book about derivatives [2015]. Professor Ian Cooper of London Business School also made a friendly review of my paper and provided me favorable comments and valuable input. I also thank my good friend Sergio Benavente, who kindly reviewed this paper and provided me with very helpful comments.

Thanks to my very good friend Tomas Fernandez M. who generously made possible the financing of my trip to present this paper at the European Financial Management Association, EFMA, meeting on June 2019 at Portugal, University of Azores at Ponta Delgada, Sao Miguel. His personal support and must say has gone far beyond this specific but very significant event.

This paper, and almost 18 years of effort working on it, is greatly dedicated to my beloved wife and to our seven children. Without their love and support this would have been impossible.

VALUE OF THE FIRM WITH CORPORATE TAXES AND OPTION PRICING THEORY

The value of a firm with risky debt can be conceptualized and also computed by using the analytical[2] option pricing model developed by Black-Sholes-Merton (B-S-M). In this context the underlying asset of the corresponding options is the value of the firm assets, V. The firm is assumed to have equity and ordinary debt as the only sources of financing, and the results of the firm in terms of payoffs to stakeholders (assets, equity, and debt) are observable only at one future time T (maturity); there are no intermediate cash flows to be discounted, but only the cash flow produced at maturity T.

[2] Here analytical is used in the sense of a model consisting of an analytical or mathematical expression (equation). Besides analytical models, also are available "numerical" solutions or models.

The firm assets value V, is assumed to follow a geometric Brownian motion type of stochastic process, between present or valuation time (t = 0) and a future time t = T. The firm equity is seen and valued as a call option[3] on the firm assets, with maturity T. The strike price[4] of this option is the final contractual value of debt, K. The firm's risky debt is valued as a riskless bond minus a put option on the value of the firm's assets. The maturity of the put option is T and its strike price is K.

Literature Review:

Most probably there exists abundant literature, papers, articles, and books, written on the tax issue when valuing a firm or even a project. This section does not represent an exhaustive review on this literature, but a sort of random scanning I have done in the process of developing my valuation model, in writing the findings using my model, and as a consequence of my professional activity as valuation consultant, and also as finance professor. But intertwined with the tax aspects, is the contingency characteristics of the model as an equally relevant aspect.

- Lutz Kruschwitz and Andreas Löffler; *Discounted cash flow: a theory of the valuation of firms*; John Wiley & Sons Ltd, 2006

A short concise book, 154 pages plus Index. Its main structure:

1 Basic Elements; 2 Corporate Income Tax; 3 Personal Income Tax; 4 Corporate and Personal Income Tax; Appendix: Proofs.

From its introduction: "*We see it as very important to systematically clarify the way in which these different variations of the DCF concept are related. Why are there several procedures and not just one? Do they all lead to the same result? If not, where do the economic differences lie? If so, for what purpose are different methods needed? And further: do the known procedures suffice? Or are there situations where none of the concepts developed up to now delivers the correct value of the firm? If so, how is the appropriate valuation formula to be found? These questions are not just interesting for theoreticians; even the practitioner who is confronted with the task of marketing his or her results has to deal with them.*"

[3] All options considered in this paper are "European" type options. This are options in which the right (of purchasing or of selling) can only be exercised at the maturity date of the option. Another type of options are the "American" options. In these, the option or right can be exercised at any date prior to the maturity date or at its maturity date.
[4] The "strike" of an option is a pre-specified price at which the buyer of the option has the right to buy (call option), or to sell (put option), the amount of the underlying asset specified in the option contract.

From 1 Basic Elements: "*The payment surpluses, which are to be discounted, are also called cash flows. Nowhere in the literature is this term clearly defined. So one can be certain that no two economists speaking about cash flows will have one and the same thing in mind.*" … "*It is relatively clear what one means when speaking of taxes while doing a business valuation.*" … "*As a rule it remains rather unclear in business valuation what is meant by cost of capital. Even those who consult the relevant literature will not find, in our opinion, any precise definition of the term.*" … "*The practically engaged evaluator must spend a considerable amount of her precious time on the prognosis of future cash flows: we already mentioned that it is not the historical payment surpluses that matter to the firm being valued, but rather the cash flows that it will yield in the future. The work of theoretically based finance experts is generally of limited use for this important activity. We will not be discussing that in this book at all.*"

Up to here I would like to through a few comments. As far as I am informed and trained, after falling into several inconsistencies during my professional life, different DCF approaches should yield identical value results. An analogy: hospital patients are often asked to rank form 0 to 10 or 1 to 10 the level of pain they currently feel. There is no "*painometer*" as there are thermometers or barometers. But no serious person would argue that pain bared by a patient is irrelevant. Maybe in the future there will exist more scientific based instruments to measure pain, and not letting the matter to a subjective patient's answer, "*I think somewhere between 6 and 7*". In valuation of a firm or project or even a financial asset (stocks and bonds), it would be stupid to argument that an exact value of such things at any specific moment in time and place exists. As Pablo Fernandez says, a valuation is an opinion, and as such is personal to the subject issuing such a value. But in my humble opinion, I think that nor academics, nor practitioners in the valuation fields, at least serious ones, may take lightly some set of existing practices, as far as they seem to go in the direction of doing the value opinion more precise and more credible. A valuation task involves several stages and activities usually related to estimations and choosing among alternative and probably competing models (especially in the discount rate). Among this facets I think that one in which academics and practitioners can be more helpful to the aim of providing a more reliable opinion of value, in in the analytical structuring of the valuation process. More specifically, equations for cost of equity and for WACC (weighted average cost of capital) should be compatible. More leeway must be left when estimating for example the future expected growth of the market (in units), future expected prices, future expected market share of the firm being valued, and a long list of other items.

I share the view that anywhere is a sound definition of the cost of capital (the rate of return used to discount to present value future expected cash flows or better free cash flows). In my paper I put a strong opposing view to the current practice of using observed discount rates for risky debt, called YTM, and argue and demonstrate that the consistent cost of debt for valuation purposes is the risk adjusted return (or expected return as professor Robert C. Merton once clarified me).

- Option Valuation of Claims on Real Assets: The Case of Offshore Petroleum Leases. James L. Paddock, Daniel R. Siegel, James L. Smith. The Quarterly Journal of Economics, Volume 103, Issue 3, August 1988, Pages 479–508.

"*This paper extends financial option theory by developing a methodology for the valuation of claims on a real asset: an offshore petroleum lease. Several theoretical and practical problems, not present in applying option pricing theory to financial assets, are addressed. Most importantly, we show the necessity of combining option pricing techniques with a model of equilibrium in the market for the underlying asset (petroleum reserves). The advantages of this approach over conventional discounted cash flow techniques are emphasized. The methodological development provides important insights for both company behavior and government policy. Promising empirical results are reported.*"

- REAL OPTIONS: STATE OF THE PRACTICE. Alex Triantis, Adam Borison. Journal of Applied Corporate Finance. Volume14, Issue2, summer 2001, Pages 8-24.

"In the mid-1980s, financial economists began building option-based models to value corporate investments in real assets, laying the foundation for an extensive academic literature in this area. The 1990s saw several books, numerous conferences, and many articles aimed at corporate practitioners, who began to experiment with these techniques. Now, as we approach the end of 2001, the real options approach to valuing real investments has established a solid, albeit limited, foothold in the corporate world. Based on their recent interviews with 39 individuals from 34 companies in seven different industries, the authors of this article attempt to answer the question, "*How is real options being practiced, and what impact is it having in the corporate setting?" The article identifies three main corporate uses of real options—as a strategic way of thinking, an analytical valuation tool, and an organization-wide process for evaluating, monitoring, and managing capital investments. For example, in some companies, real options is used as an input into an M&A process in which rigorous numerical analysis plays only a small role. In such cases, real* *options contributes as a qualitative way of thinking, with little formality either in terms of analytical rigor or organizational procedure. In other firms, real options is used in a commodity trading environment where options are clearly specified in contracts and simply need to be valued. In this case, real options functions as an analytical tool, though generally only in specialized areas of the firm and not on an organization-wide basis. In still other companies, real options is used in a technology or R&D context where the firm's success is driven by identifying and managing potential sources of flexibility. In such cases, real options functions as an organization-wide process with both a broad conceptual and analytical core. The companies that have shown the greatest interest in real options generally operate in industries where large investments with uncertain returns are commonplace, such as oil and gas, and life sciences. Major applications include the evaluation of exploration and production investments in oil and gas firms, generation plant investments in power firms, R&D portfolios in pharmaceutical and biotech firms, and technology investment portfolios in high-tech firms.*" …

<u>I apologize for my sloppiness in this section! But without any doubt, the main contributions from my personal point of view and interest, are</u>:

- Copeland, Weston, Shastri; Financial Theory and Corporate Policy, 2005; chapter on cost of capital structure and very particularly the explanation of the cost of debt through option pricing theory.

- Merton, Robert C., "On the Pricing of Corporate Debt: The Risk Structure of Interest Rate", The Journal of Finance, May 1974, pages 449 to 470.

PART 1: VALUATION WITHOUT TAXES (CORPORATE AND PERSONAL TAXES)

VALUE OF EQUITY

For simplicity, I am assuming that the firm has no non-operating assets (e.g. excess cash). This assumption is in the case of value of the firm with no taxes, and also with corporate taxes.

As already said, the value of equity, E, is equal to the value of a call option on the firm assets. Let us assume a value of assets V = $ 100.0, a riskless rate of return r = 3.0% (all returns are continuously compounded[5], unless specified something different), maturity T = 3 years, assets return volatility σ = 35.0%, assets return μ = 9.0%, and K = $ 45.0.

<u>In this first part of my work I use 2 methods to obtain the value of options</u> (call and put). The first is the well-known Black & Scholes or Black-Scholes-Merton analytical model. The second method is numerical as opposed to analytical, known as numerical integration approach or also risk neutral valuation.

In the Black-Scholes-Merton model, the formula for the European type call option is:

$$Call = V \times\ N(d_1) - K \times\ N(d_2) \times\ e^{-r\,T}$$

Omitting the multiplication signs, previous equation is:

$$Call = V\ N(d_1) - K\ N(d_2)\ e^{-r\,T} \qquad [1]$$

Recall that the call option value is equal in this context to the value of firm's equity. The formulas for d_1 and d_2 are[6]:

$$d_1 = \left[\ln(V/K) + \left(r + (\sigma^2/2)\right)T\right]/\sigma\sqrt{T} \qquad [2]$$

$$d_2 = d_1 - \sigma\sqrt{T} \qquad [3]$$

[5] If the log return r is 3.0%, then the arithmetic return is equal to *e* raised to 3.0%, minus 1.0. This is equal to 3.045453395%.

[6] ln(x) is the natural logarithm of x, where x has to be a number greater than zero. ln(x) is such that $e^{\ln(x)}$ = x. For instance, ln(4.0) = 1.386294361. Thus *e* raised to 1.386294361 is equal to 4.0. Number *e* is equal to 2.718281828459045235 (approximated to 18 decimal points).

N(y) is the cumulative probability distribution function for a variable Y with a standard normal distribution (mean equal to zero and variance equal to one). It is the probability that a variable Y with a standard normal distribution will be less than y.

$$N(y) = Pbb(Y < y) \qquad [4]$$

There are different numerical solutions to obtain the cumulative probability N(y). Fortunately, Microsoft Excel has a function, NORMDIST(y) that yields the cumulative probability for a specified y.

Plugging in the assumptions of the example into equations [2] and [3]:

$d_1 = 1.76876648443422$

$d_2 = 1.16254870178512$

Evaluating both results with Excel's NORMDIST(y) function:

$N(d_1) = 0.96153357397900$

$N(d_2) = 0.87749367292644$

With the results for $N(d_1)$ and $N(d_2)$, and assumptions for V, r, T, and K, and using [1] we obtain the value of the call option:

$Call = \$\ \mathbf{60.06475993244470}$

Thus, the value of firm's equity is \$ 60.06475993244470 (call value). In the more traditional DCF valuation context, this result is equivalent to the future expected cash flows to equity holders, FCFE, discounted by the levered risk adjusted cost of equity K_e. As I will show later, this two results, DCF and option valuation for equity value (and for any other cash flow being valued), are both numerically and conceptually compatible, and therefore should yield identical results.

Notice that for the valuation of equity in this section we assume known the market value of assets (V = \$ 100.0). This may not hold true in reality but it is not different from the assumption made to value a firm with DCF, where we commonly assume that the added independent values of debt and of equity is equal to the firm's asset value (V = D + E).

VALUE OF DEBT

In the context of the option pricing model, the firm's risky debt is valued as a riskless bond minus the value of a put option on the value of the firm's assets. The maturity of the put option is T and its strike price is K; these, T and K, are the same for the put and call options.

$$D = B - Put \qquad [5]$$

The put contract can be deemed as an insurance bought by the debt holder that would make the risky debt, riskless. This is because the put payoff is exactly equal to the possible loss to the debtholder when the firm defaults.

The formula for the value of a riskless bond, B, whose maturity is T and final contractual value is K (zero-coupon bond) is:

$$B = K\,e^{-r\,T} \qquad [6]$$

The symbol *e* in equation [6] is the base of the natural logarithm (*e* = 2.718281828459...). The analytical solution for the Black-Scholes-Merton differential equation for a European type put option is:

$$Put = Ke^{-r\,T}\,N(-d_2) - V\,N(-d_1) \qquad [7]$$

Plugging in the assumptions of the example into equations [6] and [7] and considering that N(–x) ≡ 1 – N(x):

$$B = K\,e^{-r\,T} = \$\,45.0\,e^{-3.0\%\,\times\,3} = \$\,\mathbf{41.12690333720530}$$

$$Put = \$\,\mathbf{1.19166326964994}$$

Thus, according to [5], the <u>value of the risky debt of the firm</u>, D, is equal to the value of riskless debt, minus the value of the put option:

$$D = B - Put = \$\,41.12690333720530 - \$\,1.19166326964994 = \$\,\mathbf{39.93524006755530}$$

In this example, the cost of buying insurance for the risky debt ($ 1.19166326964994) is approximately 2.99% of the value of the risky debt.

Thus, the value of firm's risky debt, using the option pricing approach, is $ 39.93524006755530. In the more traditional DCF valuation context, this result is equivalent to the future expected cash flows to debt holders, FCFD, discounted by the risk adjusted cost of debt K_d. As I will show later, this two valuation approaches are both numerically and conceptually compatible.

VALUE OF THE FIRM

According to Modigliani and Miller postulates, the value of the firm assets must be equal to the added values of the respective rights or claims on the firm's assets. In our case, firm's claims are equity and debt (later I add the taxes as a third claim on firm's pre-tax assets). This argument is based on non-arbitrage conditions[7].

$$V = E + D \qquad [8]$$

We assumed (exogenous assumption necessary for the option pricing model) a value of firm assets V = $ 100. Adding the value of equity and debt obtained in the numerical example:

$$E + D = \$\,60.06475993244470\ +\ \$\,39.93524006755530\ =\ \$\,\mathbf{100.00000000000000}$$

Thus, the valuation of debt and of equity through B-S-M[8] pricing model is consistent with the mentioned postulate of Modigliani and Miller.

ALTERNATIVE VALUATION APPROACHES AND RISK-NEUTRAL VALUATION

Up to this point I used what is known as analytic valuation approaches for derivative instruments/contracts. Analytic is the same as using mathematical formulas. The problem with analytic approaches is that not all cases are solvable with a mathematical formula. Due to this, a large number of alternative valuation approaches for derivatives have emerged in the last decades. Many of this approaches are labeled as "numerical approaches". The binomial tree approach is one example of a well-known and easy to implement method. Even Monte Carlo simulation is used to

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[7] Non arbitrage means that it is not possible to construct a portfolio consisting of assets, which perfectly replicates the future cash flows of another asset (and trading on this asset), obtaining in the process a total or almost total riskless profit (with no capital investment). The argument of non-arbitrage pricing is that, if any arbitrage opportunity would arise in any market, it will rapidly be eliminated by arbitrageurs trading in the market, making the prices to adjust to a new level, where arbitrage is not possible any more.

[8] B-S-M are the initials for Black, Scholes, & Merton option pricing model. These are financial economists Fisher Black, Myron Scholes, and Robert C. Merton.

value derivatives. Another approach, maybe less known, is numerical integration, which I intensively use in this working paper.

The analytic valuation approach already presented yields exactly the same results as the numerical integral approach, more specifically, the risk-neutral valuation approach. According to the latter, the value of a derivative is the future expected value of the derivative using risk-neutral probabilities, discounted to present value at the risk-free rate of return.

The stochastic process[9] assumed for the value of the underlying (the stock price in Black & Scholes, or the assets value V in its extension to firm valuation) variable is:

$$dV = \mu\, V\, dt + \sigma\, V\, \tilde{\epsilon}\, \sqrt{dt} \qquad [9]$$

dV is the change in V in a very small time interval *dt* (*dt* tends to zero). The first component at the right hand side of [9] is non-random (deterministic). The second component, $\sigma\, V\, \tilde{\epsilon}\, \sqrt{dt}$, is the random component of the change in V. Randomness in this equation comes uniquely from $\tilde{\epsilon}$.

This can also be expressed in instantaneous return form (the change in V, *dV*, divided by the initial level of V in the time interval *dt*):

$$dV/V = \mu\, dt + \sigma\, \tilde{\epsilon}\, \sqrt{dt} \qquad [10]$$

Variable μ, assumed constant, is the continuous annualized return of firm's assets. Variable ϵ is a random variable that has a standard normal distribution, this is, its mean is zero, and its variance is 1.0.

$$\tilde{\epsilon} \sim \phi(0, 1) \qquad [11]$$

ϕ is the symbol for a Gaussian normal distribution. The arguments inside the parenthesis are the mean and the variance, respectively, of the random variable. It is also assumed that the values of $\tilde{\epsilon}$ for any two different intervals of time *dt*, are independent or uncorrelated.

[9] A stochastic process defined in simple terms is a process through which the change of the value of a variable in time (when time changes) has a component being random and another component supposedly being not random. For instance, average temperature changes when seasons of the year change; we might think that there is an expected change in average temperature from fall to winter, but also there is an "extra" unexpected component in the change of average temperature. In colloquial language you may say, "This winter has been extraordinarily cold". In finance, changes in prices or values are supposed to be some type of random process.

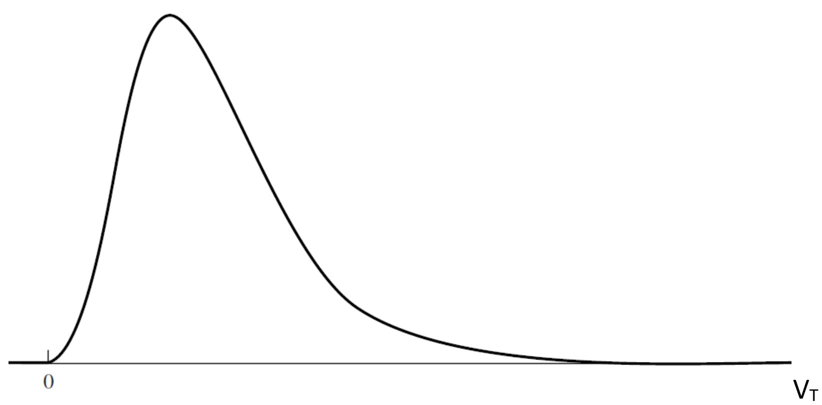
The instantaneous variance of the return of V is σ. Thus, since the expected variance of ϵ is equal to 1.0, then the second term in [10] has an expected variance equal to $\sigma^2\, dt$.

The natural logarithm of the value of firm's assets at future time T, V_T, is normally distributed as follows:

$$\ln V_T \sim \phi\left[\ln V_0 + \left(\mu \; - \; (\sigma^2/2)\right) T,\; \sigma^2 T\,\right] \qquad [12]$$

A random variable has a lognormal distribution if the natural logarithm of the variable has a normal distribution. Hence, future asset price V_T is lognormally distributed. The expected value of ln V_T is the first term inside the square brackets, and its variance is the second term (both in expression [12]). V_0 is the known value of the firm's assets today (t = 0). The next figure shows the shape (non-symmetrical) of the lognormal density function of V_T. The total area under this function and over the horizontal axis, is equal to 1.0.

Figure 1: Stylized lognormal probability density function



A variable that is lognormally distributed can only take values between zero and plus infinity. This is very convenient as a description of stock prices since it is illogic that stock prices in the market would have negative prices.

One check that can be done is about the future expected value of ln(V_T) assuming the given values of the parameters in our example. When I do the numerical Riemann integration, the resulting future expected value of V_T is[10]:

[10] All future expected values in this paper using the Riemann integration approach, both for actuarial and for risk neutral expectations, are done as follows:

$E(\ln V_T) = \$\ \mathbf{4.69142039051858}$

On the other hand, according to [12], E(lnV_T) is:

$$E(\ln V_T) = \ln V_0 + \left(\mu - (\sigma^2/2)\right) T$$

$\ln V_0 = \ln(\$\ 100) = 4.60517018598809$

$\left(\mu - (\sigma^2/2)\right) T = \left(9\% - (35\%^2/2)\right) \times 3 = 0.08625000000000$

$E(\ln V_T) = 4.60517018598809 + 0.08625000000000 = \$\ \mathbf{4.69142018598809}$

This is the analytical result of E(lnV_T). We can appreciate that the numerical integration result of E(lnV_T) only differs 0.0000043596711% of its analytical result. The reason of this small difference is twofold:

a) In my numerical integration Excel model I use 30,000 intervals for V_T. If I use more and more intervals, I would obtain more precision, but at the cost of computational efficiency (speed).
b) The number of decimal places, or mantissa[11] size, handled by Excel is limited to 15.

A second check is about the expected value of V_T. A basic premise is:

$$V_0\, e^{\mu\, T} = E(V_T) \qquad [13]$$

Or,

$$\mu = \frac{\ln(E(V_T)/V_0)}{T} \qquad [14]$$

From the numerical integral I obtain:

$E(V_T) = \$\ 130.99648079655100$

And dividing by V_0 and taking natural logarithm, and finally dividing by T = 3:

$$\ln(E(V_T)/V_0) = \frac{\ln(\$\ 130.99648079655100/\$\ 100)}{3} = \mathbf{9.00000909012497}\%$$

$$\sum_{i=1}^{30,000} V_i \times Pbb_i$$

[11] For example, for x = 3.14159, the mantissa is 0.14159. Source: Wolfram MathWorld.

Thus, the expected value of V_T obtained through numerical integration is compatible with the exogenously assumed assets return μ = 9. 00000000000000 %.

In the remainder of this working paper I use a different notation for present values obtained through numerical integration, as opposed to present values obtained through B-S-M analytic approach. In the latter, equity, debt, put option, and so forth are, E, D, Put, etc. For the numerical approach I use a super script, the symbol of integral, as follows: $E^{\int}$, $D^{\int}$, $Put^{\int}$, and so forth.

Cox-Ross-Rubinstein numerical valuation approach

A well-known numerical method for valuation of options, either "European" type or "American" type (first allows exercise of the contract only at maturity; the second can be exercised at any time of the contract until its maturity included). Merton [1995] comments about this method, called the "risk neutral" pricing method:

> "*The binomial option pricing model is elegantly simple. Its derivation does not require continuous trading and, hence, avoids the mathematical complexities of Itô stochastic integrals. It thus provides a powerful pedagogical tool for developing the economic intuitions underlying the pricing of options in arbitrage-free price systems. Of course, the practical applications of the model in a strictly discrete-time setting are severely limited by the assumption that the underlying stock price can only take on two possible values. However, as we now show, the model does provide a rather practical technique for computing approximate solutions to continuous-time option pricing models.*" "*The Cox-Ross-Rubinstein binomial option pricing formula is a necessary condition to rule out arbitrage opportunities for all sample paths of the stock price. Hence, by the proper selection of a binomial process that converges to the diffusion process posited by Black and Scholes, the binomial option pricing formula will converge to the Black-Scholes option price.*"

The numerical integration approach I use in this paper has at least two important advantages over the binomial Cox-Ross-Rubinstein approach. First, it does not require to define a specific variance for the underlying value movement in a small time interval and is not restricted to a two node movement at the small interval time. In fact there is no small interval time defined. The second advantage is computation power. In the specific Excel model I built I use 30,000 future state values for the underlying value at time T (time of expiration of the option). If binomial trees are used, to obtain 10 future state values (and thus 9 small time intervals), we need to compute (10 x 9) / 2 + 10 = 55 nodes. In general, for n time intervals (n + 1 last state nodes), we need to compute (n + 1) x n/2 + (n + 1) nodes. For 30,000 final state nodes would be (30,000 x 29,999) / 2 + 30,000 = 450,015 computations. This is a ratio of 15,000 thousand times the required number of final nodes.

The value of equity:

If the risk-free rate of return r is replaced for μ in [12], the resulting probabilities in the density function are "risk-neutral". A very special feature of moving to risk neutral probabilities is that any future expected risk neutral value can be discounted at the risk free rate of return, obtaining exactly the same value as with risk adjusted return discounting.

$$\ln V_T \sim \phi\left[\ln V_0 + \left(r - (\sigma^2/2)\right) T,\ \sigma^2 T\right] \qquad [15]$$

This change of variable and density function allows discounting future expected values at the risk-free rate of return, and obtaining correct arbitrage-free prices. Let us define the expected value of a call option at maturity in a risk-neutral world as:

$$\hat{E}[max(V_T - K; 0)] \qquad [16]$$

The symbol Ê(x), as opposed to E(x), denotes the expected value using risk-neutral probabilities, or the expected value in a risk-neutral world. This is a world where investors don't require a compensation for the risk bared[12]. The expression inside squared brackets in [16] is commonly known as the payoff function of the call contract. **The value of the call option using risk-neutral valuation approach is**:

$$Call^{\int} = e^{-r\,T}\,\hat{E}[max(V_T - K; 0)] \qquad [17]$$

[12] Bear in mind that this statement, than investors don't require extra compensation for the risk bared, is not literally correct. A better wording would be: in a risk neutral world, since we assume that probabilities are adjusted to risk neutrality, we can say that with these probabilities, they would act as if they were discounting all risky expected cash flows at the risk free rate of return. But risk neutral probabilities don't make future cash flow certain; they still have a variance. In the context of CAPM model we may think about risk neutral probabilities as assuming that every asset has a beta equal to zero.

And this is the value of the firm's equity. The upper integral symbol ∫ in $Call^{\int}$ means that the present value is obtained through risk neutral expected returns, thus discounting at the risk free rate of return r.

Using the assumptions for the example, and the Riemann integration approach (I use 30,000 equally spaced value intervals), I obtain the following results for equations [16] and [17]:

$\hat{E}[Call_T] = \$\ \mathbf{65.72134518015420}$

This is the risk-neutral future expected value of the call option in T. Discounting this value at the risk free rate r we get the call value:

$Call^{\int} = \$\ 65.72134518015420\ \ e^{-3\% \times 3} = \$\ \mathbf{60.06478689811780}$

This is the value of equity according to the numerical integration model. This result, compared with the result using Black & Scholes analytics (shown some pages before), differs in only in $ 0.00002696567316, or in relative terms in 0.0000448943327%, practically proving that both valuation methods are identical[13].

Equation [17], the value of a call option, can also be expressed as[14]:

$$Call^{\int} = e^{-r\,T} \int_0^{\infty} max(V_T - K; 0)\, f(V_T \mid V)\, dV_T \qquad [18]$$

Or also as, since the call will be worthless at maturity if V_T is smaller or equal than K:

$$Call^{\int} = e^{-r\,T} \int_K^{\infty} (V_T - K)\, f(V_T \mid V)\, dV_T \qquad [19]$$

f (V_T | V) is the log-normal, risk-neutral density function of the random variable V_T, the final random value of firm's assets at time T.

[13] It is possible getting more exactitude in computations by using a larger number of intervals in the Riemann integration, say 100,000 instead of the 30,000 y decided to use. With this we would get even closer results to the analytic solution. But this greater precision is at the cost of computational efficiency/speed.

[14] The integration has a lower bound of zero. Recall that V can't take negative values, and also that the lognormal density function lower limit in V axis (horizontal axis) is also zero.

The value of debt:

The value of debt can also be obtained through risk-neutral valuation. Let us define the expected value of a put option at maturity in a risk-neutral world as:

$$\hat{E}[max(K - V_T; 0)] \qquad [20]$$

Recall that the symbol Ê denotes the expected value using risk-neutral probabilities, or the expected value in a risk-neutral world. The value of the put option using risk-neutral valuation approach is:

$$Put^{\int} = e^{-r\,T}\,\hat{E}[max(K - V_T; 0)] \qquad [21]$$

This value, subtracted from the value of a riskless bond, B, whose final and unique payoff is equal to K, yields the value of firm's risky debt D.

Equation [21], the value of a put option, can also be stated as (recall that the put option will be worthless at maturity if V_T is larger than K):

$$Put^{\int} = e^{-r\,T} \int_0^K (K - V_T)\, f(V_T \mid V)\, dV_T \qquad [22]$$

Using the assumptions for the example, and the Riemann integration approach, I obtain the following results for equations [20] and [21]:

$\hat{E}[Put_T] = \$\,1.30388697108532$

This is the risk-neutral future expected value of a put in T. Discounting this value at the risk free rate we get the put value:

$Put^{\int} = \$\,1.30388697108532\; e^{-3\% \times 3} = \$\,\mathbf{1.19166296494372}$

The present value of a riskless bond with final promised payment of K a time T is:

$B = K\,e^{-r\,T} = \$\,45.0\; e^{-3.0\% \times 3} = \$\,\mathbf{41.12690333720530}$

And the value of risky debt is:

$$D^{\int} = B - Put^{\int} = \$\,41.12690333720530 - \$\,1.19166296494372$$
$$= \$\,\mathbf{39.93524037226160}$$

As we can appreciate in previous results, differences between numerical integration (or risk-neutral valuation) results and Black & Scholes analytics results are negligible. This is a 0.0000448943327% relative difference for the call, and a 0.0000007630008% relative difference for the put.

If we add both values of equity and debt obtained through numerical integration risk neutral valuation we obtain almost exactly the exogenously assumed total value of assets V = $ 100.0.

$$E^{\int} + D^{\int} = \$\,60.06478689811780 + \$\,39.93524037226160 = \$\,\mathbf{100.00002727037900}$$

Before moving to the second part of my work, the introduction of corporate taxes un firm valuation, I would like to explore the implications of risk adjusted returns of all claims on the assets of the firm in a world without taxes, and analyzing on the Modigliani and Miller proposition of irrelevance of financing policy in a word with no taxes.

