Active factor completion strategies*

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Abstract

Embracing the concept of factor investing we design a flexible framework for building out different factor completion strategies with respect to given traditional multi-asset allocations. Our notion of factor completion considers a maximally diversified reference portfolio anchored in a multi-asset multi-factor risk model that acknowledges both, market factors such as equity, duration and commodity as well as style factors such as carry, value, momentum and quality. Yet, the specific nature of a given factor completion strategy varies with investor preferences and constraints. In this vein, we tailor a select set of factor completion strategies that include factor-based tail-hedging, constrained factor completion and a fully diversified multi-asset multi-factor proposition. The presented framework naturally lends itself to exploiting tactical asset allocation signals while not sacrificing the notion of maximum diversification. Along these lines, we can additionally tap the common trend style that permeates many asset classes.

Keywords: Diversification, Risk Parity, Factor Completion, Multi-asset Multi-factor Investing

JEL Classification: G11; D81

The traditional asset allocation framework was severely challenged in the global financial crisis when the correlation breakdown across most asset classes led to an erosion of portfolio diversification. Given the desire for more stable portfolio building blocks, there has been considerable interest and research into structuring portfolios by relevant risk and return factors rather than by traditional asset classes alone. In this regard, style factors such as quality, value and momentum have emerged as natural candidates to meaningfully expand the investment opportunity set: Not only do they help explaining the cross-section of equity returns, these investing concepts also carry over to other asset classes such as commodities, foreign exchange, or rates. While many studies focus on generating optimal style factor portfolios there is less focus on deriving an integrated proposition that blends traditional market factors and style factors into one. Similarly, the majority of asset owners is still following a rather traditional asset allocation and there is a need for a holistic approach to integrate the notion of style factors into the asset allocator's toolkit.

We contribute to satisfying this need in the following ways: First, we put forward a global factor model that combines three market factors with four style factors. Equipped with such a risk model a given traditional asset allocation will typically be found to have little (if any) style factor exposure at all. Second, to complete a given portfolio's lack of style factor exposure a reasonable reference portfolio needs to be chosen and we suggest a maximally diversified risk parity portfolio along the salient multi-asset multi-factor drivers. Third, the nature of the factor completion strategy should accommodate investors'¡ preferences and constraints and we demonstrate likely use-cases, such as factor-based tail hedging, constrained factor completion and a fully diversified multi-asset multi-factor proposition. Fourth, though the framework rests on the notion of maximum diversification we demonstrate how to additionally include tactical asset allocation signals. Specifically, we will consider technical trading signals for the traditional asset classes, which adequately operationalizes the well-known trend style in a balanced fashion vis-à-vis the cross-sectional style factors.

The key ingredient to any of the above factor completion strategies is the maximally diversified reference portfolio. Of course, the major lesson of the seminal work of Markowitz (1952) is to diversify one's assets when aiming for an optimal risk-return trade-off. While the literature is lacking a generally accepted definition of diversification, the intuitive rationale is simple: exploit the assets' dependence structure to reduce the ensuing portfolio volatility. This notion of diversification motivates the search for uncorrelated assets that helps further diversifying a given portfolio. In this vein, Meucci (2009) resorted to principal components analysis (PCA) for synthetically constructing uncorrelated portfolios that can be formed from any set of portfolio assets. These uncorrelated principal portfolios may be considered as embedded risk factors spanning the investable asset universe. Meucci perceives a portfolio to be well-diversified when uncorrelated risk factors evenly contribute to overall portfolio volatility.

Lohre, Opfer, and Ország (2014) built on this idea and investigated a corresponding maximum diversification strategy in a classic multi-asset framework. Maximum diversification then obtains when following a risk parity strategy that is budgeting equal risk to the principal portfolios rather than the underlying assets. This approach blends in well with the work of Bruder and Roncalli (2012) and Roncalli and Weisang (2016) who advocate the use of risk parity strategies with respect to risk factors rather than single assets.

The framework of Meucci (2009) implicitly uses a purely statistical PCA-risk model. While the PCA implies orthogonality of risk factors by construction, it is also prone to some drawbacks. Most notably, risk factors (or principal portfolios) derived from a PCA often lack a sound economic intuition and its weights prove to be unstable over time. Moreover, those risk factors are omni-directional, therefore it is not straightforward whether a given principal portfolio should be bought or sold. Along these lines, Meucci, Santangelo, and Deguest (2015) consider an alternative orthogonal decomposition of the assets' variance-covariance matrix other than the eigenvector decomposition underlying the PCA. The authors advocate the use of minimum torsion factors that are extracted from an intuitive optimization procedure that seeks to minimize the tracking error between the original risk factors and the ensuing uncorrelated risk factors. This suggestion cures the above drawbacks of the PCA to a large extent: choosing the original risk factors as an anchor for factor orthogonalization closely aligns the uncorrelated minimum torsion factors to the original risk factors and, thus, speeds interpretability and stability. Also, the trade direction of a given minimum torsion factor is then in line with the corresponding original risk factor. Intuitively, this approach strives to derive the best orthogonal decomposition of a given factor model and it has recently been applied in the context of commodity factor investing (Bernardi, Leippold, and Lohre (2018)); similarly Martellini and Milhau (2017) demonstrate the benefits of minimum torsion portfolios for multi-asset allocations.

In the world of multi-asset multi-factor investing we advocate a maximum diversification strategy that pursues a diversified risk parity allocation along orthogonalized risk factors that are closely aligned with the most salient market and style factors. The paper is therefore organized as follows: Section 1 puts forward the rationale of multi-asset multi-factor investing, as well as our choice of market and style factors. Section 2 reviews the approaches of Meucci (2009) and Meucci, Santangelo, and Deguest (2015) for managing and measuring diversification and describes the mechanics of pure diversified risk parity strategies in the multi-asset multi-factor domain. Section 3 introduces our factor completion framework and its various applications including the additional use of Black-Litterman to capture the notion of trend style investing.

1 The case for multi-asset multi-factor investing

The concept of factor investing has received a lot of attention over the last decade. At its heart factor investing suggests a major shift in the investment process, away from allocating across asset classes and picking single securities towards obtaining exposure to certain factors that have emerged as relevant in describing the cross-section of asset returns. Of course, efficiently capturing factor exposure still requires to invest in single securities; yet, these investments are typically broadly diversified across many securities so as to maximize the likelihood of capitalizing a given factor premium.

1.1 The equity origins of style factor investing

Notably, factor strategies are not overly engineered strategies but often relate to certain investment styles and are thus commonly referred to as style factors. Style factor investing has a long history in theory and practice. For instance, the notion of value investing is usually traced back to the seminal book of Graham and Dodd (1934) and has found many disciples seeking to buy securities that are relative cheap compared to their fundamental value. In the academic literature the benefits of value investing have been documented in the cross-section of US stocks as early as the 1970s, see Basu (1977). The natural route to diversify value style investments is to pursue a price momentum strategy. Price momentum builds on the notion of recent winners outperforming recent losers; this effect can be verified on rather short-term look-back periods and investment horizons up to 12 months, see Jegadeesh and Titman (1993). Thus, a momentum investor has to actively monitor and rebalance his portfolio positions whereas a value investor needs to wait relatively longer for his securities to revert to fundamental value. Besides momentum and value, quality has emerged as a further relevant style factor in equities. Quality style investors are not necessarily looking to buy cheap but high quality companies with strong balance sheets; research of Novy-Marx (2013) or Ball, Gerakos, Linnainmaa, and Nikolaev (2016) has evidenced this investment style to be meaningful across a broad set of financial statement indicators such as accruals, or cash-based profitability. A related style is commonly referred to as defensive investing. Defensive assets are typically characterized in terms of their relative low-risk characteristics, as measured by their volatility or market beta. Again, defensive or low-risk assets have been documented to outperform high risk ones on a risk-adjusted basis as early as in the 1970s, see Haugen and Heins (1975) or Black (1972).

1.2 Extending style factor investing to other asset classes

As documented in the preceding section, the notion of certain style factors has long been established in the academic literature. Yet, the major promise of factor investing is not to follow a single style but to create a portfolio that allows to jointly harvest many style factor premia. Given the distinct nature of the described style factors a multi-factor proposition is expected to capitalize on style factor diversification. While traditional fund managers are often more prone to high-conviction investing according to a particular investment style, diversified multi-factor investing has been the traditional domain of quantitative investment managers.

It is natural to expect similar notions of style factor investing to also permeate other asset classes. Obviously, for a given style factor to be a meaningful building block it is a prerequisite to establish at least one compelling factor rationale. First, the candidate factor might be related to a risk-based argument that suggests the factor to compensate for systematic or genuine risk. Second, the factor could be rooted in irrational yet persistent investor behaviour allowing rational investors continued exploitation of the factor. Third, a given factor premium might be associated with a peculiar market or industry structure that promote certain return patterns. Likewise, investment constraints of certain market participants could fuel abnormal return effects that are bound to persist in the continued presence of these constraints. In looking to establish further factors there has been a proliferation of factor research in the asset pricing literature. Obviously, these developments call for safe-guarding against data-snooping biases when testing for the relevance of given factors. However, note that the goal of the present paper is not to add to the proliferation of the factor zoo but to collect and allocate style factor strategies that permeate many asset classes.

1.3 The multi-asset multi-factor universe

We build on academic research that has documented that style factors not only explain the cross-section of equity returns but also extend to other asset classes, such as commodity, foreign exchange (FX) or rates. Notable contributions are Asness, Moskowitz, and Pedersen (2013) for value and momentum and Koijen, Moskowitz, Pedersen, and Vrugt (2018) for carry. Carry investing relates to the observation that high-yielding assets outperform low-yielding assets, provided all else is equal. The most prominent example originates from FX investing where the carry trade exploits the return differential of high-yield currencies versus low-yield ones, provided that FX rate movements do not nullify the yield advantage.

