Counterparty Risk Allocation

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Abstract

We address the problem of minimizing the risk of an exposure (e.g., cash holdings) to a small number of defaultable counterparties based on spectral risk measures, in particular the expected shortfall. The resulting risk-minimal allocation turns out to be economically implausible in a number of ways: When the loss distributions is discrete, only corner solutions can be optimal, and the risk-minimal allocation does not depend continuously on the input parameters. With two counterparties, only a total allocation to one counterparty or a fifty-fifty solution can be optimal. In general, the risk-minimal allocation is not monotonic in the quantile used for calculating the expected shortfall. This non-monotonicity also holds for continuous loss distributions. These results strengthen the doubts on the appropriateness of spectral risk measures in the target function for economic decision making.

JEL Classification: C 44, D 81, G 11, G 21

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1 Introduction

Counterparty risk is an important issue not only for banks and financial service providers, but also for non-financial firms. Consider a company conducting business activities with a number of counterparties in the financial industry (termed "banks" in the following). Such activities can for instance be simple cash holdings, hedging contracts against foreign exchange risk, etc. According to these activities, the company has an exposure with respect to each bank—e.g., the balance of the cash account or the value of the hedging contract. The company faces the default risk of the counterparties, which would result in a loss (total or partial) of the respective exposures. The structure of this risk depends on the allocation of the overall exposure across the counterparties. This paper deals with the optimal allocation of the counterparty risk exposure.

In the classical sense, the allocation problem is nothing else than a portfolio selection problem. However, it is widely accepted that a μ-σ analysis, pioneered by Markowitz (1952), is not suitable for credit-risky portfolios according to the heavily skewed loss distributions (e.g., Andersson et al., 2001). Instead, tail-based risk measures, in particular the value-at-risk (VaR) and the expected shortfall (ES, sometimes referred to as the conditional value-at-risk), have found widespread use in theory and practice of credit portfolio measurement and management. While the VaR can be characterized as the maximum loss with a given confidence level, the ES is the average loss beyond this level. In timely coincidence with the evolvement of credit portfolio models in the late 1990s, a number of papers have studied optimization problems in a μ-VaR or μ-ES space, including Lucas and Klaassen (1998), Krokhmal et al. (2001), Andersson et al. (2001), Benati (2003), and Gaivoronski and Pflug (2004). Basically, all these papers study the shape of efficient frontiers or various aspects of the minimization of risk measures either for equity portfolios or for large credit portfolios.

However, very little has been said about the allocation problem when the number of defaultable counterparties is small. Compared to a bank loan portfolio with thousands of obligors, the allocation problem for a company with a few financial counterparties might be quite different. What is more, recent studies cast some doubts on the appropriateness of minimizing risk measures such as the expected shortfall for the purpose of optimal portfolio selection. Brandtner (2013) shows that for any spectral risk measure (including the expected shortfall as a special case), an optimization approach tends towards corner solutions. In a classical Markowitz setup with the presence of a risk-free asset, maximizing an expected-shortfall-based preference function always leads to a total investment in either the risk-free asset or the tangential portfolio. As discussed by Brandtner and Kürsten (2015) in greater detail, such inconsistencies arise with the application of risk measures for economic decision making, in contrast to their original purpose as a side condition or constraint for risk-taking, for example in a regulatory context.

In this paper, we discuss the allocation problem for a small number of counterparties against this background. We start with the case of $n = 2$ counterparties. This case is far from being purely academical, since many medium-sized companies are conducting their financial business with a small number of counterparties, indeed often with two distinctive relationship banks. So the question how to allocate temporary amounts of cash or hedging contracts (e.g., for foreign exchange exposure) when two relationship banks are default-risky is of high practical relevance. In particular the discrete setting when the loss distribution is singular (the loss given default is deterministic) is different from large loan or bond portfolios and yields substantially different results. We show that the discrete setting can be seen as a special case of Brandtner (2013). Discussing his results in the context of counterparty risk, we conclude that, depending on the actual specification of the risk measure, the risk-minimal solution is always a corner solution that allocates the total exposure either completely to one bank (that with lower default probability) or evenly to both banks.

Extending the discrete setting to $n > 2$ counterparties, after analyzing efficient frontiers in the μ-ES space, we focus on risk minimization instead of optimizing a preference function. In the credit risk literature, risk minimization is often performed subject to a minimum return as a side constraint, e.g. Rockafellar and Uryasev (2000), Andersson et al. (2001), Saunders et al. (2007). Some approaches consider an unconstrained risk minimization, e.g. Iscoe et al. (2012), which has also experienced growing attention in the equity portfolio selection literature (e.g., Jagannathan and Ma, 2003, and Frahm and Memmel, 2010). Taking the return dimension into account makes sense when there is a considerable and quantifiable risk premium, for instance in the case of credit-risky bonds. In our scope of counterparty risk, return aspects are less important—for example, temporary cash accounts often do not earn any return at all. We therefore concentrate on the risk-minimization problem and characterize its corner solutions in higher dimensions. As a general result, the exposure share allocated to a single bank is either 100% or does not exceed 50%. Furthermore, the risk-minimal allocation depends neither monotonically nor continuously on the chosen confidence level.

We then consider a continuous setting, in which the loss given default is stochastic. In this setting, the risk-minimal allocation depends continuously on the quantile of the expected shortfall. While for $n = 2$ this dependency is also monotonic, monotonicity does not hold in the general multivariate case. So as a joint conclusion in all settings, minimization of expected shortfall as a representative of the well-respected class of spectral risk measures (including coherent risk measures) can lead to economically implausible results. Decision makers should therefore be alerted to the unwary use of such measures for the optimization of their counterparty risk portfolio.

The remainder of the paper is organized as follows. Section 2.1 defines the basic setup of the discrete loss distribution and briefly reviews the expected shortfall. Section 2.2 deals with the 2-banks case and demonstrates the tendency to corner solutions. Section 2.3 extends the results to the multivariate case, analytically discusses properties of corner points in higher dimensions, and conducts a numerical study for 5 and 10 counterparties, respectively. Section 3 covers the setup with continuous loss distributions, again starting with the 2-banks case (Section 3.1) and proceeding to the multivariate case (Section 3.2). Section 4 concludes.

2 Discrete Loss Distribution

2.1 Basic Setup

In this subsection, we outline the basic setup in the discrete case, which holds throughout Section 2. In Section 3, the deterministic loss given default assumed here will be extended to a stochastic loss given default.

The company has a total exposure, which is normalized to 1, and which has to be allocated to $n \geqslant 2$ banks. The exposure may arise from simple cash holdings, hedging derivative positions, etc. The key task is to allocate the total exposure to the banks, that is, to find an allocation vector defined as follows.

Definition 1. *A vector* $(x_1, \ldots, x_n)$ *is an* allocation vector *if* $x_i \geqslant 0$ $\forall i$ *and* $\sum_{i=1}^{n} x_i = 1$ *holds.*

The banks offer payoff rates r_i for each allocated exposure unit. In the case of cash holdings, r_i are interest rates; for derivative contracts, $-r_i$ can be thought of as fees to be paid. The banks default with probabilities p_i. For the sake of simplicity, we assume that in the event of a default, the corresponding exposure experiences a total loss.[1] Hence, the profit from a particular exposure x_i to Bank i is

$$X_i = \begin{cases} r_i\, x_i & \text{with probability } 1 - p_i, \\ -x_i & \text{with probability } p_i. \end{cases} \tag{1}$$

We measure the corresponding loss L_i with respect to a target profit of zero, that is, $L_i = -X_i$.

The whole joint default distribution is known. For any subset $I \subset \{1, \ldots, n\}$, let π_I denote the probability that exactly those counterparties i with $i \in I$ default and those with $i \notin I$ do not default. Without loss of generalization, the banks are ordered so that $p_1 \leqslant p_2 \leqslant \cdots \leqslant p_n$, that is, Bank 1 is the least risky bank.

Under this setup, the total expected loss for a given allocation vector is

$$EL = \sum_{I} \pi_I \cdot \sum_{i \in I} x_i. \tag{2}$$

Risk is measured as the expected shortfall at a quantile α, defined as the average loss in the worst $100\alpha\%$ scenarios (Acerbi and Tasche, 2002a):

Definition 2. *If* $l^{(\alpha)}$ *denotes the upper* $(1-\alpha)$*-quantile of the loss distribution,* $l^{(\alpha)} := \inf\{l : P(L > l) \leqslant \alpha\}$*, then the* expected shortfall *with quantile* α *is defined as*

$$ES_\alpha = \frac{1}{\alpha}\left(E[L|L > l^{(\alpha)}] + (\alpha - P(L > l^{(\alpha)})) \cdot l^{(\alpha)}\right). \tag{3}$$

[1] A constant loss given default can be assumed to be 0 without loss of generalization. For any constant $LGD \in [0; 1)$, the exposure can be multiplied by $(1 - LGD)$ to achieve an equivalent setting with a loss given default of zero.

Note that in the case of a continuous loss distribution, $P(L > l^{(\alpha)}) = P(L \geqslant l^{(\alpha)}) = \alpha$, so Equation (3) simplifies to

$$ES_\alpha = \frac{1}{\alpha}\left(E[L|L > l^{(\alpha)}]\right). \tag{4}$$

This measure is sometimes referred to as the Tail Conditional Expectation (Acerbi and Tasche, 2002b), while (3) is labelled Conditional Value-at-Risk (Rockafellar and Uryasev, 2002). The latter assures the intuitive interpretation as an average loss in the worst α scenarios, as both mentioned papers discuss in greater detail. Benati (2003) provides a simple example.

In the basic setup, the loss distribution is discrete. With an ordering of the potential losses, $l_1 > l_2 > \ldots$, the expected shortfall can be calculated as (Rockafellar and Uryasev, 2002)

$$ES_\alpha = \frac{1}{\alpha}\left[\sum_{j=1}^{j(\alpha)} P(L = l_j)\, l_j + \left(\alpha - \sum_{j=1}^{j(\alpha)} P(L = l_j)\right) l_{j(\alpha)+1}\right], \tag{5}$$

where

$$j(\alpha) = \max\left\{j : \sum_{k=1}^{j} P(L = l_k) \leqslant \alpha\right\}. \tag{6}$$

2.2 Two Counterparties

2.2.1 Minimizing Expected Shortfall

We first consider the case of $n = 2$ counterparties. In this subsection, we study the general behavior of the expected shortfall, before Subsection 2.2.2 analyzes efficient frontiers in the μ-ES space. In the 2-banks case, for any allocation vector $x_2 = 1 - x_1$ holds, so x_1 is the only decision variable. As Bank 1 is not riskier than Bank 2, it is straightforward that the risk-minimal allocation requires $x_1 \geqslant x_2$, or, equivalently $x_1 \geqslant 0.5$. Imposing this restriction simplifies the following analysis. The loss distribution is given by

$$L = \begin{cases} l_1 = 1 & \text{with probability } \pi_{12}, \\ l_2 = x_1 & \text{with probability } \pi_1, \\ l_3 = 1 - x_1 & \text{with probability } \pi_2, \\ l_4 = 0 & \text{with probability } 1 - \pi_1 - \pi_2 - \pi_{12}. \end{cases} \tag{7}$$

For ease of notation, we skip the set braces in the indices, that is, $\pi_{12} := \pi_{\{1,2\}}$ and so on. Note that π_1, the probability that only Bank 1 defaults, is given by $\pi_1 = p_1 - \pi_{12}$.