This will also settle the base for analyzing risk adjusted returns in a world with corporate taxes.

RISK ADJUSTED RETURNS OF EQUITY AND DEBT IN A WORLD WITH NO TAXES

As far as we have gone, the valuation of derivatives in this working paper has been done discounting future risk-neutral expected cash flows at the risk-free rate of return, either explicitly as in the risk-neutral valuation approach, or implicitly, as in the B-S-M analytic valuation approach.

When discounting at the risk-free rate of return, some type of adjustment must be done to the true or actuarial future expected cash flows in order to obtain the true or fair price of an asset. One adjustment is using a density function assuming risk-neutrality from investors. This is what we did when using expectation in a risk-neutral world, thus risk-neutral probabilities. Another possible adjustment is the "certainty equivalent valuation approach". In it a monetary amount is deducted from the future expected cash flows, such that the resulting number can be discounted to present value at the risk-free rate of return, obtaining the true (fair) current value of the asset.

$$V_0 = [E(V_T) - CE]/(1 + r) \qquad [23]$$

CE is the monetary amount to be subtracted from the future expected value, known as certainty equivalent[15].

[15] The certainty equivalent in the single period Capital Asset Pricing Model can be computed as:

In firm valuation, three discount rates are essential in the whole valuation process:

- the cost of equity (levered);
- the cost of risky debt;
- And the weighted average cost of capital known as WACC.

All these are risk adjusted returns.[16] In general terms, a risk adjusted return will be larger or equal than the risk free return r. But there are cases where the risk adjusted return is smaller than r; for example when the asset's beta is negative. Put options have negative betas.

The following equations are the usual way to represent and compute each risk-adjusted discount rate:

$$K_e = r + \beta_e \, MRP \qquad [24]$$

$$K_d = r + \beta_d \, MRP \qquad [25]$$

$$WACC = (D/V) \, K_d(1 - \tau_c) + (E/V) \, K_e \qquad [26]$$

In the previous equations[17], r is the risk-free rate of return, MRP[18] is the return in excess to r expected to be earned when investing in the "market portfolio", and τ_c is the corporate tax percentage rate. In order that the relative weights add 100%, then it must be true that V = D + E. Equations [24] and [25] are the Capital Asset Pricing Model versions for a risk adjusted return of any financial asset. The factor that in CAPM determines the size of the risk adjustment in return, over

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$$V_0 = \frac{E(V_1)}{1 + (r + \beta_i MRP)}$$

This first expression is discounting the future expected actuarial value or cash flows at the risk adjusted rate of return for the asset being valued. Alternatively, we can obtain the same result as:

$$V_0 = \frac{E(V_1) - \lambda Cov(V_1, R_m)}{1 + r}$$

Notice that the covariance term includes dollar values vs returns. λ is defined as:

$$\lambda \equiv \frac{E(R_m - r)}{Var(R_m)} = \frac{MRP}{Var(R_m)} = \beta_i \frac{MRP}{Cov(R_V, R_m)}$$

[16] Pablo Fernandez argues that WACC as a discount rate of FCFF, is a mixture or hybrid of required and expected returns. Here I intend to show that WACC is a genuinely risk adjusted return, the same in nature as Kd and Ke.

[17] I am assuming that the CAPM model is used in practice to determine the risk adjusted return of a risky asset.

[18] Damodaran calls this variable in CAPM, "equity risk premium" (ERP) instead of market risk premium, but they both mean exactly the same.

r, is called the beta of the financial asset, βi, also known as systematic risk or non-diversifiable risk. Its mathematical expression is defined as:

$$\beta_i \equiv Cov(R_i, R_m)/Var(R_m) \qquad [26\text{-}1]$$

If we rename K_d $(1 - \tau_c)$ in WACC as $K_{d,at}$ (after tax cost of debt), then WACC is:

$$WACC = (D/V)\, K_{d,at} + (E/V)\, K_e \qquad [26\text{-}2]$$

As I will try to show later in this paper, that there is apparently no need to assume any specific theory of how a risk adjusted return is obtained. With the option pricing model applied to the firm, these adjusted returns are endogenously determined and all meet weighted adjusted criteria as I will show.

Inferred risk adjusted returns:

Instead of taking a stand on which equation or theory to use to obtain the cost of equity, K_e, and the cost of debt, K_d, we can use the present values already obtained through risk-neutral valuation for E and D ($E^{\int}$ and $D^{\int}$). If we knew the true or actuarial future expected cash flow for equity and debt, we may compute the implied risk adjusted returns as follows:

$$K_e = \ln\left[E(Call_T)/E^{\int}\right]/T \qquad [27]$$

$$K_d = \ln\left[\left(K_T - E(Put_T)\right)/D^{\int}\right]/T \qquad [28]$$

Notice that the operator E( ) means true or actuarial expected value; previously we used Ê( ) as the expected value using risk-neutral probabilities.

True or actuarial expected values for the call and put options can be obtained by using equation [12], the density function of $\ln V_T$:

$$\ln V_T \sim \phi\left[\ln V_0 + \left(\mu - (\sigma^2/2)\right) T,\ \sigma^2 T\right] \qquad [12]$$

Notice that r = 3.0% is replaced by μ = 9.0% in [12][19]. Using the values assumed in the example I obtain the following results combining equation [12] with numerical integration:

[19] Using r in [12] instead of μ, implies that the density function is adjusted to risk-neutral probabilities. Using μ implies that the density function is expressed in actuarial (true) probabilities.

$E(Call_T) = \$\,86.70115802829550$

$E(Put_T) = \$\,0.70467723174121$

These two results are computed using the same Riemann integration approach already commented in risk neutral valuation. Recall that the risk-neutral expected value of the call and of the put, as previously seen were:

$\hat{\mathrm{E}}(Call_T) = \$\,65.72134518015420$

$\hat{\mathrm{E}}(Put_T) = \$\,1.30388697108532$

These are the future actuarial expected values of the call and put options, respectively.

Notice that the risk-neutral expected value of the call is always smaller than its actuarial expected value. In the case of the put, the inverse situation occurs because the beta of a put is negative. In the latter, if you prefer, its risk adjusted discount rate is smaller than the risk free rate of return r.

Replacing these E( ) values (actuarial expectations) in formulas [27] and [28], respectively, we obtain the implied risk adjusted returns of equity and debt, K_e and K_d:

$K_e = \ln[\$\,86.70115802829550/\$\,60.06478689811780]\ /\ 3 = \mathbf{12.2347826370841}\%$

$K_d = \ln[(\$\,45 - \$\,0.70467723174121)/\$\,39.93524037226160]\ /\ 3 = \mathbf{3.4539982272085}\%$

Denominators in previous expressions add to D + E:

$\$\,60.06478689811780 + \$\,39.93524037226160 = \$\,100.00002727037900$

Numerators in both expressions, K_e and K_d, are the future actuarial expected values of E and D at time T.

We see that both risk adjusted returns have a positive risk premium over the risk-free return r = 3%. The risk premium of equity, K_e – r, is 9.23478263708410% and the risk premium of debt, K_d – r, is 0.45399797287485%[20].

[20] Since all returns are logarithmic, seems OK expressing the risk premium over r as a difference (both rates subtracted). Log returns are additive in the sense that total return is the sum of partial returns. This is not the case with arithmetic returns.

VALUE WEIGHTED AVERAGE OF RETURNS ON FIRM CLAIMS

In a world without taxes (later I see the case with corporate taxes) it should be quite simple calculating a weighted average return as follows:

$$WACC = (D/V)\,K_d + (E/V)\,K_e \qquad [29]$$

In fact this is what Robert E. Merton [Journal of Finance, may 1974] suggests as a straightforward way to compute K_e given a specific formula he proposes for K_d; same thing is done by Thomas E. Copeland et al. [2005], when demonstrating that the weighted average of risk adjusted returns analytically expressed as functions of the option pricing formula (B-S-M), is equal to the assets' return. See **Annex 1** in this paper for a detailed explanation of my point on the possible error in Merton using the linear combination of costs of capital in a discrete time framework.

Using the assumptions and partial results from our example, following is the WACC calculation and final result:

$$D^{\int}/V = \$\,39.93524037226160/\$\,100 = 39.93522948177300\%$$

$$E^{\int}/V = \$\,60.06478689811780/\$\,100 = 60.06477051822700\%$$

$$WACC = 39.93522948177300\% \times 3.45399797287485\% + 60.06477051822700\% \times 12.23478263708410\%$$

$$WACC = \mathbf{8.72815613113180}\%$$

<u>**We see that with this result for WACC, the linear equation [29] DOES NOT WORK**</u>. <u>The weighted average of returns of debt and of equity is not equal to the assumed asset return μ</u> of 9.0% in the example.

<u>**Here, in my opinion, is a misconception present in various works, including Merton's as already cited, and Chi-Chen Hsia 1981 cited in Copeland et al. 2005**</u>. In their respective works they use the linear equation to relate K_e, K_d and μ. But according to my calculations, <u>there is one particular case in which the common linear WACC is valid; when the time interval between present time t = 0, and maturity t = T tends to zero</u> (see **Annex 1** for a numerical explanation). The practical problem with these equations is that in valuing a firm, the time interval from time zero to a supposed "end of time" or maturity T (in option pricing theory applied to the firm) IS NOT approximately zero, but any discrete number, as for example 6 years. In fact, Merton does consider T as an input to determine

debt's YTM and debt's risk adjusted return. The problem is when he uses the linear weighted average of costs of debt and equity to infer the cost of equity of the firm. A simple demonstration of this inconsistency is that it doesn't meet the simple criteria:

$$E_0\, e^{K_e T} = E(E_T) \qquad [30]$$

Option pricing and TOTAL returns:

The linear expression of weighted returns DOES WORK at the level of TOTAL returns and if the returns are obtained as simple compound returns (not as logarithmic or continuously compounded returns). The TOTAL compound returns of equity and debt in our example, respectively are:

$$\left[E(E_T) - E_0^{\int}\right]/E_0^{\int} \qquad [31]$$

$$\left[E(D_T) - D_0^{\int}\right]/D_0^{\int} \qquad [32]$$

Respective results are: 44.3460678140062% and 10.9178819392453%, both 3-year returns. Their value weighted average is 30.9964450733273%. And this is exactly equal to the total return on assets:

$$[E(V_T) - V_0]/V_0 \qquad [33]$$

This yields a total implied return on assets of **30.9964450733272%**.

But notice that this other way of expressing returns as arithmetic (non-logarithmic) returns is consistent with logarithmic returns. Take for example equity's total return of 44.3460678140062%. Its yearly compound return is:

$$(1 + 44.3460678140062\%)^{1/3} - 1 = \mathbf{13.0147128398032}\%$$

This is the annualized return of equity in non-logarithmic form. But we can re express this result in logarithmic base as:

$$\ln(1 + 13.0147128398032\%) = \mathbf{12.2347826370841}\%$$

And this is exactly the same value that we previously obtained through numerical integration, deriving the implied risk adjusted return. The same reasoning applies to the total implied non-log return on assets of 30.9964450733272%.

$(1 + 30.9964450733272\%)^{1/3} - 1 = \mathbf{9.4174283705217}\%$

Previous result re expressed in logarithmic base:

$\ln(1 + 9.4174283705217\%) = \mathbf{9.0000000000006}\%$

We obtain exactly the exogenously assumed return on assets in log form. The same can be demonstrated for debt's risk adjusted return.

<u>A different (and consistent) perspective on total weighted returns</u>:

The concept of a <u>weighted average return</u> is one such when applied to the assets' expected future value, $E(V_T)$, as a discount rate, results in the current value of the assets V_0 or simply V. Algebraically:

$$V = E(V_T)\, e^{-T\ WACC} \qquad [34]$$

In words, discounting the future expected value of the firm cash flow at WACC, MUST yield as result the current value of the firm assets V, if WACC is properly defined as I will prove[21]. We should recall that WACC discounts the after-tax cash flows to shareholders in an unlevered firm. For the moment we are working under the assumption of no taxes.

Using the risk adjusted return I obtained through numerical integration when switching from risk neutral to actuarial probabilities, for firm equity:

$$E(E_T) = E^{\int}\ e^{Ke\ T} \qquad [35]$$

And for firm's debt:

$$E(D_T) = D^{\int}\ e^{Kd\ T} \qquad [36]$$

<u>This is, the future actuarial expected value of a security is equal to its current value, compounded at its risk adjusted return</u>.

[21] I think differently from Pablo Fernandez, as far as I deduct from his writing, in his idea that WACC is a mere arithmetical artifact meant just to make equal the present value of free cash flow of the firm, with the sum of 2 present values: the PV of equity discounting free cash flow to shareholders at the cost of equity K_e, and de PV of debt, discounting cash flows to debt holders at the cost of debt K_d. From this paper I think that WACC genuinely is a risk adjusted return, applicable to after tax operating or unlevered cash flows, this is FCFF.

The sum of future expected values of debt and equity must be equal to the future expected value of the assets.

$$E(V_T) = E(D_T) + E(E_T) = D^{\int} e^{Kd\,T} + E^{\int} e^{Ke\,T} \qquad [37]$$

Replacing the left hand side of the previous expression:

$$V\, e^{\mu\,T} = D^{\int} e^{Kd\,T} + E^{\int} e^{Ke\,T} \qquad [38]$$

Dividing both sides by V:

$$e^{\mu\,T} = \left(D^{\int}/V\right) e^{Kd\,T} + \left(E^{\int}/V\right) e^{Ke\,T} \qquad [39]$$

Is this a weighted average return? It is a weighted average of total returns. By total I mean the return comprehended in the total number of years, T.

$$Total\ return_i = e^{Ki\,T} \qquad [40]$$

Total return represents the future expected value of \$1 today growing at a continuous expected return K_i for T periods into the future; or, if you wish, \$1 plus its monetary return.

Taking natural logarithm at both sides of previous equation [39], and dividing by T:

$$\mu = \ln\left[\left(D^{\int}/V\right) e^{Kd\,T} + \left(E^{\int}/V\right) e^{Ke\,T}\right]/T \qquad [41]$$

The right hand side represents the weighted average total return, using K_d and K_e on an annualized basis. The left hand side is the demonstration that the right hand side is equal to the risk adjusted return of the assets of the firm μ.

Plugging in the numbers from the example we obtain:

$$WACC =$$
$$\ln\left[39.9352294817730\%\, e^{3.4539982272085\%\ \times 3} + 60.0647705182270\%\, e^{12.2347826370841\% \times 3}\right]/3$$

$$WACC = \mathbf{8.9999999999999}\% = \mu$$

Two ideas in concluding this section. The first is that the "normal" weighted averaging of returns (linear), equation [29], is better suited to non/logarithmic total period returns. When dealing with continuously compounded log returns, as in option pricing models, the weighting scheme is as previously explained. As I said, it seems to me that Robert Merton and other authors mix both things, possibly leading to erroneous conclusions to some readers when using the linear expression together with logarithmic returns, which I think is not suited for a discrete time interval T not close to zero.

Cost of debt and YTM (yield to maturity) of debt:

I decided to include this short section to connect my work with Robert Merton's [1974] development. From a valuation standpoint, is important to make the point that when discounting the free cash flow of the firm's assets at the weighted average cost of capital, WACC, the latter must include the risk-adjusted cost of debt, K_d, and NOT the YTM corresponding to the same debt. In some cases, using YTM instead of K_d won't lead to a significant error. But in other cases (in my opinion, more than expected), as I intend to show here, it will do.

Moreover, Merton develops a complex and quite complete analysis of the "Risk Structure of Interest Rates" (not the Term Structure!), entirely based on the analysis of YTM and its determinants (as far as I understand from his paper). More precisely, in the determinants of the debt spread expressed as debt-s YTM minus the risk free rate of return r. In my modest and respectful opinion, his work could have been more interesting analyzing the determinants of the risk-adjusted return on debt and corresponding determinants of the risk adjusted debt spread.

Since yield to maturity (YTM) of debt is commonly used as the "return" of debt in capital markets, even though it is not a risk adjusted return nor an expected return, I consider here its expression in log terms[22]:

$$YTM = \ln[K_T/D]/T \qquad [42]$$

YTM is nothing more than a mathematical convention. It is no different from the internal rate of return (IRR) in an investment made. Given the market price of debt, the IRR of debt is YTM using as "cash flows" the coupon promised payments. But promised doesn't mean expected payments. YTM is the best case return the investor may obtain. **Following, the answer Professor Robert C. Merton**

[22] YTM in log terms is:

$$YTM = \ln[K_T/D]/T$$

But in capital markets, YTM is not calculated in log terms but as an arithmetic compound return; assuming a single coupon, its expression is:

$$YTM = (K_T/D)^{1/T} - 1$$

kindly sent me in 2012, I think is the best explanation about how meaningless may YTM become (authorization to cite, given to me by Professor Merton):

> "It is absolutely correct that yield to maturity, even on a zero-coupon bond [in which there is no reinvestment of coupons] is NOT the expected rate of return on the bond. It is the promised rate of return, and is thus the maximum rate of return which is clearly > the expected return. Your economic rate of return on debt is I believe, its expected return. Just as the price of an option gives no indication about its expected return or the expected return on the underlying stock, so yield to maturity cannot even be used to rank order expected returns on bonds. So two zero-coupon bonds with the same maturity and different prices [and hence different yields to maturity], it cannot be inferred that the one with the higher yield to maturity will have a higher or even equal expected return , than the one with the smaller yield.
>
> My work is quite clear that the yield to maturity is nothing more than a mechanical transformation of the price of the bond. It does nothing to indicate what the expected return is on the bond [other than provide an upper bound]. Like the term structure of interest rates [which also does not contain the expected return differences of bonds of different maturities], we do plot the yield to maturity on credit risky bonds against various parameters. If you look elsewhere in the article in my book, you will find plots of expected rates of return on the firm's leveraged equity and risky debt, relative to the expected return on the firm overall, or on the firm's assets."

As I will show, YTM can significantly over estimate the risk adjusted return of debt, potentially leading to mistaken decision making in investment decision policy and in portfolio selection.

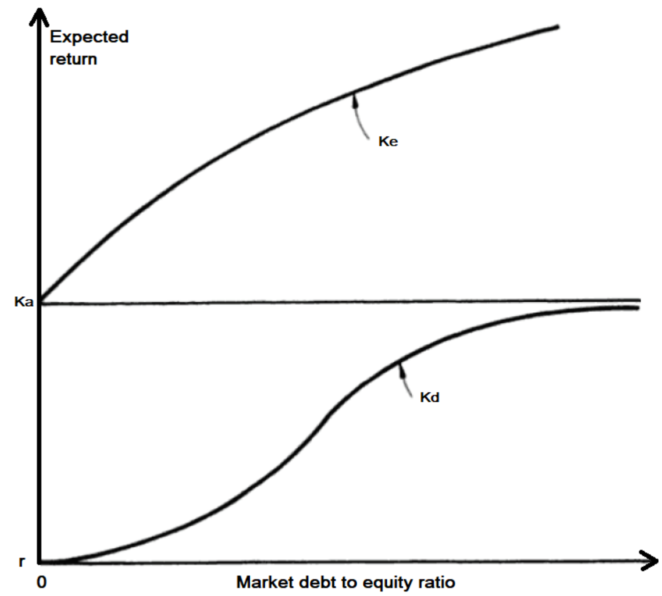
The resulting YTM of debt in my numerical example is YTM = **3.9801112777175%** calculated in logarithmic form; in non-logarithmic form is slightly different, 4.0603790811880%. Anyway, the point is that the risk premium, YTM – r, is **0.9801112777175%** being a bit more than **2 times** the risk adjusted risk premium. Recall that the risk-adjusted cost of debt in the example is 3.4539982272085% and its risk adjusted risk, K_d – r, is **0.4539982272085%**.

Previous results for YTM are with the current assumptions in the numerical example of this paper, which yields a leverage D/V = 39.9352294817730%. I recall (from many years ago) Harry DeAngelo's comment on my proposition, discouraging me since the differences between spreads using YTM versus using a risk adjusted cost of debt were negligible in practice. Changing the assumption of K (debt's final agreed value) from $ 45 to $ 15, keeping everything else constant, leverage drops to 13.7059810015279% and risk adjusted cost of debt is 3.0047712767778%. In this same case YTM is 3.0072540459503%. In this lower leverage case I agree with my MBA tutor and good friend Harry, since in both cases the risk premium is quite small relative to the risk free rate. But let us see what happens in 2 more cases with higher leverage. If leverage D/V is 49.0371914951201%, K_d is 3.8404251103101%, and YTM rises to 5.0157324939389%. Risk premium using K_d is 0.8404251103101% while risk premium using YTM rockets to 2.0157324939389%. Alternatively, if leverage D/V is 60.0170497408526%, K_d is 4.4603429661186%, and YTM rises to 6.9812045432409%. In this case, risk premium using K_d is 1.4603429661186% while risk premium using YTM sky rockets to 3.9812045432409%, almost 3 times the risk premium with risk adjusted cost of debt. The higher firm's leverage, the higher will be YTM; if D/V is 80.0147036052390%, YTM is 11.3981807572194% and K_d is 5.5509326590303%.

Capital structure and costs of capital – world without taxes – consistency with Merton's results:

In this section I compare my results using the numerical example, with the results shown by Merton [1974]. The following figure is taken from Merton's paper (I made slight changes in it to avoid copyright infringement). In the horizontal axis is leverage calculated as D/E, both D and E at market value. The upper curve is the cost of equity; the lower curve is the cost of debt; the middle horizontal line is the weighted average of these two costs where the weights are at market values. They are all risk adjusted returns. Merton calls the vertical axis "expected return" which is the same as I call here risk adjusted return. Is important to highlight that Merton's results are in a world without taxes. In such a world, the value weighted average cost of debt and equity is equal to WACC.

Figure 2: From Robert C. Merton – cost of equity and cost of debt (no taxes)



Next figure shows my own results. These are based on the numerical example that we have been following during all this paper. As you can appreciate, cost of equity (upper curve) starts equal to the assumed assets return of 9.0% when D/E tends to zero. Then increases with leverage in a concave way, same as in Merton's figure. The weighted average cost of capital in all my simulations gives a constant equal to assets returns of 9.0%. The cost of debt schedule (lower curve) in my graph is similar to Merton's. It starts at the risk free rate of return of 3% when leverage tends to zero. When leverage starts going up, the cost of debt curve is convex up to a certain level in leverage. After this inflection point, cost of debt curve still rises with leverage but is concave. A second figure with a close-up shows this latter effect; the change from convex to concave occurs in my example approximately at a leverage D/E of 0.6 to 0.7. Finally, when debt ratio D/E tends to plus infinity, cost of debt is asymptotic to asset's return of 9.0%, same as in Merton's figure.

Figure 3: My cost of equity and cost of debt functions

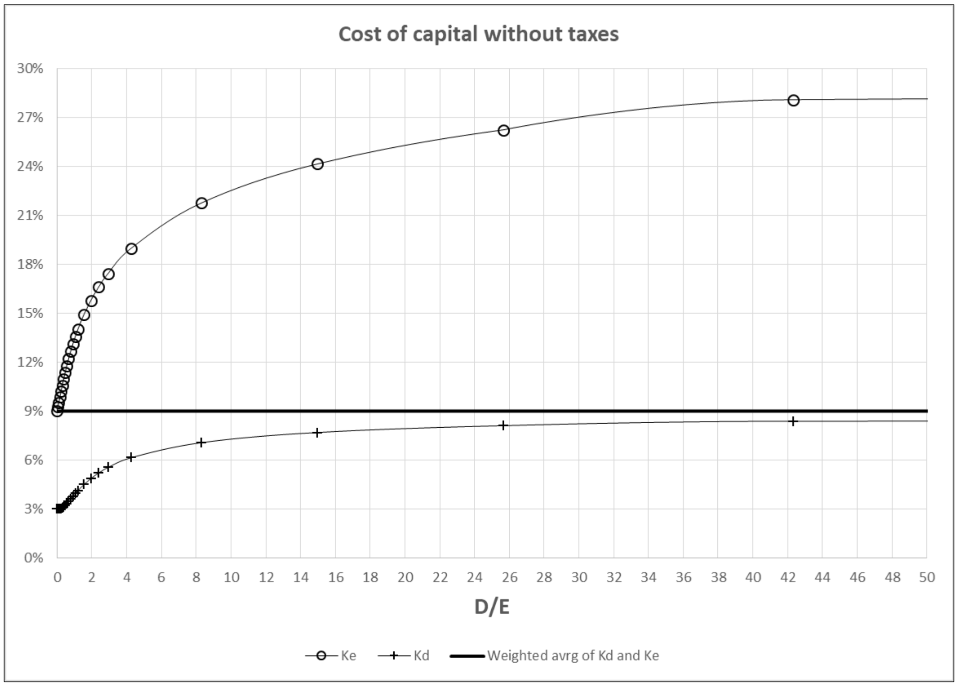


Figure 4: Close-up view of cost of equity and cost of debt in my model

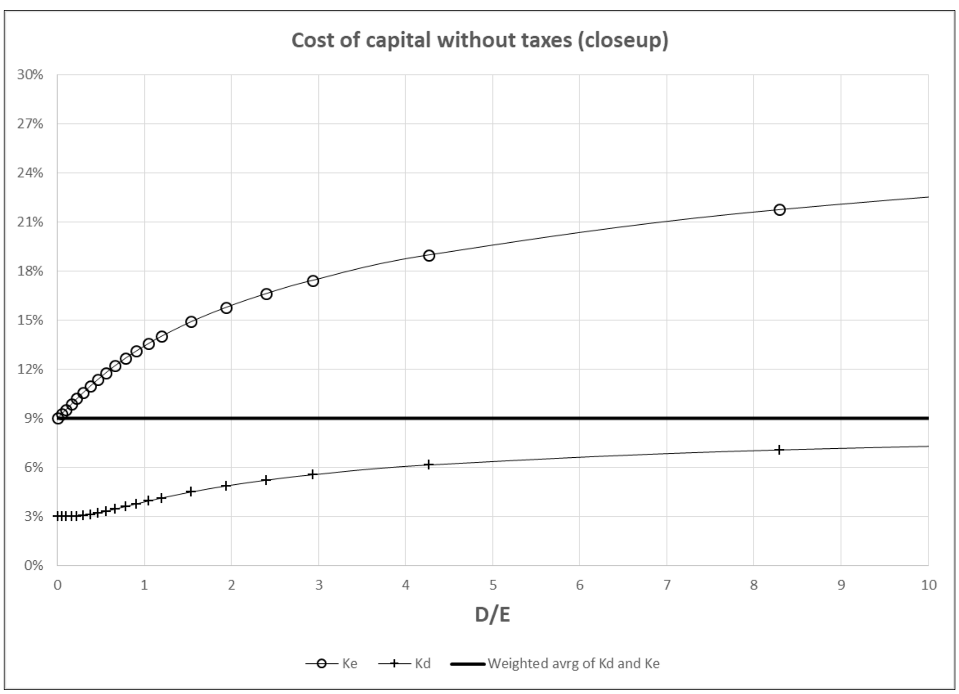
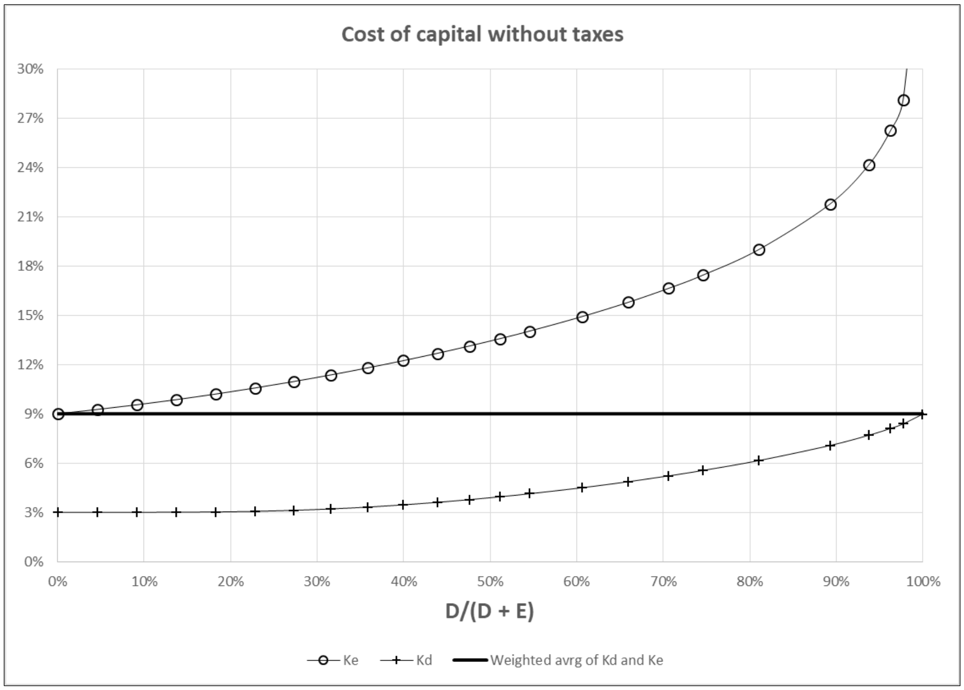


Figure 5: The same previous graphs but with the horizontal scale re expressed as D/(D + E), is:



<u>Consistency with Modigliani and Miller irrelevance of financial structure</u>:

Closing this first part of my work, I do the valuation of the firm without taxes using the same two additional leverages of the firm in the numerical example, I assumed in the previous section to argue about the difference of risk adjusted cost of debt and YTM. I show that the total value of firm's claims debt and equity, don't change with leverage, and always add up to the value of assets. I only show final results and not step by step calculations as before.