We collect carry, value, momentum and quality factor strategies within the four asset classes equities, commodities, FX, and rates. For a detailed overview of the used style factors along with their construction, please refer to Appendix A.1. Rather than utilizing single factor strategies in a kitchen sink fashion we adopt a pure style factor investing view and aggregate single factor strategies according to their underlying style factor rationale. That is, we will cluster all momentum factors into an aggregate momentum factor by applying a risk parity weighting scheme to FX, commodity, rates and equity momentum strategies. In the same manner, we synthesize aggregate carry, value and quality factors. Alongside the four aggregated style factors we consider three traditional market risk factors, whose inputs are also risk parity weighted. The first factor is global equity risk as represented by the S&P 500, EuroSTOXX 50, FTSE 100, Nikkei 225 and the MSCI Emerging Market indices. We include a duration-hedge investment grade and high yield credit component into that factor as it is highly positively correlated to equities. The second factor is duration risk represented by 10-year US T-Bills, German Bund, 10-year Japanese government bonds as well as UK Gilt. The third factor is commodity risk as captured by gold, oil, copper and agriculture indices. To actually obtain exposure to traditional market factors we consider efficient investment vehicles, such as equity index futures, bond futures and exchange traded commodities (ETC). All indices are measured in monthly local currency returns and we report total return figures from the perspective of a US investor by employing the 3-month US Treasury Rate.

Panel A of Table 1 presents the descriptive statistics of the above market factors. Over the whole sample period from January 2001 to October 2018 we observe that the MSCI Emerging

Markets index was the best performing equity market with 7.59% annualized excess return, albeit at the highest volatility with 21.55%. This translates into a Sharpe ratio of 0.35. Conversely, the EuroSTOXX50 experienced the lowest excess return (2.47%) and the lowest Sharpe ratio (0.13) across equity indices over the sample period. The bond indices' excess returns range from 1.96% (Japanese Government Bonds) to 4.32% (Bund) and have standard deviations in between 2.51% (Japanese Government Bonds) and 6.01% (Gilt). The highest dispersion in excess return can be observed for the commodity indices. The best performing commodity was Copper with 10.57% annualized excess return while Agriculture and Oil were the weakest commodity investments (1.08% and1.25%, respectively) with oil suffering from a severe drawdown of -92.40% in the sample period.

[Table 1 about here.]

Panel B of Table 1 summarizes the performance of the style factors. All style factors earned a positive premium in excess of the risk-free rate ranging from 0.89% for Rates Value to 5.77% for Commodity Carry. In general, the Rates factors show the weakest performance translating into the lowest Sharpe ratios across style factors as well. The highest Sharpe ratio obtains for Commodity Quality (1.61). Compared to the market factors the style factors show lower volatility figures in the single digits. In turn, the worst drawdown is -24.08% (Equity momentum) which is much lower than the drawdown of any equity index.

Panel C of Table 1 gives the descriptive statistics of the aggregate factors. Recall that these factors are risk parity combinations of the respective market and style factors, respectively. On the aggregate level Value posts the lowest return with 2.56% while Commodity is the strongest factor considered with 5.82% annualized excess return. All aggregated style factors have Sharpe ratios above 1: 1.98 for Quality, 1.28 for Carry, 1.08 for Momentum and 1.02 for Value. Duration has a Sharpe ratio of 0.86, Commodity 0.37 and Equity 0.33.

The upper chart in Figure 1 depicts the average correlation of single asset classes and style factors over the whole sample period. We observe high correlations within the equity bucket (reaching a maximum of 0.84 for EuroSTOXX50 and FTSE100) and within the bond bucket (maximum of 0.85 between Bund and Gilt). The cross-correlation of the equity and bond bucket is slightly negative. The commodity bucket is also discernible from the correlation analysis, yet it is evidently a more heterogenous asset class than equities or bonds.

[Figure 1 about here.]

Style factor correlations do not naturally lend themselves to cluster certain buckets as they typically range in between +/- 0.3. The highest correlation is observed for the Rates momentum factor and US 10-year Government Bonds (0.69). In addition, we find FX carry to be positively correlated to equity and credit markets (around 0.5) resonating with the severe downside co-movement over the global financial crisis. Interestingly, we find equity style factors not to be positively but negatively correlated to broader equity market indices. As expected, aggregating single asset classes and style factors gives building blocks that are fairly uncorrelated, see the lower chart of Figure 1. The highest correlation obtains for the aggregates for Carry and Equity-Credit (0.43) and the lowest one for Duration and Equity-Credit (-0.39).

2 Diversified Risk Parity for Maximum Diversification

While the mean-variance paradigm of Markowitz is the model of choice to balance expected risks and return, it leads to portfolios that are anything but efficient ex post. In particular, the ensuing portfolio allocations are fairly concentrated and thus hardly diversified. Given that the mean-variance inputs come with quite some estimation error academics and practitioners have turned to more robust allocation paradigms that do not require to estimate expected returns. The simplest approach, $1/N$, would diversify portfolio weights by equally allocating capital to each asset or factor. Depending on the heterogeneity of the investment universe this approach is prone to prompting an imbalance in risk allocation. Conversely, one could adopt a risk parity strategy that allocates such that each asset and factor contributes equally to portfolio risk. While it is intuitive to go for a balanced risk allocation there is an ad-hoc nature to this approach. In this section, we will therefore start from a definition of diversification to motivate and execute meaningful diversified risk parity strategies in a multi-asset multi-factor universe.

2.1 The search for low-correlated factors

According to standard portfolio theory diversification is geared at eliminating unsystematic risk. In this vein, a common notion of diversification is to avoid exposure to single shocks or risk factors. Naturally, diversification especially pays when combining low-correlated assets. Taking this idea to extremes, Meucci (2009) constructs uncorrelated risk sources by applying a principal component analysis (PCA) to the variance-covariance matrix of the portfolio assets. In particular, he considers a portfolio consisting of N assets with return vector $\mathbf{R}$. Given weights $\mathbf{w}$ the resulting portfolio return is $R_w = \mathbf{w}'\mathbf{R}$. According to the spectral decomposition theorem the covariance matrix $\mathbf{\Sigma}$ can be expressed as a product

$$\mathbf{\Sigma} = \mathbf{E}\mathbf{\Lambda}\mathbf{E}' \tag{1}$$

where $\mathbf{\Lambda} = \text{diag}(\lambda_1, ..., \lambda_N)$ is a diagonal matrix consisting of $\mathbf{\Sigma}$'s eigenvalues that are assembled in descending order, $\lambda_1 \geq ... \geq \lambda_N$. The columns of matrix $\mathbf{E}$ represent the eigenvectors of $\mathbf{\Sigma}$. These eigenvectors define a set of N *principal portfolios*[1] whose returns given by $\tilde{\mathbf{R}} = \mathbf{E}'\mathbf{R}$ are uncorrelated and their variances equal $\lambda_1, ..., \lambda_N$. As a consequence, a given portfolio can be either expressed in terms of its weights $\mathbf{w}$ in the original assets or in terms of its weights $\tilde{\mathbf{w}} = \mathbf{E}'\mathbf{w}$ in the principal portfolios.

More importantly, maximum diversification obtains when following a risk parity strategy along the extracted principal portfolios, see Lohre, Opfer, and Ország (2014). While the

[1] Note that Partovi and Caputo (2004) coined the term principal portfolios in their recasting of the efficient frontier in terms of these principal portfolios.

underlying rationale is plausible the ensuing PCA-driven portfolio allocation will suffer from the classic PCA critique: As principal components are statistical in nature they often lack a sound economic interpretation which complicates the buy-or-sell decision for any given principal portfolio. Moreover, principal components are known to be rather unstable which tends to translate into excessive strategy turnover, see Bernardi, Leippold, and Lohre (2018).

Recalling that a PCA is just one possible decomposition of the covariance matrix $\mathbf{\Sigma}$ Meucci, Santangelo, and Deguest (2015) conceived a new technique that uncovers alternative sets of uncorrelated, yet economically meaningful factors. Also, it is recommended to consider a parsimonious factor model as opposed to a factor model consisting of a large number of principal portfolios. In this very spirit, Meucci, Santangelo, and Deguest (2015) suggest resorting to a parsimonious factor model that can be orthogonalized such that the ensuing decorrelated factors have minimum tracking error to the original factors.

In particular, the starting point is a K-factor model $\mathbf{F}$ with which portfolio returns R_w for a portfolio with weights $\mathbf{w}$ can be represented in terms of factor returns:

$$R_w = \mathbf{w}'\mathbf{R} = \mathbf{b}'\mathbf{F} \tag{2}$$

where $\mathbf{b}$ denotes the portfolio returns' factor loadings with respect to the factor model $\mathbf{F}$. Meucci, Santangelo, and Deguest (2015) are looking to change representation (2) into one in terms of uncorrelated factors $\mathbf{F}_T$, i.e.

$$R_w = \mathbf{b}'\mathbf{F} = \mathbf{b}_T'\mathbf{F}_T \tag{3}$$

where $\mathbf{b}_T$ denotes the portfolio returns' factor loadings with respect to the factor model $\mathbf{F}_T$, respectively. Using $\mathbf{R} = \mathbf{B}'\mathbf{F}$ with $\mathbf{B} \in \mathbb{R}^{K\times N}$ the covariance matrix $\mathbf{\Sigma}$ can be decomposed as follows:

$$\mathbf{\Sigma} = \mathbf{B}'\mathbf{\Sigma}_F\mathbf{B} + \mathbf{u} \tag{4}$$

where $\mathbf{u}$ is idiosyncratic risk that is not captured by the chosen factor structure. The critical step is to construct an orthogonal decomposition of $\mathbf{F}$ using a linear transformation $\mathbf{F}_T = \mathbf{tF}$ where $\mathbf{t}$ is a $K \times K$ torsion matrix. Meucci, Santangelo, and Deguest (2015) devise an algorithm to back out an uncorrelated factor representation which closely mimics the original factor model $\mathbf{F}$. Among all decorrelating linear transformations $\mathbf{t}$ their algorithm selects the minimum torsion $\mathbf{t}_{MT}$ that minimizes the distance to the original factors:

$$\mathbf{t}_{MT} = \underset{Corr(\mathbf{tF})=\mathbf{I}_{K\times K}}{\arg\min} \sqrt{\frac{1}{K}\sum_{k=1}^{K} Var\left(\frac{\mathbf{t}'\mathbf{F}_k - \mathbf{F}_k}{\sigma_k^F}\right)} \tag{5}$$

with σ_k^F denoting the volatility of factor $\mathbf{F}_k$. Equipped with this minimum torsion $\mathbf{t}_{MT}$ we can decompose the systematic risk of a given portfolio:

$$\mathbf{B}'\mathbf{\Sigma}_F\mathbf{B} = \mathbf{B}'\mathbf{t}_{MT}^{-1}\mathbf{\Sigma}_{MT}\mathbf{t'}_{MT}^{-1}\mathbf{B} \tag{6}$$

where $\mathbf{\Sigma}_{MT} = \text{diag}(\sigma^2_{MT,1}, ..., \sigma^2_{MT,K})$ is a diagonal matrix of minimum torsion factors' variances. As with the principal portfolios we can rewrite a given portfolio with weights $\mathbf{w}$ in the original assets similarly in terms of weights $\mathbf{w}_{MT}$ in the minimum torsion factors $\mathbf{F}_{MT} = \mathbf{t}_{MT}\mathbf{F}$. Specifically, we have $\mathbf{w}_{MT} = \mathbf{t'}^{-1}_{MT}\mathbf{B}\mathbf{w}$.