For a quantile α, the expected shortfall is

- if $\alpha \leqslant \pi_{12}$:
$$ES_\alpha = 1;$$

- if $\pi_{12} < \alpha \leqslant \pi_{12} + \pi_1 = p_1$:
$$ES_\alpha = \frac{\pi_{12} + (\alpha - \pi_{12}) \cdot x_1}{\alpha};$$

- if $p_1 < \alpha \leqslant \pi_1 + \pi_2 + \pi_{12}$:
$$ES_\alpha = \frac{\pi_{12} + \pi_1 \cdot x_1 + (\alpha - \pi_1) \cdot (1 - x_1)}{\alpha} = \frac{\alpha - \pi_1 + (2\,\pi_1 + \pi_{12} - \alpha) \cdot x_1}{\alpha};$$

- if $\alpha > \pi_1 + \pi_2 + \pi_{12}$:
$$ES_\alpha = \frac{\pi_{12} + \pi_1 \cdot x_1 + \pi_2 \cdot (1 - x_1)}{\alpha} = \frac{\pi_{12} + \pi_2 + (\pi_1 - \pi_2) \cdot x_1}{\alpha}.$$

In each case, the expected shortfall is a linear function of x_1. It follows

Proposition 1. *Given the basic setup with $n = 2$. In the special cases $p_1 = p_2$, $\alpha \leqslant \pi_{12}$, or $\alpha = 2\,\pi_1 + \pi_{12}$, any choice $x_1 \in [0.5, 1]$ minimizes the expected shortfall with quantile α. In all other cases, the minimum expected shortfall is necessarily a boundary minimum either at $x_1 = 0.5$ or at $x_1 = 1$.*

Proof.

- If $\alpha \leqslant \pi_{12}$, the expected shortfall is constant and the decision maker is indifferent between any choice of x_1.

- If $\pi_{12} < \alpha \leqslant \pi_1$, the expected shortfall is linearly increasing with x_1, so the optimal choice is $x_1^* = 0.5$.

- If $p_1 < \alpha \leqslant \pi_1 + \pi_2 + \pi_{12}$, the sign of the slope depends on the quantile α: It is positive (meaning an increase of expected shortfall with x_1) if $\alpha < 2\,\pi_1 + \pi_{12}$. Hence, for $\alpha < 2\,\pi_1 + \pi_{12}$, the optimal choice is $x_1^* = 0.5$, while for $\alpha > 2\,\pi_1 + \pi_{12}$, it is $x_1^* = 1$. If exactly $\alpha = 2\,\pi_1 + \pi_{12}$ holds, the decision maker is indifferent between all choices of x_1 with $0.5 \leqslant x_1 \leqslant 1$.

- If $\alpha > \pi_1 + \pi_2 + \pi_{12}$, the expected shortfall is linearly decreasing with x_1, so the optimal choice is $x_1^* = 1$. $\square$

If the target of the decision maker is a minimization of expected shortfall, the risk-minimal allocation is always a corner solution—either a fifty-fifty allocation between the two banks or a total allocation to the bank with the lower default probability. Which solution is optimal depends on the quantile α, with the cut-off point being $\alpha = 2\,\pi_1 + \pi_{12}$. This has a severe consequence:

Corollary 1. *For $n = 2$ and $p_1 \neq p_2$, the optimization problem in the basic setup is ill-posed, as its solution does not depend continuously on the input parameters α, π_1, and π_{12}.* □

Continuity is the third Hadamard condition on the well-posedness of a problem, which is violated here. In a similar context, Alexander et al. (2006) have found that also the minimization of the expected shortfall for a portfolio of derivative contracts can be ill-posed.

Figure 1 graphically shows the dependency of the expected shortfall on the weight x_1 for different quantiles α, based on an exemplary situation with $p_1 = 1\%$, $p_2 = 2\%$, and $\pi_{12} = 0.1\%$. For very low quantiles $\alpha \leqslant \pi_{12}$, the α worst scenarios always consist of a total loss, so the expected shortfall equals 1, independent of the allocation. For $\pi_{12} \leqslant \alpha \leqslant p_1$, the allocations with $x_1 = 0$ and $x_1 = 1$ imply a total loss in the worst α scenarios, whereas for other allocations the expected shortfall decreases linearly in x_1 to the minimum value at $x_1 = 0.5$, before increasing linearly again to a value of 1 at $x_1 = 1$. For $p_1 < \alpha \leqslant 2\pi_1 + \pi_{12}$, the slope of the increasing leg for $x_1 \geqslant 0.5$ becomes smaller and reaches 0. For $\alpha > 2\pi_1 + \pi_{12}$, also the slope for $x_1 \geqslant 0.5$ becomes negative, so the optimum is reached at $x_1 = 1$. For $\alpha \geqslant \pi_1 + \pi_2 + \pi_{12}$, the α worst scenarios cover all potential loss scenarios, so the expected shortfall is simply a scaled expected loss and is thus linear in x_1 over the whole interval $[0; 1]$.

[Insert Figure 1 about here]

2.2.2 Portfolio Selection and Efficient Frontiers

In the 2-banks case, Figure 1 is transferred directly into a graph of efficient frontiers in a μ-ES-space, as the expected portfolio value μ is a linear function of the weight x_1. If we first assume that the payoff rates r_i of all banks are identical and without loss of generalization zero, the expected portfolio value equals

$$\mu = x_1 \cdot (1 - p_1) + (1 - x_1) \cdot (1 - p_2). \tag{8}$$

Figure 2 (a) shows the corresponding portfolios in a μ-ES-space. For ES quantiles above $2\pi_1 + \pi_{12}$ (here, 1.9%), only the single portfolio with $x_1 = 1$ is efficient, as μ decreases and ES increases when the second counterparty enters the portfolio. For quantiles below this cut-off point, the minimum expected shortfall portfolio is reached for $x_1 = 0.5$, so the efficient line consists of portfolios with $0.5 \leqslant x_1 \leqslant 1$ (indicated in bold in Figure 2).

[Insert Figure 2 about here]

With identical payoff rates, Bank 1 dominates Bank 2, as the expected portfolio value is larger while the expected shortfall is smaller. The situation changes when there is a compensation for default risk—for example, Bank 2 might offer more favorable conditions than Bank 1. Figure 2 (b) shows the portfolios in a μ-ES-space if Bank 1 pays $r_1 = 2\%$ interest on the exposure, while Bank 2 pays $r_2 = 4\%$. In this situation (generally speaking, if the difference in payoff rates, $r_2 - r_1$, is larger than the difference in default probabilities, $p_2 - p_1$), the maximum expected portfolio value is achieved with $x_1 = 0$. Accordingly, all portfolios are efficient if the ES quantile exceeds the cut-off point $2\pi_1 + \pi_{12}$, while for lower quantiles, portfolios are efficient with $0 \leqslant x_1 \leqslant 0.5$.

Brandtner (2013) discusses the optimal choice of a portfolio from the efficient frontier in such a situation. (Actually, our setup with discrete loss distributions can be seen as a special case of his state-space approach.) If the decision maker applies a hybrid μ-ES functional to be maximized (as introduced by Acerbi and Simonetti (2002), a convex combination $(1-\lambda)\,\mu - \lambda\, ES$, analogously to the preference function $\mu - \lambda\, \sigma^2$ in classical μ-σ analysis), then according to the linearity of the functional and the piecewise linear shape of the efficient frontier, only a corner solution can maximize the preference functional.

Instead of maximizing a preference function (whether a hybrid μ-ES functional or something different), we will focus on risk minimization in the following (which is actually a special case with $\lambda = 1$). In (equity) portfolio theory, there is growing attention for the minimum variance portfolio, which has proven superior to other optimized portfolios because of estimation and calibration issues (e.g., Ledoit and Wolf, 2003, and Jagannathan and Ma, 2003). Similarly, for the allocation problem discussed in this paper, differences in expected returns for different allocation are usually small. Especially when the offered returns of the different banks exactly offset their default probabilities, any allocation leads to the same expected portfolio return. Thus, the allocation method can concentrate on minimizing risk. For the illustration of efficient frontiers, we therefore rely on the simple assumption of zero payoff rates r_i in the following, while the key task is finding the risk-minimal portfolio.

2.3 Multiple Counterparts

2.3.1 Minimizing Expected Shortfall and Corner Points

In the general case with $n \geqslant 2$ counterparties, the discrete loss distribution is given by

$$L = \begin{cases} l_1 = 1 & \text{with probability } \pi_{I_1:=\{1,\ldots,n\}}, \\ l_2 = \sum_{i \in I_2} x_i & \text{with probability } \pi_{I_2}, \\ l_3 = \sum_{i \in I_3} x_i & \text{with probability } \pi_{I_3}, \\ \vdots & \\ l_{2^n} = 0 & \text{with probability } \pi_{\varnothing}, \end{cases} \tag{9}$$

where each I_j is a subset of $\{1, \ldots, n\}$ and $I_1, I_2, \ldots$ represent an ordering so that $\sum_{i \in I_j} x_i \geqslant \sum_{i \in I_k} x_i$ for $j < k$.

For a quantile α, the expected shortfall is calculated by means of Equation (5):

$$\alpha \cdot ES_\alpha = \sum_{j=1}^{j(\alpha)} \left(\pi_{I_j} \cdot \sum_{k \in I_j} x_k \right) + \left(\alpha - \sum_{j=1}^{j(\alpha)} \pi_{I_j} \right) \cdot \sum_{k \in I_{j(\alpha)+1}} x_k. \tag{10}$$

Obviously, the expected shortfall is locally linear in all of the weights x_i—as long as the ordering of the index sets is maintained. Accordingly, (if the expected shortfall is not locally constant) a minimum value for the risk measure can only be achieved at a point $(x_1, \ldots, x_n)$ where the ordering of the index sets changes.[2]

Definition 3. *An allocation vector $(x_1, \ldots, x_n)$ is called a* corner point *if either $x_i = 1$ for one i or*

$$\forall \Delta x = (\Delta x_1, \ldots, \Delta x_n) \neq 0 \textit{ with } \sum_{i=1}^{n} \Delta x_i = 0$$
$$\exists I, J \subset \{1, \ldots, n\}, I \neq J, \textit{ so that } \sum_{i \in I} x_i = \sum_{j \in J} x_j \textit{ and } \sum_{i \in I} x_i + \Delta x_i \neq \sum_{j \in J} x_j + \Delta x_j. \tag{11}$$

[2]This argumentation has already been brought forward by Brandtner (2013) in his Proposition 3.3 1.

In other words, an allocation vector represents such a corner point and hence a candidate for an optimum, if any zero-sum variation Δx would change the order of at least one pair of subsets I, J.[3]

The local linearity of the expected shortfall yields

Proposition 2. *Let $\alpha > \pi_{I_1}$. An allocation vector which minimizes the expected shortfall in the basic setup is either a corner point or a convex combination of two corner points. In the latter case, all respective convex combinations minimize the expected shortfall.* □

It is important to note that Proposition 2 is not special to the expected shortfall. Acerbi (2002) introduced the broad class of *spectral risk measures* as a subset of coherent risk measures defined by Artzner et al. (1999). In Appendix A we show that for arbitrary spectral risk measures, risk-minimal allocations are only obtained in corner points.

For $n = 2$, as discussed in Section 2.2.1, the only two corner points are $(1, 0)$ and $(0.5, 0.5)$ (with the restriction $x_1 \geqslant x_2$). The result of Corollary 1 also holds in higher dimensions:

Corollary 2. *In general, the optimization problem in the basic setup is ill-posed, as its solution does not depend continuously on the input parameters.* □

A similar result has been found by Brandtner (2013), who argues that the optimal portfolio is discontinuous in the level of risk aversion. One might think of using some kind of average of different quantiles α to get a smooth solution. However, any weighted average of expected shortfall measures is again a spectral risk measure, as Adam et al. (2008) have shown. Thus, according to the extension of Proposition 2 to arbitrary spectral risk measures in Appendix A, a weighted risk measure cannot heal the ill-posedness of the problem.

As we know that only corner points represent risk-minimal allocations, it is interesting to see how they look like in higher dimensions. The following result gives us an iterative construction method to build corner points in ascending dimensions.

[3] These corner point must not be confused with the corner portfolios in classical Markowitz theory, as used in the critical lines algorithm of Markowitz (1956) (see also Sharpe, 1963). A corner portfolio is a point on the efficient μ-σ line, at which one security enters or leaves the set of securities on that line.