<u>In the example with its original assumptions</u> (K = \$ 45), leverage is 39.9352294817730%, value of debt $D^{\int}$ is \$ 39.93524037226160, and value of equity $E^{\int}$ is \$ 60.06478689811780. $D^{\int}$ and $E^{\int}$ added is:

$$E^{\int} + D^{\int} = \$\,60.06478689811780 + \$\,39.93524037226160 = \$\,100.00002727037900$$

<u>When we changed K to \$ 15</u>, leverage drops to 13.7059810015279%, value of debt $D^{\int}$ is \$ 13.70598473920010, and value of equity $E^{\int}$ is \$ 86.29404253117740. $D^{\int}$ and $E^{\int}$ added is:

$$E^{\int} + D^{\int} = \$\,86.29404253117740 + \$\,13.70598473920010 = \$\,100.00002727037750$$

When we changed K to $ 57, leverage is 49.0371914951201%, value of debt $D^{\int}$ is $ 49.03720486774580, and value of equity $E^{\int}$ is $ 50.96282240263150. $D^{\int}$ and $E^{\int}$ added is:

$$E^{\int} + D^{\int} = \$\,50.96282240263150 + \$\,49.03720486774580 = \$\,100.00002727037730$$

When we changed K to $ 74, leverage is 60.0170497408526%, value of debt $D^{\int}$ is $ 60.01706610772720, and value of equity $E^{\int}$ is $ 39.98296116265020. $D^{\int}$ and $E^{\int}$ added is:

$$E^{\int} + D^{\int} = \$\,39.98296116265020 + \$\,60.01706610772720 = \$\,100.00002727037740$$

We can conclude that M-M proposition that total firm value is invariant to leverage, at least is true using our numerical example. But I must confess there is a trick in these results; total value of assets, V, is an exogenous assumption in Merton's Structural Model, thus, value of debt plus value of equity can't be different from assumed V ($ 100.00 in our example).

Trying to elaborate a more general demonstration, at least intuitively, the density function at time T remains unchanged every time we vary K, the strike price of the put option and also the strike price of the call option or equity. Recall that in risk neutral valuation we are always discounting future risk neutral expected values at the risk free rate of return. Thus, changing the strike K will obviously bring a change in relative value of debt, but, future total contingent payoffs will remain unchanged; they will only be split differently. Therefore, discounting future expected payoffs, no matter how they are split, must yield a total value of debt and equity which is invariant. Later I will show that invariance in value in a world with corporate taxes also holds, but at a pre-tax level value of the firm. **At an after tax level, as I will show, value invariance does not hold**.

PART 2: VALUATION OF THE FIRM WITH CORPORATE TAXES

I apologize to the reader of this paper for the extended length of the first part, Valuation without Corporate Taxes, since the objective of the paper is introducing corporate taxes in firm valuation. The reason for this lengthy first part is laying the ground for the second part, the consideration of corporate taxes.

In this second part of my work, firm valuation considering corporate taxes of the firm, I think emerge very significant and novel results in valuation theory.

- One is that IT IS possible, at least in the context of the B-S-M option pricing model, to obtain a well-defined function (though numerical) of the risk-adjusted discount rate for the tax savings provided by the debt of the firm, more precisely, by the interest expense of debt.
- A second relevant result is that I obtain a risk-adjusted discount rate function for the TAXES paid by the firm. This second result might have important extensions to public policy and economics, as I am able to foresee.
- Third, I also derive a weighted average cost of capital WACC that is consistent with the contingent claim characteristic of taxes, with risk adjusted returns of debt and equity, and with free cash flow after tax which also depends on different future states of nature of the firm (the pre-tax assets' value). This third point has direct relation with the determination an after tax cost of debt, as traditionally considered in WACC, but under a different formulation I propose and prove consistent.

In this part of my work I introduce corporate taxes (not including personal taxes) to the problem of the valuation of a firm with debt. The valuation problem is analyzed once again, as in the first part, in the context of the Merton Structural Model (MSM). This is, visualizing all claims to the assets of the firm as contingent claims, therefore associated to contingent payoffs to each pre-tax stakeholder. To the two claims already analyzed, risky debt and equity, now is added the Government who has a legal claim called taxes. I name this third claim on the assets as "Tax". Therefore, future PRE-TAX value of the firm, V_T, is now split into: D_T, E_T, and Tax_T.

$$V_T = D_T + E_T + Tax_T \qquad [43]$$

It is also convenient to highlight that all three claims are over the pre-tax value of the operating assets of the firm V_T at a future time T; this would ordinarily be the same as revenues minus expenses.

Figure 6: Payoff functions of debt and equity at time T in a world without corporate taxes

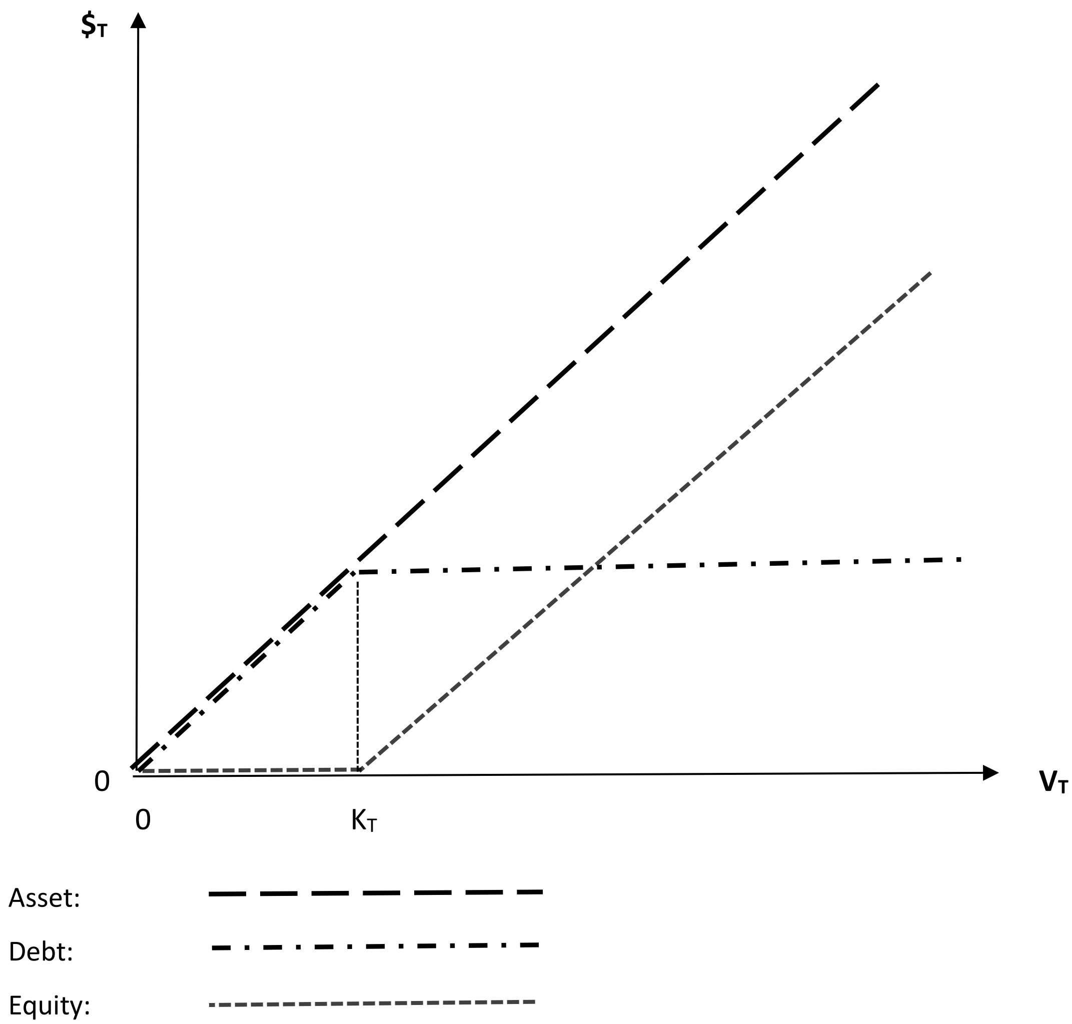
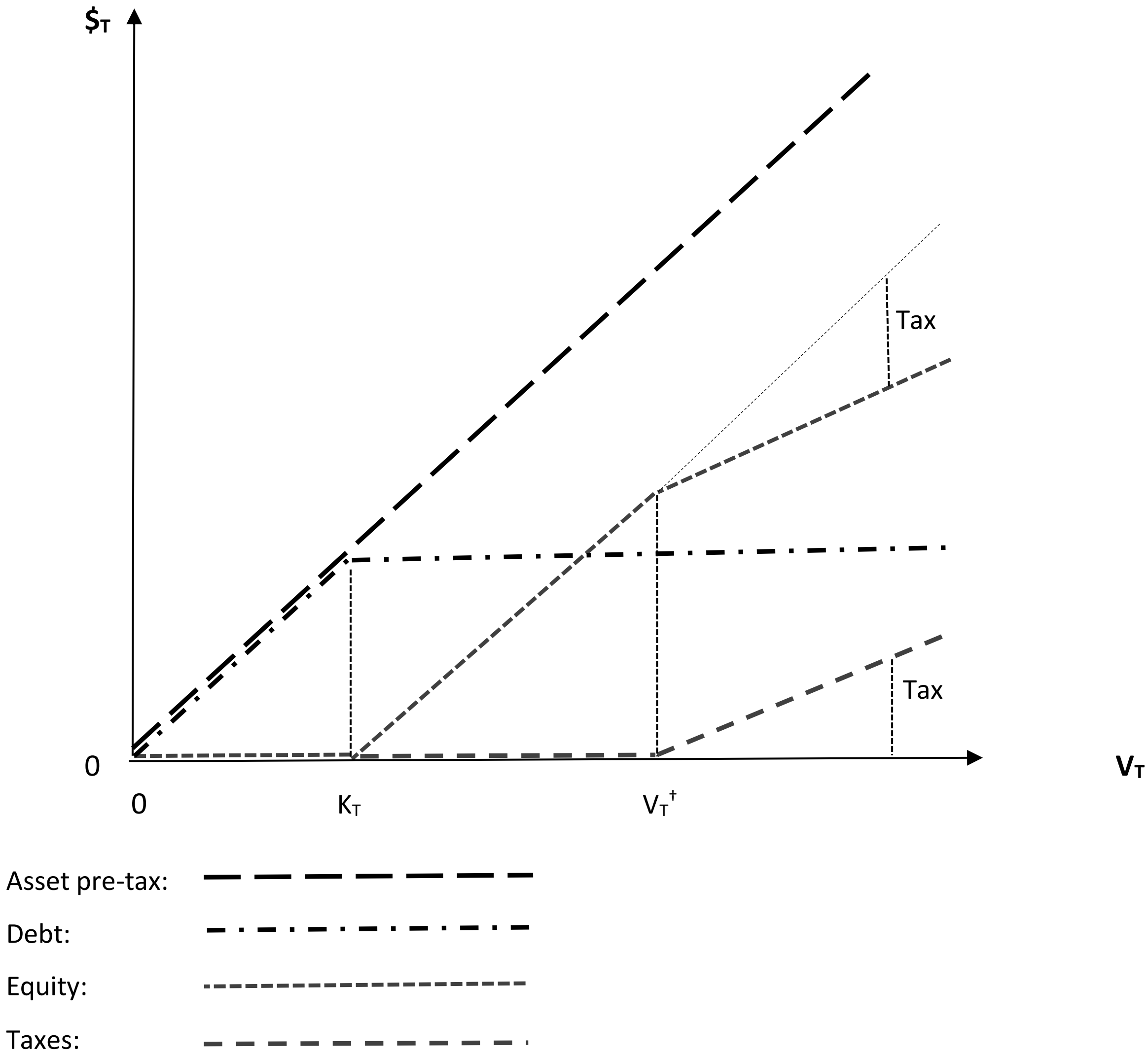


Figure 7: Payoff functions of debt, equity and taxes, at time T, in a world WITH corporate taxes



In the case of a world without taxes, Figure 6, the dashed line in 45 degrees is the value of firm's assets at maturity, V_T. This total value is shared between debt and equity as is shown in the corresponding payoff functions. If V_T is less or equal to K_T, the value of debt is V_T. In this case the firm is in bankruptcy. If V_T is larger than K_T, the value of debt is K_T (same as K). The fine dotted line is the payoff function for shareholders, assuming limited liability. If V_T is less or equal to K_T, the value of equity is zero. If V_T is larger than K_T, the value of equity is V_T - K_T.

In the case of a world with corporate taxes, Figure 7, the dashed line in black starting from the origin represents the pretax value of assets at time T. This line has 45 degrees angle from any of the two axis. Notice that payoff for equity has 45 degrees to the horizontal axis in a first leg; then the angle is smaller than 45 degrees because a percentage of this angle equal to the tax rate is paid to the government in the form of corporate taxes. The point in the horizontal axis where the firm starts paying taxes, $V_T^{\dagger}$ in the graph, is such where the pretax income, $V_T - V_0$ – Accrued interest at T, is positive[23].

$$V_T^{\dagger} = V_T - V_0 - Accrued\ interest\ at\ T > 0 \qquad [44]$$

Using the inequality at [44], the level of V_T where the firm pays taxes is where $V_T > V_0$ + Accrued interest.

The lowest line in Figure 2 is the contingent payoff function of taxes paid to (or received by) the government. Its slope is less than 45 degrees; it is equal to 45 degrees times the corporate tax rate in percentage terms, assuming a constant tax rate (not depending on the level of pretax income). Its form resembles a call option with a strike price equal to $V_T^{\dagger}$, this is the value of V_T where pretax income is exactly zero and if larger, tax is positive. Notice that V and V_T are both pre-tax value of firm assets, the first at t = 0 and the second at t = T.

The problem with analytical solutions of contingent claims value is the difficulty in handling more complex payoffs such as those generated by the existence of corporate taxes. With the numerical integration approach (and risk-neutral valuation) we can easily solve this problem, since the method is flexible enough to handle several and more complex payoff functions.

Defining taxable profits:

A basic task in this second part of my work is to properly define the taxable accounting pre-tax profit which will be multiplied by the statutory tax rate if it is positive to obtain the amount of corporate taxes paid; otherwise, corporate tax is zero.

[23] If initial book value of assets is \$ 100 and let's imagine that state contingent final value of assets V_T is \$ 135. If the initial book value of debt is \$ 45 and YTM is 10%, and T = 3 years, then final book value of debt is \$ 59.8950 (assuming compounded interest). Then accrued interest is \$ 59.8950 - \$ 45 = \$ 14.8950. Then, taxable income is equal to the increase in asset value, \$ 35, minus accrued interest, this is \$ 20.105. The level of V_T at which the firm breaks even in taxable income is equal to accrued interest plus initial book value of assets, \$ 100 + \$ 14.8950 = \$ 114.8950 = $V_T^{\dagger}$. This is the strike price of the real option I named Tax or Taxes.

I will make an **important assumption**. That the price at which assets are purchased (and booked in the balance sheet) at t = 0 is equal to their fair value. The same assumption applies for debt, and therefore for equity.[24] Taxable income is defined as:

$$Taxable\ income = \ MAX[0; (V_T - V_0) - Accrued\ interest\ at\ T] \qquad [45]$$

Also notice that the firm is assumed to pay taxes only at time T. Thus, if T is larger than 1 year, it will pay taxes for the "accumulated" taxable profits from t = 0 until t = T. In reality, taxes are paid on an annual basis.

Here an important distinction: $V_{0,\ tot}$ is the pre-tax total value of firm assets, and V_0 is the after tax current value of firm assets, such that:

$$V_{0,tot} = V_0 + Tax_0 \qquad [46]$$

This is, the current pre-tax value of firm assets $V_{0,\ tot}$ is equal to the current after-tax value of firm assets, V_0, plus the current value of taxes paid by the firm, Tax_0. Notice that $V_0 = D_0 + E_0$.

Accrued interest, under my current assumption that debt at time zero is recognized in the balance sheet at its market or fair value, is:

$$Accrued\ interest\ at\ T = K_T - D \qquad [47]$$

This is, accrued interest is the nominal change in value of debt between its initial accounting value and its final accounting value in the case that debt is fully paid. Remember that in the context of B-S-M model applied to the firm, there are no intermediate payments or cash flows. In the case that final contractual debt K_T is not fully paid, I assume that this specific "profit" from the firm's perspective is not recognized for tax purposes[25]. Accrued interest can equivalently be obtained as:

$$Accrued\ interest\ at\ T = D\ e^{YTM\ T} - D = D\ (1 - \ e^{YTM\ T}) \qquad [48]$$

Notice that accrued accounting interest is NOT calculated with the risk adjusted cost of debt, K_d, but instead at the yield to maturity or YTM rate, which is easily observable and in fact, part of the debt contract.

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[24] This assumption is the same as assuming zero NPV for any real investment. I could have assumed a positive NPV (or negative), but this wouldn't provide a better insight, but the contrary, doing calculations and explanations more difficult.

[25] When the firm goes broke there is no point in considering the benefit for tax purposes due to not paying all of the interest from debt, since this benefit won't change the total net pre-tax result, which is clearly negative.

The first component of taxable income, $V_T - V_0$, is asset appreciation. To this is subtracted the interest accrued by debt (the latter is the base for obtaining the tax benefit of having debt).

This framing of the valuation problem poses a **new and important question** which I intend to answer; **what is the risk adjusted return to value the taxes paid by the firm?** Following with the same example and assumptions I intend to explain this and other related concepts.

RISK-NEUTRAL / NUMERICAL INTEGRAL VALUATION OF EACH CLAIM ON THE PRETAX VALUE OF THE FIRM'S ASSETS

The difference with the valuation approach used in the previous section without corporate taxes, is that we incorporate a third claim holder to the pre-tax payoffs of the firm´s assets, V_T. The third claim holder is Government that holds a claim through taxes paid by the firm. The valuation of this third claim, which clearly is a contingent claim, goes exactly the same as before: using risk-neutral probabilities, numerically calculate a risk-neutral expected payoff, and discount it using the risk free rate of return. In general terms for any of the three claims (D, E and Tax), its value is:

$$f_o = e^{-r\,T}\ \hat{\mathrm{E}}(f_T) \qquad [49]$$

Where f_T is any of the payoff functions of the three contingent claims being valued (debt, equity or tax) and f_0 is its corresponding present value.

Value of claims in the numerical example:

We keep using the same assumptions as before. V_0 = \$ 100.0; r = 3.0%; T = 3 years; σ = 35.0%; K_T = \$ 45.0; and μ = 9.0%. All returns are log nominal annual returns. **We add an extra assumption, the corporate tax rate τ_c = 35.0%**. The latter is the statutory tax rate.

Following, the results using numerical integration for the value of debt, equity and taxes.

Debt value:

Future risk-neutral expected value:

$\hat{\mathrm{E}}(D_T) = \$\,\mathbf{43.69611302891220}$

Present value of future risk-neutral expected value discounting at the risk-free rate of return:

$D^{\int} = \$\, 43.69611302891220 \times e^{-3.0\% \times 3} = \$\, \mathbf{39.93524037225930}$

Debt value in the no taxes case is $ 39.93524037226160, almost exactly the same value as with taxes.

Equity value:

Future risk-neutral expected value:

$\hat{\mathrm{E}}(E_T) = \$\, \mathbf{54.40909164218390}$

Present value of future risk-neutral expected value discounting at the risk-free rate of return:

$E^{\int} = \$\, 54.40909164218390 \times e^{-3.0\% \times 3} = \$\, \mathbf{49.72616561407200}$

Equity value in the no taxes case is $ 60.06478689811780.

Up to now we have:

$D^{\int} + E^{\int} = \$\, 39.93524037225930 + \$\, 49.72616561407200 = \$\, \mathbf{89.66140598633130}$

This is the **"private sector" value of the firm** V_0 or simply V.

$$V_0^{\int} = D_0^{\int} + E_0^{\int} \qquad [50]$$

Or, dropping the time subscripts:

$$V^{\int} = D^{\int} + E^{\int} \qquad [51]$$

Now we proceed with the valuation of taxes paid by the firm using the risk-neutral numerical integration approach.

$$Tax^{\int} = e^{-r\,T}\, \hat{\mathrm{E}}(Tax_T) \qquad [52]$$

Using the assumptions of our example;

Future risk-neutral expected value of taxes paid by the firm:

$\hat{\mathrm{E}}(Tax_T) = \$\, \mathbf{11.31225353797030}$

Present value of future risk-neutral expected value discounting at the risk-free rate of return:

$Tax^{\int} = \$\, 11.31225353797030 \times e^{-3.0\% \times 3} = \$\, \mathbf{10.33862128404590}$

As we can appreciate from the numerical results in our example, **the added present values of all three claims to the pretax value of assets, is equal to the exogenously assumed pretax value of assets**, V_{tot} ($ 100.00), in the risk-neutral valuation process.

$$V_{tot} = D + E + Tax \qquad [53]$$

This result is quite logic since the violation of the previous equation would provide investors arbitrage opportunities. And as shown by Modigliani and Miller, in a world with corporate taxes only, this arbitrage opportunities are rapidly arbitraged by arbitrageurs, thus moving prices of claims to satisfy equation [53]. On the other hand, there may be counterarguments on the real possibility of arbitraging mispriced taxes, since taxes aren't traded at any market. I leave this discussion to other more experienced people.

Using the results from our example, the total private and public value of the firm is:

$$V_{tot}^{\int} = D^{\int} + E^{\int} + Tax^{\int} = \$\, 39.93524037225930 + \$\, 49.72616561407200 + \$\, 10.33862128404590 = \$\, \mathbf{100.00002727037700}$$

We can also equivalently state that the total pre-tax present value of assets is equal to the present value of the private value of assets (also equal to debt plus equity), plus the present value of public assets, this is the taxes provided by the firm (to the government and thus to society).

Notice that since the variable V_T is the pre-tax value of assets at time T, then the input V in the density function (and also in the option pricing model) has to be, for the sake of consistency, the present value of the pre-tax value of assets. Thus, V is not D + E, but rather D + E + Tax. In the same line of reasoning, volatility and assets return, σ and μ, respectively, are both pre-tax (volatility is of pre-tax assets returns).

Notice that the value of debt with and without taxes remains unchanged (previous result without taxes: D = $ 39.93524037225930). **The cost of taxes in this example is totally bared by stockholders**. Value of equity with taxes is reduced in exactly the value of taxes, compared to its value without taxes. As a matter of rigor, I checked this for a low levered firm and a high levered one, using the same numerical example in this paper, but only changing the assumption for K, the strike price or final contractual value of debt. For example, if K = $ 5.0 (everything else constant), then debt value is $ 4.56965551865211 both with and without taxes; but equity drops from $

95.43037175172570 to \$ 84.22311498664470 when considering taxes. If instead we assume K = \$ 80.0 (everything else constant), then debt value is \$ 63.35658433947270 both with and without taxes; but equity drops from \$ 36.64344293090830 to \$ 28.15087359368170 when considering taxes. So, we may safely conclude that at least at any assumed leverage of the firm, the cost of taxes is fully bared by equity holders, and that debt holders are tax shielded. Another interesting point of view is that for a medium levered firm (K = \$ 45.0), equity would drop by 17.2124497862321%; for the low levered firm (K = \$ 5.0) the drop in equity is 11.7439097840236%; finally, for the high levered firm (K = \$ 80.0) the drop in equity is 23.1762319748158%. A possible conclusion is that equity holders of high levered firms should be more concerned than equity holders of low levered firms, about possible future rises in the corporate tax rate.

As in the case of no corporate taxes, the total value of the "pie" or "pizza" is arbitrarily fixed and exogenous. In the case with no taxes, V = D + E, and in the case with corporate taxes, V_{tot} = D + E + Tax. Somebody may argue that the existence of taxes in the economy reduces the total "pie" available. Others may argue the contrary. But I think this is a matter for economists, which I am not.

In this simple example, the present value of taxes, \$ 10.33862128404590, represents approximately 10.34% of the TOTAL value, private plus public. Taxes relative to private value, D + E, represents approximately 11.53%.[26]

The circularity problem in valuation:

As is commonplace in valuation practice, some circularity problems arise in the valuation process. A classic one is the value of assets, V, necessary (depending on your assumptions) to obtain the value weighted cost of capital, WACC. To calculate the respective weights of different financing sources we need to know the total value being financed, plus the value of each financing source. But at the same time, and here arises the mathematical circularity, we need WACC to obtain the value of assets. This circularity problem can be solved by a simple iteration process that may be

[26] I did extensive work in validating my model results with the current value of debt and market cap as of December 2017 of real firms. The process is a bit complex to explain here, but overall I obtain that the value of taxes, depending mainly on leverage, ranges from approximately 7% to 12% of total private value (debt + market cap). If in the same numerical example of this paper we change T to 7 years (instead of 3), and assets' volatility to 25.0% (instead of 35.0%), taxes are worth \$ 12.30791872655430, which represents 14.03% of total private value, not so far from my testing results.

manually performed or automatically via some mathematical procedure. Fortunately Microsoft Excel can solve this circularity problem by properly setting the calculation settings in your Excel file[27].

But which is the circularity problem in the valuation I perform in this work? To obtain the present value of taxes, we need to know the taxes the firm will pay, if any, at every state of nature at time T. Then we weight each of these contingent taxes (30,000 in my Excel model) by the corresponding risk-neutral probability (using Riemann integration) to obtain the risk-neutral expected value of firm taxes. But the tax in each state of nature, as I assumed, depends on the private pre-tax value appreciation of assets, $V_T - V_0$, and thus, we need V_0 to calculate these taxes. Remember that $V_0 = D_0 + E_0$, which depends on the value of taxes. Therefore, here we have a circularity problem. In my Excel model, the problem is automatically and painlessly solved by Excel. It took me several years to fine tune and correct my model, so it may contain more circular calculations that I am currently not fully aware.

Determining taxable income and taxes:

As already said, the condition for the firm to pay taxes at time T is that taxable income is positive:

$$Taxable\ income = V_T - V_0 - Accrued\ interest\ at\ T > 0 \qquad [54]$$

Once in equilibrium, when all present values are determined, V_0, and Accrued interest at T are known and do not change across states of nature at T. The only variable that makes change pre-tax income across possible future states of nature, is V_T, the pre-tax value of assets at time T, which is a random variable. The lower limit of V_T is zero; the upper limit is plus infinity. Remember that we assumed that V_T is lognormally distributed.

Table 1 shows several possible values of V_T, pre-tax taxable income, and tax, at five different states of nature (among a total of 30,000 states) at time T. Notice that in a similar fashion of the debt and equity payoffs, tax payoff function is also a state-contingent function, and similar in nature to a call option. These numbers are obtained in my Excel model with the same assumptions of the example.

[27] *Found with Google asking "how to set up iterative calculation in excel"*: "Click the **File tab**, click **Options**, and then click the **Formulas** category. In the **Calculation** options section, select the Enable **iterative calculation check box**. To set the maximum number of times Excel will recalculate, type the number of iterations in the Maximum Iterations box."

Table 1: Contingent tax payoff function vs pre-tax value of assets – a few points

V_T	$V_T - V_0$	Interest T on debt	Pre-tax net income	Tax_T (35%)
$ 39.99466966814870	- $ 49.66673631818260	$ 5.06475962774070	- $ 54.73149594592100	$ 0
$ 49.96485505657090	- $ 39.69655092976040	$ 5.06475962774070	- $ 44.76131055749880	$ 0
$ 93.39414245000120	$ 3.73273646366994	$ 5.06475962774070	- $ 1.33202316406851	$ 0
$ 110.00886851007100	$ 20.34746252373960	$ 5.06475962774070	$ 15.28270289600120	$ 5.34894601360042
$ 120.00624701565600	$ 30.34484102932510	$ 5.06475962774070	$ 25.28008140158660	$ 8.84802849055533

Notice that V_0 in column 2 of Table 1 is the private value of the assets, this is equal to D + E. For example, in line 3, V_T is $ 93.39414245000120. We already know that current private value of the firm is:

$$D^{\int} + E^{\int} = \$\,39.93524037225930 + \$\,49.72616561407200 = \$\,\mathbf{89.66140598633130}$$

Thus, contingent asset value appreciation in this particular state of nature is:

$$V_T - V_0 = \$\,93.39414245000120 - \$\,89.66140598633130 = \$\,\mathbf{3.73273646366994}$$

In all previous scenarios we can obtain the values or payoffs at T for equity holders and for debt holders. We can also check that the sum of all payoffs must be always equal to the pre-tax asset value V_T, which I show in Table 2.

Table 2: Pre-tax value of assets at T equal to sum of all contingent claim payoffs, debt, equity and taxes

V_T	Tax_T (35%)	D_T	E_T	$Tax_T + D_T + E_T$
$ 39.99466966814870	$ 0.0	$ 39.99466966814870	$ 0.0	$ 39.99466966814870
$ 49.96485505657090	$ 0.0	$ 45.0	$ 4.96485505657090	$ 49.96485505657090
$ 93.39414245000120	$ 0.0	$ 45.0	$ 48.39414245000120	$ 93.39414245000120
$ 110.00886851007100	$ 5.34894601360042	$ 45.0	$ 59.65992249647050	$ 110.00886851007092
$ 120.00624701565600	$ 8.84802849055533	$ 45.0	$ 66.15821852510110	$ 120.00624701565643

VALUE OF THE FIRM AND TAX SAVINGS FROM DEBT

A well-known result from Modigliani and Miller is that the value of the firm, V or D + E, is equal to its unlevered value, plus the present value of tax savings –or "tax shield"– provided by debt:

$$V = V_U + VTS \qquad [55]$$

The unlevered value V_U is the value of the firm assuming it does not hold debt, supposedly being a sub-optimal value of the firm. Adding an optimal amount of debt maximizes firm value V = D + E, and the "marginal"[28] effect of using debt is the present value of the tax savings that optimal debt provides to the firm. This valuation approach is commonly labeled as "adjusted present value" or APV.

We already know V = D + E. Notice that we don´t include the value of taxes, only the private value components. I use here the classical notation for debt, D, equity, E, and firm value, V.

Using the numerical results from our example:

$$V = D^{\int} + E^{\int} = \$\,39.93524037225930 + \$\,49.72616561407200 = \$\,\mathbf{89.66140598633130}$$

Now we will value the firm in the context of V_U and VTS. We first value V_U. The future risk-neutral expected value of assets in a world with taxes, using the numerical integral and risk-neutral approach, is:

$$\hat{E}(V_U) = \$\,\mathbf{97.06184489004510}$$

Notice that this is the expected future value of Free Cash Flow to the Firm, FCFF, or better, E(FCFF), but using risk-neutral probabilities instead of real or actuarial probabilities. Discounting at the risk-free rate:

$$V_U = \$\,97.06184489004510 \times e^{-3.0\% \times 3} = \$\,\mathbf{88.70784694497100}$$

Now we value tax savings, VTS. The future risk-neutral expected value of tax savings, using the numerical integral approach, is:

$$\hat{E}(\Delta Tax_T) = \$\,\mathbf{1.04335978105073}$$

[28] Here I use the term *marginal* in the sense of comparing a situation with zero debt, to one with the optimal amount of debt (or any amount of debt managers of the firm may choose to have). Typically, marginal is used in a situation of a very small change in some variable.