2.2 Diversification distribution, entropy and diversified risk parity

Regardless of the nature of uncorrelated factors, principal portfolios or minimum torsions, the total portfolio variance emerges from simply computing a weighted average over the uncorrelated factors' variances. Focussing on minimum torsion weights $w_{MT,k}$ onwards we have:

$$Var(R_w) = \sum_{k=1}^{K} w^2_{MT,k}\sigma^2_{MT,k}. \tag{7}$$

Normalizing the minimum torsion factors' contributions by portfolio variance yields what Meucci (2009) calls the *diversification distribution*:

$$p_{MT,k} = \frac{w^2_{MT,k}\sigma^2_{MT,k}}{Var(R_w)}, \qquad k = 1, ..., K. \tag{8}$$

By design the diversification distribution is always positive and the $p_{MT,k}$s sum to one. Building on this concept Meucci (2009) conceives a portfolio to be well-diversified when the $p_{MT,k}$s are "approximately equal and the diversification distribution is close to uniform". This definition of a well-diversified portfolio coincides with allocating equal risk budgets to the uncorrelated factors; prompting Lohre, Opfer, and Ország (2014) to coin the approach *diversified risk parity*. Conversely, portfolios loading on a specific factor display a peaked diversification distribution. It is thus straightforward to apply a dispersion metric to the diversification distribution to gauge overall portfolio diversification. Meucci (2009) suggests to use the exponential of its entropy[2] to measure the *effective number of uncorrelated bets*, $\mathcal{N}_{Ent}$:

$$\mathcal{N}_{Ent} = \exp\left(-\sum_{k=1}^{K} p_{MT,k}\ln p_{MT,k}\right) \tag{9}$$

To rationalize this interpretation consider two extreme cases. For a completely concentrated portfolio we have $p_{MT,k} = 1$ for one k and $p_{MT,l} = 0$ for $l \neq k$ resulting in an entropy of 0 which implies $\mathcal{N}_{Ent} = 1$. Conversely, $\mathcal{N}_{Ent} = K$ holds for a portfolio that is completely homogeneous in terms of uncorrelated risk sources, therefore having the maximum effective number of uncorrelated bets ($\mathcal{N}_{Ent}$). In this case, $p_{MT,k} = p_{MT,l} = 1/K$ holds for all k, l implying an entropy equal to $\ln(K)$.

To achieve maximum diversification the optimal diversified risk parity strategy is ultimately an inverse volatility strategy in the minimum torsion factors and its weights can be

[2]The entropy has been used before in portfolio construction, see e.g. Woerheide and Persson (1993) or Bera and Park (2008). However, these studies consider the entropy of portfolio weights thus disregarding the dependence structure of portfolio assets.

computed analytically. For instance, the kth minimum torsion factor has weight

$$w_{MT,k}^{DRP} = \frac{1/\sigma_{MT,k}}{\sum_{k=1}^{K} 1/\sigma_{MT,k}} \tag{10}$$

Expressing the above in terms of single assets and single style factors (and not minimum torsion factors) obviously requires to account for the factors' torsion $\mathbf{t}_{MT}$ and the associated mapping $\mathbf{B}$ of single assets and single style factors on the original aggregate factors. In particular, we have

$$\mathbf{w}^{DRP_B} = \frac{\mathbf{B}^{-1}\mathbf{t}'_{MT}\mathbf{w}_{MT}^{DRP}}{\sum_{i=1}^{N}\left(\mathbf{B}^{-1}\mathbf{t}'_{MT}\mathbf{w}_{MT}^{DRP}\right)_i} \tag{11}$$

where $\mathbf{B}^{-1}$ is the Moore-Penrose inverse of the factor model sensitivities. Intuitively, to back out the physical weights one re-rotates the orthogonalized factors and reverses the mapping as given by the linear risk model coefficients $\mathbf{B}$.

Obviously, the stability of weights (11) crucially depends on the statistical fit of the chosen parsimonious risk model w.r.t. each underlying single asset or style factor. Given that the seven aggregate factors are constructed from the underlying single assets and style factors one can consider an alternative natural route to determining a diversified risk parity allocation. Specifically, we would simply invest into the minimum torsion factors based on the mapping that is implicit in the risk parity scheme used to form the aggregate factors. Collecting the risk parity weights for the seven factors in $\mathbf{w}_{ERC}^{\mathbf{F}}$ we can multiply these by the re-rotated inverse-volatility weights in the seven aggregate factors:

$$\mathbf{w}^{DRP_F} = \frac{\mathbf{w}_{ERC}^{\mathbf{F}}\mathbf{t}'_{MT}\mathbf{w}_{MT}^{DRP}}{\sum_{k=1}^{K}\left(\mathbf{t}'_{MT}\mathbf{w}_{MT}^{DRP}\right)_k} \tag{12}$$

2.3 Diversified risk parity in the multi-asset multi-factor domain

2.3.1 Rationalizing minimum torsion factors

To foster intuition about the uncorrelated risk sources synthesized from the underlying multi-asset multi-factor data, we investigate the minimum torsion factors over the whole sample period from January 2001 to October 2018. The economic nature of the minimum torsion factors is best assessed in terms of the torsion matrix $\mathbf{t}_{MT}$ that represents the minimum torsion factors' weights with respect to the original factors. Given that the correlation of the original style factors is generally low, the interpretation of the minimum torsion factors as collected in Table 2 is straightforward. Regarding the decorrelation of market factors, we observe a similar pattern for Duration that has slightly negative loadings to the Carry and the Momentum factor. Notably, MTF1 and MTF3, tracking equity and commodity market risk respectively, have somewhat higher loadings to other factors as well. For instance, MTF1 is mainly long Equity and Momentum and short Carry and Value. MTF3 is long Commodity as well as the style factors Carry and Momentum while short Quality. By and large, the ensuing minimum torsion factors are closely aligned with the underlying factors.

[Table 2 about here.]

For estimating the minimum risk factor torsions over time, we rely on 60 months rolling window estimation to adapt to potential structural breaks. To investigate the stability of these dynamic minimum torsion factor weights we collect their weights throughout time in Figure 2. Notably, the corresponding weights are indeed quite stable throughout the sample period and resonate well with the static weights over the whole sample period, see Table 2.

[Figure 2 about here.]

2.3.2 Diversified risk parity over time

Having determined the minimum torsion factors we next investigate the consequences of allocating according to the optimal diversified risk parity strategy. We thus follow an inverse volatility strategy along the minimum torsion factors which can be computed analytically. Rebalancing occurs at a monthly frequency. Given that the first estimation consumes 60 months of data the strategy performance can be assessed from January 2006 to October 2018. For benchmarking the diversified risk parity strategy we consider four alternative risk-based asset allocation strategies: $1/N$, minimum-variance, risk parity, and the most-diversified portfolio (MDP) of Choueifaty and Coignard (2008). The construction of these alternatives is detailed in Appendix A.2. Note that all allocation paradigms are applied on an aggregate factor level and not in a kitchen sink fashion. As a result, all allocation strategies generally benefit from the imposed factor structure.

[Table 3 about here.]

Table 3 gives performance and risk statistics of the DRP strategy as well as of the alternative risk-based asset allocation strategies. The reported results refer to an estimation window of 60 months. All returns are net of transaction costs, which we incorporate in three stages of the optimization: First, we consider transaction costs when constructing each portfolio representing a given style factor. This consideration does not apply for the market factor indices which are traded via futures. Second, we assume a fee of 96 bps p.a. for holding those style factors in a swap. Again, this does not apply for the market factors. Third, we assume 30 bps transaction costs for a turnover of 100% in any market factor. For the style factors, we assume 35 bps for a turnover of 100%. The optimal DRP strategy earns 3.28% at 1.45% volatility which is equivalent to a Sharpe ratio of 1.24. Among the alternative strategies the highest annualized return materializes for the $1/N$-strategy (3.53%) but comes at the price of the highest volatility (3.49%) as well. Moreover, the strategy exhibits the worst drawdown among all alternatives (-9.35%). Conversely, the minimum-variance strategy provides the lowest return (2.94%). Given that minimum-variance indeed exhibits the lowest volatility (1.31%) its Sharpe ratio of 1.15 is nevertheless comparable to the one of the DRP. Also, its drawdown statistics are the least severe amounting to a maximum loss of 1.04% during the whole sample period. Note that the DRP strategy's maximum drawdown is only slightly higher (1.92%).