Proposition 3. *Let* $(x_1, \ldots, x_n)$ *be a corner point in dimension* n*. Let* $x_{n+1} = \sum_{i \in I^*} x_i$ *for an arbitrary subset* $I^* \subset \{1, \ldots, n\}$*. Then the vector*

$$\frac{(x_1, \ldots, x_n, x_{n+1})}{1 + x_{n+1}} \tag{12}$$

is a corner point in dimension $n + 1$*.*

Proof. Let $\Delta x^{(n+1)} = (\Delta x_1, \ldots, \Delta x_{n+1})$ with $\sum_{i=1}^{n+1} \Delta x_i = 0$ be an arbitrary zero-sum variation in dimension $n+1$. It has to be shown that there are two subsets $I, J \subset \{1, \ldots, n+1\}$ which fulfill the condition of Definition 3. If $\Delta_{n+1} = 0$, then $\Delta x^{(n)} = (\Delta x_1, \ldots, \Delta x_n)$ is a zero-sum variation in dimension n, so according to the prerequisite there are such subsets of $\{1, \ldots, n\}$ and thus also of $\{1, \ldots, n+1\}$. If $\Delta_{n+1} \neq 0$, the subsets I^* and $\{n+1\}$ do the job. □

It should be noted that this construction rule does not yield points with a descending ordering of the components. However, any permutation of the components also results in a corner point. Table 1 shows the corner points up to dimension 4 (modulo permutations).

[Insert Table 1 about here]

But while the list is exhaustive for $n \leqslant 4$, not all corner points can be constructed this way in higher dimensions, according to the following proposition.

Proposition 4. *For dimension* $n \geqslant 5$*, the number of corner points is infinite.*

Proof. For any real-valued $h \geqslant 0$,

$$\frac{(2+h, 2+h, 1+h, 1+h, 1)}{7+4h}$$

is a corner point. To see this, assume that is was not. Let $(\Delta x_1, \ldots, \Delta x_5) \neq 0$. Then for all pairs of subsets I, J for which $\sum_{i \in I} x_i = \sum_{j \in J} x_j$ holds, also $\sum_{i \in I} x_i + \Delta x_i = \sum_{j \in J} x_j + \Delta x_j$ must hold. The pairs of subsets are $\{1\}$ and $\{2\}$, $\{3\}$ and $\{4\}$, $\{1\}$ and $\{3, 5\}$, $\{1\}$ and $\{4, 5\}$, $\{2\}$ and $\{3, 5\}$, and $\{2\}$ and $\{4, 5\}$. The condition $\sum_{i \in I} x_i + \Delta x_i = \sum_{j \in J} x_j + \Delta x_j$ for all these pairs leads to a system with 7 independent linear equations and 5 variables, which is not solvable.[4] So the assumption is wrong and the point must be a corner point. □

In higher dimensions, it is "more likely" to find a pair of subsets I, J according to Definition 3, so that any zero-sum variation changes the order of this pair in the loss distribution. The constructive proof of Proposition 4 gives one example of a (curved) line in the 5-dimensional unit hypercube[5] which only consists of corner points. Of course, there are many such lines. For $n \geqslant 6$, there will be (infinitely many) non-linear hyperplanes of corner points. A restriction is given by the following proposition:

Proposition 5. *Let $(x_1, \ldots, x_n)$ be a corner point with $x_1 \geqslant x_i \ \forall i$. Then either $x_1 = 1$ or $x_1 \leqslant 0.5$.*

Proof. If $1 > x_1 > 0.5$, a variation

$$\Delta x_1 = -\sum_{i=2}^{n} \Delta x_i; \quad \Delta x_i = \epsilon \cdot x_i \text{ for } i \geqslant 2 \tag{13}$$

with ϵ sufficiently small so that $x_1 + \Delta x_1 > 0.5$ leaves the ordering of subsets unchanged, so the point cannot be a corner point. □

According to this proposition, apart from the edges of the unit hypercube, all corner points or hyperplanes of corner points concentrate in the sub-hypercube $[0; 0.5]^n$. As an economic consequence, if a risk-minimal allocation does not consist of one single bank, then at most 50% of the total exposure is optimally allocated to one bank.

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[4]Writing the condition as $\sum_{i\in I}(x_i + \Delta x_i) - \sum_{j\in J}(x_j + \Delta x_j) = \sum_{i\in I} \Delta x_i - \sum_{j\in J} \Delta x_j = 0$ yields

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 1 & 0 & -1 & 0 & -1 \\ 1 & 0 & 0 & -1 & -1 \\ 0 & 1 & -1 & 0 & -1 \\ 0 & 1 & 0 & -1 & -1 \end{bmatrix} \begin{bmatrix} \Delta x_1 \\ \Delta x_2 \\ \Delta x_3 \\ \Delta x_4 \\ \Delta x_5 \end{bmatrix} = 0,$$

which is not solvable for $\Delta x \neq 0$, as the matrix has full rank.

[5]Actually, the restriction $\sum_{i=1}^{n} x_i = 1$ defines an $(n-1)$-dimensional hyperplane in the n-dimensional unit hypercube as the domain for allocation vectors.

2.3.2 Non-Monotonicity of the Risk-Minimal Allocation

While the corner points constitute the domain of possible risk-minimal allocations, it is yet unclear how the risk-minimal allocation behaves. In this section, we demonstrate that the risk-minimal allocation is a non-monotonic function of the quantile α of the expected shortfall.

Lemma 1. *In the basic setup, the expected shortfall function with respect to the quantile α decreases at a (negative) rate*

$$\frac{\partial ES_\alpha}{\partial \alpha} = \frac{l_{j(\alpha)+1} - ES_\alpha}{\alpha}. \tag{14}$$

Proof. For the function $g(\alpha) := \alpha \cdot ES_\alpha$, it follows immediately from (10) that $g'(\alpha) = \sum_{k \in I_{j(\alpha)+1}} x_k = l_{j(\alpha)+1}$. The product rule, $g'(\alpha) = ES_\alpha + \alpha \cdot ES'_\alpha$, yields the proposition. □

As $l_{j(\alpha)}$ takes only a discrete number of values in the discrete loss distribution, we have

Corollary 3. *In the basic setup, the derivative of the expected shortfall with respect to the quantile α is discontinuous at $\alpha = \sum_{i \in I_j} \pi_i$ for each j, if l_j is strictly larger than l_{j+1}. It jumps by the amount $(l_{j+1} - l_j)/\alpha$.* □

In the following, we take a look at an example with $n = 3$ counterparties, where $p_1 = 1\%$, $p_2 = 2\%$, and $p_3 = 3\%$. The dependency structure is given by joint default probabilities $\pi_{123} = 0.02\%$, $\pi_{12} = 0.08\%$, $\pi_{13} = 0.1\%$, $\pi_{23} = 0.2\%$.[6] Figure 3 (a) shows optimal weights in dependence of the quantile α.

[Insert Figure 3 about here]

At first glance, the behavior is quite counter-intuitive. With very small values of α, an equally-weighted allocation is optimal (in line with the bivariate case). Above a threshold at $\alpha = 0.26\%$, the third bank with the largest default probability drops out of the risk-minimal allocation, which now consists of 50% Bank 1 and 50% Bank 2. However, above $\alpha = 0.54\%$, Bank 3 shows up again and the risk-minimal allocation is identical to that

[6] This dependency structure is consistent with the one-factor model of Vasicek (1987) outlined in Section 2.3.3, with homogeneous asset correlations of 0.3.

with very small quantiles. Finally, when α exceeds a threshold of 2.58%, the risk-minimal allocation consists of Bank 1 only, also in line with the bivariate case.

The reason for this behavior becomes clearer with a look at efficient frontiers. Figure 3 (b) shows efficient frontiers for several values of the quantile α.[7] For very small values of α (right-most line), the frontier consists of two linear segments only, with the minimum at $\mu = 0.98$, corresponding to an equal allocation to all banks. The Min-ES allocations are indicated by bullet points in Figure 3 (b). With increasing α, the frontier exhibits a kink at $\mu = 0.985$ (corresponding to the $(0.5, 0.5, 0)$ portfolio). As this kink becomes more pronounced, with some α (here at $\alpha = 0.26\%$), the efficient frontier contains a vertical segment, so the Min-ES allocation jumps to $(0.5, 0.5, 0)$ at $\mu = 0.985$. However, the shape of the efficient frontiers continues to vary with increasing α and reaches again a vertical segment at $\alpha = 0.54\%$. The Min-ES allocation jumps back to $(0.333, 0.333, 0.333)$ at $\mu = 0.98$. When α increases further, the upper part of the efficient frontier starts to change, and at $\alpha = 2.54\%$, there is a vertical segment between the $(0.333, 0.333, 0.333)$ and the $(1, 0, 0)$ portfolio, so the Min-ES allocation jumps to the latter at $\mu = 0.99$.

Ultimately, the shapes of the efficient frontier and thus the positioning of the Min-ES allocations can be traced back to the expected shortfall values of the corner portfolios $(1, 0, 0)$, $(0.5, 0.5, 0)$, and $(0.333, 0.333, 0.333)$.[8] These values, depending on the quantile α, are displayed in Figure 3 (c). Naturally, the expected shortfall decreases with increasing α, but this decrease takes places with different speeds for different portfolios. Furthermore, the derivatives of expected shortfall with respect to α are not continuous: Each line experiences a kink at different levels of α. Accordingly, there may be several intersections between the lines, so there are different intervals of α for wich different corner portfolios reach the comparably lowest expected shortfall. These intervals refer to the discontinuous and non-monotonic behavior of the optimal weights with respect to the quantile.

The example proves

Corollary 4. *In the basic setup, the risk-minimal allocation is generally a non-monotonic function of the quantile α.* □

[7] For ease of speaking, we call the whole line "efficient frontier", although of course allocations below the minimum-expected-shortfall portfolio are not efficient in a literal sense.

[8] In the example, the portfolio $(0.5, 0.25, 0.25)$ is never optimal.

2.3.3 Numerical Analysis for $n > 3$

The example of the previous subsection refers to $n = 3$, where only four different corner points exist. Proposition 4 tells us that for $n \geqslant 5$, there are infinitely many corner points. In this subsection, we numerically analyze two examples with $n = 5$ and $n = 10$ to see whether the non-monotonic behavior of the risk-minimal allocation also appears in higher dimensions. For that purpose, we apply the Vasicek (1987) one-factor model to characterize the dependency structure between the counterparties.[9] In a nutshell, the counterparties are represented by some credit variables C_i, which have a joint standard normal distribution. A counterparty defaults if $C_i < N^{-1}(p_i)$. The dependencies between defaults are modelled via dependencies between the credit variables, which are defined by a correlation structure. In the original one-factor approach of Vasicek (1987), the credit variables share a common factor M with standard normal distribution:

$$C_i = \sqrt{\rho}\, M + \sqrt{1-\rho}\, \epsilon_i, \tag{15}$$

where the ϵ_i are independent identical standard normally distributed. With a constant ρ, correlations between each pair of two counterparties are homogenous:

$$Corr(C_i, C_j) = Corr\left(\sqrt{\rho}\, M + \sqrt{1-\rho}\, \epsilon_i, \sqrt{\rho}\, M + \sqrt{1-\rho}\, \epsilon_j\right) = \rho. \tag{16}$$

Figure 4 shows optimal weights in dependence of the quantile α for universes of $n = 5$ and $n = 10$ banks within the Vasicek model with homogenous correlations of $\rho = 0.3$. The default probabilities are defined in ascending order with $p_j = j \cdot 1\%$.

[Insert Figure 4 about here]

For $n = 5$, a similar non-monotonic behavior of the weight functions as with $n = 3$ can be observed. Most pronounced is the jump when the quantile approaches $\approx 3\%$, where the risk-minimal allocation suddenly consists of Bank 1 only.

For $n = 10$, the weight functions remain non-monotonic, although this behavior is less pronounced, as the jumps from one corner solution to another one are rather small. (Note that the seeming "flickering" of the weight function represents the actual behavior of jumps between similar, yet different corner points and is not a result of numerical instability.)