The variable ΔTax_T is defined as the difference between taxes paid by the unlevered firm and taxes paid by the levered firm, at each state of nature or contingent state (30,000 in total).

Discounting at the risk-free rate:

$$VTS = \$\, 1.04335978105073 \times e^{-3.0\% \times 3} = \$\, \mathbf{0.95355904136002}$$

Finally we add Vu and VTS, both obtained with the numerical integration approach:

$$V = V_U + VTS = \$\, 88.70784694497100 \; + \; \$\, 0.95355904136002 = \$\, \mathbf{89.66140598633102}$$

Thus, we obtain almost exactly the same value of V (private) as when valuing debt and equity independently and adding both ($89.66140598633130). So, **Modigliani and Miller proposition, V = V_U + VTS, holds under the numerical risk-neutral valuation scheme here presented**.

Pablo Fernandez Intuition on the Value of Tax Savings:

To me, Pablo Fernandez, professor at IESE (Universidad de Navarra) has been a continuous source of inspiration in the art/science of valuation. Fernandez published in July 2004 at the Journal of Financial Economics, a paper named "*The value of tax shields is NOT equal to the present value of tax shields*". From its abstract (complete):

> "*The value of tax shields is the difference between the present values of two different cash flows, each with their own risk: the present value of taxes for the unlevered company and the present value of taxes for the levered company. For constant growth companies, the value of tax shields in a world with no leverage cost is the present value of the debt, times the tax rate, times the unlevered cost of equity, discounted at the unlevered cost of equity. This result arises as the difference of two present values and does not mean that the appropriate discount for tax shields is the unlevered cost of equity.*"

After the publication of its paper, Fernandez was wildly criticized, not for its intuition into the valuation problem, which was praised, but for its specific analytical results. Its critics published articles in the opposing venue such as "*The value of tax shields IS equal to the present value of tax shields*", Cooper and Nyborg, October 2004. Another opposing paper is, *Comment on "The value of tax shields is NOT equal to the present value of tax shields"*, by Fieten, Kruschwitz, Laitenberger, Loffler, Tham, Velez-Pareja, and Wonder, 2005, in The Quarterly Review of Economics and Finance.

This last paper comments: "*If he were correct, Fernandez would indeed have a paper of great significance and practical importance in corporate finance. We show that Fernandez fails to correctly prove his result and that the earlier results have not been disproved*." "*Fernandez makes the valid point that tax shields may be valued as the difference between the discounted value of taxes paid by a levered firm and the value of taxes it would have paid as an unlevered firm.*"

Here I show that the two apparently opposing views ARE compatible. This is, the value of tax shields can be obtained as the difference in present value of two tax streams as Fernandez argues, each one having a different risk, and thus different discount rates. But this is in absolute contradictory to saying that the value of tax shields is the expected value of tax savings, discounted at an appropriate risk adjusted discount rate. I will demonstrate this through the numerical example in this paper. I demonstrate that the risk adjusted discount rate of the tax savings cash flow is NOT the unlevered cost of equity, K_U, as customary assumed in the financial literature.

If my following work on this matter proves to be correct, it may serve to solve an ongoing and unsolved discussion about the correct risk adjusted discount rate to value the tax savings from debt. Citing again Fernandez, February 2004:

> "*There is no consensus in the existing literature regarding the correct way to compute the value of tax shields. Most authors think of calculating the value of the tax shield in terms of the appropriate present value of the tax savings due to interest payments on debt, but Myers (1974) proposes to discount the tax savings at the cost of debt, whereas Harris and Pringle (1985) propose discounting these tax savings at the cost of capital for the unlevered firm. Reflecting this lack of consensus, Copeland et al. (2000, p. 482) claim that "the finance literature does not provide a clear answer about which discount rate for the tax benefit of interest is theoretically correct." In this paper, I show that a consistent way to estimate the value of the tax savings is not by thinking of them as the present value of a set of cash flows, but as the difference between the present values of two different sets of cash flows: flows to the unlevered firm and flows to the levered firm*."

Following, I show with the numerical example, using risk-neutral valuation, that Fernandez's conceptual proposition is right, both in conceptual terms, but more in the practical side, in numerical terms. This is that the value of tax savings can be obtained by discounting levered taxes and unlevered taxes, each at their own risk adjusted return, and then subtracting both present values. I also show with my numerical example, that Fernandez is wrong when he says that the value of tax savings is not the present value of a single stream of cash flow, the tax savings. I show that tax savings can be calculated either way and obtain identical results.

Using risk-neutral valuation and numerical integration in the following lines I show the results step by step, to finally obtain the value of tax savings, VTS, for the numerical example.

Value of taxes with debt:

The future risk-neutral expected value of taxes with debt, using the numerical integral approach, is:

$\hat{\mathrm{E}}(Tax_T) = \$\ \mathbf{11.31225353797030}$

Discounting at the risk-free rate:

$Tax^{\int} = \$\ 11.31225353797030 \times\ e^{-3.0\% \times 3} = \$\ \mathbf{10.33862128404590}$

This was shown in a previous section, when we obtained $V_{Tot} = V_{Priv}$ + Tax = D + E + Tax.

Value of taxes without debt:

The future risk-neutral expected value of taxes, using the numerical integral approach, is:

$\hat{\mathrm{E}}\left(Tax_{U,T}\right) = \$\ \mathbf{12.35561331902110}$

Subscript U means unlevered.

As we can appreciate, future risk-neutral expected taxes without debt are larger than future risk-neutral expected taxes with debt ($ 11.31225353797030), both values using risk-neutral probabilities.

Discounting the risk-neutral expected value of taxes at the risk-free rate:

$Tax_U^{\int} = \$\ 12.35561331902110 \times e^{-3.0\% \times 3} = \$\ \mathbf{11.29218032540600}$

Value of tax savings (VTS):

The value of tax savings, as proposed by Pablo Fernandez [2004], is the difference between the present value of taxes without debt, and the present value of taxes with debt:

$$VTS^{\int} = Tax_U^{\int} - Tax^{\int} \qquad [56]$$

Using the assumptions and results from the numerical example:

$$VTS^{\int} = \$\ 11.29218032540600 - 10.33862128404590 = \$\ \mathbf{0.95355904136010}$$

This is the value added to the firm, when it has debt instead of equity by a certain amount. It represents the lower taxes the firm pays because interest from debt is tax deductible, and dividends are not (more precisely, cost of equity is not). This is almost exactly the same result as previously obtained, when valuing tax savings discounting the difference in taxes (a single cash flow), \$ 0.95355904136002. **It is crucial to bear in mind that in this analysis of taxes saved, assets do not change in both situations (levered and unlevered); it is only an exercise of financing substitution between equity and debt**.

Notice that the value of tax savings in the example represents approximately 1.06% of the levered value of the firm D + E, \$ 89.66140598633130.

This result partially proves that Fernandez's conceptual proposition is correct. I will provide further and more solid proof in this work when obtaining risk adjusted returns for a firm in a world with corporate taxes.

RISK ADJUSTED RETURNS IN A WORLD WITH CORPORATE TAXES

The numerical integration approach provides a simple tool to estimate implied risk-adjusted returns as already shown in the case of a world with no taxes. We already know the arbitrage-free prices for debt, equity, and tax, using the numerical integral approach. We can use the future actuarial expected values in the numerical integral, to derive the risk-adjusted returns as follows (see previous section for more detail). The future actuarial expected value, as opposed to the future risk-neutral expected value, can be obtained changing in the density function of V_T the risk-free return for the pre-tax assets' return (both exogenous assumptions in Merton's model[29]). Following are the expressions for the implied risk adjusted returns of equity and of debt.

$$K_e = \ln[E(Call_T)/E]/T \qquad [57]$$

$$K_d = \ln[E(D_T)/D]/T \qquad [58]$$

In a similar fashion we can derive the risk adjusted return for taxes:

$$E(Tax_T)\, e^{-K_{tax}\,T} = Tax \qquad [59]$$

Solving for K_{tax}:

$$K_{tax} = \ln[E(Tax_T)/Tax]/T \qquad [60]$$

Notice that now all expected future values are obtained with actuarial probabilities, thus the notation is E(x) instead of Ê(x) used in risk-neutral calculations. Replacing numbers from our example we solve for risk-adjusted returns for all claims on the pre-tax value of assets:

$$K_e = \ln[\$\,69.68423497565900/\$\,49.72616561407200]/3 = \mathbf{11.24809475054090}\%$$

$$K_d = \ln[\$\,44.29532276825640/\$\,39.93524037225930]/3 = \mathbf{3.45399797287673}\%$$

$$K_{tax} = \ln[\$\,17.01692305263600/\$\,10.33862128404590]/3 = \mathbf{16.61072668317950}\%$$

Respective risk premiums (K_i – risk-free return) over r = 3.0% are: 8.24809475054091% for risk adjusted cost of equity, 0.45399797287673% for risk adjusted cost of debt, and 13.61072668317950% for risk adjusted cost of taxes.

One interesting result is that cash flow from taxes is the riskier among the three claims. In fact it is significantly riskier than equity in the example. The risk premium of taxes is 13.6107%, almost 65% larger than the risk premium of equity. An intuitive way of understanding why is the risk-adjusted return on taxes significantly higher than the risk-adjusted return on equity, is because both rights share a call option characteristic, and also share the same underlying asset, the pre-tax value of assets V_T. But there is an important difference between both options; the strike price of taxes is

[29] In strict rigor, in the Black & Scholes option pricing model, the risk free rate of return r is one of the exogenous variables/assumptions. But the risk adjusted return of the underlying asset of the option, in this paper, μ, is not an intervening variable. Said so, when extrapolating Black-Scholes-Merton pricing model to the context of firm value, μ becomes an extra exogenous variable necessary for some analytical computations, for example the actuarial probability of default of the firm.

significantly larger than the strike price for equity (look Figure 7). The latter makes the tax claim less valuable than equity, and at the same time significantly riskier. For a reasonably levered company, the equity call has a lower probability of being out of the money at maturity, but the tax call has a higher probability of being out of the money at maturity.

Riskiness of equity holders is slightly lower when considering taxes. With taxes in the example, K_e = 11.24809475054090%, and without taxes it is 12.23478263708410%. Cost of debt doesn't change when introducing corporate taxes (3.45399797287485% and 3.45399797287673%, respectively). It appears to be a shift of risk to the government through taxes. So the trade-off from shareholders perspective is: more taxes have the cost of reducing equity value as already shown; but at the same time, it appears that the risk bared by stockholders is reduced, though slightly, in our numerical example.

Probability of default:

A natural complement of the cost of debt is the probability of default by the firm. Here default is defined as the case when the value of pre-tax assets at time T, V_T, is smaller than the contractually agreed final debt payment K_T. In the numerical example, the actuarial probability of default is 7.23097426698138%, which is not negligible. It can also be computed the risk-neutral probability of default, 12.26422148611880%. I calculate the default probability simply by computing the Riemann integral from a minimum value of V_T equal to zero, to a maximum value of K, the strike price of equity holders call option.

To provide a more realistic case[30], I modified the example assumptions changing **T from 3 to 10 years**, and changing assets' **volatility from 35% to 25%**. Every other assumption remains constant. The actuarial probability of default is now **3.98276945865748%** (leverage = 37.18560434383940%). Letting **T = 5 years**, assets' volatility 25%, and final contractual debt K = $ 39.0, the actuarial probability of default is **1.35943060655236%** (leverage = 37.21304166631200%). Finally, letting **T = 15 years**, assets' volatility 25%, and final contractual debt K = $ 54.0, the actuarial probability of default is **6.09308986637064%** (leverage = 37.32149525435830%). Notice that by changing K in each case I tried to keep current firm's leverage relatively constant.

[30] I added this variant on a helpful comment from Sergio Benavente, finest friend and financial risk expert, who said me that a 7.23% probability of default at 3 years seemed unrealistically high.

I compare my results with results reported by Hamilton and Cantor [2006], of cumulative default rates at different time horizons. I compare with their "withdrawal adjusted default rates", since they argue that is a better estimator of default compared with not withdrawal adjusted. They report 20 year time horizon cumulative default rates using a sample of debt issues from 1970 to 2005. Only as a casual observation, my estimates of probability of default (by nature of the model, cumulative), at years 5, 10, and 15 are: 1.36%; 3.98%; and 6.09%, respectively, as previously shown. These are fairly close to the center (vertical wise) of Hamilton and Cantor results, of 2.06%, 4.89%, and 8.24% (years 5, 10, and 15, all for Baa rating) (See Figure 8).

Figure 8: Average Cumulative Default Rates by Whole Letter Rating, Withdrawal-Adjusted (Source: Hamilton and Cantor, 2006)

	Years After Cohort Formation Date													
	1	2	3	4	5	6	9	10	11	12	13	14	15	16
Aaa	0.00	0.00	0.00	0.03	0.11	0.18	0.45	0.56	0.66	0.78	0.90	0.97	1.04	1.12
Aa	0.01	0.02	0.05	0.12	0.19	0.29	0.51	0.58	0.65	0.76	0.92	1.06	1.16	1.29
A	0.02	0.10	0.24	0.37	0.51	0.67	1.22	1.42	1.63	1.82	2.02	2.21	2.49	2.81
Baa	0.18	0.53	0.98	1.52	2.06	2.60	4.23	4.89	5.50	6.17	6.85	7.56	8.24	8.84
Ba	1.23	3.31	5.75	8.26	10.57	12.65	18.05	19.86	21.62	23.41	25.15	26.82	28.29	29.78
B	5.65	12.35	18.65	24.09	29.06	33.50	43.59	46.12	47.56	48.77	49.65	50.51	51.26	51.77
Caa-C	21.12	33.53	43.47	51.01	56.52	61.05	71.98	74.72	75.16	75.16	75.16	75.16	75.16	75.16

Lastly, trying to stress test how my model responds and if it falls reasonably close to Hamilton and Cantor results, y tried with T = 10 years, assets' volatility of 25%, but reduced leverage to 20.08566588779910% by assuming K = $ 23.0, trying to emulate a close to AAA firm. I obtain under these assumptions a probability of default at year 10 of 0.46248398848727%, unexpectedly close to the 0.56% of Hamilton and Cantor. Now, honestly lastly, I stress tested to the opposite case, a firm close to C. I assumed K = $ 120 (making a huge effort in not cheating myself), but with a leverage in mind of something in the bulk of 70.0%. With the assumed K, leverage is 72.27166689630710%, and the resulting probability of default is 30.43640214885920%. I fall short from Hamilton and Cantor 74.72% for year 10 in their table. (I would have to assume a leverage as high as 93.72644940026900% to get a probability of default of 74.11324489156040% which is close to Hamilton and Cantor).

WEIGHTED AVERAGE RETURNS OF ALL CLAIMS WITH CORPORATE TAXES

As in the case of no taxes, the value weighted average return implied in the risk-adjusted returns of all claims can be obtained. Following, I compute the implied overall or value weighted return of debt, equity, and tax, first by computing future expected value (actuarial, NOT risk-neutral) of the three claims added, as follows:

$$D\ e^{K_d\,T} + E\ e^{K_e\,T} + Tax\ e^{K_{tax}\,T} \qquad [61]$$

Then we get the implied overall weighted average return, dividing by V and taking natural logarithm and dividing by T:

$$\ln[(D/V)\ e^{K_d\,T} + (E/V)\ e^{K_e\,T} + (Tax/V)\ e^{K_{tax}\,T}]/T = Implied\ weighted\ average\ return$$
[62]

Notice that the weights must add to 1 or 100%, since we already showed that D + E + Tax = V, and that the result in [62] is an annualized log return. Plugging in the numbers from the example we obtain:

D / V = 39.93522948177160%; E / V = 49.72615205356280%; and Tax / V = 10.33861846466560%. These are the weights of the components of the pre-tax value of the firm (or the same, the private value of the firm plus the value of taxes) and add up to 100%. Following, I develop [62] by parts:

$$(D/V)\ e^{K_d\,T} = 39.93522948177160\% \times e^{3.45399797287673\% \times 3} = 44.29531068875810\% \equiv a$$

$$(E/V)\ e^{K_e\,T} = 49.72615205356280\% \times e^{11.24809475054090\% \times 3} = 69{,}68421597251050\% \equiv b$$

$$(Tax/V)\ e^{K_{tax}\,T} = 10.33861846466560\% \times e^{16.61072668317950\% \times 3} = 17.01691841205820\% \equiv c$$

$$ln[a+b+c]/T = ln[130.99644507332700\%]/3 = \mathbf{9.00000000000052}\%$$

Thus, I prove that the implied weighted average return taking K_d, K_e, and K_{tax}, is effectively the exogenously assumed pre-tax return for the asset, μ = 9.0%.

Changing the assumption of the pre-tax assets' return. From option pricing theory we know that the present value of options (call and put) does not depend on the true expected return of the underlying asset, but only on the risk free rate of return (plus other assumptions: T, standard deviation, strike, and current value of underlying asset). In the numerical example if we change μ from 9.0% to 8.0%, keeping every other assumption constant, I get almost identical value results for debt, \$ 39.93523934086400, equity, \$ 49.72616694131050, and tax, \$ 10.33862094557670, as before, thus showing irrelevance of return expectations, value wise. BUT, risk adjusted returns change: 3.39383334429514% for debt; 9.92163602069857% for equity; and 14.43948924162770% for tax. Value weighing these under the same scheme already explained, yields 7.99999999999952%, almost exactly the new assumed assets' pretax expected return.

<u>A derived proposition may be that risk adjusted returns of the claims on the firm assets are "derivatives" or contingent on the expected return of the assets of the firm</u>. Therefore, risk premium on each claim (and respective betas), are also contingent on the underlying asset return.

<u>The weighted average return of private components D and E</u>:

$$\ln[(D/(D+E))\, e^{K_d T} + (E/(D+E))\, e^{K_e T}]/T = Implied\ weighted\ average\ return_{priv}$$

[63]

Subscript *priv* means the private part of total value of the firm, D + E. Using the results of the numerical example:

$$D/(D+E) = \$\,39.93524037225930/(\$\,39.93524037225930 + \$\,49.72616561407200)$$

$$D/(D+E) = 44.54005592812960\%$$

$$E/(D+E) = \$\,49.72616561407200/(\$\,39.93524037225930 + \$\,49.72616561407200)$$

$$E/(D+E) = 55.45994407187040\%$$

Both percentages added are equal to 100.00000000000000%.

$$\left(D/V_{priv}\right) e^{K_d T} = 49.40288665003660\% \equiv A$$

$$\left(E/V_{priv}\right) e^{K_e T} = 77.71931993379880\% \equiv B$$

$$\ln(A+B)/T = ln(127.12220658383500\%)/3 = \mathbf{7.99928981213554}\%$$

Notice that this is lower than μ = 9.0%, total pre-tax asset return. This is explained because of the higher risk adjusted return in the Tax component, 16.61072668317950% in the example. But also, the risk adjusted return of the private component is conceptually the after tax return of the firm. In the example, the effect of a statutory tax rate of 35% is lowering the pretax expected return from

9.0% to an after tax expected return on assets of 7.99928981213554%. The implied effective tax rate in this case can be derived from the following expression:

$$Pretax\ asset's\ return\ \left(1-\tau_{c,eff}\right) = After\ tax\ asset's\ return \qquad [64]$$

Replacing values from our example:

$$9.0\%\ \left(1-\tau_{c,eff}\right) = 7.99928981213554\%$$

$$\tau_{c,eff} = 1 - \frac{7.99928981213554\%}{9.0\%} = 11.11900208738290\%$$

As we can see, the effective implied tax rate is significantly lower, in the numerical example, than the assumed statutory tax rate of 35.0%.

As a further validation check of my model, I tried using a set of assumptions closer to a "real" or "average" firm and see if the implied effective tax rate gets closer to the statutory. Increasing T for example to 10 years, only worsens things; the implied effective tax rate is 7.9238367456831%, which is explained by the increased volatility of V_T at a larger maturity, which increases the number of scenarios where the firm's pre-tax result is negative. The only reasonable way in the context and limitations of the model is reducing assets' volatility. This has to be done with caution because assets' volatility can't be too low without running the risk of implicitly assuming negative betas and for example obtaining a pre-tax cost of debt lower than the risk free rate of return. The latter happens for example when assuming a volatility of 5.0%, which by the way is practically unobserved in reality. Assuming a volatility of 15.0% in pre-tax return of assets, and leaving all other assumptions of our example unchanged, yields an implied effective tax rate of 15.0647969055348%, higher than the previously obtained of 11.11900208738290%, but still far below the statutory tax rate of 35.0%. Good news to the private sector; expected tax rate is really almost half and less than the statutory tax rate (given a low volatility in assets). Besides, value of equity increases from $ 49.72616561407200 to $ 53.39522144168940, roughly a 7.4%.

<u>The weighted average return implied in D (long riskless bond plus 1 short put option)</u>:

As I already explained, the value of the firm's risky debt is equal to the present value of a riskless bond with maturity T and single coupon K. In our numerical example this is:

$$D^{\int} = B - Put^{\int} = \$\,41.12690333720530 - \$\,1.19166296494372$$
$$= \$\,\mathbf{39.93524037226160}$$

, which is almost exactly the same result obtained when discounting the risk neutral expected payoff to debtholders at the risk free rate of return, of \$ 39.93524037225930. I use this last result as my "official" result for debt value. In this section I show that the risk adjusted return of debt is equal to a value weighted average of the risk adjusted returns of: al long position in 1 riskless bond, together with a short position in 1 put option, using the exponential weighting scheme already explained in this paper.

The future actuarial expected value of the put option (implied in debt's value) is **\$ 0.70467723174121**. Using the same type of equation as [57], [58], and [60], used to obtain risk adjusted returns of debt, equity, and taxes, we obtain the risk adjusted return of the put option.

$$K_{put} = \ln(\$\,0.70467723174121/\$\,1.19166296494372)/3 = -\mathbf{17.51217298181090}\%$$

Notice that \$ 1.19166296494372 is the estimated present value of the put option through numerical integration. Also notice that the put's risk adjusted return is negative. As we already showed, the value of a riskless bond with a coupon of K and maturity T is:

$$B = K\,e^{-r\,T} = \$\,45.0\,e^{-3.0\%\times 3} = \$\,41.12690333720530$$

Since the value of risky debt is \$ 39.93524037225930, the relative weights of a portfolio of 1 riskless bond and a short position in 1 put option, are:

$$B/D = \$\,41.12690333720530/\$\,39.93524037225930 = 102.98398846191400\%$$

$$-\,Put/D = -\,\$\,1.19166296494372/\$\,39.93524037225930 = -\,2.98398846190868\%$$

The implied weighted average return in the portfolio of riskless debt and one short put is:

$$(B/D)\,e^{R_f\,T} = 102.98398846191400\% \times e^{3.0\%\times 3} = 112.68243180842100\% \equiv \alpha$$

$$(-\,Put/D)\,e^{K_{put}\,T} = -\,2.98398846190868\% \times e^{-17.51217298181090\%\times 3}$$
$$= -1.76454986916945\% \equiv \Omega$$

$$\ln(\alpha + \Omega)/T = ln(110.91788193925100\%)/3 = \mathbf{3.45399797287854}\%$$

The result obtained for the implied value weighted average return of a portfolio consisting in 1 riskless bond and a short position in 1 put option, is almost exactly the same debt risk adjusted return of 3.45399797287673% that I obtained using the actuarial future expected value of debt and its current value. Thus I demonstrate that debt and the analyzed portfolio are risk-return equivalent.

A simpler alternative of computing weighted average return:

If instead of computing log annualized returns, we use simple (arithmetic) TOTAL returns we have the following results:

$$k_d^{tot} = [E(D_T)/D] - 1 \qquad [65]$$

$$k_e^{tot} = [E(E_T)/E] - 1 \qquad [66]$$

Notice that I use lower-case ***k*** for arithmetic returns, as opposed to upper-case ***K*** for log returns.

Using our partial results:

$$k_d^{tot} = [\$\,44.29532276825640/\$\,39.93524037225930] - 1 = \mathbf{10.91788193924530}\%$$

$$k_e^{tot} = [\$\,69.68423497565900/\$\,49.72616561407200] - 1 = \mathbf{40.13595079194900}\%$$

These TOTAL returns (in 3 years) weighted by the relative value weights of debt and equity:

$$(D/(D+E))\,k_d^{tot} + (E/(D+E))\,k_e^{tot} \qquad [67]$$

Replacing the example results:

$$44.54005592812960\% \times 10.91788193924530\% + 55.45994407187040\% \times 40.13595079194900\% = \mathbf{27.12220658383540}\%$$

This is the weighted average TOTAL arithmetic return of debt and equity. This is consistent with the log form return of **7.99928981213554%** already explained.

$e^{7.99928981213554\% \times 3} - 1 = \mathbf{27.12220658383540}\%$

But more importantly, the value weighted average TOTAL risk adjusted return of debt and equity is consistent with the implied total risk adjusted return of private assets.

RISK ADJUSTED RETURNS OF UNLEVERED CASH FLOWS AND OF TAX SAVINGS

In a previous section I valued the firm, V, valuing its parts according to Modigliani Miller proposition that:

$$V = V_U + VTS \qquad [68]$$

The results obtained with risk-neutral numerical integration were:

$V_U = \$\,97.06184489004510 \times e^{-3.0\% \times 3} = \$\,88.70784694497100$

$VTS = \$\,1.04335978105073 \times e^{-3.0\% \times 3} = \$\,0.95355904136002$

$V = V_U + VTS = \$\,88.70784694497100\ +\ \$\,0.95355904136002 = \$\,\mathbf{89.66140598633102}$

In this section I obtain the corresponding risk-adjusted returns for each of the 2 value components at the right side of equation [68], V_U and VTS. The risky cash flow corresponding to V_U is what we know as the Free Cash Flow of the Firm, FCFF, also known as the unlevered cash flow of the firm. This is the after tax operating cash flow of the firm's assets. The common text-book expression for FCFF is:

$$FCFF = EBIT(1 - \tau_c) + Depreciation \,\&\, amortization - Capex - \Delta Working\ capital \qquad [69]$$

In the context of contingent claim valuation, this equation does not apply since it implicitly assumes that the tax rate τ_c is applicable to EBIT in all future states of nature. But this is not exactly true. In states of nature where EBIT is negative, and assuming as we have done during all this work, a single cash flow at T and thus no loss-carryforward possibility, the effective tax rate is more complex. When EBIT is negative, the effective tax rate is clearly zero. When EBIT is positive but smaller than accrued interest expense, again the effective tax rate is zero. Only when EBIT is positive AND larger than accrued interest expense, the effective tax rate is the same as the statutory tax rate. But this in the context of the levered firm. In the context of the unlevered firm, consistent with FCFF definition, the tax rate is zero if EBIT is negative or zero, and the tax rate is τ_c when EBIT is positive. Damodaran differentiates between the tax rate applicable in the FCFF calculation as in [69], and the tax rate applicable in obtaining the after tax cost of debt in the WACC calculation as in [26]. He claims that for the FCFF, an "effective" tax rate ought to be used, and that for the WACC, a "marginal" tax rate ought to be used. This is commonplace in the financial literature, but with slight differences and not exempt of vagueness or lack of more precise definition. In the text book

Valuation, Measuring and Managing the Value of Companies, instead of a tax rate in percentage terms, the value of operating (or adjusted) taxes is subtracted from EBIT. In this text book they argue that is the "cash tax" on operations the right tax to subtract from EBIT. I provide in this paper propositions to obtain each of the tax rate effects, for WACC and for FCFF.

There is an ongoing debate in financial literature on which is the right discount rate applicable to the tax savings from debt cash flows. Here I show that the Merton Structural Model provides an excellent opportunity to answer this question. As already explained, switching from the risk-free return assumption to the assets' return to generate the probabilities of V_T, yields future actuarial expected values with which risk adjusted returns can be easily derived. **The expression for the risk adjusted return of the tax savings cash flows is**:

$$\boldsymbol{K_{ts}} = \mathbf{ln}[\boldsymbol{E}(\Delta \boldsymbol{Tax_T})/\boldsymbol{VTS}]/\boldsymbol{T} \qquad [70]$$

The meaning of ΔTax_T is the difference of taxes paid in each state of nature (30,000 states in my model), when considering the unlevered firm vs the same levered firm. Using the actuarial density function I obtain in the example the following result:

$E(\Delta Tax_T) = \$\ 1.289547896709380$

This is the future actuarial expected value of tax savings. Notice that the risk-neutral future expected value of tax savings is, as previously shown, \$ 1.04335978105073. **With the expected actuarial value and the present value of tax savings, as already shown of VTS = \$ 0.95355904136002, we obtain the implied risk-adjusted discount rate of the tax savings cash flow**:

$$\boldsymbol{K_{ts}} = \mathbf{ln}(\$\ \mathbf{1.289547896709380}/\$\ \mathbf{0.95355904136002})/\mathbf{3} = \mathbf{10.06152081552400}\%$$

Several authors propose that, under certain assumptions, the unlevered cost of equity, K_U, is an appropriate discount rate for the tax savings from debt. Others that K_d is the proper discount rate. Even others that K_e is the right risk-adjusted discount rate. **Here I show that the risk adjusted return of tax savings is different from any of this propositions, at least in the context of the Merton Structural Model adapted to include corporate taxes**. **If this proposition may be generalized for multiple future expected cash flows, in my opinion would mean a significant advance in understanding the determination of risk adjusted discount rates, and more specifically, in the valuation of corporate cash flows in the firm**.

The risk-adjusted discount rate for the unlevered after-tax cash flows:

The risk-adjusted discount rate for the unlevered after-tax cash flows of the firm can be obtained as:

$$K_U = \ln[E(V_U)/V_U]/T \qquad [71]$$

Using the results of the numerical example:

$E(V_U) = \$\,112.69000984720600$

This is the future actuarial expected value of the unlevered after-tax cash flows. This value corresponds to what we ordinarily name as the Free Cash Flow of the Firm or FCFF. Notice that the risk-neutral expected value of V_U is, as shown, \$ 97.06184489004510.