Note that the MDP gives descriptive statistics similar to those of DRP with a return of 3.23% and a volatility of 1.57%, translating into a Sharpe ratio of 1.14. In addition to the DRP strategy, we also consider a standard risk parity strategy we applied to the original (un-orthogonalized) factors. We document standard risk parity to yield the same return as DRP yet at slightly higher risk. Thus, it seems as if the benefit of orthogonalizing the original factors is marginal (at least performance-wise) over the considered sample period, as the original factors are quite uncorrelated to begin with. Comparing the monthly two-way turnover of strategies, the DRP strategy sits in the middle with 3.66% on average. That compares to 1.53% for $1/N$, 2.99% for risk parity, 5.98% for minimum-variance, and 6.948% for MDP.[3]

As argued by Lee (2011), evaluating risk-based portfolio strategies by means of Sharpe ratios is hard to reconcile with the fact that returns are not entering their respective objective function in the first place. We rather resort to contrasting the risk characteristics of these portfolios. In fact, while all strategies look similar in terms of their overall risk and return characteristics, they do differ in terms of their effective number of bets ($\mathcal{N}_{Ent}$), hence their risk profiles are different. The DRP strategy has seven bets throughout time by design, and the risk parity strategy comes closest with 6.94 bets. While the MDP displays a sufficiently balanced risk-profile (6.20 bets), the other alternatives are more concentrated as reflected by 4.07 bets for minimum-variance and 3.11 bets for $1/N$. We next turn to an in-depth discussion of the risk-based strategies' weights and risk allocation in Figure 3. Risk is being decomposed by minimum torsion factors.

[Figure 3 about here.]

First, we examine the diversified risk parity strategy. X-raying the risk decomposition in terms of the minimum torsion factors, the DRP shows equal risk in all of these seven factors by construction. Portfolio weights are likewise diversified with market factors having around 25–35% weight in total. The risk parity strategy is fairly close in terms of portfolio weights and risk allocation. Presumably, there was no correlation breakdown across the chosen factor structure rendering orthogonalization obsolete.

As for the benchmark strategies we document up to 95% of the $1/N$-strategy's risk budget to be consumed by commodities and equities. The risk contribution attributable to equities is roughly 25%, such that commodities consume the bulk of the risk budget, rendering other asset classes and style factors irrelevant. This assessment is confirmed by the volatility decomposition along single assets and style factors, see the second column of Figure 3.

Second, we recover the archetypical weights distribution of minimum-variance that is heavily concentrated in the two style factors Quality and Value, both having the least volatile returns in the sample period. While equities hardly enter the minimum-variance portfolio, we sometimes observe a diversifying commodities position of some 5–10%. Although there is a decent position in bonds, they do not hugely impact the risk budget. Third, we notice that

[3] Note that we measure turnover based on weight changes of successive models which could understate the actual strategy turnover. We nevertheless observe turnover in the $1/N$ strategy because of embedded turnover to rebalance the underlying risk parity factor schemes.

the MDP is more balanced from a risk-perspective, where one half of the risk budget is driven by market factors and the other half by style factors, respectively. Its weights decomposition over time is in between the one of risk parity and minimum-variance.

[Figure 4 about here.]

For directly comparing the degree to which the risk-based asset allocation strategies accomplish the goal of diversifying across uncorrelated risk sources, we plot the effective number of uncorrelated bets over time in one chart in Figure 4. Reiterating our above interpretation of the associated risk contributions over time, we find the $1/N$-strategy to be mostly dominated by the other strategies. By construction, the DRP strategy is constantly maintaining the maximum number of seven bets throughout time. Unsurprisingly, standard risk parity is runner up in this statistic. In between $1/N$ and DRP we find minimum-variance and MDP having 4.07 and 6.20 bets on average, respectively. While the latter is stable throughout the sample period, we observe that minimum-variance is starting out with some 5 bets in 2006 but is slowly losing ground with some 4 bets at the end of the sample period.

3 Factor Completion Strategies

From a pure factor investing perspective, a maximum diversification strategy in the multi-asset multi-factor domain is the method of choice for optimally harvesting the available style and market factor premia. Yet, the majority of investors is rather clinging to traditional multi-asset allocations and thus might be more inclined to consider less radical steps towards integrating style factors into the overall portfolio solution. In what follows, we present a factor completion framework that caters various investment objectives and constraints whilst expanding the investment opportunity set in terms of style factors.

3.1 Traditional asset allocation through a factor lens

Before discussing potential factor completion solutions we wish to diagnose the degree of diversification of traditional multi-asset allocations along market and style factors. Such analysis obviously illustrates the scope for style factors to enhance the degree of portfolio diversification and will serve as a means to benchmarking any suggested factor completion solution.

For illustration purposes, we choose a common traditional asset allocation benchmark that invests 25% in global equities, 25% in corporate bonds, 10% in commodities and the remainder in global government bonds.[4]To examine the relevance of factors we x-ray this traditional multi-asset allocation in terms of its global asset and style factor exposures in Figure 5. Specifically, we use the three market and four style factors introduced in the preceding sections. Unsurprisingly, we find a risk allocation that is rather concentrated in traditional market factors with equity and commodities covering three quarters of the

[4]Note that we apply equal portfolio weights within any given asset class.

risk budget. Still, there are some minor implicit style factor exposures associated with the traditional asset class allocation, such as carry and quality. Invoking Meucci's concept of effective number of bets we compute the benchmark to represent 3.07 bets throughout time, see first column in Table 4. Thus, the SAA is exploiting less than half of the available spectrum of factor bets.

[Table 4 about here.]

3.2 Factor-based tail hedging

A prudent first step into multi-asset multi-factor investing is to consider style factors as a means to protect a given asset allocation against downside risk. Taking this risk-based perspective a factor completion portfolio can be determined by running a minimum-variance portfolio optimization that would fix the original asset allocation and only allow a dynamic allocation to the style factors to help reducing portfolio risk. Figure 5 depicts the ensuing factor completion portfolio over time. Notably, one finds long-short equity factors to constitute a significant part of the tail hedge factor completion portfolio. Similarly, there are allocations to defensive factors such as equity quality and FX Value. As for the primary portfolio objective the tail-hedged portfolio has a volatility of 5.85% ex post which compares to the benchmark volatility of 5.60%. Note that the risk reduction is more significant in terms of tail risk, as it leads to lowering the maximum drawdown from -17.18% to -15.15%. Of course, having included style factors in the mix we expect an increase in portfolio diversification; yet this increase is rather modest raising portfolio diversification by one extra bet to 4.11. Judging from Figure 5 this increase is not attributable to a larger style factor exposure but rather to an implicit increase in duration risk (as represented by the tail-hedge factor completion portfolio). Thus, the dominant benchmark exposure to equity and commodities can be somewhat reduced. In terms of performance, the addition of style factors leads to an increase of the Sharpe ratio to 1.02 compared to 0.72 for the benchmark portfolio.

[Figure 5 about here.]

3.3 Diversified risk parity for factor completion

3.3.1 Constrained diversified risk parity

Next, we consider a factor completion portfolio that seeks to complete the underlying benchmark asset allocation with respect to the overall risk allocation in terms of style and market factors. In absence of investment constraints, the optimal solution would follow the diversified risk parity allocation, maximizing portfolio diversification. However, the latter framework is less suited to cater constraints. We therefore propose to couch the optimization in a mean-variance setting that is fed with implied views from the optimal diversified risk parity allocation. Without constraints such optimization would naturally recover the maximally diversified portfolio with perfectly balanced risk allocation. Given a fixed asset allocation the optimizer will allocate style factors in a manner to best trade off the diversification-centric views vis-à-vis the benchmark constraints.

The ensuing factor completion portfolio is ultimately consisting of a diversified mix of style factors, see the third row of Figure 5. More importantly, the associated risk allocation is considerably less concentrated in equity and commodities and more exposed to the four style factors. The average number of effective bets is 6.00 indicating that the benchmark constraints come at the cost of one bet. Admittedly, this marginal decrease in diversification efficiency is supported by the fact that the benchmark asset allocation is fairly close to the optimal DRP allocation in the three market factors. Completing the factor profile may be less straightforward for alternative asset allocation constraints. In the current use case, the overall portfolio has a similar maximum drawdown as the tail-hedge portfolio (-15.60%) and higher returns raising the Sharpe ratio to 1.13.

3.3.2 Pure diversified risk parity

Ideally, the investor is not restricted and can consider lifting the investment constraints to fully capitalize on the maximum diversification allocation. Notably, lifting the constraints in practice does not imply to load off given physical investments but to potentially complement the existing portfolio positions in terms of a futures overlay to manage market beta exposure according to the diversified risk parity allocation. Of course, the latter needs to be further levered to cater the desired level of portfolio risk. In the specific example, we apply a leverage factor of 3.25 thus raising the DRP volatility to 5.64% annually. We refer to this factor completion solution as pure diversified risk parity as we simply lever the implied DRP views in an unconstrained mean-variance portfolio optimization. Given the moderate benchmark allocation we choose a modest risk aversion coefficient $\gamma = 5$ for this purpose. Obviously, this levered pure DRP portfolio conserves the maximum diversification properties of the original DRP allocation, as it is attaining 6.95 bets on average. Naturally, its turnover is increased but so is its net total return (7.58%) and risk-adjusted returns (1.08 in terms of Sharpe ratio, 0.74 in terms of Calmar ratio) compared to the benchmark portfolio. Increasing portfolio diversification again helps mitigating tail risk as well; the maximum drawdown of the pure DRP portfolio is further reduced to -10.29%.

3.4 Diversified risk parity with a trend

The presented framework naturally lends itself to exploiting tactical asset allocation signals while still embracing the merits of diversified risk parity. Notably, forecasting style factors is often deemed impossible, yet, there is considerable evidence that asset classes tend to exhibit time series return predictability. In particular, there is strong evidence of the efficacy of time series momentum signals often utilized by investors pursuing a trend-following investment style, see Moskowitz, Ooi, and Pedersen (2012) and more recently Hurst, Ooi, and Pedersen (2017).