[9] See Saunders et al. (2007) for an overview of factor models for credit risk optimization.

However, as in the $n = 5$ case, there is a sudden huge jump to the polar solution with $x_1 = 1$ when the quantile approaches $\approx 3\%$.

3 Continuous Loss Distributions

3.1 Two Counterparties

In this section, we introduce a more general loss distribution, which is no longer discrete. Instead of the total loss given default paradigm, the percentage loss given default is now uniformly distributed between 0 and 1. Thus, conditional on the default of only Bank j, the actual loss will be uniformly distributed between 0 and x_j:

$$P(L \leqslant l | \text{ only Bank } j \text{ defaults}) = \min\left\{\frac{l}{x_j}; 1\right\}. \tag{17}$$

The losses given default for different banks are assumed to be independent. We will refer to this setup as the "continuous setup" in the following.
In the 2-banks case, the loss given default of both banks, as a sum of two independent uniform distributions, follows a (symmetric) trapezoidal distribution with density (Kotz and van Dorp, 2004):

$$f_{Tr[0;1-x_1;x_1;1]}(x) = \begin{cases} \frac{x}{x_1\,(1-x_1)} & \text{for } 0 \leqslant x \leqslant 1 - x_1, \\ \frac{1}{x_1} & \text{for } 1 - x_1 \leqslant x \leqslant x_1, \\ \frac{1-x}{x_1\,(1-x_1)} & \text{for } x_1 \leqslant x \leqslant 1. \end{cases} \tag{18}$$

(Following our previous assumption that $p_1 \leqslant p_2$, $x_1 \geqslant 0.5$ and hence $1 - x_1 \leqslant x_1$.) Integrating this density gives the (conditional) probability function

$$P(L \leqslant l | \text{ both banks default}) = \begin{cases} \frac{l^2}{2\,x_1\,(1-x_1)} & \text{for } 0 \leqslant l \leqslant 1 - x_1, \\ \frac{1}{2} + \frac{2\,l-1}{2\,x_1} & \text{for } 1 - x_1 \leqslant l \leqslant x_1, \\ 1 - \frac{(1-l)^2}{2\,x_1\,(1-x_1)} & \text{for } x_1 \leqslant l \leqslant 1. \end{cases} \tag{19}$$

The total unconditional loss distribution is obtained by the law of total probability:

$$\begin{aligned} F_L(l) = P(L \leqslant l) = \pi_{12}\, & P(L \leqslant l | \text{ both banks default}) \\ & + \pi_1\, P(L \leqslant l | \text{ only Bank 1 defaults}) \\ & + \pi_2\, P(L \leqslant l | \text{ only Bank 2 defaults}) \end{aligned}$$

$$= \begin{cases} \pi_{12} \cdot \frac{l^2}{2\,x_1\,(1-x_1)} + \pi_1 \cdot \frac{l}{x_1} + \pi_2 \cdot \frac{l}{1-x_1} & \text{for } 0 \leqslant l \leqslant 1 - x_1, \\ \pi_{12} \cdot \left(\frac{1}{2} + \frac{2\,l-1}{2\,x_1}\right) + \pi_1 \cdot \frac{l}{x_1} + \pi_2 & \text{for } 1 - x_1 \leqslant l \leqslant x_1, \\ \pi_{12} \cdot \left(-\frac{(1-l)^2}{2\,x_1\,(1-x_1)}\right) + \pi_1 + \pi_2 & \text{for } x_1 \leqslant l \leqslant 1. \end{cases} \quad (20)$$

With this distribution function, the expected shortfall for a quantile α can be calculated by integrating the extreme losses with quantiles $1 - \alpha$ to 1 of the loss distribution (Acerbi and Tasche, 2002a):

$$ES_\alpha = \frac{1}{\alpha} \int_{1-\alpha}^{1} F_L^{-1}(q)\, dq, \quad (21)$$

where $F_L^{-1}(q) = \sup\{x \in \mathbb{R} : F_L(x) \leqslant q\}$ is the generalized inverse of F_L.[10]

In Appendix B, we derive an explicit solution for the expected shortfall as a function of the quantile α and the weight x_1. Figure 5 shows the resulting efficient frontiers in the μ-ES-space for several values of the quantile α for the same parameters as used for the discrete case in Section 2.2.1 ($\pi_{12} = 0.1\%, \pi_1 = 0.9\%, \pi_2 = 1.9\%$). In contrast to the discrete distribution, the efficient frontiers are now smooth. What is more, they reach their minimum no longer in a corner point, but at a value $x_1 \in [0.5; 1]$ which depends continuously and monotonically on the quantile α.

[Insert Figure 5 about here]

The following proposition summarizes the behavior of the 2-banks case.

Proposition 6. *In the continuous setup with $n = 2$ banks:*

1. *For very small quantiles $\alpha \leqslant \pi_{12}/2$, the risk-minimal allocation is achieved with $x_1^* = 0.5$.*

2. *For quantiles with $\pi_{12}/2 < \alpha < \pi_{12} + \pi_1 + \pi_2$, the optimal weight x_1^* grows continuously and monotonically with the quantile α from 0.5 to 1.*[11]

3. *For very large quantiles $\alpha \geqslant \pi_{12} + \pi_1 + \pi_2$, the efficient frontier is a straight line with a corner minimum at $x_1^* = 1$.*

[10] Note that for $\alpha < \pi_{12}+\pi_1+\pi_2$, the actual inverse $F_L^{-1}(\alpha)$ is well defined, whereas for $\alpha \geqslant \pi_{12}+\pi_1+\pi_2$, for the generalized inverse $F_L^{-1}(\alpha) = 0$ holds.

[11] The polar solution with $x_1 = 1$ is reached for a quantile $\alpha < \pi_{12} + \pi_1 + \pi_2$, as in the basic setup with the discrete loss distribution.

Proof. See Appendix C. □

The behavior of the risk-minimal allocation demonstrated in Figure 5 and Proposition 6 are well in line with economic intuition. This gives some hope that the implausible results of Section 2 are special to the discrete basic setup. However, as we will show in the next subsection, the property of monotonicity gets lost in the multivariate case of the continuous setup.

3.2 Multiple Counterparts

In general, the loss distribution conditional on a default of all banks in set I, $i_1, \ldots, i_{n_I}$, is obtained by a convolution of the respective single loss distributions, which are assumed to be uniformly distributed on $[0; x_{i_j}]$:

$$L|\text{exactly all banks in } I \text{ default} \sim U[0; x_{i_1}] * U[0; x_{i_2}] * \cdots * U[0; x_{i_{n_I}}] = \circledast_{i \in I} U[0; x_i]. \quad (22)$$

So the total loss distribution becomes

$$F_L(l) = P(L \leqslant l) = \sum_{I \subset \{1,\ldots,n\}} \pi_I \, F_{\circledast_{i \in I} U[0;x_i]}(l). \quad (23)$$

We demonstrate the behavior numerically for the case with $n = 3$, with the same parameters as in the basic setup with discrete loss distributions in Section 2.3.2. Figure 6 shows efficient frontiers and optimal weights in dependence of α.

[Insert Figure 6 about here]

While the continuity of the risk-minimal allocation dependent on the quantile carries over from the 2-banks case, monotonicity of the weights no longer holds. Similar to the multivariate discrete distribution, the weights in the risk-minimal allocation vary non-monotonically with the quantile α.

With regard to the convergence behavior, the results from the 2-banks case only hold under additional conditions. For small values of the quantile ($\alpha \to 0$), the risk-minimal allocation becomes equally weighted ($x_i^* = 1/n \; \forall i$) if the probability of a joint default of *all* banks, $\pi_{\{1,\ldots,n\}}$, it not zero. This is however only a sufficient, not a necessary condition. As furthermore for larger sizes n of the counterparty portfolio joint probabilities become extremely tiny, the convergence might take place for extremely small values of α, which are practically irrelevant.

For large values of the quantile ($\alpha \to 1$), the risk-minimal allocation becomes the corner solution ($x_1^* = 1$) if the sum of all default probabilities, $\sum_{i=1}^{n} p_i$, is lower than 1. This is also only a sufficient, not a necessary condition, which might nonetheless be violated when the number of counterparties becomes large. Analogously to small values of α, the convergence might take place for large values of α, beyond 10%, which are not used in practice.

We therefore refrain from formulating stricter convergence conditions, but record

Proposition 7. *In the continuous setup with $n > 2$ banks, the risk-minimal weight vector $(x_1^*, \ldots, x_n^*)$ is a continuous, but not necessarily monotonic function of the quantile α.*

Proof. The possible non-monotonicity is demonstrated in Figure 6.

Due to the convexity of expected shortfall (Föllmer and Schied, 2002), the efficient frontier for a given quantile is convex, so there are no separate local minima. We need to rule out that the efficient frontier has a vertical straight line segment, as possible in the discrete case: In these cases, any combination on the line segment is risk-minimal, and with a variation of the quantile, the minimum jumps from one end of the line segment to the other. However, the piecewise linearity stems from the fact that portfolios on the line segment are comonotonic in the discrete case. In the continuous setting, this is not possible, since the loss given default distributions are uncorrelated. So for any portfolios X, Y with $ES_\alpha(X) = ES_\alpha(Y)$,

$$ES_\alpha\left(\frac{X+Y}{2}\right) < \frac{ES_\alpha(X)}{2} + \frac{ES_\alpha(X)}{2} = ES_\alpha(X)$$

with a strict inequality. So the minimum expected shortfall is unique, and it is continuous in α. $\square$

The non-monotonicity of the risk-minimal allocation remains as a severe drawback also in the continuous setup. Although the allocation is now continuous in α, it is economically not very plausible why, dependent on the chosen quantile of the expected shortfall, an optimal allocation should move from an evenly weighted vector to an allocation $(50\%, 30\%, 20\%)$ and then back to a nearly-evenly weighted solution. Although not ill-posed in the Hadamard sense, minimizing the expected shortfall thus remains to be a formulation of the allocation problem which is hardly appropriate.

4 Conclusion

In this paper, we have shown a number of drawbacks when the expected shortfall is minimized in order to reach an optimal allocation of an exposure to a (small) number of counterparties. When loss given default is modelled as a discrete value, the optimization problem turns out to be ill-posed, as its solution neither depends continuously on the quantile used to calculate the expected shortfall nor on the default probabilities as input parameters. Instead, optimal allocations jump between particular corner points, and these jumps are not monotonic in the quantile α. (It happens that the optimal allocation jumps from point X_1 to X_2 at a certain α and then jumps back to X_1 at another value for α.) This behavior is most pronounced when the number of counterparties is small. We have shown that for $n \geqslant 5$ counterparties, there are infinitely many corner points, but still the optimal allocation is neither continuous nor monotonic.

When loss given default is modelled as a continuous distribution, also the risk-minimal allocation becomes continuous in α. While for $n = 2$, the risk-minimal allocation is also monotonic and thus in line with economic intuition, monotonicity is lost for $n \geqslant 3$. So the general behavior of the solution to the allocation problem depends on the modelling of the loss given default (discrete vs. continuous), but neither setup yields plausible results. This finding is not special to the expected shortfall but holds for the entire class of spectral risk measures.

So far, the paper can be considered "destructive", as it shows what a decision maker should *not* do. In this regard, the paper joins Brandtner and Kürsten (2015) in calling for attention when regulatory risk measures are used as a target function for decision making. Minimizing the expected shortfall or any other spectral risk measure is not necessarily a good idea when a portfolio with low "economic risk" is desired. The question is, what should be done instead? The tendency to corner solution can be traced back to the comonotic additivity of spectral risk measures in general and the expected shortfall in particular (Brandtner, 2013). So this property—meaningful in a regulatory context—is a source of implausibility in risk minimization. An alternative class of coherent risk measures which are not comonotic additive are expectiles, which have been recently proposed for the purpose of risk measurement (e.g., Bellini and Di Bernardino, 2017). However, coherent risk measures in general (to be more precise, all positive homogenous and translation-invariant risk measures) are not consistent with classical expected utility theory under risk aversion (e.g., Brandtner and Kürsten, 2015). Loosely spoken, a doubled risky position calls for a doubled capital requirement from a regulatory perspective, but it does not double (dis)utility. These results could induce a renaissance of the classical variance for risk minimization also in the context of highly-skewed credit portfolio distributions.