Then, the risk adjusted return K_U is:

$K_U = \ln[\$\,112.69000984720600/\$\,88.70784694497100]/3 = \mathbf{7.97641406077755}\%$

Is this the WACC? Clearly NO, since the present value obtained is equal to V_U when discounting the FCFF with K_U. Since V_U <= V, then K_U >= WACC.

Consistency of risk-adjusted returns with corporate taxes:

Since we have already seen and proved through our numerical example:

$$V = D + E = V_U + VTS \qquad [72]$$

$D + E = \$\,89.66140598633130 = \$\,39.93524037225930 + \$\,49.72616561407200$

$V_U + VTS = \$\,89.66140598633102 = \$\,88.70784694497100 \; + \; \$\,0.95355904136002$

Here I use V as value of private assets, this is excluding present value of taxes paid by the firm. If [72] holds, then it must be true that weighted average risk-adjusted returns using debt and equity, is equal to weighted average risk-adjusted returns using unlevered value of assets and value of tax savings, V_U and VTS, respectively. Next I prove this through the numerical example.

The weighted average return of private components V_U and VTS (in the log returns form) is:

$$\ln[(V_U/(D+E))\ e^{K_U\ T} + (VTS/(D+E))\ e^{K_{ts}\ T}]/T = Implied\ weighted\ average\ return\ _{priv}$$

[73]

Using the results of the numerical example:

$V_U/(D+E) = 98.93648885953740\%$

$VTS/(D+E) = 1.06351114046266\%$

Both percentages added are equal to 100%.

$(V_U/(D+E))\ e^{K_U\times T} = 125.68396469757100\% \equiv \theta$

$(VTS/(D+E))\ e^{K_{ts}\times T} = 1.43824188626493\% \equiv \gamma$

$ln[\theta+\gamma]/T = ln[127.12220658383593\%]/3 = \mathbf{7.99928981213572}\%$

The implied weighted average of risk-adjusted returns when considering debt and equity, as already shown, is **7.99928981213554%**, almost identical to the result considering V_U and VTS risk adjusted returns. <u>This demonstrates that the risk adjusted returns I obtain for V_U and VTS are consistent</u>.

More importantly, <u>IT DOES EXIST A UNIQUE SOLUTION FOR THE VALUE OF THE RISK ADJUSTED RETURN OF TAX SAVINGS FROM DEBT; IT IS RATHER A FUNCTION</u>. **<u>This is unique in the sense of subject to a given set of assumptions</u>**; in the context of Merton's Structural Model, these are: pre-tax value of assets V, volatility of asset returns σ, risk-free rate of return r, pre-tax expected return of firm's assets μ (extra-Merton variable), time to maturity of debt T, final contractual value of debt K_T, and corporate tax rate τ_c (extra-Merton variable). Later in this paper I provide a graphical solution to this function K_{ts}.

<u>Additional proof of the present value of tax savings and respective discount rates</u>:

I already showed that the difference between the unlevered and levered taxes paid by the firm is the same as the present value of expected tax savings from debt. But this demonstration was done obtaining present values under risk-neutral valuation (discounting future risk neutral expected values at the risk free rate of return). As already shown:

$\hat{E}(Tax_T) = \$\ 11.31225353797030$

$Tax^{\int} = \$\ 11.31225353797030 \times e^{-3.0\%\times 3} = \$\ \mathbf{10.33862128404590}$

$\hat{E}(Tax_{U,T}) = \$\ 12.35561331902110$

$Tax_U^{\int} = \$\ 12.35561331902110 \times e^{-3.0\% \times 3} = \$\ \mathbf{11.29218032540600}$

Value of tax savings (VTS):

$$VTS^{\int} = Tax_U^{\int} - Tax^{\int} \qquad [56]$$

$VTS^{\int} = \$\ 11.29218032540600 - 10.33862128404590 = \$\ \mathbf{0.95355904136010}$

Here I add an additional twist. <u>If we obtain risk adjusted returns for each one of the two streams of taxes, levered and unlevered,</u> <u>we can prove that the weighted average risk adjusted return of a long position in levered taxes and a short position in unlevered taxes, should be equal to the risk adjusted return of the expected value of tax savings cash flow</u>.

I already obtained the risk adjusted return of taxes paid by the levered firm:

$K_{tax} = \ln[\$\ 17.01692305263600/\$\ 10.33862128404590]/3 = \mathbf{16.61072668317950}\%$

Next we have to obtain the <u>risk adjusted return of taxes paid by the unlevered firm</u>. We can use for this the expression:

$$K_{tax,U} = \ln\left[E(Tax_{U,T})/Tax_U\right]/T \qquad [74]$$

We already know the present value of taxes of the unlevered firm:

$Tax_U^{\int} = \$\ 12.35561331902110 \times e^{-3.0\% \times 3} = \$\ \mathbf{11.29218032540600}$

The future expected actuarial value of taxes of the unlevered firm is, using numerical integration:

$E(Tax_{U,T}) = \$\ 18.30647094934540$

Introducing these values in [74]:

$K_{tax,U} = \ln[18.30647094934540/\$\ 11.29218032540600]/3 = \mathbf{16.10480405212300}\%$

<u>The tax savings is related to a long position in unlevered taxes and a short position in levered taxes, both from the same firm</u>. The risk adjusted return of this "portfolio" is as follows:

$$(Tax_U/VTS)\ K_{tax,U} - (Tax_L/VTS)\ K_{tax} \qquad [75]$$

$(Tax_U/VTS) = \$\ 11.29218032540600/\$\ 0.95355904136002 = 1184.21406914672000\%$

$(Tax_L/VTS) = \$\,10.33862128404590/\$\,0.95355904136002 = 1084.21406914671000\%$

Notice that both positions (first long and second short) add 100%.

$+1184.21406914672000\% - 1084.21406914671000\% = 100.00000000000700\%$

Again, if we use [75], which is a linear weighted average of returns, the result is **10.61951970721180% which IS NOT consistent** with the implied risk adjusted return already derived for tax savings cash flow TS, of 10.06152081552400%;

$K_{ts} = \ln(\$\,1.28954789670938/\$\,0.95355904136008)/3 = \mathbf{10.06152081552400}\%$

Please see previous sections where I explain two different and equivalent ways of treating weighted averages, "different" from linearly weighing. Here I use the exponential approach (not the total return approach).

$$\ln[(Tax_U/VTS)\,e^{K_{tax,u}\,T} - (Tax_L/VTS)\,e^{K_{tax}\,T}]/T = \\ Implied\ risk\ adjusted\ return\ of\ tax\ savings\ from\ debt = K_{ts} \qquad [76]$$

Replacing known values in [76]:

$$\ln[1919.8046639289000\% - 1784.5694198825400\%]/3 = \ln[135.2352440463620\%]/3 \\ = \mathbf{10.06152081552400}\%$$

This is EXACTLY the same risk adjusted implied return we already derived for the tax savings from debt (**10.06152081552400%**).

This proof shows that Pablo Fernandez intuition about the correct way of solving for the present value of tax savings is correct. It is indeed the difference in present value of two different streams of cash flows, taxes without debt and taxes with debt, in the firm. As we see here, each one of this cash streams, as Pablo Fernandez claims, has its own risk and thus its own risk adjusted return. **Here I prove that a portfolio consisting in a long position of unlevered taxes and a short position of levered taxes, each position value weighted, must yield a net risk adjusted return identical to the risk adjusted return applicable to the net cash flow consisting in the difference of unlevered and levered taxes**.

WACC IN THE CONTEXT OF MERTON'S STRUCTURAL MODEL

WACC deserves a great deal of attention. **WACC is probably the most important concept and metric in firm valuation and so is its "marriage" with the concept of "free cash flow" introduced by Michael C. Jensen [1986][31]. In this paper I intend to solidify this fruitful relationship between both concepts, from a practitioner's point of view**.

The weighted average cost of capital, WACC, is typically regarded as a discount rate which applied to the expected free cash flow of the firm (FCFF), yields the current economic or fair value of firm's operating assets. In a world with corporate taxes, the latter is equal to D + E (excludes the value of taxes). For simplicity, I am assuming that the firm has no non-operating assets (e.g. excess cash).

$$WACC = \ln[E(V_U)/(D+E)]/T \qquad [77]$$

The FCFF is equal to the actuarial expected value of unlevered cash flows from assets, after tax (only operating taxes), at time T. In the context of this work, FCFF is the same as $E(V_U)$. Thus, WACC is the discount rate that equates expected FCFF with V-private, o debt plus equity, D + E. The actuarial expected value of FCFF is calculated in my work with the Riemann integration approach, where the FCFF contingent to each future state of nature is multiplied by its actuarial probability; then all 30,000 results in my Excel model are added to get the actuarial expected value.

$FCFF = E(V_U) = \$\ \mathbf{112.69000984720600}$

Replacing in [77] known values from the numerical example:

$WACC = \ln[\$\ 112.69000984720600/\$\ 89.66140598633130]/3 = \mathbf{7.62001178107325}\%$

Recall that debt D is \$ 39.93524037225930 and equity E is \$ 49.72616561407200. Notice that this return is not the risk-adjusted return of V_U, since the denominator in WACC equation [77] is NOT meant to be the present value of V_U, but instead is the present value of the levered assets D + E, or V. Where is the trick then? Calculation of the levered value of the firm via FCFF has a "trick" consisting in adjusting the discount rate WACC to incorporate through a lower discount rate, the value effect in present value terms, of the tax savings provided by debt. This "trick" is the adjustment of the pre-tax cost of debt to an after-tax cost of debt. We can properly say that WACC

[31] Jensen, M. C. (1986) Agency Cost of Free Cash Flow, Corporate Finance, and Takeovers. *American Economic Review*, 76, (2), 323-29.

rate of return is a kind of hybrid return, which connects the unlevered dimension of value V_U (really, unlevered cash flows), with the levered dimension of value D + E.

<u>Reconciliation with the weighted average after tax cost of debt and equity</u>:

The traditional “text-book” way of expressing WACC is:

$$WACC = (D/V)\, K_{d,at} + (E/V)\, K_e \qquad [78]$$

$K_{d,at}$ is the <u>after tax cost of debt</u>, and V ≡ D + E. In many finance textbooks this is expressed as:

$$K_{d,at} = K_d(1 - \tau_c) \qquad [79]$$

τ_c is the corporate tax rate. Copeland et al [2005] say about the tax rate in WACC:

> “*The marginal tax rate for the entity being valued*”. Followed by a footnote: “*The marginal tax rate is the rate applied to a marginal dollar of interest expense. It is usually the statutory rate. If the company has substantial tax loss carry-forwards or carry-backs, or faces bankruptcy so that its tax shields may never be used, the marginal rate can be lower than the statutory rate—even zero*.”

Damodaran makes a clear differentiation between the tax rate used to obtain the “after tax cost of debt”, and the tax rate used for EBIT $(1 - \tau_c)$, used to obtain the FCFF. For the first he recommends using the statutory tax rate as a reasonable approximation to the marginal tax rate. For the latter, his recommendation is using the “effective” tax rate. Both tax rates are at the corporate level only (excluding personal taxes).

<u>**I propose the following way of computing the after tax cost of debt**</u>;

<u>The pre-tax cost of debt</u> is simply is equal to the implied risk adjusted return, given current debt value, D, and given future actuarial expected payoff to debtholders, $E(D_T)$:

$$K_d = \ln[E(D_T)/D]/T \qquad [80]$$

This was already obtained for the numerical example:

$$K_d = \ln[\$\,44.29532276825640/\$\,39.93524037225930]/3 = \mathbf{3.45399797287673}\%$$

The after-tax cost of debt must adjust the pre-tax cost of debt by an amount equal to the tax savings provided by debt, this is the future expected value of tax savings from debt; analytically:

$$\boldsymbol{K_{d,at} = \ln\left[\left(E(D_T) - E(TS_T)\right)/D\right]/T} \qquad \textbf{[81]}$$

I put equation [81] of after/tax cost of debt in bold since I consider it is a novel contribution of my paper to valuation in finance.

Calculating actuarial expected values with the numerical integral approach and replacing values:

$$\boldsymbol{K_{d,at} = \ln[(\$\,44.29532276825640 - \$\,1,28954789670938)/\$\,39.93524037225930]/3}$$

$$K_{d,at} = \mathbf{2.46917515252596}\%$$

Next I demonstrate that my proposed solution for after-tax cost of debt, a risk adjusted return as well, is consistent with other results obtained in this paper. This basically reduces to demonstrate that WACC using my proposition of after-tax cost of debt, is the discount rate which applied to expected FCFF yields as present value the same result as the value of debt and value of equity, both added.

Computing WACC with my proposition of after-tax cost of debt:

With $K_{d,at}$ we can now proceed to calculate WACC in its traditional form. But as already explained, the weighted average return formula is:

$$\ln[(D/V)\, e^{K_d\, T} + (E/V)\, e^{K_e\, T} + (Tax/V)\, e^{K_{tax}\, T}]/T = Implied\ weighted\ average\ return \qquad [82]$$

This expression is applicable to firm's total value, **including present value of taxes**. By similar reasoning, the weighted average discount return for the private value of the firm, D + E, is:

$$\boldsymbol{\ln\left[[D/(D+E)]\, e^{K_{d,at}\, T} + [E/(D+E)]\, e^{K_e\, T}\right]/T = Implied\ after\ tax\ weighted\ average\ return_p = WACC} \qquad \textbf{[83]}$$

Notice that K_d is replaced by K_d after tax or $K_{d,at}$, since the future expected cash flow to be discounted is the unlevered after tax cash flow. Plugging in the numbers of the numerical example we obtain the proposed WACC:

$\ln[44.54005592812960\% \times e^{2.46917515252596\% \times 3} + 55.45994407187040\% \times e^{11.24809475054090\% \times 3}]/3$
$=$

$= \ln[125.68396469757100\%]/3 = \mathbf{7.62001178107318}\%$

<u>This is almost exactly the same implied risk adjusted return we obtained previously, when using</u>:

$$WACC = \ln[E(V_U)/(D+E)]/T \qquad [77]$$

$WACC = \ln[\$\, 112.69000984720600/\$\, 89.66140598633130]/3 = \mathbf{7.62001178107325}\%$

<u>This proves that the after tax cost of debt formula I propose is consistent</u>.

To the reader it may remain the concern that this proof is limited to the specific example used along this paper. To help release this feeling I made the same comparison, [83] vs [77] for WACC, <u>changing two assumptions (1 at a time)</u>. First I assumed a different <u>assets' return volatility</u>; the original assumption is 35% and I modified it to 20%. Then I assumed a different <u>debt's maturity</u> T; the original assumption is T = 3 years and I modified it to 5 years. In the first case, D + E = $ 93.23109104171980, and WACC results according to [83] and [77] respectively, are 7.39197816770998% and 7.39197816771005%, respectively. In the second case, D + E = $ 86.92073640544640, and WACC results according to [83] and [77] respectively, are 7.74821696101785% and 7.74821696101776%, respectively. In each of both cases, WACC results are almost identical[32].

<u>Returning to our base case assumptions</u>, notice that the <u>after tax cost of debt obtained,</u> **<u>2.46917515252596%</u>**<u>, is not the same as what is commonly suggested</u>:

$$K_{d,at} = K_d(1 - \tau_c) \qquad [79]$$

, where τ_c is the statutory tax rate usually taken as a reasonable proxy of the marginal corporate tax rate. If we would use this formula, we obtain a wrong after tax cost of debt; using the example assumed statutory tax rate of 35% and K_d:

$3.45399797287673\% \,(1 - \mathbf{35}\%) = \mathbf{2.2450986823699}\%$

[32] Is my duty to warn the reader that in every change of assumption, I must calculate several times (iterate) in my Excel model in order that the solution converges to a stable one. Anyway, the convergence problem occurs after the 10th decimal place, so nothing to worry too much about.

Instead, the implied tax rate obtained by me is[33]:

$$3.45399797287673\% \ (1-\tau_c) = \mathbf{2.46917515252596}\%$$

$$\tau_c = 1-(2.46917515252596\%/3.45399797287673\%) = \mathbf{28.5125477225033}\%$$

The implied "marginal" tax rate is smaller than the statutory tax rate. This is because taxes are not effective in every state of nature but only in some states of nature. Only when taxable income is positive, taxes are paid, but when taxable income is negative, taxes are zero.

As a further check on the logic of the correct/implied marginal tax rate in the after tax cost of debt, I changed in the numerical example only the assumption of the assets' volatility. A priori, if volatility is larger, the implied tax rate in the after tax cost of debt should rise, since larger volatility causes a larger expected pre-tax result, conditional on positive pre-tax results. Negative pre-tax result scenarios are irrelevant, since effective tax paid will always be zero in these scenarios. Recall that V_T has a lognormal probability distribution, which in turn means that natural log of V_T is normally distributed (symmetrical). I changed assets' volatility from 35% to 40%, and the implied tax rate in the after tax cost of debt is 29.3918997861067% (larger, as expected a priori); pre-tax risk adjusted cost of debt is 3.65569376655223% (increased) and after-tax cost of debt is 2.58121591820025% (also increased).

CAPM AND BETAS

Up to this point of my work you may have noticed that plenty of risk adjusted returns have been obtained, without any reference to a particular equilibrium model of returns such as, for example, the well-known Capital Asset Pricing Model (CAPM)[34], or to any multi-factor model of returns, as for example, Fama and French Three Factor Model [1996]. Even though CAPM has been criticized

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[33] Again, this derivation of the implied tax rate in the after tax cost of debt may be flawed since it is derived from a linear expression.

[34] Copeland et al [2005] develop a whole section of their textbook in chapter 15, on the cost of risky debt using option pricing theory. They refer to a work done by Hsia C. [1981] where this author establishes a relationship between risk-adjusted returns of debt and of equity, with the option pricing model developed by Fisher Black and Myron Scholes. For example, according to Copeland et al, the risk adjusted cost of equity can be stated as:

$$K_e = R_f + (R_m - R_f)\ N(d_1)\ (V/E)\ \beta_V;\ \text{where } \beta_e = N(d_1)\ (V/E)\ \beta_V.$$

Notice that $N(d_1)$ is a component of the value of a call option.

abundantly, this paper may be a chance to partially vindicate its possible virtues[35]. Merton [1992] states a clear relationship between his development in chapter 11, "*A Dynamic General Equilibrium Model of the Asset Market and Its Application to the Pricing of the Capital Structure of the Firm*", and the classical CAPM model[36].

CAPM states a linear relationship between risk-adjusted returns (I call them also risk-adjusted required returns to differentiate from expected returns) and 3 variables: risk free rate of return; the "beta" or risk level corresponding to the asset; and the "market risk premium" (MRP).

$$K_i = r + \beta_i \times MRP \qquad [84]$$

β_i is defined as the systemic (or non-diversifiable risk of individual risky asset "i". More formally:

$$\beta_i \equiv \frac{Cov(R_i, R_M)}{Var(R_M)} \qquad [85]$$

It is the covariance between assets' "i" returns and the market portfolio, M, returns R_M, divided by the variance of M returns. Both numerator and denominator in [85] may be estimated with historical data on R_i and R_M, or with other methods (Damodaran estimates each month of the year an implied market risk premium for US market, based on current prices and current expectations). Anyway, all variables in the CAPM model are defined as expected variables.

The MRP is the market's expected excess return above r, when investing in a totally diversified portfolio called M (the market portfolio). A simple example: if the estimate of asset's j beta is 0.80 and MRP is (12% - 3%) = 9%, then j's risk adjusted return according to CAPM is 3% + 0.80 x 9% = 3% + 7.20% = 10.20%. This would be the risk adjusted return that we may use to discount future

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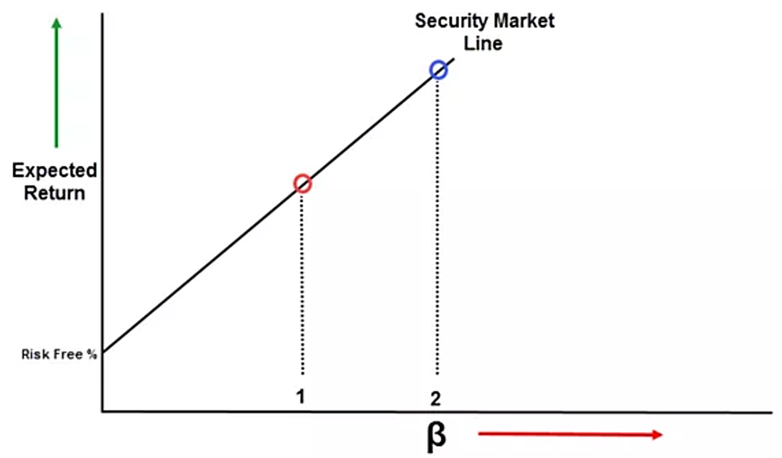
[35] Please bear in mind that I AM NOT an academic, but a finance practitioner and finance professor that intends to do his work in the most professional and serious way. So my comment about other possible equilibrium return models have nothing to do with the subtleties and complexities that I can imagine are involved in the elucidation of which is or are the best models. Being a practitioner, I have to do a leap of faith, and with criteria, use whichever model I think best fits my clients' needs and also my students at the university needs. I have for a long time stick to use CAPM as a reasonable tool to estimate risk adjusted returns of equity for valuation purposes.

[36] From 11.8, Conclusion: "*A general intertemporal equilibrium model of the asset market has been derived for arbitrary preferences, time horizon, and wealth distribution. The equilibrium relations among securities were shown to depend only on certain "observable" market aggregates, and hence are subject to empirical investigation. Under the additional assumption of a constant rate of interest, these equilibrium relations are essentially the same as those of the static CAPM of Sharpe, Lintner, and Mossin. However, these results were derived without the assumption of Gaussian distributions for security prices or quadratic utility functions. When interest rates vary, some of the intuition about "market risk" and equilibrium expected returns provided by the CAPM was shown to be incorrect.*"

expected cash flows from asset j. In practice, betas are estimated for equities (not for debt), thus, the obtained return from CAPM is given the estimated beta for a specific company stock (e.g. Coca-Cola Co.), and its correct use is for discounting future expected cash flows to shareholders of that specific company.

As shown in Figure 9 CAPM model is linear in beta, considering that the risk free rate r, and the market risk premium MRP, are both exogenous variables determined by market equilibrium.

Figure 9: asset return vs asset beta



Source: Investopedia

Implied CAPM betas in the risk adjusted returns:

The following is a summary of all risk adjusted returns (logarithmic) obtained through our unique numerical example (these are the ones corresponding to the case with corporate taxes). Let us first recall the assumptions under which these results were obtained: Pre-tax current value of assets, V = $ 100.0; maturity T = 3 years; pre-tax assets' volatility, σ = 35.0%; risk-free rate of return r = 3.0%; pre-tax risk adjusted assets' return, μ = 9.0%; corporate tax rate τ_c = 35.0%, and contractual value of debt at maturity, K = $ 45.0.

- Risk free return r: 3.0% (exogenous assumption)
- Pre-tax assets return μ: 9.0% (exogenous assumption)
- $K_e = 11.24809475054090\%$
- $K_d = 3.45399797287673\%$

- $YTM\ log = 3.9801112777175\%$
- $Weighted\ pretax\ average\ return\ of\ D\ and\ E = 7.99928981213554\%$
- $K_{ts} = 10.06152081552400\%$
- $K_U = 7.97641406077755\%$
- $K_{tax} = 16.61072668317950\%$
- $K_{tax,U} = 16.10480405212300\%$
- $WACC = 7.62001178107318\%$
- $K_{d,at} = 2.46917515252596\%$
- $Weighted\ pretax\ average\ return\ of\ D, E\ and\ Tax = 9{,}00000000000052\%$

The private leverage, D / (D + E) obtained was 44.54005592812960%

It would be too long to demonstrate all possible weighted betas by combining the previous risk adjusted returns. Thus I limit to a few relevant cases.

To compute each beta I use the following procedure:

First I assume an exogenously determined equity risk premium MRP (linear, not log, annual return). Say it is **MRP = 5.00%**. Using linear total returns as already explained in this paper, I compute the premium over the risk free rate of return as:

Recall from our previous discussion, the computation of total non-log returns;

$$k_d^{tot} = [E(D_T)/D] - 1 \qquad [65]$$

$$k_e^{tot} = [E(E_T)/E] - 1 \qquad [66]$$

Notice that I use lower-case ***k*** for arithmetic returns, as opposed to upper-case ***K*** for log returns.

Using our partial results:

$$k_d^{tot} = [\$\,44.29532276825640/\$\,39.93524037225930] - 1 = \mathbf{10.91788193924530}\%$$

$$k_e^{tot} = [\$\,69.68423497565900/\$\,49.72616561407200] - 1 = \mathbf{40.13595079194900}\%$$

These TOTAL returns (in 3 years) weighted by the relative weights of debt and equity:

$$(D/(D+E))\,k_d^{tot} + (E/(D+E))\,k_e^{tot} \qquad [67]$$

Replacing the example results:

$$44.54005592812960\% \times 10.91788193924530\% + 55.45994407187040\% \times 40.13595079194900\% = \mathbf{27.12220658383540}\%$$

This is the weighted average TOTAL arithmetic pre-tax return of debt and equity. This is consistent with the log form return of **7.99928981213554%** already explained.

$e^{7.99928981213554\% \times 3} - 1 = \mathbf{27.12220658383540}\%$

But more importantly, the value weighted average TOTAL risk adjusted return of debt and equity is consistent with the implied total risk adjusted return of private assets.

We can derive implied betas by T-year totalizing the annual risk-free return r (non-log) = 3.04545339535169% as:

$$r^{tot} \equiv \left(1 + r_{nonlog}\right)^T - 1 = (1 + 3.04545339535169\%)^3 - 1 = \mathbf{9,41742837052106}\%$$

Then do the same with the annul non-log MRP:

$$MRP^{tot} \equiv \left(1 + MRP_{nonlog}\right)^T - 1 = (1 + 5.00\%)^3 - 1 = \mathbf{15.76250000000000}\%$$

Then we can derive security j implied beta as:

$$\frac{k_j^{tot} - r^{tot}}{MRP^{tot}} = \beta_j \qquad [86]$$

From expression [86] we have that its numerator, $k_j^{tot} - r^{tot}$, is equal to $MRP^{tot} \times \beta_j$.

Weighted average of debt and equity betas:

Applying [86] to debt and equity in our numerical example, we obtain debt and equity betas:

$$\beta_d = \frac{k_d^{tot} - r^{tot}}{MRP^{tot}} = \frac{10.91788193924530\% - 9.41742837052106\%}{15.7625000000000\%} = \frac{1.50045356872421\%}{15.7625000000000\%} = \mathbf{0.09519134456617}$$

$$\beta_e = \frac{k_e^{tot} - r^{tot}}{MRP^{tot}} = \frac{40.13595079194900\% - 9.41742837052106\%}{15.7625000000000\%} = \frac{30.71852242142800\%}{15.7625000000000\%}$$
$$= \mathbf{1.94883568097878}$$

The weights of D and E in (D + E) are 44.54005592812960% and 55.45994407187040%, respectively (from previous value calculations). Thus, the weighted average of debt and equity betas is:

$$\beta_w^{d,e} = 44.54005592812960\% \times 0.09519134456617 + 55.45994407187040\%$$
$$\times 1.94883568097878 = \mathbf{1.12322145683199}$$

Next I will check that this weighted average beta of debt and equity of 1.12322145683199 is the same as weighted average beta considering V_U and VTS.

Weighted average of V_U and VTS betas:

Applying [86] to V_U and VTS in our numerical example, we obtain following betas:

$$\beta_u = \frac{k_u^{tot} - r^{tot}}{MRP^{tot}} = \frac{27.03499603266470\% - 9.41742837052106\%}{15.7625000000000\%} = \frac{17.61756766214370\%}{15.7625000000000\%}$$
$$= \mathbf{1.11768867008049}$$

$$\beta_{ts} = \frac{k_{ts}^{tot} - r^{tot}}{MRP^{tot}} = \frac{35.23524404635330\% - 9.41742837052106\%}{15.7625000000000\%}$$
$$= \frac{25.81781567583220\%}{15.7625000000000\%} = \mathbf{1.63792645048896}$$

The weights of V_U and VTS in (V_U + VTS) are 98.93648885953740% and 1.06351114046266%, respectively (from previous value calculations). The weighted average of V_U and VTS betas is:

$$\beta_w^{V_u,VTS} = 98.93648885953740\% \times 1.11768867008049 + 1.06351114046266\%$$
$$\times 1.63792645048896 = \mathbf{1.12322145683203}$$

Thus, the resulting value weighted average beta of the firm is almost exactly the same if we obtain it from debt and equity betas, or with unlevered value and tax savings betas.

With this two cases I prove that derived CAPM betas are consistent at least with the value-weighted average of betas criterion (considering non log total returns as a basis for calculations).