A naïve way to embed a trend style factor in a given portfolio would provide the portfolio optimization with the rather smooth return time series of the corresponding trend-following backtest. However, unlike the style factor "momentum" which is based on cross-sectional information, a trend-following strategy seeks to exploit time-series information by implementing directional long or short positions in the underlying asset classes. Hence, a smooth strategy backtest would most likely lead the portfolio optimizer to underestimate portfolio risk w.r.t. traditional market risk factors. Instead, one should rather build on the underlying traditional asset class time series together with the corresponding trend signal. Thus, we enhance the pure DRP strategy by providing the portfolio optimizer with explicit return forecasts for the three asset classes equity, government bonds, and commodities based on a simple trend-following signal: 3-month price momentum.

In doing so, we want to avoid driving out all of the diversification benefits associated with DRP. Thus, to incorporate trend signals in the pursuit of maximum diversification we investigate signal blending à la Black-Litterman (1990, 1992) and He and Litterman (2002). Especially, we choose the pure DRP allocation as the reference portfolio in a Black-Litterman optimization that is calibrated such that the resulting DRP trend portfolio exhibits a tracking-error of 2% relative to the pure DRP portfolio.

In particular, we simply use the standard Black-Litterman master formulae for refining return and variance-covariance estimates. For instance, the refined return input results from

$$E(R) = \left[(\tau\Sigma)^{-1} + P^T\Omega^{-1}P\right]^{-1}(\tau\Sigma)^{-1}\Pi + P^T\Omega^{-1}Q \tag{13}$$

where Σ refers to the variance-covariance matrix combining all single asset classes and factors. Specifying the objective view Π is straightforward as well. We back out the resulting expected returns coming from the DRP base allocation. The projection matrix P is then a diagonal matrix with ones along the diagonale. We shrink the respective variance-covariance matrices according to the traditional Black-Litterman formula as well

$$\Sigma_{BL} = \Sigma + \left[(\tau\Sigma)^{-1} + P^T\Omega^{-1}P\right]^{-1} \tag{14}$$

In the empirical application we use a cautious specification of Ω and τ. In particular, we set $\Omega = \text{diag}(\Sigma)$ and $\tau = 0.015$, which will prevent us from deviating too actively from the DRP base allocation.

The time-series trend signals are rescaled to live in the range of monthly returns, following the spirit of Grinold and Kahn (2000):

$$Q_{abs} = 0.5 \cdot Volatility \cdot Scores \tag{15}$$

To allow for consistently exploiting the trend signal relative to the DRP base allocation (across various leverage-gamma specifications), we construct a relative forecast centered around the implied views

$$Q_{rel} = \Pi \cdot Q_{abs} \tag{16}$$

To ensure that we minimize trading in assets where we do not have a view just because of a correlation pattern, we explicitly set Q equal to Π for those assets which have no trend signal.

As the trend signal is quite volatile, one may wish to introduce a transaction cost penalty in the Black-Litterman framework to introduce resistance in the rebalancing process. As the optimization problem is quadratic in nature it is straightforward to expand the mean-variance objective function by a quadratic transaction cost (TC) penalty[5]:

$$\max_{w} w'\mu - \frac{\gamma}{2} w'\Sigma w - \lambda_{TC}\Gamma|\Delta w|^2 \tag{17}$$

Using $\Delta w = w - w_0$ and rearranging terms we have

$$\max_{w} w'(\mu + 2\lambda_{TC}\Gamma w_0') - w'\left(\frac{\gamma}{2}\Sigma + \lambda_{TC}\Gamma\right) w \tag{18}$$

Adding the TC penalty to the objective function has two implications for the optimization: First, the transaction cost penalty leads to an increase in the expected return for the current allocation. Second, the perceived volatility is increased, therefore reducing the attractiveness of all assets. The transaction cost matrix Γ is typically populated based on estimates derived from transaction cost models, see Gârleanu and Pedersen (2013). While it is possible to impose a constant TC matrix one often assumes Γ to be linear in the variance-covariance matrix. In our empirical analysis, we choose Γ to be linear in the diagonal of the variance-covariance matrix using $\lambda_{TC} = 0.3$.[6]

As expected, the DRP trend allocation is more active relative to the pure DRP allocation, see the last row of Figure 5; its mean monthly turnover is some 13 percentage points higher. This increased turnover results in an active return of 92 bps p.a. and a Sharpe ratio of 1.17. Notably, the presence of a trend style allocation hardly leads to a loss in portfolio diversification, with the number of bets still averaging to 6.89.

[Figure 6 about here.]

Figure 6 sheds more light on the number of bets of all active factor completion strategies through time. First, we note that the pure DRP strategy consistently maintains close to 7 bets. Second, the DRP Trend portfolio is not too far off from this benchmark, with the exception of the first three years of the sample period (including the GFC). Third, the constrained DRP can compete in these three years in terms of bets but falls behind by one bet going forward. Fourth, tail hedge and benchmark portfolios represent less bets through time.

The concept of effective number of uncorrelated bets is particularly insightful in the context of calibrating the activity of the tactical asset allocation. In that regard, the lower chart of Figure 6 depicts the interplay of the aggressiveness of the TAA (as implied by an increasing τ) and the degree of diversification (as measured by $\mathcal{N}_{Ent}$). Obviously, diversification

[5] See Dichtl, Drobetz, Lohre, Rother, and Vosskamp (2018) for a detailed derivation.

[6] Ultimately, the choice of λ_{TC} is an empirical question.

is reduced for more aggressive allocations, yet a confident choice ($\tau = 0.15$) still enables to maintain at least 5 bets at all times.

4 Conclusion

Traditional asset allocations tend to be hardly balanced across the salient drivers of risk and return. Clearly, the corresponding asset classes would benefit from explicitly managing asset and style factor exposure jointly. This paper provides a viable framework to complete a given portfolio allocation in terms of a factor completion portfolio. Naturally, factor completion strategies require the specification of a meaningful reference portfolio towards which to complete to. While this paper focused on diversified risk parity strategies in the multi-asset multi-factor domain we note that the presented factor completion framework is not dependent on this choice. More importantly, we demonstrated different use-cases of factor completion customized to specific client needs. In particular, the presented framework allows gauging the associated diversification trade-offs that are implicit in the various factor completion choices.

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A Appendices

A.1 Style factor definitions

A.1.1 Foreign exchange (FX) style factors

Carry	The FX Carry strategy has historically benefited from the tendency of FX forwards of high yielding currencies to overestimate the actual depreciation of future FX spot. On a monthly basis, the strategy evaluates the implied carry rate (FX Forward vs FX Spot) of a number of currencies (G10 and EM) against the USD and ranks them based on that measure. The strategy goes long single currency indices (which roll FX forwards) for the currencies with the highest carry and short single currency indices for the currencies with the lowest carry.
Value	The FX Valuation strategy relies on exchange rates reverting back to their fair value in the medium to long-term horizons. On a monthly basis, the strategy evaluates the valuation measure (based on GS DEER, Dynamic Equilibrium Exchange Rate model) of a number of currencies (G10 and EM) against the USD and ranks them based on that measure. The strategy goes long single currency indices (which roll FX forwards) for the currencies with the highest ranking (i.e. most undervalued) and short single currency indices for the currencies with the lowest ranking (i.e. most overvalued).
Momentum	This factor capitalizes on the persistence of trends in forward exchange rate movements which are driven both by carry as well as spot movements. On a daily basis, the strategy evaluates the recent performance of 27 currencies against the USD. It then takes either a long or short position on each of those currency against the USD, depending on whether their actual performance has been positive or negative.

A.1.2 Commodity style factors

Carry	Captures tendency for commodities with tighter timespreads to outperform due to low inventories driving both backwardated futures curves and price appreciation, and buying demand from consumer hedgers for protection against price spikes in undersupplied commodities The strategy goes long the top third and short the bottom third of the 24 commodities from the S&P GSCI universe, ranked by annualized strength of front month time spreads. The strategy is rebalanced daily based on the signals over the last 10 days. The strategy is net of cost.
Value	The strategy uses the weekly Commodity Futures Trading Commission (CFTC) positioning data to determine to go long and short in commodities and will take long positions in the commodities that the culative positions are most short and short position in the commodities that speculative positions are most long.
Momentum	Momentum in commodity returns reflect initial underreaction or subsequent overreaction to changes in demand as increasing or decreasing supply take many years to implement and subsequently overshoot required changes to match demand. The strategy goes long the top third and short the bottom third of the 24 commodities from the S&P GSCI universe, ranked by rolling 1-year excess returns of each commodity. The strategy is rebalanced daily based on the signals over the last ten days. The strategy is net of cost.

Quality	This factor captures the tendency for deferred futures contracts to outperform nearer dated contracts due to producers hedging further out than consumers, and passive investors investing near the front of the curve. The strategy goes long selected points on the curve of each commodity, equally weighted amongst commodities. The strategy goes short an equally weighted basket of the nearest commodity contracts, beta-adjusted at the basket level.

A.1.3 Rates style factors

Momentum	This factor capitalizes on the persistence of trends in short and long-term interest rate movements. On a daily basis, the strategy evaluates the recent performance of a number of futures contracts for US, Germany, Japan and UK. It then takes either a long or short position on each of the futures, depending on whether their actual performance has been positive or negative.
Quality	This factor capitalizes on the observation that risk adjusted returns at the short-end of the curve tend to be higher than at the long-end. A leveraged long position on the former vs. the latter tends to capture positive excess returns as compensation for the risk premium that stems from investors having leverage constraints and favouring long-term rates. The Interest Rates Curve strategy enters a long position on 5y US Bond futures and a short position on 30y Bond futures, as well as a long position on 5y German Bond futures and a short position on 10y German Bond futures, rolling every quarter. The exposure to each future is adjusted to approximate a duration-neutral position.