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Appendix

A Spectral Risk Measures and Corner Solutions

The theoretical results of Section 2.3.1 obtained for the expected shortfall are not special for this particular risk measure. Instead, they hold for the whole class of spectral risk measures, which have been introduced by Acerbi (2002):

Definition 4. *For a portfolio distribution X, a* spectral risk measure *is given by*

$$\rho(X) = -\int_0^1 F_X^{-1}(q)\,\phi(q)\,dq, \tag{24}$$

where ϕ, the so-called risk spectrum, is a non-increasing density function, and F^{-1} is the generalized inverse of the distribution F_X.

The main result of Proposition 2 also holds for arbitrary spectral risk measures:

Proposition 8. *Let $\alpha > \pi_{I_1}$. An allocation vector which minimizes a spectral risk measure in the basic setup is either a corner point or a convex combination of two corner points. In the latter case, all respective convex combinations minimize the risk measure.*

Proof. For a loss l, the portfolio value is $x = 1 - l$. Given the discrete loss distribution (9), the portfolio value for a quantile q is

$$F_X^{-1}(q) = \sum_{i\in I_j}(1-l_i) \quad \text{for} \sum_{k=1}^{j-1}\pi_{I_{j-1}} < q \leqslant \sum_{k=1}^{j}\pi_{I_j}, \tag{25}$$

given the ordering of the index sets with $\sum_{i\in I_j} l_i \geqslant \sum_{i\in I_k} l_i$ for $j < k$ as defined in Section 2.3.1. Accordingly,

$$\rho(X) = -\sum_j \int_{Q_{j-1}}^{Q_j} \sum_{i\in I_j}(1-l_i)\,\phi(q)\,dq = -\sum_j \sum_{i\in I_j} x_i \int_{Q_{j-1}}^{Q_j} \phi(q)\,dq, \tag{26}$$

with

$$Q_j = \sum_{k=1}^{j}\pi_{I_j}. \tag{27}$$

The risk measure is linear in all of the x_i, hence, as in the case of the expected shortfall discussed in Section 2.3.1, only corner solutions can minimize the value of ρ. $\square$

As Brandtner (2013) discusses in more detail, the reason for this result lies in the linearity of spectral risk measures. Portfolios with an identical ordering in the loss distribution (9) are comonotonic in the sense of Dhaene et al. (2002); accordingly, any spectral risk measure is linear for those portfolios.

B Expected Shortfall in the Continuous Setup with $n = 2$

Let $x_1 \geqslant 0.5$. The loss distribution is given by

$$L = \begin{cases} l_1 \sim Tr[0; 1-x_1; x_1; 1] & \text{with probability } \pi_{12}, \\ l_2 \sim U[0; x_1] & \text{with probability } \pi_1, \\ l_3 \sim U[0; 1-x_1] & \text{with probability } \pi_2, \\ l_4 = 0 & \text{with probability } 1-\pi_1-\pi_2-\pi_{12}, \end{cases} \tag{28}$$

where $U[a;b]$ denotes the uniform distribution between a and b and $Tr[0; 1-x_1; x_1; 1]$ is the trapezoidal distribution with density given in (18).

The expected shortfall is the conditional expectation in the α worst cases. First we can note that for very large quantiles $\alpha \geqslant \pi_{12} + \pi_1 + \pi_2$, all potential losses lie within the tail and the expected shortfall is simply a scaled expected loss:

$$ES_\alpha = \frac{EL}{\alpha} = \frac{\pi_{12} + \pi_1\, x_1 + \pi_2\,(1-x_1)}{2\,\alpha}. \tag{29}$$

Otherwise, the α worst cases are losses beyond a level v so that $P(L \geqslant v) = \alpha$. Actually, v is the value-at-risk with quantile α. (Note that the distribution is continuous.) If $v \leqslant 1 - x_1$, the conditional loss distribution is given by

$$L|L \geqslant v = \begin{cases} \tilde{l_1} \sim Tr_v[0; 1-x_1; x_1; 1] & \text{w. prob. } \pi_{12} \cdot \left(1 - F_{Tr[0;1-x_1;x_1;1]}(v)\right), \\ \tilde{l_2} \sim U[v; x_1] & \text{w. prob. } \pi_1 \cdot \left(1 - \frac{v}{x_1}\right), \\ \tilde{l_3} \sim U[v; 1-x_1] & \text{w. prob. } \pi_2 \cdot \left(1 - \frac{v}{1-x_1}\right). \end{cases} \tag{30}$$

Here, $Tr_v[0; 1-x_1; x_1; 1]$ denotes a trapezoidal distribution which is truncated at point v. $F_{Tr[0;1-x_1;x_1;1]}$ is the distribution function of the (original) trapezoidal distribution. For $v \leqslant 1 - x_1$,

$$F_{Tr[0;1-x_1;x_1;1]}(v) = \int_0^v \frac{x}{x_1\,(1-x_1)}\, dx = \frac{v^2}{2\,x_1\,(1-x_1)}. \tag{31}$$

The value-at-risk is given by

$$
\begin{aligned}
&\pi_{12}\left(1-\frac{v^2}{2\,x_1\,(1-x_1)}\right)+\pi_1\left(1-\frac{v}{x_1}\right)+\pi_2\left(1-\frac{v}{1-x_1}\right)=\alpha \\
\Leftrightarrow\ &\frac{\pi_{12}\,v^2}{2}+(\pi_1\,(1-x_1)+\pi_2\,x_1)v-x_1\,(1-x_1)\,(\pi_{12}+\pi_1+\pi_2-\alpha)=0 \\
\Leftrightarrow\ &v=a+\sqrt{a^2+b} \qquad (32)
\end{aligned}
$$

with

$$
a=-\frac{\pi_1\,(1-x_1)+\pi_2\,x_1}{\pi_{12}} \quad ; \quad b=\frac{2\,x_1\,(1-x_1)\,(\pi_{12}+\pi_1+\pi_2-\alpha)}{\pi_{12}}. \qquad (33)
$$

With this v, the expected shortfall can be calculated as

$$
\begin{aligned}
\alpha\cdot ES_\alpha &= \pi_{12}\cdot(1-F_{Tr[0;1-x_1;x_1;1]}(v))\cdot E[\tilde{l_1}] \\
&\quad+\pi_1\cdot\left(1-\frac{v}{x_1}\right)\cdot E[\tilde{l_2}]+\pi_2\cdot\left(1-\frac{v}{1-x_1}\right)\cdot E[\tilde{l_3}] \\
&=\pi_{12}\cdot\int_v^1 x\cdot f_{Tr[0;1-x_1;x_1;1]}(x)\,dx \\
&\quad+\pi_1\cdot\left(1-\frac{v}{x_1}\right)\cdot\frac{x_1+v}{2}+\pi_2\cdot\left(1-\frac{v}{1-x_1}\right)\cdot\frac{1-x_1+v}{2} \\
&=\pi_{12}\cdot\left(\int_v^{1-x_1}\frac{x^2}{x_1\,(1-x_1)}dx+\int_{1-x_1}^{x_1}\frac{x}{x_1}dx+\int_{x_1}^{1}\frac{x\,(1-x)}{x_1\,(1-x_1)}dx\right) \\
&\quad+\frac{\pi_1}{2}\cdot\left(x_1-\frac{v^2}{x_1}\right)+\frac{\pi_2}{2}\cdot\left(1-x_1-\frac{v^2}{1-x_1}\right) \\
&=\pi_{12}\cdot\left(\frac{(1-x_1)^3-v^3}{3\,x_1\,(1-x_1)}+\frac{2\,x_1-1}{2\,x_1}+\frac{1-3\,x_1^2+2\,x_1^3}{6\,x_1\,(1-x_1)}\right) \\
&\quad+\frac{\pi_1}{2}\cdot\left(x_1-\frac{v^2}{x_1}\right)+\frac{\pi_2}{2}\cdot\left(1-x_1-\frac{v^2}{1-x_1}\right) \\
&=\pi_{12}\cdot\left(\frac{1}{2}-\frac{v^3}{3\,x_1\,(1-x_1)}\right)+\frac{\pi_1}{2}\cdot\left(x_1-\frac{v^2}{x_1}\right)+\frac{\pi_2}{2}\cdot\left(1-x_1-\frac{v^2}{1-x_1}\right). \qquad (34)
\end{aligned}
$$

However, this is only true if $v \leqslant 1-x_1$. If $v > 1-x_1$, there is no sole default of Bank 2 within the α worst scenarios (as this would induce a maximum loss of $1-x_1$). So if $1-x_1 < v \leqslant x_1$,

$$
L|L\geqslant v=\begin{cases}\tilde{l_1}\sim Tr_v[0;1-x_1;x_1;1] & \text{w. prob. } \pi_{12}\cdot\left(1-F_{Tr[0;1-x_1;x_1;1]}(v)\right), \\ \tilde{l_2}\sim U[v;x_1] & \text{w. prob. } \pi_1\cdot\left(1-\frac{v}{x_1}\right).\end{cases} \qquad (35)
$$

For $v \leqslant x_1$,

$$
F_{Tr[0;1-x_1;x_1;1]}(v)=\int_0^{1-x_1}\frac{x}{x_1\,(1-x_1)}\,dx+\int_{1-x_1}^{v}\frac{1}{x_1}\,dx
$$

$$
\begin{aligned}
&= \frac{1-x_1}{2\,x_1} + \frac{v-(1-x_1)}{x_1} \\
&= \frac{1}{2} + \frac{2\,v-1}{2\,x_1}.
\end{aligned} \tag{36}
$$

Now, the value-at-risk is given by[12]

$$
\begin{aligned}
\pi_{12}\left(\frac{1}{2} - \frac{2\,v-1}{2\,x_1}\right) + \pi_1\left(1 - \frac{v}{x_1}\right) &= \alpha \\
\Leftrightarrow v &= \frac{x_1\,(\pi_{12}+2\,\pi_1-2\,\alpha)+\pi_{12}}{2\,(\pi_{12}+\pi_1)}.
\end{aligned} \tag{37}
$$

The expected shortfall can be calculated as

$$
\begin{aligned}
\alpha \cdot ES_\alpha &= \pi_{12}\cdot(1-F_{Tr[0;1-x_1;x_1;1]}(v))\cdot E[\tilde{l_1}] + \pi_1\cdot\left(1-\frac{v}{x_1}\right)\cdot E[\tilde{l_2}] \\
&= \pi_{12}\cdot\left(\int_v^{x_1}\frac{x}{x_1}dx + \int_{x_1}^1 \frac{x\,(1-x)}{x_1\,(1-x_1)}\,dx\right) + \frac{\pi_1}{2}\cdot\left(x_1 - \frac{v^2}{x_1}\right) \\
&= \pi_{12}\cdot\left(\frac{x_1^2-v^2}{2\,x_1} + \frac{1-3\,x_1^2+2\,x_1^3}{6\,x_1\,(1-x_1)}\right) + \frac{\pi_1}{2}\cdot\left(x_1-\frac{v^2}{x_1}\right) \\
&= \pi_{12}\cdot\left(\frac{1-x_1^3}{6\,x_1\,(1-x_1)} - \frac{v^2}{2\,x_1}\right) + \frac{\pi_1}{2}\cdot\left(x_1-\frac{v^2}{x_1}\right) \\
&= \frac{\pi_{12}}{6}\cdot\left(1+x_1+\frac{1-3\,v^2}{x_1}\right) + \frac{\pi_1}{2}\cdot\left(x_1-\frac{v^2}{x_1}\right).
\end{aligned} \tag{38}
$$