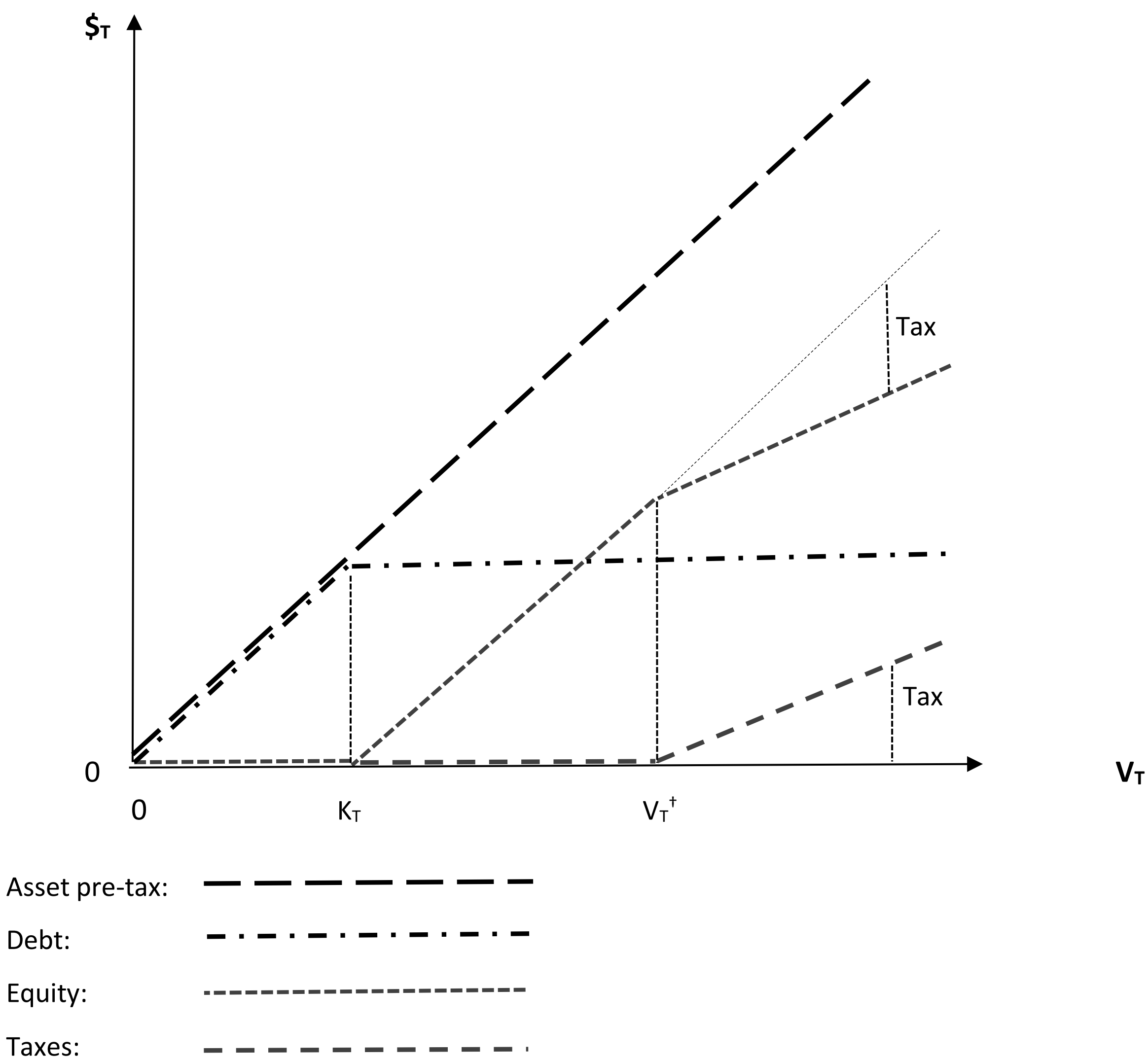
Beta of taxes paid by the firm:

Appling the same procedure as previously explained, the CAPM beta for the taxes paid by the firm, using [86], is:

$$\beta_{tax} = \frac{k_{tax}^{tot} - r^{tot}}{MRP^{tot}} = \frac{64.5956707873208\% - 9.41742837052106\%}{15.7625000000000\%}$$
$$= \frac{55.17824241679970\%}{15.7625000000000\%} = \mathbf{3.5006022151816}$$

The beta of tax cash flows is comparatively very high, reflecting the deep out-of-the-money call option that the Government has on the pre-tax cash flows of the firm.

Figure 7: Payoff functions of debt, equity and taxes, at time T, in a world WITH corporate taxes



Effect on betas of assets' volatility:

Our example in all this paper assumes an assets' volatility σ of 35.0%. Recall that this is the volatility on the pre-tax value of the firm. Analyzing Damodaran's web page where he reports firm standard deviation estimated for almost 100 industries in the USA as of December 2018, we observe that his estimation of firm's assets standard deviation is 36.98% (without financials). There exists some dispersion in volatility among industries; for example in Software Entertainment is 64.46%, and in Broadcasting is 21.09%.

Given this dispersion in volatility estimates, I tried with two assumptions: 25% and 45%, respectively. To do it simple, I did not intervened any other assumption, as for example K (to keep a constant leverage). Results are as follows:

a. Assets' volatility of **25%**:

$$\beta_d = \frac{k_d^{tot} - r^{tot}}{MRP^{tot}} = \frac{0.36137979756623\%}{15.7625000000000\%} = \mathbf{0.02292655337454}$$

$$\beta_e = \frac{k_e^{tot} - r^{tot}}{MRP^{tot}} = \frac{30.64262141318730\%}{15.7625000000000\%} = \mathbf{1.94402039100316}$$

$$\beta_w^{d,e} = 44.49344303769210\% \times 0.02292655337454 + 55.50655696230790\% \times 1.94402039100316 = \mathbf{1.08925959865726}$$

$$\beta_u = \frac{k_u^{tot} - r^{tot}}{MRP^{tot}} = \frac{17.04128101192310\%}{15.7625000000000\%} = \mathbf{1.08112805785396}$$

$$\beta_{ts} = \frac{k_{ts}^{tot} - r^{tot}}{MRP^{tot}} = \frac{30.37161464228750\%}{15.7625000000000\%} = \mathbf{1.92682725724266}$$

$$\beta_{tax} = \frac{k_{tax}^{tot} - r^{tot}}{MRP^{tot}} = \frac{72.16171850106780\%}{15.7625000000000\%} = \mathbf{4.5780630294095}$$

b. Assets' volatility of **45%**:

$$\beta_d = \frac{k_d^{tot} - r^{tot}}{MRP^{tot}} = \frac{2.83026654583849\%}{15.7625000000000\%} = \mathbf{0.17955695770585}$$

$$\beta_e = \frac{k_e^{tot} - r^{tot}}{MRP^{tot}} = \frac{29.90659606946240\%}{15.7625000000000\%} = \mathbf{1.89732568244012}$$

$$\beta_w^{d,e} = 43.53448703949060\% \times 0.17955695770585 + 56.46551296050940\% \times 1.89732568244012 = \mathbf{1.14950387960225}$$

$$\beta_u = \frac{k_u^{tot} - r^{tot}}{MRP^{tot}} = \frac{18.05875493451250\%}{15.7625000000000\%} = \mathbf{1.14567834636082}$$

$$\beta_{ts} = \frac{k_{ts}^{tot} - r^{tot}}{MRP^{tot}} = \frac{22.70865356014010\%}{15.7625000000000\%} = \mathbf{1.44067588010405}$$

$$\beta_{tax} = \frac{k_{tax}^{tot} - r^{tot}}{MRP^{tot}} = \frac{45.90964757477650\%}{15.7625000000000\%} = \mathbf{2.9125866819842}$$

The variation of volatility (base case: 35%; variation 1: 25%; variation 2: 45%) assumption didn't cause a significant change in firm's leverage so this makes next comments more reliable. Debt's beta moves from 0.09519134456617 to 0.02292655337454, and to 0.17955695770585, respectively; we may say that relative to its base case value, debt's beta changes significantly and in the same direction as assets' volatility. Equity's beta moves from 1.94883568097878 to 1.94402039100316, and to 1.89732568244012, respectively; we may say that relative to its base case value, equity's beta changes moderately and with no clear relation to the change in assets' volatility. Tax saving's beta moves from 1.63792645048896 to 1.92682725724266, and to 1.44067588010405, respectively; we may say that relative to its base case value, tax saving's beta changes less moderately than equity beta, and in opposite direction as assets' volatility. Finally, Tax beta moves from 3.5006022151816 to 4.5780630294095, and to 2.9125866819842, respectively; we may say that relative to its base case value, tax beta changes significantly, and in opposite direction as assets' volatility. As a closing comment to this section, in all cases tried, the consistency between different value weighted average betas maintained unaltered.

Linearity of CAPM

I performed a simple test about the linearity of the presumable CAPM structure/model embedded in the valuation model presented in this paper. As already shown, betas can be obtained with total T-year returns, and these betas are consistent in the sense of matching with the linear additivity of betas as argued in the finance theory. But are these betas linear? CAPM model states a linear relationship, positively sloped, between expected returns (I prefer the words: risk adjusted return) and beta risk. This relationship, according to CAPM model applies at the individual security level (e.g. Nike's stock, or Tesla's debt), and at a portfolio of securities level. The other prediction of CAPM is that if beta is zero, the asset´s expected return is equal to the risk-free rate of return r.

In my paper I obtain the return of clearly identifiable securities: firm equity, and firm risky debt. But I also obtain returns of indirect (or related) securities, such as: Taxes, value of the unlevered firm, value of tax savings, etc.

To measure the CAPM relationship, given the model variables, r, T, K, σ, μ, τ_c, and MRP, there are only a few choices of which assumption vary in order to measure, for example, if the cost of equity K_e exhibits a linear relationship with the estimated beta in this paper. Variables r and MRP are automatically discarded since they form part of CAPM equation. Let us recall:

$$K_i = r + \beta_i \times MRP \qquad [84]$$

Another two variables that may influence in the relationship return-beta, are T and τ_c, but I think is convenient to hold both of them constant, similar as controlling for one or more variables influence in a regression analysis. This leaves us with only three candidates: K, σ, and μ. It is well known, specially (and not painlessly) by practitioners in valuation, the relationship between leverage of the firm and beta; levered and unlevered betas are common calculations in DCF valuations. So, I decided to "reserve" variable K (final contractual value of debt) as a means for adjusting leverage, this is keeping it constant while measuring the return-beta relationship. Finally I decided that the most interesting and logical variable to change is assets' volatility σ (I discarded μ since it is directly related with unlevered beta, so varying μ assumption would only change my starting point for equity beta). In summary, I change the assumption of volatility σ and at the same time adjust K in order to keep constant the starting point or base case leverage (private leverage = D/(D + E)).

Figures 10 and following show the relationship obtained for various "objects", between its beta and its risk adjusted return.

Figure 10: Beta equity vs volatility (base case leverage, 44.54%)

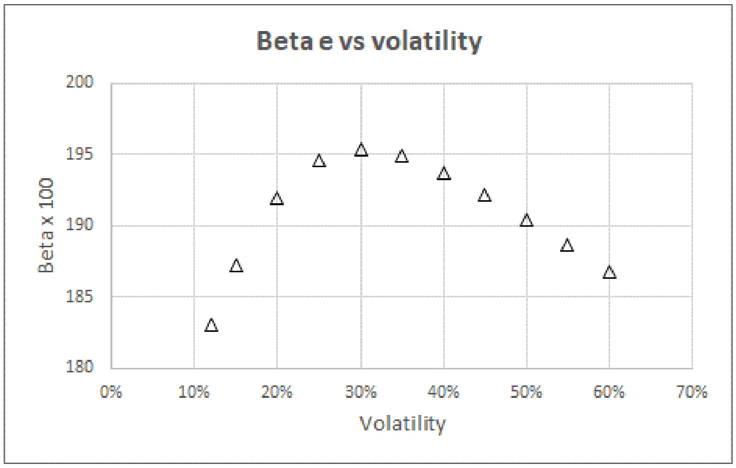
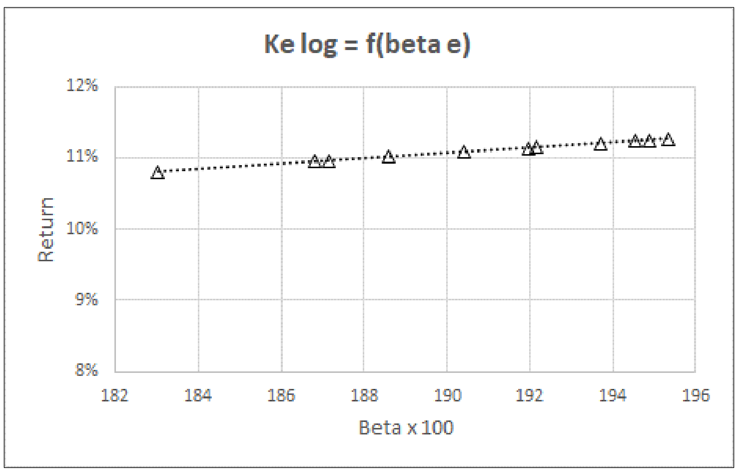
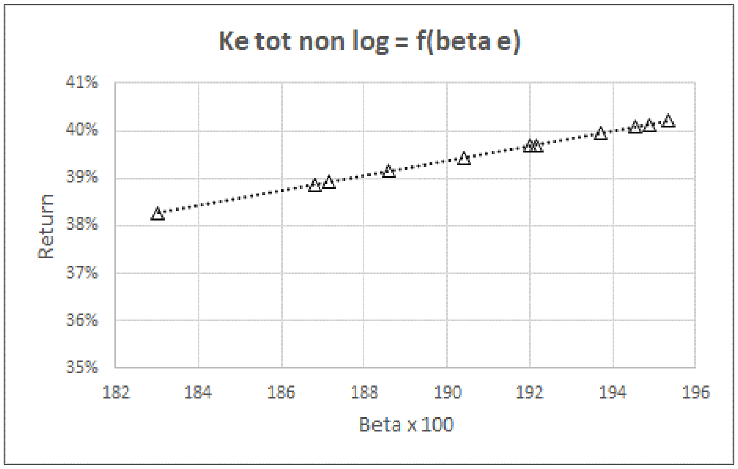


Figure 11: Beta equity vs log risk adjusted return (base case leverage, 44.54%)



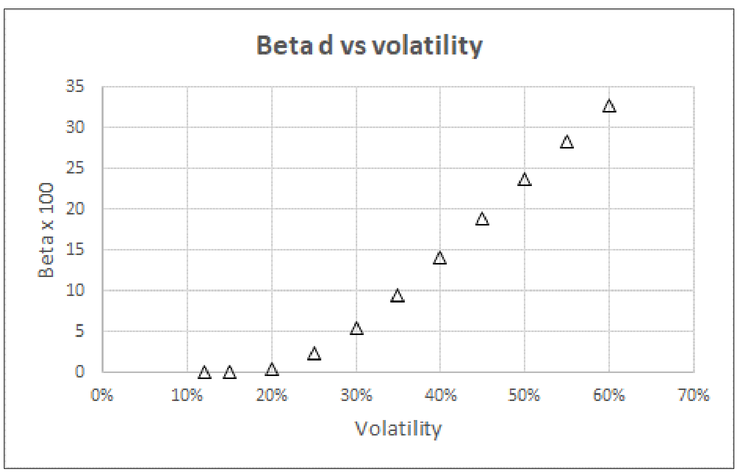
As we can appreciate, varying assets' volatility assumption (and at the same time keeping firm's leverage constant), changes equity beta, but as seen in Figure 10, this relationship is not linear, and is a concave function with a local maximum value of beta. On the other hand, when relating same obtained betas, with risk adjusted return of equity in my model (in annual log form), the relationship return-beta for equity is perfectly linear as predicted by CAPM model, and its slope is positive (higher expected return at higher beta). Figure 12 shows that this same relation holds for equity returns expressed in total T-year non log returns (used to compute beta).

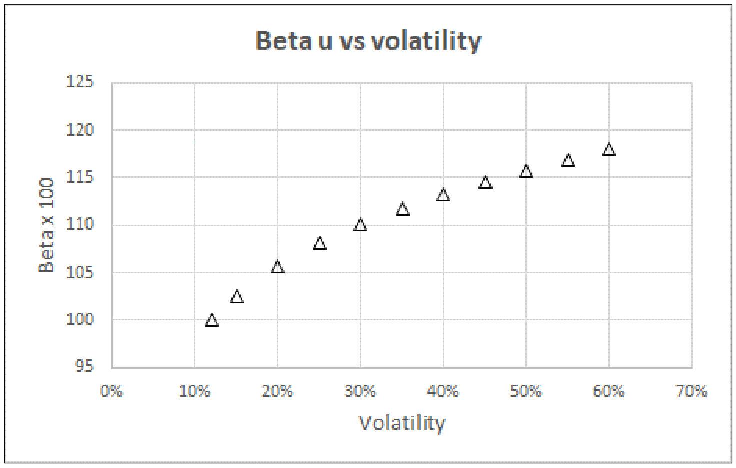
Figure12: Total non-log equity returns and its beta

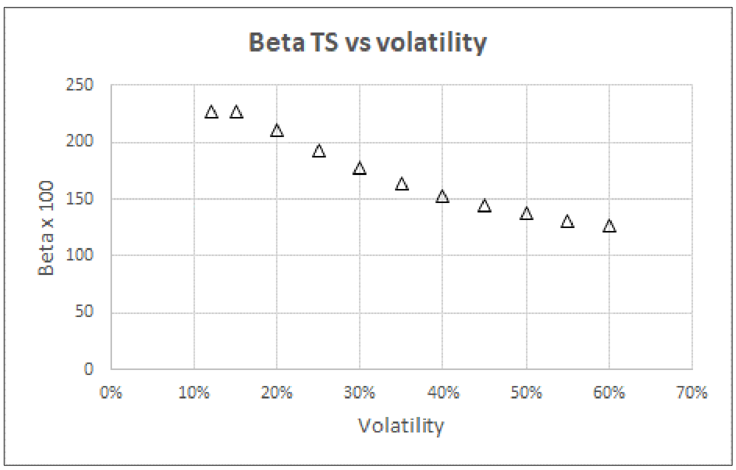


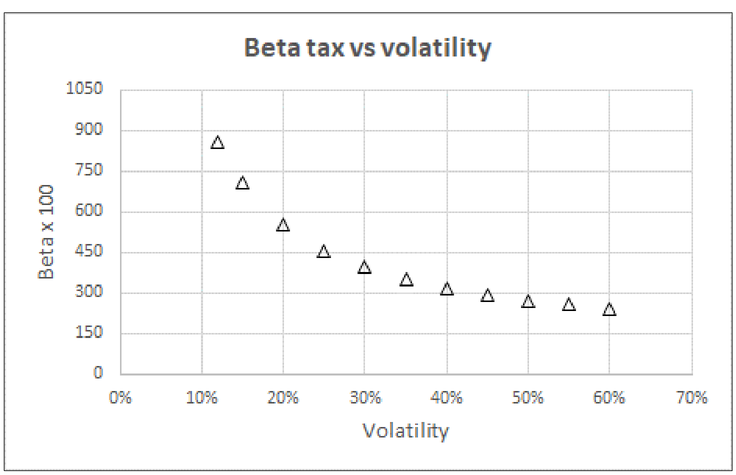
Beta equity is possibly the most "notable" beta in finance practice, but betas exist for any kind of financial security. Next Figures 13 to 16 show the relationship with assets' volatility of: debt beta, unlevered beta, beta of tax savings, and beta of taxes.

Figures 13 to 16: various betas and assets volatility




Beta u vs volatility
125
120
115
110
105
100
95
Beta x 100
0%
10%
20%
30%
40%
50%
60%
70%
Volatility


Beta TS vs volatility
250
200
150
100
50
0
Beta x 100
0%
10%
20%
30%
40%
50%
60%
70%
Volatility


Beta tax vs volatility
1050
900
750
600
450
300
150
0
Beta x 100
0%
10%
20%
30%
40%
50%
60%
70%
Volatility

As you may appreciate, beta debt tends to zero at low assets' volatility (circa 20% or less). At higher volatilities, debt's beta is as typically reported in financial literature, close to 0.3 or 0.4. The slope beta-debt-volatility is positive and is convex. Beta unlevered equity instead is concave and with positive slope beta-u-volatility. With volatility ranging from 12% to 60%, beta unlevered ranges from 1.0 to 1.18.

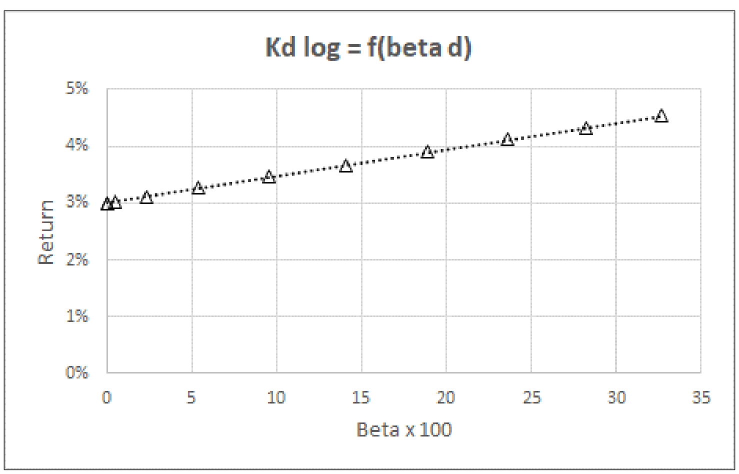
Tax related betas (this is beta of tax savings and beta of taxes) all have a negative slope relation with volatility, contrary to previous entities (all but equity beta at higher volatilities). With volatility ranging from 12% to 60%, beta of tax savings ranges from 2.27 to 1.26, and beta of taxes ranges from 8.59 to 2.53.

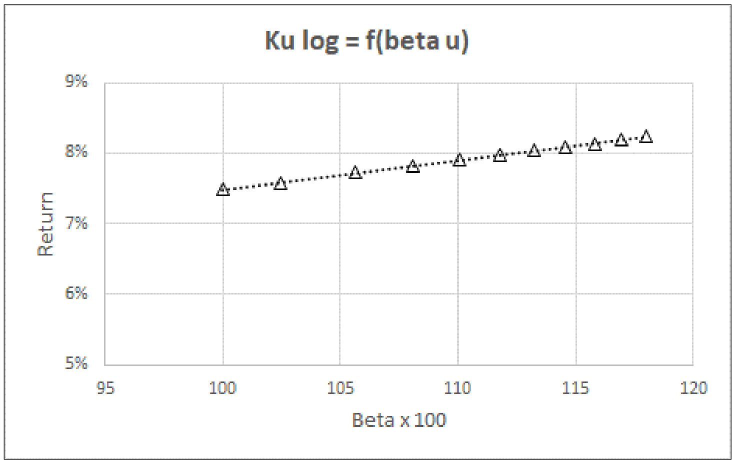
Linearity of return-beta for other entities in firm valuation:

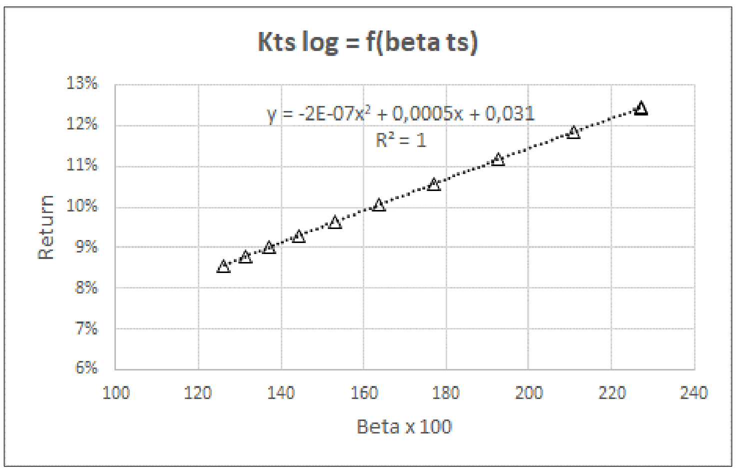
I already showed that there is a linear, positively sloped relation between equity return and its beta as calculated in this paper, inferred from Merton Structural Model considering taxes, and with my proposed numerical integration solution method. But do the other entities, debt, unlevered equity, tax savings, and taxes, exhibit the same type of behavior?

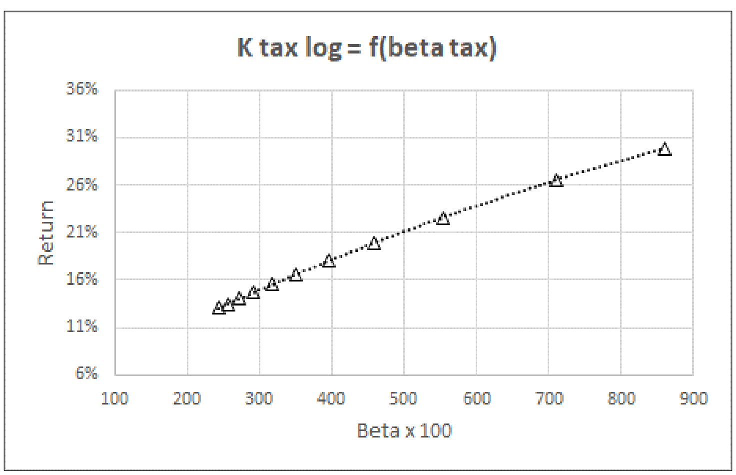
In next Figures I show the relationship between different types of beta and their respective risk adjusted returns, all obtained through my calculations explained in this paper.

Figures 17 to 20: Various expected returns and betas




Ku log = f(beta u)
Return
9%
8%
7%
6%
5%
95
100
105
110
115
120
Beta x 100


Kts log = f(beta ts)
y = -2E-07x² + 0,0005x + 0,031
R² = 1
Return
13%
12%
11%
10%
9%
8%
7%
6%
100
120
140
160
180
200
220
240
Beta x 100


K tax log = f(beta tax)
Return
36%
31%
26%
21%
16%
11%
6%
100
200
300
400
500
600
700
800
900
Beta x 100

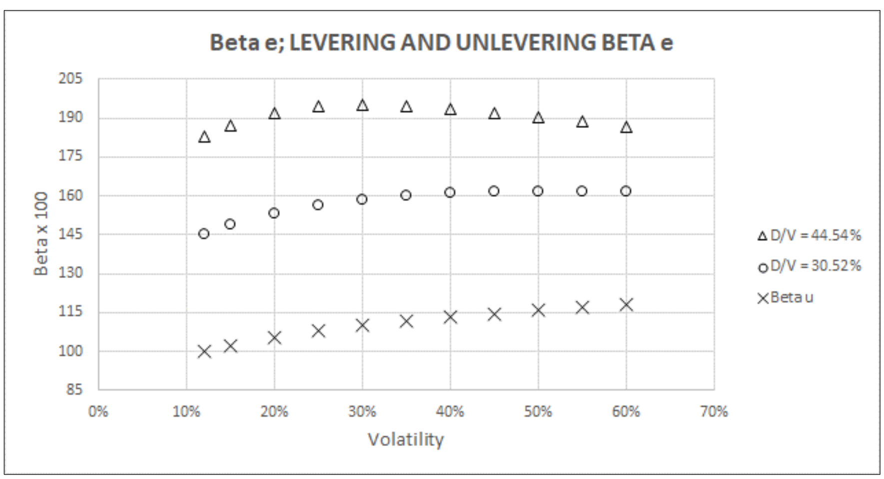
All CAPM deducted functions with my model, as seen in Figures 17 to 20, are linear, but the last two, tax savings, and taxes. They both exhibit some degree of concavity, particularly visible in the tax CAPM graph. I must warn the reader that this conclusions may be limited to the particular example developed by me in this paper, and that more work may be requires to asses a more general perspective. But up to now I think that results obtained are quite exiting.

<u>Leveraging and deleveraging betas (only equity betas)</u>:

A great deal of discussion exists not only about the correct discount rate for tax savings in firm valuation, but also about the correct way of leveraging and deleveraging betas. This last is an important practical issue in DCF valuation since a very common practice is to estimate the equity beta for the specific firm valued, starting from a pure or unlevered beta, also known as asset beta. Hamada's equation is commonplace to do this type of adjustment[37].

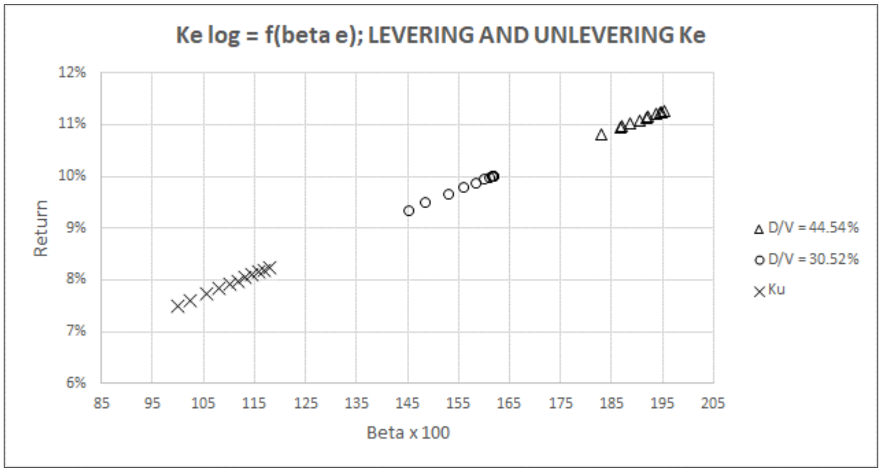
In this section I present graphical results of levered equity betas, and corresponding risk adjusted returns on equity, for three situations: unlevered firm; firm leverage equal to 30.52%; and firm leverage equal to 44.54%. See Figures 21 and 22 for these results.

Figure 21: Beta equity as a function of asset volatility



[37] There are other equations to unlever / lever betas. One is Rubinstein equation which includes debt's beta. I am indebted and grateful to Mario Inostroza, president of Emdepes Nissui for providing me with this alternative equation.

Figure 22: Beta equity and equity risk adjusted returns



We can appreciate that for equity beta, at least in the example analyzed in this paper, the leveraging/deleveraging process is linear in the sense that levered betas fall in the same straight positively sloped line relating returns with betas.

Concluding this CAPM analysis, I can say that CAPM seems to reflect in the contingent pricing model (Merton's model) at two different stages: the first is measuring the relation when varying assets' volatility, keeping firm's leverage constant; and the second is when we allow leverage to change (jointly with volatility).

CAPITAL STRUCTURE AND THE COST OF CAPITAL WITH CORPORATE TAXES

Introduction:

Even though I feel that many more aspects of valuation and corporate finance policy may be explored following my work presented here, I want to close this already lengthy paper with a simplified discussion of the effect on the cost of capital (the effect on the risk adjusted returns I already explained), when varying the assumption of the firm's capital structure, this is firm's leverage D / (D + E), and keeping every other assumption unchanged.

I do this in the limited context of the numerical example followed during all this paper. Thus the conclusions are not meant to be conclusive or even general. All I intend is to raise a few points of the possible effects of financing policy on the cost of capital.

I already compared my results of cost of capital structure (of debt and of equity) with Merton's results and showed that at least my results are quite similar in nature to Merton's [1992]. This limited to show convergence values, plus convexities, concavities, and inflection points, in cost of debt and cost of equity functions (see Figure 2). But Merton covers only the case of a firm with no taxes. In this section I present my results of the cost of capital structure in a firm WITH corporate taxes.

The practical form of changing the relative value of debt with respect to total value D + E, is by increasing the assumption of K_T, the final contractual value of debt, while keeping the value of assets V (pre-tax) and all other assumptions in my model, constant. In Figure 23 I show the relationship between firm's leverage D / (D + E), and almost every risk-adjusted return obtained in this paper. The results are based on the assumptions of the numerical example in this paper. Assumptions are: Pre-tax current value of assets, V = \$ 100.0; maturity T = 3 years; pre-tax assets' volatility, σ = 35.0%; risk-free rate of return r = 3.0%; pre-tax risk adjusted assets' return, μ = 9.0%; corporate tax rate τ_c = 35.0%, and contractual value of debt at maturity, K = \$ 45.0.