A.1.4 Equity style factors

Value	The Value factor refers to the finding that value stocks characterized by valuation ratios offer higher long-run average returns than growth stocks characterized by high valuation ratios. To optimally combine factors in a multi-factor asset allocation exercise we construct the factor to have zero exposure to other factors and does not change the exposure to other factors when added to the portfolio.
Momentum	The Momentum factor captures a medium-term continuation effect in returns by buying recent winners and selling recent losers. The factor combines price as well as earnings momentum information. To optimally combine factors in a multi-factor asset allocation exercise we construct the factor to have zero exposure to other factors and does not change the exposure to other factors when added to the portfolio.
Quality	The Quality factor combines different measures of determining financial health and operating profitability. To optimally combine factors in a multi-factor asset allocation exercise we construct the factor to have zero exposure to other factors and does not change the exposure to other factors when added to the portfolio.
Defensive	The Defensive factor refers to the finding that low volatility stocks tend to outperform high volatile stocks on a risk-adjusted basis. To capture this behaviour the factor is constructed to go long a minimum variance portfolio and short the beta-portion of the market.

A.2 Risk-based allocation methodologies

A.2.1 1/N

The $1/N$-strategy rebalances monthly to an equally weighted allocation scheme. Hence, for K factors the portfolio weights $\mathbf{w}_{1/N}$ are

$$\mathbf{w}_{1/N} = \frac{\mathbf{1}}{K} \tag{19}$$

A.2.2 Minimum-Variance

The minimum-variance (MV) portfolio weights $\mathbf{w}_{MV}$ derive from

$$\mathbf{w}_{MV} = \underset{\mathbf{w}}{\operatorname{argmin}}\, \mathbf{w}'\mathbf{\Sigma}_{\mathbf{F}}\mathbf{w} \tag{20}$$

subject to the full investment and positivity constraints, $\mathbf{w}'\mathbf{1} = 1$ and $\mathbf{w} \geq \mathbf{0}$.

A.2.3 Risk Parity

We construct the original risk parity (RP) strategy by allocating capital such that the seven aggregate factors' risk budgets contribute equally to overall portfolio risk.Since there are no closed-form solutions available, we follow Maillard, Roncalli, and Teiletche (2010) to obtain $\mathbf{w}_{RP}$ numerically via

$$\mathbf{w}_{RP} = \underset{\mathbf{w}}{\operatorname{argmin}} \sum_{i=1}^{K} \sum_{j=1}^{K} \left(w_i(\mathbf{\Sigma}_{\mathbf{F}}\mathbf{w})_i - w_j(\mathbf{\Sigma}_{\mathbf{F}}\mathbf{w})_j\right)^2 \tag{21}$$

which essentially minimizes the variance of the factors' risk contributions. Again, the full investment and positivity constraints apply.

A.2.4 Most-Diversified Portfolio of Choueifaty and Coignard (2008)

We describe the approach of Choueifaty and Coignard (2008) to building maximum diversification portfolios. To this end the authors define a portfolio diversification ratio $D(\mathbf{w})$:

$$D(\mathbf{w}) = \frac{\mathbf{w}' \cdot \boldsymbol{\sigma}_F}{\sqrt{\mathbf{w}'\mathbf{\Sigma}_{\mathbf{F}}\mathbf{w}}} \tag{22}$$

where $\boldsymbol{\sigma}_F$ is the vector of aggregate factor return volatilities. Thus, the most-diversified portfolio (MDP) simply maximizes the ratio between two distinct definitions of portfolio volatility, i.e. the ratio between the average portfolio factors' volatility and the total portfolio volatility. We obtain MDP's weights vector $\mathbf{w}_{MDP}$ by numerically computing

$$\mathbf{w}_{MDP} = \underset{\mathbf{w}}{\operatorname{argmax}}\, D(\mathbf{w}) \tag{23}$$

As before we enforce the full investment and positivity constraints.

Table 1: Descriptive statistics of market, style and aggregate factors. The table shows performance statistics of market, style and aggregate factors. Returns are annualized excess returns and are calculated using the arithmetic average of simple returns. The standard deviation and Sharpe ratio are annualized through multiplication by $\sqrt{12}$. Min and Max denote the lowest and highest monthly excess return in the sample period. MaxDD is the maximum drawdown. Return, Volatility, Min, Max and MaxDD are in percentage terms. The time period is from 01/2001 to 10/2018.

		Return	Vola	SR	t-stat	Min	Max	MaxDD
Panel A: Market factors								
SP500	*SP500*	5.04	14.43	0.35	1.47	-17.25	10.95	-53.41
EuroSTOXX50	*E50*	2.47	18.49	0.13	0.56	-18.52	16.81	-58.65
FTSE100	*FTSE*	3.31	13.86	0.24	1.01	-13.38	8.80	-47.00
Nikkei225	*Nikkei*	5.70	19.29	0.30	1.25	-25.35	12.86	-57.96
MSCI EM	*EM*	7.59	21.55	0.35	1.48	-27.50	16.66	-62.67
Credit.IG	*IG*	0.86	4.25	0.20	0.85	-7.27	4.72	-23.68
Credit.HY	*HY*	4.23	10.47	0.40	1.70	-15.81	13.49	-43.43
US10Y	*US10*	3.75	5.88	0.64	2.68	-5.81	8.68	-7.70
Bund	*Bund*	4.32	5.15	0.84	3.54	-2.90	4.78	-9.11
JGB10Y	*JGB10*	1.96	2.51	0.78	3.29	-3.14	2.26	-4.79
Gilt	*Gilt*	3.23	6.01	0.54	2.26	-4.89	5.69	-9.88
Gold	*Gold*	9.39	17.17	0.55	2.30	-18.40	13.77	-43.39
Oil	*Oil*	1.25	31.13	0.04	0.17	-32.43	27.47	-92.40
Copper	*Copper*	10.57	27.13	0.39	1.64	-36.42	31.81	-63.64
Agriculture	*Agri*	1.08	20.44	0.05	0.22	-18.97	16.21	-56.38
Panel B: Style factors								
Equity.Quality	*EQ.Q*	3.04	3.49	0.87	3.66	-1.93	4.20	-8.78
Equity.Defensive	*EQ.D*	4.00	5.12	0.78	3.30	-4.39	3.67	-11.64
Equity.Value	*EQ.V*	3.16	4.00	0.79	3.33	-2.83	4.20	-9.61
Equity.Momentum	*EQ.M*	2.77	5.60	0.49	2.08	-6.45	5.14	-24.08
FX.Carry	*FX.C*	4.94	6.44	0.77	3.23	-8.11	7.20	-15.22
FX.Value	*FX.V*	2.22	4.51	0.49	2.07	-6.25	5.06	-9.29
FX.Momentum	*FX.M*	4.89	4.95	0.99	4.16	-3.24	5.60	-11.85
Cmdty.Carry	*CM.C*	5.77	8.05	0.72	3.02	-5.79	7.41	-14.72
Cmdty.Quality	*CM.Q*	4.53	2.80	1.61	6.80	-1.55	2.93	-4.44
Cmdty.Momentum	*CM.M*	2.33	9.45	0.25	1.04	-7.80	9.53	-22.86
Cmdty.Value	*CM.V*	5.56	7.46	0.75	3.14	-5.55	6.94	-10.36
Rates.Value	*FI.V*	0.89	3.39	0.26	1.10	-3.93	3.28	-7.05
Rates.Momentum	*FI.M*	4.19	4.94	0.85	3.57	-4.26	5.29	-7.20
Rates.Quality	*FI.Q*	1.32	3.05	0.43	1.82	-2.22	3.85	-9.87
Rates.Carry	*FI.C*	2.66	3.29	0.81	3.40	-3.31	3.61	-6.33
Panel C: Aggregate factors								
Equity-Credit		3.13	9.36	0.33	1.41	-13.25	9.15	-38.37
Duration		2.90	3.36	0.86	3.63	-2.70	3.72	-5.23
Commodity		5.82	15.60	0.37	1.57	-24.00	13.05	-48.49
Carry		3.98	3.11	1.28	5.38	-3.20	3.52	-6.01
Value		2.56	2.53	1.02	4.28	-2.47	2.70	-4.86
Momentum		3.74	3.47	1.08	4.55	-2.33	3.77	-5.93
Quality		3.23	1.64	1.98	8.33	-0.85	1.67	-1.96

Table 2: Minimum torsion factor loadings to original factors. The table shows the loadings of the uncorrelated risk sources w.r.t. to original factors as represented by the minimum torsion matrix t_{MT}. The time period is from 01/2001 to 10/2018.

Original Factors	MTF1 *Equity*	MTF2 *Duration*	MTF3 *Commodity*	MTF4 *Carry*	MTF5 *Value*	MTF6 *Momentum*	MTF7 *Quality*
Equity	**1.11**	0.09	-0.13	-0.10	-0.02	0.08	-0.00
Duration	*0.40*	**1.21**	0.16	-0.28	0.24	-0.17	-0.12
Commodity	-0.03	0.01	**1.10**	0.02	0.00	0.05	-0.02
Carry	*-0.34*	-0.20	0.26	**1.11**	-0.17	-0.04	0.01
Value	-0.13	0.28	0.04	-0.27	**1.11**	-0.13	-0.07
Momentum	*0.44*	-0.20	**1.09**	-0.06	-0.13	**1.15**	-0.04
Quality	0.03	*0.58*	**-1.98**	0.09	*-0.32*	-0.15	**1.11**