Finally, for $v > x_1$, within the α worst scenarios always both banks default. We have

$$
\begin{aligned}
F_{Tr[0;1-x_1;x_1;1]}(v) &= \int_0^{1-x_1}\frac{x}{x_1\,(1-x_1)}\,dx + \int_{1-x_1}^{x_1}\frac{1}{x_1}\,dx + \int_{x_1}^{v}\frac{1-x}{x_1\,(1-x_1)}\,dx \\
&= \frac{1-x_1}{2\,x_1} + \frac{x_1-(1-x_1)}{x_1} - \frac{(1-v)^2-(1-x_1)^2}{2\,x_1\,(1-x_1)} \\
&= 1 - \frac{(1-v)^2}{2\,x_1(1-x_1)}.
\end{aligned} \tag{39}
$$

In this case, the value-at-risk is given by

$$
\pi_{12}\left(1-\left(1-\frac{(1-v)^2}{2\,x_1(1-x_1)}\right)\right) = \alpha
$$

---

[12]The condition $v \in [1-x_1; x_1]$ is equivalent to

$$
v \leqslant x_1 \quad \Leftrightarrow \quad \alpha \geqslant \frac{\pi_{12}}{2}\left(\frac{1}{x_1}-1\right)
$$

and

$$
v \geqslant 1-x_1 \quad \Leftrightarrow \quad \alpha \leqslant \frac{3\,\pi_{12}+4\,\pi_1}{2} - \frac{\pi_{12}+2\,\pi_1}{2\,x_1}.
$$

$$\implies v = 1 - \sqrt{2\,x_1\,(1-x_1)\,\frac{\alpha}{\pi_{12}}}. \tag{40}$$

The expected shortfall can be calculated as

$$\begin{aligned}
\alpha \cdot ES_\alpha &= \pi_{12} \cdot \Big(1 - F_{Tr[0;1-x_1;x_1;1]}(v)\Big) \cdot E[\tilde{l_1}] \qquad (41)\\
&= \pi_{12} \cdot \int_v^1 \frac{x\,(1-x)}{x_1\,(1-x_1)}\,dx\\
&= \pi_{12} \cdot \frac{1 - 3\,v^2 + 2\,v^3}{6\,x_1\,(1-x_1)}.
\end{aligned}$$

C Proof of Proposition 6

(*) We first show that for any given α, a choice of x_1 with $1 - x_1 \leqslant v_\alpha(x_1) \leqslant x_1$ never minimizes the expected shortfall, where

$$v_\alpha(x_1) = \frac{x_1\,(\pi_{12} + 2\,\pi_1 - 2\,\alpha) + \pi_{12}}{2\,(\pi_{12} + \pi_1)}$$

is the value-at-risk according to (37). To see this, we calculate the derivative of $\alpha\,ES_\alpha$ given by (38) with respect to x_1:

$$\begin{aligned}
\frac{\partial \alpha\,ES_\alpha(x_1)}{\partial x_1} &= \frac{\partial}{\partial x_1}\left[\frac{\pi_{12}}{6}\left(1 + x_1 + \frac{1 - 3\,v_\alpha^2(x_1)}{x_1}\right) + \frac{\pi_1}{2}\left(x_1 - \frac{v_\alpha^2(x_1)}{x_1}\right)\right]\\
&= \frac{\pi_{12}}{6}\left[1 - \frac{6\,v_\alpha(x_1)\,v'(x_1)}{x_1} - \frac{1 - 3\,v_\alpha^2(x_1)}{x_1^2}\right] + \frac{\pi_1}{2}\left[1 - \frac{2\,v_\alpha(x_1)\,v'(x_1)}{x_1} + \frac{v_\alpha^2(x_1)}{x_1^2}\right]\\
&= \frac{\pi_{12}}{6}\left[1 - \frac{3\,v_\alpha(x_1)\,(\pi_{12} + 2\,\pi_1 - 2\,\alpha)}{x_1\,(\pi_{12} + \pi_1)} - \frac{1 - 3\,v_\alpha^2(x_1)}{x_1^2}\right]\\
&\quad + \frac{\pi_1}{2}\left[1 - \frac{v_\alpha(x_1)\,(\pi_{12} + 2\,\pi_1 - 2\,\alpha)}{x_1\,(\pi_{12} + \pi_1)} + \frac{v_\alpha^2(x_1)}{x_1^2}\right]\\
&= \frac{\pi_{12}}{6} + \frac{\pi_1}{2} - \frac{v_\alpha(x_1)\,(\pi_{12} + 2\,\pi_1 - 2\,\alpha)}{2\,x_1} + \frac{3\,v_\alpha^2(x_1)\,(\pi_1 + \pi_{12}) - \pi_{12}}{6\,x_1^2}\\
&= \frac{\pi_{12}}{6} + \frac{\pi_1}{2} - \frac{(\pi_{12} + 2\,\pi_1 - 2\,\alpha)^2}{4\,(\pi_1 + \pi_{12})} - \frac{\pi_{12}\,(\pi_{12} + 2\,\pi_1 - 2\,\alpha)}{4\,x_1\,(\pi_1 + \pi_{12})}\\
&\quad + \frac{(x_1\,(\pi_{12} + 2\,\pi_1 - 2\,\alpha) + \pi_{12})^2}{8\,(\pi_{12} + \pi_1)\,x_1^2} - \frac{\pi_{12}}{6\,x_1^2}\\
&= \frac{\big(4\,(\pi_{12} + 3\,\pi_1)\,(\pi_{12} + \pi_1) - 3\,(\pi_{12} + 2\,\pi_1 - 2\,\alpha)^2\big)\,x_1^2 + 3\,\pi_{12}^2 - 4\,\pi_{12}\,(\pi_{12} + \pi_1)}{24\,(\pi_{12} + \pi_1)\,x_1^2}.
\end{aligned}$$

For this derivative to become zero,

$$x_1^2 = \frac{4\,\pi_{12}\,(\pi_{12} + \pi_1) - 3\,\pi_{12}^2}{4\,(\pi_{12} + 3\,\pi_1)\,(\pi_{12} + \pi_1) - 3\,(\pi_{12} + 2\,\pi_1 - 2\,\alpha)^2} = \frac{\pi_{12}^2 + 4\,\pi_{12}\,\pi_1}{\pi_{12}^2 + 4\,\pi_{12}\,\pi_1 + 12\,\alpha\,(\pi_{12} + 2\,\pi_1 - \alpha)} \tag{42}$$

must hold. However, this x_1 violates the condition $1 - x_1 < v_\alpha(x_1) < x_1$: The first part, $1 - x_1 < v_\alpha(x_1)$, is equivalent to

$$x_1 > \frac{\pi_{12}}{\pi_{12} + 2\,\alpha},$$

which is violated by (42) if $\alpha \leqslant \pi_{12}/2$. The second part, $v_\alpha(x_1) < x_1$, is equivalent to

$$x_1 > \frac{\pi_{12} + 2\,\pi_1}{3\,\pi_{12} + 4\,\pi_1 - 2\alpha},$$

which is violated by (42) if $\alpha \geqslant \pi_{12}/2$. These violations become evident by simple algebra.

(**) We now prove Proposition 6.1. Let $\alpha \leqslant \pi_{12}/2$. Then, for $x_1 \geqslant 0.5$,

$$\alpha \leqslant \frac{\pi_{12}}{2} + \left(\frac{\pi_{12} + 2\,\pi_1}{2 \cdot 0.5} - \frac{\pi_{12} + 2\,\pi_1}{2\,x_1}\right) = \frac{3\,\pi_{12} + 4\,\pi_1}{2} - \frac{\pi_{12} + 2\,\pi_1}{2\,x_1},$$

so $v_\alpha(x_1) \geqslant 1 - x_1$ according to Footnote 12. With (*), the minimum expected shortfall must thus be reached for $x_1 \leqslant v_\alpha(x_1)$. In this case,

$$\alpha\, ES_\alpha(x_1) = \frac{\pi_{12}}{6} \cdot \frac{1 - 3\,v_\alpha(x_1)^2 + 2\,v_\alpha(x_1)^3}{x_1\,(1 - x_1)}.$$

For any fixed v (independent of x_1),

$$\frac{\partial}{\partial x_1} \frac{1 - 3\,v^2 + 2\,v^3}{6\,x_1\,(1 - x_1)} = \frac{(1 - 3\,v^2 + 2\,v^3)\,(1 - 2\,x_1)}{x_1^2\,(1 - x_1)^2},$$

which becomes zero at $x_1 = 0.5$. As furthermore

$$\frac{\partial v_\alpha(x_1)}{\partial x_1} = \frac{\partial}{\partial x_1}\left(1 - \sqrt{2\,x_1\,(1 - x_1)\,\frac{\alpha}{\pi_{12}}}\right) = \sqrt{\frac{\alpha}{\pi_{12}}} \cdot \frac{2\,x_1 - 1}{\sqrt{x_1\,(1 - x_1)}} \geqslant 0$$

for $x_1 \geqslant 0.5$ and

$$\frac{\partial}{\partial v} \frac{1 - 3\,v^2 + 2\,v^3}{6\,x_1\,(1 - x_1)} = \frac{\pi_{12}\,v\,(v - 1)}{x_1\,(1 - x_1)} \leqslant 0$$

for $v \leqslant 1$, the minimum expected shortfall is necessarily achieved at $x_1 = 0.5$.

(***) We now prove the monotonicity claimed in Proposition 6.2. For $\alpha > \pi_{12}/2$, an optimal x_1^* must fulfill $1 - x_1^* > v_\alpha(x_1^*)$ according to (*). Obviously, the minimum value-at-risk, $v_\alpha(x_1^*)$, decreases with α: $\partial v_\alpha(x_1^*)/\partial\alpha \leqslant 0$. Furthermore, the expected shortfall increases with the value-at-risk. For the monotonicity of the optimal weight x_1^* with respect to the quantile α to hold, it is therefore sufficient to show that for a fixed level v, the optimal x_1^* which minimizes $\alpha\, ES_\alpha$ according to (34) decreases with v. To see this, note that

$$\frac{\partial}{\partial x_1}\,\pi_{12} \cdot \left(\frac{1}{2} - \frac{v^3}{3\,x_1\,(1 - x_1)}\right) + \frac{\pi_1}{2} \cdot \left(x_1 - \frac{v^2}{x_1}\right) + \frac{\pi_2}{2} \cdot \left(1 - x_1 - \frac{v^2}{1 - x_1}\right)$$

$$= \pi_{12} \frac{(1-2\,x_1)\,v^3}{x_1^2\,(1-x_1)^2} + \frac{\pi_1}{2} \cdot \left(1 + \frac{v^2}{x_1^2}\right) + \frac{\pi_2}{2} \cdot \left(1 + \frac{v^2}{(1-x_1)^2}\right)$$

$$=: \Phi(x_1, v).$$

The optimum weight x_1^* is implicitly defined by $\Phi(x_1^*(v), v) = 0$. According to the implicit function theorem,

$$\frac{\partial x_1^*}{\partial v} = -\left(\frac{\partial \Phi(x_1^*(v), v)}{\partial x_1^*}\right)^{-1} \cdot \frac{\partial \Phi(x_1^*(v), v)}{\partial v}.$$

We have

$$\frac{\partial \Phi(x_1^*(v), v)}{\partial v} = \pi_{12} \frac{(1-2\,x_1)\,v^2}{x_1^2\,(1-x_1^2)} + \pi_1 \frac{v}{x_1^2} - \pi_2 \frac{v}{(1-x_1)^2} \leqslant 0,$$

as $x_1 \geqslant 0.5$ and $\pi_1 \leqslant \pi_2$, and

$$\begin{aligned}
\frac{\partial \Phi(x_1^*(v), v)}{\partial x_1^*} &= \pi_{12}\,v^3 \frac{-\overbrace{2\,x_1^2\,(1-x_1)^2}^{\geqslant 0} - \overbrace{(1-2\,x_1)}^{\leqslant 0} \cdot \overbrace{(2\,x_1\,(1-x_1)^2 - 2\,x_1^2\,(1-x_1))}^{\leqslant 0}}{9\,x_1^4\,(1-x_1)^4} \\
&\quad \underbrace{-\pi_1 \frac{v^2}{x_1^3} - \pi_2 \frac{v^2}{(1-x_1)^3}}_{\leqslant 0} \\
&\leqslant 0.
\end{aligned}$$

It follows $\partial x_1^*/\partial v \leqslant 0$.