I provide two alternative graphical versions of capital structure: the first is relative to leverage defined as D / (D + E) in the horizontal axis of the graph; the second is relative to leverage defined as D / E in the horizontal axis of the graph. In the financial literature both options are used interchangeably.

Figure 23: Risk adjusted returns and firm leverage D/V

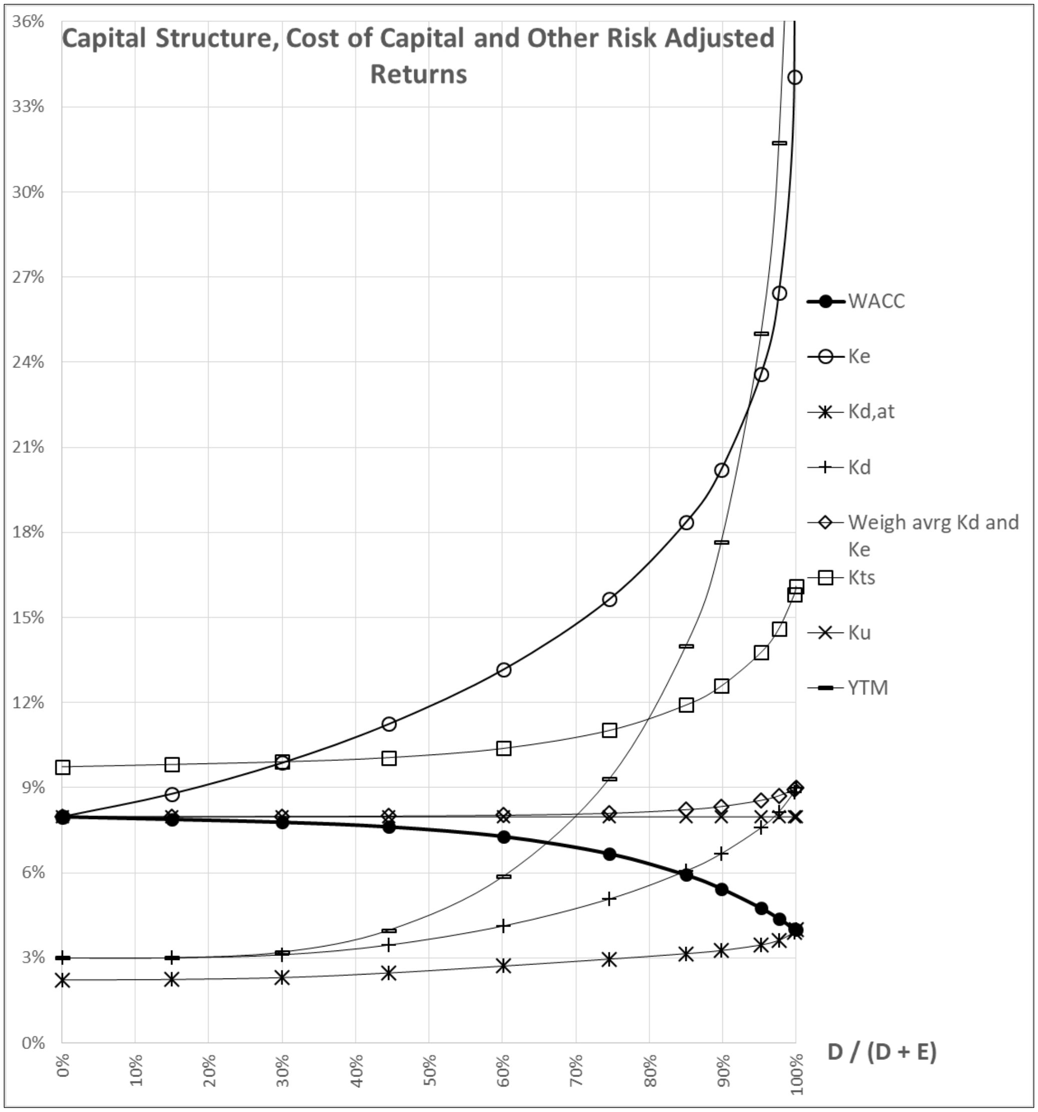


Figure 24: Risk adjusted returns and firm leverage D/E

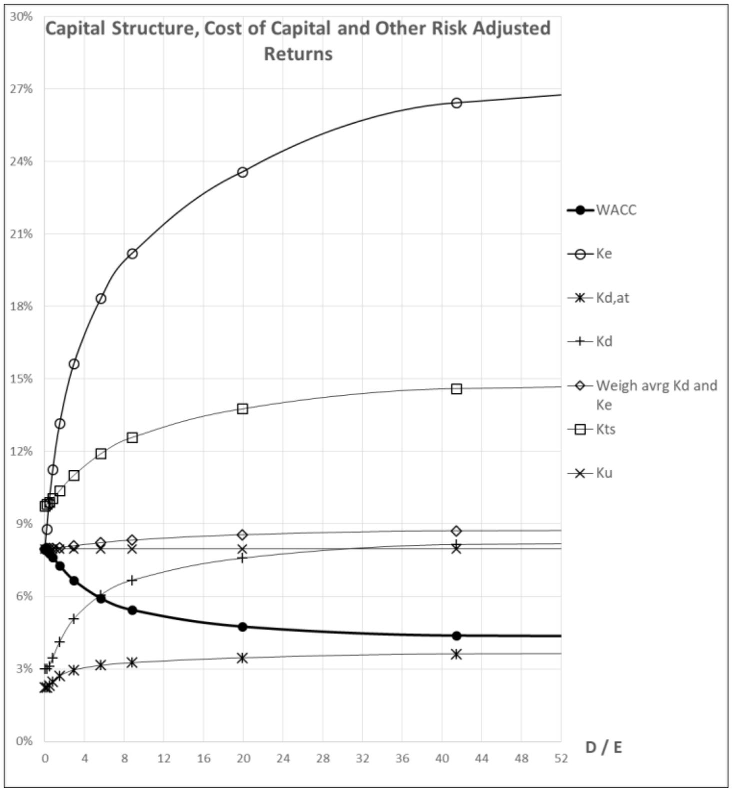
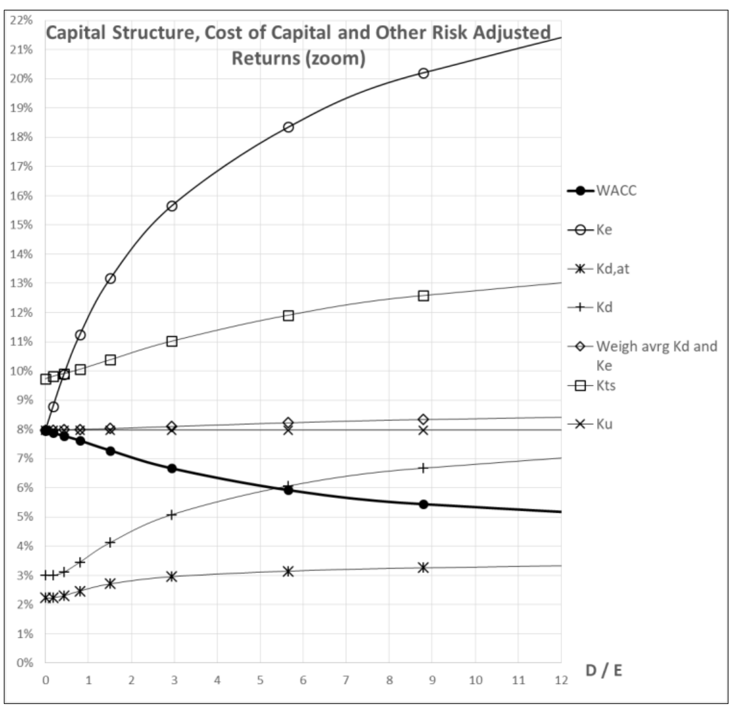


Figure 25: Risk adjusted returns and firm leverage D/E (zoom at lower leverage)

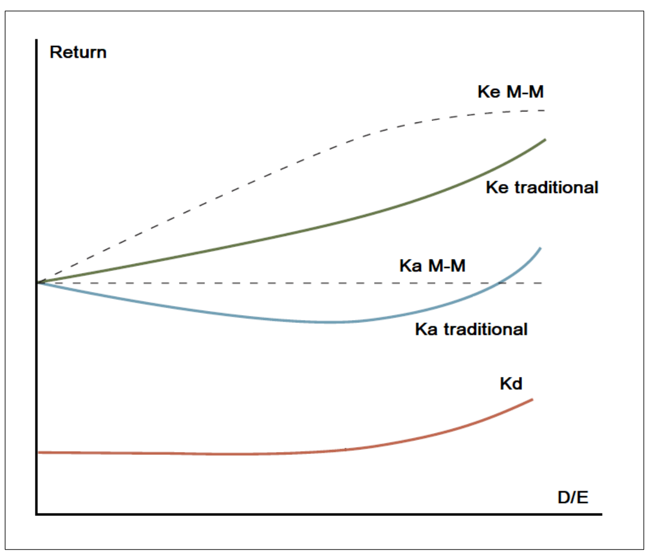


Cost of equity and cost of debt:

In previous Figure 23 we can appreciate that the cost of equity K_e, when leverage tends to zero (0.01030268976455% with K = 0.01), is 7.97689385402647% which is almost the same as the unlevered cost of equity K_U, 7.97641406077755% (the horizontal line in the same graph). If we stress still more to a leverage closer to zero, 0.00001030270957% (setting K in 0.00001), cost of equity in my model is 7.97641454052563%, even closer to the unlevered cost of equity. When leverage is increased for the same firm (assets are constant[38]), cost of equity starts rising immediately and at higher levels of leverage it reaches very high values (e.g. 15.64714841177830% when leverage is 74.57938368106090%, or 20.20938482844040% when leverage is 89.79465348073120%). In the "limit", if leverage is too high, for example 99.99999999999810%, cost of equity in my model is 159.71881227598000%.

Cost of debt instead has a different behavior, typically depicted in finance textbooks. In Figure 26 from Brealey et al [2017, page 450; slightly modified to avoid copyright infringement], we appreciate that cost of debt K_d (r_D in the graph) practically does not increase with leverage at the first part of its horizontal movement from lower to higher leverages. Notice that leverage in the Figure is expressed as D/E.

Figure 26: Cost of capital structure in literature



[38] For simplicity of computation I left Total value of assets constant, this is including the Tax present value. Maybe would be more correct to do it leaving private value of assets, D + E, constant.

In my results (see Figures 24 and 25), when leverage is virtually zero, cost of debt equals the risk-free return (3.0% in my example). Then, when leverage starts increasing, cost of debt increases, but very slowly. For example, when leverage reaches 14.99750287659670%, pre-tax cost of debt is 3.00410732112734%. At 30.03050600995160% of leverage, cost of debt rises slightly to 3.11130539457238%; at 44.54005592812960% leverage, cost of debt is 3.45399797287673%, losing initial stickiness. In this point, default risk becomes increasingly important to debt holders. For example, if leverage is 74.57938368127400%, cost of debt rises to 5.07832095033814%. Finally, when leverage is very close to 100%, 99.99999999999810%, cost of debt 8.99999999991137% is almost exactly equal to the weighted average cost of capital (pre-tax) of 8.99999999996926% (and also almost exactly equal to the exogenously assumed asset's pre-tax risk adjusted return of 9.0%).

The after-tax cost of debt, as I propose to compute it, has a flatter behavior than the pre-tax cost of debt, but still exhibits a positive slope at every level of leverage. Also notice that from the very "first cent" of debt, there is a tax savings effect reflected in a lower after-tax cost of debt of 2.22918824751340%. In the upper limit of leverage, the after-tax cost of debt is equal to the weighted average cost of capital WACC as I propose to compute it, and this is a generalized conclusion in finance.

WACC:

For me was a surprise how the weighted average cost of capital WACC behaves in my valuation model and corresponding paper (this one). There are not many sources, at least reachable to me, where to compare my results since the cost of capital structure is often presented in a rather simplistic way, and with no direct reference to the cost of debt (at least) modeled with option pricing as in this paper.

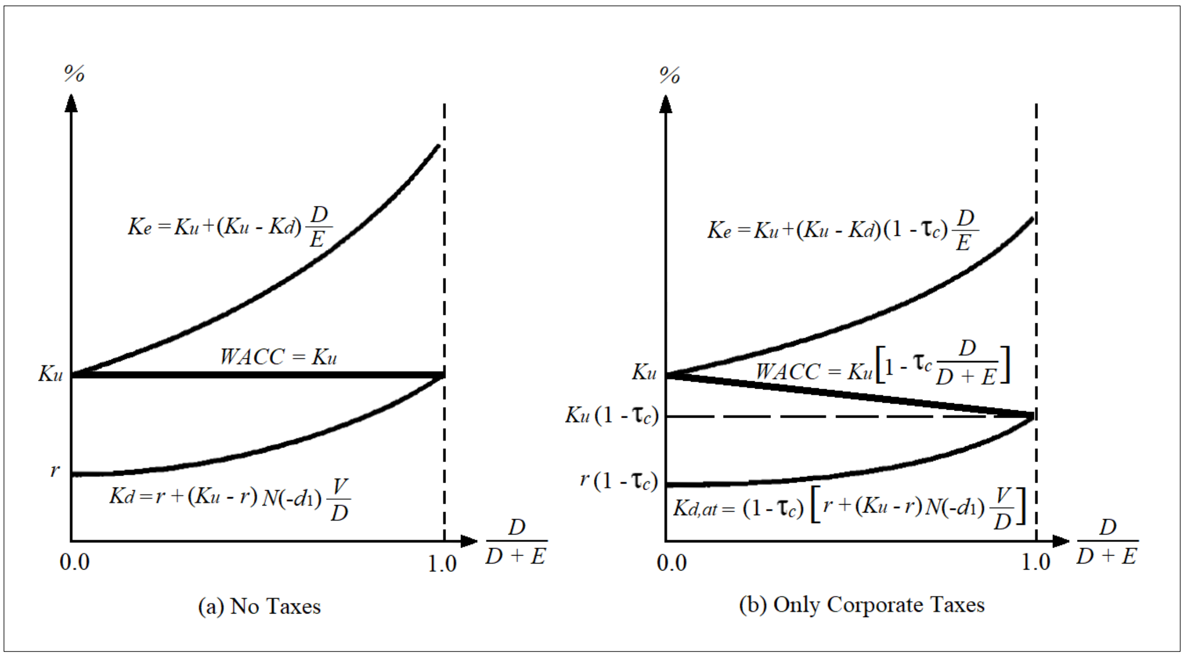
Copeland et al. [2005] is one exception, with a complete chapter devoted to cost of capital using the option pricing model. They show a step by step development of the equations for the no taxes case, and only the final results (shown in a graph) of the corporate taxes case (see Figure 27[39]). In it, the cost of equity function K_e is presented as a positively sloped function exhibiting convexity in the no

[39] I slightly modified this graph changing the variables symbols, to avoid copyright infringement.

tax case and the same in the corporate tax case; slope gets higher, the higher is the leverage expressed as D/(D + E).

But WACC in the corporate tax case, according to Copeland et al. is a negatively sloped straight line function of leverage D/(D + E). As you can appreciate in Figure 27 (b), the slope of WACC function is equal to – ρ τ_c[40]:

Figure 27: Cost of capital structure with option pricing



<u>But in my solution WACC is a concave function, NOT a straight line function</u>; this is, the negative slope is larger (in absolute terms), the larger is the leverage (see Figure 23). When leverage is close to zero, WACC is equal to the unlevered cost of equity. On the opposing side, when leverage is close to 100%, WACC is equal to the after tax cost of debt.

Copeland et al argument is based on Merton's option valuation approach. <u>Without corporate taxes</u>, cost of equity and cost of debt are:

$$K_e = K_u + (K_u - K_d)(D/E)$$

$$K_d = r + (K_u - r)\, N(-d_1)\, (D + E)/E$$

[40] This result of a negative constant slope for WACC as a function of D / (D + E) is also related to the problem already commented in this paper when considering a linear relationship in the log returns of debt, equity and assets (in a world without taxes).

And WACC is:

$$WACC = K_d\, D/(D+E) + K_e\, E/(D+E)$$

Replacing K_e and K_d:

$$WACC = D/(D+E)\,[r + (K_u - r)\, N(-d_1)\,(D+E)/E] + E/(D+E)\,[K_u + (K_u - K_d)(D/E)]$$

$$WACC = r\, D/(D+E) + (K_u - r)\, N(-d_1)\,(D/E) + E/(D+E)\, K_u + (K_u - K_d)\, D/(D+E)$$

$$WACC = r\, D/(D+E) + (K_u - r)\, N(-d_1)\,(D/E) + K_u - K_d\, D/(D+E)$$

$$WACC = K_u + r\, D/(D+E) + (K_u - r)\, N(-d_1)\,(D/E) - [r + (K_u - r)\, N(-d_1)\,(D+E)/E]\, D/(D+E)$$

$$WACC = K_u + r\, D/(D+E) + (K_u - r)\, N(-d_1)\,(D/E) - r\, D/(D+E) - [(K_u - r)\, N(-d_1)\,(D/E)]$$

$$WACC = K_u + K_u\, N(-d_1)\,(D/E) - r\, N(-d_1)\,(D/E) - K_u\, N(-d_1)\,(D/E) + r\, N(-d_1)\,(D/E)$$

Almost all terms cancel at the right hand side of previous expression, but K_u;

$$WACC = K_u$$

Same result as first graph in Figure 27.

Copeland et al. conclude that even when considering the risk adjusted cost of debt, in this case through option pricing, a constant WACC is obtained, thus replicating Modigliani and Miller's conclusion of irrelevance of financing decisions in a world without taxes.

But the solution presented in Copeland et al. is wrong because solutions for cost of equity, K_e, cost of debt K_d, and weighted average cost of capital, WACC, ARE NOT consistent. I will use the same numerical example these authors use to obtain a risk adjusted cost of debt. *"A numerical example can be used to illustrate how the cost of debt, in the absence of bankruptcy costs, increases with the firm utilization of debt. Suppose the current value of a firm, V, is \$ 3 million; the face value of debt is \$ 1.5 million; and the debt will mature in T = 8 years. The variance of returns on the firm's assets, σ^2, is 0.09; its required return on assets ρ = 0.12 (ρ* is the same as K_u in this paper*); and the riskless rate of interest is 5%."*

Copeland et al. propose the following risk adjusted cost of debt expression based on CAPM:

$$K_d = r + [E(R_m) - r]\, \beta_d$$

But they show that beta debt, β_d, can be expressed as follows in terms of option pricing:

$$\beta_d = N(-d_1)\,(D+E)/D$$

Then K_d is also:

$$K_d = r + \,[E(R_m) - r]\,N(-d_1)\,(D+E)/D$$

Given V, value of assets of the firm, value of equity is with option pricing as a call value:

$$d_1 = \frac{\ln(V/D) + R_f T}{\sigma\sqrt{T}} + \frac{1}{2}\sigma\sqrt{T}$$

$$d_1 = 1.71255037572675$$

$$N(d_1) = 0.95660235530662$$

$$N(-d_1) = 1 - N(d_1) = 0.04339764469338$$

Solution in textbook:

$$k_b = .05 + (.12 - .05)(.0434)\frac{3}{1.5}$$
$$= .05 + .0061 = .0561.$$

But this is wrong, since authors use Strike \$ 1.5 instead of debt value D in the denominator. With the given assumptions, the value of equity E, the call option value is \$ 2.05917796186387 using option pricing theory. Thus, debt value D is V – E = \$ 3.0 - \$ 2.05917796186387 = \$ 0.94082319348523. So the correct solution, different from Copeland et al. is:

$$K_d = 5\% + \,[12\% - 5\%]\,\times 0.04339764469338 \times \$\,3.0/\$\,0.94082319348523$$

$$K_d = 5.96867391859348\%$$

As you can see, correct result is 5.96867391859348% and not 5.61% as wrongly calculated in Copeland et al.[41]

<u>Copeland et al. cost of equity and weighted average cost of capital</u>:

They propose the following analytical expression for cost of equity using option pricing:

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[41] I wrote by email to Professor Tom Copeland about this possible mistake and he acknowledged the mistake and was very grateful for indicating the right calculation.

$$K_e = r + (K_u - r)\, N(d_1)\, (D + E)/E$$

Replacing known numbers:

$$K_e = 5\% + (12\% - 5\%) \times 0.95660235530662 \times \$\,3.0/\$\,2.05917796186387$$

$$K_e = 14.75566831519690\%$$

Finally, the weighted average or WACC in the no taxes case is:

$$WACC = 5.96867391859348\% \times (\$\,0.94082319348523/\$\,3.0) + 14.75566831519690\% \times (\$\,2.05917796186387/\$\,3.0)$$

$$WACC = 11.99999999999990\%$$

<u>This only shows internal consistency among the cost of equity and cost of debt equations</u>, and the linear weighted average cost of capital or WACC, since the result is almost exactly equal to the assumption of the assets' return of 12.0%. <u>But as I will show immediately, this solution does not make logical sense in a discrete time interval as the example is proposed</u>, with maturity T = 8 years.

Future expected value of equity is (or should be):

$$E_0 e^{K_e\, T} = E(E_T)$$

$$\$\,2.05917796186387 \times e^{14.75566831519690\% \times 8} = \$\,6.70437514996405$$

Future expected value of debt is (or should be):

$$D_0 e^{K_d\, T} = E(E_T)$$

$$\$\,0.94082319348523 \times e^{5.96867391859348\% \times 8} = \$\,1.51663469565560$$

Future expected value of firm assets is:

$$V_0 e^{K_V\, T} = E(V_T)$$

$$\$\,3.0 \times e^{12.0\% \times 8} = \$\,7.83508942026935$$

THE PROBLEM is that if we add expected values of debt and equity;

$$E(D_T) + E(E_T) = \$\,1.51663469565560 + \$\,6.70437514996405 = \$\,8.22100984561966$$

As you can appreciate, this last solution IS NOT consistent with the previously obtained future expected value of assets E(V_T) = $ 7.83508942026935.

As already explained, the inconsistency stems in using a linear solution for the weighted average cost of capital instead of an exponential weighted average solution. My own results under the same assumptions of Copeland et al. are:

$$K_d = 5.60867735504696\%$$

$$K_e = 14.09949318236300\%$$

Future expected value of equity is:

$$E_0 e^{K_e\, T} = E(E_T)$$

$$\$\ 2.05917796186387 \times e^{14.09949318236300\% \times 8} = \$\ 6.36151343238705$$

Future expected value of debt is (or should be):

$$D_0 e^{K_d\, T} = E(E_T)$$

$$\$\ 0.94082319348523 \times e^{5.60867735504696\% \times 8} = \$\ 1.47357900530238$$

Adding this two expected future values:

$$E(D_T) + \ E(E_T) = \$\ 1.47357900530238 + \$\ 6.36151343238705 = \$\ 7.83509243768943$$

And now we do obtain a consistent solution in terms of additivity of future expected values, present values, and corresponding risk adjusted returns.

$$\$\ 3.0 \times e^{12.0\% \times 8} = \$\ 7.83508942026935$$

<u>The case with corporate taxes</u>:

According to Copeland et al, <u>in the case with corporate taxes</u>, cost of equity and after tax cost of debt are:

$$K_e = K_u + (K_u - K_d)(1 - \tau_c)(D/E)$$

$$K_{d,at} = (1 - \tau_c)[r + (K_u - r)\, N(-d_1)\, (D + E)/E]$$

And WACC is:

$$WACC = K_{d,at}\, D/(D + E) + K_e\, E/(D + E)$$

Replacing K_e and $K_{d,at}$:

$$WACC = \{(1 - \tau_c)[r + (K_u - r)\, N(-d_1)\, (D + E)/E]\}\ D/(D + E) + [K_u + (K_u - K_d)(1 - \tau_c)(D/E)]\ E/(D + E)$$

The final solution presented is:

$$WACC = K_u[1 - \tau_c D/(D + E)]$$

One implicit assumption in Copeland et al. WACC function, as far as I can conclude from the graph since no details are given, is that they assume a constant tax rate τ_c, since WACC is a straight line function. In Figure 28 I provide the implied tax rate for WACC calculation (values in the right-hand vertical axis). Bear in mind that this tax rate is denominated the "marginal tax rate" in valuation literature, as opposed to the "effective tax rate", also in financial literature, the latter recommended to be used to obtain the free cash flow to the firm FCFF. In my results, obtaining the "implied tax rate in after-tax cost of debt", simply as:

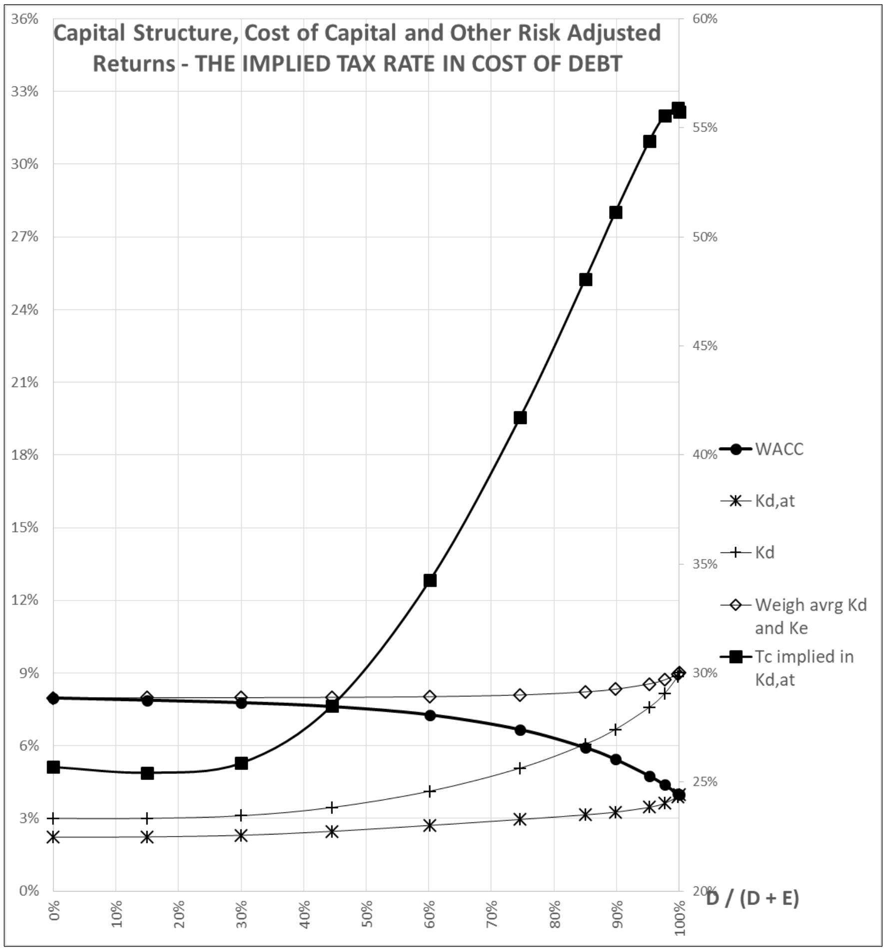
$$K_{d,at} = K_d(1 - \tau_c) \qquad [79]$$

$$(1 - \tau_c) = K_{d,at}/K_d$$

$$\tau_c = 1 - K_{d,at}/K_d$$

This is a simplified way to express the after tax cost of debt applied tax rate, since I think the right way to do it is exponentially rather than linearly.

Figure 28: Cost of capital structure and implied tax rate in after-tax cost of debt



Recall that the assumed tax rate τ_c in the example is 35.0%. In my model this variable is meant to be the statutory tax rate. As you can appreciate in Figure 28, the implied tax rate is significantly below 35.0% for leverages from 0% to approximately 35%. Only when leverage (D / (D + E)) is over 60%, the implied tax rate in the cost of debt equals the statutory tax rate of 30%. After this, when leverage keeps increasing, the implied tax rate in cost of debt rises almost exponentially, reaching levels as high as 55%. If this finding about the functional form of the tax rate implied in the after tax cost of debt proves to be right, it may suggest a radical change in the common practice of using the statutory tax rate as a reasonable proxy to the marginal tax rate. I also have serious concerns about the marginal tax rate concept being right for the cost of debt adjustment from pre-tax to after-tax.

Following with the example of Copeland et al. of cost of capital with taxes:

In a limited intent of understanding the WACC expression from Copeland et al. I continue with the same assumptions they use for the cost of debt already explained. I only add an assumed statutory tax rate of τ_c = 30.0%. Copeland et al. WACC expression is:

$$WACC = K_u[1 - \tau_c D/(D + E)]$$

Assuming that with corporate taxes, Ku, D, and E values are the same as in the no taxes case, I replace these in WACC expression:

$$WACC = 12.0\% \times [1 - 30.0\% \times (\$\,0.94082319348523/\,\$\,3.0)]$$

$$WACC = 10.87101260260920\%$$

Now let us consider cost of equity and after tax cost of debt functions separately as proposed by Copeland et al.:

$$K_e = K_u + (K_u - K_d)(1 - \tau_c)(D/E)$$

$$K_{d,at} = (1 - \tau_c)[r + (K_u - r)\,N(-d_1)\,(D + E)/E]$$

Replacing known values:

$$K_e = 12\% + (12\% - 5.96867391859348\%)(1 - 30\%)(\$\,0.94082319348523/\$\,2.05917796186387)$$

$$K_e = 13.9289678206378\%$$

And cost of debt after-tax:

$$K_{d,at} = (1 - 30\%)[5\% + (12\% - 5\%)\,0.04339764469338 \times \$\,3.0/\$\,2.05917796186387]$$

$$K_{d,at} = 4.17807174301544\%$$

And finally, the value weighted average of obtained K_e and $K_{d,at}$ is:

$$WACC = K_{d,at}\,D/(D + E) + K_e\,E/(D + E)$$

$$WACC = 4.17807174301544\% \times \,\$\,0.94082319348523/\$\,3.0 + 13.9289678206378\% \times \$\,2.05917796186387/\$\,3.0$$

$$WACC = 10.87101260260920\%$$

This is exactly the same result as the one obtained with the WACC abbreviated formula proposed in Copeland et al., once again proving internal consistency.