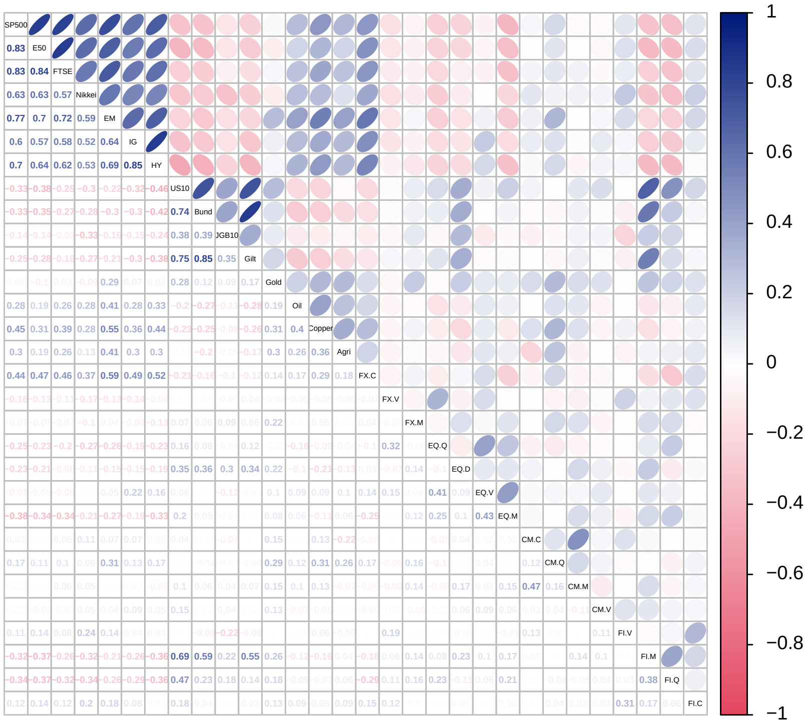
Table 3: Descriptive statistics of risk-based allocation techniques. The table shows performance statistics of risk-based allocation techniques. DRP shows the diversified risk parity performance, RP represents the risk-parity strategy, MVP mean-variance and MDP applies the maximum diversification approach by Choueifaty and Coignard (2008). Annualized excess returns are calculated using the arithmetic average of simple returns. The standard deviation and Sharpe ratio are annualized through multiplication by $\sqrt{12}$. MaxDD is the maximum drawdown. Number of bets denotes the effective number of uncorrelated bets ($\mathcal{N}_{Ent}$). Turnover is calculated as two-way turnover. Return, Volatility, MaxDD, CVaR and Turnover are in percentage terms. The time period is from 01/2001 to 10/2018.

	DRP	RP	$1/N$	MVP	MDP
Net Return	3.28	3.27	3.53	2.94	3.23
Volatility	1.45	1.48	3.49	1.31	1.57
Sharpe Ratio	1.24	1.23	0.61	1.15	1.14
MaxDD	-1.92	-1.79	-9.35	-1.04	-2.81
Calmar Ratio	1.71	1.83	0.38	2.82	1.15
CVaR	2.12	2.12	6.15	1.86	2.30
Number Of Bets	7.00	6.94	3.11	4.07	6.20
Turnover	3.66	2.97	1.53	5.98	6.94

Table 4: Descriptive statistics of factor completion strategies. The table shows performance statistics of factor completion strategies. Annualized excess returns are calculated using the arithmetic average of simple returns. The standard deviation and Sharpe ratio are annualized through multiplication by $\sqrt{12}$. MaxDD is the maximum drawdown. Number of bets denotes the effective number of uncorrelated bets ($\mathcal{N}_{Ent}$). Turnover is calculated as two-way turnover. Return, Volatility, MaxDD, CVaR and Turnover are in percentage terms. The time period is from 01/2001 to 10/2018.

	BM	Tail hedge	Constrained	Pure	Trend
Net Return	5.38	7.47	8.85	7.58	8.50
Volatility	5.60	5.85	6.44	5.64	5.95
Sharpe Ratio	0.72	1.02	1.13	1.08	1.17
MaxDD	-17.18	-15.15	-15.60	-10.29	-12.01
Calmar Ratio	0.31	0.49	0.57	0.74	0.71
CVaR	9.95	9.90	10.73	9.43	9.81
Number Of Bets	3.07	4.11	6.00	6.95	6.89
Turnover	0.00	11.32	18.94	25.93	38.66

Figure 1: Correlation Matrix. The upper chart depicts the correlation structure of the 30 market and style factors, while the lower chart shows correlation of the aggregated seven factors. Time period is from January 2001 to October 2018. Colors range from dark red (correlation of -1) to dark blue (correlation of 1).



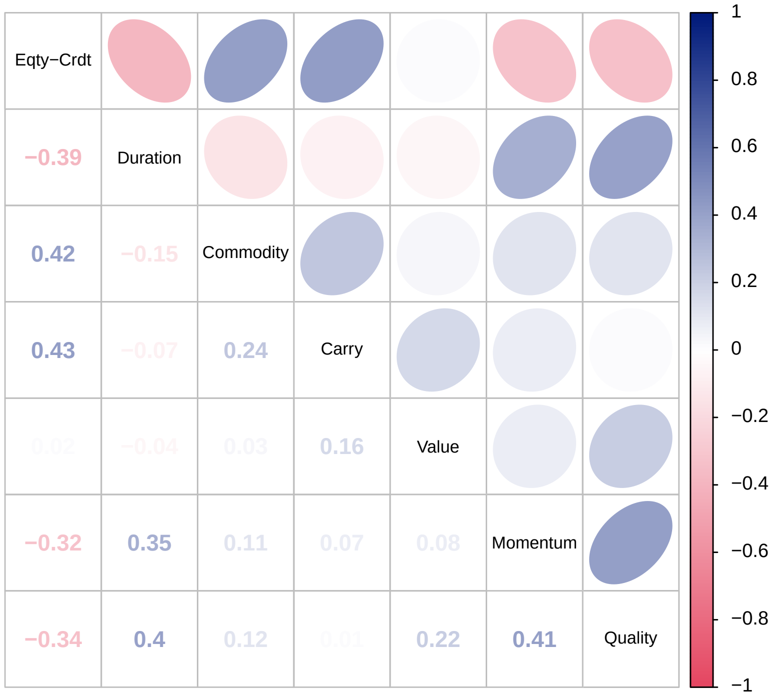


Figure 2: Minimum torsion factor loadings over time. The figure gives the minimum torsion factors loadings over time based on a rolling estimation windows of 60 months. The estimation period is from January 2006 to October 2018.

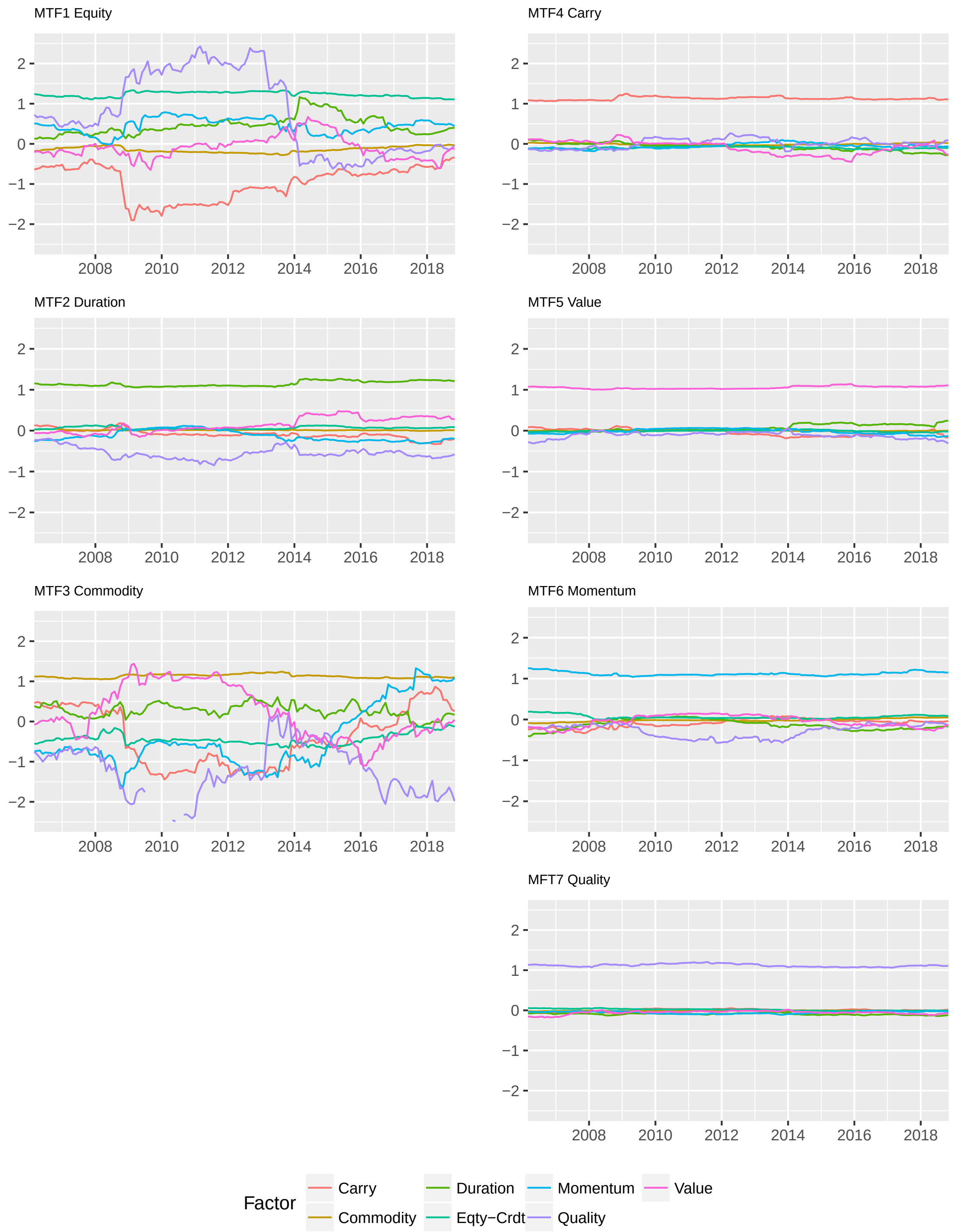


Figure 3: Weights and Risk Decompositions: Risk-Based Strategies. The figure gives the decomposition of the risk-based allocation strategies in terms of single asset and factors weights, aggregate factor allocation weights and risk. Risk is being decomposed by minimum torsion factors. The results build on rolling window estimation over 60 months. The sample period is from January 2006 to October 2018.