(****) The continuity stated in Proposition 6.2 follows from the proof of Proposition (7), which also holds in two dimensions.

(*****) Finally, for $\alpha \geqslant \pi_1 + \pi_2 + \pi_{12}$, the worst α scenarios always cover any possible loss, so the expected shortfall is merely a scaled expected loss:

$$\alpha\,ES = EL = \frac{\pi_1\,x_1 + \pi_2\,(1-x_1) + \pi_{12}}{2}.$$

This proves Proposition 6.3. $\square$

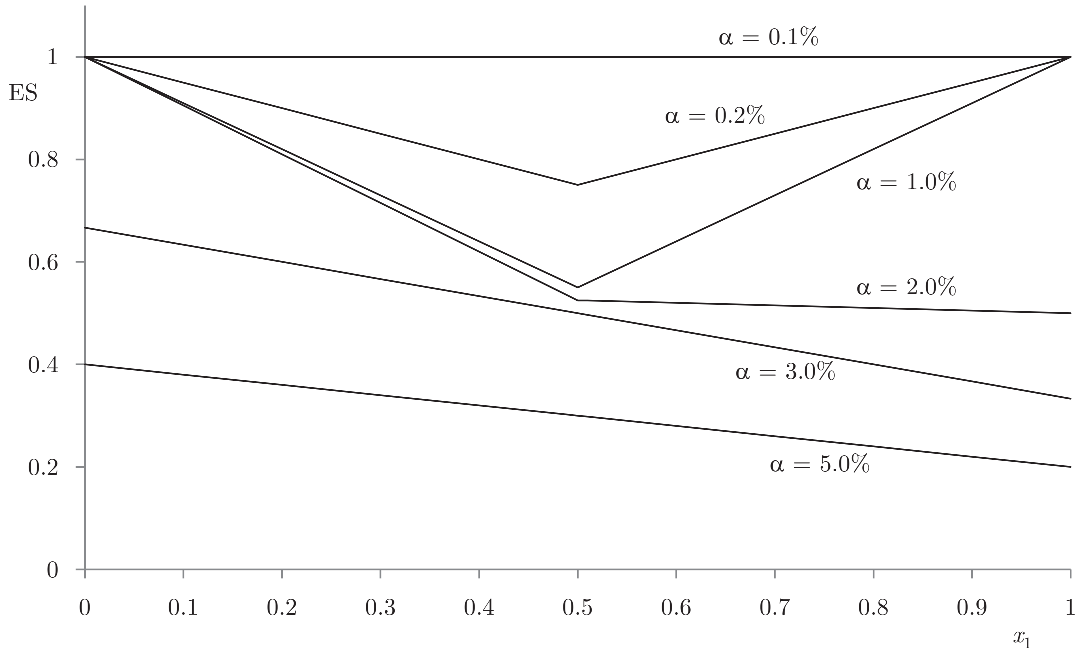
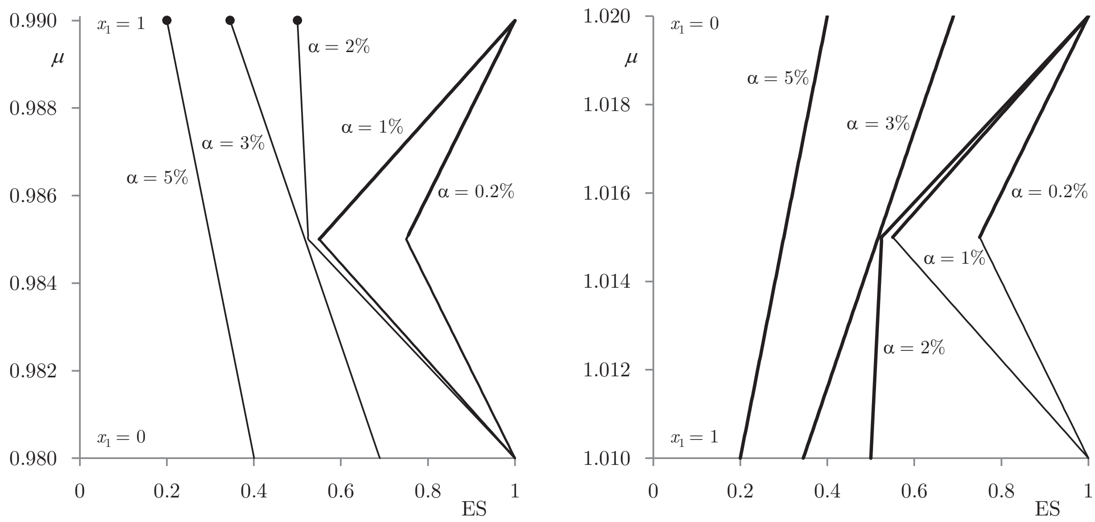
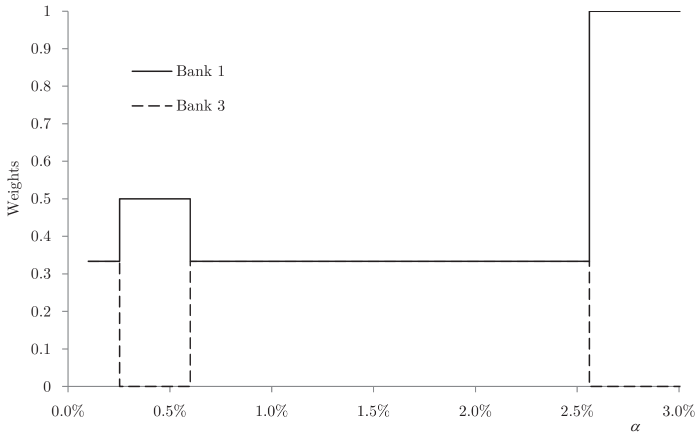


Figure 1. *Expected shortfall for counterparty risk with two counterparties for different values of the quantile* α*. The default probabilities are* $p_1 = 1\%$*,* $p_2 = 2\%$*, and* $\pi_{12} = 0.1\%$*. The* x*-axis represents the weight of the first counterparty. For quantiles up to 1.90%, expected shortfall reaches its minimum exactly at* $x_1 = 0.5$*. For larger quantiles, expected shortfall is monotonically decreasing, so the minimum is achieved for* $x_1 = 1$*.*

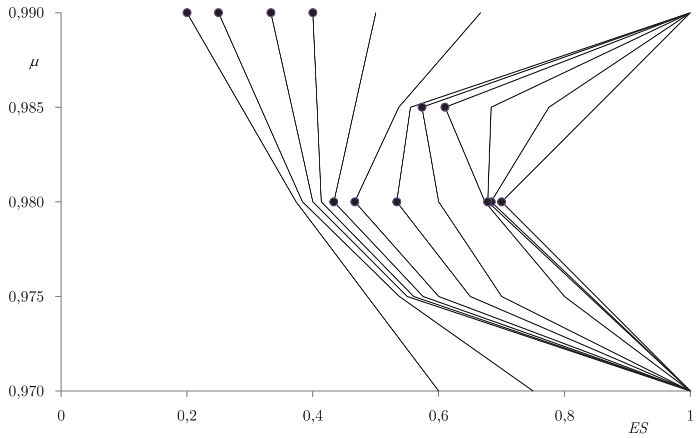


(a) *no compensation for default risk* **(b)** *compensation for default risk*

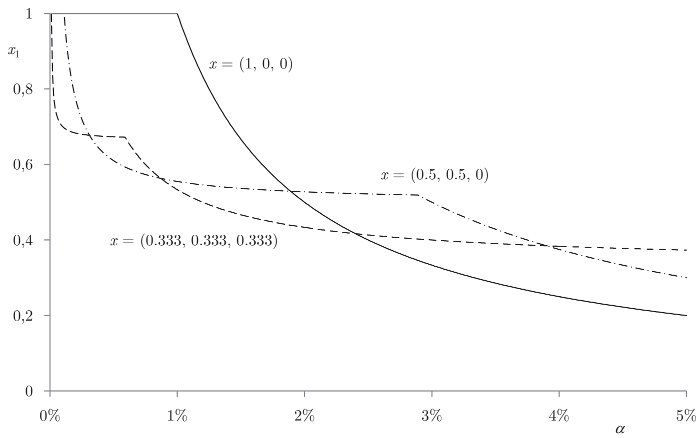
Figure 2. *Efficient frontiers in the μ-ES space for $n = 2$ counterparties. The default probabilities are (as in Figure 1) $p_1 = 1\%$, $p_2 = 2\%$, and $\pi_{12} = 0.1\%$. The left graph shows efficient frontiers when there is no compensation for default risk ($r_1 = r_2 = 0$). This graph is basically a transpose of Figure 1: For quantiles up to 1.90%, expected shortfall reaches its minimum at $x_1 = 0.5$, while both ES and μ increase for larger values of x_1, resulting in efficient portfolios (marked with bold lines). For larger quantiles, only the portfolio with $x_1 = 1$ is efficient (marked with a bold dot), as μ decreases but ES increases for smaller values of x_1. The right graph shows efficient frontiers when there is a compensation for default risk: The risky Bank 2 pays interest $r_2 = 4\pi_0$, while Bank 1 only pays $r_1 = 2\pi_0$. In this case, for larger quantiles, all portfolios are efficient, as with increasing x_1 expected shortfall decreases (as in the left graph), but now the expected return also decreases because of the lower interest rate paid by Bank 1.*