Risk adjusted return of tax savings from debt:

Since one of my important claims in this working paper is that it is possible to obtain a function for the risk adjusted return of tax savings from debt, in Figure 23 I show the form of this function obtained in my numerical example. It is positively sloped, and in some way it is similar to the pre-tax cost of debt function, but much higher in value. It can be seen that the risk adjusted return of tax savings is different from some typically proposed variables in financial literature, as unlevered cost of equity, pretax cost of debt, and levered cost of equity. Being such an important conflicting point in firm valuation in current literature, I think this matter deserves much more future research.

Some Differential Effects of Taxes on Firm Value (and adjusted returns):

The main focus of my paper is the inclusion in firm valuation of a third stakeholder, the Government, who has a contingent claim on the pre-tax value of assets in the future. In this section I briefly analyze a few situations comparing the predictions of Merton Structural Model without taxes, and the predictions of my contingent valuation model that includes corporate taxes.

1 No taxes versus taxes

As I already explained using my numerical example, the effect on the pre-tax value of the firm, $ 100.00, when switching from a case without taxes to one with taxes, is that the cost of taxes in present value is totally bared by the shareholders of the firm. If the firm has any debt, it seems that debt holders wouldn't be affected in terms of the value of debt. The results obtained were:

$$V_{tot}^{\int} = D^{\int} + E^{\int} + Tax^{\int} = \$\,39.93524037225930 + \$\,49.72616681245670 + \$\,10.33862008566110 = \$\,100.00002727037700$$

This was using exactly the same assumptions to value the firm through numerical integration without taxes, plus adding only one extra assumption, the statutory tax rate of the firm τ_c = 35.0%. Recall that the present value of taxes, $ 10.33862, comes from the value of a deep out-of-the-money call option whose underlying is the pre-tax value of the firm at time T. The results assuming no taxes were:

$$E^{\int} + D^{\int} = \$\,60.06478689811780 + \$\,39.93524037226160 = \$\,100.00002727037900$$

<u>But what happens if the firm is highly indebted</u>? I will exaggerate to make my point! Changing the assumption of the debt contractual coupon from \$ 45.0 to \$ 380.0, and keeping all other assumptions constant, the results of D and E without taxes are:

$$E^{\int} + D^{\int} = \$\,0.80003244829863 + \$\,99.19999482209700 = \$\,100.00002727037700$$

With taxes assuming again τ_c = 35.0%, the results for debt, equity, and tax values are:

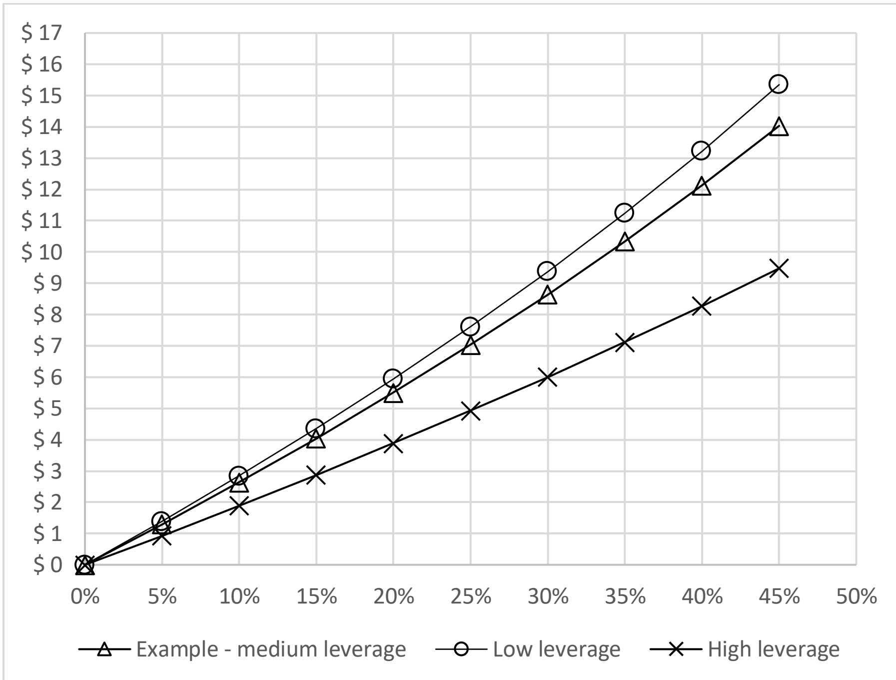
$$V_{tot}^{\int} = D^{\int} + E^{\int} + Tax^{\int} = \$\,99.19999482209700 + \$\,0.52155370305575 + \$\,0.27847874524288 = \$\,100.00002727037700$$

We can appreciate that in this highly levered case, the value of taxes is still "expropriated" from the firm stockholders, same as in the case of a less levered firm. In the highly levered case, value of debt does not change with taxes vs without taxes. <u>A final word of caution</u>; in both cases (with and without taxes) I assume that the total pre-tax value of assets, in present value, is \$ 100.0 (exogenous assumption). Trying to apply ideas I recall from reading one of my favorite economists, Ronald Coase (1910 - †2013), it may be the case that if society for some reason chooses not to tax corporations (and assuming that only corporations can be taxed; not individuals, workers, etc.), the net benefits to corporations (and for the rest of the society) when having taxes wouldn't exist in a tax-less society. If this may be the case, this net benefits, if are effectively demanded by corporations (e.g. public infrastructure, police, military defense, etc.), may reduce the present value of pre-tax assets due to increased "operating" expenses bared by corporations in order to count with at least some of this lost public benefits. If this is the case, I think that debtholders will suffer a loss in value when the firm is taxed.

2 <u>Level of tax rate</u>

Next figure shows the relationship between the level of the statutory tax rate and the present value of taxes paid by the firm. Three cases appear in the graph: a medium leverage firm, this is the example as presented in this paper (K = \$ 45.0); a low leverage case where K is assumed equal to \$

3.0; and a high leverage case, where K is assumed equal to \$ 100.0. In the two new cases, leverage D / (D + E) is 3.089035% and 78.231609%, respectively (when statutory tax rate is 35%).



This figure gives us a hint about the larger incentive for a firm to increase its leverage, the higher the statutory tax rate is.

3 <u>Increased assets volatility</u>

It is a known fact that the value of a call option will increase when expected asset's volatility (the option's underlying) increases, and vice versa. In our example, if instead of 35.0% volatility we assume 37.0%, the resulting values of firm's components are:

$$V_{tot}^{\int} = D^{\int} + E^{\int} + Tax^{\int} = \$\,39.62946086348170 + \$\,49.59519546555440 + \$\,10.77537271917340 = \$\,100.00002904820700$$

Original results with the lower volatility of 35.0% are:

$$V_{tot}^{\int} = D^{\int} + E^{\int} + Tax^{\int} = \$\,39.93524037225930 + \$\,49.72616681245670 + \$\,10.33862008566110 = \$\,100.00002727037700$$

In this case the value of equity does not increase with higher volatility, but instead falls. We can also appreciate that the value of debt is lower, thus expropriated to bondholders, when volatility increases. But the winner of this game in shifting assets' volatility is the government, since the value of taxes is the only component of the pre-tax value of the firm that rises in value. Is this a general conclusion? Following I check the same effect in all values changing the assumption of volatility from 35.0% to 37.0%, but assuming a much lower leverage (K = $ 5.0).

$$V_{tot}^{\int} = D^{\int} + E^{\int} + Tax^{\int} = \$\,4.56965551865211 + \$\,84.22311498664450 + \$\,11.20725676508100 = \$\,100.00002727037700$$

And with higher volatility:

$$V_{tot}^{\int} = D^{\int} + E^{\int} + Tax^{\int} = \$\,4.56965407373493 + \$\,83.75353853617550 + \$\,11.67683643829710 = \$\,100.00002904820700$$

In this case we see that the value of debt remains virtually unchanged when volatility increases. On the side of equity, almost 100% of the value shift is from shareholders to the government. Following I check the same effect in all values of the firm changing the assumption of volatility from 35.0% to 37.0%, but assuming a higher leverage (K = $ 100.0).

$$V_{tot}^{\int} = D^{\int} + E^{\int} + Tax^{\int} = \$\,72.65934669178890 + \$\,20.21787695508290 + \$\,7.12280362351011 = \$\,100.00002727037700$$

And with higher volatility:

$$V_{tot}^{\int} = D^{\int} + E^{\int} + Tax^{\int} = \$\,71.41392928921280 + \$\,21.07283866132120 + \$\,7.51326109767838 = \$\,100.00002904820800$$

In this case (high leverage) the only looser are debtholders. Value of equity and of debt, both increase with higher volatility. But in relative terms, equity rises 4.23%, less than the tax value rise of 5.48%. It seems as a competition between two stakeholders, shareholders and government, to capture value from debt holders. Will this effect be exacerbated with very high leverage, e.g. financial industry firms? I next assume a firm with leverage D / (D + E) equal to 90.10928014% by changing the assumption of K to $ 148.0. This looks like a bank. But not quite, since a bank may have assets with a lower volatility on the aggregate. Let us see the effect of this “high-assets-risk” bank when increasing volatility from 35.0% to 37.0%.

$$V_{tot}^{\int} = D^{\int} + E^{\int} + Tax^{\int} = \$\,86.33861950688480 + \$\,9.47683853312452 + \$\,4.18456923037498 = \$\,100.00002727037700$$

And with higher volatility:

$$V_{tot}^{\int} = D^{\int} + E^{\int} + Tax^{\int} = \$\, 84.97721948457310 + \$\, 10.42879189172320 + \$\, 4.59401767191854 = \$\, 100.00002904820800$$

In this case we see that depositors are expropriated of 1.58% of its value (0.80% approximately for every 1% in extra assets volatility). Now I adjust assets risk of this "bank" to a more realistic level of say 15.0%. Again we check the effect when increasing volatility of assets from 15.0% to 16.0%. I also lower debt (K = $ 104.0), since when lowering volatility, leverage rises significantly. With the lower debt that I assume, leverage is almost unchanged for the bank, being now 89.99211494%.

$$V_{tot}^{\int} = D^{\int} + E^{\int} + Tax^{\int} = \$\, 87.25375132251810 + \$\, 9.70335584700369 + \$\, 3.04290930299837 = \$\, 100.00001647251700$$

And with higher volatility:

$$V_{tot}^{\int} = D^{\int} + E^{\int} + Tax^{\int} = \$\, 86.59798545568030 + \$\, 10.17592807108640 + \$\, 3.22610323983014 = \$\, 100.00001676659400$$

In this lower asset risk "bank" we see that depositors lose 0.75% per 1% of extra asset risk. This looks pretty close to the 0.8% obtained in the high-risk-asset bank! Then who is the winner in this game of lowering assets risk? In the context of this example, it seems that is in the advantage of government to keep banking business assets with low risk. In this example, given a 1% increase in assets volatility results in a 6.02% in Tax value, and only a 4.87% increase in equity value. Otherwise, if assets' risk is high, government is benefitted in 9.78% and equity with 10.04%. Government increases its relative expropriating power relative to bank stockholders keeping assets volatility at low levels. Thus, it has an incentive to regulate toward this end. The relevant conclusion from this simple exercise is that this might not be in the interest or coverage of depositors' wealth.

4 <u>Maturity of debt</u>

Now in my base case example I change the assumption of maturity, from 3 to 6 years. All other assumptions, including K, are kept constant. With K constant, firm's leverage will always decrease. The results under this new assumptions are:

$$V_{tot}^{\int} = D^{\int} + E^{\int} + Tax^{\int} = \$\, 34.46133455879740 + \$\, 51.32598678909700 + \$\, 14.21272332783050 = \$\, 100.00004467572300$$

Results with T = 3 years were:

$$V_{tot}^{\int} = D^{\int} + E^{\int} + Tax^{\int} = \$ \, 39.93524037225930 + \$ \, 49.72616681245670 + \$ \, 10.33862008566110 = \$ \, 100.00002727037700$$

It is obvious that if the promised debt payment K is constant, increasing maturity will yield a lower present value of debt. Now let us see the effect of increasing volatility from 35.0% to 37.0% (as before).

The results with higher volatility are:

$$V_{tot}^{\int} = D^{\int} + E^{\int} + Tax^{\int} = \$ \, 33.87413107138680 + \$ \, 51.37966216952140 + \$ \, 14.74625602567410 = \$ \, 100.00004926658000$$

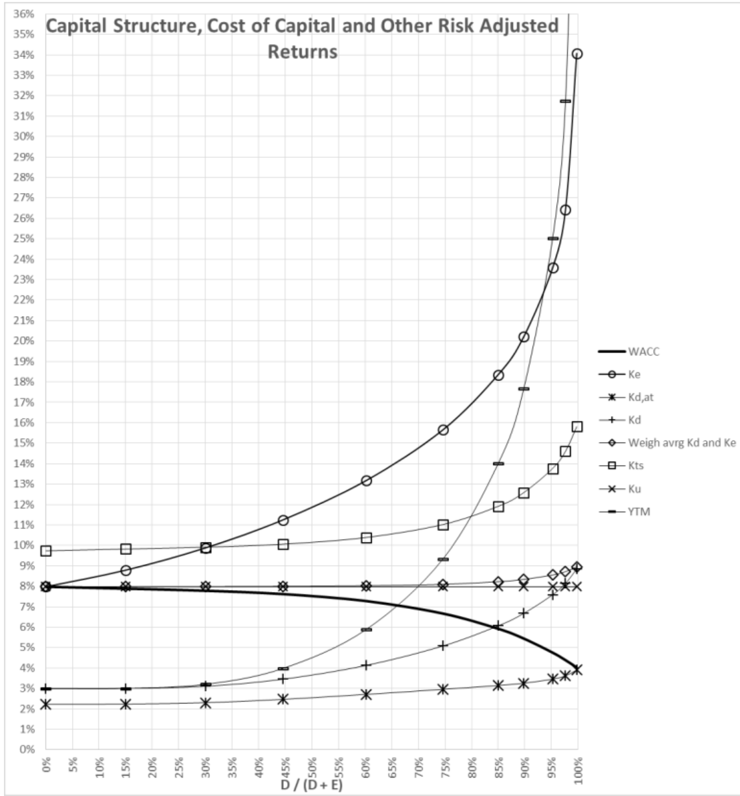
The value of taxes increases 3.75% with increased volatility of assets. Since volatility increased 2%, from 35% to 37%, on average tax value increases 1.87% per each 1% increase in volatility.

But another important issue is how much of the total pre-tax value of the firm is reasonably explained by taxes. With T = 6 years and final contractual debt K constant ($ 45.0), leverage is 40.17066161%; when T = 3 leverage is 44.54005593%. With 6 years assumption the value of taxes as already shown is $ 14.21272332783050, equivalent to 14.21% of total pre-tax value of the firm. It also represents 16.57% of total private value of the firm (D + E). When assuming T = 3 these percentages are 10.34% and 11.53%, respectively. Thus, in principle we may assume that the longer the term, taxes paid by the firm (in present value) are a larger, and a larger portion of total firm value. This is in line with option pricing theory that states that a call option increases in value, the longer is its time to maturity[42]. Therefore, is in the interest of the government that firms last the longest possible.

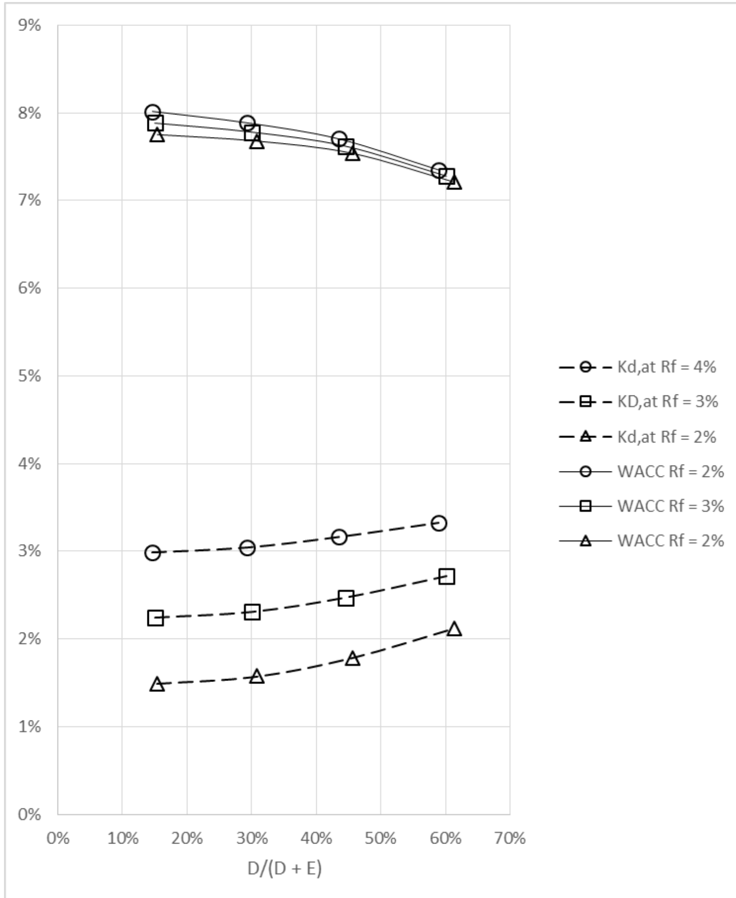
5 Leverage and level of interest rate

In financial theory, the cost of equity and the cost of debt, both risk adjusted returns, are thought to be increasing functions of firm leverage. I already showed how in Merton's model including corporate taxes as I do here in this paper, this assertion is true.

[42] That a longer maturity option is always more valuable than a shorter maturity, everything else constant, is always true for "American" type options. For "European" type options there are very specific circumstances in which a longer maturity option may be worth less than its shorter maturity equivalent. But in the majority of cases it will be true for European type options, that longer maturity implies a larger value of the option.



In Merton's Model very few exogenous assumptions are needed to obtain a robust set of detailed results as already shown in my paper. Besides the amount of final debt K, payable at time T, the other assumptions are risk free rate of return, maturity, underlying's volatility, and corporate tax rate. One interesting question is how does the level of interest rates affects this conclusions about the cost of capital. Next figures show how the cost of debt (after tax), cost of equity, and WACC functions vary when changing the assumption of the risk-free rate of return, from 3.0%, to 2.0% and 4.0%. To vary leverage in each case I assumed K = \$ 14.6, \$ 29.5, \$ 45.0 (original example), and \$ 65.0.



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50%
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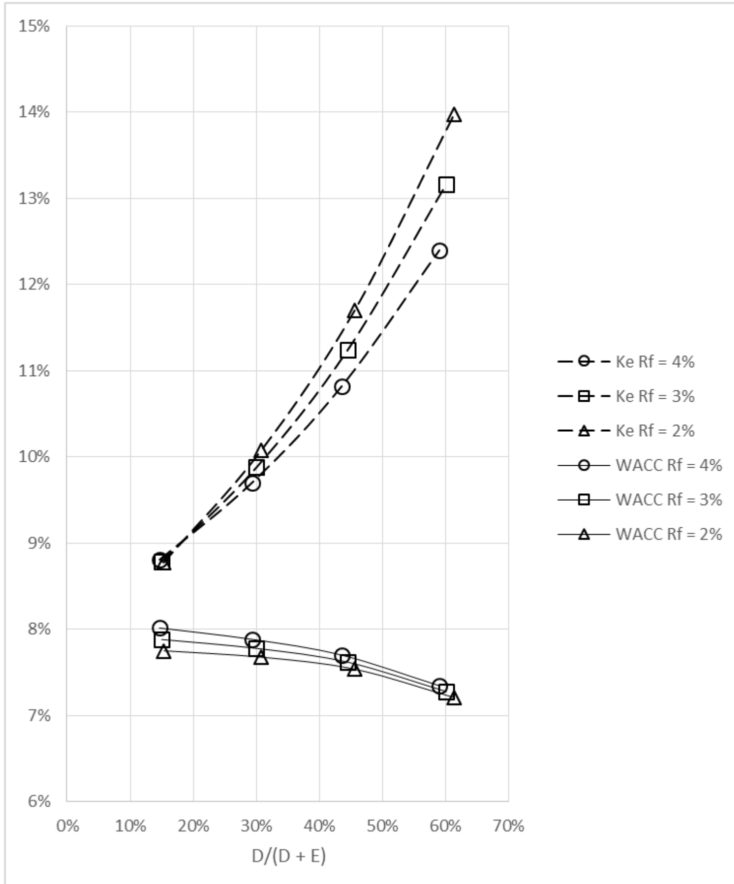
70%
D/(D + E)
Kd,at Rf = 4%
KD,at Rf = 3%
Kd,at Rf = 2%
WACC Rf = 2%
WACC Rf = 3%
WACC Rf = 2%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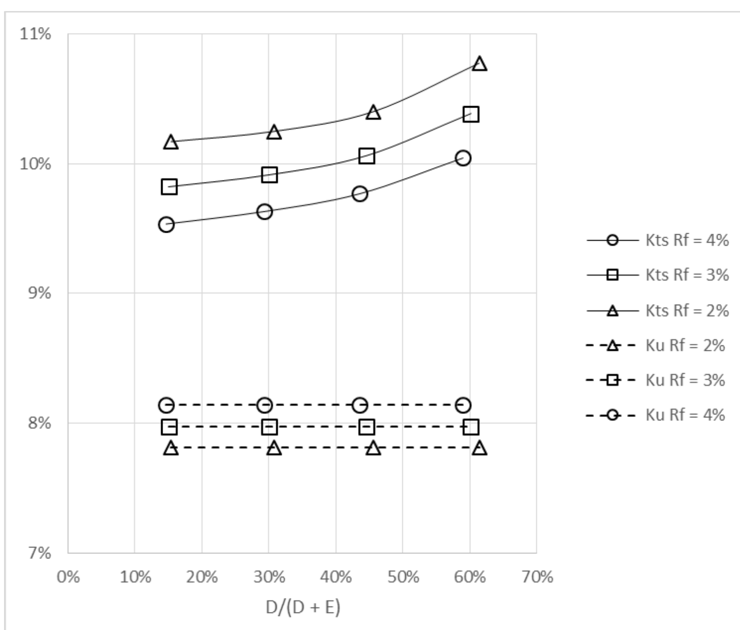
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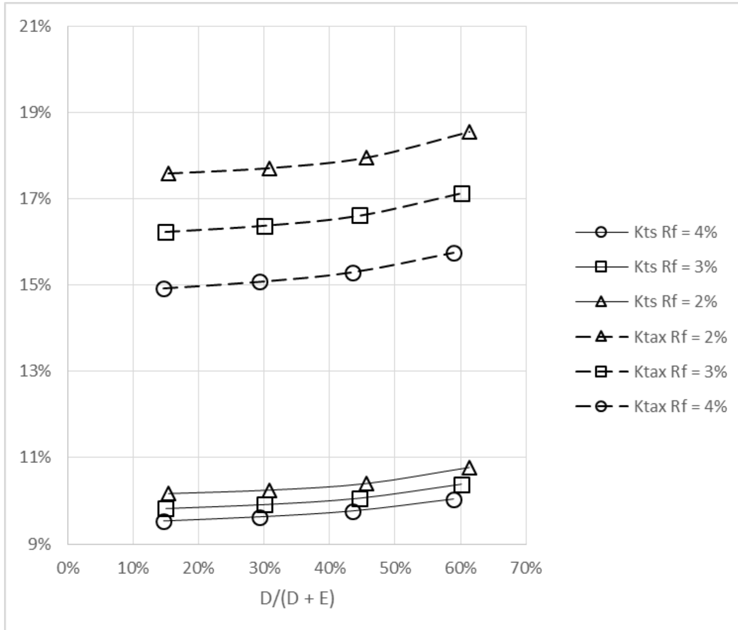

CONCLUSIONS

This working paper presents a step by step guide to the valuation of a firm, more precisely, of the valuation of the claims involved in a firm, in the context of what is known as Merton's Structural Model. The novelty is that I show a numerical integration valuation approach, similar in nature with the valuation through binomial trees, and solve for the risk adjusted returns of every and each component of the firm value with corporate taxes. Risk adjusted returns are obtained for: debt, equity, unlevered free cash flow, tax savings, and taxes (with and without debt).

This may represent a solution to the apparently unsolved problem in valuation, which is the appropriate risk adjusted discount rate for tax savings from debt. If this is the case, this work would be a first important step to come with a solution for this risk adjusted return in a multi-period setting. Unfortunately, I do not have the training nor the time to devote myself to this task.

Another important contribution is a consistent expression of the weighted average cost of capital, WACC, where I consider, besides endogenously calculated cost of equity and pretax cost of debt, an expression that I haven't seen in the financial literature to obtain the tax adjustment resulting in a consistent after tax cost of debt.

Bottom line, this is a paper where I try to better explain valuation (present value and cost of capital), using the well-known Merton Structural Model, which is a "real options" application among many types. Brealey, Myers and Allen [2017, pages 590, 591] comment on real options applicability:

"*The challenges in applying real-options analysis are not conceptual but practical. It isn't always easy. We can tick off some of the reasons why.*" These are: complexity and thus, time to develop a solution; lack of structure (for example, which is exactly the underlying of the real option); value of RO may depend on competitor´s moves. But they close thoughts with:

"*Given these hurdles, you can understand why systematic, quantitative valuation of real options is restricted mostly to well-structured problems like the examples in this chapter. The qualitative implications of real options are widely appreciated, however. Real options give the financial manager a conceptual framework for strategic planning and thinking about capital investments. If you can identify and understand real options, you will be a more sophisticated consumer of DCF analysis and better equipped to invest your company's money wisely.*"

The inspiration to carry on this work for more than a decade (almost two), I owe it to Professor Thomas E. Copeland (and Fred Weston as well) reading in its well-known text-book Financial Theory and Corporate Policy, about the consistency between the valuation of the firm through option pricing, and the valuation grounded on the Capital Asset Pricing Model, CAPM. Also I feel greatly indebted to Aswath Damodaran, Ignacio Velez-Pareja, Harry De-Angelo, and Pablo Fernandez, as great friends and constant sources of inspiration.

ANNEXES

ANNEX 1: Robert C. Merton – the arguably linear combination of cost of debt and cost of equity

From Merton's text:

> To determine the path of the required return on equity, α_e, as X moves between zero and infinity, we use the well known identity that the equity return is a weighted average of the return on debt and the return on the firm. I.e.,
>
> $$\alpha_e = \alpha + X(\alpha - \alpha_y) = \alpha + (1 - g)X(\alpha - r). \qquad (12.39)$$

X is D/E according to Merton, thus:

Ke = Ka + D/E (Ka – Kd)   / x (E/V)

(E/V) Ke = (E/V) Ka + (D/V) (Ka – Kd)

(E/V) Ke = (E/V) Ka + (D/V) Ka – (D/V) Kd

(E/V) Ke = Ka ((E/V) + (D/V)) – (D/V) Kd

(E/V) Ke = Ka (V/V) – (D/V) Kd

Ka = (E/V) Ke + (D/V) Kd   This is OK!!

Analysis of the last part of equation 12.39 from Merton:

Ke = Ka + (1 – g) (D/E) (Ka – r)

r is in Merton the risk free rate of return. What is g?

> Define the market debt-to-equity ratio to be X which is equal to $(F/f) = F/(V - F)$. From (12.20), the required expected rate of return on the debt, α_y, will equal $r + (\alpha - r)g$. Thus, for a fixed investment policy,

From Merton's previous paragraph:

Kd = r + (Ka – r) g

In Section 12.5, the characteristics of g are examined in detail. For the purposes of this section, we simply note that g is a function of d and T only, and that from the "no-arbitrage" condition, (12.6), we have that

$$\frac{\alpha_y - r}{\alpha - r} = g[d, T] \tag{12.20}$$

$$d \equiv \frac{B \exp(-r\tau)}{V}$$

From (12.12) and $F = V - f$, we can write the value of the debt issue as

$$F[V, \tau] = B \exp(-r\tau) \left\{ \Phi\left[h_2(d, \sigma^2\tau)\right] + \frac{1}{d}\Phi\left[h_1(d, \sigma^2\tau)\right] \right\} \tag{12.13}$$

equity case and hence, to the debt. From Black-Scholes equation (13) when σ^2 is a constant, we have that

$$f(V, \tau) = V\Phi(x_1) - B\exp(-r\tau)\Phi(x_2) \tag{12.12}$$

$\tau \equiv T - t$ is length of time until maturity

This very last quote form Merton leaves quite clear that he is not thinking in valuing within an infinitesimal time interval.

Replacing g in Ke first:

Ke = Ka + (1 – g) (D/E) (Ka – r)

Ke = Ka + (1 – (Kd – r)/(Ka – r)) (D/E) (Ka – r)

Ke = Ka + ((Ka – r - Kd + r)/(Ka – r)) (D/E) (Ka – r)

Ke = Ka + (Ka - Kd) (D/E) THIS IS THE TRADITIONAL FORMULATION FOR Ke!

Ke = Ku + (Ku – Kd) (D/E) ; Letting Ka = Ku

Now let us see Kd according to Merton:

Kd = r + (Ka – r) g

Kd = r + (Ka – r) (Kd – r)/(Ka – r)

Kd = r + (Kd – r), WHICH IS A TAUTOLOGICAL STATEMENT…AND TO KNOW g WE NEED KNOWING Kd.

IN MY OPINION, MERTON'S EXPRESSION FOR Kd IS NOT A REAL SOLUTION FOR Kd.

Make a comparison of Ke as Merton and Ke with my model without taxes and then with taxes.

Ke according to Merton's formula Ke = (Ku – Kd) (D/E) in a world without taxes, using my same example: Ku = 9.0000000000005%; Kd = 3.4539979728767%; D = $ 39.93524037226160; E = $ 60.06478689811780; D/E = 66.4869425742%.

Ke = (Ku – Kd) (D/E) = 12.6873671829%

THIS IS DIFFERENT FROM THE VALUE OF Ke I OBTAIN, 12.2347826370841%. The cost of debt I obtain is 3.4539979728767%, but Merton's solution yields 4.1347098966479%.

<u>Leaving everything else constant, but changing **my assumption of T to a value close to zero**</u>, T = 0.01 (3.65 days), then Ke in my model is 13.9037388371396% and Merton's equation Ke is 13.9064129982251%. Kd in my model is 2.9999999872028%, and Merton's Kd is 3.00000000000000%. With T getting closer to zero, my numerical integration solution of both Ke and Kd get quite close to Merton's analytical solutions.

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