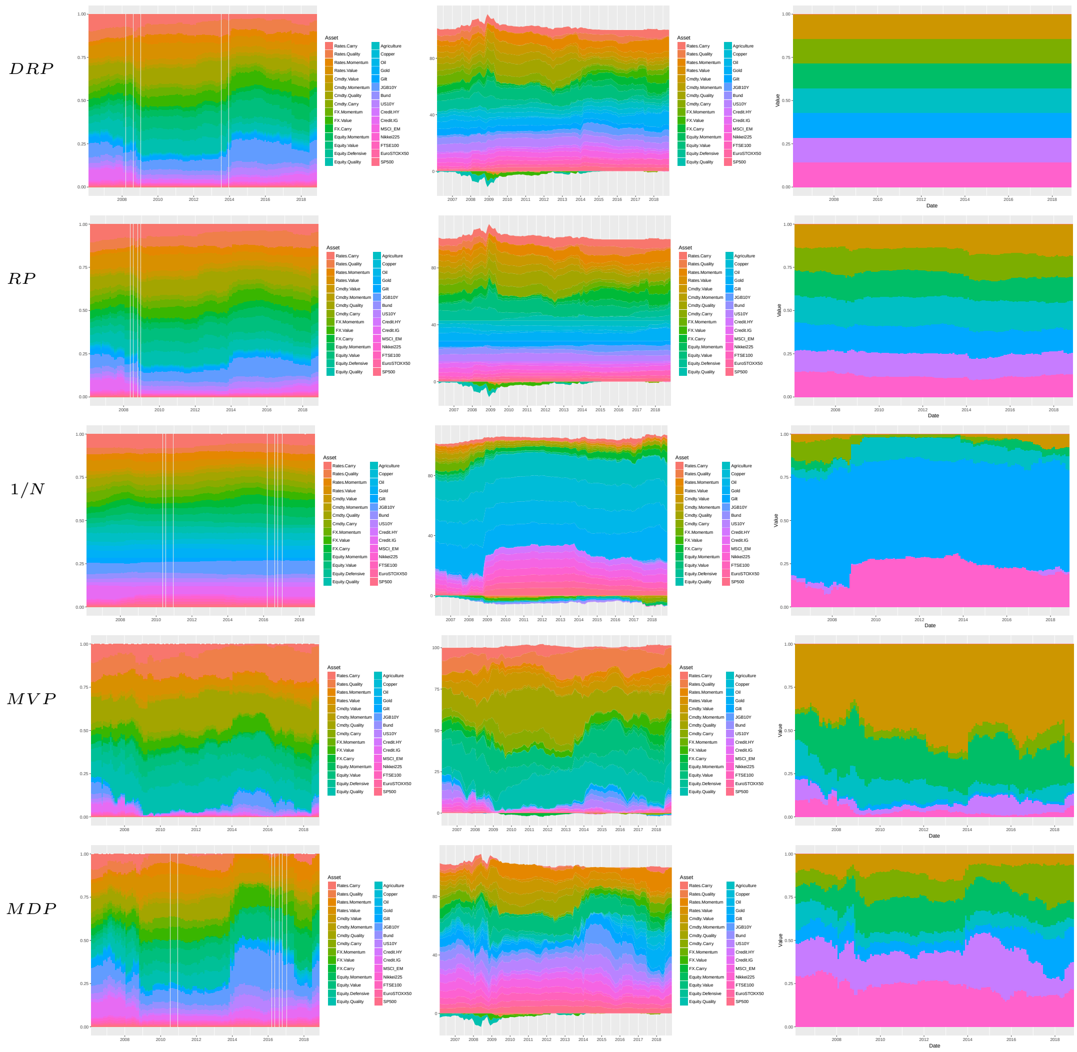


Figure 4: Effective Number of Uncorrelated Bets. We plot the number of uncorrelated bets for the risk-based asset allocation strategies when using rolling window estimation of 60 months. The sample period is January 2006 to October 2018.

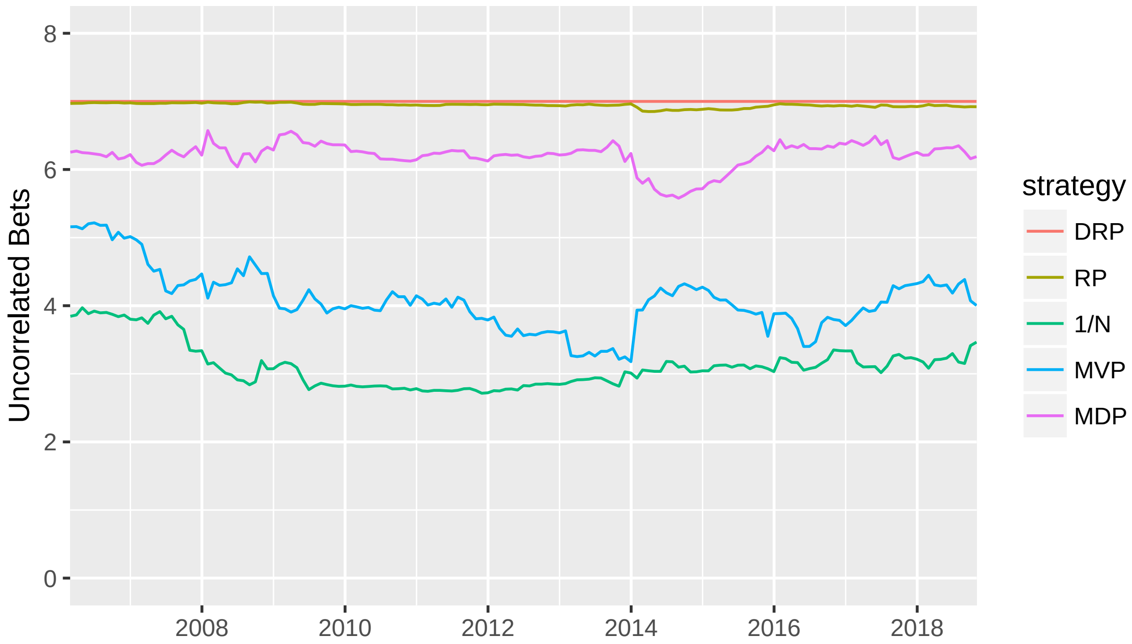


Figure 5: Weights and Risk Decompositions: Active factor completion strategies. The figure gives the decomposition of the active factor completion strategies in terms of single asset and factors weights, aggregate factor allocation weights and risk. Risk is being decomposed by minimum torsion factors. The results build on rolling window estimation over 60 months. The sample period is from January 2006 to October 2018.

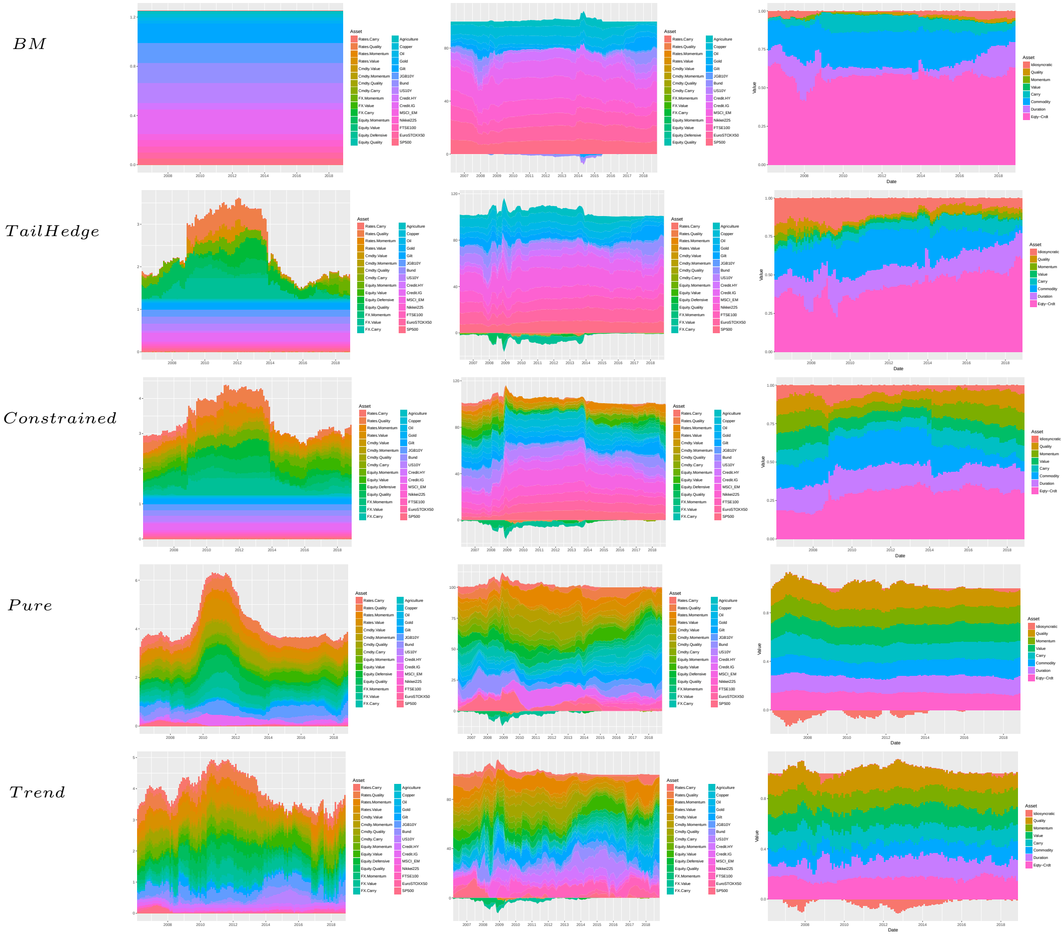


Figure 6: Effective Number of Uncorrelated Bets. The upper chart depicts the number of uncorrelated bets for the active factor completion strategies when using rolling window estimation of 60 months. The lower chart compares the effective number of bets when choosing different τ while τ is ranging from zero (Pure) to 15% (tau15). The sample period is January 2006 to October 2018.