(a) *risk-minimal allocations*

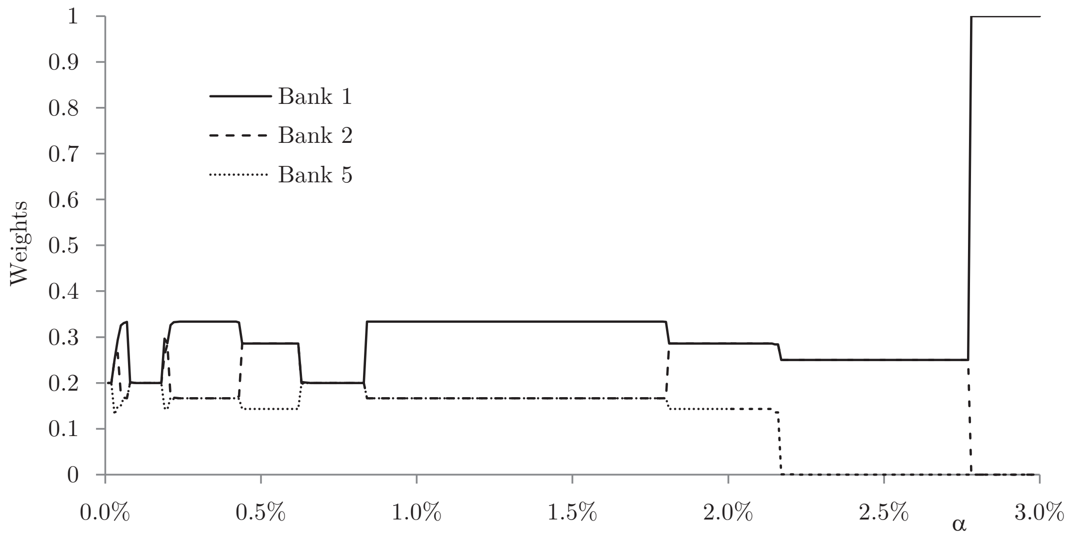


(b) *efficient frontiers*

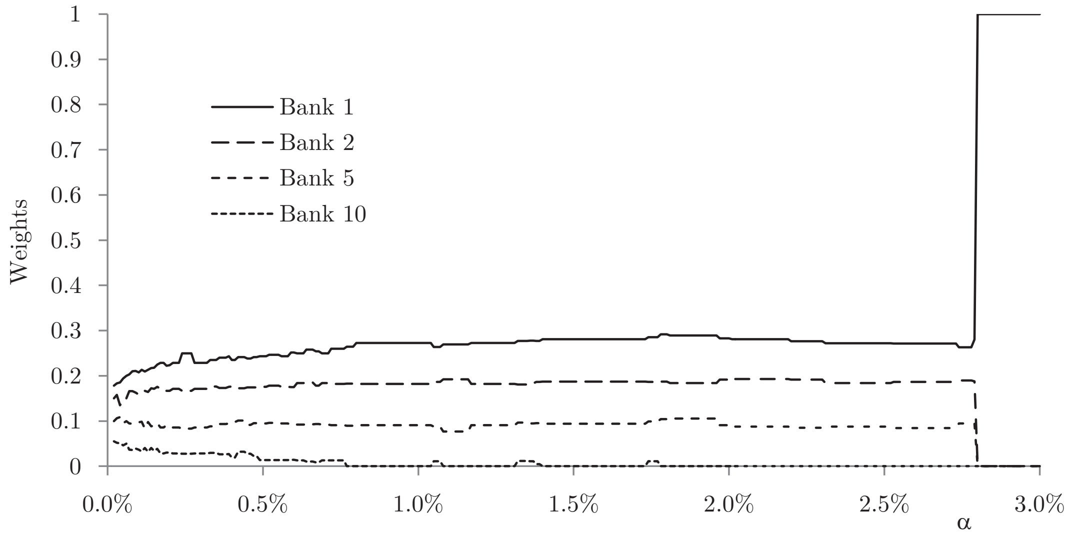


(c) *expected shortfall for corner portfolios*

Figure 3. *Illustration of the 3-banks case. Parameters are $p_1 = 1\%$, $p_2 = 2\%$, and $p_3 = 3\%$, $\pi_{123} = 0.02\%$, $\pi_{12} = 0.08\%$, $\pi_{13} = 0.1\%$, $\pi_{23} = 0.2\%$. Subfigure (a) shows the risk-minimal allocation with respect to the quantile α (Banks 1 and 3). Obviously, the allocation is non-monotonic. Subfigure (b) shows efficient frontiers in the μ-ES space for several values of α. Subfigure (c) shows the expected shortfall with respect to α for three corner portfolios, which are candidates for risk-minimal allocations.*



(a) $n = 5$



(b) $n = 10$

Figure 4. *Risk-minimal allocations in higher dimensions ($n = 5$ and $n = 10$) in dependence of the quantile α. The bank default probabilities are given by $p_j = j\%$. Dependencies between the banks are modelled with the Vasicek (1987) one-factor model with homogeneous correlations of 0.3. For both cases, a non-monotonic behavior of the risk-minimal allocation is observed.*

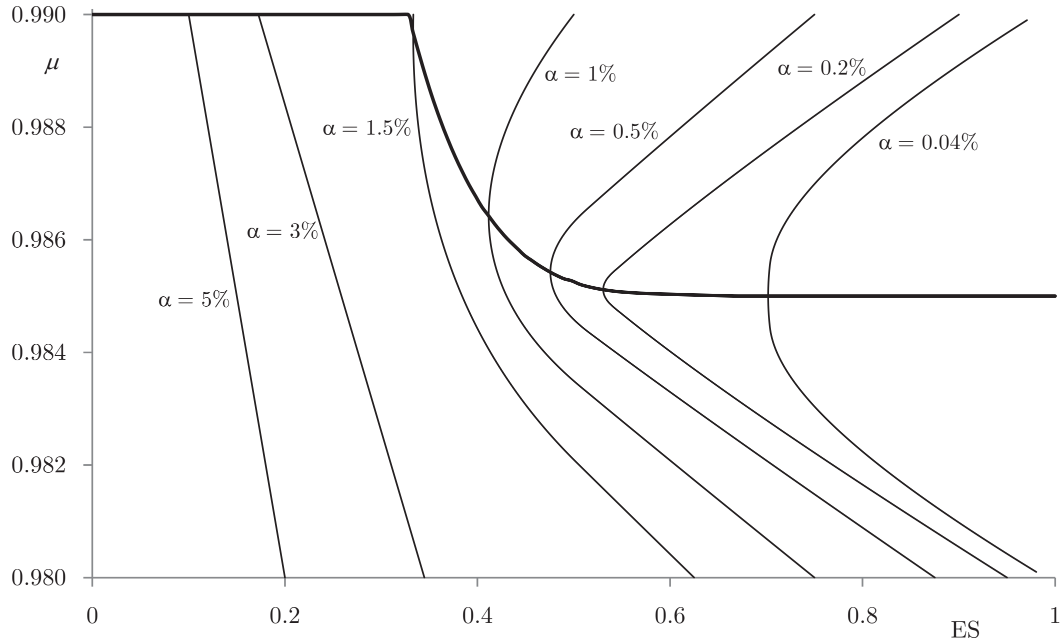
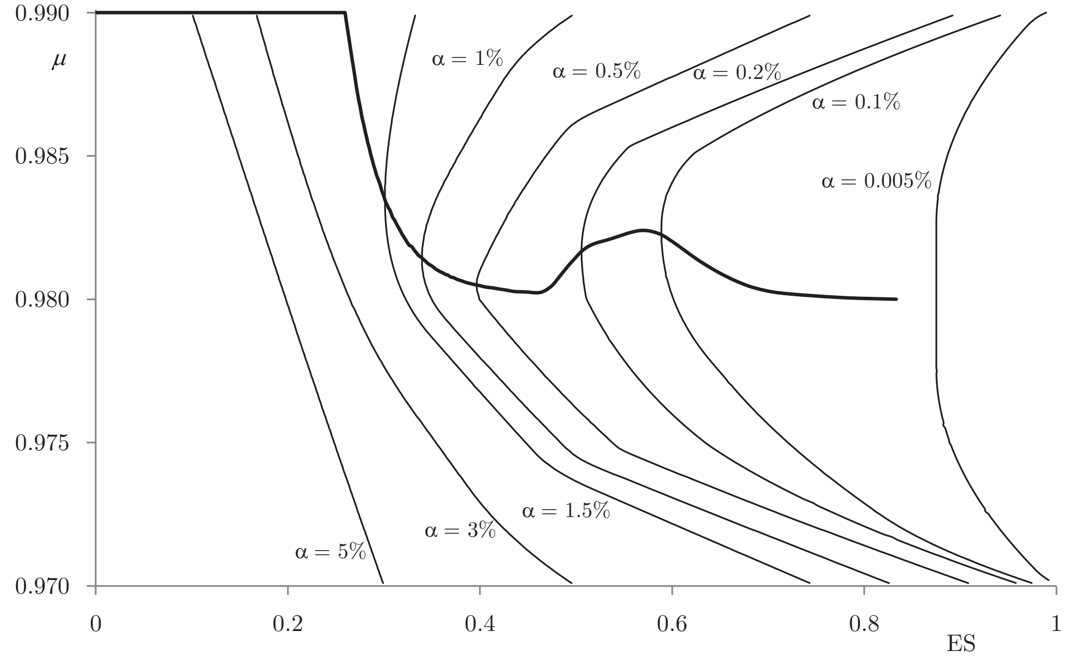
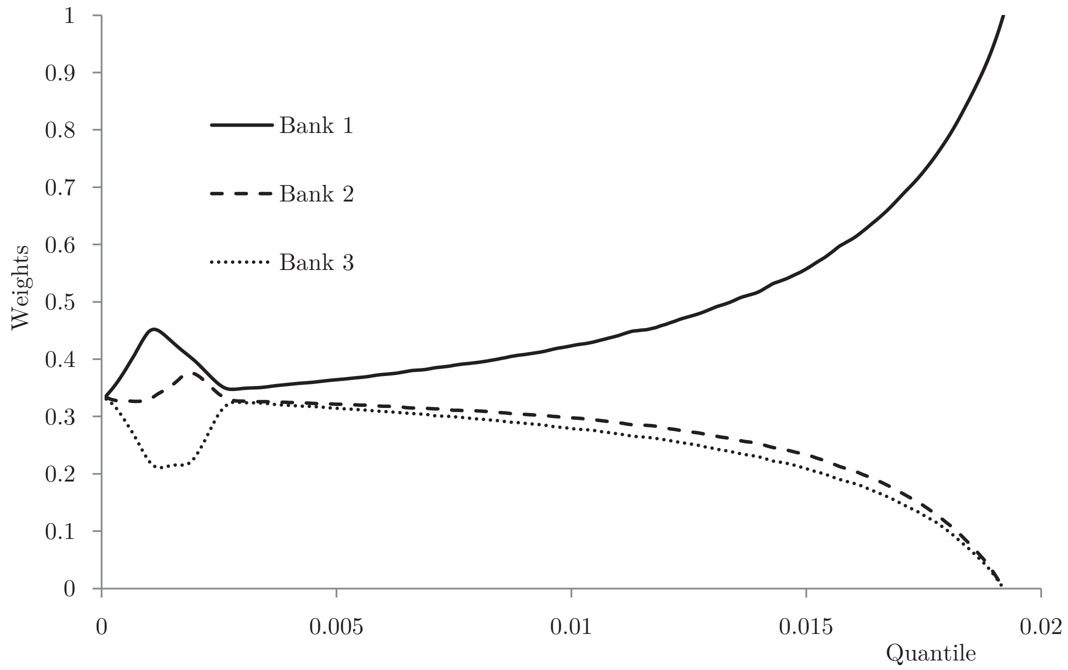


Figure 5. *Efficient frontiers and risk-minimal portfolios in the μ-ES space in the continuous setup for two counterparties. Parameters are $p_1 = 1\%$, $p_2 = 2\%$, and $\pi_{12} = 0.1\%$, with loss given defaults independent uniformly distributed between 0 and 1. The thin lines show efficient frontiers for several values of the quantile α. The bold line marks the location of the risk-minimal portfolio, with respect to α.*



(a) *efficient frontiers*



(b) *weights*

Figure 6. *Illustration of the continuous setup with $n = 3$ counterparties. Parameters are $p_1 = 1\%$, $p_2 = 2\%$, and $p_3 = 3\%$, $\pi_{123} = 0.02\%$, $\pi_{12} = 0.08\%$, $\pi_{13} = 0.1\%$, $\pi_{23} = 0.2\%$, with loss given defaults independent uniformly distributed between 0 and 1. Subfigure (a) shows efficient frontiers (thin lines) and risk-minimal portfolios (bold line) in the μ-ES space. Subfigure (b) shows the weights of the risk-minimal allocation in dependence of the quantile α.*

$n=2$			$n=3$				$n=4$			
$\frac{1}{2}$	$\frac{1}{2}$		$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$		$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$
							$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{2}{5}$
							$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{2}$
			$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{2}$		$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{3}$
							$\frac{1}{7}$	$\frac{1}{7}$	$\frac{2}{7}$	$\frac{3}{7}$
							$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$
0	1		0	$\frac{1}{2}$	$\frac{1}{2}$		0	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$
							0	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{2}$
			0	0	1		0	0	$\frac{1}{2}$	$\frac{1}{2}$
							0	0	0	1

Table 1. *Corner points in dimensions 2, 3, 4. The table illustrates the construction of corner points according to Proposition 2. For example, from* $(1/3, 1/3, 1/3)$ *in dimension 3 one can construct the points* $\frac{(1/3,1/3,1/3,1/3)}{4/3}$, $\frac{(1/3,1/3,1/3,2/3)}{5/3}$, *and* $\frac{(1/3,1/3,1/3,3/3)}{6/3}$ *in dimension 4. Note that this construction method yields only points with* $x_i \geqslant x_j$ *for* $i > j$*. Any point with permutated components of a corner point is also a corner point. With the restriction* $x_i \leqslant x_j$ *for* $i > j$*, the corner points suitable for a solution are actually the points from the table in reversed order of the components.*